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Probabilities Review

Measure Theory, Probabilities Theory, Random Variables and Observations

Definitions

σ-Algebra:

We say a collection ζ of subsets of Ω is a σ-Algebra if:

1. $\emptyset \in \zeta$
2. If $E \in \zeta \Rightarrow E^C \in \zeta$
3. If $\{E_i\}_{i\in\mathbb{N}} \subseteq \zeta$ then $\bigcup_{i\in\mathbb{N}} E_i \in \zeta$

Example: 2^Ω, $\{\emptyset, \Omega\}$.

Note: All $\sigma-$algebras are algebras.

Observation: If $\{F_i\}_{i\in I}$ are σ-algebras on Ω:

$$\bigcap_{i\in I} F_i \quad \text{are a } \sigma\text{-algebra too}$$

If C is a arbitrary collection on Ω we define:

$$\sigma(C) := \bigcap_{C\subseteq\zeta} \zeta \quad \text{where } \zeta \text{ are } \sigma\text{-algebras}$$

$\sigma(C)$ is the smallest σ-algebra which contains C.

The **borelians** are σ(open sets).

Algebra:

We say a collection $\mathcal{A}$ of subsets of Ω is an Algebra if:

1. $\emptyset \in \mathcal{A}$
2. If $E \in \mathcal{A} \Rightarrow E^C \in \mathcal{A}$
3. If $E_1, E_2, ..., E_n \in \mathcal{A}$ then $\bigcup_{i=1}^{n} E_i \in \mathcal{A}$

Example: The set of all finite disjoint unions of intervarls.

Set Function:

A set function is a function whose domain is a family of subsets of some given set (Ω for example) and that takes its values in the extended real number line $\mathbb{R} \cup \{\pm\infty\}$.

Pre-Measure:

Let's consider $\Omega \neq \emptyset$ and $\mathcal{A}$ an algebra on it. We say $\lambda : \mathcal{A} \to [0, +\infty]$ is a pre-measure if:

1. $\lambda(\emptyset) = 0$
2. If $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ is a collection of disjoint sets and if their union is contained in $\mathcal{A}$ then:

$$\lambda\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda(A_i)$$

Measure:

Let's consider $\Omega \neq \emptyset$ and s ζ a σ-algebra on it. We say $\mu : \zeta \to [0, +\infty]$ is a measure if:

1. $\mu(\emptyset) = 0$
2. If $\{E_i\}_{i \in \mathbb{N}} \subseteq \zeta$ is a collection of disjoint set then:

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

Examples :

- Counting Measure:

$$\mu(A) = \begin{cases} |A| & if \quad A \text{ is countable} \\ +\infty & else \end{cases}$$

- If we consider any function $f : X \to [0, \infty)$ we can defines a measure μ on (Ω, ζ) vía:

$$\mu(A) := \sum_{a \in A} f(a)$$

- Let B be a set on ζ and m other measure on (Ω, ζ) such that $0 < m(B) < \infty$:

$$\mu(A) = \frac{m(A \cap B)}{m(B)}$$

Measurable Space:

Consider a set Ω and a σ-algebra ζ on Ω. Then the tuple (Ω, ζ) is called a measurable space.

Note that in contrast to a **measure space**, no measure is needed for a measurable space.

Note: In probability theory we call (Ω, ζ) **event space**.

Measure Space:

A measure space is a triple (Ω, ζ, μ), where:

1. Ω is a set.
2. ζ is a σ-algebra on the set Ω.
3. μ is a measure on (Ω, ζ)

In other words, a measure space consists of a **measurable space** (Ω, ζ) together with a measure on it.

Probability Measure:

Let's consider (Ω, ζ) an event space. We say $\mathbb{P} : \zeta \to [0, 1]$ is a probability measure if:

1. $\mathbb{P}(\Omega) = 1$
2. If $\{E_i\}_{i\in\mathbb{N}} \subseteq \zeta$ is a collection of disjoint set then:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

We call $(\Omega, \zeta, \mathbb{P})$ **probability space**.

Notes:

a From this definition we can deduce $\mathbb{P}(\emptyset) = 0$.

$E = \Omega \cup \emptyset$. It's clear $\Omega \cap \emptyset = \emptyset$. By (2): $\mathbb{P}(E) = \mathbb{P}(\Omega) + \mathbb{P}(\emptyset)$. It's clear $E = \Omega$, then: $\mathbb{P}(\Omega) = \mathbb{P}(\Omega) + \mathbb{P}(\emptyset)$. How $\mathbb{P}(\Omega)$ is finite (in particular it's value is 1) then: $\mathbb{P}(\emptyset) = 0$.

b If instead of the σ-algebra ζ we have the algebra $\mathcal{A}$ and $\mathbb{P}$ satisfies (2) as long as $\bigsqcup_{i=1}^{\infty} E_i \in \mathcal{A}$, we say $\mathbb{P}$ is a probability measure in $\mathcal{A}$.

c If instead of the σ-algebra ζ we have the algebra $\mathcal{A}$ and $\mathbb{P}$ satisfies $(2')$ instead (2):

$$2'.\ \ \mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) \ \text{ if } \ E_1, E_2 \in \mathcal{A} \ \text{ and } \ E_1 \cap E_2 = \emptyset$$

We say $\mathbb{P}$ is a probability measure **finitely additive**.

Example:

a Let be (Ω, ζ, μ) a measure space where Ω its composed by a finite number of elements and where μ is the **counting measure** then we can define the next **probabily measure**:

$$\mathbb{P}(A) = \frac{\mu(A \cap \Omega)}{\mu(\Omega)} = \frac{\mu(A)}{\mu(\Omega)}$$

We usually use this probability measure when we work with **discrete random variables**.

Random Variable:

A random variable is a mathematical formalization of a quantity or object which depends on random events. It is a mapping or a function from possible outcomes in a sample space to a measurable space, often the real numbers.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a **probability space** and $(E, \mathcal{E})$ a **measurable space**. Then an $(E, \mathcal{E})$ - **value random variable** is a measurable function $X : \Omega \to E$, which means that, for every subset $B \in \mathcal{E}$, its preimage is $\mathcal{F}$-measurable i.e.

$$\forall B \in \mathcal{E},\ X^{-1}(B) \in \mathcal{F} \text{ where } X^{-1}(B) = \{w \in \Omega : X(w) \in B\}$$

The probability that X takes on a value in a measurable set $S \in E$ is written as:

$$\mathbb{P}(X \in S) = \mathbb{P}(\{w \in \Omega | X(w) \in S\})$$

If E is countable, then X is called a **discrete random variable**.

Example:

1. $\Omega = \{head, tail\}$

   $$X(w) = \begin{cases} 1 & if \quad w = head \\ 0 & if \quad w = tail \end{cases}$$

   Note $E = \{0, 1\}$.

   $$\mathbb{P}(X \in \{1\}) = \mathbb{P}(\{w \in \Omega : X(w) \in \{1\}\}) = \mathbb{P}(\{w \in \Omega : X(w) = 1\}) = \mathbb{P}(\{head\})$$

   If we use the probability measure we **introduced previously** then we have:

   $$\mathbb{P}(\{\text{head}\}) = \frac{\mu(\{head\})}{\mu(\{head, tail\})} = \frac{1}{2}$$

2. $\Omega = \{Alberto, Gustavo, Franco\}$

   $$H(w) = \begin{cases} 1{,}77 & if \quad w = Alberto, Franco \\ 1{,}79 & if \quad w = Gustavo \end{cases}$$

   Note $E = \{1{,}77, 1{,}79\}$.

$\mathbb{P}(H \in \{1{,}77\}) = \mathbb{P}(\{w \in \Omega : H(w) \in \{1{,}77\}\}) = \mathbb{P}(\{w \in \Omega : H(w) = 1{,}77\}) = \mathbb{P}(\{Alberto, Franco\})$

If we use the probability measure we **introduced previously** then we have:

$$\mathbb{P}(\{\text{Alberto, Franco}\}) = \frac{\mu(\{Alberto, Franco\})}{\mu(\{Alberto, Franco, Gustavo\})} = \frac{2}{3}$$

Distribution Functions:

If a random variable $X : \Omega \to \mathbb{R}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given, we can ask questions like "How likely is it that value of X is equal to 2?". This is the as the probability of the event $\{w \in \Omega : X(w) = 2\}$ which is often written as $\mathbb{P}(X = 2)$ or $p_X(2)$ for short.

If X is real-valued, we can always captured its cumulative distribution function:

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\{w \in \Omega : X(w) \leq x\})$$

Observation or realization:

Observation, realization or observed value of a random variable is the value that is actually observed (what actually happened). The random variable itself is the process dictating how the observation comes about.

$$x = X(w)$$

When we have an observation X_i from a random variable $X \sim D(u, \sigma)$ then $X_i \overset{\text{iid}}{\sim} D(\mu, \sigma)$ that is **every observation have the same distribution of the random variable.**

We can also say every observation is also a random variable.

Bayes' Theorem:

In probability theory Bayes' theorem (alternatively Bayes' law or Bayes' rule) describes the probability of an event, based on prior knowledge of conditions that might be related to the event.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)}$$

Where A and B are events ($A, B \subseteq \Omega$) and $\mathbb{P}(B) \neq 0$.

Applications:

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad f_{Y|X=x}(x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{X|Y=y}(x) = \frac{f_{Y|X=x}(x) \cdot f_X(x)}{f_Y(y)}$$

Law of total probability:

Discrete case:

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i)$$

Continuous case:

$$\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A|X = x) \cdot f_X(x) dx$$

Applications ML:

Linear regression:

We consider X as data, $\epsilon \sim N(0, \sigma^2)$ as a random variable. We propose the next **poblational modeling**:

$$Y = \beta \cdot X + \alpha + \epsilon$$

$Y \sim N(\beta \cdot X + \alpha, \sigma^2)$

Note Y and ϵ are **random variables**.

Also from the previous equation we can deduce:

$$\mathbb{E}[Y|X] = \beta \cdot \mathbb{E}[X] + \alpha$$

Note because the **poblational modeling** we assume known the values of β and α.

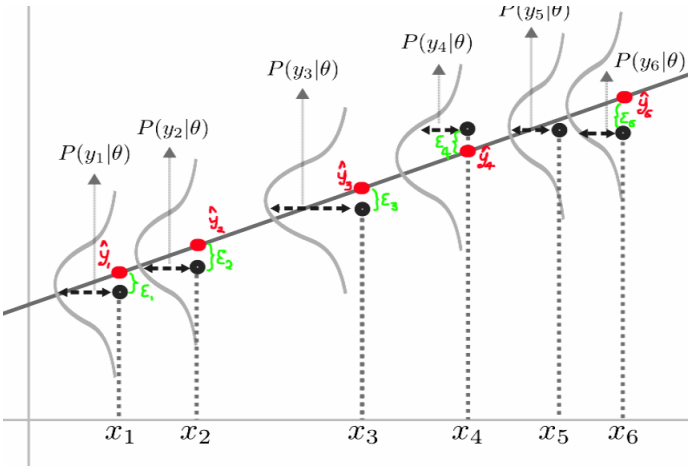
If we have n **observations**, we have the next:

$$y_i = \beta \cdot x_i + \alpha + \epsilon_i$$

Where $\epsilon_i \overset{iid}{\sim} N(0, \sigma^2), y_i \overset{iid}{\sim} N(\beta \cdot x_i + \alpha, \sigma^2)$ and x_i are data.

Note in this case β and α are unknown and we have to **estimate it**.

We can see normal distributions over the straight. Of course assuming known the values β and α, for this reason on the image uses $P(y_1|\theta)$.



Again, we have:

$$\mathbb{E}[y_i|x_i, \alpha, \beta] = \beta \cdot \mathbb{E}[x_i] + \alpha$$

Note: How ϵ_i are observation from ϵ we use it to estimate σ^2.

Time series:

In time series when we talk about $\{X_t\}_{t=1,\dots,T}$ these **aren't observations**, but **random variables** and for each random variable generally **we only have** 1 **observation**.

For example let's consider the autoregressive model:

$$X_t = \phi \cdot X_{t-1} + \epsilon_t$$

where $\epsilon_t \sim N(0, \sigma^2)$.

We can see clearly that $\{\epsilon_t\}_{t=1,...,T}$ are different random variables too.

Notes:

- Despite $\epsilon_1, , ..., \epsilon_T$ are different random variables and not observations, similarly we use it to estimate σ^2 because all these random variables have the **same distribution** and also are **independent**.
- Analogously, when we have a stationary time series we use for example the observations $X_1^1, X_2^1, X_3^1, X_4^1, X_5^1$ from the random variables X_1, X_2, X_3, X_4, X_5 to estimate $corr(X_{t+1}, X_t)$ because:

$$corr(X_5, X_4) = corr(X_4, X_3) = corr(X_3, X_2) = corr(X_2, X_1)$$

By the way, we estimate $corr(X_{t+1}, X_t)$ in the following way:

$$corr\left(\begin{bmatrix} X_2^1 \\ X_3^1 \\ X_4^1 \\ X_5^1 \end{bmatrix} \begin{bmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \\ X_4^1 \end{bmatrix}\right)$$
