Lund University

Center for Mathematical Sciences

FMAN95

Computer Vision

Assignment 1

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VT24

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2 Points in Homogeneous Coordinates

Exercise 1

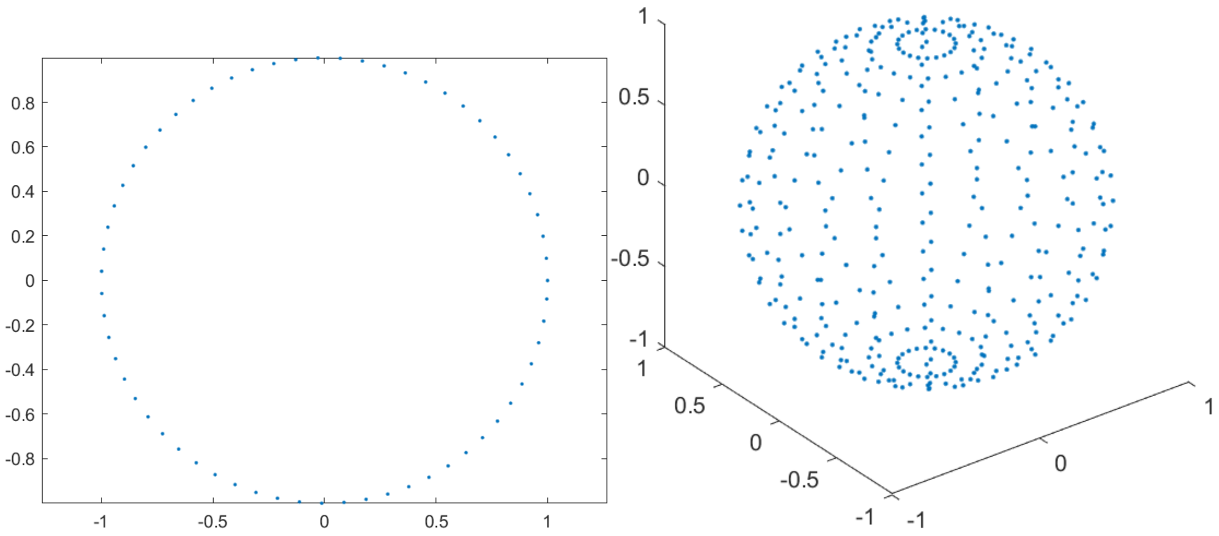
To get the 2D Cartesian coordinates, we divide by the third coordinate.

$$\hat{\mathbf{x}}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \hat{\mathbf{x}}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \quad \hat{\mathbf{x}}_3 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

For a point with 0 in the third coordinate, like in $\mathbf{x_4}$, we can not interpret the result by dividing with the third coordinate, since this one is 0.

To interpret $x_4 = \Big(4, -2, 0\Big)$ geometrically, we instead consider the vector $x_4' = \Big(4, -2, \varepsilon\Big)$, where ε is a small positive number. This point has a non-zero third coordinate and is equivalent to $\Big(\frac{4}{\varepsilon}, -\frac{2}{\varepsilon}, 1\Big)$, that is, it is a point where the x coordinate tends to positive infinity and the y coordinate tends to negative infinity. Making ε smaller we see that $\Big(4, -2, 0\Big)$ can be interpreted as a point infinitely far away in the direction $\Big(4, -2\Big)$. These points (with zero third coordinates) are either called **ideal points** (or **points of infinity**) or (**vanishing points**).

Computer Exercise 1



(a) x2D plot. (b) x3D plot.

Figure 1: Plots for Computer Exercise 1.

3 Lines

Exercise 2

Let's denote the intersection point for l_1 and l_2 (in $\mathbb{P}^2$) by x_{12}. Since x_{12} is on both lines we have

$$\begin{cases} l_1^\top x_{12} &= 0 \\ l_2^\top x_{12} &= 0 \end{cases} \Leftrightarrow \begin{cases} x+y+z &= 0 \\ 3x+2y+z &= 0 \end{cases} \Leftrightarrow \begin{cases} x &= s \\ y &= -2s \\ z &= s \end{cases}, \ s \in \mathbb{R}. \tag{1}$$

Therefore the intersection point is $x_{12} \sim \Big(1, -2, 1\Big)$ (which can be interpreted as $\Big(1, -2\Big)$ in $\mathbb{R}^2$).

Similarly, let's denote the intersection point for l_3 and l_4 (in $\mathbb{P}^2$) by x_{34}. Since x_{34} is on both lines we have

$$\begin{cases} l_3^\top x_{34} &= 0 \\ l_4^\top x_{34} &= 0 \end{cases} \Leftrightarrow \begin{cases} x+2y+3z &= 0 \\ x+2y+z &= 0 \end{cases} \Leftrightarrow \begin{cases} x &= -2s \\ y &= s \\ z &= 0 \end{cases}, \ s \in \mathbb{R}. \tag{2}$$

Here the intersection point becomes $x_{34} \sim \Big(-2, 1, 0\Big)$, however, since the third coordinate is 0 we can not divide by it. If we look at it in $\mathbb{R}^2$ we can see that l_3 and l_4 are parallel and have no intersection, but they have an intersection in images. Hence, $\mathbb{P}^2 =$ "$\mathbb{R}^2 \cup \mathrm{l}_\infty$", where $\mathbb{R}^2$ represents the points, and l_∞ represents directions.

Lastly, to compute a line going through two points we do the same operations since **points are dual to lines in** $\mathbb{P}^2$. Let's denote the line going through given x_1 and x_2 by l. The third parameter for both points is 1 when we switch to homogeneous coordinates. Since l passes through both points we get

$$\begin{cases} l^\top x_1 &= 0 \\ l^\top x_2 &= 0 \end{cases} \Leftrightarrow \begin{cases} a+b+c &= 0 \\ 3a+2b+c &= 0 \end{cases} \Leftrightarrow \begin{cases} a &= s \\ b &= -2s \\ c &= s \end{cases}, \ s \in \mathbb{R}. \tag{3}$$

Therefore the line is $l = \Big(1, -2, 1\Big)$.

Exercise 3

As we saw from the previous exercise the intersection point of l_1 and l_2 is $x_{12} = \begin{pmatrix} 1, -2, 1 \end{pmatrix}$ (in homogeneous coordinates). If we do matrix multiplication with M we can see that $Mx_{12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and the reason for this is that matrix $M = \begin{pmatrix} l_1^\top \\ l_2^\top \end{pmatrix}$, hence x_{12} is in null space of M, as it is the intersection of the lines that M consists of.

If we interpret the non-zero vectors in the nullspace of M as points in $\mathbb{P}^2$ (projective space), then besides the intersection point, there are no other distinct points formed by the vectors in the nullspace. Since the intersection point has a scaling factor $s \neq 0$ it does not matter what we input instead of s to get the intersection point in $\mathbb{P}^2$.

Computer Exercise 2

From the analysis performed on Figure 2 we can see that the lines going through image points appear parallel in 3D, but have an intersection in the image. The distance between the first line and the intersection point is $d = 8.1950$. It is not close to 0, since the three lines do not intersect at one point.

(a) compEx2 plot.

Figure 2: Plot for Computer Exercise 2.

Exercise 4

$$\mathbf{y}_1 \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{y}_2 \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

l_1 goes through points x_1 and x_2, we get

$$\begin{cases} l_1^\top x_1 & = 0 \\ l_1^\top x_2 & = 0 \end{cases} \Leftrightarrow \begin{cases} a + c & = 0 \\ b + c & = 0 \end{cases} \Leftrightarrow \begin{cases} a & = -s \\ b & = -s \\ c & = s \end{cases}, s \in \mathbb{R}. \tag{4}$$

Therefore $l_1 = \Big(-1, -1, 1\Big)$.
Similarly, for l_2 we get

$$\begin{cases} l_2^\top y_1 & = 0 \\ l_2^\top y_2 & = 0 \end{cases} \Leftrightarrow \begin{cases} a & = 0 \\ a + b + c & = 0 \end{cases} \Leftrightarrow \begin{cases} a & = 0 \\ b & = -s \\ c & = s \end{cases}, s \in \mathbb{R}. \tag{5}$$

Therefore $l_2 = \Big(0, -1, 1\Big)$.

$$(H^{-1})^T l_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}^T \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = l_2$$

As we can see $(H^{-1})^T l_1 = l_2$. That is because projective transformations preserve lines. Now let's prove that, for each line l_1, there is a corresponding line l_2 such that if x belongs to l_1 then the transformation $y \sim Hx$ belongs to l_2.
We know that $l_1^\top x = 0$. Then $l_1^\top H^{-1} H x = 0$, since $H^{-1}H = I$. We get

$$0 = l_1^\top x = l_1^\top H^{-1} H x = ((H^{-1})^T l_1)^T H x \sim l_2^\top y,$$

which means that $l_1^\top x = 0$ if and only if $l_2^\top y = 0$.

Computer Exercise 3

As we can see from the plots **(on page 6)**, H_1 transformation preserves distances and is a **Euclidean transformation**. H_2 preserves angles and is a **Similarity transformation**. Besides being projective, H_3 transformation has the special property that parallel lines are mapped to parallel lines. Hence it is an **Affine transformation**. And lastly, H_4 is a **General projective** transformation that preserves lines.
Another way to classify the transformations is to look at the transformation matrices.

$$H_1 = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}, H_2 = \begin{pmatrix} sR & t \\ 0 & 1 \end{pmatrix}, H_3 = \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix},$$

And H_4 is a general projective.

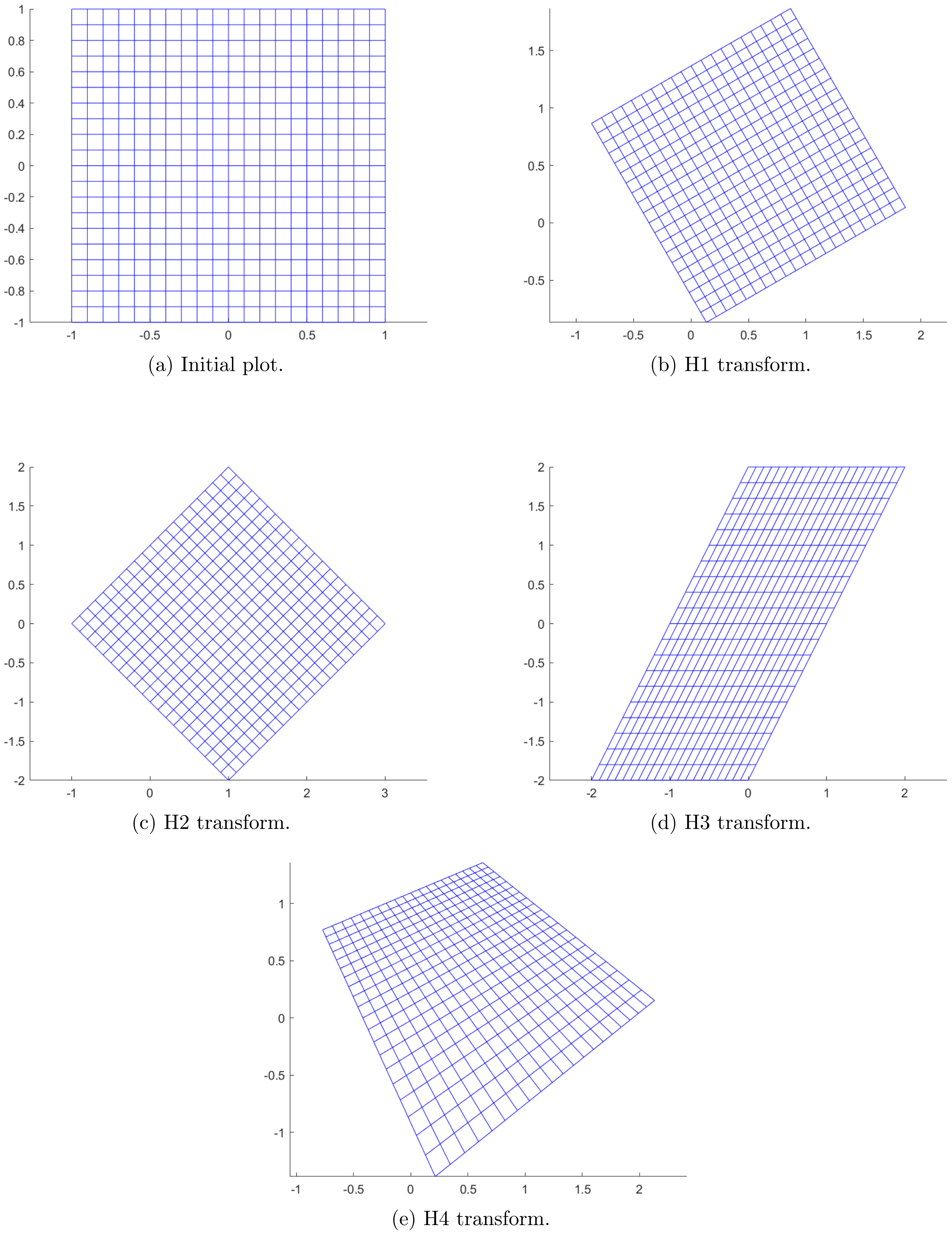


Figure 3: Plots for Computer Exercise 3.

Exercise 5

To get the projection of a point in 3D with homogeneous coordinates we multiply it with the camera matrix.

$$\mathbf{v}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Division with the third coordinate now gives the projection $\left(\frac{1}{4}, \frac{1}{2}\right)$.
Similarly,

$$\mathbf{v}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Division with the third coordinate now gives the projection $\left(\frac{1}{2}, \frac{1}{2}\right)$.
And lastly,

$$\mathbf{v}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

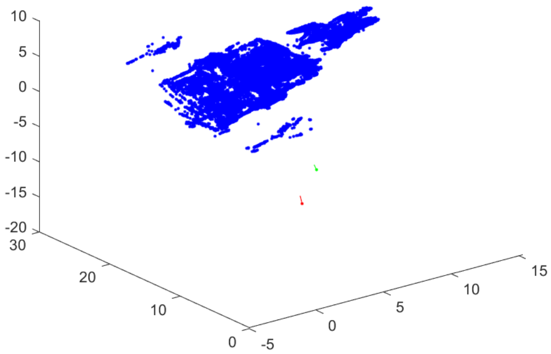
Which is a vanishing point that we can interpret as a point infinitely far away in the direction $\left(1, 1\right)$.
To compute the center of the camera we need to compute the nullspace of the P matrix.

$$\begin{cases} X & = 0 \\ Y & = 0 \\ Z + W & = 0 \end{cases} \Leftrightarrow \begin{cases} X & = 0 \\ Y & = 0 \\ Z & = -s \\ W & = s \end{cases}, \quad s \in \mathbb{R}. \tag{6}$$

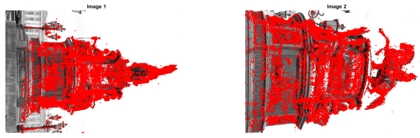
This gives the camera center $C = \left(0, 0, -1\right)$. To compute the viewing direction of the camera we take the last row of R rotation matrix, which in this case is $v = \left(0, 0, 1\right)$.

Computer Exercise 4

The projected points in the image appear to reasonably capture the main motive. Such is motivated by the coverage of the points, which appears to cover most of the statue.



(a) 3D Points with Camera centers and view directions.



(b) Projected Points in images.

Figure 4: Plots for Computer Exercise 4.