1 Derivation of the Constriction Factor of PSO with EM

We follow the derivation of the constriction factor according to [1]. Consider a *determinisitc* version of Vanilla PSO equations with $[x, v]$ as a 2-D discrete-time map (equations obtained from [1])

$$v(t+1) = v(t) + \phi(g - x(t)) \tag{1}$$
$$x(t+1) = x(t) + v(t+1) \tag{2}$$

This is deterministic because we have fixed $p_{best} = g_{best} = g$. Let $y(t) = g - x(t)$ and we choose $w = 1$ for simplicity. If we introduce a momentum time-series $m(t)$, we get a 3-D discrete-time map in $[v, y, m]$ for EMPSO as follows

$$v(t+1) = (1-\beta)v(t) + \phi y(t) + \beta m(t) \tag{3}$$
$$y(t+1) = (\beta-1)v(t) + (1-\phi)y(t) - \beta m(t) \tag{4}$$
$$m(t+1) = (1-\beta)v(t) + \beta m(t) \tag{5}$$

The evolution matrix of this system is

$$U = \begin{bmatrix} 1-\beta & \phi & \beta \\ \beta-1 & 1-\phi & -\beta \\ 1-\beta & 0 & \beta \end{bmatrix} \tag{6}$$

The eigenvalues of the matrix U provide important information about the swarm dynamics [2]. According to [1], we are interested in eigenvalues $|\lambda| > 1$ for deriving a constriction factor

$$\lambda_{\pm} = \frac{(2-\phi) \pm \sqrt{\phi^2 - 4(1-\beta)\phi}}{2} \tag{7}$$

[1] mentions that constriction entails finding the scaling factor χ for the eigenvalues λ such that setting $\lambda' = \chi\lambda$ gives $|\lambda'| \leq 1$. Hence it is sufficient to set

$$\chi = \frac{1}{max(|\lambda_+|, |\lambda_-|)} \tag{8}$$

where $\lambda_{\pm}$ is either of the roots defined in eq (7). Based on the discriminant $\Delta = \phi^2 - 4(1-\beta)\phi$, we obtain two cases that give us real/complex roots for λ.
Case 1 ($\Delta \leq 0$) —

$$\begin{aligned} |\lambda| &= \sqrt{\frac{(2-\phi)^2 - \Delta}{4}} \\ &= \sqrt{1 - \beta\phi} \end{aligned} \tag{9}$$

Since $\sqrt{1-\beta\phi}$ is an absolute modulus, we must have

$$\phi \leq \frac{1}{\beta} \tag{10}$$

Moreover, $\Delta \leq 0 \implies \phi \leq 4(1-\beta)$. Also, eq (10) must be satisfied simultaneously. To check whether this is true, construct a function in the range $\beta \in (0,1)$

$$f(\beta) = \frac{1}{\beta} - 4(1-\beta) \tag{11}$$

and its derivative

$$f'(\beta) = 4 - \frac{1}{\beta^2} \tag{12}$$

The critical point is $\beta = \frac{1}{2}$ which is also the global minimum in $\mathbb{R}^+$ due to $f''(\beta) = \frac{2}{\beta^3} > 0$. Moreover, from $f(\frac{1}{2}) = 0$, we have $f(\beta) \geq 0$ in its domain and

$$\begin{aligned} \frac{1}{\beta} &\geq 4(1-\beta) \\ \implies \phi \leq 4(1-\beta) &\leq \frac{1}{\beta} \end{aligned}$$

Eq (10) is thus satisfied and $|\lambda| = \sqrt{1-\beta\phi} \leq 1$, hence we can set $\chi = 1$.
Case 2 ($\Delta > 0$) — Define $\lambda_m = max(|\lambda_\pm|)$. It can be shown that it simplifies to

$$\lambda_m = \frac{|\phi - 2| + \sqrt{\phi^2 - 4(1-\beta)\phi}}{2} \tag{13}$$

Since we are interested in $\lambda_m > 1$, we can do

$$\begin{aligned} \lambda_m^2 &> 1 \\ \implies \quad 4\beta\phi + 4|\phi - 2| &> 8 \end{aligned} \tag{14}$$

Subcase 2.1 ($\phi > 2$) — Eq (14) simplifies to

$$\phi > 4(1+\beta)^{-1} \tag{15}$$

For simplicity, we denote $\omega = 4(1+\beta)^{-1}$. β exists in the interval $0 < \beta < 1$. This implies that $2 < \omega < 4$. Thus, there exists an interval of β, ϕ where eq (15) is satisfied for a suitable interval of ϕ. If we choose $\phi \in [2,4]$, this condition can be met.
Subcase 2.2 ($\phi \leq 2$) — It can be shown that eq (14) leads to $\beta > 1$ which is a contradiction. Hence, this subcase can be ignored.

For the sake of implementation, we adopt a negative constriction co-efficient as has been used in SMPSO for our formulation. Hence, we combine eq (8, 13) and eq (15)

$$\chi^{(m)}(\phi,\beta) = \begin{cases} \frac{2}{2-\phi-\sqrt{\phi^2-4(1-\beta)\phi}} & \phi > 4(1+\beta)^{-1} \\ 1 & \text{otherwise} \end{cases} \tag{16}$$

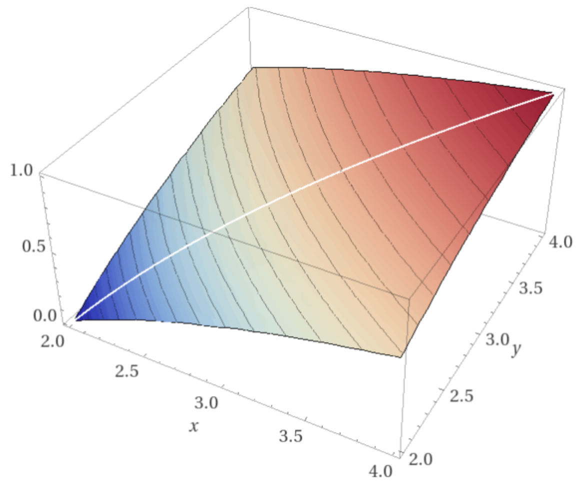

Fig. 1: X-axis $\rightarrow \phi_1$, Y-axis $\rightarrow \phi_2$, Z-axis $\rightarrow P(E)$

2 Fairness Analysis

2.1 Unfairness Value of Vanilla PSO

We have $\phi \sim U(3,5)$ and $\beta \sim U(0,1)$. Hence

$$\begin{aligned}\phi_l &= max(3,2) = 3 \\ \phi_g &= min(4,5) = 4\end{aligned}$$

And the probability integral with $p_\beta(\beta) = 1$ and $p_\phi(\phi) = \frac{1}{2}$

$$\begin{aligned}P(E) &= \int_3^4 \int_{4\phi^{-1}-1}^1 \mathrm{d}\beta \, \frac{\mathrm{d}\phi}{2} + \int_3^4 \frac{\mathrm{d}\phi}{2} \\ &= \frac{3 - 4\ln(4/3)}{2} \end{aligned} \tag{17}$$

Hence, the unfairness value

$$\mu = 1 - 2\ln(4/3) \approx 0.42 \tag{18}$$

2.2 Deriving a Fairly Constricting Parameter Set

We start with the probability integral stated in the main text

$$\begin{aligned}P(E) &= \int_{\phi_1}^{\phi_2} \int_{4/\phi-1}^1 \mathrm{d}\beta \, \frac{\mathrm{d}\phi}{\phi_2 - \phi_1} \\ &= 2 - 4\frac{\ln(\phi_2/\phi_1)}{\phi_2 - \phi_1} \end{aligned} \tag{19}$$

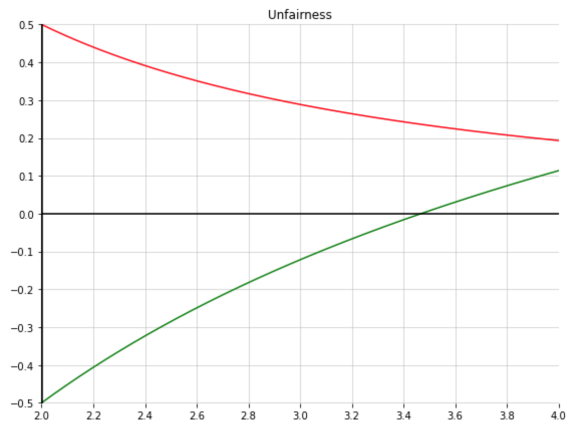

Fig. 2: Green - $\mu(\phi)$, Red - $\frac{\mathrm{d}\mu}{\mathrm{d}\phi}$

Let us arbitrarily assign $\phi_1 = 2$. This is reasonable as per the previous parameter sets studied. We obtain an the unfairness as a function of ϕ_2

$$\mu(\phi_2) = \frac{3}{2} - 4\frac{\ln(\phi_2/2)}{\phi_2 - 2} \tag{20}$$

We transform $\frac{\phi_2}{2} \rightarrow x$ and set $\mu(x) = 0$ to obtain the following transcendental equation in the variable x

$$\psi(x) \equiv \frac{x-1}{\ln x} - \frac{4}{3} = 0 \tag{21}$$

where $\psi(x)$ has been defined for convenience. A solution $\bar{x}$ to this equation must lie in $1 < \bar{x} \leq 2$. Note that $\lim_{x\to 1} \psi(x) = \frac{-1}{3} < 0$ and $\psi(2) = \frac{1}{\ln 2} - \frac{4}{3} > 0$. It is well known that $\psi(x)$ is monotonically increasing (*it is of the form of the asymptotic prime counting function* [3]) and thus a unique solution exists in the range $(1, 2]$. This can also be visually confirmed from the plot of eq (20) in figure (2) [1]

Wolfram Alpha [4] outputs the solution as $\bar{x} \approx 1.7336$ and we obtain $\phi_2 = 2\bar{x} \approx 3.4672$. Hence using $c_1, c_2 \sim U(1, 1.7336)$ and $\beta \sim U(0, 1)$ would result in a fairly constricted algorithm. We call it *Fairly Constricted Particle Swarm Optimization* (**FCPSO**). The surface plot of eq (19) in the (ϕ_1, ϕ_2) plane is plotted in figure (1). Note that there may exist other parameter sets that are also fairly constricting. In this work, we have derived only one fairly constricting set and subsequently used it for benchmarking.

[1] In the plot, the independent variable of the X-axis is the same as ϕ_2 from eq 20. In other words, $\phi \equiv \phi_2$.

References

1. M. Clerc and J. Kennedy, “The particle swarm - explosion, stability, and convergence in a multidimensional complex space,” *IEEE Transactions on Evolutionary Computation*, vol. 6, no. 1, pp. 58–73, 2002.
2. S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering*. Westview Press, 2000.
3. D. Zagier, “Newman’s short proof of the prime number theorem,” *The American Mathematical Monthly*, vol. 104, no. 8, pp. 705–708, 1997. [Online]. Available: https://doi.org/10.1080/00029890.1997.11990704
4. “Wolfram alpha,” https://www.wolframalpha.com/, accessed: 2020-02-09.