Higher Pro-Arrows: Towards a Model for Naturality Pretype Theory

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https://anuyts.github.io/files/2024/natpt-hottest-pres.pdf

Introduction

Why I care

Not out of an intrinsic interest in

- (directed) algebraic topology,
- synthetic (∞, ∞)-category theory.

Consequences

- Types stratified by finite dimensions. (Cf. Haskell but less weird.)
- I'm not afraid of strict equality. I am afraid of coherence obligations.
- I don't mind if my model doesn't present spaces. But I want it to compute!
- Factorization systems are not my native language.

I want better languages for **verified functional programming**!

Programs should be categorically structured.

With native support for relations/morphisms/isomorphisms:

- Parametricity for free!
- Functoriality for free!
- Naturality for free!
- Variance of dependent multi-argument functions sorted out for free!

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So how is **Directed TT** relevant to **verified functional programming**?

An example problem

$$
\begin{tikzcd}
 & & M & N & & \mathrm{Monad} \\
A \arrow[d, "f"'] & \mathrm{List}\,A & \mathrm{WriterT}\,(\mathrm{List}\,A)\,M & \mathrm{WriterT}\,(\mathrm{List}\,A)\,N & & \\
B & \mathrm{List}\,B & \mathrm{WriterT}\,(\mathrm{List}\,B)\,M & \mathrm{WriterT}\,(\mathrm{List}\,B)\,N & & \\
\pitchfork & & & & \mathrm{Monad} & \\
\mathrm{Type} & \mathrm{Monoid} & & & & \times
\end{tikzcd}
$$

$$
\begin{array}{ccccc}
 & & M & N & \mathrm{Monad} \\
A & \mathrm{List}\,A & \mathrm{WriterT}\,(\mathrm{List}\,A)\,M & \mathrm{WriterT}\,(\mathrm{List}\,A)\,N & \\
\big\downarrow f & & & & \\
B & \mathrm{List}\,B & \mathrm{WriterT}\,(\mathrm{List}\,B)\,M & \mathrm{WriterT}\,(\mathrm{List}\,B)\,N & \\
\pitchfork & & & \mathrm{Monad} & \\
\mathrm{Type} \xrightarrow{\mathrm{List}} \mathrm{Monoid} & & & & \times
\end{array}
$$

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\mathrm{Type} \arrow[r, "\mathrm{List}"] & \mathrm{Monoid} & & & & \times
\end{tikzcd}
$$

$$M \xrightarrow{g} N \quad \in \quad \mathrm{Monad}$$

$$\begin{tikzcd}
A \arrow[d, "f"'] & \mathrm{List}\,A \arrow[d, "\mathrm{List}\,f"'] & \mathrm{WriterT}\,(\mathrm{List}\,A)\,M & \mathrm{WriterT}\,(\mathrm{List}\,A)\,N \\
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\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & & \mathrm{Monad}
\end{tikzcd}$$

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$$
\begin{tikzcd}
 & & M \arrow[r, "g"] & N & \in & \mathsf{Monad} \arrow[dddd, dash] \\
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\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & M \arrow[r, "g"] & N & \in & \mathsf{Monad} \arrow[dddd, no head] \\
A \arrow[d, "f"'] & \mathsf{List}\,A \arrow[d, "\mathsf{List}\,f"'] & \mathsf{WriterT}\,(\mathsf{List}\,A)\,M \arrow[r, bend left, "\mathsf{WriterT}\,(\mathsf{List}\,A)\,g"] \arrow[d, "\mathsf{WriterT}\,(\mathsf{List}\,f)\,M"] & \mathsf{WriterT}\,(\mathsf{List}\,A)\,N \arrow[d, "\mathsf{WriterT}\,(\mathsf{List}\,f)\,N"] & & \\
B \arrow[dd, phantom, sloped, "\in"] & \mathsf{List}\,B \arrow[dd, phantom, sloped, "\in"] & \mathsf{WriterT}\,(\mathsf{List}\,B)\,M \arrow[r, bend right, "\mathsf{WriterT}\,(\mathsf{List}\,B)\,g"'] & \mathsf{WriterT}\,(\mathsf{List}\,B)\,N \arrow[dr, phantom, sloped, "\in"] & & \\
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In plain DTT

Functoriality of List : Type $\to$ Monoid:

- Object action: $(\mathrm{List}\,A, [], ++)$
- Functorial action:
  - $\mathrm{List}\,f : \mathrm{List}\,A \to \mathrm{List}\,B$ **(by recursion)**
  - $\mathrm{List}\,f$ is a monoid morphism:
    - $\mathrm{List}\,f$ preserves $[]$ (trivial)
    - $\mathrm{List}\,f$ preserves $++$ **(by induction)**

  + functor laws **(by induction)**

Functoriality of
WriterT : Monoid $\to$ MonadTrans

- Object action: $\mathrm{WriterT}\,W \in \mathrm{MonadTrans}$
  - Object action: $\mathrm{WriterT}\,W\,M \in \mathrm{Monad}$
    - Object action: Define $\mathrm{WriterT}\,W\,M\,A$
    - Functorial action $\mathrm{WriterT}\,W\,M\,f$
      + functor laws
    - return & bind + naturality

... Object action: $\mathrm{WriterT}\,W \in \mathrm{MonadTrans}$
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  - lift : $M \to \mathrm{WriterT}\,W\,M$ + naturality
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In parametric DTT

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$$
\begin{tikzcd}
 & & M \arrow[r, "g", "\cong"'] & N & \in & \mathrm{Monad} \arrow[dddd, no head] \\
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\end{tikzcd}
$$

$$
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 & & M \arrow[r, equal] & N & \in & \mathsf{Monad} \arrow[dddd, no head] \\
A \arrow[d, equal] & \mathsf{List}\,A \arrow[d, equal] & \mathsf{WriterT}\,(\mathsf{List}\,A)\,M \arrow[r, equal, bend left] \arrow[d, equal] \arrow[dr, phantom, "\textcolor{purple}{\mathbf{?}}"] & \mathsf{WriterT}\,(\mathsf{List}\,A)\,N \arrow[d, equal] & & \\
B & \mathsf{List}\,B & \mathsf{WriterT}\,(\mathsf{List}\,B)\,M \arrow[r, equal, bend right] & \mathsf{WriterT}\,(\mathsf{List}\,B)\,N & \in & \\
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In HoTT (assuming f, g and $h = \mathsf{List}\, f$ are isos)

Functoriality of $\mathsf{List} : \mathsf{Type} \to \mathsf{Monoid}$:

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In Naturality TT

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  \+ functor laws

Variance and modalities

WriterT $W\,M\,A := M(A \times W)$ is **covariant** w.r.t.

- W : Monoid
- M : Monad
- A : Type

ReaderT $R\,M\,A := R \to MA$ is **contravariant** w.r.t.

- R : Type

return : $A \to$ WriterT $W\,M\,A$ is **natural** w.r.t.

- W : Monoid
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Ignoring variance

- HoTT: only consider **isomorphisms**
  ☹ Not everything is an isomorphism.
- Param'ty: **relations**, not morphisms
  ☹ Don't know how to compute fmap.

Naturality TT

- Preserve isomorphisms
- Preserve relations
- Keep track of action on morphisms

Hence:

- Use functoriality/naturality when possible
- Use HoTT when applicable
- Use param'ty when necessary

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- Use functoriality/naturality when possible
- Use HoTT when applicable
- Use param'ty when necessary

Variance and modalities

WriterT $W\,M\,A := M(A \times W)$ is **covariant** w.r.t.

- W : Monoid
- M : Monad
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- HoTT: only consider **isomorphisms**
  ☹ Not everything is an isomorphism.
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Naturality TT

- Preserve isomorphisms
- Preserve relations
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Pretypes: A Note on Fibrancy

A note on fibrancy

A **presheaf model** of DTT can account for:

- The **existence** of shapes (point, path, morphism, bridge, . . .)
- **Unary operations** on shapes (src, rfl)
- **Unary equations** on shapes ($\mathrm{src} \circ \mathrm{rfl} = \mathrm{id}$)

Fibrancy allows for:

- Other arities (composition, . . .)
- Specific geometries (transport, . . .)

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Naturality **Pre**type Theory

We **ignore** fibrancy for now:

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  $\Rightarrow$ need to consider pretypes anyway
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  - Contextual fibrancy [BT17, Nuy20]
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Definition

A CwF is **locally democratic** if every arrow $\sigma : \Delta \to \Gamma$ is isomorphic to some $\pi : \Gamma.T \to \Gamma$.

Internalizing an AWFS [§8.5 of my PhD thesis]

- A CwF is exactly a model of the **structural rules** of DTT.
- On a **locally democratic CwF**, the following **correspond**:
  - Defining an **AWFS** whose right replacement monad RR **preserves pullbacks**,
  - Modelling an **internal monad** RR on types with a **functorial action** on **dependent functions** (+ equations):

$$\frac{\Gamma, rx : \mathrm{RR}\,A \vdash T\ \mathrm{type} \qquad \Gamma \vdash f : (x : A) \to T(\eta_{\mathrm{RR}}(x))}{\Gamma \vdash \mathrm{RR}\,f : (rx : \mathrm{RR}\,A) \to (\mathrm{RR}\,T)(rx)}$$

(Aside) Actually, I'd like your feedback

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Model-first Approach

Separation of concerns:

We need **modalities** to keep track of **variance**.

- ➔ Instantiate **MTT** (Multimodal Type Theory) [GKNB21]
- ☺ The syntax is **their problem**!

We need **substructural intervals** for **bridges** / **morphisms** / **paths**.

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Towards MTT (Multimod[e/al] Type Theory)

Let $R : \mathscr{C} \to \mathscr{D}$ be a functor.

$$\frac{\Gamma \text{ ctx} \,@\, \mathscr{C}}{R\Gamma \text{ ctx} \,@\, \mathscr{D}} \qquad \frac{\tau : \Gamma \to \Gamma' \,@\, \mathscr{C}}{R\tau : R\Gamma \to R\Gamma' \,@\, \mathscr{D}} \qquad \frac{\Gamma \vdash T \text{ type} \,@\, \mathscr{C}}{R\Gamma \vdash RT \text{ type} \,@\, \mathscr{D}} \qquad \frac{\Gamma \vdash t : T \,@\, \mathscr{C}}{R\Gamma \vdash Rt : RT \,@\, \mathscr{D}}$$

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$$\frac{?}{\Delta \vdash RT \text{ type}}$$

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BUT: Don't bother the user. Synthesize Γ and σ.

$\Gamma \in \mathscr{C}$ should be the **universal** context Γ such that $\sigma : \Delta \to R\Gamma$ exists.
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MTT [GKNB21] is parametrized by a **2-category** called the **mode theory**:

- modes p, q, r, ...
- modalities $\mu : p \to q$

$$\frac{\Gamma\ \mathsf{ctx} \ @\ q}{\Gamma, \text{🔒}_\mu\ \mathsf{ctx}\ @\ p} \qquad \frac{\Gamma, \text{🔒}_\mu \vdash T\ \mathsf{type}\ @\ p}{\Gamma \vdash \langle \mu \mid T \rangle\ \mathsf{type}\ @\ q} \qquad \frac{\Gamma, \text{🔒}_\mu \vdash t : T\ @\ p}{\Gamma \vdash \mathsf{mod}_\mu\, t : \langle \mu \mid T \rangle\ @\ q}$$

- (2-cells $\alpha : \mu \Rightarrow \nu$).

Semantics:

- $[\![p]\!]$ is a (often presheaf) category modelling all of DTT,
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Note: If codomain $\mathscr{D}$ is democratic, then DRA = right adjoint that is a CwF morphism.

MTT [GKNB21] is parametrized by a **2-category** called the **mode theory**:

- modes p, q, r, ...
- modalities $\mu : p \to q$

$$\frac{\Gamma \; \mathsf{ctx} \; @ \; q}{\Gamma, 🔒_\mu \; \mathsf{ctx} \; @ \; p} \qquad \frac{\Gamma, 🔒_\mu \vdash T \; \mathsf{type} \; @ \; p}{\Gamma \vdash \langle \mu \mid T \rangle \; \mathsf{type} \; @ \; q} \qquad \frac{\Gamma, 🔒_\mu \vdash t : T \; @ \; p}{\Gamma \vdash \mathsf{mod}_\mu \, t : \langle \mu \mid T \rangle \; @ \; q}$$

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MTT (Multimod[e/al] Type Theory)

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Semantics of MTraS (Modal Transpension System) [ND24]

Idea: Treat

$$\begin{array}{ccccccc} \exists(u:\mathbb{U}) & \dashv & Ⅎ[u] & \dashv & \forall(u:\mathbb{U}) & \dashv & \between[u] \\ \Sigma(u:\mathbb{U}) & \dashv & \Omega[u] & \dashv & \Pi(u:\mathbb{U}) & & \end{array}$$

as modalities.

Problem: They bind / depend on variables.
(Not supported by MTT.)

Solution: Put **shape context** Ξ in the **mode**.

- $\Xi \in \mathrm{Psh}(\mathscr{W})$
- Pick **any old functor** $\sqcup \ltimes \mathbb{U} : \mathscr{W} \to \mathscr{W}$
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$$\begin{array}{ccccccc} (\exists^{\int\Xi}_{\mathbb{U}})_! & \dashv & (\exists^{\int\Xi}_{\mathbb{U}})^* & \dashv & (\exists^{\int\Xi}_{\mathbb{U}})_* & & \\ & & \| \wr & & \| \wr & & \\ & & (Ⅎ^{\int\Xi}_{\mathbb{U}})_! & \dashv & (Ⅎ^{\int\Xi}_{\mathbb{U}})^* & \dashv & (Ⅎ^{\int\Xi}_{\mathbb{U}})_* \end{array}$$

$$\exists^{\Xi}_{\mathbb{U}} \dashv Ⅎ^{\Xi}_{\mathbb{U}} \dashv \forall^{\Xi}_{\mathbb{U}} \dashv \between^{\Xi}_{\mathbb{U}}$$

$$\begin{array}{ccccccc} [\![\text{🔒}^{\exists u}_{Ⅎ u}]\!] & \dashv & [\![\text{🔒}^{Ⅎ u}_{\forall u}]\!] & \dashv & [\![\text{🔒}^{\forall u}_{\between u}]\!] & & \\ & & [\![Ⅎ\, u]\!] & \dashv & [\![\forall\, u]\!] & \dashv & [\![\between\, u]\!] \end{array}$$

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 & & \| \wr & & \| \wr & & \\
 & & (\Finv_{\mathbb{U}}^{\int \Xi})_! & \dashv & (\Finv_{\mathbb{U}}^{\int \Xi})^* & \dashv & (\Finv_{\mathbb{U}}^{\int \Xi})_* \\
\hline\hline
\exists_{\mathbb{U}}^{\Xi} & \dashv & \Finv_{\mathbb{U}}^{\Xi} & \dashv & \forall_{\mathbb{U}}^{\Xi} & \dashv & \between_{\mathbb{U}}^{\Xi} \\
\hline\hline
\llbracket \text{🔒}^{\exists u}_{\Finv u} \rrbracket & \dashv & \llbracket \text{🔒}^{\Finv u}_{\forall u} \rrbracket & \dashv & \llbracket \text{🔒}^{\forall u}_{\between u} \rrbracket & & \\
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& & \| \wr & & \| \wr & & & & \\
& & (\Finv^{\int \Xi}_{\mathbb{U}})_! & \dashv & (\Finv^{\int \Xi}_{\mathbb{U}})^* & \dashv & (\Finv^{\int \Xi}_{\mathbb{U}})_* & &
\end{array}
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 & & \| \wr & & \| \wr & & \\
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\end{array}
$$

Idea: Treat

$$
\begin{array}{ccccccc}
\exists(u:\mathbb{U}) & \dashv & \reflectbox{$\exists$}[u] & \dashv & \forall(u:\mathbb{U}) & \dashv & \between[u] \\
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$$
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$$
\begin{array}{ccccccc}
\left[\!\left[\text{🔒}^{\exists u}_{\reflectbox{$\exists$} u} \right]\!\right] & \dashv & \left[\!\left[\text{🔒}^{\reflectbox{$\exists$} u}_{\forall u} \right]\!\right] & \dashv & \left[\!\left[\text{🔒}^{\forall u}_{\between u} \right]\!\right] & & \\
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\end{array}
$$

Introduction: Wrapping up

- We want to preserve **relations**, **morphisms** *and* **isomorphisms**.
- We need **variance** ➔ MTT
- We need **intervals** ➔ MTraS
- We need **fibrancy** ➔ future work (internal)
- For now, we care about:
  - a **mode theory**,
  - a **presheaf model** for each mode,
  - an **adjunction** for each modality,
  - a **functor** for each interval.

Three Approaches to the Model

$$\begin{tikzcd}
\begin{array}{c}\text{Tamsamani \& Simpson's}\\ \text{model of 2-categories}\\ \text{(2-simplicial sets)}\end{array} \arrow[r, "\text{heterogenize}"'] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"] & \begin{array}{c}\text{Pro-arrow}\\ \text{(pre-)equipments}\\ \text{[Woo82,Woo85]}\end{array} \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"'] & \begin{array}{c}\text{Bridge/path}\\ \text{cubical sets}\\ \text{(ParamDTT)}\\ \text{[NVD17]}\end{array} \arrow[l, "\text{directify}"] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"'] \\
\begin{array}{c}\text{Tamsamani \& Simpson's}\\ \text{model of } n\text{-categories}\\ (n\text{-simplicial sets})\\ \text{[CL04,Tam99,Sim97]}\end{array} \arrow[r, "\text{heterogenize}"'] & \begin{array}{c}\textbf{Naturality}\\ \textbf{(Pre)type Theory}\end{array} & \begin{array}{c}\text{Degrees of}\\ \text{Relatedness}\\ \text{(RelDTT)}\\ \text{[ND18]}\end{array} \arrow[l, "\text{directify}"]
\end{tikzcd}$$

Tamsamani & Simpson's model of n-Categories

Tamsamani (1999)
Simpson (1997)
see Cheng & Lauda (2004)

A **reflexive graph** Γ has:

- A set of **nodes** Γ_0
- A set of **edges** Γ_1
- $\Gamma_{\mathrm{src}}, \Gamma_{\mathrm{tgt}} : \Gamma_1 \to \Gamma_0$ and $\Gamma_{\mathrm{rfl}} : \Gamma_0 \to \Gamma_1$

It is a diagram in Set:

$$
\begin{tikzcd}
\Gamma_0 \arrow[r, "\Gamma_{\mathrm{rfl}}"] & \Gamma_1 \arrow[l, bend right=50, "\Gamma_{\mathrm{src}}"'] \arrow[l, bend left=50, "\Gamma_{\mathrm{tgt}}"]
\end{tikzcd}
$$

A **simplicial set** Γ has:

- For each n, a set of ***n*-simplices** Γ_n (nodes, edges, triangles, tetrahedra, ...)
- For each monotonic $f : \{0..m\} \hookrightarrow \{0..n\}$, a **face map** $\Gamma_f : \Gamma_n \to \Gamma_m$ (vertices of, edges of, faces of, ...)
- For each monotonic $f : \{0..m\} \twoheadrightarrow \{0..n\}$, a **degeneracy map** $\Gamma_f : \Gamma_n \to \Gamma_m$ (flat tetrahedra)

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\begin{tikzcd}
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0 \arrow[r, bend left=50, "0\mapsto 0"] \arrow[r, bend right=50, "0\mapsto 1"'] & 1 \arrow[l, "_\mapsto 0"'] \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] & 2 \arrow[l, bend right=25] \arrow[l, bend left=25] & \cdots
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Simplex category Δ

Δ is a skeleton of NonEmptyFinLinOrd

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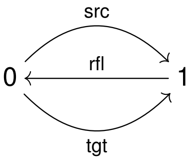

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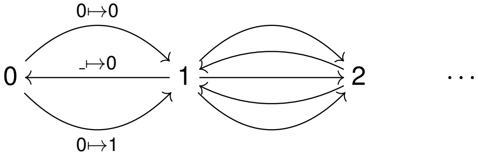

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Nerve $N(\mathscr{C})$ of a category $\mathscr{C}$

Simplicial set whose:

- **nodes** are objects
- **edges** are morphisms
- **triangles** are commutative diagrams
- $(n \geq 3)$**-simplices** uniquely exist

Segal condition

Q: When is a simplicial set the nerve of a category?

A: If every chain of n edges

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is the **spine** (Hamiltonian path) of a unique n-simplex. I.e. if compositions uniquely exist.

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Let $(\mathscr{V}, I, \otimes)$ be a **monoidal category**.

A $\mathscr{V}$-enriched category $\mathscr{C}$ has:

- A (big) set of **objects**
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- $\mathrm{id}_x : I \to \mathrm{Hom}(x,x)$
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Strict n-category

- A 0-category is a **set**.
- An $(n+1)$-category is a **category enriched over n-categories**.

Q: Can we understand higher categories via simplicial sets?

Cheng & Lauda's Guidebook: [CL04]
A thousand times **yes!**

Tamsamani & Simpson: [Sim97,Tam99]
One such time **yes!**
➔ using **double** / **n-fold** categories

Let $(\mathscr{V}, \top, \times)$ be a **cartesian category**.

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A **double category** $\mathscr{C}$ has:

- objects
- horiz. arrows / (1)-arrows (1-cells)
- vertical arrows / (2)-arrows (trivial)
- squares (2-cells)

and can be defined as a **bisimplicial set** $\mathscr{C} \in \mathrm{Psh}(\Delta \times \Delta)$ satisfying the **Segal condition** in each dimension.

A **T&S 2-category** is:

a **double category** whose **vertical arrows** are **trivial**.

$$\begin{tikzcd} \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}$$

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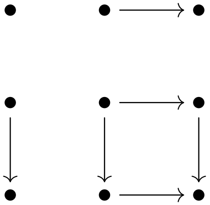

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$$\bullet \qquad \bullet \longrightarrow \bullet$$

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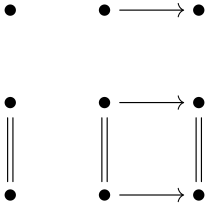

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an ***n*-fold** category where only

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- (1,2)-squares, (2-cells)
- (1,2,3)-cubes, (3-cells)
- $\ldots$

can be non-trivial.

Pretypes!

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Pro-arrow
Equipments

Richard J. Wood (1982, 1985)

(Pro-arrow) Equipment

An **equipment** $\mathscr{C}$ is a **double category** with

- objects
- arrows ($\to$)
- pro-arrows ($\nrightarrow$)
- squares

such that every arrow $\varphi : x \to y$ has **"graph"** pro-arrows

$$\varphi^{\ddagger} : x \nrightarrow y, \qquad \varphi^{\dagger} : y \nrightarrow x$$

such that $(\ldots)$.

Example (Set)

Set is an **equipment** with:

- sets
- functions
- relations
  - identity relation: equality
  - $(R; S)(x, z) = \exists y . R(x, y) \wedge S(y, z)$
- proofs that $R(a, b) \Rightarrow S(f\,a, g\,b)$

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "S"', "|" marking] & D \end{tikzcd}$$

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Example (Cat)

Cat is an **equipment** with:

- categories
- functors
- profunctors $\mathscr{P} : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathrm{Set}$
  - identity profunctor: Hom
  - $(\mathscr{P};\mathscr{Q})(x,z) = \overset{\text{coend}}{\exists}\ y.\mathscr{P}(x,y) \times \mathscr{Q}(y,z)$
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$$\begin{tikzcd}
\mathscr{A} \arrow[r, "\mathscr{P}", "\shortmid"{marking}] \arrow[d, "F"'] & \mathscr{B} \arrow[d, "G"] \\
\mathscr{C} \arrow[r, "\mathscr{Q}"', "\shortmid"{marking}] & \mathscr{D}
\end{tikzcd}$$

Example (Cat)

Cat is a **T&S 2-category** with:

- categories
- functors
- trivial (2)-arrows
- nat. transformations $\overset{\text{end}}{\forall}\, a.\mathrm{Hom}(F\,a, G\,a)$

  i.e. $\overset{\text{end}}{\forall}\, a, b.\mathrm{Hom}(a, b) \Rightarrow \mathrm{Hom}(F\,a, G\,b)$

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathrm{Hom}", "|" marking] \arrow[d, "F"'] & \mathscr{A} \arrow[d, "G"] \\ \mathscr{C} \arrow[r, "\mathrm{Hom}"', "|" marking] & \mathscr{C} \end{tikzcd}$$

To get **heterogeneous** nat. transformations:
drop T&S's triviality condition!

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$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathscr{P}", "|" marking] \arrow[d, "F"'] & \mathscr{B} \arrow[d, "G"] \\ \mathscr{C} \arrow[r, "\mathscr{Q}"', "|" marking] & \mathscr{D} \end{tikzcd}$$

Example (Cat)

Cat is a **T&S 2-category** with:

- categories
- functors
- trivial (2)-arrows
- nat. transformations $\overset{\text{end}}{\forall} a.\mathrm{Hom}(F\,a, G\,a)$
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To get **heterogeneous** nat. transformations: **drop** T&S's triviality condition!

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  - identity profunctor: Hom
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Three Approaches to the Model

$$\begin{tikzcd}
\text{Tamsamani \& Simpson's model of 2-categories (2-simplicial sets)} \arrow[r, "\text{heterogenize}"'] \arrow[d, "\text{higher dimension}"] & \text{Pro-arrow (pre-)equipments [Woo82,Woo85]} \arrow[d, "\text{higher dimension}"'] & \text{Bridge/path cubical sets (ParamDTT) [NVD17]} \arrow[l, "\text{directify}"'] \arrow[d, "\text{higher dimension}"'] \\
\text{Tamsamani \& Simpson's model of n-categories (n-simplicial sets) [CL04,Tam99,Sim97]} \arrow[r, "\text{heterogenize}"'] & \textbf{Naturality (Pre)type Theory} & \text{Degrees of Relatedness (RelDTT) [ND18]} \arrow[l, "\text{directify}"']
\end{tikzcd}$$

Higher

Pro-arrow

Equipments

Higher Pro-arrow Equipments

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- ☹ A large set
- 😐 A category
- ☺ An equipment

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Eqmnt is . . .

- ☹ An equipment
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Eqmnt has:

Objects	Equipments
Arrows	Equipment functors
Pro-arrows	Equipment profunctors: Contain arrows and pro-arrows
Pro-pro-arrows	Equipment **pro-profunctors:** Contain pro-arrows
Squares	. . .
Cubes	. . .

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An n**-equipment** is an n**-fold category** (. . .)

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☺ **Directed layer** on top of your favorite TT!

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Against self-classification

By nature, classifiers (typically) do **NOT** contain themselves:

- **All of mankind** is **not** an example of a **human**.
- **The world's literature** is **not** an example of a **book**.

Forcing things to be otherwise is (a priori) **unreasonable**.

Classifiers of **collection-like** objects:

- **Set** is **more than** a (large) set.
- **Cat** is **more than** a (large) category.

It's not because you **can truncate** to achieve **self-classification**, that you **should!**

➔ Provide the user with the **unscathed classifier** *and* the **truncation modality**.
➔ Use **multimode** type theory.

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. . . and while I am ranting . . .

Against the Grothendieck Construction as "the" Σ-type for categories

Grothendieck Construction

Given a **category** $\mathscr{C}$ and a **functor** $\mathscr{D} : \mathscr{C} \to \mathrm{Arws}(\mathrm{Cat})$,
i.e. **eqmnt functor** $\mathscr{H} : \mathrm{FPro}(\mathscr{C}) \to \mathrm{Cat}$,
the category $\int_{\mathscr{C}} \mathscr{D}$ has:

- objects $(c, d \in \mathscr{D}(c))$
- morphisms

$$\left(c_1 \xrightarrow{\gamma} c_2, \mathscr{D}(\gamma)(d_1) \xrightarrow{\delta} d_2\right)$$

$\mathrm{Arws}(\mathrm{Cat}) \in \mathrm{Cat}$ is **truncated**.

$$\mathrm{FPro} \dashv \mathrm{Arws} : \mathrm{Eqmnt} \to \mathrm{Cat}$$

Arws	Discards pro-arrows
FPro	Freely adds "graph" pro-arrows
Pros	Discards arrows

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Against the Grothendieck Construction as “the” Σ-type for categories

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Three Approaches to the Model

$$
\begin{tikzcd}
\begin{array}{c}\text{Tamsamani \& Simpson's}\\ \text{model of 2-categories}\\ \text{(2-simplicial sets)}\end{array} \arrow[r, "\text{heterogenize}"'] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"] & \begin{array}{c}\text{Pro-arrow}\\ \text{(pre-)equipments}\\ \text{[Woo82,Woo85]}\end{array} \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"'] & \begin{array}{c}\text{Bridge/path}\\ \text{cubical sets}\\ \text{(ParamDTT)}\\ \text{[NVD17]}\end{array} \arrow[l, "\text{directify}"] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"'] \\
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Degrees of Relatedness
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Nuyts and Devriese (2018) @ LICS

- **Relational version** of what NatTT intends to be
- Perhaps **alienating**:
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  (Semantically, prop. eq. = def. eq.)
- **Degradation:** $((a : A) \frown_i^R (b : B)) \to ((a : A) \frown_{i+1}^R (b : B))$
- **Dependency:** $(a : A) \frown_i^R (b : B)$ **presumes** $R : A \frown_{i+1}^{\mathsf{U}} B$
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$(a : A) \frown_0^A (b : A)$ Equality.

$(a : A) \frown_0^R (b : B)$ Heterogeneous equality along ...

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Understanding modalities: Parametricity

par : types $\to$ values

$$\mathit{if} : (\mathbf{par} \mid X : \mathrm{U}^0) \to B\,X$$

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X = Y \arrow[r] \arrow[d] & \mathit{if}_X = \mathit{if}_Y \arrow[dd] \\
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The Mode Theory

- Modes are **depths** $p \in \mathbb{Z}_{\geq -1}$
- Modalities $\mu : p \to q$ are
  functions $\{0 \leq \ldots \leq q\} \to \{(=) \leq 0 \leq \ldots \leq p \leq \top\} : i \mapsto i \cdot \mu$
  where $f : (\mu \mid x : A) \to B(x)$ sends

$$(r : x \frown^{A}_{i \cdot \mu} y) \quad \to \quad f(x) \frown^{B(r)}_{i} f(y).$$

  Modal types:

$$\mathrm{mod}_\mu\, x \frown^{\langle \mu \mid A \rangle}_{i} \mathrm{mod}_\mu\, y \quad = \quad x \frown^{A}_{i \cdot \mu} y$$

- 2-cells are degree-wise inequalities.

Depth p is modelled in cubical sets with $p + 1$ different dimension flavours.

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Three Approaches to the Model

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{Tamsamani \& Simpson's}\\ \text{model of 2-categories}\\ \text{(2-simplicial sets)}\end{array} \arrow[r, "\text{heterogenize}"'] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"] & \begin{array}{c}\text{Pro-arrow}\\ \text{(pre-)equipments}\\ \text{[Woo82,Woo85]}\end{array} \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"'] & \begin{array}{c}\text{Bridge/path}\\ \text{cubical sets}\\ \text{(ParamDTT)}\\ \text{[NVD17]}\end{array} \arrow[l, "\text{directify}"] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"'] \\
\begin{array}{c}\text{Tamsamani \& Simpson's}\\ \text{model of } n\text{-categories}\\ (n\text{-simplicial sets})\\ \text{[CL04,Tam99,Sim97]}\end{array} \arrow[r, "\text{heterogenize}"'] & \begin{array}{c}\textbf{Naturality}\\ \textbf{(Pre)type Theory}\end{array} & \begin{array}{c}\text{Degrees of}\\ \text{Relatedness}\\ \text{(RelDTT)}\\ \text{[ND18]}\end{array} \arrow[l, "\text{directify}"]
\end{tikzcd}
$$

Higher Pro-arrows:
Directifying
Degrees of Relatedness

- Equip types with **multiple, proof-relevant relations** $s \twoheadrightarrow_i t$ indexed by **degree** i
  - Proofs called i**-jets** (pro^{i-1}-arrows).
- Describe **function behaviour** by saying how functions **influence degree** and **orientation** of jets.

Degrees of Relatedness: **Four Laws**

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Pretypes!

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Understanding degrees

$(a : A) \rightarrow^{f}_{0} (b : B)$	a maps to b along ...
$f : (A : \mathrm{U}^0) \rightarrow^{\mathrm{U}^0}_{1} (B : \mathrm{U}^0)$	Any function f.
$\varphi : (G : \mathrm{Grp}) \rightarrow^{\mathrm{Grp}}_{1} (H : \mathrm{Grp})$	Any morphism φ.
$\psi : (G : \mathrm{Grp}) \rightarrow^{\mathscr{P}}_{1} (M : \mathrm{Monoid})$	Any heterogeneous morphism ψ along ...
$\mathscr{P} : (\mathrm{Grp} : \mathrm{U}^1) \rightarrow^{\mathrm{U}^1}_{2} (\mathrm{Monoid} : \mathrm{U}^1)$	Any profunctor $\mathscr{P}$ e.g. $\mathscr{P} = \mathrm{Hom}_{\mathrm{Monoid}}(U_{\mathrm{Grp}} \sqcup, \sqcup)$

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$\varphi : (G : \mathsf{Grp}) \multimap^{\mathsf{Grp}}_{1} (H : \mathsf{Grp})$	Any morphism φ.
$\psi : (G : \mathsf{Grp}) \multimap^{\mathscr{P}}_{1} (M : \mathsf{Monoid})$	Any heterogeneous morphism ψ along ...
$\mathscr{P} : (\mathsf{Grp} : \mathsf{U}^1) \multimap^{\mathsf{U}^1}_{2} (\mathsf{Monoid} : \mathsf{U}^1)$	Any profunctor $\mathscr{P}$ e.g. $\mathscr{P} = \mathrm{Hom}_{\mathsf{Monoid}}(U_{\mathsf{Grp}} \sqcup, \sqcup)$

Understanding degrees

$(a : A) \rightarrow\!\!\triangleright^{f}_{0} (b : B)$	a maps to b along ...
$f : (A : \mathsf{U}^0) \rightarrow\!\!\triangleright^{\mathsf{U}^0}_{1} (B : \mathsf{U}^0)$	Any function f.
$\varphi : (G : \mathrm{Grp}) \rightarrow\!\!\triangleright^{\mathrm{Grp}}_{1} (H : \mathrm{Grp})$	Any morphism φ.
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Understanding degrees

$(a : A) \rightarrow\!\!\triangleright_0^{f} (b : B)$	a maps to b along ...
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Understanding degrees

$(a : A) \to_0^{f} (b : B)$ — a maps to b along ...

$f : (A : \mathrm{U}^0) \to_1^{\mathrm{U}^0} (B : \mathrm{U}^0)$ — Any function f.

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$\mathscr{P} : (\mathrm{Grp} : \mathrm{U}^1) \to_2^{\mathrm{U}^1} (\mathrm{Monoid} : \mathrm{U}^1)$ — Any profunctor $\mathscr{P}$

e.g. $\mathscr{P} = \mathrm{Hom}_{\mathrm{Monoid}}(U_{\mathrm{Grp}} \sqcup, \sqcup)$

Understanding degrees

$(a : A) \rightharpoondown^{f}_{0} (b : B)$	a maps to b along ...
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Understanding modalities: Limits

$\mathbf{lim}^{\oplus}$

$$e : (\mathbf{lim}^{\oplus} \shortmid X : \mathrm{Grp}) \to |X|$$

$$\begin{tikzcd}
G = H \arrow[rr] \arrow[d] & & e_G = e_H \arrow[dd] \\
p : G \xrightarrow[0]{\mathrm{id}_{\mathrm{Grp}}} H \arrow[d, "\ddagger"'] & & \\
\varphi : G \xrightarrow[1]{\mathrm{Hom}_{\mathrm{Grp}}} H \arrow[r] \arrow[d] & & e_G \xrightarrow[0]{|\varphi|} e_H \arrow[d] \\
\top \arrow[rr] & & \top
\end{tikzcd}$$

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$$|\sqcup| : (\mathbf{ftr}^{\oplus} \shortmid \mathrm{Grp}) \to \mathrm{U}^0$$

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G = H \arrow[r] \arrow[d] & |G| = |H| \arrow[d] \\
G \xrightarrow[0]{\mathrm{id}_{\mathrm{Grp}}} H \arrow[r, dotted, no head] \arrow[d, "\ddagger"'] & |G| \xrightarrow[0]{\mathrm{id}_{\mathrm{U}^0}} |H| \arrow[d, "\ddagger"] \\
G \xrightarrow[1]{\mathrm{Hom}_{\mathrm{Grp}}} H \arrow[r, dotted, no head] \arrow[d] & |G| \xrightarrow[1]{\mathrm{Hom}_{\mathrm{U}^0}} |H| \arrow[d] \\
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\varphi : G \overset{\mathrm{Hom}_{\mathsf{Grp}}}{\mathrel{-\!\!\triangleright}}_1 H \arrow[rr] \arrow[d] & & e_G \overset{|\varphi|}{\mathrel{-\!\!\triangleright}}_0 e_H \arrow[d] \\
\top \arrow[rr] & & \top
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G \overset{\mathrm{Hom}_{\mathsf{Grp}}}{\mathrel{-\!\!\triangleright}}_1 H \arrow[r, dotted, no head] \arrow[d] & |G| \overset{\mathrm{Hom}_{\mathsf{U}^0}}{\mathrel{-\!\!\triangleright}}_1 |H| \arrow[d] \\
\top \arrow[r] & \top
\end{tikzcd}$$

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\top \arrow[r] & \top
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G \xrightarrow[1]{\mathrm{Hom}_{\mathrm{Grp}}} H \arrow[r] \arrow[d] & {|G| \xrightarrow[1]{\mathrm{Hom}_{\mathrm{U}^0}} |H|} \arrow[d] \\
\top \arrow[r] & \top
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G \overset{\mathrm{Hom}_{\mathsf{Grp}}}{\rightarrowtriangle_1} H \arrow[r] \arrow[d] & |G| \overset{\mathrm{Hom}_{\mathsf{U}^0}}{\rightarrowtriangle_1} |H| \arrow[d] \\
\top \arrow[r] & \top
\end{tikzcd}$$

$\mathbf{lim}^{\ominus}$

$$\mathsf{hd} : (\mathbf{lim}^{\ominus} \shortmid X : \mathsf{Coalg}_{\mathbb{N}\times\sqcup}) \to |X| \to \mathbb{N}$$

$$\begin{tikzcd}
X = Y \arrow[rr] \arrow[d] & & \mathit{hd}_X = \mathit{hd}_Y \arrow[dd] \\
p : X \xrightarrow[0]{\mathrm{id}_{\mathsf{Coalg}_{\mathbb{N}\times\sqcup}}} Y \arrow[d, "\ddagger"'] & & \\
\varphi : X \xrightarrow[1]{\mathrm{Hom}_{\mathsf{Coalg}_{\mathbb{N}\times\sqcup}}} Y \arrow[r, "\ominus"] \arrow[d] & \mathit{hd}_X \xleftarrow[0]{\sqcup\circ|\varphi|} \mathit{hd}_Y \arrow[r, phantom] & \arrow[d] \\
\top \arrow[rr] & & \top
\end{tikzcd}$$

$\mathbf{ftr}^{\ominus}$

$$\lambda X.(|X| \to \mathbb{N}) : (\mathbf{ftr}^{\ominus} \shortmid \mathsf{Coalg}_{\mathbb{N}\times\sqcup}) \to \mathsf{U}^0$$

$$\begin{tikzcd}
X = Y \arrow[r] \arrow[d] & \mathbb{N}^{|X|} = \mathbb{N}^{|Y|} \arrow[d] \\
X \xrightarrow[0]{\mathrm{id}_{\mathsf{Coalg}_{\mathbb{N}\times\sqcup}}} Y \arrow[r, dotted, no head] \arrow[d, "\ddagger"'] & \mathbb{N}^{|X|} \xrightarrow[0]{\mathrm{id}_{\mathsf{U}^0}} \mathbb{N}^{|Y|} \arrow[d, "\dagger"] \\
G \xrightarrow[1]{\mathrm{Hom}_{\mathsf{Grp}}} H \arrow[r, dotted, no head] \arrow[d] & \mathbb{N}^{|X|} \xleftarrow[1]{\mathrm{Hom}_{\mathsf{U}^0}} \mathbb{N}^{|Y|} \arrow[d] \\
\top \arrow[r] & \top
\end{tikzcd}$$

Understanding modalities: Limits

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\varphi : X \overset{\mathrm{Hom}_{\mathrm{Coalg}_{\mathbb{N}\times\sqcup}}}{\rightarrowtriangle_1} Y \arrow[r, "\ominus"] \arrow[d] & hd_X & hd_Y \arrow[l, "\sqcup\circ|\varphi|"', "0"] \arrow[d] \\
\top \arrow[rr] & & \top
\end{tikzcd}$$

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\top \arrow[r] & \top
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\top \arrow[rr] & & \top
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G \rightarrowtriangle_1^{\mathrm{Hom}_{\mathrm{Grp}}} H \arrow[r, "\ominus"] \arrow[d] & \mathbb{N}^{|X|} \leftarrowtriangle_1^{\mathrm{Hom}_{\mathrm{U}^0}} \mathbb{N}^{|Y|} \arrow[d] \\
\top \arrow[r] & \top
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The mode theory

NatPT instantiates MTT (Multimode Type Theory) with:

- Modes are **dimensions** $p \in \mathbb{N}$ (+ you can mark a degree $i < n$ as symmetric)
- Modalities $\mu : p \to q$ are certain functions

  $\{0, \ldots, q-1\} \to \{(=), 0, \ldots, p-1, \top\} \times \{\circledast, \oplus, \ominus, \otimes\}$

  where $f : (\mu \mid x : A) \to B(x)$ sends

  $$(r : x \twoheadrightarrow^{A}_{i \cdot \mu} y) \quad \to \quad f(x) \twoheadrightarrow^{B(r)}_{i} f(y).$$

  Modal types:

  $$\mathrm{mod}_\mu\, x \twoheadrightarrow^{\langle \mu | A \rangle}_{i} \mathrm{mod}_\mu\, y \quad = \quad x \twoheadrightarrow^{A}_{i \cdot \mu} y$$

- 2-cells are degree-wise inequalities.

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- 2-cells are degree-wise inequalities.

Three Approaches to the Model

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{Tamsamani \& Simpson's}\\ \text{model of 2-categories}\\ \text{(2-simplicial sets)}\end{array} \arrow[r, "\text{heterogenize}"'] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"] & \begin{array}{c}\text{Pro-arrow}\\ \text{(pre-)equipments}\\ \text{[Woo82,Woo85]}\end{array} \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"'] & \begin{array}{c}\text{Bridge/path}\\ \text{cubical sets}\\ \text{(ParamDTT)}\\ \text{[NVD17]}\end{array} \arrow[l, "\text{directify}"] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"'] \\
\begin{array}{c}\text{Tamsamani \& Simpson's}\\ \text{model of } n\text{-categories}\\ (n\text{-simplicial sets})\\ \text{[CL04,Tam99,Sim97]}\end{array} \arrow[r, "\text{heterogenize}"'] & \begin{array}{c}\textbf{Naturality}\\ \textbf{(Pre)type Theory}\end{array} & \begin{array}{c}\text{Degrees of}\\ \text{Relatedness}\\ \text{(RelDTT)}\\ \text{[ND18]}\end{array} \arrow[l, "\text{directify}"]
\end{tikzcd}
$$

The Model

The Twisted Prism Functor

Δ is a skeleton of (hence $\simeq$) NEFinLinOrd.

Twisted Prism Functor [PK20]

$\sqcup \ltimes \mathbb{I} : \mathrm{NEFinLinOrd} \to \mathrm{NEFinLinOrd} :$
$W \mapsto W^{\mathrm{op}} \uplus_{<} W$

$$a \longrightarrow b \qquad \mapsto \qquad \begin{tikzcd} (a,0) \arrow[d] & (b,0) \arrow[l] \arrow[d] \\ (a,1) \arrow[r] & (b,1) \end{tikzcd}$$

MTraS shape modelled by $\sqcup \ltimes \mathbb{I}$ reconciles:

- Hom as a **contra-/covariant bifunctor**,
- Hom as a **constrained function type**.

$\mathbb{I}$ as an MTraS-shape is better behaved on $\bowtie$:

Twisted Cube Category $\bowtie$ [PK20]

(Roughly) the subcategory of NEFinLinOrd (or Δ) generated by $\top$ and $\sqcup \ltimes \mathbb{I}$.

➔ Use $\bowtie$ instead of Δ.

ℹ Pinyo & Kraus carve $\bowtie$ out of **graph** category.

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$\sqcup \ltimes \mathbb{I}$: NEFinLinOrd $\to$ NEFinLinOrd :
$W \mapsto W^{\mathrm{op}} \uplus_{<} W$

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MTraS shape modelled by $\sqcup \ltimes \mathbb{I}$ reconciles:

- Hom as a **contra-/covariant bifunctor**,
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$\mathbb{I}$ as an MTraS-shape is better behaved on $\bowtie$:

Twisted Cube Category $\bowtie$ [PK20]

(Roughly) the subcategory of NEFinLinOrd (or Δ) generated by $\top$ and $\sqcup \ltimes \mathbb{I}$.

➔ Use $\bowtie$ instead of Δ.

ℹ Pinyo & Kraus carve $\bowtie$ out of **graph** category.

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Jet Set of dimension n

Set equipped with n Prop-valued **jet-relations** $\rightarrowtriangle_i$ such that:

- $\rightarrowtriangle_i$ is **reflexive**
- $\rightarrowtriangle_i$ implies $\leftarrowtriangle\!\!\rightarrowtriangle_{i+1}$

- **Intervals** $(\!|\rightarrowtriangle_i|\!) = \{0 \rightarrowtriangle_i 1\}$
- **Twisted prism** functor $\sqcup \ltimes (\!|\rightarrowtriangle_i|\!)$ only **op**s degree i
- **Jet cubes** are generated by $\top$ and $\sqcup \ltimes (\!|\rightarrowtriangle_i|\!)$
  - What is a morphism of jet cubes?

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The Category of **Jet Cubes**

$$\begin{tikzcd}
\mathrm{JetCube}_n \arrow[rr, bend left] \arrow[dr, dotted, "\mathrm{ForgetDeg}"'] & & \mathrm{JetSet}_n \arrow[d, "U"] \\
& \mathrm{Cube}^{\text{⌑}}_M \arrow[r, dotted, "\mathrm{Endpoints}"'] & \mathrm{Set}
\end{tikzcd}$$

- What **interval operations** do you want? ➔ $\mathrm{Cube}^{\boxslash}_M \cong \mathrm{Kleisli}(M)^{\mathrm{op}}$
- Do you want diagonals? ➔ $\text{⌑} \in \{\square, \boxslash\}$
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When is a morphism of **cubes** a morphism of **jet cubes**?

$$\dfrac{}{\vdash () : V \to ()}\ \textsc{Terminal}$$

$$\dfrac{\vdash \varphi : V \to \mathsf{Op}^{\boxdot}_i(W)}{\vdash (\varphi, 0/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis)}\ \textsc{Src:Fwd} \qquad \dfrac{\vdash \varphi : V \to \mathsf{Op}^{\boxdot}_i(W)}{\vdash (\varphi, 1/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis)}\ \textsc{Src:Bck}$$

$$\dfrac{\vdash \varphi : V \to W}{\vdash (\varphi, 1/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis)}\ \textsc{Tgt:Fwd} \qquad \dfrac{\vdash \varphi : V \to W}{\vdash (\varphi, 0/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis)}\ \textsc{Tgt:Bck}$$

$$\dfrac{\vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis)}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis)}\ \textsc{Inv:Fwd} \qquad \dfrac{\vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis)}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis)}\ \textsc{Inv:Bck}$$

$$\dfrac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis)}\ \textsc{Prism:Fwd} \qquad \dfrac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis)}\ \textsc{Prism:Bck}$$

$$\dfrac{\vdash \varphi : \mathsf{FSym}^{\boxdot}_i V \to W}{\vdash \varphi : V \to \mathsf{USym}^{\boxdot}_i W}\ \textsc{Symmetrize}$$

$$\dfrac{\vdash \varphi : \mathsf{SymCl}^{\boxdot}_i(V) \to W \qquad R \in \{\multimap, \multimapinv, \multimapboth\}}{\vdash (\varphi, \mathbf{i}/\oslash) : (V, \mathbf{i} : \llparenthesis R_i \rrparenthesis) \to W}\ \textsc{Wkn} \qquad \dfrac{\vdash \varphi : (V, \mathbf{j} : \llparenthesis \multimapboth_i \rrparenthesis, U_1, \mathbf{i} : \llparenthesis \multimapboth_i \rrparenthesis, U_2) \to W}{\vdash \varphi : (V, \mathbf{i} : \llparenthesis \multimapboth_i \rrparenthesis, U_1, \mathbf{j} : \llparenthesis \multimapboth_i \rrparenthesis, U_2) \to W}\ \textsc{Exchange}$$

$$\dfrac{P \in \{\multimap, \multimapinv, \multimapboth\} \qquad Q \in \{\multimap, \multimapinv\} \qquad j > i \\ \vdash \varphi : \mathsf{SymCl}^{\boxdot}_i(\mathsf{SymCl}^{\boxdot}_j U, V) \to W}{\vdash (\varphi, \mathbf{j}/\mathbf{i}) : (U, \mathbf{j} : \llparenthesis P_j \rrparenthesis, V) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \textsc{Concursor}$$

$$\dfrac{(Q, \Diamond) \in \boxed{\{(\multimap, \vee), (\multimapinv, \wedge)\}} \\ \vdash \varphi : \mathsf{SymCl}^{\boxdot}_i V \to W \\ \vdash (\varphi, t/\mathbf{i}) : \boxed{\mathsf{Op}^{\boxdot}_i V} \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \Diamond \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \mathrm{Boo} \quad \textsc{Conn:Prism:Src-Neutral}$$

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$$\dfrac{Q \in \{\multimap, \multimapinv\} \qquad \Diamond \in \{\vee, \wedge\} \\ \vdash (\varphi, s/\mathbf{i}) : \mathsf{SymCl}^{\boxdot}_i V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \\ \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \Diamond s/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \mathrm{Boo} \quad \textsc{Conn:Degree-Symmetric}$$

Semantic Modalities

⌛ In progress. . .

Three Approaches to the Model

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\begin{array}{c}\text{Tamsamani \& Simpson's}\\ \text{model of 2-categories}\\ \text{(2-simplicial sets)}\end{array}
\arrow[r, "\text{heterogenize}"'] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"]
& \begin{array}{c}\text{Pro-arrow}\\ \text{(pre-)equipments}\\ \text{[Woo82,Woo85]}\end{array}
\arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"']
& \begin{array}{c}\text{Bridge/path}\\ \text{cubical sets}\\ \text{(ParamDTT)}\\ \text{[NVD17]}\end{array}
\arrow[l, "\text{directify}"] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"']
\\
\begin{array}{c}\text{Tamsamani \& Simpson's}\\ \text{model of } n\text{-categories}\\ (n\text{-simplicial sets})\\ \text{[CL04,Tam99,Sim97]}\end{array}
\arrow[r, "\text{heterogenize}"']
& \begin{array}{c}\textbf{Naturality}\\ \textbf{(Pre)type Theory}\end{array}
& \begin{array}{c}\text{Degrees of}\\ \text{Relatedness}\\ \text{(RelDTT)}\\ \text{[ND18]}\end{array}
\arrow[l, "\text{directify}"]
\end{tikzcd}
$$

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Thanks!

Questions?

Depth n types

- ***i*-edge** relations $\frown_i$
- **Dependency:**
  $r : a \frown_i^R b$ presumes $R : A \frown_{i+1}^{\mathrm{U}} B$
- **Degradation:**
  $a \frown_i b \Rightarrow a \frown_{i+1} b$
- Modalities change indices:

$$\begin{tikzcd}
a \frown_0 b \arrow[d] & fa \frown_0 fb \arrow[d] \\
a \frown_1 b \arrow[d] \arrow[ur] & fa \frown_1 fb \\
a \frown_2 b \arrow[ur] &
\end{tikzcd}$$

$\boxed{\mathbf{lim}^{\oplus}}$

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n-equipments

- ***i*-jet** (pro^{i-1}-arrow) relations $\rightarrowtriangle_i$
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- **Companion / conjoint:**
  $(\ddagger, \dagger) : a \rightarrowtriangle_i b \Rightarrow a \leftrightarrowtriangle_{i+1} b$
- Modalities change indices & orientation:

$$\begin{tikzcd}
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$\boxed{\mathbf{lim}^{\ominus}}$

$$\begin{tikzcd}
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a \rightarrowtriangle_1 b \arrow[r, "\ominus"] \arrow[d, "\circledast"'] & fa \rightarrowtriangle_1 fb \arrow[d, "\circledast"] \\
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a \frown_1 b \arrow[d] \arrow[ur] & fa \frown_1 fb \\
a \frown_2 b \arrow[ur] &
\end{tikzcd}$$

$\boxed{\mathbf{lim}^{\oplus}}$

$$\begin{tikzcd}
a \frown_0 b \arrow[r] \arrow[d] & fa \frown_0 fb \arrow[d] \\
a \frown_1 b \arrow[r] \arrow[d] & fa \frown_1 fb \arrow[d] \\
a \frown_2 b \arrow[r] & fa \frown_2 fb
\end{tikzcd}$$

$\boxed{\mathbf{ftr}^{\oplus}}$

n-equipments

- ***i*-jet** (pro^{i-1}-arrow) relations $\rightarrowtriangle_i$
- **Dependency:**
  $j : a \rightarrowtriangle_i^J b$ presumes $J : A \rightarrowtriangle_{i+1}^{\mathsf{U}} B$
- **Companion / conjoint:**
  $(\ddagger, \dagger) : a \rightarrowtriangle_i b \Rightarrow a \leftrightarrowtriangle_{i+1} b$
- Modalities change indices & orientation:

$$\begin{tikzcd}
a \rightarrowtriangle_0 b \arrow[d, "\circledast"'] & fa \rightarrowtriangle_0 fb \arrow[d, "\circledast"] \\
a \rightarrowtriangle_1 b \arrow[d, "\circledast"'] \arrow[ur, "\oplus"] & fa \rightarrowtriangle_1 fb \\
a \rightarrowtriangle_2 b \arrow[ur, "\oplus"] &
\end{tikzcd}$$

$\boxed{\mathbf{lim}^{\ominus}}$

$$\begin{tikzcd}
a \rightarrowtriangle_0 b \arrow[r, "\oplus"] \arrow[d, "\circledast"'] & fa \rightarrowtriangle_0 fb \arrow[d, "\circledast"] \\
a \rightarrowtriangle_1 b \arrow[r, "\ominus"] \arrow[d, "\circledast"'] & fa \rightarrowtriangle_1 fb \arrow[d, "\circledast"] \\
a \rightarrowtriangle_2 b \arrow[r, "\oplus"] & fa \rightarrowtriangle_2 fb
\end{tikzcd}$$

$\boxed{\mathbf{ftr}^{\ominus}}$

Depth n types

- ***i*-edge** relations $\frown_i$
- **Dependency:**
  $r : a \frown_i^{R} b$ presumes $R : A \frown_{i+1}^{\mathrm{U}} B$
- **Degradation:**
  $a \frown_i b \Rightarrow a \frown_{i+1} b$
- Modalities change indices:

$$\begin{tikzcd}
a \frown_0 b \arrow[d] & fa \frown_0 fb \arrow[d] \\
a \frown_1 b \arrow[d] \arrow[ur] & fa \frown_1 fb \\
a \frown_2 b \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
a \frown_0 b \arrow[r] \arrow[d] & fa \frown_0 fb \arrow[d] \\
a \frown_1 b \arrow[r] \arrow[d] & fa \frown_1 fb \arrow[d] \\
a \frown_2 b \arrow[r] & fa \frown_2 fb
\end{tikzcd}$$

$\boxed{\mathbf{lim}^{\oplus}}$ $\boxed{\mathbf{ftr}^{\oplus}}$

n-equipments

- ***i*-jet** (pro^{i-1}-arrow) relations $\rightarrowtriangle_i$
- **Dependency:**
  $j : a \rightarrowtriangle_i^{J} b$ presumes $J : A \rightarrowtriangle_{i+1}^{\mathrm{U}} B$
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a \rightarrowtriangle_2 b \arrow[ur, "\oplus"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
a \rightarrowtriangle_0 b \arrow[r, "\oplus"] \arrow[d, "\circledast"'] & fa \rightarrowtriangle_0 fb \arrow[d, "\circledast"] \\
a \rightarrowtriangle_1 b \arrow[r, "\ominus"] \arrow[d, "\circledast"'] & fa \rightarrowtriangle_1 fb \arrow[d, "\circledast"] \\
a \rightarrowtriangle_2 b \arrow[r, "\oplus"] & fa \rightarrowtriangle_2 fb
\end{tikzcd}$$

$\boxed{\mathbf{lim}^{\ominus}}$ $\boxed{\mathbf{ftr}^{\ominus}}$

Understanding modalities: Irrelevance

irr : values $\to$ values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\, A$$

shi : values $\to$ types

$$\lambda n.\mathrm{List}_n\, A : (\mathbf{shi} \mid n : \mathbb{N}) \to \mathrm{U}^0$$

Understanding modalities: Irrelevance

irr : values $\to$ values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\ A$$

$$\begin{tikzcd}
m = n \arrow[rr] \arrow[d] & & {[\,]_m = [\,]_n} \arrow[d] \\
r : m \frown^{\mathbb{N}}_{0} n \arrow[d] & & {[\,]_m \frown^{\mathrm{List}_\bullet\ A}_{0} [\,]_n} \arrow[d] \\
\bullet : \top \arrow[urr] & & \top
\end{tikzcd}$$

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 & \mathrm{List}_m\ A \frown^{\mathrm{U}^0}_{0} \mathrm{List}_n\ A \arrow[d] \\
m \frown^{\mathbb{N}}_{0} n \arrow[d] \arrow[ur, dotted] & \mathrm{List}_m\ A \frown^{\mathrm{U}^0}_{1} \mathrm{List}_n\ A \arrow[d] \\
\top \arrow[ur, dotted] & \top
\end{tikzcd}$$

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m \frown^{\mathbb{N}}_{0} n \arrow[d] \arrow[ur, dotted, no head] & \mathrm{List}_m\, A \frown^{\mathrm{U}^0}_{1} \mathrm{List}_n\, A \arrow[d] \\
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m \frown^{\mathbb{N}}_{0} n \arrow[d] & \mathrm{List}_m\ A \frown^{\mathrm{U}^0}_{1} \mathrm{List}_n\ A \arrow[d] \arrow[dl, dotted, no head] \\
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m \frown^{\mathbb{N}}_{0} n \arrow[d] \arrow[ur, dotted, no head] & \mathsf{List}_m\ A \frown^{\mathsf{U}^0}_{1} \mathsf{List}_n\ A \arrow[d] \\
\top \arrow[ur] & \top
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