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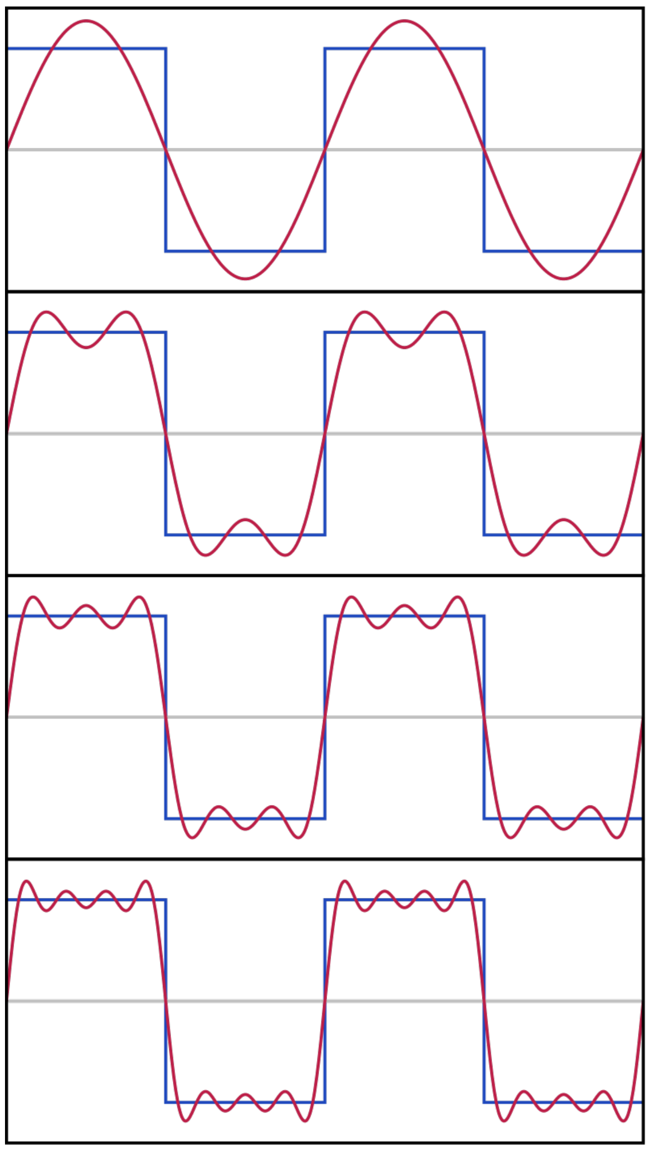
Fourier Series

A Fourier series is an expansion of a periodic function into a sum of trigonometric functions. The Fourier series is an example of a trigonometric series, but not all trigonometric series are Fourier series.



$$\boxed{f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cdot cos\left(\frac{2n\pi t}{T}\right) + b_n \cdot sin\left(\frac{2n\pi t}{T}\right) \right]}$$

$$\boxed{a_0 = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt}$$

$$\boxed{a_n = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot cos\left(\frac{2n\pi t}{T}\right) dt}$$

$$\boxed{b_n = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot sin\left(\frac{2n\pi t}{T}\right) dt}$$

Integration by parts:

$$\boxed{\int_a^b h_1'(t) \cdot h_2(t)dt = h_1(t) \cdot h_2(t)\Big|_{t=a}^{t=b} - \int_a^b h_1(t) \cdot h_2'(t)dt}$$

Example: Find the fourier series for $f(t) = t$ for $t \in [-\pi, \pi]$.

<u>Solution:</u>

We have $T = 2\pi$.

$$a_0 = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt = \frac{2}{2\pi} \cdot \int_{-\pi}^{\pi} t\ dt = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} t\ dt = \frac{1}{\pi} \cdot \frac{t^2}{2}\Big|_{t=-\pi}^{t=\pi} = 0$$

$\Rightarrow$

$$\boxed{a_0 = 0} \quad (1)$$

$$a_n = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot cos\left(\frac{2n\pi t}{T}\right) dt = \frac{2}{2\pi} \cdot \int_{-\pi}^{\pi} t \cdot cos\left(\frac{2n\pi t}{2\pi}\right) dt$$

$$= \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} t \cdot cos\,(nt)\,dt$$

You can see that $g(t) = t \cdot cos(nt)$ is odd function:

$$g(-t) = (-t) \cdot cos(-nt) = -t \cdot cos(nt) = -g(t)$$

$$\boxed{g(t) = -g(-t)}$$

For this reason:

$$\boxed{a_n = 0} \quad (2)$$

$$b_n = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot sin\left(\frac{2n\pi t}{T}\right) dt = \frac{2}{2\pi} \cdot \int_{-\pi}^{\pi} t \cdot sin\left(\frac{2n\pi t}{2\pi}\right) dt$$
$$= \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} t \cdot sin\,(nt)\, dt$$

$$\boxed{b_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} t \cdot sin\,(nt)\, dt}$$

Using integral by parts:

$$h_1(t) = t \;\Rightarrow\; h_1'(t) = 1$$

$$h_2'(t) = sin(nt) \;\Rightarrow\; h_2(t) = -\frac{cos(nt)}{n}$$

$\Rightarrow$

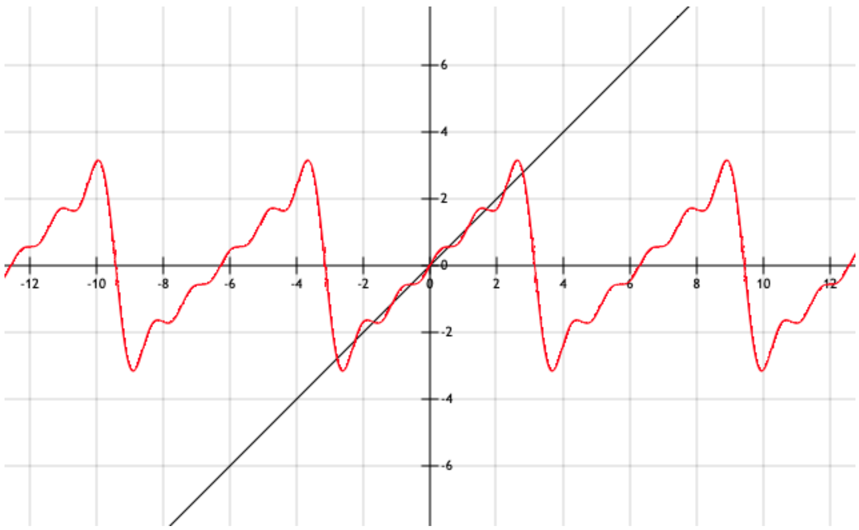
$$b_n = \frac{1}{\pi} \cdot \left[-\frac{t \cdot cos(nt)}{n} - \left(-\frac{sin(nt)}{n^2}\right)\right] \Big|_{t=-\pi}^{t=\pi} = \frac{1}{\pi} \cdot \left[-\frac{t \cdot cos(nt)}{n} + \frac{sin(nt)}{n^2}\right] \Big|_{t=-\pi}^{t=\pi}$$
$$= \frac{1}{\pi} \cdot \left[-\frac{\pi \cdot cos(n \cdot \pi)}{n} + \frac{sin(n \cdot \pi)}{n^2} + \frac{(-\pi) \cdot cos(-n \cdot \pi)}{n} - \frac{sin(-n \cdot \pi)}{n^2}\right]$$
$$= \frac{1}{\pi} \cdot \left[-\frac{\pi \cdot cos(n \cdot \pi)}{n} + \frac{(-\pi) \cdot cos(-n \cdot \pi)}{n}\right] = \frac{1}{\pi} \cdot \left[-\frac{\pi \cdot cos(n \cdot \pi)}{n} - \frac{\pi \cdot cos(n \cdot \pi)}{n}\right]$$
$$= -\frac{2 \cdot \pi \cdot cos(n \cdot \pi)}{n \cdot \pi} = -\frac{2 \cdot cos(n \cdot \pi)}{n} = -\frac{2 \cdot (-1)^n}{n} = \frac{2 \cdot (-1)^{(n+1)}}{n}$$

$\Rightarrow$

$$\boxed{b_n = \frac{2 \cdot (-1)^{(n+1)}}{n}} \quad (3)$$

$$\boxed{f(t) = \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^{(n+1)}}{n} \cdot sin(n \cdot t)}$$

$$\boxed{f(t) = 2sin(t) - sin(2t) + \frac{2}{3} \cdot sin(3t) - \frac{1}{2} \cdot sin(4t) + \frac{2}{5} \cdot sin(5t) + ...}$$



Example: Find the fourier series for $f(t) = t^2$ for $t \in [-\pi, \pi]$.

<u>Solution:</u>

We have $T = 2\pi$.

$$a_0 = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt = \frac{2}{2\pi} \cdot \int_{-\pi}^{\pi} t^2 \; dt = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} t^2 \; dt = \frac{1}{\pi} \cdot \frac{t^3}{3}\Big|_{t=-\pi}^{t=\pi}$$

$$= \frac{1}{\pi} \cdot \frac{\pi^3}{3} - \frac{1}{\pi} \cdot \frac{(-\pi)^3}{3} = \frac{1}{\pi} \cdot \frac{\pi^3}{3} + \frac{1}{\pi} \cdot \frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$\Rightarrow$

$$\boxed{a_0 = \frac{2\pi^2}{3} \quad , \quad \frac{a_0}{2} = \frac{\pi^2}{3}} \qquad (1)$$

$$a_n = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot cos\left(\frac{2n\pi t}{T}\right) dt = \frac{2}{2\pi} \cdot \int_{-\pi}^{\pi} t^2 \cdot cos\left(\frac{2n\pi t}{2\pi}\right) dt$$

$$= \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} t^2 \cdot cos\,(nt)\,dt$$

$\Rightarrow$

$$\boxed{a_n = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} t^2 \cdot cos\,(nt)\,dt}$$

Now we going to compute the integral using integration by parts.

$$h_1(t) = t^2 \;\Rightarrow h_1'(t) = 2t$$

$$h_2'(t) = cos(nt) \;\Rightarrow h_2(t) = \frac{sin(nt)}{n}$$

$\Rightarrow$

$$\int t^2 cos(nt)dt = \frac{t^2 \cdot sin(nt)}{n} - \int \frac{2t \cdot sin(nt)}{n} dt = \frac{t^2 \cdot sin(nt)}{n} - \frac{2}{n}\int t \cdot sin(nt)dt$$

$\Rightarrow$

$$\boxed{\int t^2 cos(nt)dt = \frac{t^2 \cdot sin(nt)}{n} - \frac{2}{n} \cdot \int t \cdot sin(nt)dt} \quad (i)$$

Now we going to use again integral by parts:

$$h_1(t) = t \;\Rightarrow h_1'(t) = 1$$

$$h_2'(t) = sin(nt) \;\Rightarrow h_2(t) = -\frac{cos(nt)}{n}$$

$$\int t \cdot sin(nt)dt = -\frac{t \cdot cos(nt)}{n} - \int -\frac{cos(nt)}{n}\,dt$$

$$-\frac{t \cdot cos(nt)}{n} + \frac{1}{n} \cdot \int cos(nt)\;dt = -\frac{t \cdot cos(nt)}{n} + \frac{1}{n} \cdot \frac{sin(nt)}{n}$$

$$= -\frac{t \cdot cos(nt)}{n} + \frac{sin(nt)}{n^2}$$

$\Rightarrow$

$$\int t^2 cos(nt)dt = \frac{t^2 \cdot sin(nt)}{n} - \frac{2}{n} \cdot \left(-\frac{t \cdot cos(nt)}{n} + \frac{sin(nt)}{n^2} \right)$$

$$= \frac{t^2 \cdot sin(nt)}{n} + \frac{2 \cdot t \cdot cos(nt)}{n^2} - \frac{2 \cdot sin(nt)}{n^3}$$

$\Rightarrow$

$$\boxed{\int t^2 cos(nt)dt = \frac{t^2 \cdot sin(nt)}{n} + \frac{2 \cdot t \cdot cos(nt)}{n^2} - \frac{2 \cdot sin(nt)}{n^3}}$$

$$a_n = \frac{1}{\pi} \cdot \left[\frac{\pi^2 sin(n\pi)}{n} + \frac{2\pi \cdot cos(n\pi)}{n^2} - \frac{2 \cdot sin(n\pi)}{n^3} - \frac{(-\pi)^2 \cdot sin(-n\pi)}{n} - \frac{2(-\pi) \cdot cos(-n\pi)}{n^2} + \frac{2sin(-n\pi)}{n^3} \right]$$

$$= \frac{1}{\pi} \cdot \left[\frac{2\pi \cdot cos(n\pi)}{n^2} + \frac{2\pi \cdot cos(-n\pi)}{n^2} \right] = \frac{1}{\pi} \cdot \left[\frac{2\pi \cdot cos(n\pi)}{n^2} + \frac{2\pi \cdot cos(n\pi)}{n^2} \right]$$

$$= \frac{1}{\pi} \cdot \frac{4\pi \cdot cos(n\pi)}{n^2} = \frac{4cos(n\pi)}{n^2} = \frac{4 \cdot (-1)^n}{n^2}$$

$\Rightarrow$

$$\boxed{a_n = \frac{4 \cdot (-1)^n}{n^2}} \quad (2)$$

$$b_n = \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot sin\left(\frac{2n\pi t}{T}\right) dt = \frac{2}{2\pi} \cdot \int_{-\pi}^{\pi} t^2 \cdot sin\left(\frac{2n\pi t}{2\pi}\right) dt$$

$$= \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} t^2 \cdot sin\,(nt)\, dt$$

We can define $g(t) = t^2 \cdot sin(nt)$. This function is odd:

$$g(-t) = (-t)^2 \cdot sin(-nt) = -t^2 \cdot sin(nt) = -g(t)$$

For this is clear that:

$$\boxed{b_n = 0} \quad (3)$$

Thus:

$$\boxed{f(t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4 \cdot (-1)^n}{n^2} \cdot cos(nt)}$$

$$\boxed{f(t) = \frac{\pi^2}{3} - 4cos(t) + cos(2t) - \frac{4}{9} \cdot cos(3t) + \frac{1}{4} \cdot cos(4t) - \frac{4}{25} \cdot cos(5t) + ...}$$