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# Recurrence Relations

In mathematics, a recurrence relation is an equation according to which the $n$th-term of a sequence of numbers is equal to some combination of the previous terms.

### How to solve?

We can solve it in a similar way that ODE.

**Example:**

$$F_n = F_{(n-1)} + F_{(n-2)}$$

$$F_0 = 0, F_1 = 1$$

### Solution:

We going to propose a solution of the form:

$$F_n = r^n$$

Substituting we have:

$$r^n = r^{(n-1)} + r^{(n-2)}$$

$\Rightarrow$

$$r^n = \frac{r^n}{r} + \frac{r^n}{r^2}$$

$\Rightarrow$

$$1 = \frac{1}{r} + \frac{1}{r^2}$$

$\Rightarrow$

$$r^2 = r + 1$$

$\Rightarrow$

$$r^2 - r - 1 = 0$$

This is a second grade equation for $r$ and we going to solve it:

Using $a = 1, b = -1$ and $c = -1$.

$$r_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\Rightarrow$

$$\boxed{r_1 = \frac{1+\sqrt{5}}{2} \quad , \quad r_2 = \frac{1-\sqrt{5}}{2}}$$

The solution will be of the form:

$$F_n = c_1 \cdot (r_1)^n + c_2 \cdot (r_2)^n$$

We going to use the initial conditions:

$$F_0 = c_1 + c_2 = 0 \Rightarrow \boxed{c_2 = -c_1}$$

$$F_1 = c_1 \cdot r_1 + c_2 \cdot r_2 = 1 \Rightarrow c_1 \cdot r_1 - c_1 \cdot r_2 = 1 \Rightarrow \boxed{c_1 = \frac{1}{r_1 - r_2}}$$

Now we can see that:

$$r_1 - r_2 = \left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right) = \sqrt{5}$$

$\Rightarrow$

$$c_1 = \frac{1}{\sqrt{5}}$$

$\Rightarrow$

$$\boxed{c_1 = \frac{1}{\sqrt{5}} \quad , \quad c_2 = -\frac{1}{\sqrt{5}}}$$

Thus:

$$F_n = \frac{1}{\sqrt{5}} \cdot [(r_1)^n - (r_2)^n]$$

We can define the golden number:

$$\phi = \frac{1+\sqrt{5}}{2}$$

We can see that:

$$r_1 = \phi$$

On the other hand:

$$r_2 = \frac{1-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right) \cdot \left(\frac{1+\sqrt{5}}{1+\sqrt{5}}\right) = \frac{1^2 - \sqrt{5}^2}{2 \cdot (1+\sqrt{5})} = -\frac{4}{2 \cdot (1+\sqrt{5})} = -\frac{2}{1+\sqrt{5}} = -\frac{1}{\phi} = -\phi^{-1}$$

Thus:

$$\boxed{F_n = \frac{(\phi)^n - (-\phi)^n}{\sqrt{5}}}$$

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