# *Introduction to Robotics*

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# Recap

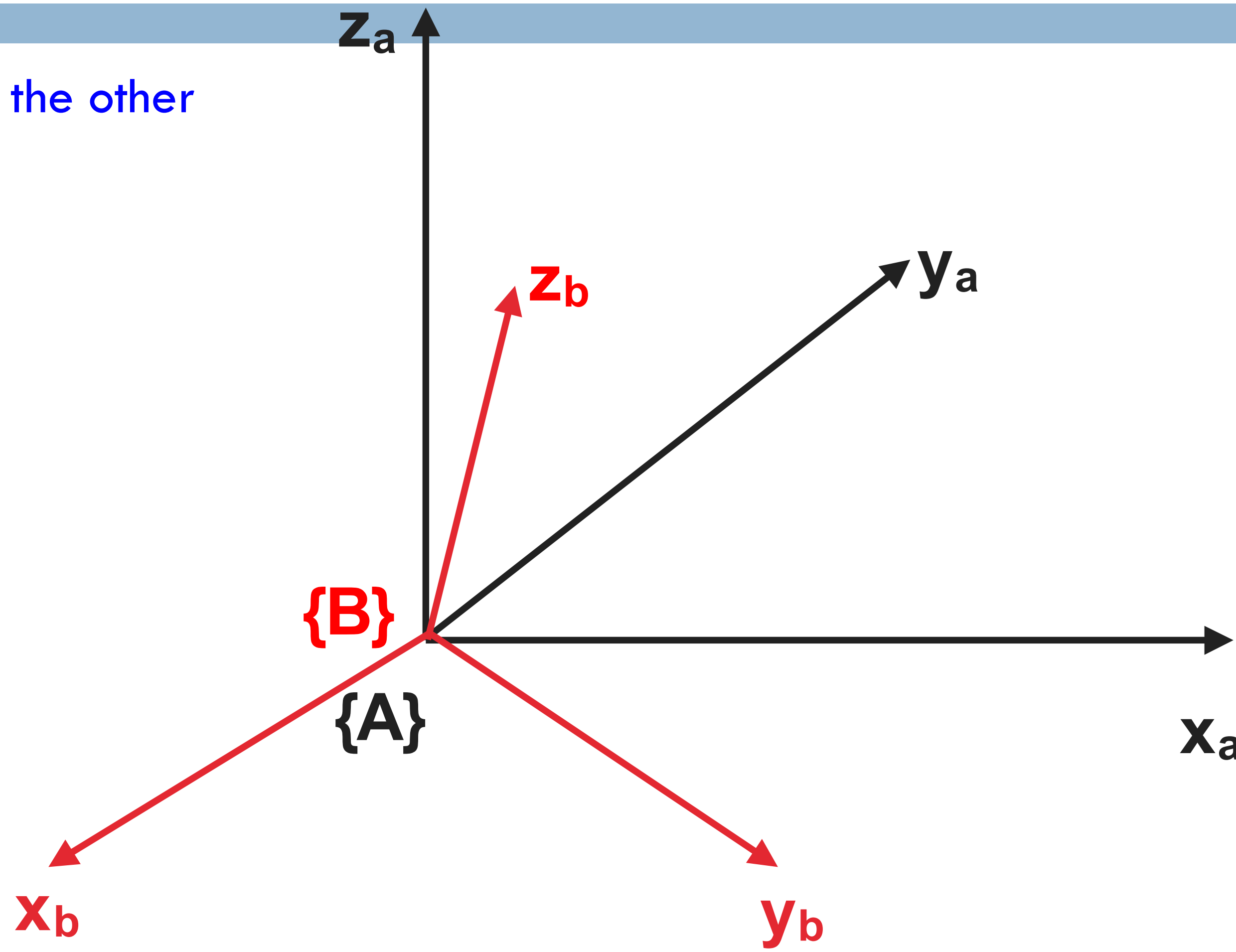
- Representation of “Point” in 3D.
- Pose in 3D.
- Homogenous Transformation Matrix in 3D.

# Today's Discussion

- Three Angle Representation:
  - Euler Angle,
  - Carden Angle.
- 2-Axis Representation.
- Axis Angle Representation.
- Quaternion Representation.
- Comparison.

# Three Angle Representation

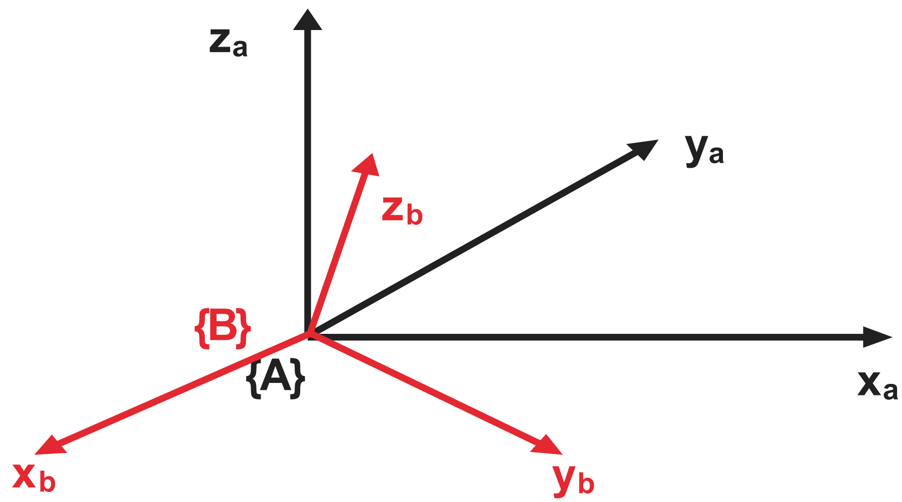
Describing a frame with respect to the other



# Three Angle Representation

- Representing rotation with sequence of elementary rotation.
- **Euler's Rotation Theorem:** Any two independent orthonormal coordinate frame can be related by a sequence of rotations (not more that 3) about coordinate axes, where no two successive rotations may be about same axis.

| | | | |
|---|---|---|---|
| XYX | XYZ | XZY | XZX |
| YXY | YXZ | YZX | YZY |
| ZXY | ZXZ | ZYX | ZYZ |



# Euler Angles

Contain two rotation about same axis but not sequential

| | | | |
|---|---|---|---|
| **XYX** | XYZ | XZY | **XZX** |
| **YXY** | YXZ | YZX | **YZY** |
| ZXY | **ZXZ** | ZYX | **ZYZ** |

# Euler Angles

Contain two rotation about same axis but not sequential

| | | | |
|---|---|---|---|
| **XYX** | XYZ | XZY | **XZX** |
| **YXY** | YXZ | YZX | **YZY** |
| ZXY | **ZXZ** | ZYX | **ZYZ** |

- Total 6 Euler Angles.
- Depending on the application, the sequence is selected. For robotics, aerospace, etc., ZYZ is used.

# Euler Angles

Contain two rotation about same axis but not sequential

$$R = R_z(\phi)R_y(\theta)R_z(\psi)$$

$$\Gamma = (\phi, \theta, \psi)$$

For representing rotation about z- 45, y- 90 and z- 30

$$\Gamma = (45^\circ, 90^\circ, 30^\circ)$$

This can be later converted into rotation matrix if required

# Euler Angles – Special Case

$$\Gamma = (\phi, 0, \psi)$$

$$R = R_z(\phi) I R_z(\psi)$$

$$R_y(\theta) = \begin{pmatrix} \mathrm{Cos}\,\theta & 0 & \mathrm{Sin}\,\theta \\ 0 & 1 & 0 \\ -\mathrm{Sin}\,\theta & 0 & \mathrm{Cos}\,\theta \end{pmatrix}$$

$$R = R_z(\phi) R_z(\psi) = R(\phi + \psi)$$

1 DOF is Lost, this problem is called as **Singularity**.

Popularly called as **Gimbal Lock** problem.

# Carden Angles

| | | | |
|---|---|---|---|
| XYX | **XYZ** | **XZY** | XZX |
| YXY | **YXZ** | **YZX** | YZY |
| **ZXY** | ZXZ | **ZYX** | ZYZ |

- **Roll – Pitch – Yaw Angles – XYZ or ZYX**
  - Roll – Rotation about the forward axis
  - Pitch – elevation of the front with respect to horizontal
  - Yaw – direction of travel
- Depends on the application axis is considered

# Roll – Pitch – Yaw Angles

$$R = R_z(\phi)R_y(\theta)R_x(\psi)$$ Describing the attitude of vehicles such as ships, aircraft and cars

$$R = R_x(\phi)R_y(\theta)R_z(\psi)$$ Describing the attitude of a robot gripper

# Roll – Pitch – Yaw Angles – SPECIAL CASE

$$\Gamma = \left( \phi, \pm\frac{\pi}{2}, \psi \right)$$

$$\Gamma = \left( \phi, +\frac{\pi}{2}, \psi \right)$$

$$R = R_x(\phi) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} R_z(\psi)$$

$$R = R_z(\phi + \psi)\,*$$

1 DoF is lost, this problem is called **Singularity**.

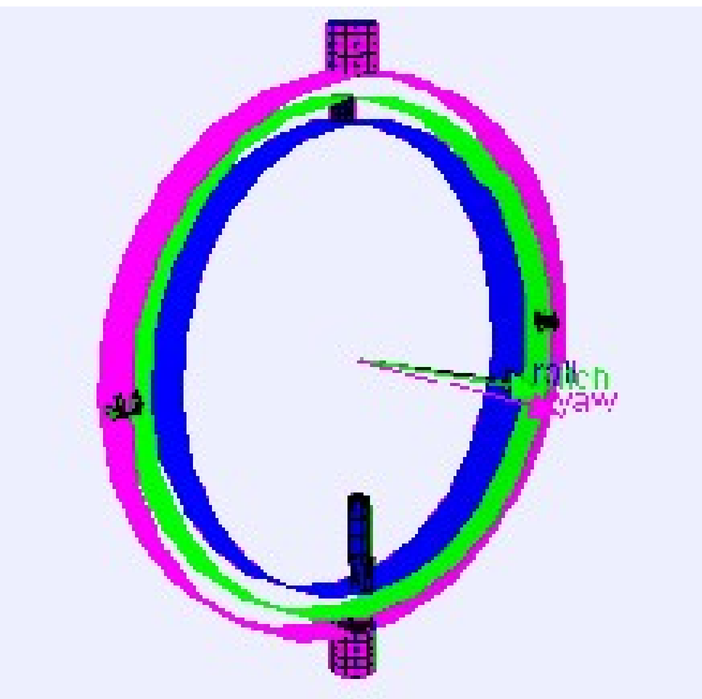
Popularly called as **Gimbal Lock** Problem.

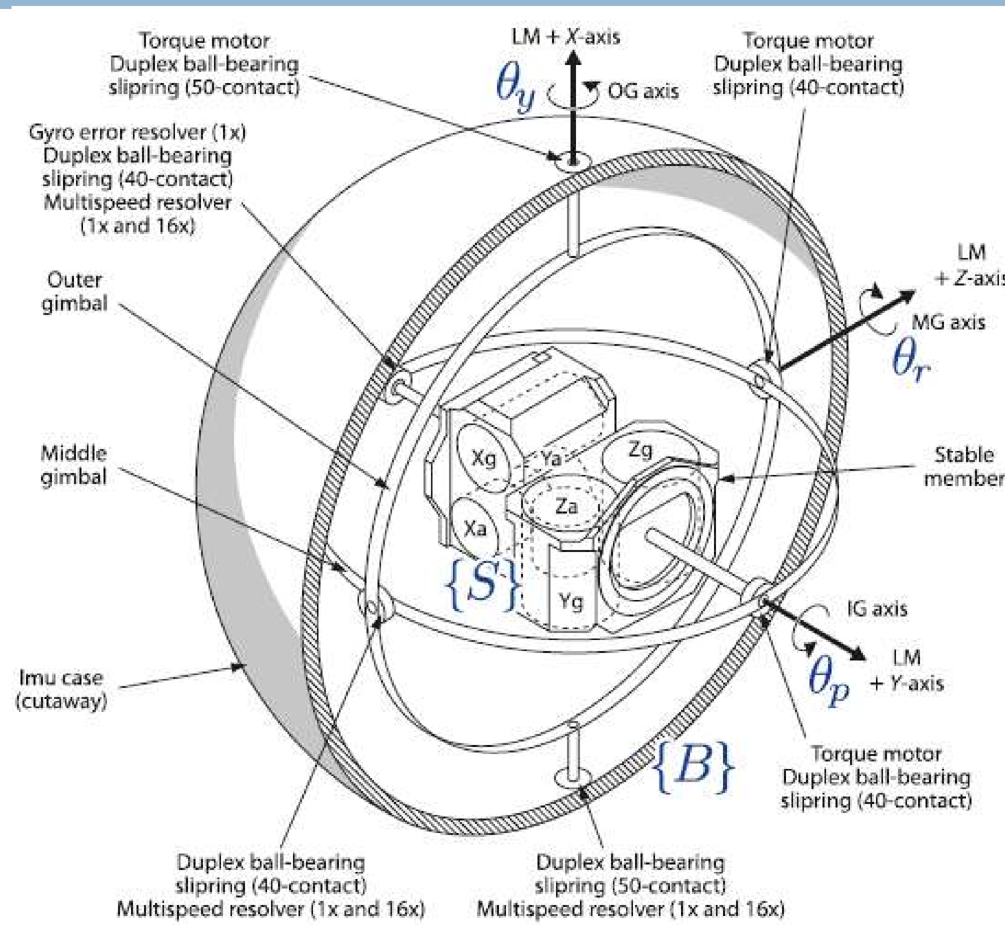
***Proof:** https://en.wikipedia.org/wiki/Gimbal_lock

# Singularity

Also called as **Gimbal Lock.**

What is **Gimbal**?



# Singularity

One DoF is lost.

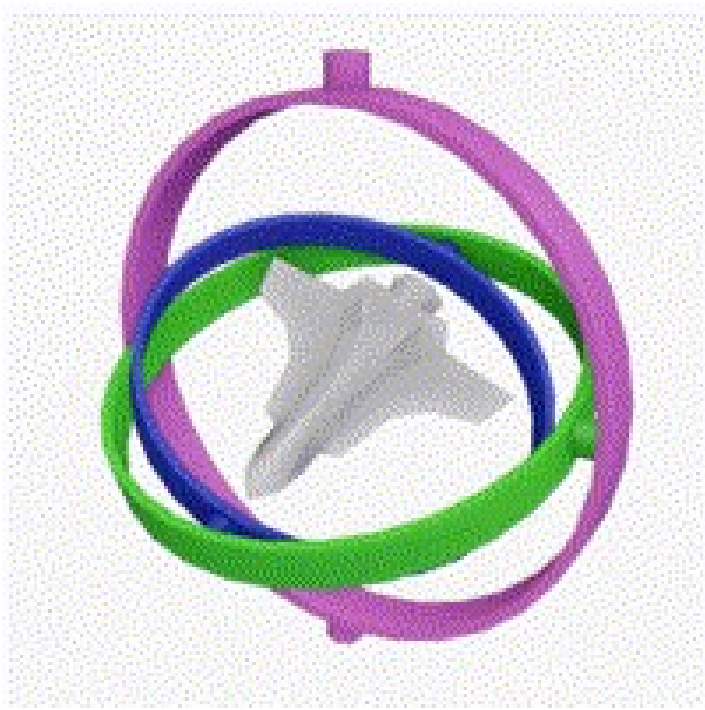
Two rotational axis becomes parallel.

Rotations obey the cyclic rotation rules

$$Rx(\tfrac{\pi}{2})\ Ry(\theta)\ Rx(\tfrac{\pi}{2})^T \equiv Rz(\theta)$$
$$Ry(\tfrac{\pi}{2})\ Rz(\theta)\ Ry(\tfrac{\pi}{2})^T \equiv Rx(\theta)$$
$$Rz(\tfrac{\pi}{2})\ Rx(\theta)\ Rz(\tfrac{\pi}{2})^T \equiv Ry(\theta)$$

and anti-cyclic rotation rules

$$Ry(\tfrac{\pi}{2})^T\ Rx(\theta)\ Ry(\tfrac{\pi}{2}) \equiv Rz(\theta)$$
$$Rz(\tfrac{\pi}{2})^T\ Ry(\theta)\ Rz(\tfrac{\pi}{2}) \equiv Rx(\theta).$$



# Singularity – Solution

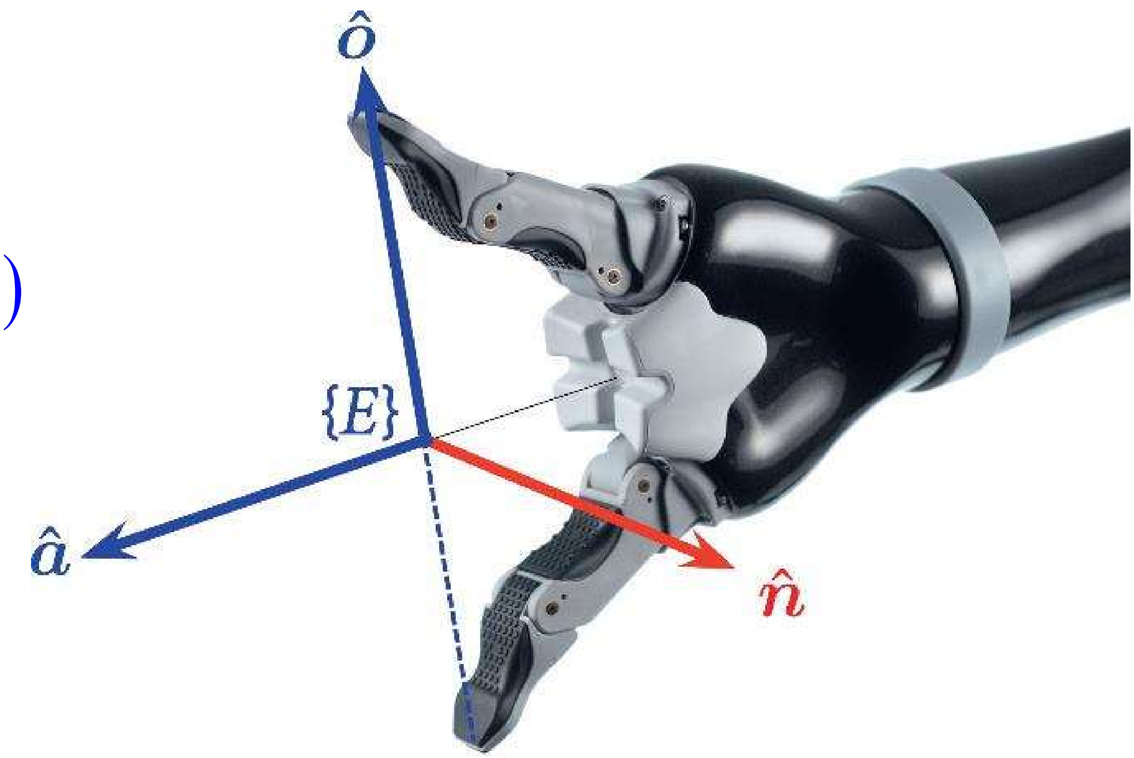
- To add one more active axis
- Avoid locking by moving the gimbals arbitrarily
- Avoid gimbal mechanism
- Use alternate representation

# Two Vector Representation

- **Approach Vector** $\hat{a} = (a_x, a_y, a_z)$
- **Orientation Vector** $o = (o_x, o_y, o_z)$
- **Normal Vector** $n = (n_x, n_y, n_z)$

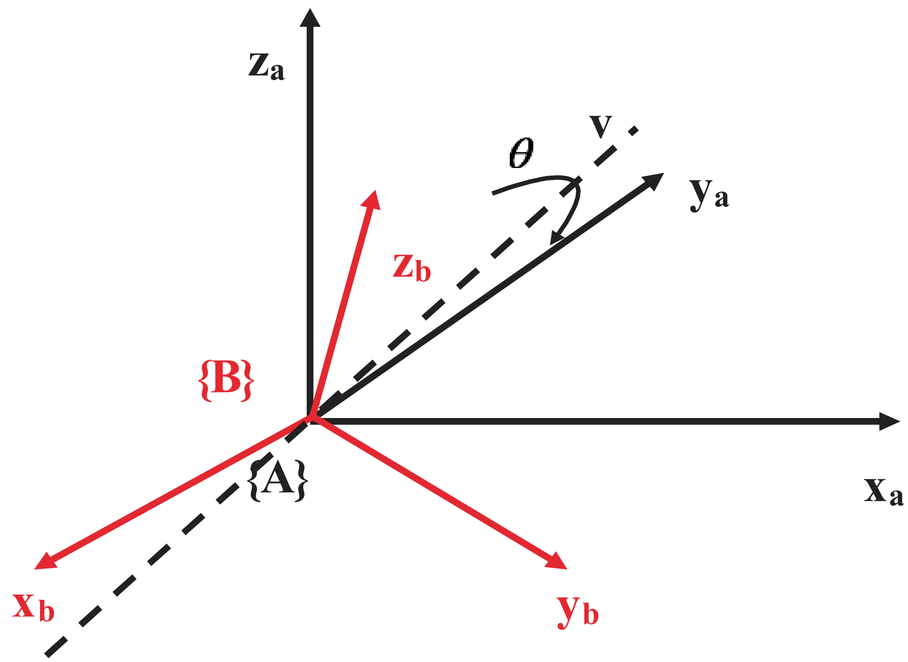
$$R = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}$$

$$n = o \times \hat{a}$$



# Rotation about an Arbitrary Vector

Two coordinate frames of arbitrary orientation are related by a single rotation about some axis in space.




Also called as Axis Angle Representation.

# Rotation about an Arbitrary Vector

- The axis around which the rotation occurs must be unchanged by the rotation
- Therefore the rotation axis must be an Eigen vector of R
- A rotation matrix has three Eigen vector
  - Always one real Eigen vector corresponding to Eigen value of 1
  - Plus two complex Eigen vectors with Eigen values

$$\lambda = cos\theta \pm isin\theta$$

  - Finding rotation matrix is by Rodrigues' rotation formula

$$R = I_{3\times3} + \sin\theta\left[\hat{v}\right]_\times + (1 - \cos\theta)\left[\hat{v}\right]_\times^2$$

# Quaternion Representation

Ordinary Complex Number

$$a + ib \qquad\qquad i^2 = -1$$

Hyper-Complex Number

$$a + ib + jc + kd \qquad\qquad i^2 + j2 + k2 = ijk = -1$$

Quaternion is a hyper-complex number, which is written as scaler and vector

$$s + iv_x + jv_y + kv_z$$

$$s + v$$

# Quaternion Representation

Quaternions were discovered by Sir William Hamilton over 150 years ago and, while initially controversial, have great utility for robotics.

$$\overset{\circ}{q} = s + v$$

$$\overset{\circ}{q} = s\left\langle v_1, v_2, v_3 \right\rangle$$

$$\left|\overset{\circ}{q}\right| = \sqrt{s^2 + v_1^2 + v_2^1 + v_3^2}$$

# Quaternion Representation

Unit quaternion is 1: $\left|\overset{\circ}{q}\right| = 1$

Can be used to encode 3D rotation: $\overset{\circ}{q} = s + v$

Angle and axis representation: $s = cos\dfrac{\theta}{2}$ $v = nsin\dfrac{\theta}{2}$

Compounding: $\overset{\circ}{q}_1 \cdot \overset{\circ}{q}_2 = s_1 \cdot s_2 - v_1 \cdot v_2, \left\langle s_1 v_1 + s_2 v_1 + v_1 \times v_2 \right\rangle$

Hamilton Product Rule

Inverse: $\left(\overset{\circ}{q}\right)^{-1} = s\left\langle -v \right\rangle$

Identity Quaternion: $\overset{\circ}{q} = 1\left\langle 0,0,0 \right\rangle$

# Summary

| Format | Number of Parameters | Singularity | Compounding |
|---|---|---|---|
| Rotation Matrix | 9 | No | matrix multiplication |
| Euler Angle, RPY | 3 | Yes | non-trivial |
| 2 Vector | 6 | No | non-trivial |
| Angle axis | 3(1+2) | No | non-trivial |
| Unit Quaternion | 4 | No | quaternion multiplication |