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Laplace Transform

In mathematics, the Laplace transform, named after its discoverer Pierre-Simon Laplace is an integral transform that converts a function of a real variable (usually t in the time domain) to a function of a complex variable s (in the complex frequency domain, also known as s-domain, or s-plane). The transform has many applications in science and engineering, mostly as a tool for solving linear differential equations.

$$\mathscr{L}\{f\}(s) = \int_0^\infty f(t) \cdot e^{-st} dt$$

Properties:

$$(i) \quad \boxed{\mathscr{L}\{\alpha \cdot f + \beta \cdot g\}(s) = \alpha \cdot \mathscr{L}\{f\}(s) + \beta \cdot \mathscr{L}\{g\}(s)}$$

$$(ii) \quad \boxed{\mathscr{L}\{f'\}(s) = s \cdot \mathscr{L}\{f\}(s) - f(0)}$$

$$(iii) \quad \boxed{\mathscr{L}\{f * g\}(s) = \mathscr{L}\{f\}(s) \cdot \mathscr{L}\{g\}(s)}$$

Convolution:

$$\boxed{(f * g)(t) = \int_0^t f(u) \cdot g(t-u) du}$$

Integration by parts:

$$\boxed{\int_a^b h_1'(t) \cdot h_2(t) dt = h_1(t) \cdot h_2(t)\Big|_{t=a}^{t=b} - \int_a^b h_1(t) \cdot h_2'(t) dt}$$

Proof:

(i)

$$\mathscr{L}\{\alpha \cdot f + \beta \cdot g\}(s) = \int_0^\infty [\alpha \cdot f(t) + \beta \cdot g(t)] \cdot e^{-st} dt$$

$$= \alpha \cdot \int_0^\infty [f(t)] \cdot e^{-st} dt + \beta \cdot \int_0^\infty [g(t)] \cdot e^{-st} dt$$

$$= \alpha \cdot \mathscr{L}\{f\}(s) + \beta \cdot \mathscr{L}\{g\}(s)$$

(ii)

$$\mathscr{L}\{f\}(s) = \int_0^\infty f(t) \cdot e^{-st} dt$$

We can say that:

$$h_1(t) = f(t) \ \Rightarrow h_1'(t) = f'(t)$$

$$h_2'(t) = e^{-st} \ \Rightarrow h_2(t) = -\frac{e^{-st}}{s}$$

Using integration by parts:

$$\int_0^\infty f(t) \cdot e^{-st} dt = \left(-\frac{1}{s}\right) \left[\lim_{t \to \infty} f(t) \cdot e^{-st} - f(0)\right] - \int_0^\infty f'(t) \cdot -\frac{e^{-st}}{s} dt$$

$$= \frac{1}{s} \cdot f(0) + \frac{1}{s} \cdot \int_0^\infty f'(t) \cdot e^{-st} dt$$

$\Rightarrow$

$$\int_0^\infty f(t) \cdot e^{-st} dt = \frac{1}{s} \cdot f(0) + \frac{1}{s} \cdot \int_0^\infty f'(t) \cdot e^{-st} dt$$

$\Rightarrow$

$$\mathscr{L}\{f\}(s) = \frac{1}{s} \cdot f(0) + \frac{1}{s} \cdot \mathscr{L}\{f'\}(s)$$

$\Rightarrow$

$$s \cdot \mathscr{L}\{f\}(s) = f(0) + \mathscr{L}\{f'\}(s)$$

$\Rightarrow$

$$\mathscr{L}\{f'\}(s) = s \cdot \mathscr{L}\{f\}(s) - f(0)$$

(iii)

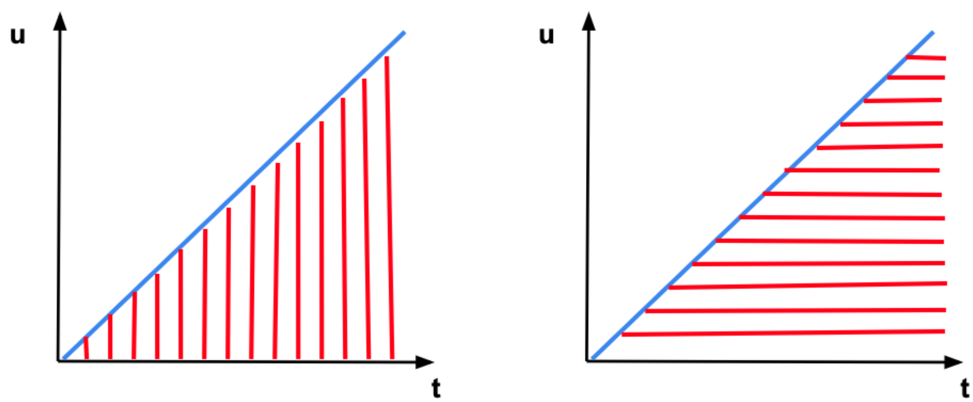
$$h(t) = \int_0^t f(u) \cdot g(t-u) du$$

$$\mathscr{L}\{f * g\}(s) = \mathscr{L}\{h\}(s) = \int_0^\infty h(t) \cdot e^{-st} dt = \int_0^\infty \int_0^t f(u) \cdot g(t-u) du \cdot e^{-st} dt$$

$$= \int_0^\infty \int_0^t f(u) \cdot g(t-u) \cdot e^{-st} du \; dt = \int_D f(u) \cdot g(t-u) \cdot e^{-st} du \; dt$$

We can see that:

$$D = \{(t,u) : 0 \leq t, 0 \leq u \leq t\} = \{(u,t) : 0 \leq u, u \leq t \leq \infty\}$$



$$= \int_0^\infty \int_u^\infty f(u) \cdot g(t-u) \cdot e^{-st} dt \; du = \int_0^\infty f(u) \cdot \left[\int_u^\infty g(t-u) \cdot e^{-st} dt\right] \; du$$

Now we going to compute:

$$\int_u^\infty g(t-u) \cdot e^{-st} dt$$

Consider the change of variable $z = t - u \Rightarrow (dz = dt \wedge t = u + z)$

$$\int_u^\infty g(t-u) \cdot e^{-st} dt = \int_0^\infty g(z) \cdot e^{-s \cdot [u+z]} dz = \int_0^\infty g(z) \cdot e^{-s \cdot z} \cdot e^{-s \cdot u} dz$$

$$= e^{-s \cdot u} \cdot \int_0^\infty g(z) \cdot e^{-s \cdot z} dz = e^{-s \cdot u} \cdot \mathscr{L}\{g\}(s)$$

$\Rightarrow$

$$\mathscr{L}\{f * g\}(s) = \int_0^\infty f(u) \cdot \left[\int_u^\infty g(t-u) \cdot e^{-st} dt\right] \, du$$

$$= \int_0^\infty f(u) \cdot e^{-s \cdot u} \cdot \mathscr{L}\{g\}(s) \, du = \mathscr{L}\{g\}(s) \cdot \int_0^\infty f(u) \cdot e^{-s \cdot u} du$$

$$= \mathscr{L}\{g\}(s) \cdot \mathscr{L}\{f\}(s)$$

Solving ODEs:

$$\boxed{y''(t) = -4 \cdot y(t)}$$

$$\boxed{y(0) = 0 \quad , \quad y'(0) = 2}$$

Solution:

Note that:

$$\mathscr{L}\{y''\}(s) = s \cdot \mathscr{L}\{y'\}(s) - y'(0) = s \cdot [s \cdot \mathscr{L}\{y\}(s) - y(0)] - y'(0)$$

$$= s^2 \cdot \mathscr{L}\{y\}(s) - s \cdot y(0) - y'(0) = s^2 \cdot \mathscr{L}\{y\}(s) - 2$$

$\Rightarrow$

$$\boxed{\mathscr{L}\{y''\}(s) = s^2 \cdot \mathscr{L}\{y\}(s) - 2} \quad (1)$$

Now using the equation:

$$\boxed{\mathscr{L}\{y''\}(s) = -4 \cdot \mathscr{L}\{y\}(s)} \quad (2)$$

Combining (1) and (2) we have:

$$s^2 \cdot \mathscr{L}\{y\}(s) - 2 = 4 \cdot \mathscr{L}\{y\}(s)$$

$\Rightarrow$

$$(s^2 + 4) \cdot \mathscr{L}\{y\}(s) = 2$$

$\Rightarrow$

$$\boxed{\mathscr{L}\{y\}(s) = \frac{2}{s^2 + 2^2}}$$

$\Rightarrow$

$$\boxed{y(t) = sin(2t)}$$
