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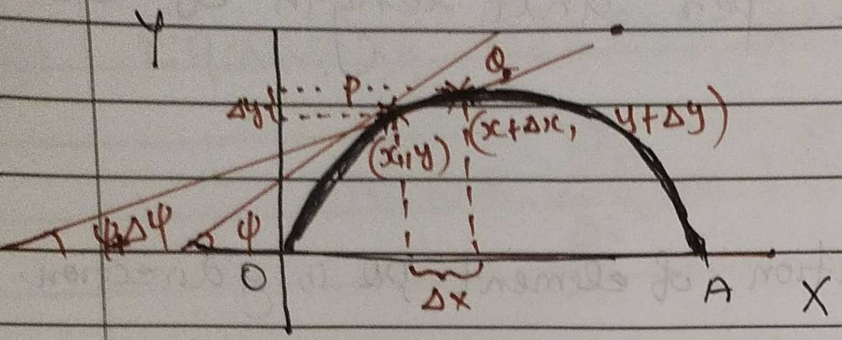
Derivation of One dimensional wave equation



* Consider an elastic string of length l tightly stretched between two points O and A.
* The string is then disturbed at time $t=0$ and allowed to vibrate.

Keep the following assumptions.

* The motion takes place completely in one plane. and in this plane the particles vibrate $\perp^{lr}$ to the equilibrium position of the string.
* Tension T in the string is constant.
* Gravitational force is negligible
* The string is perfectly elastic, uniform and doesnot resist to bending
* Friction is negligible.

* Let $P(x, y)$ and $Q(x+\Delta x, y+\Delta y)$ be any two points in OA. that makes angles ψ and $\psi+\Delta\psi$ with x axis (through tangents drawn to P and Q)

* Let m is macs per unit length of the string.

* $\frac{\partial^2 y}{\partial t^2}$ → acceleration of element PQ in y direction

* vertical component of force acting on this element

$$= T\sin(\psi+\Delta\psi) - T\sin\psi$$

$$= T(\sin(\psi+\Delta\psi) - \sin\psi)$$

Here $\psi, \psi+\Delta\psi$ are negligibly small angle.

$\therefore$

$$= T(\tan(\psi+\Delta\psi) - \tan\psi)$$

$\left(\tan\theta = \text{slope of the curve ie; } \frac{dy}{dx}\right)$

$$= T\left(\left(\frac{\partial y}{\partial x}\right)_Q - \left(\frac{\partial y}{\partial x}\right)_P\right) \quad \text{—— (1)}$$

By Newton's second law

$$F = ma$$

mass of PQ $= m \times \Delta x$

$$F = m \times \Delta x \frac{\partial^2 y}{\partial t^2} \quad \text{—— (2)}$$

Equating ① and ②

$$m \cdot \Delta x \cdot \frac{\partial^2 y}{\partial t^2} = T\left[\left(\frac{\partial y}{\partial x}\right)_Q - \left(\frac{\partial y}{\partial x}\right)_P\right]$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m} \frac{\left[\left(\frac{\partial y}{\partial x}\right)_Q - \left(\frac{\partial y}{\partial x}\right)_P\right]}{\Delta x}$$

$$\boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

$Q : \varphi + \Delta \varphi$
$P : \varphi$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m} \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad c^2 = \frac{T}{m}$$

always a +ve qty.

This pde is

$$\boxed{\text{One Dimensional ware Equation: } \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$

Solution of One dimensional wave equation by method of separation of variable.

wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—} (1)$$

Let $y = XT$ — (2)

Diff. (2) p. w.r.t x and t

$$\frac{\partial y}{\partial t} = XT' \qquad \frac{\partial y}{\partial x} = X'T$$

$$\frac{\partial^2 y}{\partial t^2} = XT'' \qquad \frac{\partial^2 y}{\partial x^2} = X''T$$

Sub. these values in (1)

$$XT'' = c^2 X'' T$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = K$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$\cancel{m^2 - K = 0} \quad [D^2 - K]X = 0$$

$$\cancel{m = \pm\sqrt{K}}$$

$$\frac{1}{c^2}\frac{T''}{T} = K$$

$$T'' = c^2 TK$$

$$T'' - Kc^2 T = 0$$

$$[D^2 - Kc^2]T = 0$$

3 In this case we can expect three different cases. (Three different values for k)

Case I $(k = p^2)$ k is +ve

$(D^2 - p^2)X = 0$

A.E → $m^2 - p^2 = 0$

$m^2 = p^2$

$m = \pm p$

$X = c_1 e^{px} + c_2 e^{-px}$

$(D^2 - p^2c^2)T = 0$

$m^2 - p^2c^2 = 0$

$m^2 = p^2c^2$

$m = \pm pc$

$T = c_3 e^{pct} + c_4 e^{-pct}$

$$\boxed{y = XT \Rightarrow y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pct} + c_4 e^{-pct})}$$ — 1

Case II k is −ve $(k = -p^2)$

$(D^2 + p^2)X = 0$

$D^2 = -p^2$

$m^2 = -p^2$

$m = \pm ip$ $\alpha = 0$ $\beta = p$

when m is imaginary $\alpha \pm i\beta$

$CF = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

$X = e^{0}[c_5 \cos px + c_6 \sin px]$

ie; $X = [c_5 \cos px + c_6 \sin px]$

$(D^2 + p^2c^2)T = 0$

$m^2 + p^2c^2 = 0$

$m^2 = -p^2c^2$

$m = \pm ipc$

$T = [c_7 \cos pct + c_8 \sin pct]$

Therefore the solution is

$$\boxed{y(x,t) = (C_5 \cos px + C_6 \sin px)(C_7 \cos pct + C_8 \sin pct)} \quad \text{--- (2)}$$

Case 3 $K = 0$

$D^2X = 0$	$D^2T = 0$
$m^2 = 0,$	$m^2 = 0$
$m = 0, 0$	$m = 0, 0$
$CF = (C_1 + C_2 x) e^{mx}$	
$X = (C_9 + C_{10} x)$	$T = (C_{11} + C_{12} t)$

$$\boxed{y(x,t) = (C_9 + C_{10} x)(C_{11} + C_{12} t)} \quad \text{--- (3)}$$

These are the possible solutions. But the suitable solution is — (2)

Rewriting:

$$\boxed{y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos pct + C_4 \sin pct)}$$