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Distance between two rectangles with no rotation

In differents problems is very useful measure the distance between two rectangles with no rotation. For instance when we work with **Bounding Boxes** is useful understand the distance between two Bounding Boxes i.e. two rectangles with no rotation.

To start we going to present the definition of distance between two sets in a metric space from a metric distance.

$$\boxed{d(A,B) = \inf_{x \in A} \inf_{y \in B} \{d(x,y)\}}$$

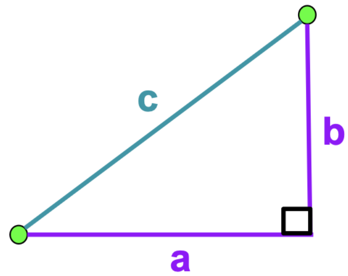
For our particular case:

- A = One Rectangle.
- B = Another Rectangle.
- d = Eucledian distance on $\mathbb{R}^2$.

We going to analize four cases based by the relation between the two rectangles.

a. No intersection on both X-axis and Y-axis.
b. Intersection on X-axis but No intersection on Y-axis.
c. Intersection on Y-axis but No intersection on X-axis.
d. Intersection on both X-axis and Y-axis.

Before start we going to do a good remember of the Pythagoras Theorem.

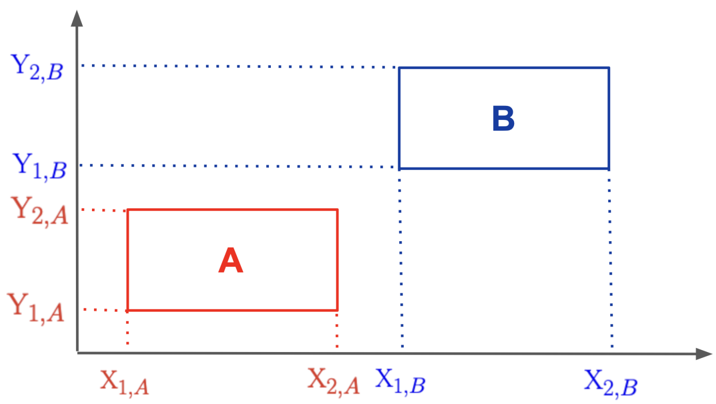


$$\boxed{a^2 + b^2 = c^2}$$

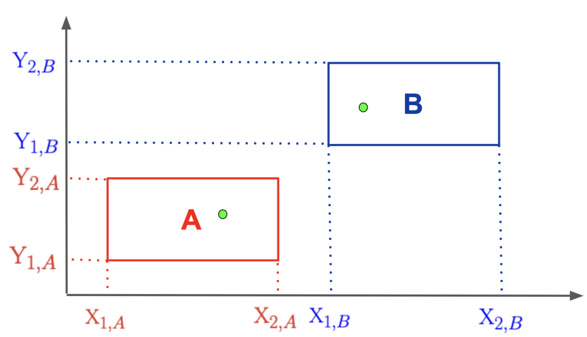
Also we should remember the distance between two vertex corresponds the diagonal of the rectangle triangle formed by the two vertex.

Case a.

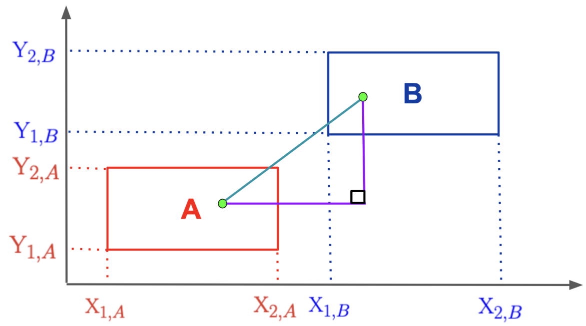
Step 1: We have the two rectangles.



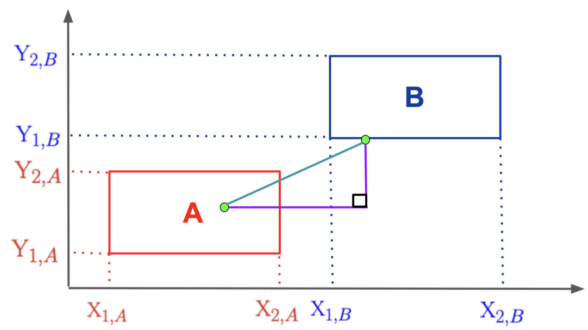
Step 2: We put two random points. One belongs to the A rectangle and the other belongs to the B rectangle.



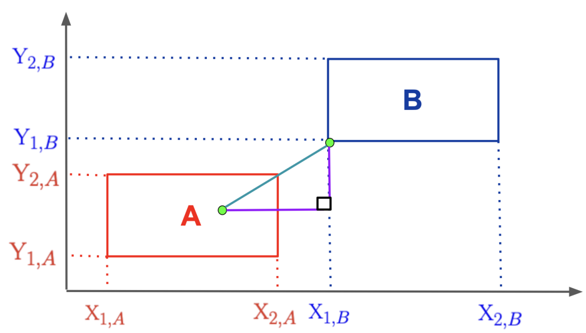
Step 3: We draw a rectangle triangle between the two points.



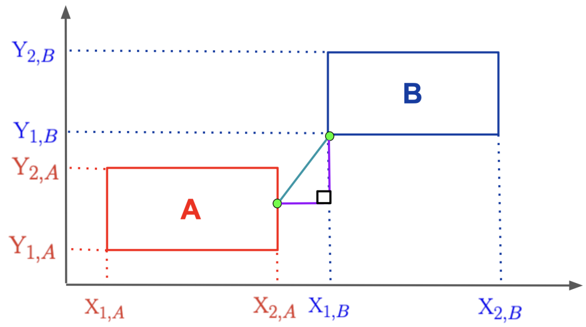
Step 4: In orden of decrease the length of the diagonal we can keep the same value of a and reduce the value b moving down the point of the B rectangle.



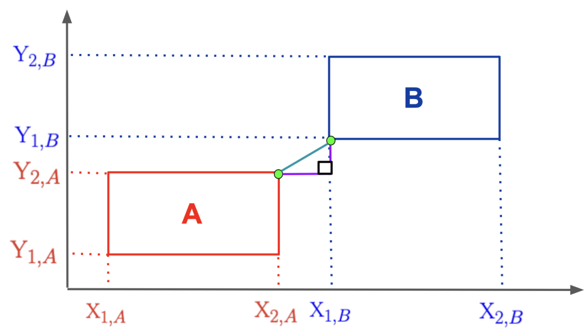
Step 5: In orden of decrease the length of the diagonal we can keep the same value of b and reduce the value a moving the point of the B rectangle to the left. Now the point is one vertex of B and also there is no direction to decrease the diagonal moving the point of the B rectangle.



Step 6: In orden of decrease the length of the diagonal we can keep the same value of b and reduce the value a moving the point of the A rectangle to the right.



Step 7: In orden of decrease the length of the diagonal we can keep the same value of a and reduce the value b moving up the point of the A rectangle. Now the point is one vertex of the A rectangle and also there is no direction to decrease the diagonal moving the point of the A rectangle.



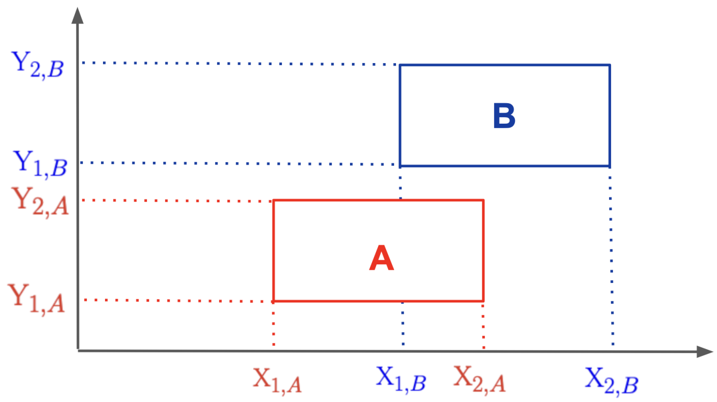
Step 8: Now we reach the optimal distance. With the previous reasoning and generalizing the result we can conclude that:

$$\boxed{d(A,B) = min_{x\in\{v_A^1,v_A^2,v_A^3,v_A^4\},y\in\{v_B^1,v_B^2,v_B^3,v_B^4\}} d(x,y)}$$

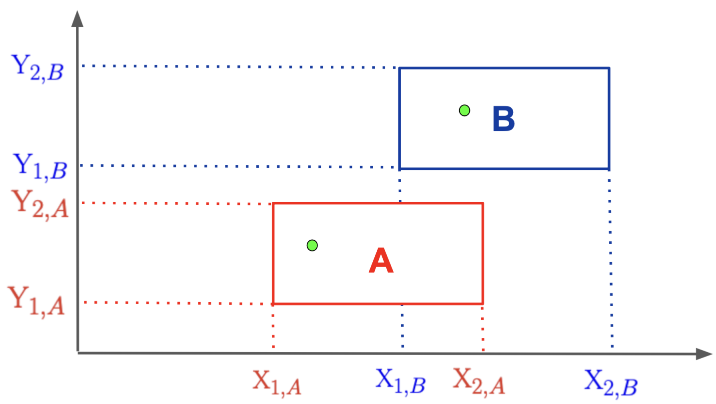
$\{v_A^1, v_A^2, v_A^3, v_A^4\}$ are the vertices of the A rectangle and $\{v_B^1, v_B^2, v_B^3, v_B^4\}$ are the vertices of the B rectangle.

Case b.

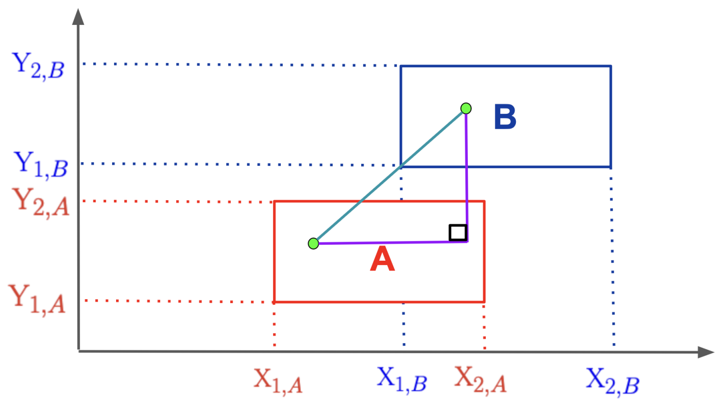
Step 1: We have the two rectangles.



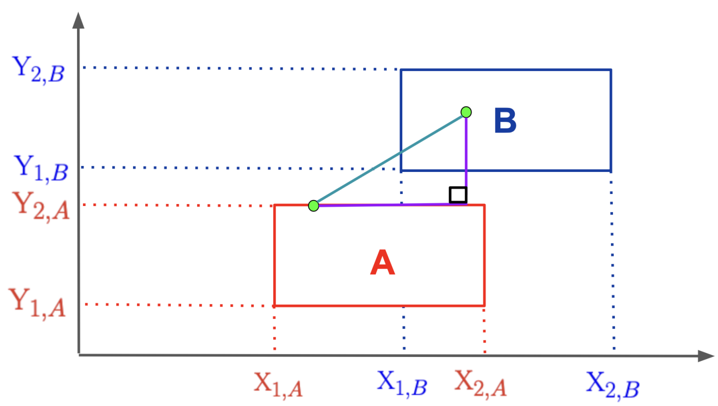
Step 2: We put two random points. One belongs to the A rectangle and the other belongs to the B rectangle.



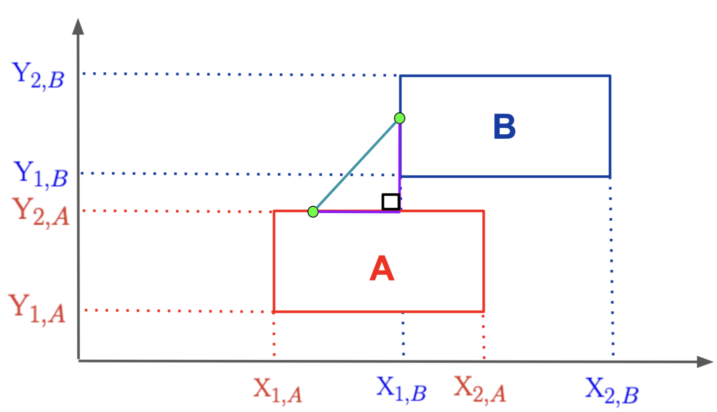
Step 3: We draw a rectangle triangle between the two points.



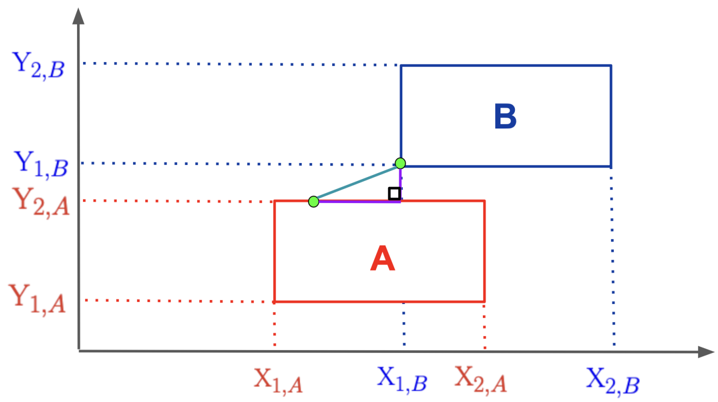
Step 4: In orden of decrease the length of the diagonal we can keep the same value of a and reduce the value b moving up the point of the A rectangle.



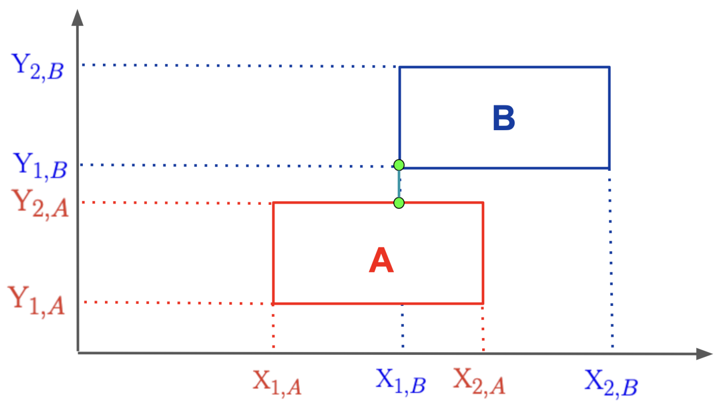
Step 5: In orden of decrease the length of the diagonal we can keep the same value of b and reduce the value a moving the point of the B rectangle to the left.



Step 6: In orden of decrease the length of the diagonal we can keep the same value of a and reduce the value b moving down the point of the B rectangle. Now there is no direction to reduce the lenght of the diagonal moving the point of the B rectangle.



Step 7: In orden of decrease the length of the diagonal we can keep the same value of b and reduce the value a moving the point of the A rectangle to the right. Now there is no direction to reduce the lenght of the diagonal moving the point of the A rectangle.

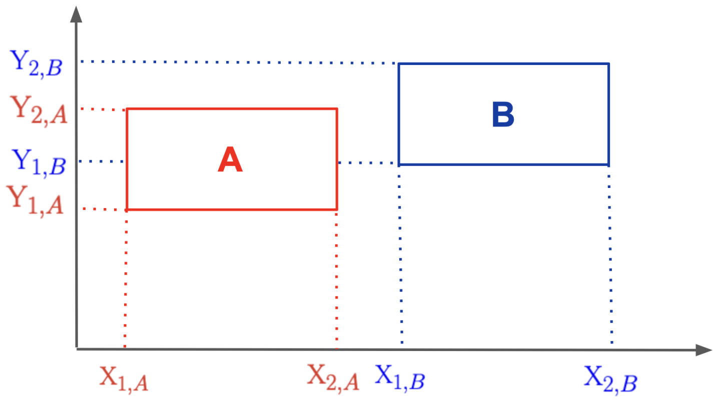


Step 8: Now we reach the optimal distance. With the previous reasoning and generalizing the result we can conclude that:

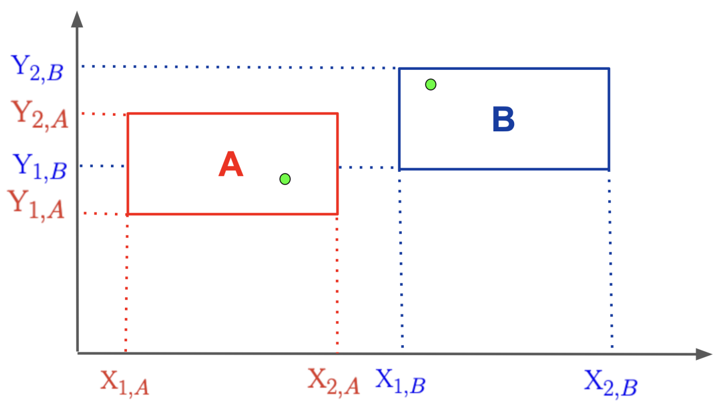
$$\boxed{d(A,B) = min\{|Y_{1,A} - Y_{1,B}|, |Y_{1,A} - Y_{2,B}|, |Y_{2,A} - Y_{1,B}|, |Y_{2,A} - Y_{2,B}|\}}$$

Case c.

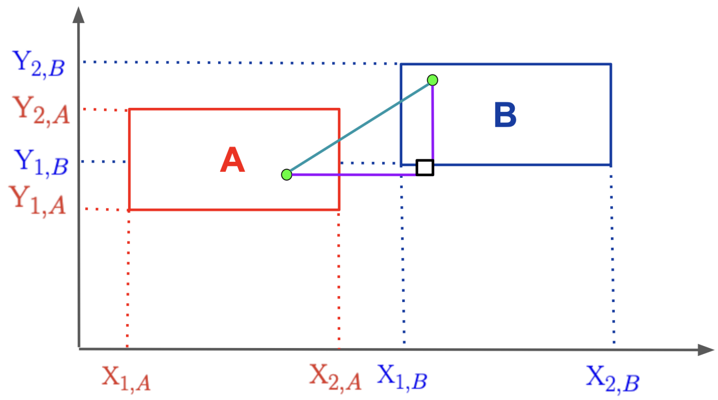
Step 1: We have the two rectangles.



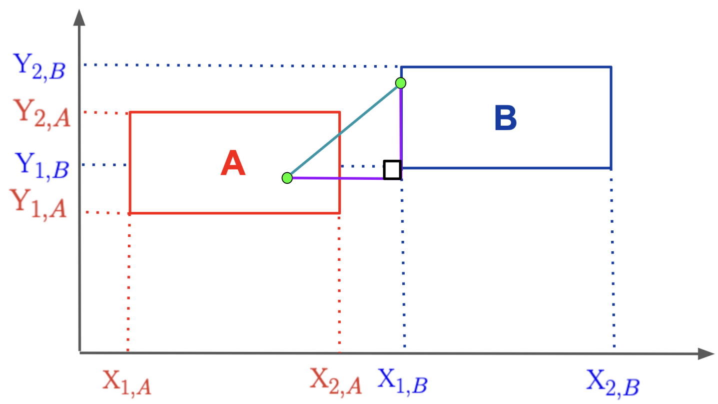
Step 2: We put two random points. One belongs to the A rectangle and the other belongs to the B rectangle.



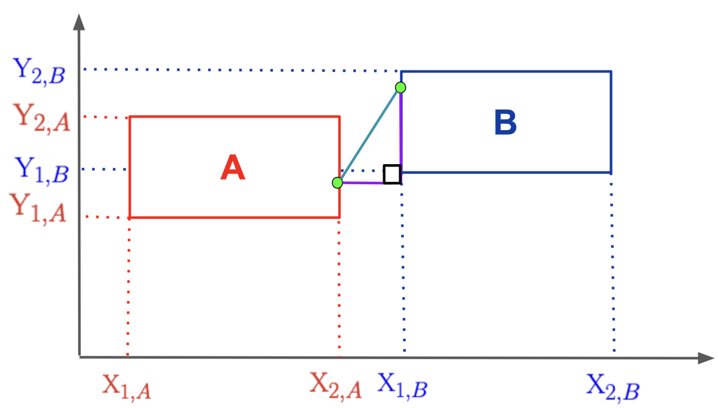
Step 3: We draw a rectangle triangle between the two points.



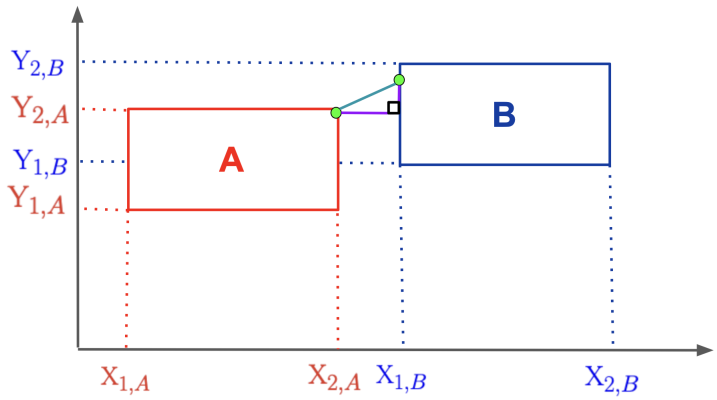
Step 4: In orden of decrease the length of the diagonal we can keep the same value of b and reduce the value a moving the point of the B rectangle to the left.



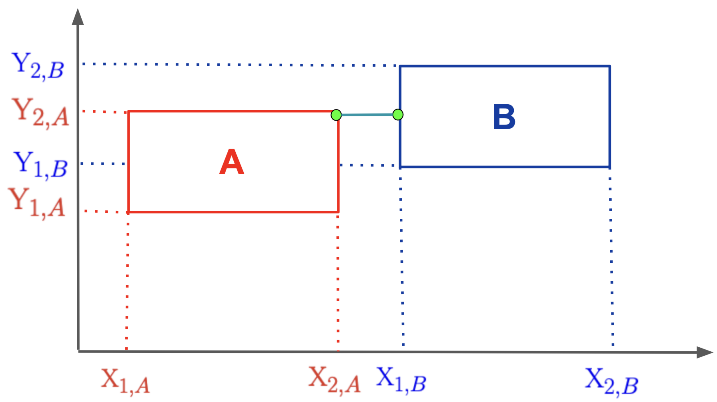
Step 5: In orden of decrease the length of the diagonal we can keep the same value of b and reduce the value a moving the point of the A rectangle to the right.



Step 6: In orden of decrease the length of the diagonal we can keep the same value of a and reduce the value b moving up the point of the A rectangle. Now there is no direction to reduce the lenght of the diagonal moving the point of the A rectangle.



Step 7: In orden of decrease the length of the diagonal we can keep the same value of a and reduce the value b moving down the point of the B rectangle. Now there is no direction to reduce the lenght of the diagonal moving the point of the B rectangle.

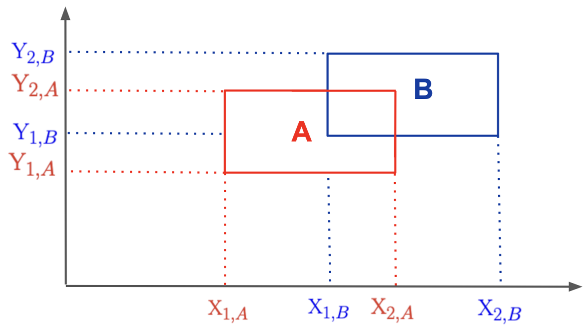


Step 8: Now we reach the optimal distance. With the previous reasoning and generalizing the result we can conclude that:

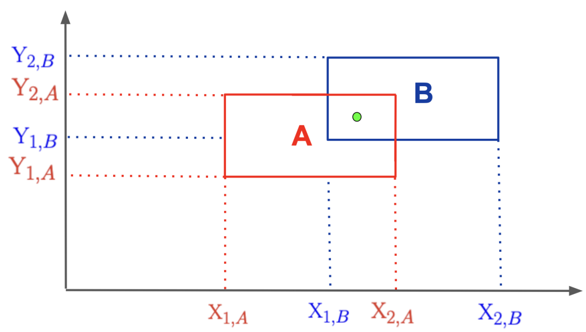
$$\boxed{d(A,B) = min\{|X_{1,A} - X_{1,B}|, |X_{1,A} - X_{2,B}|, |X_{2,A} - X_{1,B}|, |X_{2,A} - X_{2,B}|\}}$$

Case d.

Step 1: We have the two rectangles.



Step 2: We put and unique random point which belongs both to the A rectangle and B rectangle.



By definition of metric distance we have that:

$$0 \leq d(x, y) \qquad \forall x, \in A, y \in B$$

$\Rightarrow$

$$0 \leq \inf_{x \in A} \inf_{y \in B} d(x, y)$$

$\Rightarrow$

$$\boxed{0 \leq d(A, B) \quad (1)}$$

By definition of distance between two sets we know that:

$$d(A,B) = \inf_{x \in A} \inf_{y \in B} d(x,y) \leq d(u,v) \qquad \forall u \in A, v \in B$$

$\Rightarrow$

$$d(A,B) \leq d(u,v) \qquad \forall u \in A, v \in B$$

We calls c the point on the image which belongs both to the A rectangle and the B rectangle. Note the previous inequality is valid in particular for $u = v = c$ then:

$$d(A,B) \leq d(c,c) \overset{\text{definition of metric distance}}{=} 0$$

$\Rightarrow$

$$\boxed{d(A,B) \leq 0 \quad (2)}$$

Joining the equations (1) and (2) we have that:

$$\boxed{d(A,B) = 0}$$

Finally we have the distance between two rectangles with no rotation is:

$$\boxed{d(A,B) = \begin{cases} min_{x \in \{v_A^1, v_A^2, v_A^3, v_A^4\}, y \in \{v_B^1, v_B^2, v_B^3, v_B^4\}} d(x,y) & \text{Case a.} \\ min\{|Y_{1,A} - Y_{1,B}|, |Y_{1,A} - Y_{2,B}|, |Y_{2,A} - Y_{1,B}|, |Y_{2,A} - Y_{2,B}|\} & \text{Case b.} \\ min\{|X_{1,A} - X_{1,B}|, |X_{1,A} - X_{2,B}|, |X_{2,A} - X_{1,B}|, |X_{2,A} - X_{2,B}|\} & \text{Case c.} \\ 0 & \text{Case d.} \end{cases}}$$
