

We have:

$$y^3 + py + q = 0$$

We want to transform this equation to a quadratic one.

Develop:

$$y = z + \frac{k}{z}$$

$$\begin{aligned} y^3 &= \left(z + \frac{k}{z}\right)^3 = z^3 + 3 \cdot z \left(\frac{k}{z}\right)^2 + 3 \cdot z^2 \left(\frac{k}{z}\right) + \frac{k^3}{z^3} \\ &= z^3 + 3 \cdot \frac{k^2}{z} + 3 \cdot k \cdot z + \frac{k^3}{z^3} \end{aligned}$$

\Rightarrow

$$z^3 + \frac{3k^2}{z} + 3kz + \frac{k^3}{z^3} + pz + p\frac{k}{z} + q = 0$$

$$\Rightarrow z^3 + \frac{k^3}{z^3} + \underbrace{[3k^2 + pk]}_{k \cdot [3k + p]} \cdot \frac{1}{z} + \underbrace{[3k + p]}_{[3k + p]} \cdot z + q = 0$$

$$k \cdot [3k + p] = 0 \quad [3k + p] = 0$$

$$\Rightarrow 3k = -p \Rightarrow \boxed{k = -\frac{p}{3}}$$

$$\Rightarrow z^3 - \frac{p^3}{27} \cdot \frac{1}{z^3} + q = 0$$

$$\text{Let } \boxed{w = z^3}$$

$$\Rightarrow w - \frac{p^3}{27} \cdot \frac{1}{w} + q = 0 \quad / \cdot 27w$$

$$27 \cdot w^2 - p^3 + 27w \cdot q = 0$$

$$27 \cdot w^2 + [27q] \cdot w - p^3 = 0$$

$$\Rightarrow \boxed{w^2 + q \cdot w - \frac{p^3}{27} = 0}$$

Remind: $w^2 + p' \cdot w + q' = 0$

$$\Rightarrow \boxed{w_{1,2} = \frac{-p' \pm \sqrt{(p')^2 - 4q'}}{2}}$$

$$p' = q \quad q' = -\frac{p^3}{27}$$

$$\begin{aligned} w_{1,2} &= \frac{-q \pm \sqrt{q^2 - 4 \cdot \left(-\frac{p^3}{27}\right)}}{2} \\ &= \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} \end{aligned}$$

\Rightarrow

$$\boxed{z^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}}$$