Kinematics

ME 416 - Prof. Tron

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1 Parametrized curves

In robotics, the quantities of interest (e.g., a point in the environment, or the pose of a robot) more often than not vary with time. To represent these quantities we will use *parametrized curves*. Parametrized curves are functions that take in a parameter (typically time t), and return a value for the quantity of interest.

For instance, we can represent the trajectory of a moving point x as $x(t)$. To give a more specific example, $x(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ represents the motion of a point in a 2-D circle.

By using functions, we can apply the same method to any quantity, and define the "trajectories" of quantities that are not simple geometric points. For instance, we can define a time-varying rotation $R(t)$, or pose $(R(t), T(t))$, or some scalar function $V(t)$ (which could represent, for instance, some kind of "energy" in the system).

Note that the argument "(t)" is often omitted, and the fact that a quantity depends on time or not is deduced from the context.

2 Velocities

The velocity of a quantity is defined as the time derivative of a parametrized curve. It is customary to use the notation: $\dot{x} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$.

Note that the time derivative is applied entry-wise; if is a point (more precisely, a parametrized curve in, say, $\mathbb{R}^2$), then is a vector of the same dimensions (i.e., another parametrized curve).

This definition of velocity can be then applied consistently on any quantity; in particular, we can apply it to vectors and rotations.

2.1 ★★: *Geometrical meaning of velocities for vectors.*

For any given path or curve, the velocity at a given time, $\dot{x}(t)$ is the tangent to the curve at that time t. As shown in Figure 1 the velocity $v = \dot{x}$ at any point x along the path is the tangent to the curve.

★: Rigid body transformations for velocities The technical definition of a (right) derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lim_{h \to \infty} \frac{x(t+h) - x(t)}{h}, \tag{1}$$

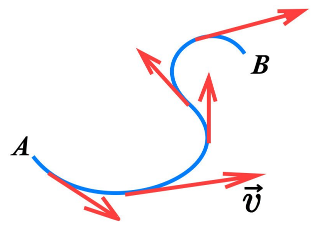

Figure 1: Example of velocity of a curve. The velocity is always tangent to the parametrized curve.

i.e., is a limit of a difference between two points. Hence, velocities are vectors, and are affected only by the rotation component of a rigid body transformation. More precisely, if $({}^{\mathcal{W}}R_{\mathcal{B}}, {}^{\mathcal{W}}T_{\mathcal{B}})$ represents the rigid body transformation from a frame $\mathcal{B}$ to a frame $\mathcal{W}$, we have that the velocity of a point expressed in $\mathcal{W}$ is related to the one expressed in $\mathcal{B}$ by the relation:

$$ {}^{\mathcal{W}}\dot{x} = {}^{\mathcal{W}}R_{\mathcal{B}}\, {}^{\mathcal{B}}\dot{x}. \tag{2} $$

2.2 ★★: Velocities for rotations

Our definition of velocity can be then applied consistently on any quantity; in particular, we can apply it to rotations.

Given that rotations are always orthonormal matrices, their velocities have a particular structure, which can be written as: $\dot{R}(t) = \hat{w}(t)R(t)$, where $\hat{w}$ (pronounced "hat w") is always a skew-symmetric matrix (i.e., $\hat{w} = -\hat{w}^{\mathrm{T}}$). The matrix $\begin{smallmatrix} 0 & 2 & -1 \\ -2 & 0 & -4 \\ 1 & 4 & 0 \end{smallmatrix}$ is an example of a skew-symmetric matrix.

2.2.1 2-D rotations

For 2-D rotations, the matrix $\hat{w}$ takes the form of: $\hat{w} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$. This can be written as $\hat{w} = \omega S$ where S is the matrix $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The scalar variable ω represents the 2-D *angular velocity*

Note: the matrix S is itself a rotation (of 90°); this is a coincidence, but it is useful.

2.2.2 3-D rotations

For 3-D rotations, the matrix $\hat{w}$ takes the form of: $\hat{w} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ -\omega_2 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$. The 3-D vector $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$ represents the 3-D *angular velocity*. This vector represents the instantaneous axis of rotation, weighted with the angular speed (this is similar to the angle/axis representation of rotations). Each entry represents a rate of rotation along one of the axis of the frame (see next subsection).

2.3 Notes on the dimension and reference frames for angular velocities

As discussed before, 2-D rotations can be represented with a single angle, and so the velocity is also one-dimensional. On the other hand, 3-D rotations need to be represented with three numbers, collected in the vector w. Since w is a vector, it requires the specification of a reference frame in order to be meaningful (e.g., angular velocities in body coordinates are in general different from angular velocities in world coordinates). With the definition $\dot{R} = \hat{w}R$ used above, the reference frame of is the "left superscript"; for instance if ${}^{\mathcal{W}}R_{\mathcal{B}}$ represents a rotation from $\mathcal{B}$ to $\mathcal{W}$, then w is expressed in the reference frame $\mathcal{B}$, i.e., we have $\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{W}}R_{\mathcal{B}} = {}^{\mathcal{W}}\dot{R}_{\mathcal{B}} = {}^{\mathcal{W}}\hat{w}\,{}^{\mathcal{W}}R_{\mathcal{B}}$.

3 Accelerations

The acceleration of a quantity is the second order derivative of the parametrized curve, denoted as: $\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) = \ddot{x}(t)$.

Consider a second order ODE of the form $\ddot{x}(t) = f(t)$. We can write this (and any) 2nd order ODE as a 1st order ODE. In equation form, this is can be done by introducing the variable $v(t) = \dot{x}$, and the new overall state $z = \begin{bmatrix} x \\ v \end{bmatrix}$. We then have $\dot{z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$, which is a first-order ODE.