Lifting Profunctors to Presheaf Categories
v0.1

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Abstract

It is well-known that a functor between base categories, leads to an adjoint triple between presheaf categories. Here, we discuss how a profunctor between base categories, leads to an adjoint pair between presheaf categories. The former result can be recovered from the latter: a functor has a companion and a conjoint profunctor, each of which leads to an adjoint pair between the presheaf categories. It turns out that these pairs have one functor in common and as such constitute an adjoint triple. We give explicit constructions, but the result also follows from well-known categorical facts, so no particular novelty is claimed.

Notation 0.1. We use the presheaf notations from earlier work [Nuy20, §2.3.1], concretely:

- The application of a presheaf $\Gamma \in \mathsf{Psh}(\mathcal{W})$ to an object $W \in \mathcal{W}$ is denoted $W \Rightarrow \Gamma$. An element $\gamma : W \Rightarrow \Gamma$ is called a **cell** of Γ of **shape** W.
- The **restriction** of $\gamma : W \Rightarrow \Gamma$ by $\varphi : V \to W$, i.e. the action of Γ on φ applied to γ, is denoted $\gamma \circ \varphi$ or $\gamma\varphi : V \Rightarrow \Gamma$.
- The application of a presheaf morphism $\sigma : \Gamma \to \Delta$ to $\gamma : W \Rightarrow \Gamma$ is denoted $\sigma \circ \gamma$ or $\sigma\gamma$. By naturality of σ, we have $\sigma \circ (\gamma \circ \varphi) = (\sigma \circ \gamma) \circ \varphi$.

If $\mathcal{W}$ has a terminal object, then we call a cell $\top \Rightarrow \Gamma$ a **point**.

Notation 0.2. We will always regard **profunctors** as functors to presheaf categories, i.e. a profunctor $\mathcal{W}^{\mathsf{op}} \times \mathcal{V} \to \mathsf{Set}$ will be studied in swapped-curried form $\mathcal{V} \to \mathsf{Psh}(\mathcal{W})$.

1 Lifting Functors

This section is taken verbatim from [Nuy20, §2.3.8].

Theorem 1.1. Any functor $F : \mathcal{V} \to \mathcal{W}$ gives rise to functors $F_! \dashv F^* \dashv F_*$, with a natural isomorphism $F_! \circ \mathbf{y} \cong \mathbf{y} \circ F : \mathcal{V} \to \mathsf{Psh}(\mathcal{W})$. We will call $F_! : \mathsf{Psh}(\mathcal{V}) \to \mathsf{Psh}(\mathcal{W})$ the **left lifting** of F to presheaves, $F^* : \mathsf{Psh}(\mathcal{W}) \to \mathsf{Psh}(\mathcal{V})$ the **central** and $F_* : \mathsf{Psh}(\mathcal{V}) \to \mathsf{Psh}(\mathcal{W})$ the **right lifting**.[1] [Sta19, nLa24b]

The operation $\sqcup^* : \mathsf{Cat}^{\mathsf{coop}} \to \mathsf{Cat}$ sending $F : \mathcal{V} \to \mathcal{W}$ to $F^* : \mathsf{Psh}(\mathcal{W}) \to \mathsf{Psh}(\mathcal{V})$ is a strict 2-functor. Hence $\sqcup_!, \sqcup_* : \mathsf{Cat} \to \mathsf{Cat}$ are pseudofunctors.

Proof. Using quantifier symbols for ends and co-ends, we can define:

$$\begin{aligned} W \Rightarrow F_!\Gamma \quad &:= \quad \exists V.(W \to FV) \times (V \Rightarrow \Gamma), \\ V \Rightarrow F^*\Delta \quad &:= \quad FV \Rightarrow \Delta \\ W \Rightarrow F_*\Gamma \quad &:= \quad \forall V.(FV \to W) \to (V \Rightarrow \Gamma) \quad = \quad (F^*\mathbf{y}W \to \Gamma). \end{aligned}$$

[1] The central and right liftings are also sometimes called the inverse image and direct image of F, but these are actually more general concepts and as such could perhaps cause confusion or unwanted connotations in some circumstances. The left-central-right terminology is very no-nonsense.

By the co-Yoneda lemma, we have, naturally in W:

$$\begin{array}{rcl} W \Rrightarrow F_!\mathbf{y}V & = & \exists V'.(W \to FV') \times (V' \to V) \\ & \cong & (W \to FV) \quad = \quad (W \Rrightarrow \mathbf{y}FV), \end{array}$$

i.e. $F_!\mathbf{y}V \cong \mathbf{y}FV$.

Adjointness also follows from applications of the Yoneda and co-Yoneda lemmas. [Sta19]

It is evident from the definition of $\sqcup^*$ that it preserves identity and composition on the nose, and easy to check that it turns around natural transformations. By uniqueness of the left/right adjoint, $\sqcup_!$ and $\sqcup_*$ are then pseudofunctors. $\square$

2 Lifting Profunctors

Theorem 2.1. Any profunctor $F : \mathcal{V} \to \mathsf{Psh}(\mathcal{W})$ gives rise to functors $F_\odot \dashv F^\odot$, with a natural isomorphism $F_\odot \circ \mathbf{y} \cong F$. We will call $F_\odot : \mathsf{Psh}(\mathcal{V}) \to \mathsf{Psh}(\mathcal{W})$ the **left lifting** of F to presheaves and $F^\odot : \mathsf{Psh}(\mathcal{W}) \to \mathsf{Psh}(\mathcal{V})$ the **central lifting**.[2] [Sta19, nLa24b]

Proof. Using quantifier symbols for ends and co-ends, we define:

$$\begin{array}{rcl} W \Rrightarrow F_\odot\Gamma & := & \exists V.(W \Rrightarrow FV) \times (V \Rrightarrow \Gamma), \\ V \Rrightarrow F^\odot\Delta & := & \forall W.(W \Rrightarrow FV) \to (W \Rrightarrow \Delta). \end{array}$$

By the co-Yoneda lemma, we have, naturally in W:

$$\begin{array}{rcl} W \Rrightarrow F_\odot\mathbf{y}V & = & \exists V'.(W \Rrightarrow FV') \times (V' \to V) \\ & \cong & (W \Rrightarrow FV), \end{array}$$

i.e. $F_\odot\mathbf{y}V \cong FV$.

To see adjointness, we have a chain of natural isomorphisms:

$$\begin{array}{rcl} F_\odot\Gamma \to \Delta & = & \forall W.(W \Rrightarrow F_\odot\Gamma) \to (W \Rrightarrow \Delta) \\ & = & \forall W.(\exists V.(W \Rrightarrow FV) \times (V \Rrightarrow \Gamma)) \to (W \Rrightarrow \Delta) \\ & \cong & \forall W.\forall V.(W \Rrightarrow FV) \to (V \Rrightarrow \Gamma) \to (W \Rrightarrow \Delta) \\ & \cong & \forall V.(V \Rrightarrow \Gamma) \to \forall W.(W \Rrightarrow FV) \to (W \Rrightarrow \Delta) \\ & = & \forall V.(V \Rrightarrow \Gamma) \to (V \Rrightarrow F^\odot\Delta) \\ & = & \Gamma \to F^\odot\Delta. \end{array}$$

$\square$

Remark 2.2. When we view profunctors $\mathcal{W} \nrightarrow \mathcal{V}$ as functors $\mathcal{W}^{\mathsf{op}} \times \mathcal{V} \to \mathsf{Set}$, then

- the identity profunctor $\mathcal{V} \nrightarrow \mathcal{V}$ is the functor $\mathrm{Hom} : \mathcal{V}^{\mathsf{op}} \times \mathcal{V} \to \mathsf{Set}$,
- the composite of $\mathcal{Q} : \mathcal{W} \nrightarrow \mathcal{V}$ and $\mathcal{P} : \mathcal{V} \nrightarrow \mathcal{U}$ is given by:

$$(\mathcal{Q} \otimes \mathcal{P})(U, W) = \exists V.\mathcal{Q}(W, V) \times \mathcal{P}(V, U).$$

If instead we view profunctors $\mathcal{W} \nrightarrow \mathcal{V}$ as functors $\mathcal{V} \to \mathsf{Psh}(\mathcal{W})$, then

- the identity profunctor $\mathcal{V} \nrightarrow \mathcal{V}$ is the functor $\mathbf{y} : \mathcal{V} \to \mathsf{Psh}(\mathcal{V})$,
- the composite of $G : \mathcal{V} \to \mathsf{Psh}(\mathcal{W})$ and $F : \mathcal{U} \nrightarrow \mathsf{Psh}(\mathcal{V})$ is given by:

$$\begin{array}{rcl} W \Rrightarrow (F \ggg G)U & = & \exists V.(W \Rrightarrow GV) \times (V \Rrightarrow FU) \\ & = & W \Rrightarrow G_\odot FU. \end{array}$$

[2] The central and right liftings are also sometimes called the inverse image and direct image of F, but these are actually more general concepts and as such could perhaps cause confusion or unwanted connotations in some circumstances. The left-central-right terminology is very no-nonsense.

On a side note: yes, Psh is a 2-monad[3] with unit $\mathbf{y}$ and yes, the 2-category of categories and profunctors is the Kleisli-2-category of Psh, just like the category of sets and relations is the Kleisli-category of the powerset monad.

Theorem 2.3. The operations $\sqcup_\odot$ and $\sqcup^\odot$ are pseudofunctors w.r.t. identity and composition of profunctors.

Proof. By adjointness, it suffices to show this for $\sqcup_\odot$. For the identity, we have:

$$\begin{array}{rcl} V \Rightarrow \mathbf{y}_\odot \Gamma & = & \exists V'.(V \Rightarrow \mathbf{y}V') \times (V' \Rightarrow \Gamma) \\ & = & \exists V'.(V \to V') \times (V' \Rightarrow \Gamma) \\ & = & V \Rightarrow \Gamma \end{array}$$

For composition, we have:

$$\begin{array}{rcl} W \Rightarrow (G_\odot F)_\odot \Theta & = & \exists U.(W \Rightarrow G_\odot FU) \times (U \Rightarrow \Theta) \\ & = & \exists U.(\exists V.(W \Rightarrow GV) \times (V \Rightarrow FU)) \times (U \Rightarrow \Theta) \\ & \cong & \exists V.(W \Rightarrow GV) \times (\exists U.(V \Rightarrow FU) \times (U \Rightarrow \Theta)) \\ & = & \exists V.(W \Rightarrow GV) \times (V \Rightarrow F_\odot \Theta) \\ & = & W \Rightarrow G_\odot F_\odot \Theta. \end{array}$$

We do not consider the associahedra. □

3 Lifting Companion and Conjoint Profunctors

Remark 3.1. To a functor $F : \mathcal{V} \to \mathcal{W}$, we can generally associate

- a companion profunctor $\mathrm{Hom}(F\sqcup, \sqcup) : \mathcal{V}^{\mathsf{op}} \times \mathcal{W} \to \mathsf{Set}$,
- a conjoint profunctor $\mathrm{Hom}(\sqcup, F\sqcup) : \mathcal{W}^{\mathsf{op}} \times \mathcal{V} \to \mathsf{Set}$.

Viewing profunctors as functors to the presheaf category, we get instead:

- as the companion, the functor $F^* \circ \mathbf{y} : \mathcal{W} \to \mathsf{Psh}(\mathcal{V})$,
- as the conjoint, the functor $\mathbf{y} \circ F : \mathcal{V} \to \mathsf{Psh}(\mathcal{W})$.

Theorem 3.2. For a functor $F : \mathcal{V} \to \mathcal{W}$, we have:

$$(\mathbf{y} \circ F)_\odot = F_!, \qquad (\mathbf{y} \circ F)^\odot \cong F^* \cong (F^* \circ \mathbf{y})_\odot, \qquad F_* = (F^* \circ \mathbf{y})^\odot.$$

Proof. We have

$$\begin{array}{rcl} W \Rightarrow (\mathbf{y} \circ F)_\odot \Gamma & = & \exists V.(W \Rightarrow \mathbf{y}FV) \times (V \Rightarrow \Gamma) \\ & = & \exists V.(W \to FV) \times (V \Rightarrow \Gamma) \\ & = & F_! \Gamma, \\ V \Rightarrow (\mathbf{y} \circ F)^\odot \Delta & = & \forall W.(W \Rightarrow \mathbf{y}FV) \to (W \Rightarrow \Delta) \\ & = & \forall W.(W \to FV) \to (W \Rightarrow \Delta) \\ & \cong & FV \Rightarrow \Delta \quad = \quad V \Rightarrow F^* \Delta, \\ V \Rightarrow (F^* \circ \mathbf{y})_\odot \Delta & = & \exists W.(V \Rightarrow F^* \mathbf{y} W) \times (W \Rightarrow \Delta) \\ & = & \exists W.(FV \to W) \times (W \Rightarrow \Delta) \\ & \cong & FV \Rightarrow \Delta \quad = \quad V \Rightarrow F^* \Delta, \\ W \Rightarrow (F^* \circ \mathbf{y})^\odot \Gamma & = & \forall V.(V \Rightarrow F^* \mathbf{y} W) \to (V \Rightarrow \Gamma) \\ & = & F^* \mathbf{y} W \to \Gamma \quad = \quad W \Rightarrow F_* \Gamma. \end{array}$$ □

[3] If we ignore size issues, otherwise it is a relative 2-monad

4 At a Higher Level

I personally like to brute-force presheaf details, but the existence of pseudofunctorial operations $\sqcup_\odot$ and $\sqcup^\odot$ can also be argued at a higher level. It is known that $\mathsf{Psh}(\mathcal{W})$ is the free cocompletion of $\mathcal{W}$, with $\mathbf{y} : \mathcal{W} \to \mathsf{Psh}(\mathcal{W})$ embedding the original. Then it is clear that a profunctor $F : \mathcal{V} \to \mathsf{Psh}(\mathcal{W})$ can be extended cocontinuously along $\mathbf{y} : \mathcal{V} \to \mathsf{Psh}(\mathcal{V})$ to $F_\odot : \mathsf{Psh}(\mathcal{V}) \to \mathsf{Psh}(\mathcal{W})$, since $\mathsf{Psh}(\mathcal{W})$ is cocomplete. Now $F_\odot$ is a cocontinuous functor between cocomplete categories and thus [nLa24a] has a right adjoint $F^\odot$.

If we want to define $\sqcup_!$ at a high level, we would simply specialize the above reasoning to a *pure* functor, i.e. we *define* $F_! = (\mathbf{y} \circ F)_\odot$. We could similarly *define* $F_* = (F^* \circ \mathbf{y})^\odot$, where F^* is defined by precomposition, as usual.

Finally, we need to show

$$(\mathbf{y} \circ F)^\odot \cong F^* \cong (F^* \circ \mathbf{y})_\odot.$$

Since colimits in presheaf categories are taken pointwise, F^* is clearly cocontinuous, so we can prove that $F^* \cong (F^* \circ \mathbf{y})_\odot$ by proving that they extend the same functor $\mathcal{W} \to \mathsf{Psh}(\mathcal{V})$ along $\mathbf{y} : \mathcal{W} \to \mathsf{Psh}(\mathcal{W})$. Now by construction, we have $(F^* \circ \mathbf{y})_\odot \circ \mathbf{y} \cong F^* \circ \mathbf{y}$, so this is in fact trivial.

Finally, using just a small amount of brute force, the construction of right adjoint functors to cocontinuous functors between presheaf categories, works by setting $(W \Rightarrow R\Gamma) := (L\mathbf{y}W \to \Gamma)$. When we apply this to $(\mathbf{y} \circ F)^\odot$, we find:

$$\begin{aligned} W \Rightarrow (\mathbf{y} \circ F)^\odot \Gamma &= (\mathbf{y} \circ F)_\odot \mathbf{y} W \to \Gamma \\ &\cong \mathbf{y} F W \to \Gamma \\ &\cong W \Rightarrow F^* \Gamma. \end{aligned}$$

References

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