**Alberto Andrés Valdés González.**
**Degree:** Mathematical Engineer.
**Work position:** Data Scientist.
**Mail:** anvaldes@uc.cl/alberto.valdes.gonzalez.96@gmail.com
**Location:** Santiago, Chile.

---

# Distributions Review

## Review of contents

**Median:** In statistics and probability theory, the median is the value separating the higher half from the lower half of a data sample, a population, or a probability distribution.

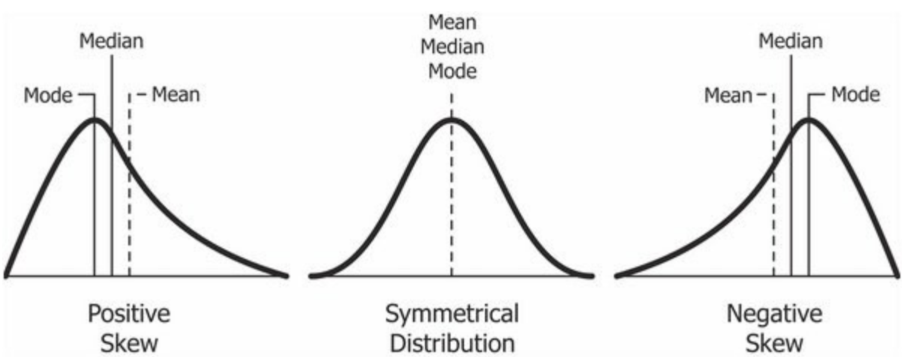
$$\boxed{F(t_{median}) = 0{,}5}$$

**Mode:** The mode is the value that appears most often in a set of data values.

$$\boxed{f(t_{mode}) \geq f(x) \quad \forall x \in X}$$

**Skewness:** In probability theory and statistics, skewness is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean. The skewness value can be positive, zero, negative, or undefined.

$$K = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$$




---

# Review of distributions

---

**1. Name:** Exponential.

**2. Explanation:** Is the probability distribution of the time between events in a Poisson point process.

**3. Density function:**

$$\boxed{f(x) = \begin{cases} \lambda \cdot e^{-\lambda \cdot x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}}$$

**4. Distribution function:**

By definition:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

<u>i. If $x < 0$ :</u>

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} 0dt = 0$$

<u>ii. If $x \geq 0$ :</u>

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{x} f(t)dt = \int_{-\infty}^{0} 0dt + \int_{0}^{x} \lambda \cdot e^{-\lambda \cdot t}dt$$

$$= (-e^{-\lambda \cdot t})\Big|_{t=0}^{t=x} = (-e^{-\lambda \cdot x}) - (-e^{-\lambda \cdot 0}) = 1 - e^{-\lambda \cdot x}$$

$$\boxed{F(x) = \begin{cases} 1 - e^{-\lambda \cdot x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}}$$

**5. Mean:**

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} t \cdot f(t)dt = \int_{-\infty}^{0} t \cdot f(t)dt + \int_{0}^{+\infty} t \cdot f(t)dt = \int_{-\infty}^{0} t \cdot 0dt + \int_{0}^{+\infty} t \cdot \lambda \cdot e^{-\lambda \cdot t}dt$$

$$= \int_{0}^{+\infty} t \cdot \lambda \cdot e^{-\lambda \cdot t}dt$$

$h_1(t) = t \;\Rightarrow\; h_1'(t) = 1$

$h_2'(t) = \lambda \cdot e^{-\lambda \cdot t} \;\Rightarrow\; h_2(t) = -e^{-\lambda \cdot t}$

$\Rightarrow$

$$\mathbb{E}[X] = \int_0^{+\infty} t \cdot \lambda \cdot e^{-\lambda \cdot t} dt = \left(-t \cdot e^{-\lambda \cdot t}\right)\Big|_{t=0}^{t=+\infty} - \int_0^{+\infty} -e^{-\lambda \cdot t} \cdot 1 dt$$

$$= -\lim_{t \to +\infty} \frac{t}{e^{\lambda \cdot t}} + \frac{1}{\lambda} \cdot \int_0^{+\infty} \lambda \cdot e^{-\lambda \cdot t} dt = -0 + \frac{1}{\lambda} \cdot \int_0^{+\infty} f(t) dt = \frac{1}{\lambda} \cdot 1 = \frac{1}{\lambda}$$

$\Rightarrow$

$$\boxed{\mathbb{E}[X] = \frac{1}{\lambda}}$$

**6. Variance:**

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} t^2 \cdot f(t) dt = \int_{-\infty}^{0} t^2 \cdot f(t) dt + \int_0^{+\infty} t^2 \cdot f(t) dt = \int_{-\infty}^{0} t^2 \cdot 0 dt + \int_0^{+\infty} t^2 \cdot \lambda \cdot e^{-\lambda \cdot t} dt$$

$$= \int_0^{+\infty} t^2 \cdot \lambda \cdot e^{-\lambda \cdot t} dt$$

$h_1(t) = t^2 \;\Rightarrow\; h_1'(t) = 2 \cdot t$

$h_2'(t) = \lambda \cdot e^{-\lambda \cdot t} \;\Rightarrow\; h_2(t) = -e^{-\lambda \cdot t}$

$$\mathbb{E}[X^2] = \int_0^{+\infty} t^2 \cdot \lambda \cdot e^{-\lambda \cdot t} dt = \left(-t^2 \cdot e^{-\lambda \cdot t}\right)\Big|_{t=0}^{t=+\infty} - \int_0^{+\infty} -e^{-\lambda \cdot t} \cdot 2t dt$$

$$= -\lim_{t \to +\infty} \frac{t^2}{e^{\lambda \cdot t}} + 2 \cdot \int_0^{+\infty} t \cdot e^{-\lambda \cdot t} dt = 0 + \frac{2}{\lambda} \cdot \int_0^{+\infty} t \cdot \lambda \cdot e^{-\lambda \cdot t} dt$$

$$= \frac{2}{\lambda} \cdot \int_0^{+\infty} t \cdot f(t) dt = \frac{2}{\lambda} \cdot \mathbb{E}[X] = \frac{2}{\lambda} \cdot \frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$\Rightarrow$

$$\mathbb{E}[X^2] = \frac{2}{\lambda^2}$$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

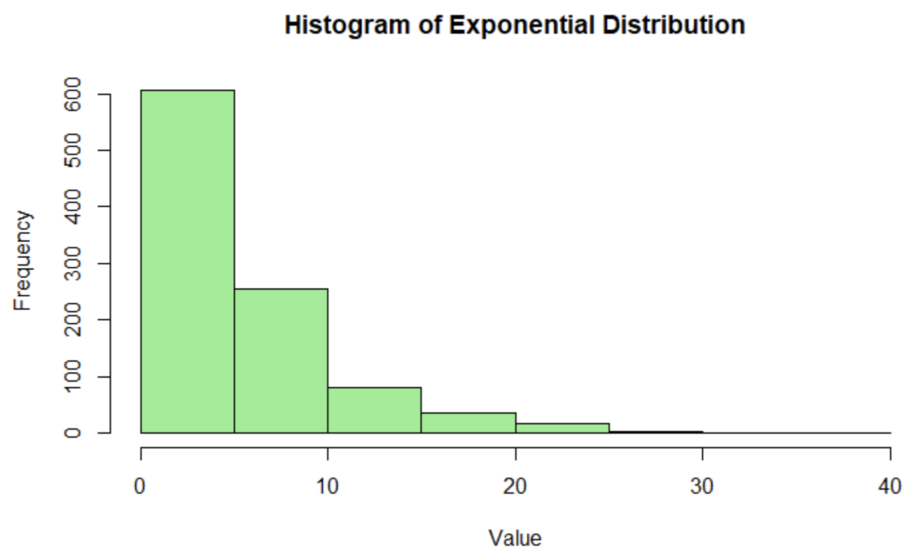
$\Rightarrow$

$$\boxed{\mathbb{V}[X] = \frac{1}{\lambda^2}}$$

**7. Relation between other variables:**

- Sum of $k$ independent exponentials random variables is $Gamma(\lambda, k)$.
- Exponentials represents the time between events in poisson point process.

**8. Histogram:**




---

**1. Name:** Poisson.

**2. Explanation:** In probability theory and statistics, the Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant mean rate and independently of the time since the last event.

**3. Density function:**

$$\boxed{\mathbb{P}(X = n) = \frac{e^{-\lambda \cdot t} \cdot (\lambda \cdot t)^n}{n!} \qquad \text{or} \qquad \mathbb{P}(X = n) = \frac{e^{-\lambda} \cdot \lambda^n}{n!} \qquad \forall n \in \mathbb{N}}$$

**4. Distribution function:**

By definition:

$$\boxed{F(k) = \mathbb{P}(X \leq k) = \sum_{n=0}^{k} \frac{e^{-\lambda} \cdot \lambda^n}{n!}}$$

**5. Mean:**

$$\mathbb{E}[X] = \sum_{n=0}^{\infty} n \cdot \mathbb{P}(X = n) = \sum_{n=1}^{\infty} n \cdot \mathbb{P}(X = n) = \sum_{n=1}^{\infty} n \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n!} = \sum_{n=1}^{\infty} n \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n \cdot (n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{e^{-\lambda} \cdot \lambda^n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{[n+1]}}{([n+1]-1)!} = \sum_{n=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^n \cdot \lambda}{n!} = \lambda \cdot \sum_{n=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^n}{n!} = \lambda \cdot 1 = \lambda$$

$\Rightarrow$

$$\boxed{\mathbb{E}[X] = \lambda}$$

**6. Variance:**

$$\mathbb{E}[X^2] = \sum_{n=0}^{\infty} n^2 \cdot \mathbb{P}(X = n) = \sum_{n=1}^{\infty} n^2 \cdot \mathbb{P}(X = n) = \sum_{n=1}^{\infty} n \cdot n \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n!} = \sum_{n=1}^{\infty} n \cdot n \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n \cdot (n-1)!}$$

$$= \sum_{n=1}^{\infty} n \cdot \frac{e^{-\lambda} \cdot \lambda^n}{(n-1)!} = \sum_{n=0}^{\infty} (n+1) \cdot \frac{e^{-\lambda} \cdot \lambda^{[n+1]}}{([n+1]-1)!} = \sum_{n=0}^{\infty} (n+1) \cdot \frac{e^{-\lambda} \cdot \lambda^n \cdot \lambda}{n!}$$

$$= \lambda \cdot \left( \sum_{n=0}^{\infty} (n+1) \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n!} \right) = \lambda \cdot \left( \sum_{n=0}^{\infty} n \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n!} + \sum_{n=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^n}{n!} \right)$$

$$= \lambda \cdot \left( \sum_{n=0}^{\infty} n \cdot \mathbb{P}(X = n) + \sum_{n=0}^{\infty} \mathbb{P}(X = n) \right) = \lambda \cdot (\mathbb{E}[X] + 1) = \lambda \cdot (\lambda + 1) = \lambda^2 + \lambda$$

$\Rightarrow$

$$\mathbb{E}[X] = \lambda^2 + \lambda$$

$\Rightarrow$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda^2 + \lambda - (\lambda)^2 = \lambda$$

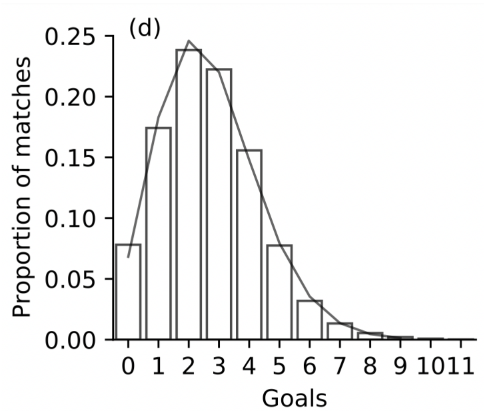
$\Rightarrow$

$$\boxed{\mathbb{V}[X] = \lambda}$$

**7. Relation between other variables:**

- Exponentials represents the time between events in poisson point process.

**8. Histogram:**




---

**1. Name:** Gamma.

**2. Explanation:** In probability theory and statistics, the gamma distribution is a two-parameter family of continuous probability distributions. The exponential distribution, Erlang distribution, and chi-squared distribution are special cases of the gamma distribution.

**3. Density function:**

$$\boxed{f(x) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-\beta \cdot x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}}$$

**4. Distribution function:**

No computed.

**5. Mean:**

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} t \cdot f(t)dt = \int_{-\infty}^{0} t \cdot f(t)dt + \int_{0}^{+\infty} t \cdot f(t)dt = \int_{-\infty}^{0} t \cdot 0dt + \int_{0}^{+\infty} t \cdot \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot t^{\alpha-1} \cdot e^{-\beta \cdot t}dt$$

$$= \int_{0}^{+\infty} t \cdot \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot t^{\alpha-1} \cdot e^{-\beta \cdot t}dt = \int_{0}^{+\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot t^{(\alpha+1)-1} \cdot e^{-\beta \cdot t}dt$$

$$= \frac{\alpha}{\beta} \cdot \int_{0}^{+\infty} \frac{\beta}{\alpha} \cdot \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot t^{(\alpha+1)-1} \cdot e^{-\beta \cdot t}dt = \frac{\alpha}{\beta} \cdot \int_{0}^{+\infty} \frac{\beta^{(\alpha+1)}}{\alpha \cdot \Gamma(\alpha)} \cdot t^{(\alpha+1)-1} \cdot e^{-\beta \cdot t}dt$$

$$= \frac{\alpha}{\beta} \cdot \int_{0}^{+\infty} \frac{\beta^{(\alpha+1)}}{\Gamma(\alpha+1)} \cdot t^{(\alpha+1)-1} \cdot e^{-\beta \cdot t}dt$$

Using $\alpha' = \alpha + 1$

$$\mathbb{E}[X] = \frac{\alpha}{\beta} \cdot \int_{0}^{+\infty} \frac{\beta^{\alpha'}}{\Gamma(\alpha')} \cdot t^{\alpha'-1} \cdot e^{-\beta \cdot t}dt = \frac{\alpha}{\beta} \cdot \int_{0}^{+\infty} f(t)dt = \frac{\alpha}{\beta}$$

$\Rightarrow$

$$\boxed{\mathbb{E}[X] = \frac{\alpha}{\beta}}$$

**6. Variance:**

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} t^2 \cdot f(t)dt = \int_{-\infty}^{0} t^2 \cdot f(t)dt + \int_{0}^{+\infty} t^2 \cdot f(t)dt = \int_{-\infty}^{0} t^2 \cdot 0dt + \int_{0}^{+\infty} t^2 \cdot \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot t^{\alpha-1} \cdot e^{-\beta \cdot t}dt$$

$$= \int_{0}^{+\infty} t^2 \cdot \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot t^{\alpha-1} \cdot e^{-\beta \cdot t}dt = \int_{0}^{+\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot t^{(\alpha+2)-1} \cdot e^{-\beta \cdot t}dt$$

$$= \frac{\alpha \cdot (\alpha+1)}{\beta^2} \cdot \int_{0}^{+\infty} \frac{\beta^2}{\alpha \cdot (\alpha+1)} \cdot \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot t^{(\alpha+2)-1} \cdot e^{-\beta \cdot t}dt$$

$$= \frac{\alpha \cdot (\alpha+1)}{\beta^2} \cdot \int_{0}^{+\infty} \frac{\beta^{(\alpha+2)}}{(\alpha+1) \cdot \alpha \cdot \Gamma(\alpha)} \cdot t^{(\alpha+2)-1} \cdot e^{-\beta \cdot t}dt$$

$$= \frac{\alpha \cdot (\alpha+1)}{\beta^2} \cdot \int_{0}^{+\infty} \frac{\beta^{(\alpha+2)}}{\Gamma(\alpha+2)} \cdot t^{(\alpha+2)-1} \cdot e^{-\beta \cdot t}dt$$

Using $\alpha' = \alpha + 2$

$$= \frac{\alpha \cdot (\alpha+1)}{\beta^2} \cdot \int_0^{+\infty} \frac{\beta^{\alpha'}}{\Gamma(\alpha')} \cdot t^{\alpha'-1} \cdot e^{-\beta \cdot t} dt = \frac{\alpha \cdot (\alpha+1)}{\beta^2} \cdot \int_0^{+\infty} f(t) dt = \frac{\alpha \cdot (\alpha+1)}{\beta^2}$$

$$= \frac{\alpha^2}{\beta^2} + \frac{\alpha}{\beta^2}$$

$\Rightarrow$

$$\mathbb{E}[X^2] = \frac{\alpha^2}{\beta^2} + \frac{\alpha}{\beta^2}$$

$\Rightarrow$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{\alpha^2}{\beta^2} + \frac{\alpha}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha^2}{\beta^2} + \frac{\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$$

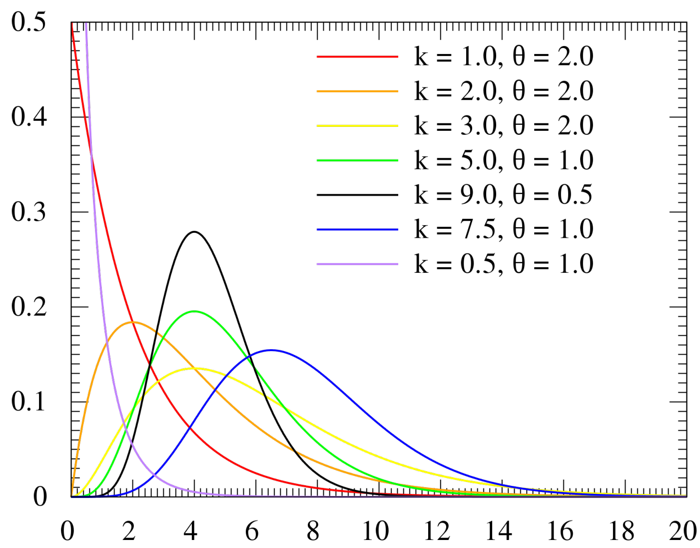
$\Rightarrow$

$$\boxed{\mathbb{V}[X] = \frac{\alpha}{\beta^2}}$$

**7. Relation between other variables:**

- Sum of $k$ independent exponentials random variables is $Gamma(\alpha = k, \beta = \lambda)$.
- Chi-square is a particular version of $Gamma\left(\alpha = \frac{k}{2}, \beta = \frac{1}{2}\right)$.

**8. Histogram:**




$$\alpha = k, \ \theta = \frac{1}{\beta}$$

**1. Name:** Bernoulli.

**2. Explanation:** In probability theory and statistics, the Bernoulli distribution, named after Swiss mathematician Jacob Bernoulli, is the discrete probability distribution of a random variable which takes the value 1 with probability $p$ and the value 0 with probability $q = 1 - p$.

**3. Density function:**

$$\boxed{f(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}}$$

**4. Distribution function:**

No computed.

**5. Mean:**

$$\mathbb{E}[X] = 0 \cdot (1 - p) + 1 \cdot p = p$$

$\Rightarrow$

$$\boxed{\mathbb{E}[X] = p}$$

**6. Variance:**

$$\mathbb{E}[X^2] = 0^2 \cdot (1 - p) + 1^2 \cdot p = p$$

$\Rightarrow$

$$\mathbb{E}[X^2] = p$$

$\Rightarrow$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2 = p \cdot (1 - p)$$

$\Rightarrow$

$$\boxed{\mathbb{V}[X] = p \cdot (1 - p)}$$

**7. Relation between other variables:**

The probability of getting exactly $k$ successes in $n$ independent Bernoulli trials is the binomial distribution.

**1. Name:** Binomial.

**2. Explanation:** Is the probability of getting exactly $k$ successes in $n$ independent Bernoulli trials.

**3. Density function:**

$$\boxed{\mathbb{P}(X=k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \qquad \forall k \in \{0, 1, ..., n\}}$$

**4. Distribution function:**

No computed.

**5. Mean:**

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \mathbb{P}(X=k) = \sum_{k=1}^{n} k \cdot \mathbb{P}(X=k) = \sum_{k=1}^{n} k \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \frac{n!}{k! \ \cdot (n-k)!} \cdot p^k \cdot (1-p)^{n-k} = \sum_{k=1}^{n} k \cdot \frac{n!}{k \cdot (k-1)! \ \cdot (n-k)!} \cdot p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)! \ \cdot (n-k)!} \cdot p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=0}^{n-1} \frac{n!}{([k+1]-1)! \ \cdot (n-[k+1])!} \cdot p^{[k+1]} \cdot (1-p)^{n-[k+1]}$$

$$= \sum_{k=0}^{n-1} \frac{n \cdot (n-1)!}{k! \ \cdot ([n-1]-k)!} \cdot p^k \cdot p \cdot (1-p)^{[n-1]-k}$$

$$= n \cdot p \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \ \cdot ([n-1]-k)!} \cdot p^k \cdot (1-p)^{[n-1]-k}$$

Using $n' = n-1$

$$= n \cdot p \cdot \sum_{k=0}^{n'} \frac{n'!}{k! \ \cdot (n'-k)!} \cdot p^k \cdot (1-p)^{n'-k}$$

$$= n \cdot p \cdot \sum_{k=0}^{n'} \binom{n'}{k} \cdot p^k \cdot (1-p)^{n'-k} = n \cdot p \cdot 1 = n \cdot p$$

$\Rightarrow$

$$\boxed{\mathbb{E}[X] = n \cdot p}$$

**6. Variance:**

$$\mathbb{E}[X^2] = \sum_{k=0}^{n} k^2 \cdot \mathbb{P}(X = k) = \sum_{k=1}^{n} k^2 \cdot \mathbb{P}(X = k) = \sum_{k=1}^{n} k^2 \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot k \cdot \frac{n!}{k! \ \cdot (n-k)!} \cdot p^k \cdot (1-p)^{n-k} = \sum_{k=1}^{n} k \cdot k \cdot \frac{n!}{k \cdot (k-1)! \ \cdot (n-k)!} \cdot p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \frac{n!}{(k-1)! \ \cdot (n-k)!} \cdot p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=0}^{n-1} (k+1) \cdot \frac{n!}{([k+1]-1)! \ \cdot (n-[k+1])!} \cdot p^{[k+1]} \cdot (1-p)^{n-[k+1]}$$

$$= \sum_{k=0}^{n-1} (k+1) \cdot \frac{n \cdot (n-1)!}{k! \ \cdot ([n-1]-k)!} \cdot p^k \cdot p \cdot (1-p)^{[n-1]-k}$$

$$= n \cdot p \cdot \left( \sum_{k=0}^{n-1} (k+1) \cdot \frac{(n-1)!}{k! \ \cdot ([n-1]-k)!} \cdot p^k \cdot (1-p)^{[n-1]-k} \right)$$

Using $n' = n - 1$

$$= n \cdot p \cdot \left( \sum_{k=0}^{n'} (k+1) \cdot \frac{n'!}{k! \ \cdot (n'-k)!} \cdot p^k \cdot (1-p)^{n'-k} \right)$$

$$= n \cdot p \cdot \left( \sum_{k=0}^{n'} (k+1) \cdot \binom{n'}{k} \cdot p^k \cdot (1-p)^{n'-k} \right)$$

$$= n \cdot p \cdot \left( \sum_{k=0}^{n'} k \cdot \binom{n'}{k} \cdot p^k \cdot (1-p)^{n'-k} + \sum_{k=0}^{n'} \binom{n'}{k} \cdot p^k \cdot (1-p)^{n'-k} \right)$$

$$= n \cdot p \cdot (\mathbb{E}[X'] + 1) = n \cdot p \cdot (n' \cdot p + 1) = n \cdot p \cdot ([n-1] \cdot p + 1)$$

$$= n \cdot p \cdot (n \cdot p - p + 1) = (n \cdot p)^2 - n \cdot p^2 + n \cdot p$$

$\Rightarrow$

$$\mathbb{E}[X^2] = (n \cdot p)^2 - n \cdot p^2 + n \cdot p$$

$\Rightarrow$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = (n \cdot p)^2 - n \cdot p^2 + n \cdot p - (n \cdot p)^2 = n \cdot p - n \cdot p^2 = n \cdot p \cdot (1-p)$$

$\Rightarrow$

$$\boxed{\mathbb{V}[X] = n \cdot p \cdot (1-p)}$$

**7. Relation between other variables:**

The sum of $n$ independent Bernoulli variables is a Binomial variable.

---

**1. Name:** Beta.

**2. Explanation:**

**3. Density function:**

$$\boxed{f(x) = \begin{cases} \dfrac{x^{\alpha-1} \cdot (1-x)^{\beta-1}}{B(\alpha,\beta)} & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}}$$

$$\boxed{B(\alpha,\beta) = \frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$

**4. Distribution function:**

No computed.

**5. Mean:**

$$\begin{aligned} \mathbb{E}[X] = \int_0^1 x \cdot \frac{x^{\alpha-1} \cdot (1-x)^{\beta-1}}{B(\alpha,\beta)} dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \cdot \Gamma(\beta)} \cdot \int_0^1 x \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} dx \\ = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \cdot \Gamma(\beta)} \cdot \int_0^1 x^{(\alpha+1)-1} \cdot (1-x)^{\beta-1} dx \\ = \frac{(\alpha+\beta)}{(\alpha+\beta)} \cdot \frac{\alpha}{\alpha} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \cdot \Gamma(\beta)} \cdot \int_0^1 x^{(\alpha+1)-1} \cdot (1-x)^{\beta-1} dx \\ = \frac{\alpha}{\alpha+\beta} \cdot \frac{(\alpha+\beta) \cdot \Gamma(\alpha+\beta)}{\alpha \cdot \Gamma(\alpha) \cdot \Gamma(\beta)} \cdot \int_0^1 x^{(\alpha+1)-1} \cdot (1-x)^{\beta-1} dx \end{aligned}$$

$$= \frac{\alpha}{\alpha+\beta} \cdot \frac{\Gamma([\alpha+1]+\beta)}{\Gamma(\alpha+1)\cdot\Gamma(\beta)} \cdot \int_0^1 x^{(\alpha+1)-1} \cdot (1-x)^{\beta-1} dx$$

Using $\alpha' = \alpha + 1$

$$= \frac{\alpha}{\alpha+\beta} \cdot \frac{\Gamma(\alpha'+\beta)}{\Gamma(\alpha')\cdot\Gamma(\beta)} \cdot \int_0^1 x^{\alpha'-1} \cdot (1-x)^{\beta-1} dx$$

$$= \frac{\alpha}{\alpha+\beta} \cdot \frac{1}{B(\alpha',\beta)} \cdot \int_0^1 x^{\alpha'-1} \cdot (1-x)^{\beta-1} dx$$

$$= \frac{\alpha}{\alpha+\beta} \cdot \int_0^1 \frac{x^{\alpha'-1} \cdot (1-x)^{\beta-1}}{B(\alpha',\beta)} dx = \frac{\alpha}{\alpha+\beta}$$

$\Rightarrow$

$$\boxed{\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}}$$

**6. Variance:**

$$\mathbb{E}[X^2] = \int_0^1 x^2 \cdot \frac{x^{\alpha-1} \cdot (1-x)^{\beta-1}}{B(\alpha,\beta)} dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot \int_0^1 x^2 \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot \int_0^1 x^{(\alpha+2)-1} \cdot (1-x)^{\beta-1} dx$$

$$= \frac{(\alpha+\beta+1)\cdot(\alpha+\beta)}{(\alpha+\beta+1)\cdot(\alpha+\beta)} \cdot \frac{(\alpha+1)\cdot\alpha}{(\alpha+1)\cdot\alpha} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot \int_0^1 x^{(\alpha+2)-1} \cdot (1-x)^{\beta-1} dx$$

$$= \frac{(\alpha+1)\cdot\alpha}{(\alpha+\beta+1)\cdot(\alpha+\beta)} \cdot \frac{(\alpha+\beta+1)\cdot(\alpha+\beta)\cdot\Gamma(\alpha+\beta)}{(\alpha+1)\cdot\alpha\cdot\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot \int_0^1 x^{(\alpha+2)-1} \cdot (1-x)^{\beta-1} dx$$

$$= \frac{(\alpha+1)\cdot\alpha}{(\alpha+\beta+1)\cdot(\alpha+\beta)} \cdot \frac{(\alpha+\beta+1)\cdot\Gamma(\alpha+\beta+1)}{(\alpha+1)\cdot\Gamma(\alpha+1)\cdot\Gamma(\beta)} \cdot \int_0^1 x^{(\alpha+2)-1} \cdot (1-x)^{\beta-1} dx$$

$$= \frac{(\alpha+1)\cdot\alpha}{(\alpha+\beta+1)\cdot(\alpha+\beta)} \cdot \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\cdot\Gamma(\beta)} \cdot \int_0^1 x^{(\alpha+2)-1} \cdot (1-x)^{\beta-1} dx$$

Using $\alpha' = \alpha + 2$

$\Rightarrow$

$$= \frac{(\alpha+1)\cdot\alpha}{(\alpha+\beta+1)\cdot(\alpha+\beta)} \cdot \frac{\Gamma(\alpha'+\beta)}{\Gamma(\alpha')\cdot\Gamma(\beta)} \cdot \int_0^1 x^{\alpha'-1} \cdot (1-x)^{\beta-1} dx$$

$$= \frac{(\alpha+1)\cdot\alpha}{(\alpha+\beta+1)\cdot(\alpha+\beta)} \cdot \frac{1}{B(\alpha',\beta)} \cdot \int_0^1 x^{\alpha'-1}\cdot(1-x)^{\beta-1}dx$$

$$= \frac{(\alpha+1)\cdot\alpha}{(\alpha+\beta+1)\cdot(\alpha+\beta)} \cdot \int_0^1 \frac{x^{\alpha'-1}\cdot(1-x)^{\beta-1}}{B(\alpha',\beta)}dx$$

$$= \frac{(\alpha+1)\cdot\alpha}{(\alpha+\beta+1)\cdot(\alpha+\beta)}$$

$\Rightarrow$

$$\mathbb{E}[X^2] = \frac{(\alpha+1)\cdot\alpha}{(\alpha+\beta+1)\cdot(\alpha+\beta)}$$

$\Rightarrow$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{(\alpha+1)\cdot\alpha}{(\alpha+\beta+1)\cdot(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha}{(\alpha+\beta)} \cdot \left(\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha}{\alpha+\beta}\right)$$

$$= \frac{\alpha}{(\alpha+\beta)} \cdot \left(\frac{(\alpha+1)\cdot(\alpha+\beta)}{(\alpha+\beta+1)\cdot(\alpha+\beta)} - \frac{\alpha\cdot(\alpha+\beta+1)}{(\alpha+\beta)\cdot(\alpha+\beta+1)}\right)$$

$$= \frac{\alpha}{(\alpha+\beta)^2\cdot(\alpha+\beta+1)} \cdot ([\alpha^2+\alpha\cdot\beta+\alpha+\beta] - [\alpha^2+\alpha\cdot\beta+\alpha]) = \frac{\alpha\cdot\beta}{(\alpha+\beta)^2\cdot(\alpha+\beta+1)}$$

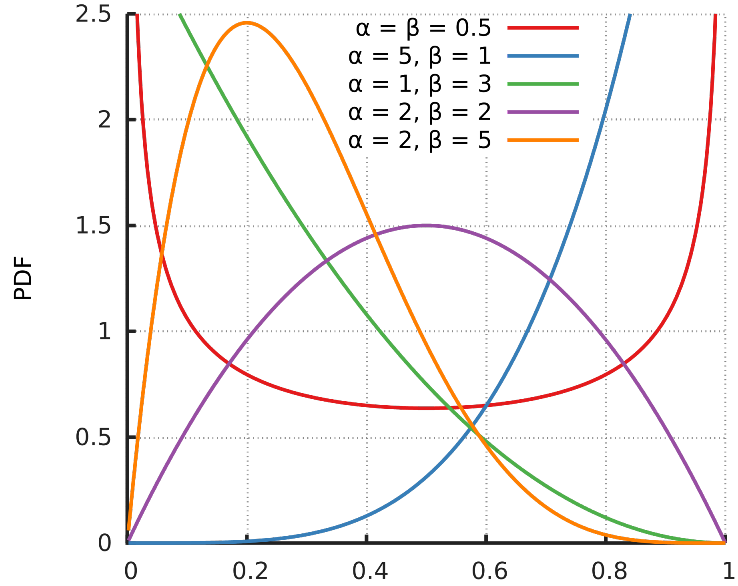
$\Rightarrow$

$$\boxed{\mathbb{V}[X] = \frac{\alpha\cdot\beta}{(\alpha+\beta)^2\cdot(\alpha+\beta+1)}}$$

**7. Relation between other variables:**

- No relations.

**8. Histogram:**




---

**1. Name:** Normal.

**2. Explanation:** Normal distributions are important in statistics and are often used in the natural and social sciences to represent real-valued random variables whose distributions are not known. Their importance is partly due to the central limit theorem.

**3. Density function:**

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left(-\frac{1}{2} \cdot \left[\frac{x-\mu}{\sigma}\right]^2\right)}$$

**4. Distribution function:**

$$\boxed{\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-\frac{1}{2} \cdot \left[\frac{t-\mu}{\sigma}\right]^2\right) dt}$$

**5. Mean:**

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-\frac{1}{2} \cdot \left[\frac{x-\mu}{\sigma}\right]^2\right) dx$$

Change of variables:

$$t = \frac{x-\mu}{\sigma} \Rightarrow \left(dt = \frac{dx}{\sigma} \quad \wedge \quad x = t \cdot \sigma + \mu \quad \wedge \quad dx = \sigma \cdot dt\right)$$

$\Rightarrow$

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{+\infty} (t \cdot \sigma + \mu) \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) \sigma \cdot dt \\
&= \int_{-\infty}^{+\infty} (t \cdot \sigma + \mu) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt \\
&= \sigma \cdot \left[\int_{-\infty}^{+\infty} t \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt\right] + \mu \cdot \left[\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt\right] \\
&= \sigma \cdot I + \mu \cdot 1 = \sigma \cdot I + \mu
\end{aligned}$$

with:

$$I = \int_{-\infty}^{+\infty} t \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt = \int_{-\infty}^{+\infty} h(t) dt$$

Note $h(t) = -h(-t)$ i.e. $h(\cdot)$ is odd function thus $I = 0$.

$\Rightarrow$

$$\boxed{\mathbb{E}[X] = \mu}$$

**6. Variance:**

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-\frac{1}{2} \cdot \left[\frac{x-\mu}{\sigma}\right]^2\right) dx$$

Change of variables:

$$t = \frac{x-\mu}{\sigma} \Rightarrow \left( dt = \frac{dx}{\sigma} \quad \wedge \quad x = t\cdot\sigma + \mu \quad \wedge \quad dx = \sigma \cdot dt \right)$$

$\Rightarrow$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} (t\cdot\sigma+\mu)^2 \cdot \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot \exp\left(-\frac{t^2}{2}\right) \cdot \sigma \cdot dt$$

$$= \int_{-\infty}^{+\infty} [t^2\cdot\sigma^2 + 2\cdot t\cdot\sigma\cdot\mu + \mu^2] \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{t^2}{2}\right) dt$$

$$= \sigma^2 \cdot \int_{-\infty}^{+\infty} t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{t^2}{2}\right) dt + 2\cdot\sigma\cdot\mu\cdot \int_{-\infty}^{+\infty} t \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{t^2}{2}\right) dt$$

$$\mu^2 \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{t^2}{2}\right) dt = \sigma^2 \cdot I_1 + 2\sigma\cdot\mu\cdot I_2 + \mu^2 \cdot I_3$$

with:

$$I_1 = \int_{-\infty}^{+\infty} t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\cdot t^2\right) dt$$

$$I_2 = \int_{-\infty}^{+\infty} t \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\cdot t^2\right) dt$$

$$I_3 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\cdot t^2\right) dt$$

Note $I_3 = 1$ because its a integral over a distribution function $[N(0,1)]$. From the computation of mean of a normal distribution we have that $I_2 = 0$. Now we have to focus on compute $I_1$.

$\Rightarrow$

$$\mathbb{E}[X^2] = \sigma^2 \cdot I_1 + \mu^2$$

We going to use integration by parts, but first:

$$I_1 = \int_{-\infty}^{+\infty} t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\cdot t^2\right) dt =$$

$$\int_{-\infty}^{0} t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\cdot t^2\right) dt + \int_{0}^{+\infty} t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\cdot t^2\right) dt$$

$$= I_1' + I_1''$$

with:

$$I_1' = \int_{-\infty}^{0} t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt$$

$$I_2' = \int_{-\infty}^{0} t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt$$

Now we going to really use integration by parts:

$$I_1' = \int_{-\infty}^{0} t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt = \lim_{c\to\infty} \int_{-c}^{0} t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt$$

$$= \lim_{c\to\infty} \int_{-c}^{0} t \cdot t \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt = \lim_{c\to\infty} \int_{-c}^{0} h_1(t) \cdot h_2'(t) dt$$

with:

$$h_1(t) = t \Rightarrow h_1'(t) = 1$$

$$h_2'(t) = t \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) \Rightarrow h_2(t) = -\frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right)$$

$\Rightarrow$

$$I_1' = \lim_{c\to\infty} \left[ h_1(t) \cdot h_2(t) \Big|_{t=-c}^{t=0} - \int_{-c}^{0} h_1(t) \cdot h_2'(t) dt \right]$$

$$= \lim_{c\to\infty} [h_1(0) \cdot h_2(0) - h_1(-c) \cdot h_2(-c)] - \lim_{c\to\infty} \left[ \int_{-c}^{0} h_1(t) \cdot h_2'(t) dt \right]$$

$$= \lim_{c\to\infty} \left[ -\sqrt{2\pi} \cdot \frac{c}{exp\left(\frac{c^2}{2}\right)} \right] - \lim_{c\to\infty} \left[ \int_{-c}^{0} h_1(t) \cdot h_2'(t) dt \right]$$

$$= 0 - \lim_{c\to\infty} \left[ \int_{-c}^{0} h_1(t) \cdot h_2'(t) dt \right] = \lim_{c\to\infty} \left[ \int_{-c}^{0} \frac{1}{\sqrt{2\pi}} \cdot exp\left(-\frac{t^2}{2}\right) dt \right]$$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \cdot exp\left(-\frac{t^2}{2}\right) dt = \Phi(0) = \frac{1}{2}$$

$\Rightarrow$

$$I_1' = \frac{1}{2}$$

Now we going to use integration by parts for second time:

$$I_1'' = \int_0^{+\infty} t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt = \lim_{c\to\infty} \int_0^c t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt$$
$$= \lim_{c\to\infty} \int_0^c t \cdot t \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) dt = \lim_{c\to\infty} \int_0^c h_1(t) \cdot h_2'(t) dt$$

with:

$$h_1(t) = t \Rightarrow h_1'(t) = 1$$
$$h_2'(t) = t \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right) \Rightarrow h_2(t) = -\frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot t^2\right)$$

$\Rightarrow$

$$I_1'' = \lim_{c\to\infty} \left[ h_1(t) \cdot h_2(t) \Big|_{t=0}^{t=c} - \int_0^c h_1(t) \cdot h_2'(t) dt \right]$$
$$= \lim_{c\to\infty} [h_1(c) \cdot h_2(c) - h_1(0) \cdot h_2(0)] - \lim_{c\to\infty} \left[ \int_0^c h_1(t) \cdot h_2'(t) dt \right]$$
$$= \lim_{c\to\infty} \left[ -\sqrt{2\pi} \cdot \frac{c}{exp\left(\frac{c^2}{2}\right)} \right] - \lim_{c\to\infty} \left[ \int_0^c h_1(t) \cdot h_2'(t) dt \right]$$
$$= 0 - \lim_{c\to\infty} \left[ \int_0^c h_1(t) \cdot h_2'(t) dt \right] = \lim_{c\to\infty} \left[ \int_0^c \frac{1}{\sqrt{2\pi}} \cdot exp\left(-\frac{t^2}{2}\right) dt \right]$$
$$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot exp\left(-\frac{t^2}{2}\right) dt = 1 - \Phi(0) = 1 - \frac{1}{2} = \frac{1}{2}$$

$\Rightarrow$

$$I_1'' = \frac{1}{2}$$

$\Rightarrow$

$$I_1 = I_1' + I_1'' = \frac{1}{2} + \frac{1}{2} = 1$$

$\Rightarrow$

$$\mathbb{E}[X^2] = \sigma^2 + \mu^2$$

$\Rightarrow$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

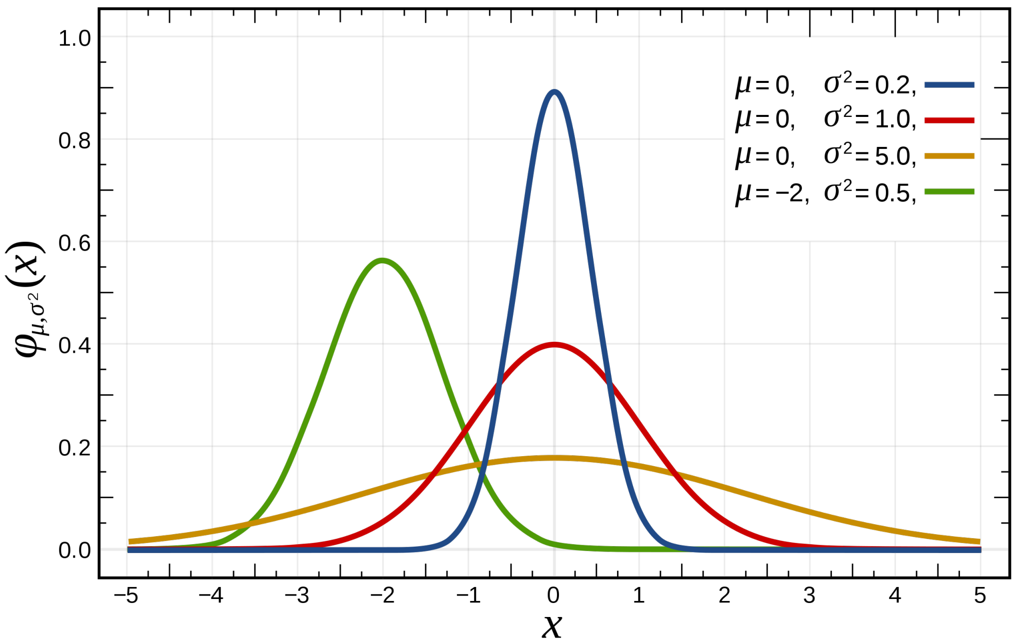
$\Rightarrow$

$$\boxed{\mathbb{V}[X] = \sigma^2}$$

**7. Relation between other variables:**

- The central limit theorem (CLT) establishes that, in many situations, for independent and identically distributed random variables, the sampling distribution of the standardized sample mean tends towards the standard normal distribution even if the original variables themselves are not normally distributed.

**8. Histogram:**




---

**1. Name:** Lognormal.

**2. Explanation:** In probability theory, a log-normal (or lognormal) distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed.

**3. Density function:**

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi} \cdot \sigma \cdot x} \exp\left( -\frac{1}{2} \cdot \left[ \dfrac{ln(x) - \mu}{\sigma} \right]^2 \right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

**4. Distribution function:**

$$\mathbb{P}(X \leq x) = \mathbb{P}(ln(X) \leq ln(x)) = \mathbb{P}\left(\frac{ln(X) - \mu}{\sigma} \leq \frac{ln(x) - \mu}{\sigma}\right)$$

$$= \mathbb{P}\left(Z \leq \frac{ln(x) - \mu}{\sigma}\right) = \Phi\left(\frac{ln(x) - \mu}{\sigma}\right)$$

$\Rightarrow$

$$\boxed{F(x) = \Phi\left(\frac{ln(x) - \mu}{\sigma}\right)}$$

**5. Mean:**

$$\mathbb{E}[X] = \int_0^{+\infty} x \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma \cdot x} \cdot \exp\left(-\frac{1}{2} \cdot \left[\frac{ln(x) - \mu}{\sigma}\right]^2\right) dx$$

$$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-\frac{1}{2} \cdot \left[\frac{ln(x) - \mu}{\sigma}\right]^2\right) dx$$

Change of variables:

$$t = \frac{ln(x) - \mu}{\sigma} \Rightarrow \left(dt = \frac{dx}{x \cdot \sigma} \quad \wedge \quad x = exp\left(t \cdot \sigma + \mu\right) \quad \wedge \quad dx = \sigma \cdot x \cdot dt\right)$$

$\Rightarrow$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-\frac{t^2}{2}\right) \cdot \exp\left(t \cdot \sigma + \mu\right) \cdot \sigma \cdot dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{t^2}{2}\right) \cdot \exp\left(t \cdot \sigma + \mu\right) \cdot dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{t^2}{2} + t \cdot \sigma + \mu\right) \cdot dt$$

Now we can see that:

$$-\frac{t^2}{2} + t \cdot \sigma + \mu = -\frac{1}{2} \cdot \left[t^2 - 2 \cdot t \cdot \sigma - 2 \cdot \mu\right] = -\frac{1}{2} \cdot \left[t^2 - 2 \cdot t \cdot \sigma - 2 \cdot \mu + \sigma^2 - \sigma^2\right]$$

$$= -\frac{1}{2} \cdot \left[t^2 - 2 \cdot t \cdot \sigma + \sigma^2\right] + \mu + \frac{\sigma^2}{2}$$

$$= -\frac{1}{2} \cdot [t - \sigma]^2 + \mu + \frac{\sigma^2}{2}$$

$\Rightarrow$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot [t - \sigma]^2 + \mu + \frac{\sigma^2}{2}\right) \cdot dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot [t - \sigma]^2\right) \cdot exp\left(\mu + \frac{\sigma^2}{2}\right) \cdot dt$$

$$= exp\left(\mu + \frac{\sigma^2}{2}\right) \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot [t - \sigma]^2\right) dt$$

$$\overset{N(\sigma,1)}{=} exp\left(\mu + \frac{\sigma^2}{2}\right)$$

$\Rightarrow$

$$\boxed{\mathbb{E}[X] = exp\left(\mu + \frac{\sigma^2}{2}\right)}$$

**6. Variance:**

$$\mathbb{E}[X^2] = \int_{0}^{+\infty} x^2 \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma \cdot x} \cdot \exp\left(-\frac{1}{2} \cdot \left[\frac{ln(x) - \mu}{\sigma}\right]^2\right) dx$$

$$= \int_{0}^{+\infty} x \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-\frac{1}{2} \cdot \left[\frac{ln(x) - \mu}{\sigma}\right]^2\right) dx$$

Change of variables:

$$t = \frac{ln(x) - \mu}{\sigma} \Rightarrow \left(dt = \frac{dx}{x \cdot \sigma} \quad \wedge \quad x = exp\left(t \cdot \sigma + \mu\right) \quad \wedge \quad dx = \sigma \cdot x \cdot dt\right)$$

$\Rightarrow$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(t \cdot \sigma + \mu\right) \cdot \exp\left(-\frac{t^2}{2}\right) \cdot \exp\left(t \cdot \sigma + \mu\right) \cdot \sigma \cdot dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(t \cdot \sigma + \mu\right) \cdot \exp\left(-\frac{t^2}{2}\right) \cdot \exp\left(t \cdot \sigma + \mu\right) \cdot \sigma \cdot dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{t^2}{2} + 2 \cdot t \cdot \sigma + 2 \cdot \mu\right) \cdot dt$$

Now we can see that:

$$-\frac{t^2}{2} + 2 \cdot t \cdot \sigma + 2 \cdot \mu = -\frac{1}{2} \cdot \left[t^2 - 4 \cdot t \cdot \sigma - 4 \cdot \mu\right]$$

$$= -\frac{1}{2} \cdot \left[t^2 - 4 \cdot t \cdot \sigma - 4 \cdot \mu + 4 \cdot \sigma^2 - 4 \cdot \sigma^2\right]$$

$$= -\frac{1}{2} \cdot [t^2 - 4 \cdot t \cdot \sigma + 4 \cdot \sigma^2] + 2 \cdot \mu + 2 \cdot \sigma^2$$

$$= -\frac{1}{2} \cdot [t - 2 \cdot \sigma]^2 + 2 \cdot \mu + 2 \cdot \sigma^2$$

$\Rightarrow$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot [t - 2 \cdot \sigma]^2 + 2 \cdot \mu + 2 \cdot \sigma^2\right) \cdot dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot [t - 2 \cdot \sigma]^2\right) \cdot exp\left(2 \cdot \mu + 2 \cdot \sigma^2\right) \cdot dt$$

$$= exp\left(2 \cdot \mu + 2 \cdot \sigma^2\right) \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot [t - 2 \cdot \sigma]^2\right) dt$$

$$\overset{N(2 \cdot \sigma, 1)}{=} exp\left(2 \cdot \mu + 2 \cdot \sigma^2\right)$$

$\Rightarrow$

$$\mathbb{E}[X^2] = exp\left(2 \cdot \mu + 2 \cdot \sigma^2\right)$$

$\Rightarrow$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = e^{2 \cdot \mu + 2 \cdot \sigma^2} - \left[e^{\mu + \frac{\sigma^2}{2}}\right]^2 = e^{2 \cdot \mu + 2 \cdot \sigma^2} - e^{2 \cdot \mu + \sigma^2} =$$

$$= e^{2 \cdot \mu + \sigma^2} \cdot \left(e^{\sigma^2} - 1\right)$$

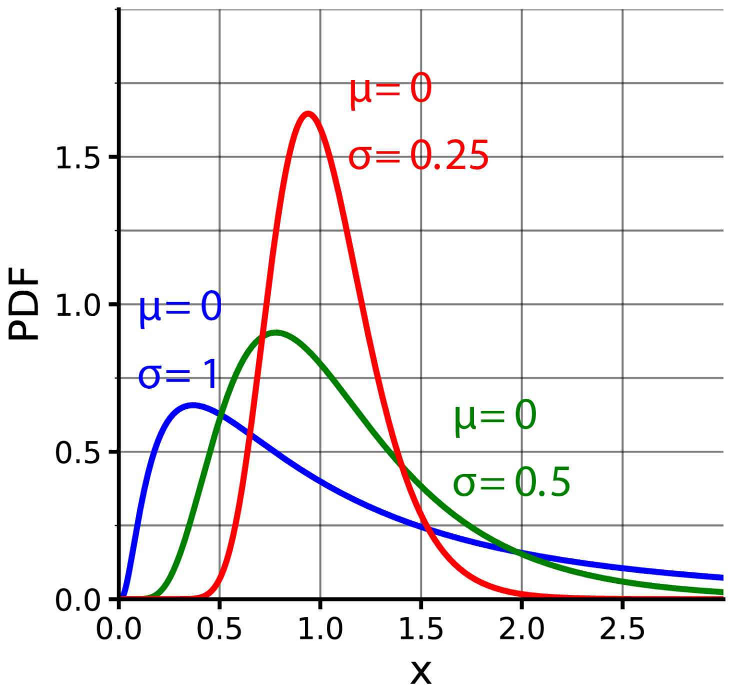
$\Rightarrow$

$$\boxed{\mathbb{V}[X] = e^{2 \cdot \mu + \sigma^2} \cdot \left(e^{\sigma^2} - 1\right)}$$

**7. Relation between other variables:**

- The central limit theorem (CLT) establishes that, in many situations, for independent and identically distributed random variables, the sampling distribution of the standardized sample mean tends towards the standard normal distribution even if the original variables themselves are not normally distributed.

**8. Histogram:**




---

**1. Name:** Chi-square.

**2. Explanation:** In probability theory and statistics, the chi-squared distribution with $k$ degrees of freedom is the distribution of a sum of the squares of $k$ independent standard normal random variables.

**3. Density function:**

$$\boxed{f(x) = \begin{cases} \dfrac{1}{2^{\frac{k}{2}} \cdot \Gamma\left(\frac{k}{2}\right)} \cdot x^{\frac{k}{2}-1} \cdot e^{-\frac{x}{2}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}}$$

**4. Distribution function:**

No computed.

**5. Mean:**

$$\mathbb{E}[X] = \frac{\alpha}{\beta} = \frac{\frac{k}{2}}{\frac{1}{2}} = k$$

$\Rightarrow$

$$\boxed{\mathbb{E}[X] = k}$$

**6. Variance:**

$$\mathbb{V}[X] = \frac{\alpha}{\beta^2} = \frac{\frac{k}{2}}{\left(\frac{1}{2}\right)^2} = \frac{\frac{k}{2}}{\frac{1}{4}} = 2 \cdot k$$

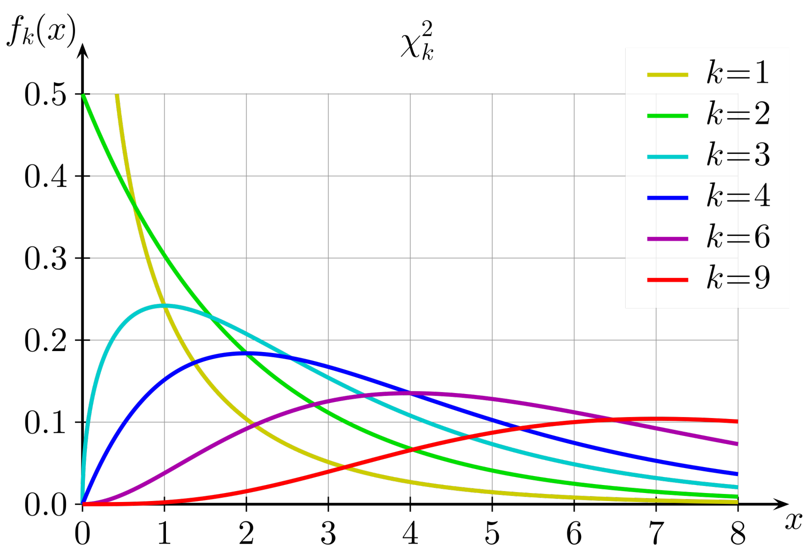
$\Rightarrow$

$$\boxed{\mathbb{V}[X] = 2 \cdot k}$$

**7. Relation between other variables:**

- Chi-square is a particular version of $Gamma\left(\alpha = \frac{k}{2}, \beta = \frac{1}{2}\right)$.
- Sum of square of $k$ independent standard normal random variables is $\chi_k$.

**8. Histogram:**



**1. Name:** t-Student.

**2. Explanation:** In probability and statistics, Student's $t-$distribution (or simply the t-distribution) is a continuous probability distribution that generalizes the standard normal distribution. Like the latter, it is symmetric around zero and bell-shaped.

**3. Density function:**

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n \cdot \pi} \cdot \Gamma\left(\frac{n}{2}\right)} \cdot \left(1 + \frac{x^2}{n}\right)^{-\frac{(n+1)}{2}}$$

**4. Distribution function:**

No computed.

**5. Mean:**

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f(x) dx$$

We can define $h(x) = x \cdot f(x)$ and we have that:

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} h(x) dx$$

Note that $f(-x) = f(x)$ thus:

$$h(-x) = (-x) \cdot f(-x) = -x \cdot f(x) = -h(x)$$

$\Rightarrow$

$$h(x) = -h(-x)$$

$\Rightarrow$

$$h(\cdot) \text{ is odd function}$$

$\Rightarrow$

$$\boxed{\mathbb{E}[X] = 0}$$

**6. Variance:**

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f(x)dx$$

If we define $h(x) = x^2 \cdot f(x)$ and remembering that $f(x) = f(-x)$ then we have:

$$h(-x) = (-x)^2 \cdot f(-x) = x^2 \cdot f(x) = h(x)$$

Then we have that:

$$\mathbb{E}[X^2] = 2 \cdot \int_{0}^{+\infty} x^2 \cdot f(x)dx = 2 \cdot \int_{0}^{+\infty} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n \cdot \pi} \cdot \Gamma\left(\frac{n}{2}\right)} \cdot \left(1 + \frac{x^2}{n}\right)^{-\frac{(n+1)}{2}} \cdot x^2 dx$$

Change of variables:

$$t = \frac{x^2}{n} \Rightarrow \left(dt = \frac{2 \cdot x \cdot dx}{n} \quad \wedge \quad x = \sqrt{n} \cdot \sqrt{t}\right)$$

$$\begin{aligned}
\mathbb{E}[X^2] &= 2 \cdot \int_{0}^{+\infty} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n \cdot \pi} \cdot \Gamma\left(\frac{n}{2}\right)} \cdot \left(1 + \frac{x^2}{n}\right)^{-\frac{(n+1)}{2}} \cdot x \cdot x dx \\
&= 2 \cdot \int_{0}^{+\infty} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n \cdot \pi} \cdot \Gamma\left(\frac{n}{2}\right)} \cdot (1+t)^{-\frac{(n+1)}{2}} \cdot \sqrt{n} \cdot \sqrt{t} \cdot \frac{n}{2} dt \\
&= \frac{n}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot \int_{-\infty}^{+\infty} (1+t)^{-\frac{(n+1)}{2}} \cdot t^{\frac{1}{2}} \cdot dt
\end{aligned}$$

Remind the integral representation of Beta function we have that:

$$B(x, y) = \int_{0}^{+\infty} t^{x-1} \cdot (1+t)^{-[x+y]} dt$$

Using $x = \frac{3}{2}$ and $y = \frac{n}{2} - 1$ we have that:

$$B\left(\frac{3}{2}, \frac{n-2}{2}\right) = \int_{0}^{+\infty} t^{\frac{1}{2}} \cdot (1+t)^{-\frac{(n+1)}{2}} dt$$

$\Rightarrow$

$$\mathbb{E}[X^2] = \frac{n}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot B\left(\frac{3}{2}, \frac{n-2}{2}\right) = \frac{n}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\Gamma\left(\frac{3}{2}\right) \cdot \Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$$

Note that:

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\cdot\sqrt{\pi}$$

$\Rightarrow$

$$\mathbb{E}[X^2] = \frac{n}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\cdot\frac{\frac{\sqrt{\pi}}{2}\cdot\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \frac{n}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\cdot\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$= \frac{n}{2}\cdot\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

Note that:

$$\left(\frac{n-2}{2}\right)\cdot\Gamma\left(\frac{n-2}{2}\right) = \Gamma\left(\frac{n}{2}\right)$$

$\Rightarrow$

$$\mathbb{E}[X^2] = \frac{n}{2}\cdot\frac{\Gamma\left(\frac{n-2}{2}\right)}{\left(\frac{n-2}{2}\right)\cdot\Gamma\left(\frac{n-2}{2}\right)} = \frac{n}{n-2}$$

$\Rightarrow$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{n}{n-2}$$

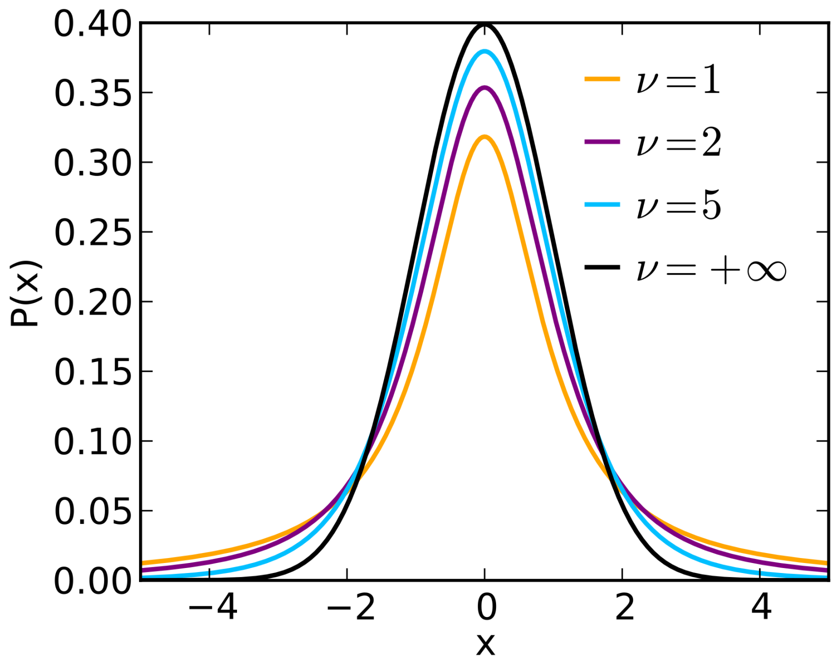
$\Rightarrow$

$$\boxed{\mathbb{V}[X] = \frac{n}{n-2}}$$

**7. Relation between other variables:**

- Continuous probability distribution that generalizes the standard normal distribution.

**8. Histogram:**




---

**1. Name:** Negative binomial.

**2. Explanation:** Measure the probability of get $k$ failures before complete $r$ success on a bernoulli process.

**3. Density function:**

$$\mathbb{P}(X = k) = \binom{k+r-1}{k} \cdot (1-p)^k \cdot p^r$$

**4. Distribution function:**

No computed.

**5. Mean:**

First of all note that:

$$k_1 \cdot \binom{k_1+r_1-1}{k_1} = \frac{k_1 \cdot (k_1+r_1-1)!}{(r_1-1)! \ \cdot k_1!} = \frac{(k_1+r_1-1)!}{(r_1-1)! \ \cdot (k_1-1)!} = \frac{r_1}{r_1} \cdot \frac{(k_1+r_1-1)!}{(r_1-1)! \ \cdot (k_1-1)!}$$

$$= r_1 \cdot \frac{(k_1+r_1-1)!}{r_1! \ \cdot (k_1-1)!} = r_1 \cdot \binom{k_1+r_1-1}{k_1-1}$$

$\Rightarrow$

$$\boxed{k_1 \cdot \binom{k_1+r_1-1}{k_1} = r_1 \cdot \binom{k_1+r_1-1}{k_1-1}} \quad (*)$$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \cdot \mathbb{P}(X=k) = \sum_{k=1}^{\infty} k \cdot \mathbb{P}(X=k) = \sum_{k=1}^{\infty} k \cdot \binom{k+r-1}{k} \cdot (1-p)^k \cdot p^r$$

$$\overset{\text{using } (*)}{=} \sum_{k=1}^{\infty} r \cdot \binom{k+r-1}{k-1} \cdot (1-p)^k \cdot p^r = \sum_{k=0}^{\infty} r \cdot \binom{k+r}{k} \cdot (1-p)^k \cdot (1-p) \cdot p^r$$

$$= r \cdot (1-p) \cdot \sum_{k=0}^{\infty} \binom{k+r}{k} \cdot (1-p)^k \cdot p^r \overset{\text{using } r'=r+1}{=}$$

$$= r \cdot (1-p) \cdot \sum_{k=0}^{\infty} \binom{k+r'-1}{k} \cdot (1-p)^k \cdot \frac{p^{r+1}}{p}$$

$$= r \cdot \frac{(1-p)}{p} \cdot \sum_{k=0}^{\infty} \binom{k+r'-1}{k} \cdot (1-p)^k \cdot p^{r'} = r \cdot \frac{(1-p)}{p}$$

$\Rightarrow$

$$\boxed{\mathbb{E}[X] = r \cdot \frac{(1-p)}{p}}$$

**6. Variance:**

$$\mathbb{E}[X^2] = \sum_{k=0}^{\infty} k^2 \cdot \mathbb{P}(X=k) = \sum_{k=1}^{\infty} k^2 \cdot \mathbb{P}(X=k) = \sum_{k=1}^{\infty} k \cdot k \cdot \mathbb{P}(X=k)$$

$$= \sum_{k=1}^{\infty} k \cdot \left[k \cdot \binom{k+r-1}{k}\right] \cdot (1-p)^k \cdot p^r \overset{\text{using } (*)}{=} \sum_{k=1}^{\infty} k \cdot \left[r \cdot \binom{k+r-1}{k-1}\right] \cdot (1-p)^k \cdot p^r$$

$$= r \cdot \sum_{k=1}^{\infty} k \cdot \binom{k+r-1}{k-1} \cdot (1-p)^k \cdot p^r = r \cdot \sum_{k=0}^{\infty} (k+1) \cdot \binom{k+r}{k} \cdot (1-p)^{k+1} \cdot p^r$$

$$= r \cdot (1-p) \cdot \sum_{k=0}^{\infty} (k+1) \cdot \binom{k+r}{k} \cdot (1-p)^k \cdot p^r$$

$$= r \cdot (1-p) \cdot \left[\sum_{k=0}^{\infty} k \cdot \binom{k+r}{k} \cdot (1-p)^k \cdot p^r + \sum_{k=0}^{\infty} \binom{k+r}{k} \cdot (1-p)^k \cdot p^r\right]$$

$$\overset{\text{using } r'=r+1}{=} r \cdot (1-p) \cdot \left[\sum_{k=0}^{\infty} k \cdot \binom{k+r'-1}{k} \cdot (1-p)^k \cdot \frac{p^{r'}}{p} + \sum_{k=0}^{\infty} \binom{k+r'-1}{k} \cdot (1-p)^k \cdot \frac{p^{r'}}{p}\right]$$

$$= r \cdot \frac{(1-p)}{p} \cdot \left[ \sum_{k=0}^{\infty} k \cdot \binom{k+r'-1}{k} \cdot (1-p)^k \cdot p^{r'} + \sum_{k=0}^{\infty} \binom{k+r'-1}{k} \cdot (1-p)^k \cdot p^{r'} \right]$$

$$\overset{\text{using } \mathbb{E}[X]}{=} r \cdot \frac{(1-p)}{p} \cdot \left[ \frac{(r+1) \cdot (1-p)}{p} + 1 \right] = r \cdot \frac{(1-p)}{p} \cdot \left[ \frac{r \cdot (1-p)}{p} + \frac{(1-p)}{p} + 1 \right]$$

$$= r \cdot \frac{(1-p)}{p} \cdot \left[ \frac{r \cdot (1-p)}{p} + \frac{1}{p} \right] = \mathbb{E}[X] \cdot \left( \mathbb{E}[X] + \frac{1}{p} \right) = (\mathbb{E}[X])^2 + \frac{\mathbb{E}[X]}{p}$$

$\Rightarrow$

$$\mathbb{E}[X^2] = (\mathbb{E}[X])^2 + \frac{\mathbb{E}[X]}{p}$$

$\Rightarrow$

$$\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{\mathbb{E}[X]}{p}$$

$\Rightarrow$

$$\mathbb{V}[X] = \frac{\mathbb{E}[X]}{p} = \frac{r \cdot (1-p)}{p^2}$$

$\Rightarrow$

$$\boxed{\mathbb{V}[X] = \frac{r \cdot (1-p)}{p^2}}$$

**7. Relation between other variables:**

- Measure the probability of get $k$ failures before complete $r$ success on a bernoulli process.

---