Lecture 4-Completing a Rank One Matrix, Circulants

Agenda

- Given m+n-1 nonzero entries in A. Can we compute to rank 1 matrix (mxn). This leads to bipartite graphs

- Convolution and Cyclic convolution eigen values/eigen vectors of Circulant matrices

Source: Section IV.8 and IV.2 in Linear Algebra and Learning from Data (2019) by **Gilbert Strang**

Completing rank 1 matrix

Take rank 1 matrix $A = uv^T$ this means that $a_{ij} = u_i v_j$, i=1, m and j=1,n

Let us take m=n=3 and m+n-1 nonzero entries in A(mxn) matrix, i.e., m+n-1 = 5 nonzero entries in which positions in A

$$A = \begin{pmatrix} x & x & x \\ x & \bullet & \bullet \\ x & \bullet & \bullet \end{pmatrix}$$

In any rank 1 matrix, every (2x2) determinant must be zero.
So (2, 2) entry of A is decided by $a_{22}a_{33} = a_{23}a_{32}$

Check the failure of A

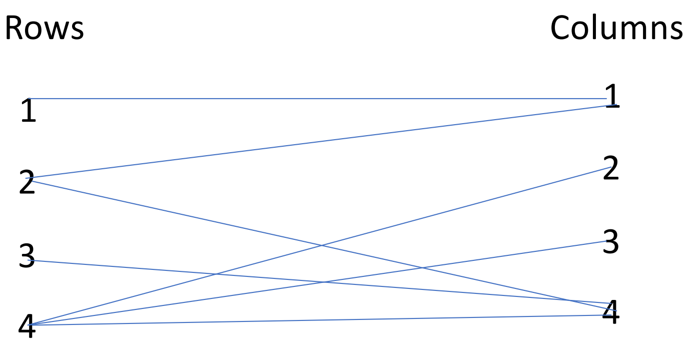
$$A = \begin{pmatrix} x & x & \bullet \\ x & \bullet & x \\ \bullet & \bullet & x \end{pmatrix}$$

In this matrix, any (2x2) determinant fails to zero
A fails to have rank 1

Completing rank 1 matrix

Take an example of combinatorics. To explain this, by constructing a graph with mxn nodes

Take m=n=4 and m+n-1=7 entries in matrix A



Matrix A $_{(4x4)}$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \left(\begin{array}{cccc} x & & & \\ x & & & x \\ & & & x \\ & x & x & x \end{array} \right) \end{array}$$

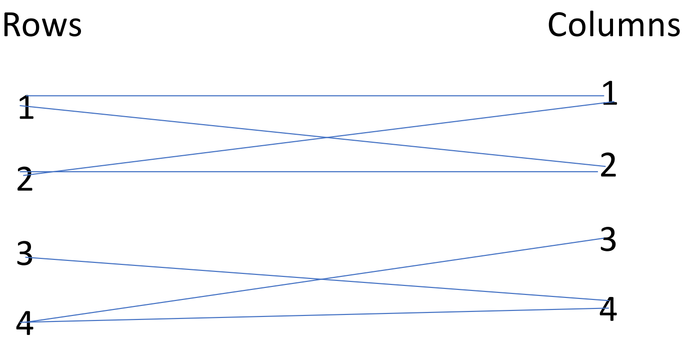
There are seven x's, as seven edges are there. And it is a bipartite graph

Four nodes here, four nodes another end. Total eight nodes. It's a bipartite graph because one part of nodes over there, one part of nodes here

Bipartite means two parts

Completing rank 1 matrix

Complete rank 1 matrix there? what's the rule?
Can't complete the matrix as we have 2 by 2
To avoid this , a 2 by 2, in other four entries



Matrix A $_{(4x4)}$

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\left(\begin{array}{cccc}
x & x & & \\
x & x & & \\
 & & & x \\
 & & x & x
\end{array}\right)
\end{array}
$$

We have seven altogether entries, that's a failure
we do not have a zero determinant here. We won't have rank 1

It is a Cycle graph = failure to have rank 1 matrix. Rank 1 matrix has no cyclic graph

Convolution and Circulant matrices

Convolution matrix-- a **cyclic convolution matrix**
cyclic convolution matrix is **circulant**

A circulant has constant diagonals. Convolution means constant down each diagonal

Cyclic means complete, circle around again, the diagonals circle around

$$C = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 2 & 5 & 1 & 0 \\ 0 & 2 & 5 & 1 \\ 1 & 0 & 2 & 5 \\ 5 & 1 & 0 & 2 \end{pmatrix} \end{array}$$

1st column, 2, 0, 1, 5. 2nd column is 5, 2, 0, 1 shifted by one.
3rd column again shifted by one, 1, 5, 2, 0
Last column is 0, 1, 5, 2.

They're all the same columns after a cyclic shift

Convolution and Circulant matrices

Take 0, 1, 0, 0, 0, 0-- it has one non-zero diagonal. And then it's cyclic

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad P^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

If it is square matrix, this is a shift by one again and is multiplying it again by P i.e., P^2

It's a shift by two

$$P^2x = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_0 \\ x_1 \end{pmatrix}$$

It's shifted it by two and cyclically

x_2, x_3 got shifted off the bottom come back to the top

Convolution and Circulant matrices

Every Circulant matrix is polynomial of P, i.e., $C=C_0I+C_1P+C_2P^2+C_3P^3$ (I is an identity matrix)

$$C=C_0I+C_1P+C_2P^2+C_3P^3=\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} C_0 & C_1 & C_2 & C_3 \\ C_3 & C_0 & C_1 & C_2 \\ C_2 & C_3 & C_0 & C_1 \\ C_1 & C_2 & C_3 & C_0 \end{pmatrix}
\end{array}$$

Every circulant matrix is C_0 times the identity circulant plus C_1 times the single shift plus C_2 times the double shift plus C_3 times the triple shift. It takes to put C_0, C_1, C_2, and C_3 on those diagonals

Suppose C and D are Circulant matrices

Fact 1: C is polynomial in P and D is polynomial in P

CD= Polynomial in P and question is how to get degree 3?

$CD=(C_0I+C_1P+C_2P^2+C_3P^3)(D_0I+D_1P+D_2P^2+D_3P^3)$

P is 4x4 circular shift. So $P^4=I$

P^6 is really a P^2. P^5 is really a P term. P^4 is really a P^0

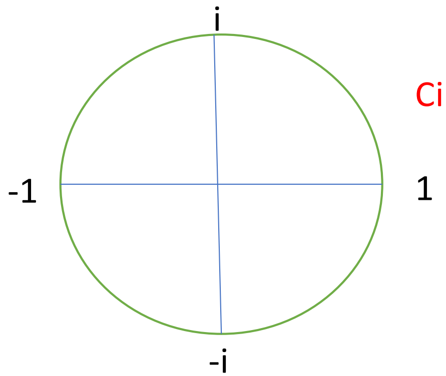
Convolution and Circulant matrices

Eigen Vector and Eigen Values of C with $Px=\lambda x$

$$
Px= \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_0 \end{bmatrix} = \lambda \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{Gives} \quad \begin{aligned} x_1 &= \lambda x_0 \\ x_2 &= \lambda x_1 \\ x_3 &= \lambda x_2 \\ x_0 &= \lambda x_3 \end{aligned}
$$

Now, $x_0 = \lambda x_3 = \lambda^2 x_2 = \lambda^3 x_1 = \lambda^4 x_0$, So $\lambda^4 = 1$ This gives evalues of P which are i,-1, -i, 1

In a circulant world, draw a circle.

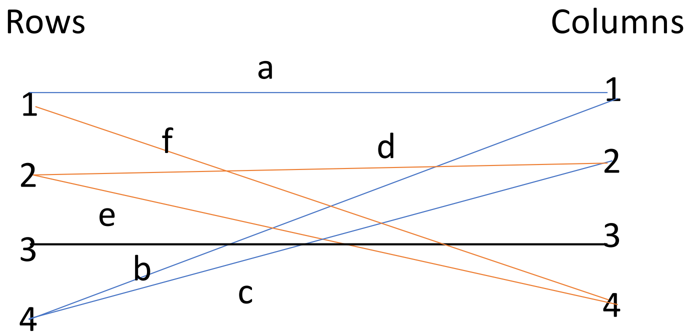


Circulant has same evectors

Back-up slides

Completing rank 1 matrix

Let us take another graph



a-b-c-d-e-f (1-1-4-2-2-4-1)

Matrix A $_{(4x4)}$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \left(\begin{array}{cccc} x & & & x \\ & x & & x \\ & & x & \\ x & x & & \end{array} \right) \end{array}$$

Seven positions or any m+ n-1 positions like that, Create a graph like following the rule

And it's a bipartite graph because every edge goes from this part over to this part. And that's a failure

Because we have here a cycle, That would be a cycle equals failure

In combinatorics, only compute complete rank 1 matrix if and only if no cycles

Convolution and Circulant matrices

Convolution: Let us take the convolution of $(3,1,2) \ast (4,6,1)$
we have got vectors (3,1,2)and (4,6,1) and there is a polynomial
$(3+x+2x^2) \ast (4+6x+x^2)$
Hence, $(3,1,2) \ast (4,6,1)= (12,22,17,13,2)$

It is non-cyclic convolution

Ordinary convolution is the multiplication you learned in second grade
(made easier because there is no "carrying" to the next column) :

```
          4  6  1
          3  1  2
       -----------
          8 12  2
       4  6  1
   12 18  3
   ----------------
   12 22 17 13 2
```

Cyclic Convolution: $(3,1,2) \otimes (4,6,1))= (12,22,17,13,2)= (12+13,22+2,13)= (25,24,17)$

So this represents 12, 22P, $17P^2$, $13P^3$. And what's $13P^3$? If n is 3 and we are handling 3 by 3 matrices, then $13P^3$ cubed is the same 13, right? P^3 is I.
So the 13 will go back there. And the 2 will be P^4.

And it will go back as P. So now with convolution, cyclic convolution gives 12 and 13, 25; 22 and 2, 24; and 17

So we are getting back a vector of length 3 just as we wanted to. Check with $25+24+17=66$ and $(3+1+2) \ast (4+6+1)=66$