Partitons in Combinatorial Computing

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1 Introduction

We calculate the number of Partitions of a natural number n with $k \in \{1, 2, ..., n\}$

R(n,k) denote the number of partitions with maximum number of elements equal to k. Q(n,k) denote the number of partitions with maximum value of an element equal to k.

Using the bit-pattern representation for sets, which uses a compact difference representation of consecutive elements of a partition.

For example, for the partition 5,4,3, its bit pattern representation will be 00100010000.

The number of bits in the representation is n-1, where n is the sum of elements and k is the number of elements

Number of 1's= $k - 1$

Number of 0's= $n - 1 - (k - 1)$

In order to find all the partitions of a number, we organise the n-k 0's into k bins.

Let the number of 0's in the i^{th} bin be p_i

$p_1 + p_2 + ... + p_k = n - k$

also, $p_1 \geq p_2 .... \geq p_k \geq 0$

If $p_k = 0$ and all the other elements are non-zero, the number of elements in the partition will be $k - 1$

With similar argument the number of elements vary from 1 to k.

Let P(n,k) represent the number of partitions of n with k elements

$\Rightarrow \; P(n, k) = P(n - k, 1) + P(n - k, 2) + ... + P(n - k, k)$

$\Rightarrow \; P(n, k) = \sum_1^k P(n - k, i)$

Finding a recursive relation,

$P(n, k) + P(n - k, k + 1) = \sum_{i=1}^{k+1} P(n - k, i)$

$P(n, k) + P(n - k, k + 1) = P(n + 1, k + 1)$

$\Rightarrow$ $\boxed{\text{P(n,k)=P(n-1,k-1)+P(n-k,k)}}$ (1)

Also, $P(n + k, k) = \sum_1^k P(n, i)$

$R(n, k) = \sum_1^k P(n, i)$

$\Rightarrow$ $\boxed{\text{P(n+k,k)=R(n,k)}}$ (2)

$R(n, k) + P(n, k + 1) = R(n, k + 1)$

Using (2) we get,

$\Rightarrow$ $\boxed{\text{R(n,k)=R(n-k,k)+R(n,k-1)}}$(3)

We know,

Q(n,k)=number of partitions with at least one k as the maximum element+ number of partitions containing maximum element k-1

$\Rightarrow$ $\boxed{\text{Q(n,k)=Q(n-k,k)+Q(n,k-1)}}$(4)

We observe from equation (3) and (4)

$Q(n, k) = R(n, k)$

Also, using Ferror's diagram for a partition, R(n,k)=number of vertical cells, whereas when we rotate the diagram about its diagonal we obtain another partition which has maximum value k. Since for every partition of a natural number n, we can rotate its ferror diagram to obtain another partition, R(n,k)=Q(n,k).