Homogeneous space

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Homogeneous co-ordinates

- Represent coordinates in 2 dimensions

with a 3-vector

Add a 3rd co-ordinate to every 2D point

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

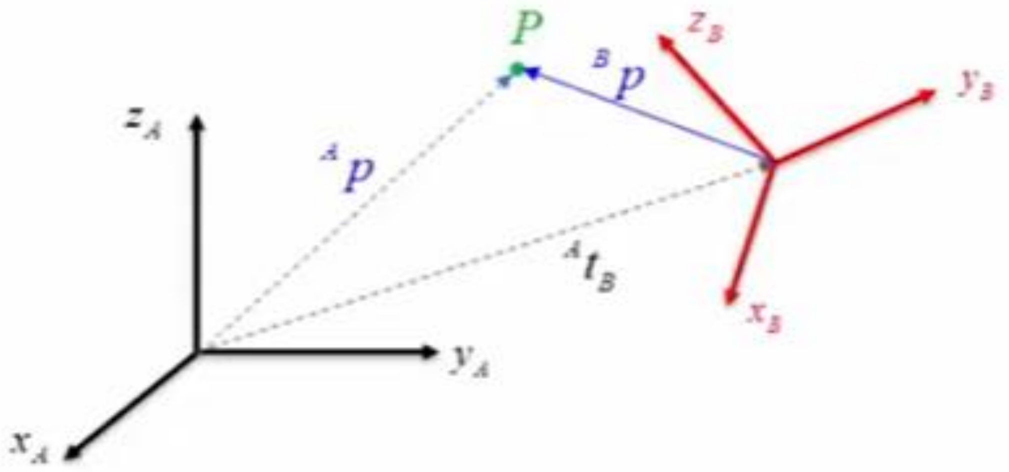
$$(x, y, w) \Rightarrow (x/w, y/w)$$

This states that for a point (x,y) there exists a homogeneous point (xt,yt,t) where t is an arbitrary number

Eg: point (3,4) has homogeneous coordinates (6,8,2) because 3=6/2 and 4=8/2

(12,16,4) , (15,20,5) , (300,400,100)

Homogeneous transformations



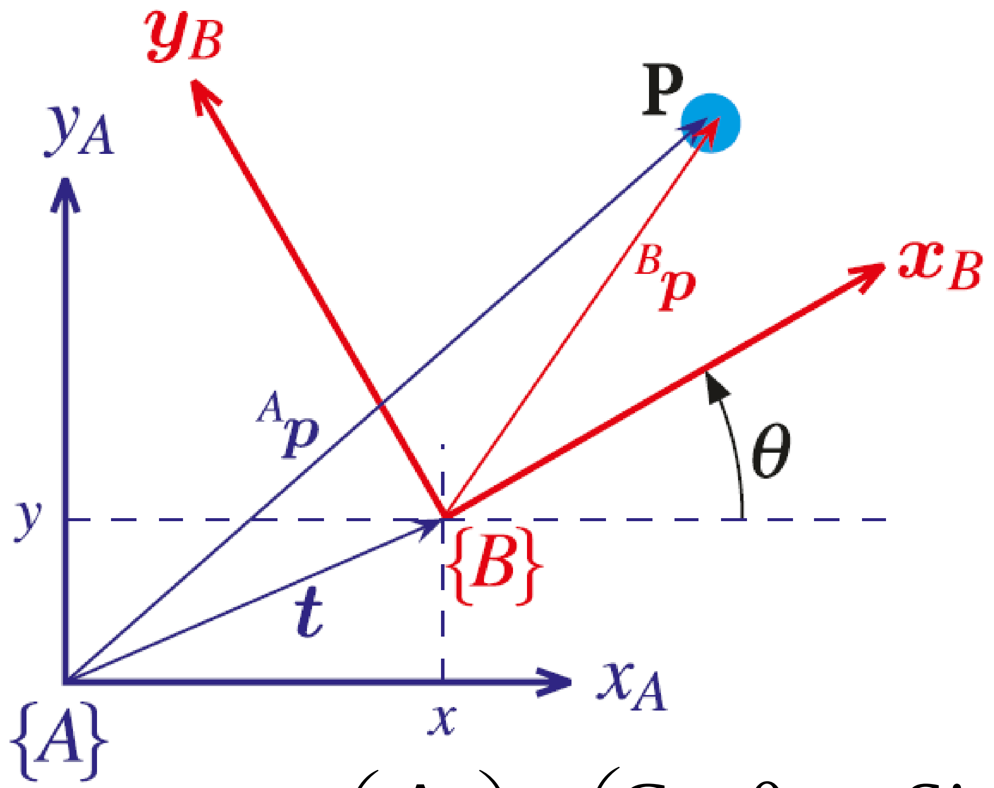
${}^A t_B$: Origin of frame B with respect to frame A

- Point P of frame $\{B\}$ in frame $\{A\}$: $${}^A p = {}^A t_B + {}^A R_B \, {}^B p$$
- In a more compact way:

$$\underbrace{\begin{bmatrix} {}^A p \\ 1 \end{bmatrix}}_{{}^A\tilde{p}} = \underbrace{\begin{bmatrix} {}^A R_B & {}^A t_B \\ 0 & 1 \end{bmatrix}}_{{}^A T_B} \underbrace{\begin{bmatrix} {}^B p \\ 1 \end{bmatrix}}_{{}^B\tilde{p}}$$

$\left.\begin{matrix} {}^A\tilde{p} \\ {}^B\tilde{p} \end{matrix}\right\}$ Homogeneous coordinates

Homogenous Matrix – Derivation explanation



← Consider this figure

$$A_p = {}^A t_B + {}^A R_B \bullet B_p$$

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix} + \begin{pmatrix} {}^A x_B \\ {}^A y_B \end{pmatrix}$$

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & {}^A x_B \\ \sin\theta & \cos\theta & {}^A y_B \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} A_x \\ A_y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & {}^A x_B \\ \sin\theta & \cos\theta & {}^A y_B \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ 1 \end{pmatrix}$$

$$A_p = \begin{pmatrix} {}^A R_B & {}^A t_B \\ 0,0 & 1 \end{pmatrix} \cdot B_p$$

Homogeneous Matrix

$$ {}^{A}P = {}^{A}t_B + {}^{A}R_B \cdot {}^{B}P \quad \text{— (1)} $$

Column matrix

$$ \begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{bmatrix} A_{x_B} \\ A_{y_B} \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} B_x \\ B_y \end{bmatrix} \quad \text{— (2)} $$

$$ {}^{A}\tilde{P} = \begin{bmatrix} {}^{A}R_B & {}^{A}t_B \\ 0, 0 & 1 \end{bmatrix} \cdot {}^{B}\tilde{P} \quad \text{— (3)} $$

Homogeneous vector

Homogeneous transform

Homogeneous vector

$$\tilde{{}^{A}p} = {}^{A}\xi_{B} \cdot \tilde{{}^{B}p}$$

$\hookrightarrow$ describes relative POSE as 3x3 matrix

$$\underbrace{{}^{A}\xi_{B} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}}$$

Mathematical representation of POSE

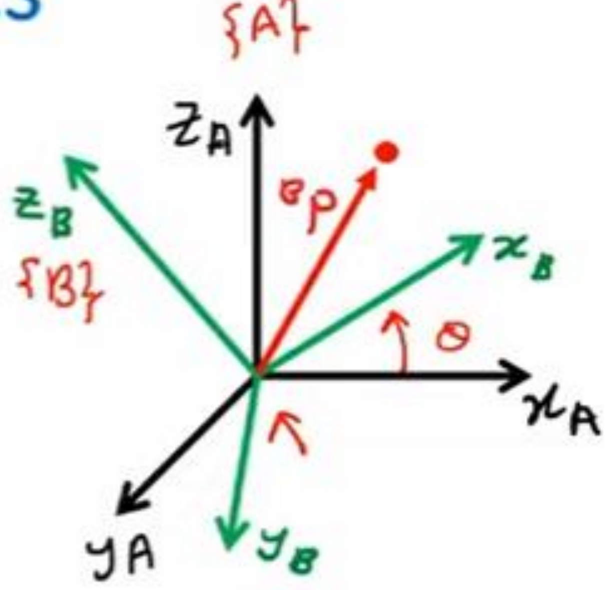
Mapping Rotated Frames

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

${}^{A}_{B}R$

$$R_y = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

$$R_z = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Lab - Matlab

We create a homogeneous transformation which represents a translation of (1, 2) followed by a rotation of 30°

```
>> T1 = transl2(1, 2) * trot2(30, 'deg')
T1 =
    0.8660   -0.5000    1.0000
    0.5000    0.8660    2.0000
         0         0    1.0000
```

The function `transl2` creates a relative pose with a finite translation but zero rotation, while `trot2` creates a relative pose with a finite rotation but zero translation. We can plot this, relative to the world coordinate frame, by

```
>> plotvol([0 5 0 5]);
>> trplot2(T1, 'frame', '1', 'color', 'b')
```

Planar Rotation About a Point

A rotation through an angle θ about a point $\binom{a}{b}$ is obtained by performing a translation which maps $\binom{a}{b}$ to the origin, followed by a rotation through an angle θ about the origin, and followed by a translation which maps the origin to $\binom{a}{b}$. The rotation matrix is

$$\begin{aligned}
\mathrm{Rot}_{(a,b)}(\theta) &= \mathrm{Trans}(a,b)\,\mathrm{Rot}(\theta)\,\mathrm{Trans}(-a,-b) \\
&= \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta & -a\cos\theta + b\sin\theta + a \\ \sin\theta & \cos\theta & -a\sin\theta - b\cos\theta + b \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Certain things to remember

1) A **point** is described by a bound coordinate vector.
2) Points and Vectors are two different things: **(a)** we can add vectors, but not points; **(b)** difference of two points $\rightarrow$ vector.
3) A rigid object can be represented by set of points.
4) Position + Orientation of object's coordinate frame $\rightarrow$ **Pose**.
5) Relative pose $\rightarrow$ ξ.
6) The $\bullet$ operator.
7) The $\oplus$ operator.