A Lock Calculus for Multimode Type Theory (MTT)

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Crash Course on Multimode Type Theory (MTT)

Let $R:\mathscr{C}\to\mathscr{D}$ be a functor.

$$\begin{array}{c}
\tau: \Gamma \to \Gamma' @ \mathscr{C} \\
\hline
B\tau: B\Gamma \to B\Gamma' @ \mathscr{D}
\end{array}$$

$$\Gamma \vdash T \text{ type } @ \mathscr{C}$$

Ok, so how do we check

?
$$\triangle \vdash RT$$
 type

We check $\Gamma dash au$ type $@\mathscr{C}$ and substitute with $\sigma : \Delta o R \Gamma$

BUT: Don't bother the user. Synthesize Γ and σ

 $\Gamma \in \mathscr{C}$ should be the **universal** context Γ such that $\sigma : \Delta \to R\Gamma$ exists

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$$\begin{array}{c|c} \Gamma \operatorname{ctx} @ \mathscr{C} & \tau : \Gamma \to \Gamma' @ \mathscr{C} \\ \hline R\Gamma \operatorname{ctx} @ \mathscr{D} & R\tau : R\Gamma \to R\Gamma' @ \mathscr{D} \\ \end{array}$$

$$\frac{\Gamma \vdash T \text{ type } @ \mathscr{C}}{R\Gamma \vdash RT \text{ type } @ \mathscr{D}} \qquad \frac{\Gamma}{F}$$

$$\frac{\Gamma \vdash t : T @ \mathscr{C}}{B\Gamma \vdash Bt : BT @ \mathscr{D}}$$

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 $\Gamma \in \mathscr{C}$ should be the **universal** context $L\Delta$ such that $\eta_{\Delta} : \Delta \to RL\Delta$ exists.

MTT [GKNB21] is parametrized by a 2-category (the mode theory):

- modes *p*, *q*, *r*, . . .
- modalities $\mu : p \rightarrow q$

• 2-cells $\alpha : \mu \Rightarrow \nu$.

- [p] is a CwF (category with families) modelling all of DTT,
- $\llbracket \mu \rrbracket$ is a (weak) dependent right adjoint (DRA) [BCMMPS20] to $\llbracket \mathbf{A}_{\mu} \rrbracket : \llbracket q \rrbracket \to \llbracket p \rrbracket$,
- $\llbracket \mathbf{Q}_{\alpha} \rrbracket : \llbracket \mathbf{A}_{\nu} \rrbracket \to \llbracket \mathbf{A}_{\mu} \rrbracket$ and corresponding $\llbracket \alpha \rrbracket : \llbracket \mu \rrbracket \to \llbracket \mathbf{v} \rrbracket$

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Plain DTT

 $\frac{\Gamma \operatorname{ctx}}{()\operatorname{ctx}} \qquad \frac{\Gamma \operatorname{ctx}}{\Gamma \vdash T \operatorname{type}}$

 \Rightarrow Contexts are lists of types.

MTT: Modal variables

MTT: Locks

$$\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q \\
\Gamma, \mathbf{A}_{\mu} \operatorname{ctx} @ p$$

$$ullet$$
 $\Gamma, lacksquare$ $V \circ \mu = \Gamma, lacksquare$ $V, lacksquare$

Plain DTT

 $\frac{\Gamma \cot x}{\Gamma \vdash T \text{ type}}$ $\frac{\Gamma \cot x}{\Gamma, x : T \cot x}$

 \Rightarrow Contexts are lists of types.

MTT: Modal variables

 $\begin{array}{c} \rho \, \text{mode} \\ \hline () \, \text{ctx} \, @ \, \rho \end{array} \qquad \begin{array}{c} \Gamma \, \text{ctx} \, @ \, g & \mu : \rho \to g \\ \hline \Gamma, \, \blacksquare_{\mu} \vdash T \, \text{type} \, @ \, \rho \end{array}$

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$$\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q \\
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$$\begin{array}{c|c} & \Gamma \operatorname{ctx} @ g & \mu : p \to q \\ \hline p \operatorname{mode} & \Gamma, \blacksquare_{\mu} \vdash T \operatorname{type} @ p \\ \hline () \operatorname{ctx} @ p & \Gamma, \mu \vdash x : T \operatorname{ctx} @ g \end{array}$$

MTT: Locks

$$\frac{\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q}{\Gamma, \triangle_{\mu} \operatorname{ctx} @ p}$$

$$\bullet$$
 Γ , $\triangle_{id} = \Gamma$,

$$\bullet$$
 $\Gamma, \triangle_{V \circ \mu} = \Gamma, \triangle_V, \triangle_{\mu}$

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Plain DTT

$$\frac{\Gamma \operatorname{ctx}}{\Gamma \vdash T \operatorname{type}} = \frac{\Gamma \operatorname{ctx}}{\Gamma, x : T \operatorname{ctx}}$$

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Plain DTT

$$\Delta \operatorname{ctx}$$
 (): $\Delta \to$ ()

$$\frac{\sigma : \Delta \to \Gamma}{\Delta \vdash t : T[\sigma]} \\
\frac{\Delta \vdash t : T[\sigma]}{(\sigma, t/x) : \Delta \to (\Gamma, x : T)}$$

⇒ Substitutions are lists of terms,

MTT: Modal variables

$$\frac{\triangle \operatorname{ctx} @ \rho \qquad \rho \operatorname{mode}}{() : \triangle \to () @ \rho}$$

$$\sigma : \triangle \to \Gamma @ q \qquad \mu : \rho \to q$$

$$\triangle, \blacksquare_{\mu} \vdash t : T[\sigma, \blacksquare_{\mu}] @ \rho$$

$$(\sigma, t/x) : \triangle \to (\Gamma, \mu \mid x : T) @ q$$

MTT: Locking is bifunctorial

$$\frac{\sigma: \Delta \to \Gamma @ \rho}{\alpha: \mu \Rightarrow \nu} \\
(\sigma, \mathbf{A}_{\alpha}): (\Delta, \mathbf{A}_{\nu}) \to (\Gamma, \mathbf{A}_{\mu})$$

Plain DTT

$$\frac{\Delta \operatorname{ctx}}{():\Delta \to ()}$$

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$$\Delta, \mathbf{A}_{\mu} \vdash t : T[\sigma, \mathbf{A}_{\mu}] @ \rho$$

$$(\sigma, t/x) : \Delta \to (\Gamma, \mu + x : T) @ q$$

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\end{array}$$

The Lock Calculus

Let $\kappa : p \to p$ be **copointed** by $\epsilon : \kappa \Rightarrow id$.

Then \triangle_{κ} is **pointed** by $\triangleleft_{\varepsilon}$: Id $\rightarrow \triangle_{\kappa}$.

The following is ambiguous:

$$\frac{\Gamma, \triangle_{\kappa}, \triangle_{\kappa}, \triangle_{\kappa} \vdash t : T}{\Gamma, \triangle_{\kappa}, \triangle_{\kappa} \vdash t[\triangle_{\varepsilon}] : T[\triangle_{\varepsilon}]}$$

Did we use

$$\bullet$$
 $(id_{\Gamma}, \mathbf{A}_{\kappa}, \mathbf{A}_{\kappa}, \mathbf{A}_{\varepsilon})$ or

•
$$(\mathrm{id}_{\Gamma}, \mathbf{A}_{\kappa}, \mathbf{A}_{\varepsilon}, \mathbf{A}_{\kappa})$$
 or

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$$(\mathrm{id}_{\Gamma}, \mathbf{Q}_{\varepsilon}, \mathbf{Q}_{\kappa}, \mathbf{Q}_{\kappa})$$
?

This smells like de Bruijn indices.

Let's use names instead.

MTT: Locks

$$\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q$$

$$\Gamma, \mathfrak{m} : \mathbf{A}_{\mu} \operatorname{ctx} @ p$$

MTT: Modal variables

$$\frac{\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q}{\Gamma, \mathfrak{m} : \mathbf{A}_{\mu} \vdash T \operatorname{type} @ p}$$
$$\overline{\Gamma, x : \{\mathfrak{m} : \mathbf{A}_{\mu}\} \triangleright T \operatorname{ctx} @ q}$$

$$\Gamma, i: \mathbf{\Omega}_{K}, j: \mathbf{\Omega}_{K}, \mathfrak{k}: \mathbf{\Omega}_{K} \vdash t: T$$

$$\Gamma, i: \mathbf{\Omega}_{K}, \mathfrak{k}: \mathbf{\Omega}_{K} \vdash t[\mathbf{Q}_{\mathfrak{E}}()/j]: T[\mathbf{Q}_{\mathfrak{E}}()/j]$$

Let $\kappa: p \to p$ be copointed by $\epsilon: \kappa \Rightarrow \mathrm{id}$. Then \mathbf{A}_{κ} is pointed by $\mathbf{A}_{\epsilon}: \mathrm{Id} \to \mathbf{A}_{\kappa}$.

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\hline
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- $(id_{\Gamma}, \mathbf{A}_{\varepsilon}, \mathbf{A}_{\kappa}, \mathbf{A}_{\kappa})$?

This smells like de Bruijn indices. Let's use names instead

MTT: Locks $\frac{\Gamma \cot @ \ q \qquad \mu : p \to q}{\Gamma, \mathfrak{m} : \mathbf{\Omega}_{\mu} \cot @ \ p}$

$$\Gamma, i: \mathbf{\Omega}_{\kappa}, j: \mathbf{\Omega}_{\kappa}, \mathfrak{k}: \mathbf{\Omega}_{\kappa} \vdash t: T$$

$$\Gamma, i: \mathbf{\Omega}_{\kappa}, \mathfrak{k}: \mathbf{\Omega}_{\kappa} \vdash t[\mathbf{Q}_{\varepsilon}()/j]: T[\mathbf{Q}_{\varepsilon}()/j]$$

Let $\kappa: p \to p$ be copointed by $\epsilon: \kappa \Rightarrow \mathrm{id}$. Then \mathbf{A}_{κ} is pointed by $\mathbf{A}_{\epsilon}: \mathrm{Id} \to \mathbf{A}_{\kappa}$.

The following is ambiguous:

$$\frac{\Gamma, \mathbf{\triangle}_{\kappa}, \mathbf{\triangle}_{\kappa}, \mathbf{\triangle}_{\kappa} \vdash t : T}{\Gamma, \mathbf{\triangle}_{\kappa}, \mathbf{\triangle}_{\kappa} \vdash t[\mathbf{A}_{\epsilon}] : T[\mathbf{A}_{\epsilon}]}$$

Did we use

- $(\mathrm{id}_{\Gamma}, \triangle_{\kappa}, \triangle_{\kappa}, \diamondsuit_{\varepsilon})$ or
- $(id_{\Gamma}, \triangle_{\kappa}, \triangleleft_{\varepsilon}, \triangle_{\kappa})$ or
- $(id_{\Gamma}, \mathbf{A}_{\varepsilon}, \mathbf{A}_{\kappa}, \mathbf{A}_{\kappa})$?

This smells like de Bruijn indices. Let's use names instead

MTT: Locks

$$\frac{\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q}{\Gamma, \mathbf{m} : \mathbf{A}_{\mu} \operatorname{ctx} @ p}$$

MTT: Modal variables

$$\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q \\
\Gamma, \mathfrak{m} : \mathbf{A}_{\mu} \vdash T \operatorname{type} @ p \\
\overline{\Gamma, x} : \{\mathfrak{m} : \mathbf{A}_{\mu}\} \triangleright T \operatorname{ctx} @ q$$

$$\Gamma, \mathfrak{i}: \mathbf{\Omega}_{\kappa}, \mathfrak{j}: \mathbf{\Omega}_{\kappa}, \mathfrak{k}: \mathbf{\Omega}_{\kappa} \vdash t: T$$

$$\Gamma, \mathfrak{i}: \mathbf{\Omega}_{\kappa}, \mathfrak{k}: \mathbf{\Omega}_{\kappa} \vdash t[\mathbf{Q}_{\mathfrak{e}_{\kappa}}()/\mathfrak{j}]: T[\mathbf{Q}_{\mathfrak{e}_{\kappa}}()/\mathfrak{j}]$$

Let $\kappa: p \to p$ be copointed by $\epsilon: \kappa \Rightarrow \mathrm{id}$. Then \mathbf{A}_{κ} is pointed by $\mathbf{A}_{\epsilon}: \mathrm{Id} \to \mathbf{A}_{\kappa}$.

The following is ambiguous:

$$\frac{\Gamma, \triangle_{\kappa}, \triangle_{\kappa}, \triangle_{\kappa} \vdash t : T}{\Gamma, \triangle_{\kappa}, \triangle_{\kappa} \vdash t[\triangle_{\varepsilon}] : T[\triangle_{\varepsilon}]}$$

Did we use

- \bullet $(id_{\Gamma}, \triangle_{\kappa}, \triangle_{\kappa}, A_{\epsilon})$ or
- $(id_{\Gamma}, \triangle_{\kappa}, \triangle_{\varepsilon}, \triangle_{\kappa})$ or
- $(id_{\Gamma}, \mathbf{Q}_{\varepsilon}, \mathbf{\Delta}_{\kappa}, \mathbf{\Delta}_{\kappa})$?

This smells like de Bruijn indices. Let's use names instead

MTT: Locks

$$\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q$$

$$\Gamma, \mathfrak{m} : \mathbf{A}_{\mu} \operatorname{ctx} @ p$$

MTT: Modal variables

$$\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q
\Gamma, \mathfrak{m} : \mathbf{\Delta}_{\mu} \vdash T \operatorname{type} @ p
\overline{\Gamma, x : \{\mathfrak{m} : \mathbf{\Delta}_{\mu}\}} \triangleright T \operatorname{ctx} @ q$$

$$\Gamma, \mathfrak{i}: \mathbf{\Omega}_{K}, \mathfrak{j}: \mathbf{\Omega}_{K}, \mathfrak{k}: \mathbf{\Omega}_{K} \vdash t: T$$

$$\Gamma, \mathfrak{i}: \mathbf{\Omega}_{K}, \mathfrak{k}: \mathbf{\Omega}_{K} \vdash t[\mathbf{Q}_{\mathfrak{e}_{\mathcal{E}}}()/\mathfrak{j}]: T[\mathbf{Q}_{\mathfrak{e}_{\mathcal{E}}}()/\mathfrak{j}]$$

Let $\kappa: \rho \to \rho$ be copointed by $\epsilon: \kappa \Rightarrow \mathrm{id}$. Then \mathbf{A}_{κ} is pointed by $\mathbf{A}_{\epsilon}: \mathrm{Id} \to \mathbf{A}_{\kappa}$.

The following is ambiguous:

$$\frac{\Gamma, \triangle_{\kappa}, \triangle_{\kappa}, \triangle_{\kappa} \vdash t : T}{\Gamma, \triangle_{\kappa}, \triangle_{\kappa} \vdash t[\triangle_{\varepsilon}] : T[\triangle_{\varepsilon}]}$$

Did we use

- \bullet $(id_{\Gamma}, \triangle_{\kappa}, \triangle_{\kappa}, A_{\epsilon})$ or
- $(id_{\Gamma}, \triangle_{\kappa}, \triangle_{\varepsilon}, \triangle_{\kappa})$ or
- $(id_{\Gamma}, \mathbf{Q}_{\varepsilon}, \mathbf{\Delta}_{\kappa}, \mathbf{\Delta}_{\kappa})$?

This smells like de Bruijn indices. Let's use names instead

MTT: Locks

$$\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q$$

$$\Gamma, \mathfrak{m} : \mathbf{A}_{\mu} \operatorname{ctx} @ p$$

MTT: Modal variables

$$\frac{\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q}{\Gamma, \mathfrak{m} : \mathbf{A}_{\mu} \vdash T \operatorname{type} @ p}$$
$$\overline{\Gamma, x : \{\mathfrak{m} : \mathbf{A}_{\mu}\} \triangleright T \operatorname{ctx} @ q}$$

$$\frac{\Gamma, \mathbf{i} : \mathbf{\Delta}_{\kappa}, \mathbf{j} : \mathbf{\Delta}_{\kappa}, \mathbf{t} : \mathbf{\Delta}_{\kappa} \vdash t : T}{\Gamma, \mathbf{i} : \mathbf{\Delta}_{\kappa}, \mathbf{t} : \mathbf{\Delta}_{\kappa} \vdash t[\mathbf{\Delta}_{\kappa}()/\mathbf{j}] : T[\mathbf{\Delta}_{\kappa}()/\mathbf{j}]}$$

Let $\kappa: p \to p$ be copointed by $\epsilon: \kappa \Rightarrow \mathrm{id}$. Then \mathbf{A}_{κ} is pointed by $\mathbf{A}_{\epsilon}: \mathrm{Id} \to \mathbf{A}_{\kappa}$.

The following is ambiguous:

$$\frac{\Gamma, \triangle_{\kappa}, \triangle_{\kappa}, \triangle_{\kappa} \vdash t : T}{\Gamma, \triangle_{\kappa}, \triangle_{\kappa} \vdash t[\triangle_{\varepsilon}] : T[\triangle_{\varepsilon}]}$$

Did we use

- \bullet $(id_{\Gamma}, \triangle_{\kappa}, \triangle_{\kappa}, A_{\epsilon})$ or
- $(id_{\Gamma}, \triangle_{\kappa}, \triangle_{\varepsilon}, \triangle_{\kappa})$ or
- $(id_{\Gamma}, \mathbf{A}_{\varepsilon}, \mathbf{A}_{\kappa}, \mathbf{A}_{\kappa})$?

This smells like de Bruijn indices. Let's use names instead

MTT: Locks

$$\frac{\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q}{\Gamma, \mathbf{m} : \mathbf{A}_{\mu} \operatorname{ctx} @ p}$$

MTT: Modal variables

$$\frac{\Gamma \operatorname{ctx} @ q \qquad \mu : p \to q}{\Gamma, \mathfrak{m} : \mathbf{A}_{\mu} \vdash T \operatorname{type} @ p}$$
$$\overline{\Gamma, x : \{\mathfrak{m} : \mathbf{A}_{\mu}\} \triangleright T \operatorname{ctx} @ q}$$

$$\frac{\Gamma, \mathbf{i} : \mathbf{\Delta}_{\kappa}, \mathbf{j} : \mathbf{\Delta}_{\kappa}, \mathbf{\ell} : \mathbf{\Delta}_{\kappa} \vdash t : T}{\Gamma, \mathbf{i} : \mathbf{\Delta}_{\kappa}, \mathbf{\ell} : \mathbf{\Delta}_{\kappa} \vdash t[\mathbf{\Delta}_{\kappa}()/\mathbf{j}] : T[\mathbf{\Delta}_{\kappa}()/\mathbf{j}]}$$

Lock Calculus

Ψ Itele $@\ q o p$		$ig \ \llbracket\Psi rbracket$ is a functor $\llbracket q rbracket o \llbracket p rbracket$.
$\Psi \vdash \mathbf{t} : \mathbf{A}_{\mu} @ q \rightarrow p$	t is a lock term in ctx. Ψ	$\llbracket \mathfrak{t} \rrbracket : \llbracket \Psi \rrbracket \to \llbracket oldsymbol{eta}_{\mu} \rrbracket$ is a nat. transf.
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	$\mathfrak T$ is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} \rrbracket : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$ is a nat. transf.

Lock telescopes

$$\begin{array}{c|c} p \, \mathsf{mode} & & & \Psi \, \mathsf{Itele} \, @ \, r \to q & \mu : p \to q \\ \hline () \, \mathsf{Itele} \, @ \, p \to p & & & \\ \hline \Psi, \mathfrak{m} : \, \blacksquare_{\mu} \, \mathsf{Itele} \, @ \, r \to p & & \\ \end{array}$$

Lock terms (**no** weakening, **no** exchange, **no** contraction):

$$\begin{array}{c} \underline{\mu: p \rightarrow q} \\ \overline{\mathfrak{m}: \blacksquare_{\mu} \vdash \mathfrak{m}: \blacksquare_{\mu} @ q \rightarrow p} \end{array} \qquad \begin{array}{c} \underline{\Psi \vdash \mathfrak{t}: \blacksquare_{\nu}} \\ \underline{\Psi \vdash \mathfrak{t}: \blacksquare_{\nu} & \alpha: \mu \Rightarrow \nu: p \rightarrow q} \\ \underline{\Psi \vdash \mathfrak{c}_{\alpha}(\mathfrak{t}): \blacksquare_{\mu} @ q \rightarrow p} \end{array}$$

$$\underline{\Phi \vdash \mathfrak{s}: \blacksquare_{\nu} @ r \rightarrow q} \qquad \underline{\Psi \vdash \mathfrak{t}: \blacksquare_{\mu} @ q \rightarrow p}$$

$$\underline{\Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}): \blacksquare_{\nu \circ \mu} @ r \rightarrow p}$$

$$\underline{\Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}): \blacksquare_{\nu \circ \mu} @ r \rightarrow p}$$

$$\underline{\Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}): \blacksquare_{\nu \circ \mu} @ r \rightarrow p}$$

Lock Calculus

Ψ Itele $@\ q o p$		$ig \llbracket \Psi bracket$ is a functor $racket{ \llbracket q bracket} o racket{ \llbracket p bracket}.$
$\Psi \vdash \mathbf{t} : \mathbf{A}_{\mu} @ q \rightarrow p$	t is a lock term in ctx. Ψ	$\llbracket \mathfrak{t} \rrbracket : \llbracket \Psi \rrbracket o \llbracket oldsymbol{eta}_{\mu} \rrbracket$ is a nat. transf.
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	$\mathfrak T$ is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} \rrbracket : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$ is a nat. transf.

Lock telescopes:

$$\begin{array}{c} \rho \operatorname{mode} & \qquad \qquad \Psi \operatorname{Itele} @ \, r \to q \qquad \mu : p \to q \\ \hline \text{() Itele} @ \, \rho \to \rho & \qquad \qquad \Psi, \mathfrak{m} : \mathbf{A}_{\mu} \operatorname{Itele} @ \, r \to \rho \end{array}$$

Lock terms (no weakening, no exchange, no contraction):

$$\begin{array}{c} \underline{\mu: p \rightarrow q} \\ \overline{\mathfrak{m}: \blacksquare_{\mu} \vdash \mathfrak{m}: \blacksquare_{\mu} @ q \rightarrow p} \end{array} \qquad \begin{array}{c} \underline{\Psi \vdash \mathfrak{t}: \blacksquare_{\nu}} \\ \overline{\Psi \vdash \mathfrak{e}_{\alpha}(\mathfrak{t}): \blacksquare_{\mu} @ q \rightarrow p} \end{array}$$

$$\underline{\Phi \vdash \mathfrak{s}: \blacksquare_{\nu} @ r \rightarrow q} \qquad \underline{\Psi \vdash \mathfrak{t}: \blacksquare_{\mu} @ q \rightarrow p}$$

$$\underline{\Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}): \blacksquare_{\nu \circ \mu} @ r \rightarrow p} \qquad () \vdash (): \blacksquare_{\mathrm{id}} @ p \rightarrow p}$$

Ψ Itele @ $q ightarrow ho$		$ig \ \llbracket\Psi rbracket$ is a functor $\llbracket q rbracket o \llbracket p rbracket$.
$\Psi \vdash \mathbf{t} : \mathbf{A}_{\mu} @ q \rightarrow p$	t is a lock term in ctx. Ψ	$\llbracket \mathfrak{t} rbracket : \llbracket \Psi rbracket o \llbracket oldsymbol{eta}_{\mu} rbracket$ is a nat. transf.
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	$\mathfrak T$ is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} \rrbracket : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$ is a nat. transf.

Lock telescopes:

$$\begin{array}{c} \rho \operatorname{mode} & \qquad \qquad \Psi \operatorname{Itele} @ \, r \to q \qquad \mu : p \to q \\ \hline \text{() Itele} @ \, \rho \to \rho & \qquad \qquad \Psi, \mathfrak{m} : \mathbf{A}_{\mu} \operatorname{Itele} @ \, r \to \rho \end{array}$$

$$\begin{array}{c} \mu: p \rightarrow q \\ \hline \mathfrak{m}: \P_{\mu} \vdash \mathfrak{m}: \P_{\mu} @ q \rightarrow p \end{array} \qquad \begin{array}{c} \Psi \vdash \mathfrak{t}: \P_{\nu} & \alpha: \mu \Rightarrow \nu: p \rightarrow q \\ \hline \Psi \vdash \mathfrak{a}_{\alpha}(\mathfrak{t}): \P_{\mu} @ q \rightarrow p \end{array}$$

$$\begin{array}{c} \Phi \vdash \mathfrak{s}: \P_{\nu} @ r \rightarrow q & \Psi \vdash \mathfrak{t}: \P_{\mu} @ q \rightarrow p \\ \hline \Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}): \P_{\nu \circ \mu} @ r \rightarrow p \end{array} \qquad () \vdash (): \P_{\mathrm{id}} @ p \rightarrow p \end{array}$$

Ψ Itele @ $q ightarrow ho$	Ψ is a lock telescope $q o p$	$ig \ \llbracket \Psi rbracket$ is a functor $\llbracket oldsymbol{q} rbracket ig = racket{p}.$
$\Psi \vdash \mathbf{t} : \mathbf{A}_{\mu} @ q \rightarrow p$	t is a lock term in ctx. Ψ	$ig igl[oldsymbol{t} igl] : igl[oldsymbol{\Psi} igr] ightarrow igl[oldsymbol{eta}_{\mu} igr] $ is a nat. transf.
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	$\mathfrak T$ is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} \rrbracket : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$ is a nat. transf.

Lock telescopes:

$$\begin{array}{c} \rho \, \mathsf{mode} & \qquad \qquad \Psi \, \mathsf{Itele} \, @ \, r \to q \qquad \mu : \rho \to q \\ \hline () \, \mathsf{Itele} \, @ \, \rho \to \rho & \qquad \qquad \Psi \, \mathsf{Itele} \, @ \, r \to \rho \end{array}$$

$$\begin{array}{c} \underline{\mu: p \rightarrow q} \\ \overline{\mathfrak{m}: \blacksquare_{\mu} \vdash \mathfrak{m}: \blacksquare_{\mu} @ q \rightarrow p} \end{array} \qquad \begin{array}{c} \underline{\Psi \vdash \mathfrak{t}: \blacksquare_{\nu}} \qquad \alpha: \underline{\mu} \Rightarrow \underline{\nu: p \rightarrow q} \\ \overline{\Psi \vdash \mathbf{Q}_{\alpha}(\mathfrak{t}): \blacksquare_{\mu} @ q \rightarrow p} \end{array}$$

$$\underline{\Phi \vdash \mathfrak{s}: \blacksquare_{\nu} @ r \rightarrow q} \qquad \underline{\Psi \vdash \mathfrak{t}: \blacksquare_{\mu} @ q \rightarrow p}$$

$$\underline{\Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}): \blacksquare_{\nu \circ \mu} @ r \rightarrow p} \qquad () \vdash (): \blacksquare_{\mathrm{id}} @ p \rightarrow p}$$

Ψ Itele @ $q ightarrow ho$		$\llbracket \Psi rbracket$ is a functor $\llbracket q rbracket o \llbracket p rbracket$.
$\Psi \vdash \mathbf{t} : \mathbf{A}_{\mu} @ q \rightarrow p$	t is a lock term in ctx. Ψ	$\llbracket \mathfrak{t} rbracket : \llbracket \Psi rbracket o \llbracket oldsymbol{eta}_{\mu} rbracket$ is a nat. transf.
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	$\mathfrak T$ is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} \rrbracket : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$ is a nat. transf.

Lock telescopes:

$$\begin{array}{c} \rho \, \mathsf{mode} & \qquad \qquad \Psi \, \mathsf{Itele} \, @ \, r \to q \qquad \mu : \rho \to q \\ \hline () \, \mathsf{Itele} \, @ \, \rho \to \rho & \qquad \qquad \Psi \, \mathsf{Itele} \, @ \, r \to \rho \end{array}$$

$$\frac{\mu: \rho \to q}{\mathfrak{m}: \bigoplus_{\mu} \vdash \mathfrak{m}: \bigoplus_{\mu} @ q \to \rho} \qquad \frac{\Psi \vdash \mathfrak{t}: \bigoplus_{\nu} \qquad \alpha: \mu \Rightarrow \nu: \rho \to q}{\Psi \vdash \mathfrak{a}_{\alpha}(\mathfrak{t}): \bigoplus_{\mu} @ q \to \rho}$$

$$\Phi \vdash \mathfrak{s}: \bigoplus_{\nu} @ r \to q \qquad \Psi \vdash \mathfrak{t}: \bigoplus_{\mu} @ q \to \rho$$

$$\Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}): \bigoplus_{\nu \circ \mu} @ r \to \rho \qquad () \vdash (): \bigoplus_{\mathrm{id}} @ \rho \to \rho$$

Ψ Itele @ $q ightarrow ho$		$ig \ \llbracket\Psi rbracket$ is a functor $\llbracket q rbracket o \llbracket p rbracket$.
$\Psi \vdash \mathbf{t} : \mathbf{A}_{\mu} @ q \rightarrow p$	t is a lock term in ctx. Ψ	$\llbracket \mathfrak{t} rbracket : \llbracket \Psi rbracket o \llbracket oldsymbol{eta}_{\mu} rbracket$ is a nat. transf.
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	$\mathfrak T$ is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} \rrbracket : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$ is a nat. transf.

Lock telescopes:

$$\begin{array}{c} \rho \, \mathsf{mode} & \qquad \qquad \Psi \, \mathsf{Itele} \, @ \, r \to q \qquad \pmb{\mu} : p \to q \\ \hline () \, \mathsf{Itele} \, @ \, \rho \to \rho & \qquad \qquad \qquad \Psi \, \mathsf{m} : \, \pmb{\triangleq}_{\pmb{\mu}} \, \mathsf{Itele} \, @ \, r \to \rho \end{array}$$

$$\frac{\mu: p \to q}{\mathfrak{m}: \bigoplus_{\mu} \vdash \mathfrak{m}: \bigoplus_{\mu} @ q \to p} \qquad \frac{\Psi \vdash \mathfrak{t}: \bigoplus_{\nu} \qquad \alpha: \mu \Rightarrow \nu: p \to q}{\Psi \vdash \bigoplus_{\alpha} (\mathfrak{t}): \bigoplus_{\mu} @ q \to p}$$

$$\frac{\Phi \vdash \mathfrak{s}: \bigoplus_{\nu} @ r \to q \qquad \Psi \vdash \mathfrak{t}: \bigoplus_{\mu} @ q \to p}{\Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}): \bigoplus_{\nu \circ \mu} @ r \to p} \qquad () \vdash (): \bigoplus_{\mathrm{id}} @ p \to p}$$

Ψ Itele @ $q ightarrow ho$		$ig \llbracket \Psi bracket$ is a functor $racket{ \llbracket q bracket} o racket{ \llbracket p bracket}.$
F		$\llbracket \mathfrak{t} rbracket : \llbracket \Psi rbracket o \llbracket oldsymbol{eta}_{\mu} rbracket$ is a nat. transf.
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	$\mathfrak T$ is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} \rrbracket : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$ is a nat. transf.

Lock telescopes:

$$\begin{array}{c} \rho \operatorname{mode} & \qquad \qquad \Psi \operatorname{Itele} @ \, r \to q \qquad \mu : p \to q \\ \hline \text{() Itele} @ \, \rho \to \rho & \qquad \qquad \Psi, \mathfrak{m} : \mathbf{A}_{\mu} \operatorname{Itele} @ \, r \to \rho \end{array}$$

$$\frac{\mu: p \to q}{\mathfrak{m}: \mathbf{A}_{\mu} \vdash \mathfrak{m}: \mathbf{A}_{\mu} @ q \to p} \qquad \frac{\Psi \vdash \mathfrak{t}: \mathbf{A}_{\nu} \qquad \alpha: \mu \Rightarrow \nu: p \to q}{\Psi \vdash \mathbf{A}_{\alpha}(\mathfrak{t}): \mathbf{A}_{\mu} @ q \to p}$$

$$\Phi \vdash \mathfrak{s}: \mathbf{A}_{\nu} @ r \to q \qquad \Psi \vdash \mathfrak{t}: \mathbf{A}_{\mu} @ q \to p$$

$$\Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}): \mathbf{A}_{\nu \circ \mu} @ r \to p \qquad () \vdash (): \mathbf{A}_{\mathrm{id}} @ p \to p$$

Ψ Itele @ $q ightarrow ho$	Ψ is a lock telescope $q o p$	$\llbracket \Psi rbracket$ is a functor $\llbracket q rbracket o \llbracket p rbracket$.
$\Psi \vdash \mathbf{t} : \mathbf{A}_{\mu} @ q \rightarrow p$	t is a lock term in ctx. Ψ	$ig egin{bmatrix} bluet^* bluet^*$
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	$\mathfrak T$ is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} \rrbracket : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$ is a nat. transf.

Lock telescopes:

$$\begin{array}{c} \rho \, \mathsf{mode} & \qquad \qquad \Psi \, \mathsf{Itele} \, @ \, r \to q \qquad \pmb{\mu} : p \to q \\ \hline () \, \mathsf{Itele} \, @ \, \rho \to \rho & \qquad \qquad \qquad \Psi \, \mathsf{m} : \, \pmb{\triangleq}_{\pmb{\mu}} \, \mathsf{Itele} \, @ \, r \to \rho \end{array}$$

$$\begin{array}{c} \underline{\mu: p \rightarrow q} \\ \underline{\mathfrak{m}: \blacksquare_{\mu} \vdash \mathfrak{m}: \blacksquare_{\mu} @ q \rightarrow p} \end{array} \qquad \begin{array}{c} \underline{\Psi \vdash \mathfrak{t}: \blacksquare_{\nu}} \qquad \alpha: \underline{\mu} \Rightarrow \underline{\nu: p \rightarrow q} \\ \underline{\Psi \vdash \square_{\alpha} (\mathfrak{t}): \blacksquare_{\mu} @ q \rightarrow p} \end{array}$$

$$\begin{array}{c} \underline{\Phi \vdash \mathfrak{s}: \blacksquare_{\nu} @ r \rightarrow q} \qquad \underline{\Psi \vdash \mathfrak{t}: \blacksquare_{\mu} @ q \rightarrow p} \\ \underline{\Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}): \blacksquare_{\nu \circ \mu} @ r \rightarrow p} \end{array} \qquad \begin{array}{c} \underline{() \vdash (): \blacksquare_{id} @ p \rightarrow p} \end{array}$$

Ψ Itele @ $q ightarrow ho$	Ψ is a lock telescope $q ightarrow p$	$ig \llbracket \Psi bracket$ is a functor $rall q bracket o rall p bracket$.
$\Psi \vdash \mathbf{t} : \mathbf{A}_{\mu} @ q \rightarrow p$	t is a lock term in ctx. Ψ	$\llbracket \mathfrak{t} rbracket : \llbracket \Psi rbracket ightarrow \llbracket oldsymbol{eta}_{\mu} rbracket$ is a nat. transf.
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	T is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} rbracket : \llbracket \Psi rbracket o \llbracket \Phi rbracket$ is a nat. transf.

Lock telescopes:

$$\begin{array}{c|c} p \, \mathsf{mode} & \qquad & \Psi \, \mathsf{Itele} \, @ \, r \to q & \mu : p \to q \\ \hline () \, \mathsf{Itele} \, @ \, p \to p & \qquad & \Psi, \mathfrak{m} : \, \blacksquare_{\mu} \, \mathsf{Itele} \, @ \, r \to p \end{array}$$

$$\frac{\mu: \rho \to q}{\mathfrak{m}: \blacksquare_{\mu} \vdash \mathfrak{m}: \blacksquare_{\mu} @ q \to \rho} \qquad \frac{\Psi \vdash \mathfrak{t}: \blacksquare_{\nu} \qquad \alpha: \mu \Rightarrow \nu: \rho \to q}{\Psi \vdash \mathfrak{a}_{\alpha}(\mathfrak{t}): \blacksquare_{\mu} @ q \to \rho}$$

$$\frac{\Phi \vdash \mathfrak{s}: \blacksquare_{\nu} @ r \to q \qquad \Psi \vdash \mathfrak{t}: \blacksquare_{\mu} @ q \to \rho}{\Phi, \Psi \vdash (\mathfrak{s}, \mathfrak{t}): \blacksquare_{\nu \circ \mu} @ r \to \rho} \qquad () \vdash (): \blacksquare_{\mathrm{id}} @ \rho \to \rho$$

Ψ Itele @ $q ightarrow ho$		$\llbracket \Psi rbracket$ is a functor $\llbracket q rbracket o \llbracket p rbracket$.
$\Psi \vdash \mathbf{t} : \mathbf{A}_{\mu} @ q \rightarrow p$	t is a lock term in ctx. Ψ	$\llbracket \mathfrak{t} rbracket : \llbracket \Psi rbracket o \llbracket oldsymbol{eta}_{oldsymbol{\mu}} rbracket$ is a nat. transf.
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	$\mathfrak T$ is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} rbracket : \llbracket \Psi rbracket o \llbracket \Phi rbracket$ is a nat. transf.

Lock substitutions:

• Arise from terms:
$$\frac{\Psi \vdash \mathfrak{t} : \mathbf{\Delta}_{\mu} @ q \to p}{\Psi \vdash (\mathfrak{t}/\mathfrak{m}) : (\mathfrak{m} : \mathbf{\Delta}_{\mu}) @ q \to p}$$

$$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$$

• Compose horizontally: $\psi', \psi \vdash \mathfrak{S}, \mathfrak{T} : \Phi', \Phi @ r \rightarrow p$

$$\Psi \vdash \mathfrak{T} : \Xi @ q \rightarrow p$$

$$\Phi \vdash \mathfrak{S} : \Psi @ q \rightarrow p$$

• Compose vertically (by substitution): $\frac{}{\Phi \vdash \mathcal{T}[\mathfrak{S}] : \Xi \otimes q \to q}$

Ψ Itele @ $q ightarrow ho$		$ig \ \llbracket\Psi rbracket$ is a functor $\llbracket q rbracket o \llbracket p rbracket$.
$\Psi \vdash \mathbf{t} : \mathbf{A}_{\mu} @ q \rightarrow p$	t is a lock term in ctx. Ψ	$\llbracket \mathfrak{t} rbracket : \llbracket \Psi rbracket o \llbracket oldsymbol{eta}_{oldsymbol{\mu}} rbracket$ is a nat. transf.
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	$\mathfrak T$ is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} rbracket : \llbracket \Psi rbracket o \llbracket \Phi rbracket$ is a nat. transf.

Lock substitutions:

• Arise from terms:
$$\frac{\Psi \vdash \mathfrak{t} : \mathbf{\Delta}_{\mu} @ q \to p}{\Psi \vdash (\mathfrak{t}/\mathfrak{m}) : (\mathfrak{m} : \mathbf{\Delta}_{\mu}) @ q \to p}$$
$$\Psi' \vdash \mathfrak{S} : \Phi' @ r \to q$$

$$\Psi \vdash \mathbf{\mathfrak{T}} : \Phi @ q \rightarrow p$$

• Compose horizontally: $\frac{}{\Psi',\Psi \vdash \mathfrak{S},\mathfrak{T}:\Phi',\Phi@r \rightarrow p}$

$$\Psi \vdash \mathfrak{L} : \pm \mathfrak{Q} q \rightarrow p$$

 $\Phi \vdash \mathfrak{S} : \Psi \mathfrak{Q} q \rightarrow p$

• Compose vertically (by substitution): $\frac{}{\Phi \vdash \mathfrak{T}[\mathfrak{S}] : \Xi @ q \rightarrow p}$

Ψ Itele @ $q ightarrow ho$	Ψ is a lock telescope $q o p$	$ig \llbracket \Psi bracket$ is a functor $racket{ \llbracket q bracket} o racket{ \llbracket ho bracket}.$
$\Psi \vdash \mathbf{t} : \mathbf{A}_{\mu} @ q \rightarrow p$	t is a lock term in ctx. Ψ	$\llbracket \mathfrak{t} rbracket : \llbracket \Psi rbracket o \llbracket oldsymbol{eta}_{oldsymbol{\mu}} rbracket$ is a nat. transf.
$\Psi \vdash \mathfrak{T} : \Phi @ q \rightarrow p$	$\mathfrak T$ is a lock subst. from Ψ to Φ	$\llbracket \mathfrak{T} rbracket : \llbracket \Psi rbracket o \llbracket \Phi rbracket$ is a nat. transf.

Lock substitutions:

• Arise from terms:
$$\frac{\Psi \vdash \mathfrak{t} : \mathbf{\triangle}_{\mu} @ q \to p}{\Psi \vdash (\mathfrak{t/m}) : (\mathfrak{m} : \mathbf{\triangle}_{\mu}) @ q \to p}$$
$$\Psi' \vdash \mathfrak{S} : \Phi' @ r \to q$$
$$\Psi \vdash \mathfrak{T} : \Phi @ q \to p$$

• Compose horizontally: $\overline{\Psi',\Psi \vdash \mathfrak{S},\mathfrak{T}:\Phi',\Phi@r \rightarrow p}$

$$\Psi \vdash \mathfrak{T} : \Xi @ q \rightarrow p$$

$$\Phi \vdash \mathfrak{S} : \Psi @ q \rightarrow p$$

• Compose vertically (by substitution): $\frac{\Phi \vdash \mathfrak{S} : \Psi @ q \to p}{\Phi \vdash \mathfrak{T}[\mathfrak{S}] : \Xi @ q \to p}$

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$$\Psi \vdash J @ q \rightarrow p$$

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• Compose vertically (by substitution): $\frac{\Phi \vdash \mathfrak{S} : \Psi @ q \rightarrow p}{\Phi \vdash J[\mathfrak{S}] @ q \rightarrow p}$

Intermezzo: Modal Type Notation

MTT

$$\frac{\Gamma, \mathbf{\Delta}_{\mu} \vdash T \text{ type}}{\Gamma \vdash \langle \mu \mid T \rangle \text{ type}}$$

$$\frac{\Gamma, \mathbf{\Delta}_{\mu} \vdash t : T}{\Gamma \vdash \mathsf{mod}_{\mu} t : \langle \mu \mid T \rangle}$$

Named MT1

$$\frac{\Gamma,\mathfrak{m}: \mathbf{\triangle}_{\mu} \vdash T \operatorname{type}}{\Gamma \vdash (\mathfrak{m}: \mathbf{\triangle}_{\mu}) \to T \operatorname{type}}$$

$$\frac{\Gamma,\mathfrak{m}: \mathbf{\triangle}_{\mu} \vdash t: T}{\Gamma \vdash \lambda(\mathfrak{m}: \mathbf{\triangle}_{\mu}).t: (\mathfrak{m}: \mathbf{\triangle}_{\mu}) \to T}$$

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$$\Gamma, \triangle_{\nu \circ \mu} = \Gamma, \triangle_{\nu}, \triangle_{\mu}$$

$$\Gamma, \mathfrak{o}: \mathbf{\Omega}_{V \circ \mu} = \Gamma, \mathfrak{n}: \mathbf{\Omega}_{V}, \mathfrak{m}: \mathbf{\Omega}_{\mu}$$

Crucially used e.g. in projection for internal right adjoint modalities.

Internal adjunction

Assume $\kappa \dashv \mu$ as witnessed by $\eta : \mathrm{id} \Rightarrow \mu \circ \kappa$ and $\varepsilon : \kappa \circ \mu \Rightarrow \mathrm{id}$

MTT: Projection

$$(\kappa : \langle \mu \mid T \rangle) \to T[\mathbf{Q}_{\varepsilon}]$$

$$(\Gamma, \mathbf{Q}_{\kappa}, \mathbf{Q}_{\mu}) = (\Gamma, \mathbf{Q}_{\kappa \circ \mu}) \stackrel{\mathbf{Q}_{\kappa \varepsilon}}{\longleftarrow} \Gamma$$

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$$\big(\{\mathfrak{k}: \mathbf{A}_{\kappa}\} \triangleright (\mathfrak{m}: \mathbf{A}_{\mu}) \to T\big) \to T[\mathbf{A}_{\mathfrak{k}_{\mathcal{E}}}()/\mathfrak{o}]$$

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- Lock calculus is the internal language of the mode theory.
- Mode theory is an arbitrary 2-category.
- Single-object 2-category is a monoidal category.

What do internal languages for monoida categories do?

They have eliminators:

•
$$let((n,m) = t)$$
 in ...

•
$$let(() = t)$$
 in ... (no weakening!)

(Jaskelioff & Moggi, 2010; Shulman, 2016)

$$\Gamma, \mathfrak{o}: \triangle_{\nu \circ \mu} \quad \stackrel{\neq}{\approx} \quad \Gamma, \mathfrak{n}: \triangle_{\nu}, \mathfrak{m}: \triangle_{\mu}$$

Nice: Many models don't have = → no longer need to strictify!

$$(\{\mathfrak{k}: \mathbf{A}_{\kappa}\} \triangleright (\mathfrak{m}: \mathbf{A}_{\mu}) \to T) \to$$
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The Metamode

Systems with infinite mode theory:

- Degrees of Relatedness ($p \in \mathbb{Z}_{\geq -2}$)
- Transpension System (modes are shape contexts)
- ⇒ Code ∞plication unsustainable, need internal mode/modality/2-cell polymorphism.

$$\mathbf{par}: (p:\mathsf{Mode}) \to \mathsf{Modty}(p+1,p)$$
$$\mathsf{id}: (p:\mathsf{Mode}) \to (\mathbf{par}\,p:X:\mathscr{U}^p_\ell) \to X \to X$$

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Assume

- Γ , \mathbf{m} : $\mathbf{\Delta}_{\mu} \vdash T$ type @p
- $\Psi \vdash \mathfrak{s}, \mathfrak{t} : \mathbf{\Omega}_{\mu} @ q \rightarrow p$

Then $\Gamma, \Psi \vdash T[\mathfrak{s}/\mathfrak{m}], T[\mathfrak{t}/\mathfrak{m}]$ type

When are lock terms equal?

Need internal reasoning about lock term equality

2-posetal mode theories: Always

- Guarded type theory (later, always = constantly o forever)
- Degrees of Relatedness
 (parametricity, irrelevance, algebra, . . .

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... is not a mode but a metamode:

- Full copy of DTT,
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 - $\Theta \mid \Psi \mid \text{tele } @ g \rightarrow p$
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- Make substitution a replacement operation again,
- Omit modal strictification,
- Have internal mode/modality/2-cell polymorphism,
- Give up decidability of 2-cell equality without problems.

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