Alberto Andrés Valdés González.
Degree: Mathematical Engineer.
Work position: Data Scientist.
Mail: anvaldes@uc.cl/alberto.valdes.gonzalez.96@gmail.com
Location: Santiago, Chile.

Relation between two bernoulli random variables

Consider two events: $A, B \subseteq \Omega$. We define the next bernoulli random variables:

$$X_A(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A \end{cases}$$

$$X_B(w) = \begin{cases} 1 & \text{if } w \in B \\ 0 & \text{if } w \notin B \end{cases}$$

You can see that $X_A \cdot X_B = X_C$ with $C = A \cap B$.

Properties of bernoulli variables:

If $X \sim Bernoulli(p)$:

$$\mathbb{E}[X] = p \qquad \mathbb{V}[X] = p \cdot (1 - p)$$

Deduction of correlation:

$$Cov(X_A, X_B) = \mathbb{E}[X_A \cdot X_B] - \mathbb{E}[X_A] \cdot \mathbb{E}[X_B] = \mathbb{E}[X_C] - \mathbb{P}(A) \cdot \mathbb{P}(B) =$$

$$\mathbb{P}(C) - \mathbb{P}(A) \cdot \mathbb{P}(B) = \mathbb{P}(A \cap B) - \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$\Rightarrow$

$$Cov(X_A, X_B) = \mathbb{P}(A \cap B) - \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$\Rightarrow$

$$corr(X_A, X_B) = \frac{\mathbb{P}(A \cap B) - \mathbb{P}(A) \cdot \mathbb{P}(B)}{\sigma_A \cdot \sigma_B}$$

$\Rightarrow$

$$\rho(X_A, X_B) = \frac{\mathbb{P}(A \cap B) - \mathbb{P}(A) \cdot \mathbb{P}(B)}{\sqrt{\mathbb{P}(A) \cdot [1 - \mathbb{P}(A)]} \cdot \sqrt{\mathbb{P}(B) \cdot [1 - \mathbb{P}(B)]}}$$

$$\boxed{\rho(X_A, X_B) = \frac{\mathbb{P}(A \cap B) - \mathbb{P}(A) \cdot \mathbb{P}(B)}{\sqrt{\mathbb{P}(A) \cdot [1 - \mathbb{P}(A)]} \cdot \sqrt{\mathbb{P}(B) \cdot [1 - \mathbb{P}(B)]}}} \quad (1)$$

$\Rightarrow$

$$\boxed{\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) + \rho(X_A, X_B) \cdot \sqrt{\mathbb{P}(A) \cdot [1 - \mathbb{P}(A)]} \cdot \sqrt{\mathbb{P}(B) \cdot [1 - \mathbb{P}(B)]}} \quad (2)$$

Examples:

i. $\underline{\rho(X_A, X_B) = 0}$: From (2) we have that:

$$\boxed{\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)}$$

ii. $\underline{\rho(X_A, X_B) > 0}$: From (2) we have that:

$$\boxed{\mathbb{P}(A \cap B) > \mathbb{P}(A) \cdot \mathbb{P}(B)}$$

iii. $\underline{\rho(X_A, X_B) < 0}$: From (2) we have that:

$$\boxed{\mathbb{P}(A \cap B) < \mathbb{P}(A) \cdot \mathbb{P}(B)}$$

iv. $\underline{A = B}$: How $A = B$ then $\rho(X_A, X_B) = 1$ and $\mathbb{P}(A) = \mathbb{P}(B)$.

From (2) we have that:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) + \rho(X_A, X_B) \cdot \sqrt{\mathbb{P}(A) \cdot [1 - \mathbb{P}(A)]} \cdot \sqrt{\mathbb{P}(B) \cdot [1 - \mathbb{P}(B)]}$$

$$= \mathbb{P}(A) \cdot \mathbb{P}(A) + 1 \cdot \sqrt{\mathbb{P}(A) \cdot [1 - \mathbb{P}(A)]} \cdot \sqrt{\mathbb{P}(A) \cdot [1 - \mathbb{P}(A)]}$$

$$= \mathbb{P}(A) \cdot \mathbb{P}(A) + \mathbb{P}(A) \cdot [1 - \mathbb{P}(A)] = \mathbb{P}(A) \cdot [\mathbb{P}(A) + 1 - \mathbb{P}(A)] = \mathbb{P}(A)$$

$$\boxed{\mathbb{P}(A \cap B) = \mathbb{P}(A)}$$

v. $A = \overline{B}$: How $A = \overline{B}$ then $\rho(X_A, X_B) = -1$ and $\mathbb{P}(A) = 1 - \mathbb{P}(B)$.

From (2) we have that:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) + \rho(X_A, X_B) \cdot \sqrt{\mathbb{P}(A) \cdot [1 - \mathbb{P}(A)]} \cdot \sqrt{\mathbb{P}(B) \cdot [1 - \mathbb{P}(B)]}$$

$$= \mathbb{P}(A) \cdot [1 - \mathbb{P}(A)] - 1 \cdot \sqrt{\mathbb{P}(A) \cdot [1 - \mathbb{P}(A)]} \cdot \sqrt{\mathbb{P}(A) \cdot [1 - \mathbb{P}(A)]}$$

$$= \mathbb{P}(A) \cdot [1 - \mathbb{P}(A)] - \mathbb{P}(A) \cdot [1 - \mathbb{P}(A)] = 0$$

$$\boxed{\mathbb{P}(A \cap B) = 0}$$
