Lax-Idempotent 2-Monads, Degrees of Relatedness, and Multilevel Type Theory

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Degrees of Relatedness (RelDTT)

Nuyts and Devriese (2018) @ LICS

Degrees of Relatedness: Overview

- Parametricity is about **relations**,
- Equip types with **multiple, proof-relevant relations** $s \frown_i t$ indexed by **degree** i:
  - Just **one** for **small types** (Bool, $\mathbb{N} \to \mathbb{N}$, ...),
  - **More** for **larger** types ($\mathrm{U}_0 \to \mathrm{U}_0$, Grp, ...).
  - Proofs called *i*-**edges**.
- Describe **function behaviour** by saying how functions **influence degree** of relatedness,
- This explains
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  - **ad hoc polymorphism**
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Degrees of Relatedness: Four Laws

- **Reflexivity:** $(a : A) \frown_i^A (a : A)$
  (Semantically, prop. eq. = def. eq.)
- **Degradation:** $((a : A) \frown_i^R (b : B)) \to ((a : A) \frown_{i+1}^R (b : B))$
- **Dependency:** $(a : A) \frown_i^R (b : B)$ **presumes** $R : A \frown_{i+1}^{\mathrm{U}} B$
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Understanding degrees

$(a : A) \frown_0^{A} (b : A)$	Equality.
$(a : A) \frown_0^{R} (b : B)$	Heterogeneous equality along ...
$R : (A : \mathsf{U}_0) \frown_1^{\mathsf{U}_0} (B : \mathsf{U}_0)$	Any relation R.
$P : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$	Any logical/algebraic relation P.
$Q : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Monoid})$	Any logical/algebraic relation Q along ...
$V : (\mathsf{Grp} : \mathsf{U}_1) \frown_2^{\mathsf{U}_1} (\mathsf{Monoid} : \mathsf{U}_1)$	V could specify that Q must respect e **and** $*$ (but it could ask Q to be anything).

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Understanding modalities (2)

par : types → values

$$\mathit{if} : (\mathbf{par} \mid X : \mathrm{U}_0) \to B\,X$$

$$\begin{tikzcd}
X = Y \arrow[rr] \arrow[d] & & \mathit{if}_X = \mathit{if}_Y \arrow[dd] \\
R : X \frown^{\mathrm{U}_0}_{0} Y \arrow[d] & & \\
R : X \frown^{\mathrm{U}_0}_{1} Y \arrow[rr] \arrow[d] & & \mathit{if}_X \frown^{B\,R}_{0} \mathit{if}_Y \arrow[d] \\
\top \arrow[rr] & & \top
\end{tikzcd}$$

con : types → types

$$B : \mathrm{U}_0 \to \mathrm{U}_0$$

$$B\,X = \mathrm{Bool} \to X \to X \to X$$

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X = Y \arrow[r] \arrow[d] & B\,X = B\,Y \arrow[d] \\
X \frown^{\mathrm{U}_0}_{0} Y \arrow[r, dotted, no head] \arrow[d] & B\,X \frown^{\mathrm{U}_0}_{0} B\,Y \arrow[d] \\
X \frown^{\mathrm{U}_0}_{1} Y \arrow[r, dotted, no head] \arrow[d] & B\,X \frown^{\mathrm{U}_0}_{1} B\,Y \arrow[d] \\
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Understanding modalities (1)

irr : values → values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\, A$$

$$\begin{tikzcd}
m = n \arrow[rr] \arrow[d] & & {[\,]_m = [\,]_n} \arrow[d] \\
r : m \frown_0^{\mathbb{N}} n \arrow[d] & & {[\,]_m \frown_0^{\mathrm{List}_\bullet A} [\,]_n} \arrow[d] \\
\bullet : \top \arrow[urr] & & \top
\end{tikzcd}$$

shi : values → types

$$\lambda n.\mathrm{List}_n\, A : (\mathbf{shi} \mid n : \mathbb{N}) \to \mathrm{U}_0$$

$$\begin{tikzcd}
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r : m \frown^{\mathbb{N}}_{0} n \arrow[d] & {[\,]_m \frown^{\mathrm{List}_\bullet\, A}_{0} [\,]_n} \arrow[d] \\
\bullet : \top \arrow[ur] & \top
\end{tikzcd}$$

shi : values → types

$$\lambda n.\mathrm{List}_n\, A : (\mathbf{shi} \mid n : \mathbb{N}) \to \mathrm{U}_0$$

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 & \mathrm{List}_m\, A \frown^{\mathrm{U}_0}_{0} \mathrm{List}_n\, A \arrow[d] \\
m \frown^{\mathbb{N}}_{0} n \arrow[d] \arrow[ur, dash, dotted] & \mathrm{List}_m\, A \frown^{\mathrm{U}_0}_{1} \mathrm{List}_n\, A \arrow[d] \\
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\end{tikzcd}$$

Understanding modalities (1)

irr : values $\to$ values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\, A$$

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RelDTT can be built on an instance of MTT (Multimode Type Theory) with mode theory DoR:

- Modes are **depths** $p \in \mathbb{Z}_{\geq -1}$
- Modalities $\mu : p \to q$ are

  functions $\{0 \leq \ldots \leq q\} \to \{(=) \leq 0 \leq \ldots \leq p \leq \top\} : i \mapsto i \cdot \mu$

  where $f : (\mu \mid x : A) \to B(x)$ sends

$$(r : x \frown^{A}_{i \cdot \mu} y) \quad \to \quad f(x) \frown^{B(r)}_{i} f(y).$$

  Modal types:

$$\mathrm{mod}_\mu\, x \frown^{\langle \mu | A \rangle}_{i} \mathrm{mod}_\mu\, y \quad = \quad x \frown^{A}_{i \cdot \mu} y$$

- 2-cells are degree-wise inequalities.

Depth p is modelled in cubical sets with $p + 1$ different dimension flavours.

The mode theory DoR

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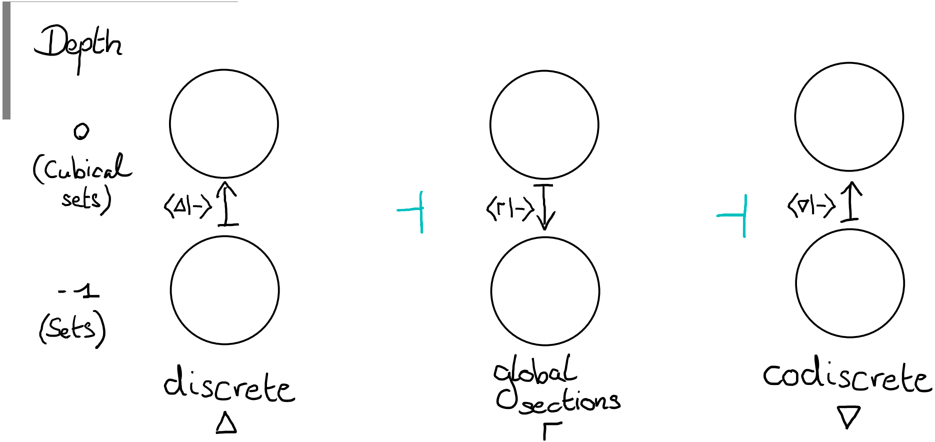
Cohesion in RelDTT

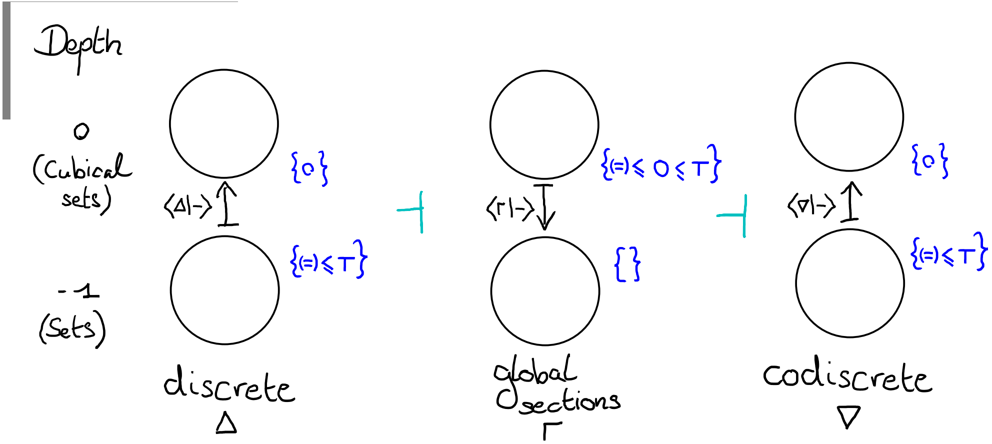
Depth

0
(Cubical sets)

-1
(Sets)

Cohesion in RelDTT




Depth
0
(Cubical sets)
-1
(Sets)
⟨Δ|-⟩
{0}
{(=) ≤ ⊤}
discrete
Δ
⊣
⟨Γ|-⟩
{(=) ≤ 0 ≤ ⊤}
{}
global sections
Γ
⊣
⟨∇|-⟩
{0}
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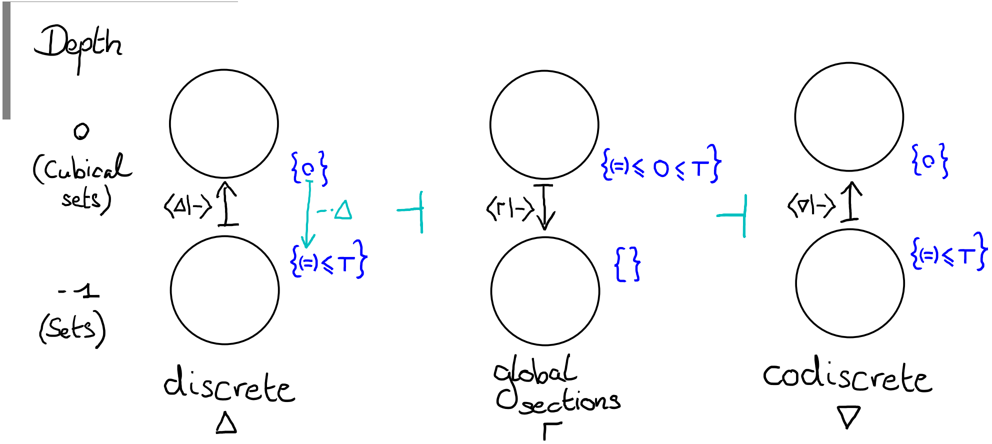
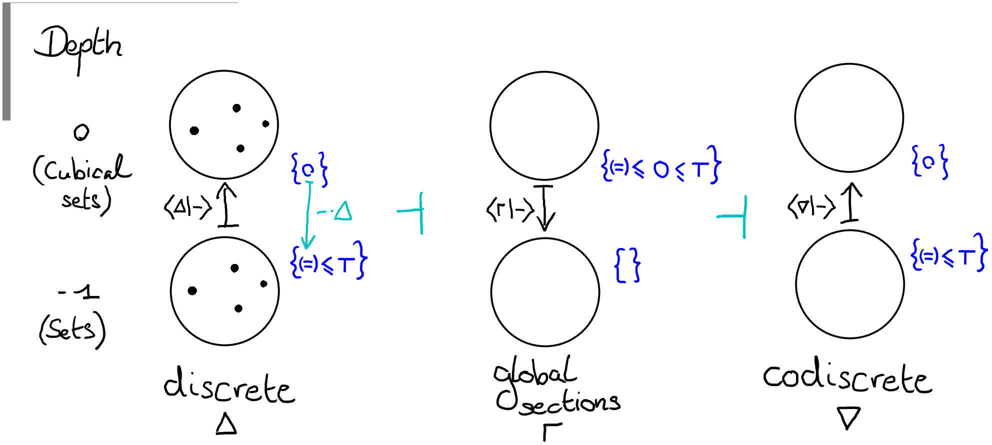
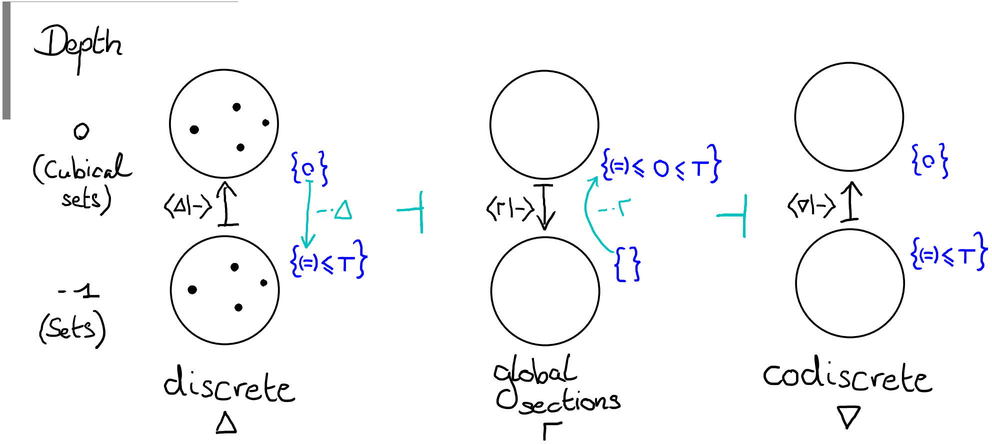
codiscrete
∇

Cohesion in RelDTT



Cohesion in RelDTT




Depth
0
(Cubical sets)
-1
(Sets)
⟨Δ|−⟩
{0}
−·Δ
{(=)≤T}
discrete
Δ
⟨Γ|−⟩
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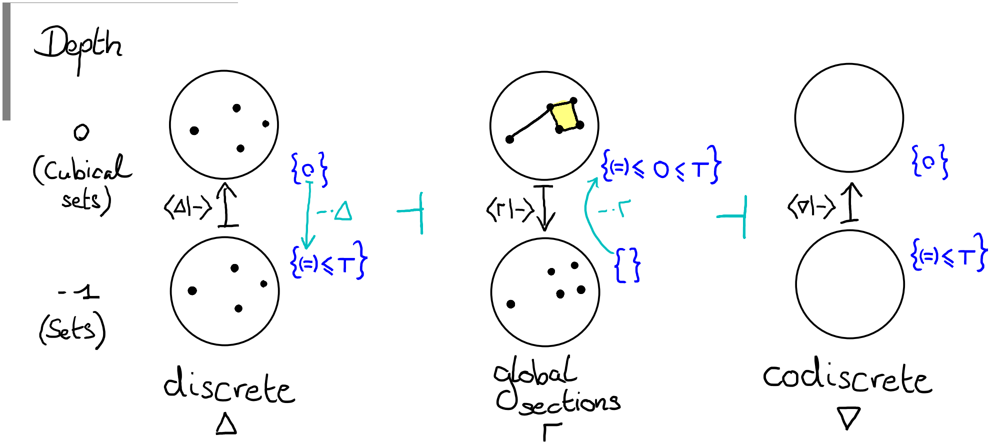

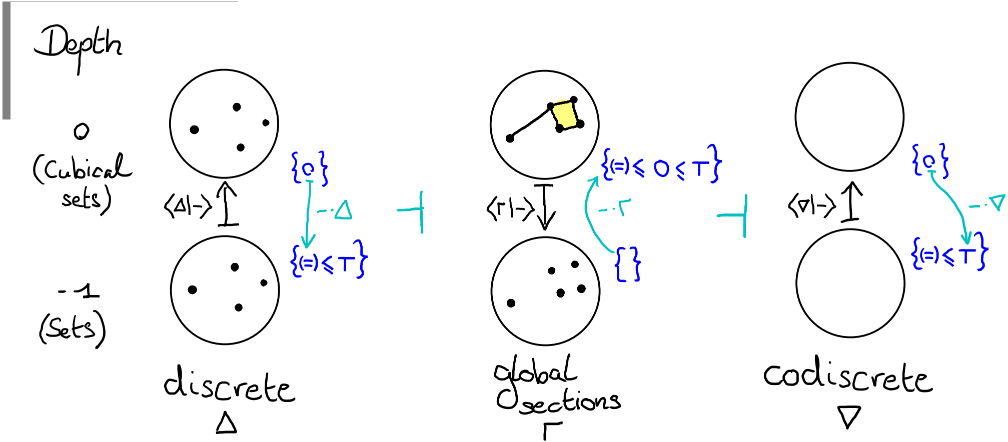
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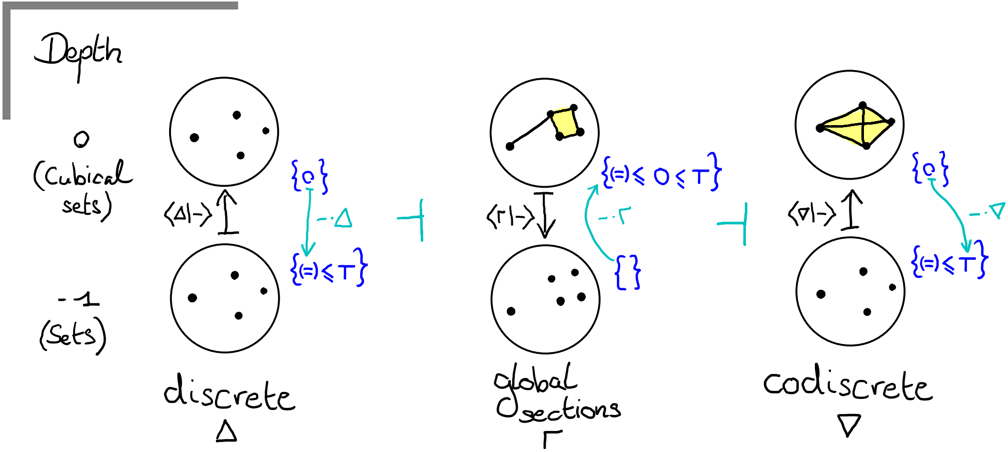
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Two-Level Type Theory (2LTT)

Voevodsky (2013)
Altenkirch, Capriotti & Kraus (2016)
Annenkov, Capriotti, Kraus & Sattler (2017)

More specifically, **its general presheaf model**

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More specifically, **its general presheaf model**

Inner Level

CwF $(\mathscr{C}, \mathrm{Ty}^i, \mathrm{Tm}^i)$

$\mathrm{Ty}^i \in \mathrm{Psh}(\mathscr{C})$

$\mathrm{Tm}^i \in \mathrm{Psh}\left(\int_{\mathscr{C}} \mathrm{Ty}^i\right)$

$\Gamma\ \mathsf{ctx}^i$

$\Gamma \vdash^i T\ \mathsf{type}$

$\Gamma \vdash^i t : T$

E.g inner system = HoTT

Outer Level

CwF $(\mathrm{Psh}(\mathscr{C}), \mathrm{Ty}^o, \mathrm{Tm}^o)$

$\vdash^o \mathrm{Ty}^i\ \mathsf{type}$

$X : \mathrm{Ty}^i \vdash^o \mathrm{Tm}^i(X)\ \mathsf{type}$

$\Rightarrow \mathbf{y}\Gamma\ \mathsf{ctx}^o$

$\Leftrightarrow \mathbf{y}\Gamma \vdash^o T : \mathrm{Ty}^i$

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Talk about strict equality

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$\Gamma \vdash^i T$ type	$\Leftrightarrow$ $\mathbf{y}\Gamma \vdash^o T : \mathrm{Ty}^i$
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E.g inner system = HoTT	Talk about strict equality

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$\Gamma \vdash^i t : T$	$\Leftrightarrow \mathbf{y}\Gamma \vdash^o t : \mathrm{Tm}^i(T)$
E.g inner system = HoTT	Talk about strict equality

Lifting Functors

$$\begin{tikzcd} V \arrow[d, "F"] \\ W \end{tikzcd}$$

- $F^* =$ precomposition by F.
- $F_!$ extends F along $\mathbf{y}$, i.e. $F_!\mathbf{y} \cong \mathbf{y}F$.

Multilevel Type Theory

Lifting Functors

$$
\begin{tikzcd}
\mathcal{V} \arrow[d, "F"] \\
\mathcal{W}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{Psh}(\mathcal{V}) \arrow[d, bend right=60, "F_!"'] \arrow[d, bend left=60, "F_*"] \arrow[d, phantom, shift right=4, "\dashv"] \arrow[d, phantom, shift left=4, "\dashv"] \\
\mathrm{Psh}(\mathcal{W}) \arrow[u, "F^*"]
\end{tikzcd}
$$

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Multilevel Type Theory

Lifting Functors

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\begin{tikzcd}
\mathcal{V} \arrow[r, hook, "\mathbf{y}"'] \arrow[d, "F"] & \mathrm{Psh}(\mathcal{V}) \arrow[d, bend right=50, "F_!"'] \arrow[d, phantom, shift right=3, "\dashv"] \arrow[d, phantom, shift left=3, "\dashv"] \arrow[d, bend left=50, "F_*"] \\
\mathcal{W} \arrow[r, hook, "\mathbf{y}"'] & \mathrm{Psh}(\mathcal{W}) \arrow[u, "F^*"']
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Multilevel Type Theory

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Multilevel Type Theory

$$\begin{tikzcd}
\text{Level 1} & \mathrm{Psh}(\mathcal{C}) \\
\text{Level 0} & \mathcal{C} \arrow[u, "\mathbf{y}"']
\end{tikzcd}$$

Lifting Functors

$$
\begin{tikzcd}
\mathcal{V} \arrow[r, hook, "\mathbf{y}"'] \arrow[d, "F"] & \mathrm{Psh}(\mathcal{V}) \arrow[d, bend right=60, "F_!"'] \arrow[d, bend left=60, "F_*"] \\
\mathcal{W} \arrow[r, hook, "\mathbf{y}"'] & \mathrm{Psh}(\mathcal{W}) \arrow[u, "F^*"']
\end{tikzcd}
$$

- $F^* =$ precomposition by F.
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Multilevel Type Theory

$$
\begin{tikzcd}
\text{Level 2} & \mathrm{Psh}(\mathrm{Psh}(\mathcal{C})) \arrow[d, "\mathbf{y}^*"] \\
\text{Level 1} & \mathrm{Psh}(\mathcal{C}) \arrow[u, bend left=60, "\mathbf{y}_!"] \arrow[u, bend right=60, "\mathbf{y}_*"'] \\
\text{Level 0} & \mathcal{C} \arrow[u, "\mathbf{y}"']
\end{tikzcd}
$$

Lifting Functors

$$
\begin{tikzcd}
\mathcal{V} \arrow[r, hook, "\mathbf{y}"'] \arrow[d, "F"'] & \mathrm{Psh}(\mathcal{V}) \arrow[d, bend right=50, "F_!"'] \arrow[d, bend left=50, "F_*"] \\
\mathcal{W} \arrow[r, hook, "\mathbf{y}"'] & \mathrm{Psh}(\mathcal{W}) \arrow[u, "F^*"']
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Multilevel Type Theory

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\text{Level 1} & \mathrm{Psh}(\mathcal{C}) \arrow[u, bend left=60, "\mathbf{y}_!"] \arrow[u, bend right=60, "\mathbf{y}_* \cong \mathbf{y}"'] \\
\text{Level 0} & \mathcal{C} \arrow[u, "\mathbf{y}"']
\end{tikzcd}
$$

Multilevel TT: Case Study

- Level 0: $\star$ (point category is a CwF)
- Level 1: $\mathrm{Psh}(\star) \cong \mathsf{Set}$
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What is $\mathrm{Psh}(\mathsf{Set})$?

Presheaves over
finite
non-empty
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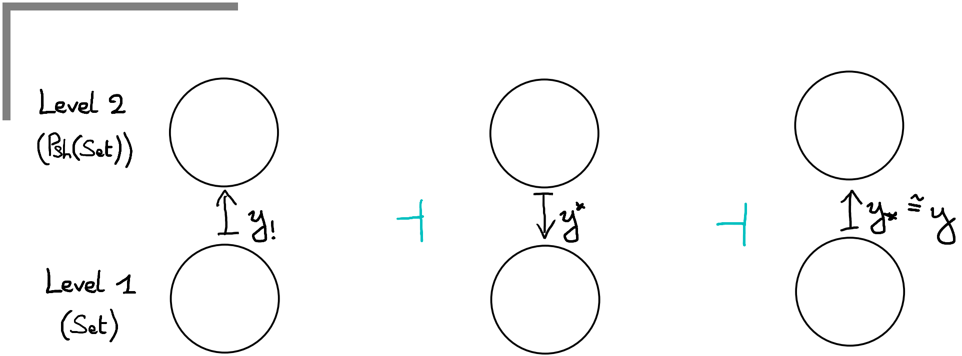
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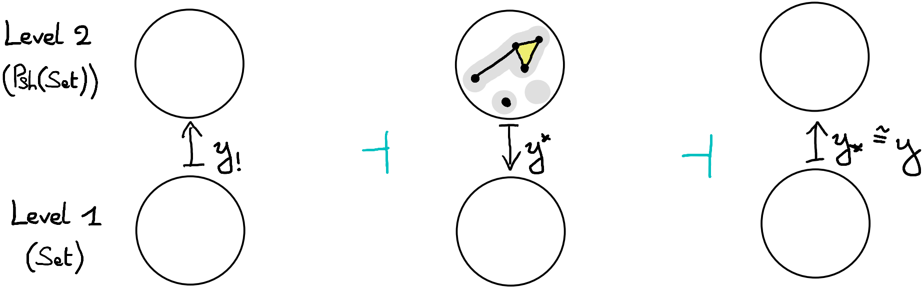
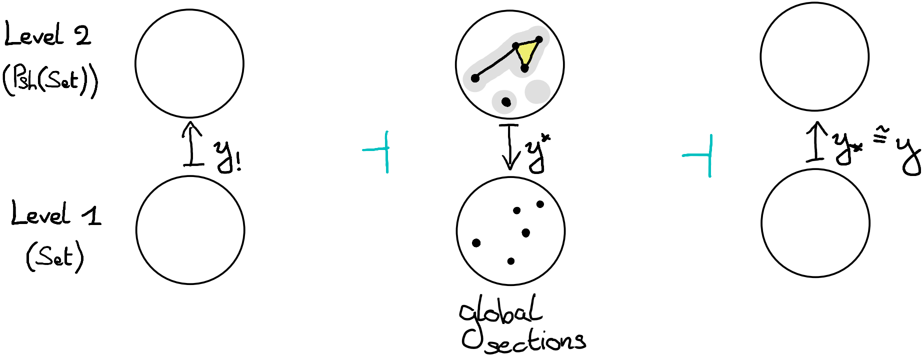
y^*
$y_* \cong y$

Cohesion in Multilevel TT





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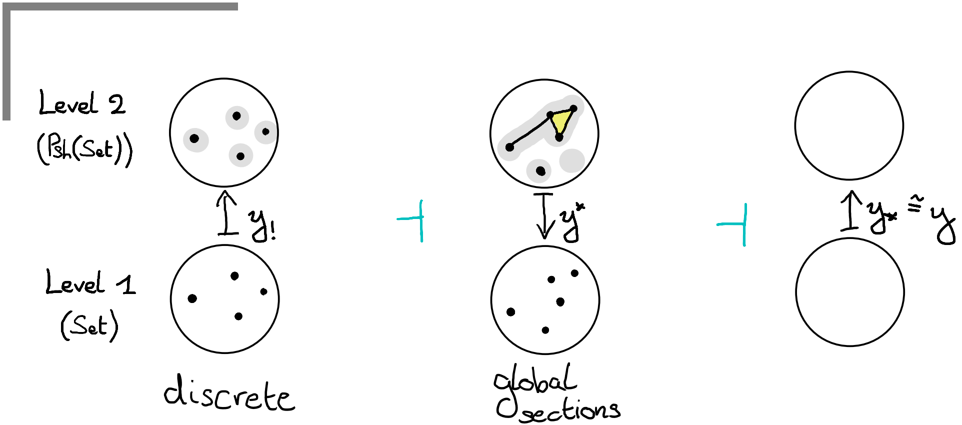

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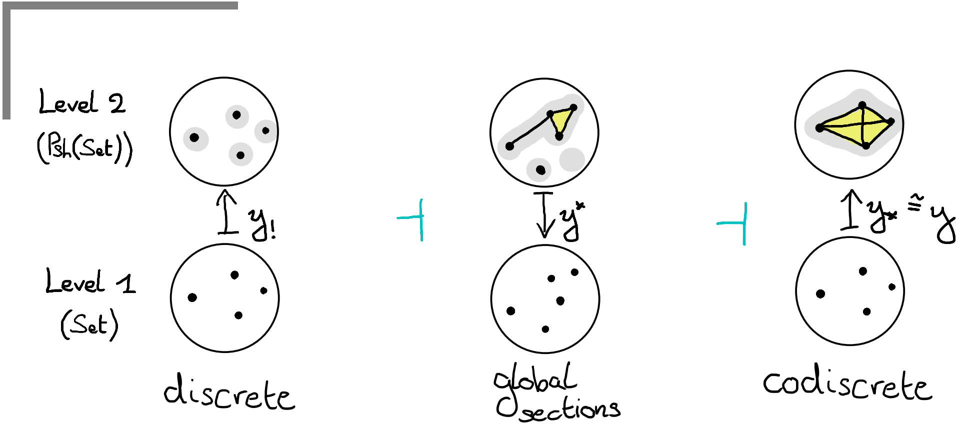

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Goal: Formalize this correspondence, using **interface** of

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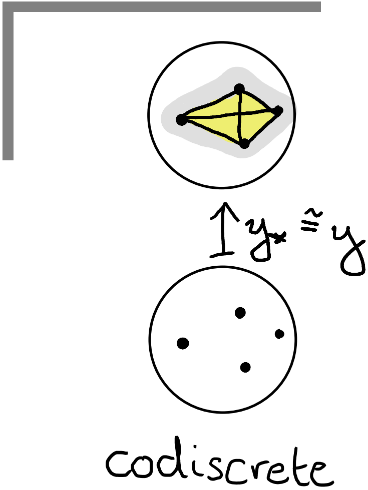

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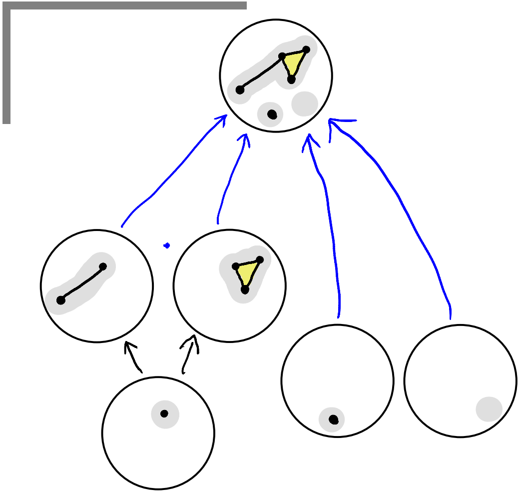

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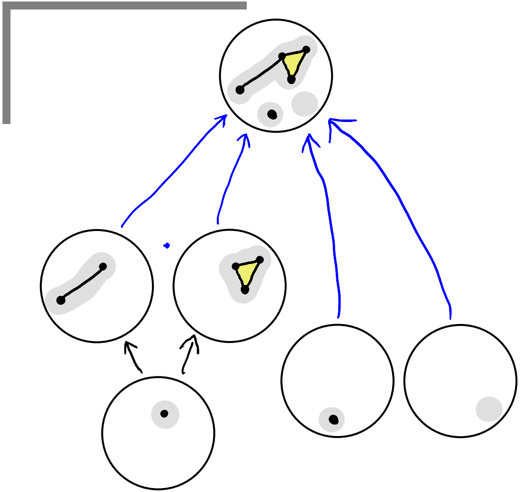

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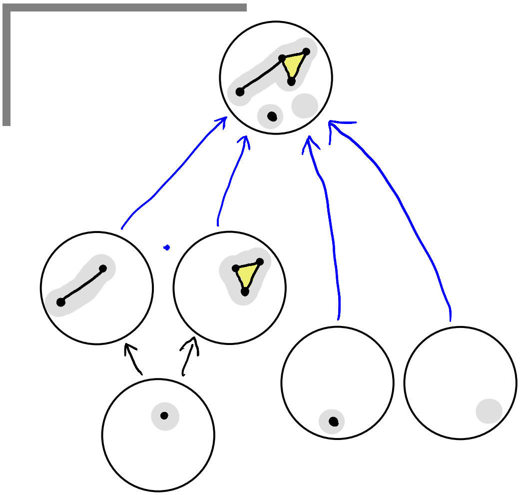

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$\dashv \quad \dashv$

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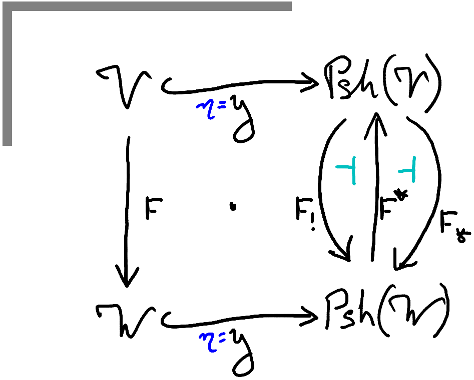

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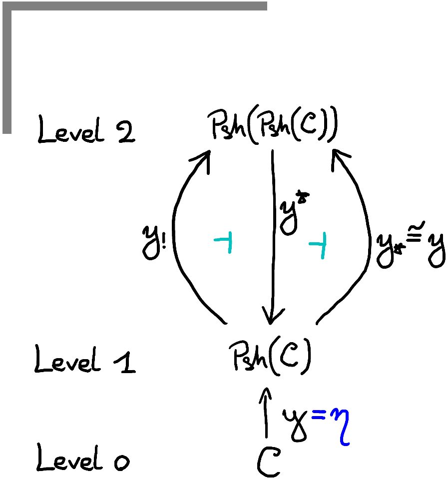

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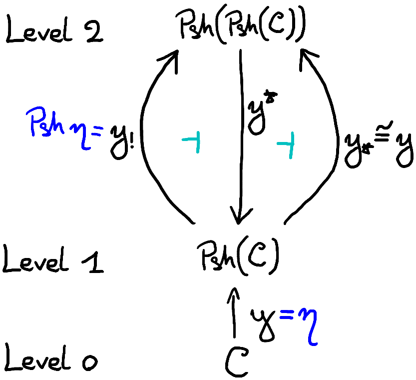

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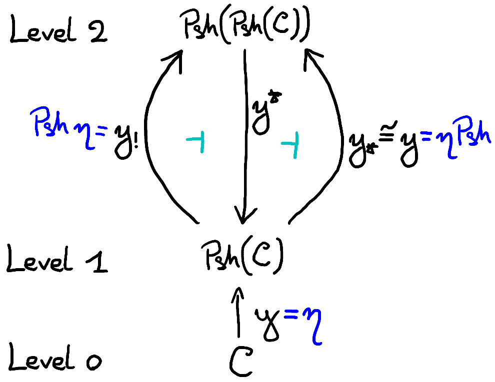

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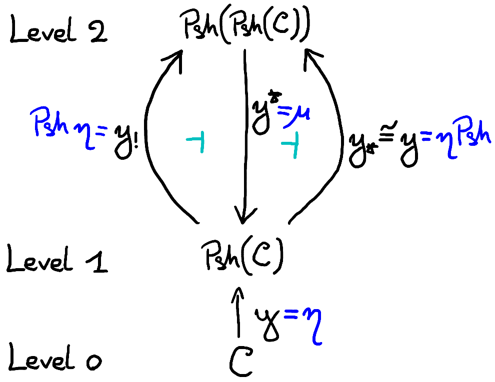

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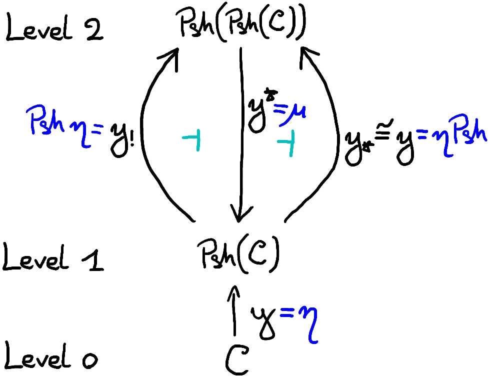

Definition

Define 2-category DoR:

- Objects are $p \in \mathbb{Z}_{\geq -1}$
- Morphisms $\mu : p \to q$ are functions $\{0 \leq \ldots \leq q\} \to \{(=) \leq 0 \leq \ldots \leq p \leq \top\} : i \mapsto i \cdot \mu$
- 2-cells are degree-wise inequalities.
- Freely add $\perp =: -2$.

Definition

Let LIM be the 2-category freely generated by:

- $\mathbf{C} \in \mathrm{Obj}(\mathrm{LIM})$,
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Clearly, we get $[\![-]\!]_{\mathscr{C}} : \mathrm{LIM} \to \mathrm{Cat}$ for any category $\mathscr{C}$:

- $\mathbf{C} \mapsto \mathscr{C}$,
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Main theorem (formal proof WIP)

$$\begin{array}{ccccc} \mathrm{DoR} & \cong & \mathrm{LIM} & \to & \mathrm{Cat} \\ p & \mapsto & \mathbf{M}^{p+2}(\mathbf{C}) & \mapsto & \mathrm{Psh}^{p+2}(\mathscr{C}) \end{array}$$

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Sketch of Proof

$$
\begin{tikzcd}
\top \arrow[d] & 5 \arrow[dl] & 4 \arrow[dl] & 3 \arrow[dll] & 2 \arrow[d] & 1 \arrow[dl] & 0 \arrow[dll] & = \arrow[dl] \\
\top & 4 & 3 & 2 & 1 & 0 & =
\end{tikzcd}
$$

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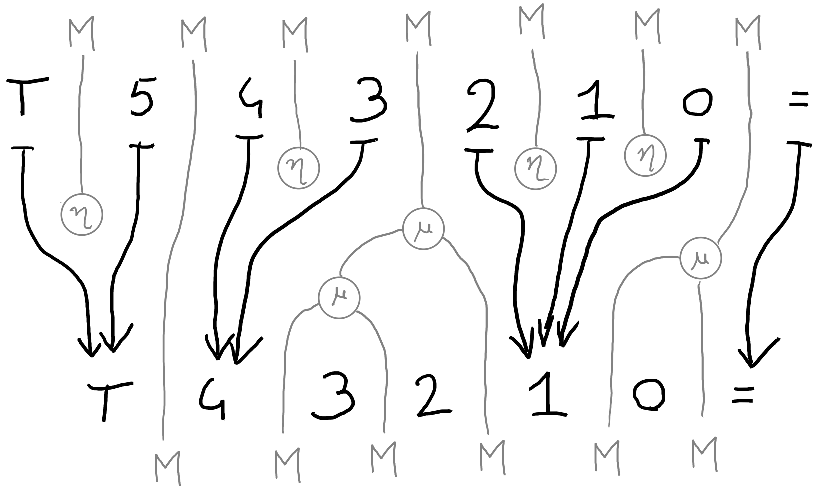

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Implications of main theorem:

- **Degrees of Relatedness*** can serve as an **internal language** for **multilevel type theory**,
- A model for **parametricity, irrelevance, . . .** found **in the wild**.

To do:

- Formalize proof.
- Study **discreteness** and **internal parametricity** in this setting.
- Directify :-)

*Or a reasonable adaptation.

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Thanks!

Questions?