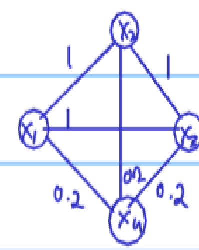
Quiz 3 (Questions 1 and 3)

1. (10 points) Consider a gray-scale image I of size 2×2 where the pixel values are given as $I = \begin{array}{|c|c|} \hline x_1 & x_2 \\ \hline x_3 & x_4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 0.9 & 0.9 \\ \hline 0.9 & 0.1 \\ \hline \end{array}$. Construct a graph representing this image where each pixel denotes a vertex and weight of an edge between two vertices is determined by the similarity between the corresponding pixels and defined as $w_{ij} = \begin{cases} 1 - |x_i - x_j| & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$. Here, x_i and x_j are the intensities of the i-th and the j-th pixels, respectively. Find the Laplacian matrix of the resulting graph. Find the optimal clustering for two cluster case using the ratio-cut approach. One of the cluster should have three vertices and other cluster should have only one vertex.

Ans 1 $\quad W_{ij} = 1 - |x_i - x_j|$

Adjacency Matrix $\quad A = \begin{bmatrix} 0 & 1 & 1 & 0.2 \\ 1 & 0 & 1 & 0.2 \\ 1 & 1 & 0 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0 \end{bmatrix}$



Degree Matrix $\quad D = \begin{bmatrix} 2.2 & 0 & 0 & 0 \\ 0 & 2.2 & 0 & 0 \\ 0 & 0 & 2.2 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix}$

Laplacian Matrix $\quad L = D - A = \begin{bmatrix} 2.2 & -1 & -1 & -0.2 \\ -1 & 2.2 & -1 & -0.2 \\ -1 & -1 & 2.2 & -0.2 \\ -0.2 & -0.2 & -0.2 & 0.6 \end{bmatrix}$

Case 1 $\quad C_1 = \{x_1, x_2, x_3\}, \; C_2 = \{x_4\}$

RatioCut $= \sum_{i=1}^{2} \frac{1}{|C_i|} \sum_{r \in C_i} \sum_{s \notin C_i} w_{rs} = \frac{1}{3}(0.2+0.2+0.2) + \frac{1}{1}(0.2+0.2+0.2) = \frac{0.6}{3} + 0.6 = 0.8$

Case 2 $\quad C_1 = \{x_2, x_3, x_4\}, \; C_2 = \{x_1\}$

Ratio Cut $= \frac{1}{3}(1+1+0.2) + \frac{1}{1}(1+1+0.2) = \frac{2.2}{3} + 2.2 = \frac{8.8}{3}$

Case 3 $\quad C_1 = \{x_3, x_4, x_1\}, \; C_2 = \{x_2\}$

Ratio cut $= \frac{1}{3}(1+1+0.2) + \frac{1}{1}(1+1+0.2) = \frac{8.8}{3}$

Case 4 $\quad C_1 = \{x_4, x_1, x_2\}, \; C_2 = \{x_3\}$

Ratio Cut $= \frac{1}{3}(1+1+0.2) + \frac{1}{1}(1+1+0.2) = \frac{8.8}{3}$

optimal clustering : $C_1 = \{x_1, x_2, x_3\}, \; C_2 = \{x_4\}$

3. (10 points) On performing SVD on an observation matrix you get the following U and V matrices and the diagonal matrix obtained in SVD contains the singular values: 1,2,3,4.

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, V = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

a) Find the observation matrix (**W**) that satisfies the properties of the observation matrix of the orthographic structure from motion problem.
b) How many points are being tracked, and how many camera frames are there?

a) **W** should have a rank 3, so we will rebuild the diagonal matrix and zero out the 4th singular value

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\mathbf{W} = U\Sigma V^T$$

$$\mathbf{W} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \times \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

$$\mathbf{W} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \times \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

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$$\mathbf{W} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & \frac{5}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{5}{2} \end{bmatrix}$$

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b) No. of points being tracked = 4
   No. of frames = 2

Quiz 4 (Questions 2 and 4)

2. (10 points) Consider the problem of clustering the points $\{\mathrm{x}_1, \ldots, \mathrm{x}_n\}$ into k clusters $\mathcal{C}_1, \ldots, \mathcal{C}_k$ using the k-means clustering algorithm. In k-means clustering algorithm, we minimize the below cost function to find the optimal cluster centers.

$$\sum_{i=1}^{k} \sum_{\mathrm{x} \in \mathcal{C}_i} \|\mathrm{x} - \mu_i\|_2^2$$

Show that for the optimal cluster centers would be

$$\mu_i = \frac{1}{|\mathcal{C}_i|} \sum_{\mathrm{x} \in \mathcal{C}_i} \mathrm{x}.$$

Here, $|\mathcal{C}_i|$ represents the number of data points in cluster $\mathcal{C}_i$.

Ans 2:

$$f(\mu_1, \ldots \mu_k) = \sum_{i=1}^{k} \sum_{x \in C_i} \|x - \mu_i\|_2^2 = \sum_{i=1}^{k} \sum_{x \in C_i} (x - \mu_i)^T (x - \mu_i)$$

$$= \sum_{i=1}^{k} \sum_{x \in C_i} \left(x^T x - 2 x^T \mu_i + \mu_i^T \mu_i \right)$$

$$\nabla_{\mu_i} f = \sum_{x \in C_i} (-2x + 2\mu_i) = \vec{0}$$

$$+2 \sum_{x \in C_i} x = 2 \sum_{x \in C_i} \mu_i$$

$$\sum_{x \in C_i} x = \mu_i \sum_{x \in C_i} 1 = \mu_i |C_i|$$

$$\Rightarrow \mu_i = \frac{1}{|C_i|} \sum_{x \in C_i} x$$

4. (10 points) Note that for the following questions, $a \times b - c$ constitutes 2 multiplication/addition/subtraction operations. Do not make any additional assumptions
a) Count the exact number of multiplication/addition/subtraction operations in computing an integral image from an image of size 10×10 using the raster scan method.
b) Given a Haar filter of size 4×10, consisting of 2 rectangles (pos & neg) of size 2×10 and 2×10, respectively, count the exact number of multiplication/addition/subtraction operations needed to apply it to the original image of size 10×10.
c) Given a Haar filter of size 4×10, consisting of 2 rectangles (pos & neg) of size 2×10 and 2×10, respectively, count the exact number of multiplication/addition/subtraction operations needed to apply it to the integral image computed above.

Ans:

a) First row costs = 9 and First col costs = 9
   Remaining costs = 81*3
   Total = 261

b) Two answers possible

   Sol 1 - if the multiplication of filter values with image pixels is done
   For one application of the filter = 20 + 20 + 39
For the entire image = 7 x (20 + 20 + 39)
   Sol 2 - If multiplication of filter values with image pixels is not done, simply add the positive region and negative region pixels separately and find the difference of the sums

   For one application of the filter = 39
For the entire image = 7 x 39

c) For first application of the filter the cost is = 2
   For subsequent application of the filter the cost is = 3
   For the entire image = 2+7x3=20