Coordinate vectors, coordinate bases, change of coordinates

ME 416 - Prof. Tron

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1 Coordinate vectors and coordinate bases

Consider a *coordinate vector* representing a position in 3-D space, say $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Each entry of the vector represents a *coordinate* in a direction (e.g., up, right, forward).

★★Note: $\mathbb{R}^d$ represents the space of all vectors of d real numbers.

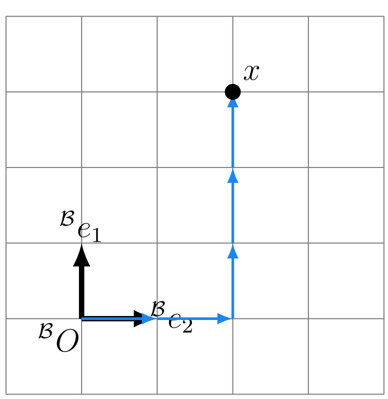
However, the coordinates in x are physically meaningless unless we provide also a *coordinate system* $\mathcal{B}$ in which they can be expressed.

★★Note: there is a subtle difference between *points* and *vectors*, intended as geometric entities. Points identify a single location in space. Vectors represent the displacement of point to another point, i.e., they need to be attached to a "base point".

A coordinate system $\mathcal{B}$ is defined by an *origin* point ${}^{\mathcal{B}}O$ (or *center*) and a set of linearly independent (geometric) vectors $\{{}^{\mathcal{B}}e_1, {}^{\mathcal{B}}e_2, \dots, {}^{\mathcal{B}}e_d\}$.

★★★Note: Two vectors v_1, v_2 are said to be *linearly independent* if the only coefficients $a_1, a_2 \in \mathbb{R}^d$ such that $a_1 v_1 + a_2 v_2 = 0$ are $a_1 = a_2 = 0$. This notion can be extended to any number of vectors. More technically, vectors $\{v_i\}_{i=1}^d$ are linearly independent if $\sum_{i=1}^d a_i v_i = 0 \iff a_i = 0 \forall i$.

The physical meaning of a coordinate vector ${}^{\mathcal{B}}x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is then given by ${}^{\mathcal{B}}x \equiv {}^{\mathcal{B}}O + 2\,{}^{\mathcal{B}}e_1 + 3\,{}^{\mathcal{B}}e_2$, where the scalings and the summations are intended in a geometric sense.



Left superscripts: We use the left superscript to denote the reference frame in which a quantity is expressed. This can be omitted if we are dealing with a single coordinate system, but it helps when we have two or more coordinate systems (e.g., one for a fixed environment, and one for a robot).

2 Euclidean changes of coordinates

A single physical point can be seen under different perspectives, i.e., it can be expressed in different coordinate systems. With a mobile robot, this situation happens commonly, as it is usual to have an "absolute" reference system $\mathcal{W}$ which is fixed on the environment (world), and another one $\mathcal{B}$ that is attached to the mobile robot. For instance, a physical point can have coordinates ${}^{\mathcal{B}}x = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ in $\mathcal{B}$, and coordinates ${}^{\mathcal{W}}x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in $\mathcal{W}$ (note: for simplicity, we are considering here a 2-D world, and a_1, a_2, b_1, b_2 are all real numbers).

2.1 Changing coordinates: rigid body transformations and poses

The question now is: what is the relation between the numbers (more precisely, coordinates) in ${}^{\mathcal{W}}x$ and ${}^{\mathcal{B}}x$? From now on, we assume that the axes of both $\mathcal{W}$ and $\mathcal{B}$ are *orthonormal* (i.e., they all have norm one and they are orthogonal to each other) and right-oriented (i.e., the ordering of the axes follows the right-hand rule). Then, the coordinates of a same physical point are related one to the other by the *rigid body transformation*:

$$ {}^{\mathcal{W}}x = {}^{\mathcal{W}}R_{\mathcal{B}}\, {}^{\mathcal{B}}x + {}^{\mathcal{W}}T_{\mathcal{B}}, \tag{1} $$

where ${}^{\mathcal{W}}R_{\mathcal{B}}$ is a $d \times d$ *rotation matrix* (or simply, *rotation*) and ${}^{\mathcal{W}}T_{\mathcal{B}}$ is a *translation vector* (or simply, *translation*). The name *rigid body transformation* comes from the fact that this transformation preserves all the angles and distances between points.

A *pose* is the tuple $({}^{\mathcal{W}}R_{\mathcal{B}}, {}^{\mathcal{W}}T_{\mathcal{B}})$ containing the parameters (rotation and translation) defining a rigid body transformation.

In the example above, this rotation and translation pair represents unambiguously the pose of the robot in the environment.

★★★★: TODO: discuss equivalence between rigid transformations and the requirement that all distances are preserved (see Ma's book).

In summary, you can think of rigid transformations as a "machine" (function) where you feed in coordinates in one system (frame) and you get out coordinates (of the same point) in the other system. The pair of rotation and translation in the pose are "settings" (parameters) of this function that completely describe how to perform the transformation.

★★★TODO: homogeneous coordinates and matrix representation of poses.

2.2 Physical interpretation of a pose

Given the pose $({}^{\mathcal{W}}R_{\mathcal{B}}, {}^{\mathcal{W}}T_{\mathcal{B}})$, it is possible to use some particular cases and test points to understand what is the physical meaning of the rotation and translation components of the pose.

contains the coordinates of the origin of expressed in (you can see this by setting to be the vector of all zeros. Similarly, the -th column of is contains the coordinates of one of the -th axis, , expressed in (you can see this by setting to be a vector of all zeros except for the -th entry, which is set to one).

2.2.1 Identity transformation

Assume that $({}^{\mathcal{W}}R_{\mathcal{B}}, {}^{\mathcal{W}}T_{\mathcal{B}}) = (I, 0)$, where I is the *identity matrix*, i.e., the matrix that contains ones on its diagonal and zeros everywhere else. Then the relation between ${}^{\mathcal{W}}x$ and ${}^{\mathcal{B}}x$ reduces to

$$ {}^{\mathcal{W}}x = {}^{\mathcal{W}}R_{\mathcal{B}}\,{}^{\mathcal{B}}x + {}^{\mathcal{W}}T_{\mathcal{B}} = {}^{\mathcal{B}}x, \tag{2} $$

i.e., points have the same coordinates in either $\mathcal{B}$ or $\mathcal{W}$. From the considerations above, this means that the two reference frames coincide (same origin, same axes).

2.3 More about rotations

Assume for the moment that ${}^{\mathcal{W}}T_{\mathcal{B}} = 0$ (i.e., no translation). The relation between ${}^{\mathcal{W}}x$ and ${}^{\mathcal{B}}x$ then reduces to ${}^{\mathcal{W}}x = {}^{\mathcal{W}}R_{\mathcal{B}}\,{}^{\mathcal{B}}x$. To understand better what the entries of ${}^{\mathcal{W}}R_{\mathcal{B}}$ mean, let us set ${}^{\mathcal{B}}x$ to be a vector of all zeros, except for a 1 in the i-th element (e.g., ${}^{\mathcal{B}}x = \begin{bmatrix}1\\0\end{bmatrix}$, or ${}^{\mathcal{B}}x = \begin{bmatrix}0\\1\end{bmatrix}$ for a 2-D environment). This implies two things:

1) ${}^{\mathcal{B}}x$ corresponds to the i-th local axis ${}^{\mathcal{B}}e_i$.

2) By the properties of matrix multiplication, we have that ${}^{\mathcal{W}}x$ is equal to the i-th column of ${}^{\mathcal{W}}R_{\mathcal{B}}$.

As a result, *the i-th column of ${}^{\mathcal{W}}R_{\mathcal{B}}$ corresponds to the i-th axis ${}^{\mathcal{B}}e_i$ of the "source" frame expressed in the "target" frame $\mathcal{W}$.*

With the assumption that the frames are orthonormal, any generic rotation R will have have the following properties:

1) $RR^{\mathrm{T}} = R^{\mathrm{T}}R = I$

2) The inverse is simply the transpose, i.e., $R^{-1} = R^{\mathrm{T}}$.

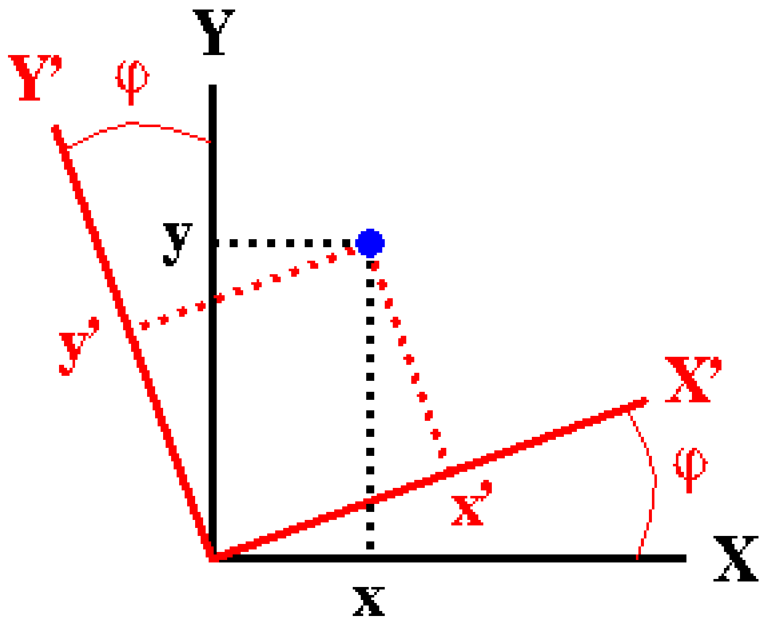
3) By assuming that both reference frames are right-handed, the determinant is one, i.e., $\det(R) = 1$.

4) The identity matrix corresponds to the "no rotation" case.

★★★: in the definition above, both the definition of "norm" and "orthogonal" are associated to the notion of angles between vectors, and, more precisely, a definition of an *inner product* on the space.

2.4 2-D rotations

A 2-D rotation can be written as $R = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ sin(\theta) & \cos(\theta)\end{bmatrix}$. The angle θ represents the angle of rotations of the axes. Hence, we can uniquely pass from a rotation matrix R to an angle θ. As a special case $R = I$ corresponds to $\theta = 0$.



Commutativity Note that 2-D rotations always commute, that is $R_1 R_2 = R_2 R_1$. This is not true for general matrices or for 3-D rotations.

2.5 3-D rotations

A 3-D rotation can be expressed as generalization of a 2-D rotation by first specifying an axis ω (assumed to have norm one, $\|w\| = 1$), and then an angle of rotation θ as in the 2-D case. For the case of no rotation ($R = I$), the angle is zero, $\theta = 0$, and the axis is not well defined (i.e., it can be any norm-one vector).

★★★: The definition of the axis w is not unique when $\theta = \pi$.

Commutativity Note that 3-D rotations in general do not commute, that is, $R_1 R_2 \neq R_2 R_1$. However, they always commute if the have the same axis of rotations (i.e., all rotations happen in the same plane).

2.6 Inverse of a rigid body transformation

Given a pose $({}^{\mathcal{W}}R_{\mathcal{B}}, {}^{\mathcal{W}}T_{\mathcal{B}})$, its inverse is $({}^{\mathcal{W}}R_{\mathcal{B}}, {}^{\mathcal{W}}T_{\mathcal{B}})^{-1} = ({}^{\mathcal{B}}R_{\mathcal{W}}, {}^{\mathcal{B}}T_{\mathcal{W}}) = ({}^{\mathcal{W}}R_{\mathcal{B}}^{\mathrm{T}}, -{}^{\mathcal{W}}R_{\mathcal{B}}^{\mathrm{T}}\, {}^{\mathcal{W}}T_{\mathcal{B}})$ (with a stripped down notation, this is $(R, T)^{-1} = (R^{\mathrm{T}}, -R^{\mathrm{T}}T)$).

2.7 ★★Composition of rigid body transformations

Given three reference frames $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{W}$, consider the two rigid body transformations from $\mathcal{B}_2$ to $\mathcal{B}_1$, $({}^{\mathcal{B}_1}R_{\mathcal{B}_2}, {}^{\mathcal{B}_1}T_{\mathcal{B}_2})$, and from to $\mathcal{B}_1$ to $\mathcal{W}$, $({}^{\mathcal{W}}R_{\mathcal{B}_1}, {}^{\mathcal{W}}T_{\mathcal{B}_1})$. The transformation from $\mathcal{B}_2$ to $\mathcal{W}$, $({}^{\mathcal{W}}R_{\mathcal{B}_2}, {}^{\mathcal{W}}T_{\mathcal{B}_2})$, can be obtained with the composition:

$$\begin{aligned} {}^{\mathcal{W}}R_{\mathcal{B}_2} &= {}^{\mathcal{W}}R_{\mathcal{B}_1}\, {}^{\mathcal{B}_1}R_{\mathcal{B}_2}, \\ {}^{\mathcal{W}}T_{\mathcal{B}_2} &= {}^{\mathcal{W}}T_{\mathcal{B}_2} + {}^{\mathcal{W}}R_{\mathcal{B}_1}\, {}^{\mathcal{B}_1}T_{\mathcal{B}_2}. \end{aligned} \tag{3}$$

As a sanity check on the formula, note that the superscripts and subscripts "cancel out" in the right way to give the answer. Please see also in-class activity.

TODO: figure