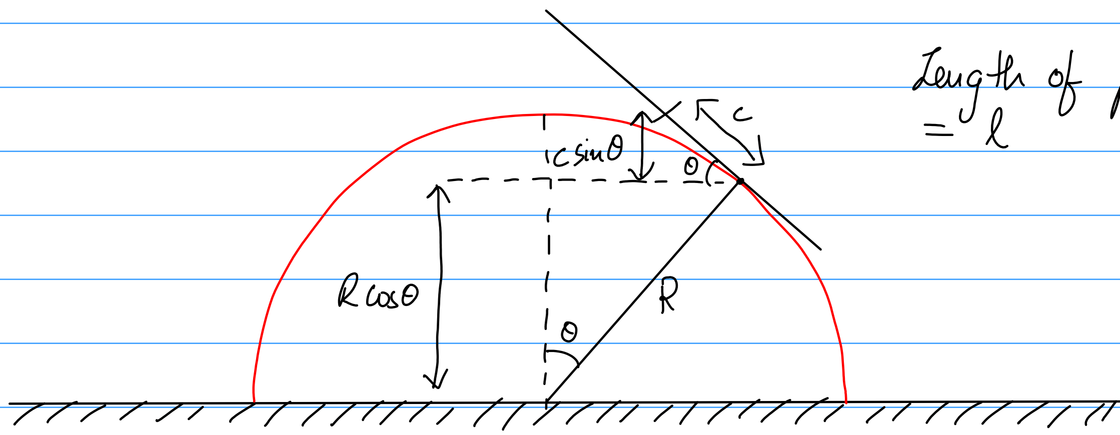
Plank on a Hemisphere



Length of plank $= l$

The θ system (L_1) –

Instantaneous rotation about the point of contact

$$I = m\left[\frac{l^2}{12} + c^2\right]$$

$$V = mg(R\cos\theta + \sin\theta)$$

$$T = \frac{m}{2}\left[\frac{l^2}{12} + c^2\right]\dot{\theta}^2$$

However, we know that $c = R\theta$ because plank was centrally placed & no slipping occurs!

$$\Rightarrow L_1 \equiv L = T - V$$

$$= \frac{m}{2}\left[\frac{l^2}{12} + R^2\theta^2\right]\dot{\theta}^2 - mgR(\cos\theta + \theta\sin\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

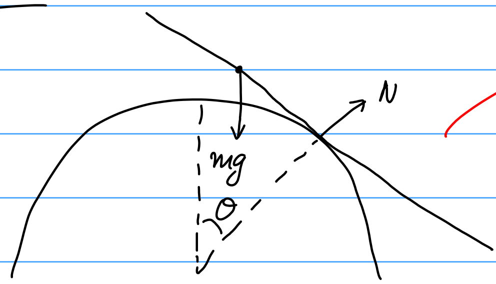
$$\frac{d}{dt}\left[m\left(\frac{l^2}{12} + R^2\theta^2\right)\dot{\theta}\right] - \frac{m}{2}\left[\frac{l^2}{12} + 2R^2\theta\right]\dot{\theta}^2 - mgR\left(\cancel{-\sin\theta} + \cancel{\sin\theta} + \theta\cos\theta\right) = 0$$

$$m\left(\frac{l^2}{12} + R^2\theta^2\right)\ddot{\theta} + 2mR^2\theta\dot{\theta}^2 - mR^2\theta\dot{\theta}^2 - mgR\theta\cos\theta = 0$$

$$\left(\frac{l^2}{12} + R^2\theta^2\right)\ddot{\theta} + R^2\theta\dot{\theta}^2 + gR\theta\cos\theta = 0$$

$$\ddot{\theta} = \frac{-\theta\left(\dot{\theta}^2 + g/R\cos\theta\right)}{\frac{(l/R)^2}{12} + \theta^2} \quad \text{—— I}$$

High School Treatment -



We would also say that since $\theta << 1$, $I \approx \frac{ml^2}{12}$

$$\tau = I\omega$$

$$R\theta \cancel{m}g\cos\theta = \cancel{m}\left[\frac{l^2}{12} + (R\theta)^2\right]\ddot{\theta} \qquad Rg\theta = \frac{l^2\ddot{\theta}}{12}$$

For $\theta << 1$, $R\theta << l$, $\cos\theta = 1$

$$\omega = \sqrt{\frac{12Rg}{l^2}}$$

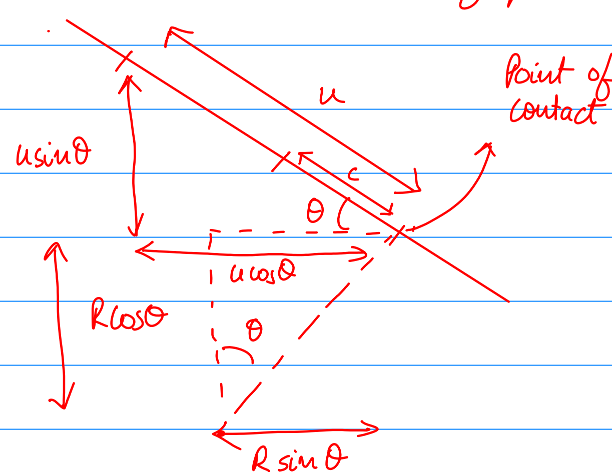
Kinetic Energy of the Plank -

We have assumed instantaneous rotation of the plank about its point of contact on the hemisphere. Let's examine the kinetic energy of the plank in the most general case.

$c \equiv$ distance to COM of plank

Let u be the length up/down the plank from its point of contact. The position coordinates of it are —

$$x = R\sin\theta - u\cos\theta$$
$$y = R\cos\theta + u\sin\theta$$



$$\dot{x} = (R\dot{\theta} - \dot{u})\cos\theta + (u\dot{\theta})\sin\theta$$
$$\dot{y} = (\dot{u} - R\dot{\theta})\sin\theta + (u\dot{\theta})\cos\theta$$

Since the plank is a rigid body, $\dot{u} = \dot{c}$

$$\dot{x} = (R\dot{\theta} - \dot{c})\cos\theta + (u\dot{\theta})\sin\theta \quad \& \quad \dot{y} = (\dot{c} - R\dot{\theta})\sin\theta + (u\dot{\theta})\cos\theta$$

$$v^2 = \dot{x}^2 + \dot{y}^2 = (R\dot{\theta} - \dot{c})^2 \cos^2\theta + (u\dot{\theta})^2 \sin^2\theta + \cancel{2\cos\theta\sin\theta\,(R\dot{\theta}-\dot{c})}(u\dot{\theta})$$
$$+ (R\dot{\theta} - \dot{c}) \sin^2\theta + (u\dot{\theta})^2 \cos^2\theta - \cancel{2\cos\theta\sin\theta\,(R\dot{\theta}-\dot{c})}(u\dot{\theta})$$
$$= (R\dot{\theta} - \dot{c})^2 + (u\dot{\theta})^2$$

$$\int_{\text{Plank}} v^2 \, du = \int_{c-\ell/2}^{c+\ell/2} (R\dot{\theta} - \dot{c})^2 + (u\dot{\theta})^2 \, du$$

$$= (R\dot{\theta} - \dot{c})^2 u \Big|_{c-\ell/2}^{c+\ell/2} + \frac{\dot{\theta}^2 u^3}{3}\Big|_{c-\ell/2}^{c+\ell/2}$$

$$= (R\dot{\theta} - \dot{c})^2 \ell + \frac{\dot{\theta}^2}{3}\left[(c+\ell/2)^3 - (c-\ell/2)^3\right]$$

$$= (R\dot{\theta} - \dot{c})^2 \ell + \frac{\dot{\theta}^2}{3}\left[3c^2\ell + \ell^3/4\right]$$

$$= \ell\left[(R\dot{\theta} - \dot{c})^2 + \dot{\theta}^2\left(c^2 + \ell^2/12\right)\right]$$

Kinetic energy of sliding

Instantaneous rotation energy

$$T = \int_{\text{Plank}} \frac{1}{2}\left(\frac{m}{\ell}\right) v^2 \, du = \frac{m}{2}\left[(R\dot{\theta} - \dot{c})^2 + \dot{\theta}^2\left(c^2 + \ell^2/12\right)\right]$$

$$V = mg(R\cos\theta + c\sin\theta)$$

From this, we can construct a Lagrangian that captures the motion of the plank in two dimensions — rotation about hemisphere and slipping at contact. We can also seen by careful integration that if $R\dot{\theta} = \dot{c}$ (no slipping), an instantaneous rotation of the plank occurs.

The (θ, c) system (L_2) -

In this system, the hemisphere offers the necessary normal force to sustain the plank on its surface. However, there is no dissipation of energy due to sliding friction.

$$L_2 \equiv L = T - V$$
$$= \frac{m}{2}\left[(R\dot{\theta}-\dot{c})^2 + \dot{\theta}^2\left(c^2+\ell^2/12\right)\right] - mg\left(R\cos\theta + c\sin\theta\right)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\left[R(R\dot{\theta}-\dot{c}) + \dot{\theta}\left(c^2+\ell^2/12\right)\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\left[R(R\ddot{\theta}-\ddot{c}) + \ddot{\theta}\left(c^2+\ell^2/12\right) + 2c\dot{c}\dot{\theta}\right]$$

$$\frac{\partial L}{\partial \theta} = mg\left(R\sin\theta - c\cos\theta\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \Rightarrow \quad \cancel{m}\left[R(R\ddot{\theta}-\ddot{c}) + \ddot{\theta}\left(c^2+\ell^2/12\right) + 2c\dot{c}\dot{\theta}\right] - \cancel{m}g\left(R\sin\theta - c\cos\theta\right) = 0$$

$$\Rightarrow \left(R^2+c^2+\ell^2/12\right)\ddot{\theta} - R\ddot{c} + 2c\dot{c}\dot{\theta} - g\left(R\sin\theta - c\cos\theta\right) = 0$$
$$\Rightarrow \ddot{\theta} - \left(\frac{R\ddot{c} - 2c\dot{c}\dot{\theta} + g\left(R\sin\theta - c\cos\theta\right)}{R^2+c^2+\ell^2/12}\right) = 0 \quad \text{—} \quad \text{II.a}$$

$$\frac{\partial L}{\partial \dot{c}} = m\left(\dot{c} - R\dot{\theta}\right) \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{c}}\right) = m\left(\ddot{c} - R\ddot{\theta}\right)$$

$$\frac{\partial L}{\partial c} = m\dot{\theta}^2 c - mg\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{c}}\right) - \frac{\partial L}{\partial c} = 0 \quad \Rightarrow \quad \ddot{c} - R\ddot{\theta} - \dot{\theta}^2 c + g\sin\theta = 0 \quad \text{—} \quad \text{II.b}$$

We can recover (I) from (II.a) and (II.b) by introducing the constraint of no slipping $dc = R d\theta$

$$\left.\begin{aligned} \Rightarrow \quad \dot{c} &= R\dot{\theta} \\ \Rightarrow \quad \ddot{c} &= R\ddot{\theta} \end{aligned}\right\} \quad - (1)$$

This non-holonomic constraint could be integrated –

$$\int dc = \int R\, d\theta$$

$$c = R\theta + c_0$$

c_0 gives the initial placement of the plank. $c_0 = 0$ means the plank was centrally placed. We now have a holonomic constraint equation –

$$K \equiv c - R\theta - c_0 = 0$$

(II.b) now gets modified to – ($\lambda \leftarrow$ Lagrange Multiplier)

$$\cancel{\ddot{c}} - \cancel{R\ddot{\theta}} - \dot{\theta}^2 c + g\sin\theta + \frac{\lambda}{m}\frac{\partial K}{\partial c} = 0 \qquad [\text{Eq } (1)]$$

$$\lambda = m(\dot{\theta}^2 c - g\sin\theta) \qquad \left[\frac{\partial K}{\partial c} = 1\right]$$

This value of λ can be used in (II.a) as the lagrange multiplier –

$$(R^2 + c^2 + \ell^2/12)\ddot{\theta} - R\ddot{c} + 2c\dot{c}\dot{\theta} - g(R\sin\theta - c\cos\theta) + \frac{\lambda}{m}\frac{\partial K}{\partial \theta} = 0$$

Use the fact that $\dot{c} = R\dot{\theta}$, $\ddot{c} = R\ddot{\theta}$ and $c_0 = 0$ –

$$(\cancel{R^2} + R^2\theta^2 + \ell^2/12)\ddot{\theta} - \cancel{R^2\ddot{\theta}} + 2R^2\theta\dot{\theta}^2 - gR(\sin\theta - \theta\cos\theta) - \frac{\lambda R}{m} = 0 \qquad \left[\frac{\partial K}{\partial \theta} = -R\right]$$

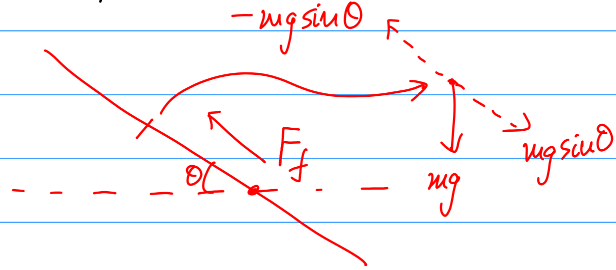
$$(R^2\theta^2 + \ell^2/12)\ddot{\theta} + 2R^2\theta\dot{\theta}^2 - gR(\sin\theta - \theta\cos\theta) - \frac{R}{\cancel{m}}\cancel{m}(\dot{\theta}^2 c - g\sin\theta) = 0$$

$$(R^2\theta^2 + \ell^2/12)\ddot{\theta} + 2R^2\theta\dot{\theta}^2 - \cancel{gR\sin\theta} + gR\theta\cos\theta - R^2\theta\dot{\theta}^2 + \cancel{gR\sin\theta} = 0$$

$$(R^2\theta^2 + \ell^2/12)\ddot{\theta} + R^2\theta\dot{\theta}^2 + gR\theta\cos\theta = 0$$

$$\ddot{\theta} = \frac{-\theta(\dot{\theta}^2 + g/R\cos\theta)}{\theta^2 + \frac{(\ell/R)^2}{12}} \longrightarrow \text{Eq (I) recovered.}$$

The integrated non-holonomic constraint entails that we endow the hemisphere with static friction (of infinite capacity). This is the lagrange multiplier –



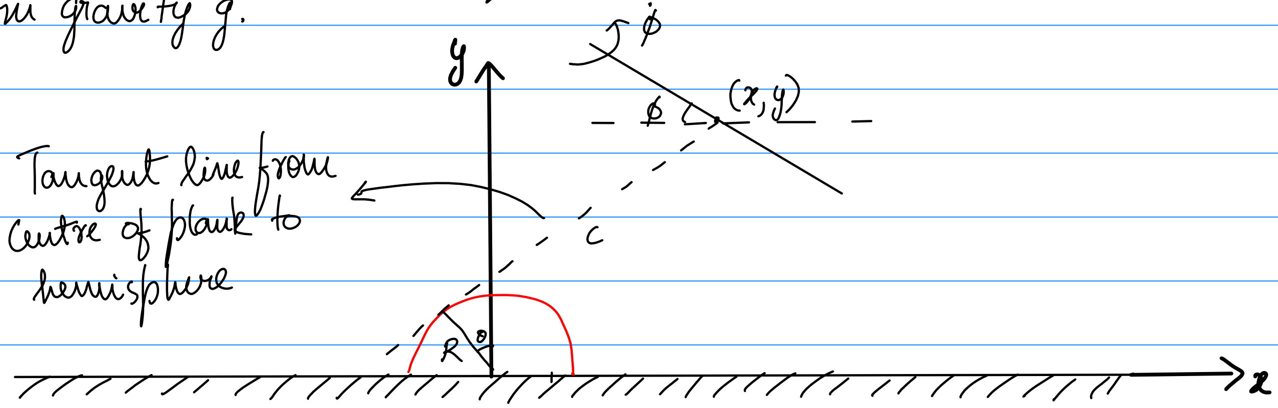
$\lambda = m(\dot{\theta}^2 c - g\sin\theta)$

$\lambda = m(R\theta\dot{\theta}^2 - g\sin\theta)$

$\hookrightarrow$ Force of friction F_f.

Complete Motion of the Plank –

Imagine the plank had complete freedom to move above the ground in uniform gravity g.



$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\phi}^2$ $\qquad$ $V = mgy$

$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\phi}^2 - mgy$

We would like to recover the (θ, c) system from our formulation with 3 generalized co-ordinates. Instead of using (x, y) to denote the COM of the plank, we use (θ, c) as it is more amenable to imposing constraints.

θ $\equiv$ Angle which the line of tangency from the COM to the hemisphere makes with the horizontal

c $\equiv$ Distance from point of tangency to COM

It is easy to see that the (θ, c, ϕ) system is equivalent to the (x, y, ϕ) system as it can describe all possible configurations of the freely moving (and rotating) plank.

$$x = R\sin\theta - c\cos\theta \qquad \Longrightarrow \qquad \dot{x}^2 + \dot{y}^2 = (R\dot{\theta} - \dot{c})^2 + (c\dot{\theta})^2 \quad \text{— } (2)$$
$$y = R\cos\theta + c\sin\theta$$

$$\mathcal{L} = \frac{m}{2}\left[(R\dot{\theta} - \dot{c})^2 + (c\dot{\theta})^2\right] + \frac{1}{2}I\dot{\phi}^2 - mg(R\cos\theta + c\sin\theta)$$

If the plank is on the hemisphere, then $\theta = \phi$ at all ti

$K \equiv \theta - \phi = 0$ and we also have $\dot{\theta} = \dot{\phi}$, $\ddot{\theta} = \ddot{\phi}$

$$\frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = I\ddot{\phi}, \quad \frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} + \lambda\frac{\partial K}{\partial \phi} = 0$$

$$I\ddot{\phi} - \lambda = 0$$

$$\lambda = I\ddot{\phi} \quad \text{—————} \quad (3)$$

Physically this describes the fact that if we constrain the plank to slide on the hemisphere, it will experience a torque. The Euler-Lagrange equation for c remains unchanged $\Rightarrow \ddot{c} - R\ddot{\theta} - \dot{\theta}^2 c + g\sin\theta = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = m\left[R(R\dot{\theta} - \dot{c}) + c^2\dot{\theta}\right] \qquad \frac{\partial L}{\partial \theta} = mg(R\sin\theta - c\cos\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\left[R^2\ddot{\theta} - R\ddot{c} + 2c\dot{c}\dot{\theta} + c^2\ddot{\theta}\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \lambda\frac{\partial K}{\partial \theta} = 0$$

$$m\left[R^2\ddot{\theta} - R\ddot{c} + 2c\dot{c}\dot{\theta} + c^2\ddot{\theta}\right] - mg(R\sin\theta - c\cos\theta) + \lambda = 0$$
$$(R^2+c^2)\ddot{\theta} - R\ddot{c} + 2c\dot{c}\dot{\theta} - gR\sin\theta + gc\cos\theta + \frac{\lambda}{m} = 0$$

But $\dfrac{\lambda}{m} = \dfrac{I\ddot{\phi}}{m} = \dfrac{\cancel{m}l^2}{12.\cancel{m}}\ddot{\theta} = l^2/12\,\ddot{\theta}$ $\qquad [\ddot{\phi} = \ddot{\theta}]$

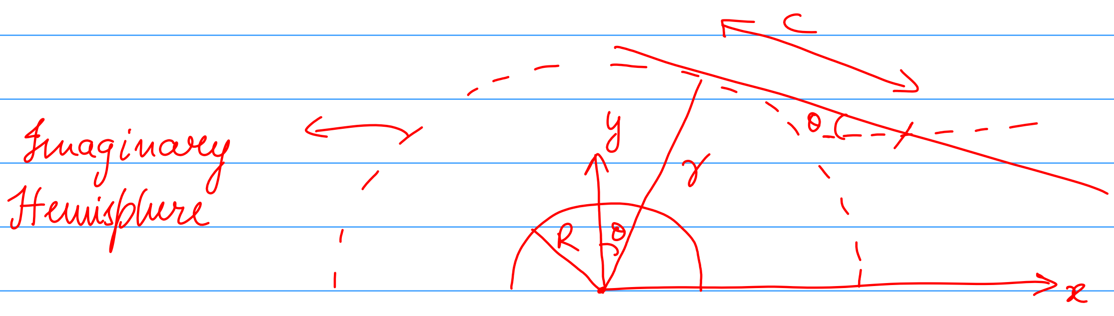
$$(R^2+c^2+l^2/12)\ddot{\theta} - R\ddot{c} + 2c\dot{c}\dot{\theta} - g(R\sin\theta - c\cos\theta) = 0$$
$$\ddot{\theta} - \left[\frac{R\ddot{c} - 2c\dot{c}\dot{\theta} + g(R\sin\theta - c\cos\theta)}{R^2+c^2+l^2/12}\right] = 0 \longrightarrow$$ Eq II.a is recovered.

Thus the (θ, c, ϕ) system is consistent with our previous results.
We would like to obtain the normal force on the plank by the hemisphere.
However, this can't be obtained from the present generalized co-ordinates as it's not the most suitable.

If we take the transformation equations 2 and change R to a free parameter r —

$$\left.\begin{aligned} x &= r\sin\theta - c\cos\theta \\ y &= r\cos\theta + c\sin\theta \end{aligned}\right\} \text{—— } 4$$

Physically, this means that at any angular orientation ϕ of the plank, it will be tangential to some suitable hemisphere of radius r. In this orientation $\phi = \theta$, and hence our coordinates are (r, c, θ).



If u is the distance to a point on the plank from the point of contact to the imaginary hemisphere, then the coordinates of the point are –

$x = r\sin\theta - u\cos\theta$

$y = r\cos\theta + u\sin\theta$

Differentiating –

$\dot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta - \dot{u}\cos\theta + u\dot{\theta}\sin\theta = (\dot{r}+u\dot{\theta})\sin\theta + (r\dot{\theta}-\dot{u})\cos\theta$

$\dot{y} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta + \dot{u}\sin\theta + u\dot{\theta}\cos\theta = (\dot{u}-r\dot{\theta})\sin\theta + (\dot{r}+u\dot{\theta})\cos\theta$

$$\begin{aligned}
v^2 &= \dot{x}^2 + \dot{y}^2 \\
&= (\dot{r}+u\dot{\theta})^2\sin^2\theta + (r\dot{\theta}-\dot{u})^2\cos^2\theta + 2\cancel{(\dot{r}+u\dot{\theta})(r\dot{\theta}-\dot{u})\sin\theta\cos\theta} \\
&\quad + (\dot{r}+u\dot{\theta})^2\cos^2\theta + (r\dot{\theta}-\dot{u})^2\sin^2\theta - 2\cancel{(\dot{r}+u\dot{\theta})(r\dot{\theta}-\dot{u})\sin\theta\cos\theta} \\
&= (\dot{r}+u\dot{\theta})^2 + (r\dot{\theta}-\dot{u})^2 \\
&= (\dot{r}+u\dot{\theta})^2 + (r\dot{\theta}-\dot{c})^2
\end{aligned}$$

[$\dot{u} = \dot{c}$ as plank is rigid]

$$\int_{\text{plank}} v^2\,du = \int_{c-l/2}^{c+l/2} (\dot{r}+u\dot{\theta})^2 + (r\dot{\theta}-\dot{c})^2\,du$$

$$= \left.\frac{(\dot{r}+u\dot{\theta})^3}{3\dot{\theta}}\right|_{c-l/2}^{c+l/2} + \left.(r\dot{\theta}-\dot{c})^2 u\right|_{c-l/2}^{c+l/2}$$

$$= \frac{1}{3\dot{\theta}}\left\{\left[\dot{r}+(c+l/2)\dot{\theta}\right]^3 - \left[\dot{r}+(c-l/2)\dot{\theta}\right]^3\right\} + (r\dot{\theta}-\dot{c})^2 l$$

$$= \frac{1}{3\dot{\theta}}\left\{\left[(\dot{r}+c\dot{\theta}) + l/2\,\dot{\theta}\right]^3 - \left[(\dot{r}+c\dot{\theta}) - l/2\,\dot{\theta}\right]^3\right\} + (r\dot{\theta}-\dot{c})^2 l$$

$$= 6(\dot{r}+c\dot{\theta})^2 l/2\,\dot{\theta} + 2(l/2)^3\dot{\theta}^3 = 3(\dot{r}+c\dot{\theta})^2 l\dot{\theta} + (l^3/4)\dot{\theta}^3$$

$$= l\left[(\dot{r}+c\dot{\theta})^2 + l^2/12\,\dot{\theta}^2 + (r\dot{\theta}-\dot{c})^2\right]$$

$$T = \frac{m}{2l}\int_{\text{Plank}} v^2 du = m\left[(\dot{r}+c\dot{\theta})^2 + l^2/12\,\dot{\theta}^2 + (r\dot{\theta}-\dot{c})^2 \right]$$

$$V = mg(r\cos\theta + c\sin\theta)$$

$$L = T - V$$

$$= \frac{m}{2}\left[(\dot{r}+c\dot{\theta})^2 + l^2/12\,\dot{\theta}^2 + (r\dot{\theta}-\dot{c})^2 \right] - mg(r\cos\theta + c\sin\theta)$$

To be on a hemisphere of fixed radius R, we set $r = R$

$$K \equiv r - R = 0 \longrightarrow \dot{r} = \ddot{r} = 0 \quad \text{—— (5)}$$

$$\frac{\partial L}{\partial \dot{r}} = m(\dot{r} + c\dot{\theta}) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m(\ddot{r} + c\ddot{\theta})$$

$$\frac{\partial L}{\partial r} = m\dot{\theta}(r\dot{\theta} - \dot{c}) - mg\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} + \lambda\frac{\partial K}{\partial r} = 0$$

$$m(\ddot{r} + c\ddot{\theta}) - m\dot{\theta}(r\dot{\theta} - \dot{c}) + mg\cos\theta + \lambda = 0$$

$$mc\ddot{\theta} - m\dot{\theta}(R\dot{\theta} - \dot{c}) + mg\cos\theta + \lambda = 0$$

$$\lambda = -mg\cos\theta + m\dot{\theta}(R\dot{\theta} - \dot{c}) - mc\ddot{\theta}$$ $\longrightarrow$ Normal Force in the Sliding centre case

$\longrightarrow$ III.a

$$\frac{\partial L}{\partial \dot{\theta}} = m\left[c(\dot{r} + c\dot{\theta}) + (l^2/12)\dot{\theta} + r(r\dot{\theta} - \dot{c})\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\left[\dot{c}(\dot{r} + c\dot{\theta}) + c(\ddot{r} + \dot{c}\dot{\theta} + c\ddot{\theta}) + (l^2/12)\ddot{\theta} + \dot{r}(r\dot{\theta} - \dot{c}) + r(\dot{r}\dot{\theta} + r\ddot{\theta} - \ddot{c})\right]$$

$$= m\left[c\dot{c}\dot{\theta} + c(\dot{c}\dot{\theta} + c\ddot{\theta}) + (l^2/12)\ddot{\theta} + R(R\ddot{\theta} - \ddot{c})\right] \qquad [\text{Eq } (5)]$$

$$= m\left[2c\dot{c}\dot{\theta} + (R^2 + c^2 + l^2/12)\ddot{\theta} - R\ddot{c}\right]$$

$$\frac{\partial L}{\partial \theta} = mg(r\sin\theta - c\cos\theta) = mg(R\sin\theta - c\cos\theta) \qquad [\text{Eq } (5)]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} + \lambda \frac{\partial K}{\partial \theta} = 0 \qquad \left[\frac{\partial K}{\partial \theta} = 0\right]$$

$$\Rightarrow \ddot\theta - \left(\frac{R\ddot c - 2c\dot c\dot\theta + mg(R\sin\theta - c\cos\theta)}{R^2 + c^2 + \ell^2/12}\right) = 0 \longrightarrow \text{Eq (II.a) is recovered!}$$

$$\frac{\partial L}{\partial \dot c} = m(r\dot\theta - \dot c) \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot c}\right) = m(\dot r\dot\theta + r\ddot\theta - \ddot c) = m(R\ddot\theta - \ddot c) \longrightarrow [\text{Eq (5)}]$$

$$\frac{\partial L}{\partial c} = m\dot\theta(\dot r + c\dot\theta) - mg\sin\theta = mc\dot\theta^2 - mg\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot c}\right) - \frac{\partial L}{\partial c} + \lambda\frac{\partial K}{\partial c} = 0$$

$$\Rightarrow \ddot c - R\ddot\theta - \dot\theta^2 c + g\sin\theta = 0 \longrightarrow \text{Eq (II.b) is recovered.}$$

We would impose the additional constraint of $R\theta = c$ to obtain the normal force in the no-slipping case –

$$\left.\begin{aligned} K_1 &\equiv r - R = 0 \quad \Rightarrow \dot r = \ddot r = 0 \\ K_2 &\equiv c - R\theta = 0 \quad \Rightarrow \dot c = R\dot\theta \;\; \& \;\; \ddot c = R\ddot\theta \end{aligned}\right\} \longrightarrow (6)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} + \lambda\frac{\partial K_1}{\partial r} + \mu\frac{\partial K_2}{\partial r} = 0 \qquad \left[\frac{\partial K_2}{\partial r} = 0\right]$$

$$\lambda = -mg\cos\theta + m\dot\theta(R\dot\theta - \dot c) - mc\ddot\theta \longrightarrow \text{Eq (III.a) is recovered}$$

$$= -mg\cos\theta - mR\theta\ddot\theta$$

$$= -m(g\cos\theta + R\theta\ddot\theta) \longrightarrow \text{We have finally obtained the normal force in the non-sliding case}$$

We now need to see whether Eq (I) is recovered by imposing the additional constraint K_2 with multiplier μ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{c}}\right) - \frac{\partial L}{\partial c} + \lambda \frac{\partial K_1}{\partial c} + \mu \frac{\partial K_2}{\partial c} = 0$$

$$m(r\dot{\theta} - \ddot{c}) - mc\dot{\theta}^2 + mg\sin\theta + \mu = 0 \qquad \left[\frac{\partial K_1}{\partial c} = 0, \quad \frac{\partial K_2}{\partial c} = 1\right]$$

$$\mu = m(R\theta\dot{\theta}^2 - g\sin\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \lambda \frac{\partial K_1}{\partial \theta} + \mu \frac{\partial K_2}{\partial \theta} = 0 \qquad \left[\frac{\partial K_1}{\partial \theta} = 0, \quad \frac{\partial K_2}{\partial \theta} = -R\right]$$

$$2c\dot{c}\dot{\theta} + (R^2 + c^2 + l^2/12)\ddot{\theta} - R\ddot{c} - g(R\sin\theta - c\cos\theta) - (\mu/m)R = 0$$

$$(\cancel{R^2} + R^2\theta^2 + l^2/12)\ddot{\theta} + 2R^2\theta\dot{\theta}^2 - \cancel{R\ddot{c}} - \cancel{gR\sin\theta} + gR\theta\cos\theta - \theta(R\dot{\theta})^2 + \cancel{gR\sin\theta} = 0$$

$$\left[\theta^2 + \frac{(l/R)^2}{12}\right]\ddot{\theta} + \theta\dot{\theta}^2 + gR\theta\cos\theta = 0$$

$$\ddot{\theta} = \frac{-\theta(\dot{\theta}^2 + gR\cos\theta)}{\theta^2 + \frac{(l/R)^2}{12}}$$

Eq (I) recovered.

Since the equations are recoverable, the following formulations are consistent with each other –

i) (θ)

ii) $(\theta, c) \longrightarrow$ Static Friction ; $Rc = \theta$

iii) $(\theta, c, \phi) \longrightarrow$ Torque Equation ; $\theta = \phi$

iv) $(r, c, \theta) \longrightarrow$ Normal Equation ; $r = R$

$\Big\}$ Constraints

Linearization of the θ equation–

Consider eqn (I), if $\theta \ll 1 \Rightarrow \cos\theta \approx 1$

$\Rightarrow \theta^2 \approx 0$ or $\frac{(l/R)^2}{12} \gg \theta^2$

$$\ddot{\theta} = -\frac{\theta(\dot{\theta}^2 + gR)}{(l/R)^2/12} \Rightarrow \ddot{\theta} + \alpha(\dot{\theta}^2 + \beta)\theta = 0 \qquad \left[\alpha = \frac{12}{(l/R)^2}, \quad \beta = gR\right]$$

Let $\dot{\theta} = \omega \qquad \equiv f(\theta, \omega)$

$\Rightarrow \quad \dot{\omega} = -\alpha(\omega^2 + \beta)\theta \qquad \equiv g(\theta, \omega)$ —— 7

Linearize about the fixed point $(\theta, \omega) = (0,0)$

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \partial f/\partial\theta & \partial f/\partial\omega \\ \partial g/\partial\theta & \partial g/\partial\omega \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\alpha(\omega^2+\beta) & -2\alpha\omega\theta \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \qquad \text{at } (0,0)$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha\beta & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -\alpha\beta & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda^2 + \alpha\beta = 0$$
$$\lambda = \pm\sqrt{-\alpha\beta}$$

Both the eigenvalues are imaginary and hence there is a centre-like behaviour. However, this does not prove the stability of the centre. We seek a Lyapunov function $F(\theta,\omega)$ such that $F(0,0) = 0$, $F(\theta,\omega) > 0$ for $(\theta,\omega) \neq (0,0)$ and $\frac{dF}{dt} = 0$

Consider eq 7,

$\dot{\omega} = -\alpha(\omega^2+\beta)\theta$

$\frac{d\omega}{d\theta}\frac{d\theta}{dt} = -\alpha(\omega^2+\beta)\theta$

$\dot{\theta}\, d\omega = -\alpha(\omega^2+\beta)\theta\, d\theta$

$\int \frac{\omega\, d\omega}{\omega^2+\beta} = -\int \alpha\theta\, d\theta$

$\frac{1}{2}\ln(\omega^2+\beta) = -\frac{\alpha\theta^2}{2} + E$

$E = \frac{\alpha}{2}\theta^2 + \frac{1}{2}\ln(\omega^2+\beta)$

Choose $F = \frac{\alpha\theta^2}{2} + \frac{1}{2}\ln\left(\frac{\omega^2+\beta}{\beta}\right)$

This was actually obvious.

$F(0,0) = 0$

$F(\theta,\omega) > 0 \quad \forall\ (\theta,\omega) \neq (0,0).$

$$\frac{dF}{dt} = \alpha\theta\dot{\theta} + \frac{\omega\dot{\omega}}{\omega^2+\beta} = \alpha\theta\omega - \frac{\alpha\theta\omega\cancel{(\omega^2+\beta)}}{\cancel{\omega^2+\beta}} = 0$$

Hence, we have a stable centre about $(\theta,\omega) = (0,0)$

Stability of the Complete Equation -

$$\ddot{\theta} = \frac{-\theta\left(\dot{\theta}^2 + gR\cos\theta\right)}{\theta^2 + (\ell R)^2/12} \implies \dot{\omega} = \frac{-\theta(\omega^2+\beta\cos\theta)}{\theta^2+\alpha^{-1}}$$

Let $\alpha^{-1} = \gamma$ and $\dot{\omega} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$

$$\omega\frac{d\omega}{d\theta} = -\frac{\theta(\omega^2+\beta\cos\theta)}{\theta^2+\gamma}$$

$$\omega(\theta^2+\gamma)d\omega = -\theta(\omega^2+\beta\cos\theta)d\theta$$

$$\theta(\omega^2+\beta\cos\theta)d\theta + \omega(\theta^2+\gamma)d\omega = 0$$

$$M\,d\theta + N\,d\omega = 0$$

$M = \theta(\omega^2+\beta\cos\theta)$ $\qquad \frac{\partial M}{\partial \omega} = 2\theta\omega$ and $\frac{\partial N}{\partial \theta} = 2\theta\omega$

$N = \omega(\theta^2+\gamma)$

We have an exact differential and hence we can easily find a function $F(\theta,\omega)$ that satisfies Lyapunov stability conditions.

$$\frac{\partial E}{\partial \omega} = N \qquad\qquad \frac{\partial E}{\partial \theta} = M$$

$$E = \int N\,d\omega + h(\theta) \qquad\qquad \cancel{\omega^2\theta} + h'(\theta) = \theta(\cancel{\omega^2} + \beta\cos\theta)$$

$$h'(\theta) = \beta\theta\cos\theta$$

$$= \int \omega(\theta^2+\gamma)\,d\omega + h(\theta) \qquad\qquad \int dh = \int \beta\theta\cos\theta\,d\theta$$

$$= \frac{\omega^2(\theta^2+\gamma)}{2} + h(\theta) \qquad\qquad h(\theta) = \beta[\theta\cos\theta + \sin\theta]$$

Thus $E = \frac{\omega^2}{2}(\theta^2+\gamma) + \beta[\theta\cos\theta + \sin\theta]$

$$= \frac{\omega^2}{2}\left[\theta^2 + \frac{(\ell/R)^2}{12}\right] + gR[\theta\cos\theta + \sin\theta]$$

Does it look familiar?

We didn't have reinvent the wheel to arrive to this point.
The fact that the differential equation was derived from a conservative system makes $M\,d\theta + N\,d\omega$ an exact differential by default.

For the approximated case of $\theta \approx 0$ & $\cos\theta \approx 1$, we needed to test the stability of its linearization and find a suitable Lyapunov function. However for the exact equation I, its stability i granted as it's derived from a Lagrangian, but its centre-like behaviour needs to be confirmed.

$$\dot{\theta} = \omega \qquad \equiv f(\theta,\omega)$$

$$\dot{\omega} = \frac{-\theta(\omega^2+\beta\cos\theta)}{\theta^2+\gamma} \qquad \equiv g(\theta,\omega)$$

$(\theta,\omega) = (0,0)$ is still a fixed point.

$$\left.\frac{\partial f}{\partial \theta}\right|_{(0,0)} = 0 \qquad \left.\frac{\partial f}{\partial \omega}\right|_{(0,0)} = 1$$

$$\left.\frac{\partial g}{\partial \theta}\right|_{(0,0)} = \left.\frac{\left[-(\omega^2+\beta\cos\theta)+\beta\theta\sin\theta\right](\theta^2+\gamma)+2\theta^2(\omega^2+\beta\cos\theta)}{\theta^2+\gamma}\right|_{(0,0)}$$

$$= \frac{[-(0+\beta)]\cancel{\gamma}}{\cancel{\gamma}} = -\beta$$

$$\left.\frac{\partial g}{\partial \omega}\right|_{(0,0)} = \left.\frac{-2\theta\omega}{\theta^2+\gamma^2}\right|_{(0,0)} = 0$$

$$\begin{bmatrix}\dot\theta\\ \dot\omega\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\beta & 0\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix} \longrightarrow \text{Clearly a centre}$$

Its stability follows. Hence, we have a system with oscillatory behaviour, provided sufficient static friction!

Analysis of the Sliding Centre Case -

$$\ddot\theta - \left(\frac{R\ddot c - 2c\dot c\dot\theta + g(R\sin\theta - c\cos\theta)}{R^2+c^2+\ell^2/12}\right) = 0 \quad \text{—— II.a}$$

$$\ddot c - R\ddot\theta - \dot\theta^2 c + g\sin\theta = 0 \quad \text{—— II.b}$$

II.a $\times R$ + II.b

$$\Rightarrow \ddot c - \dot\theta^2 c + g\sin\theta - R\left[\frac{R\ddot c - 2c\dot c\dot\theta + g(R\sin\theta - c\cos\theta)}{R^2+c^2+\ell^2/12}\right] = 0$$

$$(R^2+c^2+\ell^2/12)(\ddot c - \dot\theta^2 c + g\sin\theta) - R^2\ddot c + 2Rc\dot c\dot\theta - gR^2\sin\theta + gRc\cos\theta = 0$$

$$\cancel{R^2\ddot c} - R^2\dot\theta^2 c + \cancel{gR^2\sin\theta} + c^2\ddot c - c^3\dot\theta^2 + gc^2\sin\theta + \ell^2/12\,\ddot c - (\ell\dot\theta)^2/12\,c + (\ell^2/12)g\sin\theta$$

$$- \cancel{R^2\ddot c} + 2Rc\dot c\dot\theta - \cancel{gR^2\sin\theta} + gRc\cos\theta = 0$$

$$(c^2+\ell^2/12)\ddot c - c\dot\theta^2(R^2+c^2) + g\sin\theta(c^2+\ell^2/12) + c\left(2R\dot c\dot\theta + gR\cos\theta - (\ell\dot\theta)^2/12\right) = 0$$

$$\ddot{c} = \frac{c\dot{\theta}^2(R^2+c^2) - g\sin\theta(c^2+\ell^2/12) - c(2R\dot{c}\dot{\theta} + gR\cos\theta - (\ell\dot{\theta})^2/12)}{c^2+\ell^2/12} \quad \text{—— } 8$$

Multiply II·b with R and substitute in II·a

$$R\ddot{c} = R^2\ddot{\theta} + Rc\dot{\theta}^2 - gR\sin\theta$$

$$\ddot{\theta} - \left[\frac{R^2\ddot{\theta} + Rc\dot{\theta}^2 - \cancel{gR\sin\theta} - 2c\dot{c}\dot{\theta} + \cancel{gR\sin\theta} - gc\cos\theta}{R^2+c^2+\ell^2/12}\right] = 0$$

$$(\cancel{R^2}+c^2+\ell^2/12)\ddot{\theta} - \cancel{R^2}\ddot{\theta} - Rc\dot{\theta}^2 + 2c\dot{c}\dot{\theta} + gc\cos\theta = 0$$

$$(c^2+\ell^2/12)\ddot{\theta} - Rc\dot{\theta}^2 + 2c\dot{c}\dot{\theta} + gc\cos\theta = 0$$

$$\ddot{\theta} = \frac{Rc\dot{\theta}^2 - 2c\dot{c}\dot{\theta} - gc\cos\theta}{c^2+\ell^2/12} \quad \text{—— } 9$$

Let $\dot{\theta} = \omega$

$\dot{c} = v$

$$\dot{\omega} = \frac{Rc\omega^2 - 2cv\omega - g\,c\cos\theta}{c^2+\ell^2/12}$$

$$\dot{v} = \frac{c\omega^2(R^2+c^2) - g\sin\theta\,(c^2+\ell^2/12) - c(2Rv\omega + gR\cos\theta - \ell^2\omega^2/12)}{c^2+\ell^2/12}$$

Fixed point at $(\theta, c, \omega, v) = (0,0,0,0) = \bar{0}$

Let $X = (\theta, c, \omega, v)$

$\dot{X} = (h_1, h_2, h_3, h_4) \longrightarrow \frac{\partial h_1}{\partial \omega} = \frac{\partial h_2}{\partial v} = 1$

$$\left.\frac{\partial h_3}{\partial \theta}\right|_{\bar{0}} = \frac{gc\sin\theta}{c^2+\ell^2/12} = 0 \qquad \left.\frac{\partial h_3}{\partial c}\right|_{\bar{0}} = \frac{(R\omega^2 - 2v\omega - g\cos\theta)(c^2+\ell^2/12) - 2c(\ )}{(c^2+\ell^2/12)^2}$$

$$= \frac{\ell^2/12}{(\ell^2/12)^{\cancel{2}}}(-g) = -12g/\ell^2$$

$$\left.\frac{\partial h_3}{\partial \omega}\right|_{\bar{0}} = \frac{2Rc\omega - 2cv}{c^2 + l^2/12} = 0 \qquad \left.\frac{\partial h_3}{\partial v}\right|_{\bar{0}} = \frac{-2c\omega}{c^2+l^2/12} = 0$$

$$\left.\frac{\partial h_4}{\partial \theta}\right|_{\bar{0}} = \frac{-g\cos\theta\left(c^2+l^2/12\right) + gRc\sin\theta}{c^2+l^2/12} = \frac{-gl^2/\cancel{12}}{l^2/\cancel{12}} = -g$$

$$\left.\frac{\partial h_4}{\partial c}\right|_{\bar{0}} = \frac{\left[\omega^2(R^2+c^2) + 2c^2\omega^2 - 2gc\sin\theta - (2Rv\omega + gR\cos\theta - l^2\omega^2/12)\right]\left(c^2+l^2/12\right) - 2c(\ldots)}{\left(c^2+l^2/12\right)^2}$$

$$= \frac{\left[0+0-0-(0+gR-0)\right]l^2/12 - 0}{\left(l^2/12\right)^2} = \frac{-gR}{l^2/12} = \frac{-12gR}{l^2}$$

$$\left.\frac{\partial h_4}{\partial \omega}\right|_{\bar{0}} = \frac{2c\omega(R^2+c^2) - c(2Rv - l^2\omega/6)}{c^2+l^2/12} = 0$$

$$\left.\frac{\partial h_4}{\partial v}\right|_{\bar{0}} = \frac{-2Rc\omega}{c^2+l^2/12} = 0$$

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -12g/l^2 & 0 & 0 \\ 0 & -12gR/l^2 & 0 & 0 \end{bmatrix} \Rightarrow \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & -12g/l^2 & -\lambda & 0 \\ 0 & -12gR/l^2 & 0 & -\lambda \end{vmatrix} = 0$$

$$\lambda \begin{vmatrix} -\lambda & 0 & 1 \\ -12g/l^2 & -\lambda & 0 \\ -12gR/l^2 & 0 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda\left[-\lambda^3 - \lambda\frac{12gR}{l^2}\right] = 0$$

$$\Rightarrow \lambda^2\left(\lambda^2 + 12gR/l^2\right) = 0$$

$$\lambda = 0 \text{ or } \pm \frac{i\sqrt{12gR}}{l}$$

Linearization fails!