Proportional-Integral-Derivative (PID) control

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The most common goal of feedback control is to create a controller $u(x)$ such that the closed loop $\dot{x} = f(x, u(x))$ has an asymptotically stable equilibrium at x_0.

1 Proportional controller

Assume a system with simple integrator dynamics, $\dot{x} = u$ and a one-dimensional state $x \in \mathbb{R}$, and assume that we want to asymptotically stabilize the system to a constant reference point r (i.e., the point $x_0 = r$ should be an asymptotic equilibrium).

Intuitively, if $x > r$, then we would like $u < 0$, and if $x < x_0$, we would like $u > 0$.

Define the **error signal** $e = x - r$. Then one possible choice for u is $u = -k_p e$, where $k_p > 0$ is a **proportional control gain** (or simply *gain* for short).

We now show that $x_0 = r$ is an asymptotically stable equilibrium of the closed-loop system, that is, $\lim_{t\to\infty} x(t) = r$. More precisely, we will show the equivalent statement $\lim_{t\to\infty}(x(t) - r) = \lim_{t\to\infty} e(t) = 0$.

Computing the derivative of the error signal, we have $\dot{e} = \dot{x}$ (since $\dot{r} = 0$, because r is assumed constant).

Substituting $\dot{x} = u$, we have $\dot{e} = -k_p e$. The solution to this ODE is $e(t) = E_0 \exp -k_p t$ (where E_0 is a constant depending on the initial state $x(0)$). It is easy to verify $\lim_{t\to\infty} e(t) = 0$, as long as $k_p > 0$.

1.1 Considerations on the choice of k_p

Notice that a large k_p makes the system converge faster, but the magnitude of the control signal (sometimes called the *control effort*) is also increased.

TODO: figures of $e(t)$ and $\|u\|(t)$ for high/low values of k_p.

Additionally, if there are delays in the system (e.g., if the reference r is sensed with some delay due to a sensor processing algorithm), the system might become unstable for high values of k_p.

1.2 Moving reference

If the reference signal r is not constant, in general $e(t)$ will not converge to $r(t)$, but it will always "lag behind".

However, if we try to still obtain the relation $\dot{e} = -k_p e$ (to achieve convergence), we obtain $\dot{x} - \dot{r} = -k_p e$, which suggests the control law $u = -k_p e + \dot{r}$, where $\dot{r}$ is a *feed-forward* term. Notice that this technique requires knowing the derivative of the reference signal.

Intuitively, the feed-forward signal helps the controller to perfectly anticipate the changes in the reference signal r.

2 Proportional-Derivative controller

If we do not have access to the derivative $\dot{r}$, we can try to approximate it with the derivative of the error signal. This gives rise to a **proportional-derivative controller** where $u = -k_p e - k_d \dot{e}$, where . In practice, $\dot{e}$ is often approximated using finite differences, as in the Euler method.

Intuitively, the derivative makes the controller more "reactive" to changes in the error. This can be particularly effective to counteract the effects of delays in the system. At the same time, however, any spurious deviation due to noise will be amplified.

3 Proportional-Integral controller

Assume again that we have a constant reference frame. However, at the same time, assume also that we have an unknown constant offset in the control, i.e., the control actually perceived by the system u is given by the control that we design u' plus an offset Δ, $u = u' + \Delta$.

TODO: give example with ROSBot

Integral control is a "trick" that allows us to control the system to the reference constant signal r despite not knowing Δ.

We introduce a new variable ξ, and we set $\dot{x}i = e$. In practice, ξ keeps track of the integral of the error (if $\xi(0) = 0$, then $\xi(t) = \int_0^t e(\tau)\tau$). We then modify our control law to $u = -k_p e - k_i \xi$, where k_i is the *integral control gain*. The dynamics of the closed loop system is then given by: \$ $\dot{z} = \begin{bmatrix} -k_p e + \Delta - k_i \xi \\ e \end{bmatrix}$, where $z = \begin{bmatrix} x \\ \xi \end{bmatrix}$.

Consider now the equilibria of the system, given by the condition $\dot{z} = 0$. This condition implies, in particular, that $\dot{x}i = 0$, i.e., $e = 0$ (therefore, at equilibrium, the $x(t) = r$). In addition, we have $\dot{x} = 0$, which implies $-k_p e + \Delta - k_i \xi = 0$, which leads to $\xi = \frac{\Delta}{k_i}$.

Intuitively, the integrator automatically discovers the offset, which is then compensated so that the error is zero.

Analysis of integral control is not trivial, but there are a few qualitative considerations that we can make:

- Since the our controller has dynamics, it might introduce instabilities.
- A common problem is integrator windup, where the temporary large changes in Δ are accumulated and then released, leading to large overshoots in the controlled system.

TODO: insert example of water heater

4 Proportional-Integrative-Derivative control

The three strategies are often implemented together into a Proportional-Integrative-Derivative (PID) controller.

TODO: image with block diagram of PID control.

PID control is one of the most common approaches used in industrial processes, and it can usually be tuned to work with reasonably good performance. There are some guidelines on how to tune (adjust the gains) of the controller, however in practiceHowever, it performs best with simple systems with linear or "almost-linear" dynamics. For more complex systems (e.g., large state spaces, or non-linear dynamics), more advanced control paradigms are necessary.