Boundary Conditions of wave equation

i) $y(0,t) = 0 \quad \forall t$ (displacement at starting point)

ii) $y(l,t) = 0 \quad \forall t$ (displacement at final point)

Initial conditions of wave equation

iii) $\frac{\partial y}{\partial t}(x,0) = g(x)$ (initial velocity)

iv) $y(x,0) = f(x)$ (initial displacement)

To solve problems you have to write all the four conditions. write all default conditions first and then write the given condition since the order applying the conditions will be in that order.

$y(x,t) \rightarrow$ displacement

$x \rightarrow$ distance $\quad t \rightarrow$ time

$l \rightarrow$ length of string

$\frac{\partial^2 y}{\partial t^2} \rightarrow$ acceleration

$\frac{\partial y}{\partial t} \rightarrow$ velocity.

Problem 1:

A tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$. At time $t=0$, the string is given a shape defined by $F(x)=\mu x(l-x)$ where μ is a constant and then released. Find the displacement of any point x of the string at any time $t>0$.

Solution:

(we have to find the displacement. Now understand it is a problem based on wave equation).

Wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

shape is defined by $F(x)=\mu x(l-x)$ means it is displacement

Boundary conditions are

i) $y(0,t)=0 \quad \forall t$

ii) $y(l,t)=0 \quad \forall t$

iii) $\frac{\partial y}{\partial t}(x,0)=0$ (velocity not given, so 0)

iv) $y(x,0)=\mu x(l-x)$ (displacement function is given write east)

Suitable solution is

$$y(x,t)=(C_1\cos px + C_2\sin px)(C_3\cos pct + C_4\sin pct) \quad \text{—①}$$

Apply each the conditions one by one in —①

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Applying (i) in — 1

$y(0,t) = 0$ ie; put $x=0$, $y=0$

$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)$

$(c_1 \cos 0 + c_2 \sin 0)(c_3 \cos pct + c_4 \sin pct) = 0$

$c_1 (c_3 \cos pct + c_4 \sin pct) = 0$

$\boxed{c_1 = 0}$ { $c_1 = 0$, other part can't 0 since it has variable t }

update solution by sub. $c_1 = 0$ — 1

$$\boxed{y(x,t) = (c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)} \quad — 2$$

new solution.

Applying (ii) in 2

$y(l,t) = 0$ ie; $x = l$ $y = 0$

$c_2 \cdot \sin pl \cdot [c_3 \cos pct + c_4 \sin pct] = 0$

$\sin pl = 0$ $\sin n\pi = 0$

ie; $n\pi = pl$

$\boxed{p = \frac{n\pi}{l}}$

c_2 can't be zero since — 2 will become zero. called trivial solution

Variable part also can't zero.

sub. p value in 2.

$$\boxed{y(x,t) = c_2 \cdot \sin \frac{n\pi x}{l} \cdot \left[c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right]} \quad — 3$$

~~new solution~~

modify.

$$y(x,t) = \sin\frac{n\pi x}{l}\left[c_2c_3\cos\frac{n\pi ct}{l} + c_2c_4\sin\frac{n\pi ct}{l}\right]$$

$$\boxed{y(x,t) = \sin\frac{n\pi x}{l}\left[A\cos\frac{n\pi ct}{l} + B\sin\frac{n\pi ct}{l}\right]}$$

New solution — (3)

Applying (iii) it is $\frac{\partial y}{\partial t}(x,0) = 0$

but $\frac{\partial y}{\partial t}$ is not there. find and than apply.

D. (3) p. w.r.t.t

$$\frac{\partial y}{\partial t}(x,t) = \sin\frac{n\pi x}{l}\left[-A\cdot\frac{n\pi c}{l}\sin\frac{n\pi ct}{l} + B\cdot\frac{n\pi c}{l}\cos\frac{n\pi ct}{l}\right]$$

— (4)

Now apply (iii) in (4)

$$\frac{\partial y}{\partial t}(x,0) = 0, \quad t=0$$

$$\sin\frac{n\pi x}{l}\left[-A\frac{n\pi c}{l}\sin 0 + B\frac{n\pi c}{l}\cos 0\right] = 0$$

$$\left(\sin\frac{n\pi x}{l}\right)\cdot B\left(\frac{n\pi c}{l}\right) = 0$$

$\sin\frac{n\pi x}{l} \neq 0$ variable is there

$\frac{n\pi c}{l} \neq 0$ it is a const.

$B = 0$

$$\boxed{B=0}$$

(6)

New suitable solution is

apply $B = 0$ in (3)

$$y(x,t) = \sin\frac{n\pi x}{l}\left[A\cos\frac{n\pi ct}{l} + 0\right]$$

$$\boxed{y(x,t) = \sin\frac{n\pi x}{l}\left[A\cos\frac{n\pi ct}{l}\right]. \quad — (5)}$$

New solution.

This is the most general solution. Rewrite the eqn.

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad — (6)}$$

applying (iv) in (6)

$$y(x,0) = \mu(lx - x^2) \quad ; t = 0.$$

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos 0 = \mu(lx - x^2)$$

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = \mu(lx - x^2) \quad — (7)$$

Here we use $f(x)$ to find the A_n value only, and A_n will be sub. in (6)

This is Half range Sine series

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{l}\int_0^l \mu(lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\mu}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\mu}{l}\left[(lx-x^2)\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (l-2x)\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2 a^2}{l^2}}\right.$$

$$\left. + (-2)\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$= \frac{2\mu}{l}\left[\left[(l\cdot l-l^2)\frac{-\cos\frac{n\pi l}{l}}{\frac{n\pi}{l}} - (l-2l)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right.\right.$$

$$\left. + (-2)\frac{\cos\frac{n\pi l}{l}}{\frac{n^3\pi^3}{l^3}}\right] - \left[0 - 0 - 2\frac{\cos 0}{\frac{n^3\pi^3}{l^3}}\right]$$

$$= \frac{2\mu}{l}\left[\left[0-0-2\frac{l^3}{n^3\pi^3}\cos n\pi\right] - \left[-\frac{2l^3}{n^3\pi^3}\right]\right]$$

$$= \frac{2\mu}{l}\left[-2\frac{l^3}{n^3\pi^3}(-1)^n + 2\frac{l^3}{n^3\pi^3}\right]$$

$$= \frac{2\mu}{\cancel{l}}\cdot\frac{2l^{\cancel{3}}}{n^3\pi^3}\left[-(-1)^n+1\right]$$

$$= \frac{4\mu l^{\cancel{3}2}}{n^3\pi^3}\left[-(-1)^n+1\right]$$

(7) $$= \frac{4\mu l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

Sub. A_n in (6)

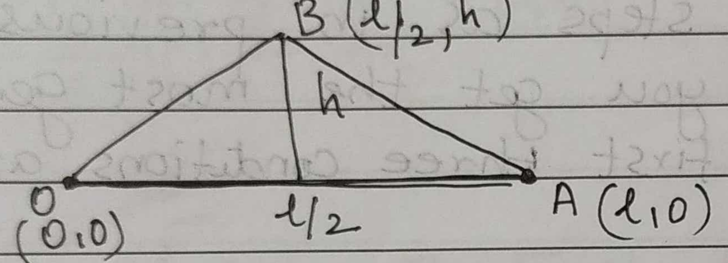
Therefore, the solution is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} \frac{4\mu l^2}{n^3\pi^3}\left[1-(-1)^n\right] \sin\frac{n\pi}{l}x \cos\frac{n\pi ct}{l}}$$

①

Problem 2: A string of length l is fastened at both ends. The mid-points of the string is taken to a height 'h'. Write the analytic expression for the function and also find displacement at any point at any time.

Solution:



Equation of a line joining two points (x_1, y_1) and (x_2, y_2) is $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$

Consider OB

O $(x_1,y_1) = (0,0)$ B $(x_2, y_2) = (l/2, h)$

$$\frac{y-0}{h-0} = \frac{x-0}{l/2-0} \Rightarrow y = \frac{2hx}{l} \quad 0<x<l/2$$

Along BA $(l/2, h), (l,0)$

$$\frac{y-h}{0-h} = \frac{x-l/2}{l-l/2} \Rightarrow y = \frac{2h}{l}[l-x] \quad l/2<x<l$$

$$y(x,0) = f(x) = \begin{cases} \dfrac{2hx}{l}, & 0<x<l/2 \\ \dfrac{2h}{l}(l-x), & l/2<x<l \end{cases}$$

This is Analytic expression (called initial displacement)

The boundary conditions are

(i) $y(0,t) = 0 \quad \forall t$

(ii) $y(l,t) = 0 \quad \forall t$

(iii) $\frac{\partial y}{\partial t}(x,0) = 0$

(iv) $y(x,0) = f(x) = \begin{cases} \frac{2hx}{l}, & 0 < x < l/2 \\ \frac{2h}{l}(l-x), & l/2 < x < l \end{cases}$

Do the same steps as in previous problem until you get the most general solution. since first three conditions are same.

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \cos\frac{n\pi c}{l}t \quad — ①$$

applying B.C. (iv) in ①

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l}x = f(x) \quad — ②$$ Half range Sine series.

$$A_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2h}{l}x \sin\frac{n\pi}{l}x\,dx + \int_{l/2}^{l} \frac{2h}{l}(l-x) \sin\frac{n\pi}{l}x\,dx\right]$$

$$= \frac{2}{l}\cdot\frac{2h}{l}\left[\int_0^{l/2} x \sin\frac{n\pi}{l}x\,dx + \int_{l/2}^{l}(l-x) \sin\frac{n\pi}{l}x\,dx\right]$$

$$= \frac{4h}{l^2}\left[\int_0^{l/2} x \sin\frac{n\pi}{l}x\,dx + \int_{l/2}^{l}(l-x) \sin\frac{n\pi}{l}x\,dx\right]$$

$$= \frac{4h}{\ell^2}\left[\left[x\left(\frac{-\cos\frac{n\pi}{\ell}x}{\frac{n\pi}{\ell}}\right) - (1)\left(\frac{-\sin\frac{n\pi}{\ell}x}{\frac{n^2\pi^2}{\ell^2}}\right)\right]_0^{\ell/2} + \left[(\ell-x)\left(\frac{-\cos\frac{n\pi}{\ell}x}{\frac{n\pi}{\ell}}\right) - (-1)\left(\frac{-\sin\frac{n\pi}{\ell}x}{\frac{n^2\pi^2}{\ell^2}}\right)\right]_{\ell/2}^{\ell}\right]$$

$$= \frac{4h}{\ell^2}\left[\left[\cancel{\ell/2 \cdot \frac{\ell}{n\pi}\cos\frac{n\pi}{2}} + \frac{\ell^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] - \left[0+0\right] + \left[0-0\right] - \left[\cancel{-\ell/2 \cdot \ell/n\pi\cos\frac{n\pi}{2}} - \frac{\ell^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\right]$$

$$= \frac{4h}{\cancel{\ell^2}}\left[2\frac{\cancel{\ell^2}}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

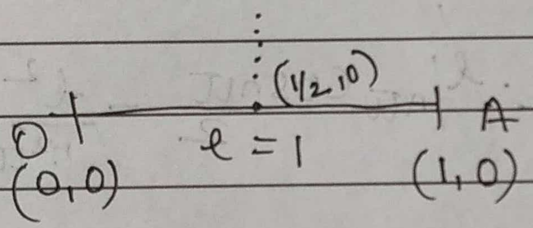
sub. A_n in (1)

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{\ell}\cos\frac{n\pi c}{\ell}t}$$

Problem 3: A tightly stretched string with fixed end points $x=0$ and $x=1$ is initially in a position given by $f(x) = \begin{cases} x, & 0 \le x \le 1/2 \\ (1-x), & 1/2 \le x \le 1 \end{cases}$

If it is released from this position with velocity zero. find the displacement.

Solution:



Boundary conditions are

(i) $y(0,t) = 0$

(ii) $y(1,t) = 0$

(iii) $\frac{\partial y}{\partial t}(x,0) = 0$

(iv) $y(x,0) = f(x)$

Rest of all are same as previous problems.