Alberto Andrés Valdés González.
Degree: Mathematical Engineer.
Work position: Data Scientist.
Mail: anvaldes@uc.cl/alberto.valdes.gonzalez.96@gmail.com
Location: Santiago, Chile.

Transformation of Random Variables

Sum of random variables

Exercise: Consider $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$. X and Y are independent.

We define $Z = X + Y$, then:

$$\mathbb{P}(Z = n) = \mathbb{P}(X + Y = n) \overset{\text{total probabilities}}{=} \sum_{j=0}^{\infty} \mathbb{P}(X + Y = n | Y = j) \cdot \mathbb{P}(Y = j)$$

$$\overset{X \text{ and } Y \text{ are independents}}{=} \sum_{j=0}^{\infty} \mathbb{P}(X + j = n) \cdot \mathbb{P}(Y = j) = \sum_{j=0}^{\infty} \mathbb{P}(X = n - j) \cdot \mathbb{P}(Y = j)$$

Note $\mathbb{P}(X = n - j) = 0 \quad \forall j \geq (n+1)$

$\Rightarrow$

$$\mathbb{P}(Z = n) = \sum_{j=0}^{\infty} \mathbb{P}(X = n - j) \cdot \mathbb{P}(Y = j)$$

$$= \sum_{j=0}^{n} \mathbb{P}(X = n - j) \cdot \mathbb{P}(Y = j) + \sum_{j=n+1}^{\infty} \mathbb{P}(X = n - j) \cdot \mathbb{P}(Y = j)$$

$$= \sum_{j=0}^{n} \mathbb{P}(X = n - j) \cdot \mathbb{P}(Y = j) + \sum_{j=n+1}^{\infty} 0 \cdot \mathbb{P}(Y = j) = \sum_{j=0}^{n} \mathbb{P}(X = n - j) \cdot \mathbb{P}(Y = j)$$

$$= \sum_{j=0}^{n} \frac{e^{-\lambda_1} \cdot (\lambda_1)^{n-j}}{(n-j)!} \cdot \frac{e^{-\lambda_2} \cdot (\lambda_2)^{j}}{j!} = e^{-(\lambda_1 + \lambda_2)} \cdot \frac{n!}{n!} \cdot \sum_{j=0}^{n} \frac{(\lambda_1)^{n-j}}{(n-j)!} \cdot \frac{(\lambda_2)^{j}}{j!}$$

$$= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} \cdot \sum_{j=0}^{n} \left(\frac{n!}{(n-j)! \ \cdot j!} \right) (\lambda_1)^{n-j} \cdot (\lambda_2)^{j}$$

$$= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} \cdot \sum_{j=0}^{n} \binom{n}{j} \cdot (\lambda_1)^{n-j} \cdot (\lambda_2)^{j} \overset{\text{Binomial Theorem}}{=} \frac{e^{-(\lambda_1 + \lambda_2)} \cdot (\lambda_1 + \lambda_2)^{n}}{n!}$$

$\Rightarrow$

$$\boxed{Z \sim Poisson(\lambda_1 + \lambda_2)}$$

Exercise: Consider $X \sim Exp(\lambda_1)$ and $Y \sim Exp(\lambda_2)$. X and Y are independent. $\lambda_1 \neq \lambda_2$.

We define $Z = X + Y$, then:

$$F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(X + Y \leq z) \overset{\text{total probabilities}}{=} \int_0^\infty \mathbb{P}(X + Y \leq z | Y = y) \cdot f_Y(y) dy$$

$$\overset{X \text{ and } Y \text{ are independents}}{=} \int_0^\infty \mathbb{P}(X + y \leq z) \cdot f_Y(y) dy = \int_0^\infty \mathbb{P}(X \leq z - y) \cdot f_Y(y) dy$$

Note $\mathbb{P}(Z \leq z - y) = 0 \quad \forall y \geq z$

$\Rightarrow$

$$F_Z(z) = \int_0^\infty \mathbb{P}(X \leq z - y) \cdot f_Y(y) dy = \int_0^z \mathbb{P}(X \leq z - y) \cdot f_Y(y) dy + \int_z^\infty \mathbb{P}(X \leq z - y) \cdot f_Y(y) dy$$

$$= \int_0^z \mathbb{P}(X \leq z - y) \cdot f_Y(y) dy + \int_z^\infty 0 \cdot f_Y(y) dy = \int_0^z \mathbb{P}(X \leq z - y) \cdot f_Y(y) dy$$

$$= \int_0^z [1 - e^{-\lambda_1 \cdot (z-y)}] \cdot \lambda_2 \cdot e^{-\lambda_2 \cdot y} dy = \lambda_2 \cdot \left(\int_0^z [e^{-\lambda_2 \cdot y} - e^{-\lambda_1 \cdot z} \cdot e^{(\lambda_1 - \lambda_2) \cdot y}] dy \right)$$

$$= \lambda_2 \cdot \left(\left[\frac{e^{-\lambda_2 \cdot y}}{-\lambda_2} \right] \Big|_{y=0}^{y=z} - e^{-\lambda_1 \cdot z} \cdot \left[\frac{e^{(\lambda_1 - \lambda_2) \cdot y}}{\lambda_1 - \lambda_2} \right] \Big|_{y=0}^{y=z} \right)$$

$$= \lambda_2 \cdot \left(\left[\frac{e^{-\lambda_2 \cdot z}}{-\lambda_2} - \frac{1}{-\lambda_2} \right] - e^{-\lambda_1 \cdot z} \cdot \left[\frac{e^{(\lambda_1 - \lambda_2) \cdot z}}{\lambda_1 - \lambda_2} - \frac{1}{\lambda_1 - \lambda_2} \right] \right)$$

$$= -e^{-\lambda_2 \cdot z} + 1 - \left(\frac{\lambda_2}{\lambda_1 - \lambda_2} \right) \cdot e^{-\lambda_2 \cdot z} + \left(\frac{\lambda_2}{\lambda_1 - \lambda_2} \right) \cdot e^{-\lambda_1 \cdot z}$$

$$= 1 - \left(\frac{\lambda_1}{\lambda_1 - \lambda_2} \right) \cdot e^{-\lambda_2 \cdot z} + \left(\frac{\lambda_2}{\lambda_1 - \lambda_2} \right) \cdot e^{-\lambda_1 \cdot z}$$

$$= 1 - \left(\frac{1}{\lambda_1 - \lambda_2} \right) \cdot [\lambda_1 \cdot e^{-\lambda_2 \cdot z} - \lambda_2 \cdot e^{-\lambda_1 \cdot z}]$$

$\Rightarrow$

$$\boxed{F_Z(z) = 1 - \left(\frac{1}{\lambda_1 - \lambda_2}\right) \cdot [\lambda_1 \cdot e^{-\lambda_2 \cdot z} - \lambda_2 \cdot e^{-\lambda_1 \cdot z}]}$$

Now:

$$\begin{aligned}
f_Z(z) = \frac{dF_Z(z)}{dz} &= \frac{d}{dz}\left(1 - \left[\frac{1}{\lambda_1 - \lambda_2}\right] \cdot [\lambda_1 \cdot e^{-\lambda_2 \cdot z} - \lambda_2 \cdot e^{-\lambda_1 \cdot z}]\right) \\
&= -\left[\frac{1}{\lambda_1 - \lambda_2}\right] \cdot [-\lambda_1 \cdot \lambda_2 \cdot e^{-\lambda_2 \cdot z} + \lambda_1 \cdot \lambda_2 \cdot e^{-\lambda_1 \cdot z}] \\
&= \left(\frac{\lambda_1 \cdot \lambda_2}{\lambda_1 - \lambda_2}\right) \cdot [e^{-\lambda_2 \cdot z} - e^{-\lambda_1 \cdot z}]
\end{aligned}$$

$\Rightarrow$

$$\boxed{f_Z(z) = \left(\frac{\lambda_1 \cdot \lambda_2}{\lambda_1 - \lambda_2}\right) \cdot [e^{-\lambda_2 \cdot z} - e^{-\lambda_1 \cdot z}]}$$
