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Determination coefficient

We going to define determination coefficient:

$$\boxed{R^2 = \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} = 1 - \frac{\sigma_T^2}{\sigma_Y^2}}$$

Relation with correlation: On linear regression.

Remind on linear regression:

$$\boxed{\hat{y}_i = \beta_1 \cdot x_i + \beta_0} \quad (1)$$

$$\boxed{\beta_1 = \frac{S_Y}{S_X} \cdot \rho_{X,Y}} \quad (2)$$

with:

$$S_Y^2 = \frac{1}{(n-1)} \cdot \sum_{i=1}^{n}(y_i - \bar{y})^2$$

$$S_X^2 = \frac{1}{(n-1)} \cdot \sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\boxed{\beta_0 = \bar{y} - \beta_1 \cdot \bar{x}} \quad (3)$$

Now we going to determine the relation between R^2 and correlation on linear regression:

$$R^2 \overset{\text{definition}}{=} \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} \overset{(1)}{=} \frac{\sum_{i=1}^{n}(\beta_1 \cdot x_i + \beta_0 - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} \overset{(3)}{=} \frac{\sum_{i=1}^{n}(\beta_1 \cdot x_i + \bar{y} - \beta_1 \cdot \bar{x} - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

$$= \frac{\sum_{i=1}^{n}(\beta_1 \cdot x_i - \beta_1 \cdot \bar{x})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} = \frac{\sum_{i=1}^{n}[\beta_1]^2 \cdot (x_i - \bar{x})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} = [\beta_1]^2 \cdot \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

$$= [\beta_1]^2 \cdot \frac{\frac{1}{(n-1)} \cdot \sum_{i=1}^{n}(x_i - \bar{x})^2}{\frac{1}{(n-1)} \cdot \sum_{i=1}^{n}(y_i - \bar{y})^2} = [\beta_1]^2 \cdot \frac{S_X^2}{S_Y^2} = \left(\rho_{X,Y} \cdot \frac{S_Y}{S_X}\right)^2 \cdot \frac{S_X^2}{S_Y^2}$$

$$= \left(\rho_{X,Y}\right)^2 \cdot \frac{S_Y^2}{S_X^2} \cdot \frac{S_X^2}{S_Y^2} = \left(\rho_{X,Y}\right)^2$$

$\Rightarrow$

$$\boxed{R^2 = \left(\rho_{X,Y}\right)^2}$$
