1. Stiff ODEs and Astrochemical Reactions

Consider an element X whose concentration $y(t)$ is evolved with time using the following equation:

$$\frac{dy}{dt} = 5.0y - 5t^2, \quad y(0) = \frac{2}{25} \tag{1}$$

Derive the analytic solution and show that it is a parabola.

Solve the above equation numerically over the time interval $0 \leq t \leq 2$. Try to solve this problem thrice, getting an approximately parabolic answer each time. The first time, use the Euler method, second time RK2, and the third time use the backward Euler method. For each solution attempt, try the values of n = 25,250, and then n = 2500, where n is the number of grid points within the time interval.

2. Classic 1D Advection Equation

Consider a 1D domain with x as the spatial coordinate that ranges from −10.0 to 10.0 and is discretized into 200 points. At an initial time $t = 0$, consider a normalized Gaussian pulse that is centered at $x = 0$ with $\sigma = 1.0$ given as:

$$u(x, t = 0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{2}$$

- Develop a 1D advection solver to solve the following equation:

  $$\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} = 0 \tag{3}$$

  up to a maximum time $t = 25$ using the following schemes:

  a) FTCS
  b) Upwind
  c) Lax Method
  d) MacCormack method
  e) Lax-Wendroff Scheme

  For each scheme, adopt values of $\lambda = 3.0, -3.0, 10.0$. Adopt a periodic boundary condition for all cases and $\Delta t = 10^{-3}$. Explain in each case how the solution behaves compared to its expected value.

- Vary the chosen time steps for the Lax-Wendroff scheme $\Delta t = 10^{-1}, 10^{-2}, 10^{-4}$ and check if the solution remains stable. Based on your inference, discuss the CFL condition.

- Repeat the above exercise only using the MacCormack method for a rectangular wave pulse of height 2.0 and width 1.0, centered at $x = 0$ at time $t = 0$ and in the same spatial domain. Explain the features you observe in your solutions. Do they appear physically correct?