Canonical Transformations

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1 Action-Angle Variables

The transformation from the action-angle variables to the usual co-ordinates is as follows -

$$q = \sqrt{\frac{2I}{m\omega}} \sin\theta \tag{7.1}$$

$$p = \sqrt{2Im\omega}\cos(\theta) \tag{7.2}$$

This can be inverted to obtain the canonical transformation -

$$\theta = \tan^{-1}\left(\frac{m\omega q}{p}\right) \tag{7.3}$$

$$I = \frac{p^2}{2m\omega} + \frac{m\omega q^2}{2} \tag{7.4}$$

For the F_1 generating function, the following condition must be satisfied -

$$\begin{aligned}\frac{\partial F_1(q,\theta)}{\partial q} &= p = qm\omega\cot\theta \\ F_1 &= \int qm\omega\cot\theta dq \\ F_1 &= \frac{q^2 m\omega}{2}\cot\theta + h(\theta) \qquad (7.5)\end{aligned}$$

F_1 is variable upto a function of θ. Using the second equation of the generating function -

$$\begin{aligned}\frac{\partial F_1(q,\theta)}{\partial\theta} &= -I \\ -\frac{q^2 m\omega}{2}\csc^2\theta + h'(\theta) &= -I\end{aligned}$$

But from (7.1), we have $I = \frac{q^2 m\omega}{2}\csc^2\theta$, and thus $h'(\theta) = 0$. Since constants do not play a role in the generating function, we have the result -

$$F_1 = \frac{q^2 m\omega}{2}\cot\theta$$

2 A Particular Generating Function

The generating function $F_1 = cq^2 \cot Q$ gives us the relations -

$$p = \frac{\partial F_1}{\partial q} = 2cq\cot Q \tag{8.1}$$

$$P = -\frac{\partial F_1}{\partial Q} = cq^2\csc^2 Q \implies q^2 = \frac{P}{c}\sin^2 Q \tag{8.2}$$

Equation (8.2) does not describe the canonical transformation entirely as it depends on Q. The transformation is correctly described as -

$$Q = \tan^{-1}\left(\frac{2cq}{p}\right) \tag{8.3}$$

$$P = \frac{p^2 + 4c^2q^2}{4c} \tag{8.4}$$

Equation (8.4) was obtained using (8.3), the pythagorean theorem and substituting into (8.2). Substituting these transformation equations into the hamiltonian $H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$, we get -

$$\begin{aligned}
K(Q,P) &= \frac{4c^2q^2}{2m}\cot^2 Q + \frac{Pm\omega^2}{2c}\sin^2 Q \\
&= \left(\frac{4c^2}{2m}\cot^2 Q \times \frac{P}{c}\sin^2 Q\right) + \frac{Pm\omega^2}{2c}\sin^2 Q \\
&= \frac{2Pc}{m}\cos^2 Q + \frac{Pm\omega^2}{2c}\sin^2 Q
\end{aligned} \tag{8.5}$$

Under this canonical transformation, the Hamilton's equations from (8.5) are -

$$\dot{P} = -\frac{\partial K}{\partial Q} = \frac{4Pc}{m}\sin Q\cos Q - \frac{Pm\omega^2}{c}\sin Q\cos Q \tag{8.6}$$

$$\dot{Q} = \frac{\partial K}{\partial P} = \frac{2c}{m}\cos^2 Q + \frac{m\omega^2}{2c}\sin^2 Q \tag{8.7}$$

Observing that $\dot{Q} = \frac{K}{P}$, where K is the value of the Hamiltonian at the initial condition (Q_0, P_0), the equations of motions simplify to -

$$\dot{P} = \frac{P \sin 2Q}{2} \left(\frac{4c}{m} - \frac{m\omega^2}{c} \right) \tag{8.7}$$

$$\dot{Q} = \frac{K}{P} \tag{8.8}$$

The above two equations is a system of differential equations in (Q,P) and is highly non-linear. However, a simplification can be observed by setting $\dot{P} = 0$ which gives $c = \frac{m\omega}{2}$. For this particular value of the constant c, the new variables reduce to the action-angle variables.

2.1 Solving the Systems in the Old and New Variables

In the old variables, the solution is trivial -

$$q = A \cos(\omega t + \phi) + B \sin(\omega t + \phi) \tag{8.9}$$

$$p = m\omega \left[B \cos(\omega t + \phi) - A \sin(\omega t + \phi) \right] \tag{8.10}$$

The values of A, B are obtained from the initial conditions (q_0, p_0). For the purposes of demonstration, let us set $m = 1$ and $\omega = 2$. In the old variables, the phase plot is as follows for $(q_0, p_0) = (0, 1)$ -

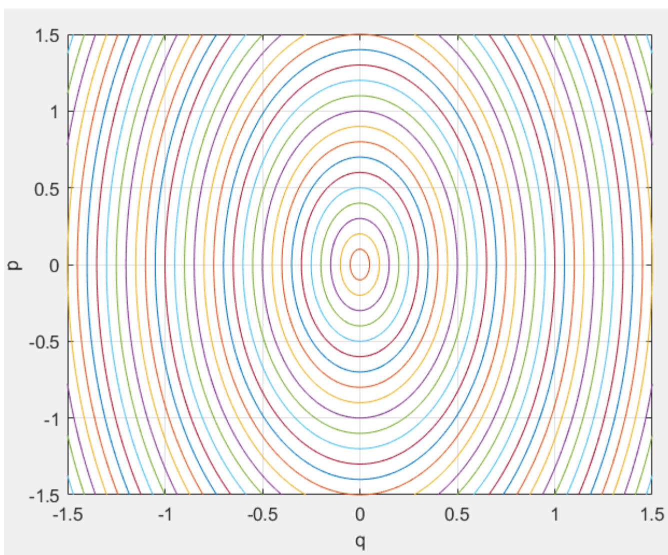

Figure 1: Phase potrait of harmonic oscillator

According to the equations (8.3) and (8.4), the initial conditions under the canonical transformation are $(Q_0, P_0) = (0, \frac{1}{4c})$. The value of c such that this transformation degeneratoes to the action-angle variables is $c = \dfrac{m\omega}{2} = 1$. The initial value of the Hamiltonian is $H = K = \dfrac{p^2}{2m} + \dfrac{m\omega^2 q^2}{2}\Big|_{(0,1)} = \dfrac{1}{2}$. The range of the Q-axis is $[0, 2\pi]$.

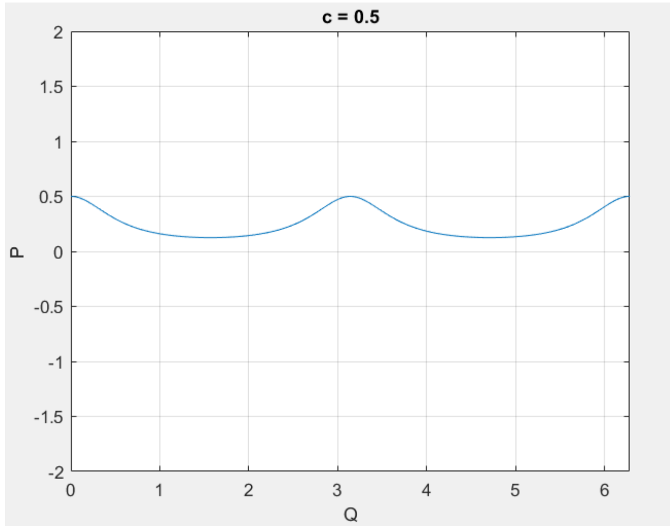

Figure 2: c = 0.5

The next plot is for $c = 1$. This is expected to boil down to the action-angle transformation (*and it does!*)

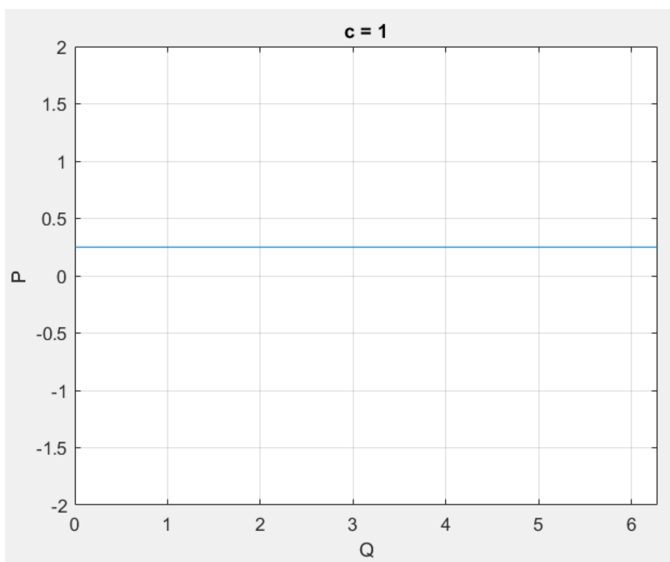

Figure 3: c = 1.0

The last figure is for $c = 2$ -

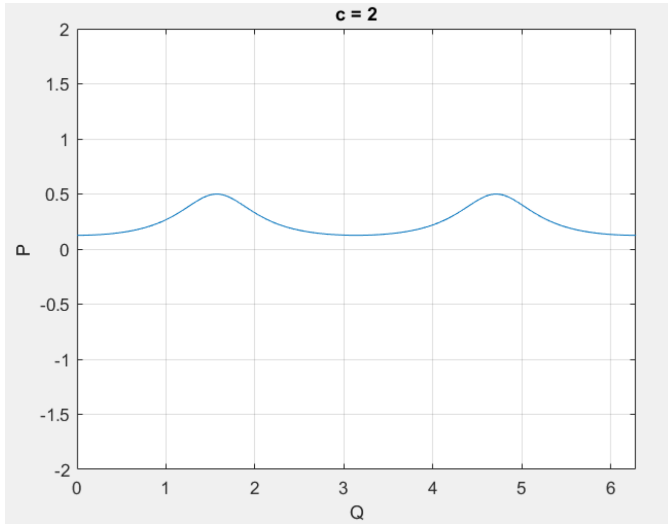

Figure 4: c = 2.0

It is also interesting to observe the phase potraits for various initial conditions $(q_0, p_0) \to (Q_0, P_0)$, for particular values of c. In the case of $c = \frac{1}{2}$, it will be a bunch of horizontal lines at various heights from the Q-axis. The following plot is the phase potrait for $c = 0.9$.

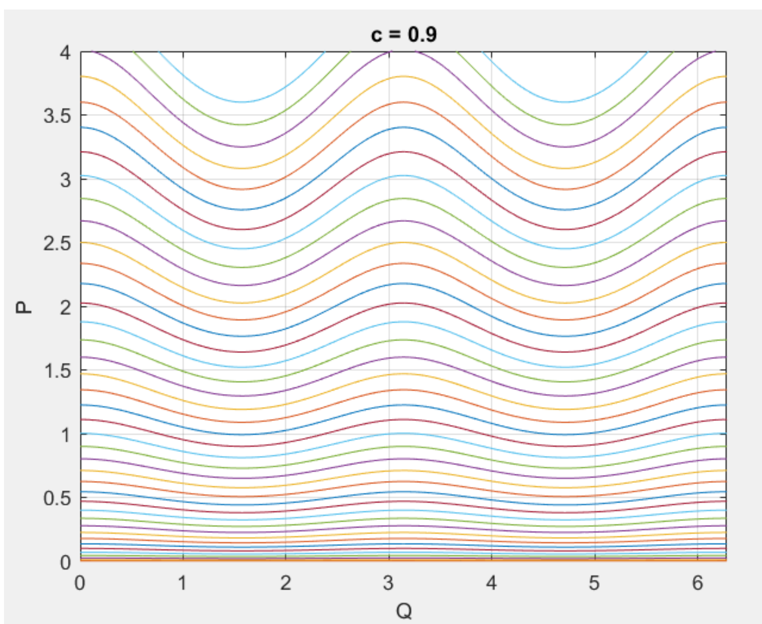

Figure 5: Phase potrait for C=0.9

Finally, the following is the phase potrait for $c = 3.0$ (*next page*)

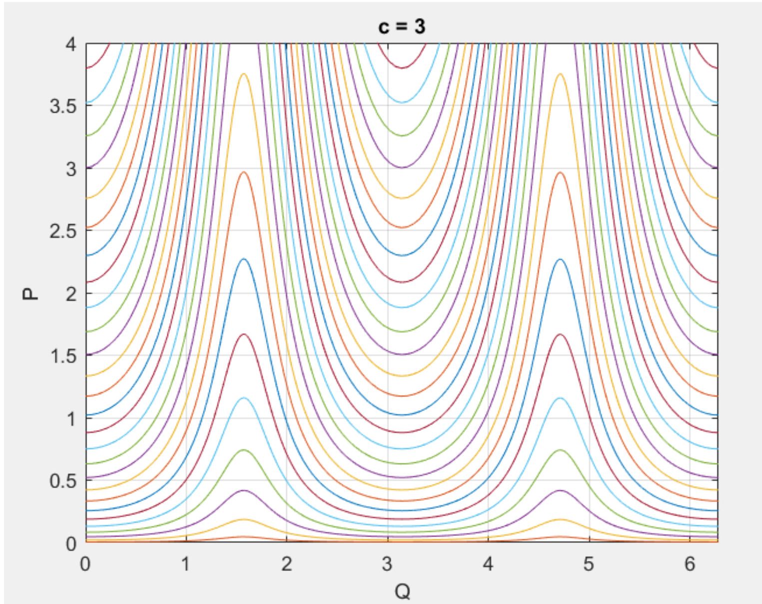

Figure 6: Phase potrait for C=3.0

It is easily noticeable the difference in the contours this image and the previous one. One is for $c < 2$ and the other is $c > 2$.