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Multidimensional Scaling

We have n vectors $X_1, X_2, ..., X_n$ of dimension r.

We want a representation of every vector by a vector of dimension t.

How will we do that?

First of all we have to define the distance matrix $\delta \in \mathbb{R}^{n \times n}$.

$$\boxed{B = H \cdot \delta^{(2)} \cdot H} \qquad \boxed{\delta_{i,j}^{(2)} = (\delta_{i,j})^2} \qquad \boxed{H = I_n - \frac{1}{n} \cdot J_n} \qquad \boxed{J_{n(i,j)} = 1}$$

You can see that B is symmetric and positive definite. We can do a descomposition on its singular values.

$$\boxed{B = V \cdot \Delta \cdot V^T}$$

$$V \in \mathbb{R}^{n \times n}, \Delta \in \mathbb{R}^{n \times n}$$

$$V = \begin{bmatrix} | & | & \vdots & | \\ v_1 & v_2 & \vdots & v_n \\ | & | & \vdots & | \end{bmatrix} \qquad \Delta = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

If $rg(B) = \hat{r}$ then we can define:

$$B = V_{\hat{r}} \cdot \Delta_{\hat{r}} \cdot (V_{\hat{r}})^T$$

$$V \in \mathbb{R}^{n \times \hat{r}}, \Delta \in \mathbb{R}^{\hat{r} \times \hat{r}}$$

$$V_{\hat{r}} = \begin{bmatrix} | & | & \vdots & | \\ v_1 & v_2 & \vdots & v_{\hat{r}} \\ | & | & \vdots & | \end{bmatrix} \qquad \Delta_{\hat{r}} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{\hat{r}} \end{bmatrix}$$

$$Y = V_{\hat{r}} \cdot \Delta_{\hat{r}}^{1/2}$$

$$Y = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{bmatrix}$$

$y_1 \in \mathbb{R}^{\hat{r}}, y_2 \in \mathbb{R}^{\hat{r}}, ..., y_n \in \mathbb{R}^{\hat{r}}$

We want to find $\widetilde{B}$ of range $t \leq \hat{r}$ very similar to B.

$$\boxed{tr_{rg(\widetilde{B})=t} \left[(B - \widetilde{B}) \cdot (B - \widetilde{B})^T \right] \approx 0}$$

The solution of this problem is:

$$V_t = \begin{bmatrix} | & | & \vdots & | \\ v_1 & v_2 & \vdots & v_t \\ | & | & \vdots & | \end{bmatrix} \qquad \Delta_t = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_t \end{bmatrix}$$
