Example applications of kinematics

ME 416 - Prof. Tron

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1 Translating and spinning disk

Consider a 2-D disk, with radius r, that rotates and translates, as depicted in Figure **??** for a given time (say, $t = 0$). Consider a world reference frame $\mathcal{W}$ (inertial), and a body-fixed reference frame $\mathcal{B}$ attached at the center of the disk. Let P be the point at the edge of the disk directly below ${}^{\mathcal{B}}O$ at the depicted time. Let ${}^{\mathcal{W}}R_{\mathcal{B}} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}$, and ${}^{\mathcal{W}}T_{\mathcal{B}} = \begin{bmatrix} st \\ 0 \end{bmatrix}$, where ω, s are two scalar constants.

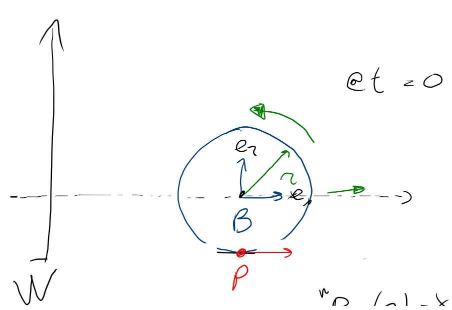
Angular and linear velocities of the disk Taking derivatives, we can compute the rotation and translation velocities

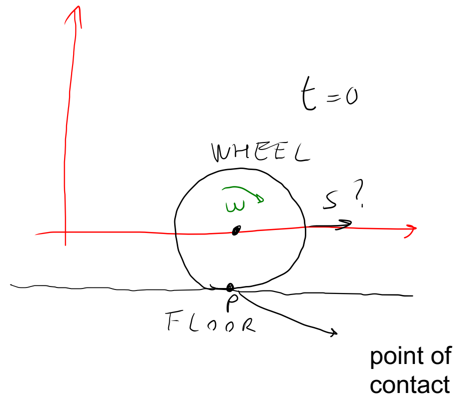
$$ {}^{\mathcal{W}}\dot{R}_{\mathcal{B}} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} {}^{\mathcal{W}}R_{\mathcal{B}} = \omega \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S} {}^{\mathcal{W}}R_{\mathcal{B}}, \tag{1} $$

$$ {}^{\mathcal{W}}\dot{T}_{\mathcal{B}} = \begin{bmatrix} s \\ 0 \end{bmatrix}. \tag{2} $$

Hence, ω and s are, respectively, the angular and linear speeds of the disk.

Velocity of the point P expressed in $\mathcal{W}$ The coordinates of the point P in the body frame $\mathcal{B}$ are ${}^{\mathcal{B}}P = \begin{bmatrix} 0 \\ -r \end{bmatrix}$. Using the rigid body transformation formula, the coordinates of





the same point in P are given by:

$$
{}^{\mathcal{W}}\dot{P} = {}^{\mathcal{W}}\dot{R}_{\mathcal{B}}\,{}^{\mathcal{B}}P + {}^{\mathcal{W}}R_{\mathcal{B}}\,{}^{\mathcal{W}}\dot{P} + {}^{\mathcal{W}}\dot{T}_{\mathcal{B}} = \omega S\,{}^{\mathcal{W}}R_{\mathcal{B}}\,{}^{\mathcal{B}}P + \begin{bmatrix} s \\ 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -r \end{bmatrix} + \begin{bmatrix} s \\ 0 \end{bmatrix}
$$
$$
= \omega \begin{bmatrix} r \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \end{bmatrix} = \begin{bmatrix} \omega r + s \end{bmatrix}. \quad (3)
$$

Note that in the second equality we used the fact that P has constant coordinates in $\mathcal{B}$ (i.e., ${}^{\mathcal{B}}\dot{P} = 0$); in the third equality, we substituted $t = 0$. The velocity of the point P at time $t = 0$ is therefore $\omega r + s$ along the x-axis of $\mathcal{W}$. This makes sense: if $\omega = 0$, $s > 0$, the disk is translating forward, and so is P; if $\omega > 0$, $s = 0$, the disk is rotating in place and, since P is on the lowest point of P, its tangential velocity is in the direction of the the x axis.

Wheel with no-slip constraints We can use the result above to determine the linear speed of the disk (s) when it is constrained to rotate, without slipping, on a ground plane placed at $y = -r$ (see Figure **??**).

At the depicted time $t = 0$, the point P represents the point of contact between the ground and the disk. The no-slip constraints imposes that ${}^{\mathcal{W}}\dot{P} = 0$. Comparing this with (**??**), we have:

$$
\begin{bmatrix} \omega r + s \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (4)
$$

from which we obtain $s = -\omega r$.

2 Kinematics of a differential drive robot

In this section, we will obtain the kinematic model for a differential drive robot (a robot with two wheels whose speed can be controlled separately). Figure **??** illustrates the model that we will use (see the caption for definitions of the symbols used. The final goal is to compute the angular and linear velocities of the robots from the commanded speeds given to the motors.

We will use an inertial world reference frame $\mathcal{W}$ and a body-fixed $\mathcal{B}$. The wheels are placed on the y-axis of $\mathcal{B}$, at locations ${}^{\mathcal{B}}x_{LW} = \begin{bmatrix} 0 \\ d \end{bmatrix}$ and ${}^{\mathcal{B}}x_{RW} = \begin{bmatrix} 0 \\ -d \end{bmatrix}$ for, respectively, the left and right wheels.

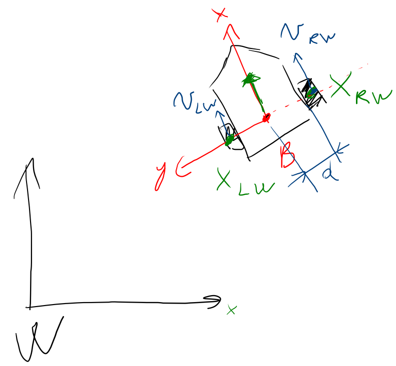


Figure 1: Model of the robot. $\mathcal{W}$, $\mathcal{B}$: world and body reference frames. $x_{LW}, x_{RW} \in \mathbb{R}^2$: center of the left and right wheel. $v_{LW}, v_{RW} \in \mathbb{R}$: velocity of left and right wheel. $d \in \mathbb{R}$: distance between ${}^{\mathcal{B}}O$ and each wheel.

Each wheel is modeled as a point that moves with a velocity that has constant direction in the body frame $\mathcal{B}$ (parallel to the body x-axis); more precisely, the velocities are defined as ${}^{\mathcal{B}}v_{LW} = \begin{bmatrix} k_W s_{LW} \\ 0 \end{bmatrix}$ and ${}^{\mathcal{B}}v_{RW} = \begin{bmatrix} k_W s_{RW} \\ 0 \end{bmatrix}$ for the left and right wheel, respectively, where $s_{LW}, s_{RW} \in \mathbb{R}$ are speeds controlled by the onboard computer (assumed to be known), and k_W is a known constant lumping together the wheel radius and motor constants (i.e., the relation between commanded speed and actual speed). The rigid transformation $({}^{\mathcal{W}}R_{\mathcal{B}}, {}^{\mathcal{W}}T_{\mathcal{B}})$ from the body to the world frame is assumed to be known (that is, we can transform points and vectors from $\mathcal{B}$ to $\mathcal{W}$).

Velocity of the wheels in the world frame Our first step is to express the velocity vectors of the two wheels in the world frame. Since these are vectors, their transformation requires only the rotation component.

$$ {}^{\mathcal{W}}v_{LW} = {}^{\mathcal{W}}R_{\mathcal{B}} {}^{\mathcal{B}}v_{LW} = k s_{LW} {}^{\mathcal{W}}R_{\mathcal{B}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{5} $$

$$ {}^{\mathcal{W}}v_{RW} = {}^{\mathcal{W}}R_{\mathcal{B}} {}^{\mathcal{B}}v_{RW} = k s_{RW} {}^{\mathcal{W}}R_{\mathcal{B}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{6} $$

However, we have an alternative way to compute these same quantities from the angular and linear velocities of the robot. First, we compute the positions of the wheels in the world reference frame

$$ {}^{\mathcal{W}}x_{LW} = {}^{\mathcal{W}}R_{\mathcal{B}} {}^{\mathcal{B}}x_{LW} + {}^{\mathcal{W}}T_{\mathcal{B}}, \tag{7} $$
$$ {}^{\mathcal{W}}x_{RW} = {}^{\mathcal{W}}R_{\mathcal{B}} {}^{\mathcal{B}}x_{RW} + {}^{\mathcal{W}}T_{\mathcal{B}}. \tag{8} $$

We can then take their derivative:

$$ {}^{\mathcal{W}}\dot{x}_{LW} = {}^{\mathcal{W}}v_{LW} = {}^{\mathcal{W}}\dot{R}_{\mathcal{B}} {}^{\mathcal{B}}x_{LW} + {}^{\mathcal{W}}\dot{T}_{\mathcal{B}}, \tag{9} $$
$$ {}^{\mathcal{W}}\dot{x}_{RW} = {}^{\mathcal{W}}v_{RW} = {}^{\mathcal{W}}\dot{R}_{\mathcal{B}} {}^{\mathcal{B}}x_{RW} + {}^{\mathcal{W}}\dot{T}_{\mathcal{B}}. \tag{10} $$

Equating (**??**) with (**??**) and (**??**) with (**??**), we obtain a set of two pairs of equations:

$$k s_{LW}\, {}^{\mathcal{W}}R_{\mathcal{B}} \begin{bmatrix}1\\0\end{bmatrix} = {}^{\mathcal{W}}\dot{R}_{\mathcal{B}}\, {}^{\mathcal{B}}x_{LW} + {}^{\mathcal{W}}\dot{T}_{\mathcal{B}}, \tag{11}$$

$$k s_{RW}\, {}^{\mathcal{W}}R_{\mathcal{B}} \begin{bmatrix}1\\0\end{bmatrix} = {}^{\mathcal{W}}\dot{R}_{\mathcal{B}}\, {}^{\mathcal{B}}x_{RW} + {}^{\mathcal{W}}\dot{T}_{\mathcal{B}}. \tag{12}$$

This is a set of four equations (two vectors with two elements each) in three unknowns (one for the angular velocity, and two for the linear velocity).

Linear velocity Summing (**??**) with (**??**), and noticing that ${}^{\mathcal{B}}x_{LW} = -\,{}^{\mathcal{B}}x_{RW}$, we obtain:

$${}^{\mathcal{W}}\dot{T}_{\mathcal{B}} = \frac{k}{2}(s_{RW} + s_{LW})\, {}^{\mathcal{W}}R_{\mathcal{B}} \begin{bmatrix}1\\0\end{bmatrix}, \tag{13}$$

which is the expression of the linear velocity of the robot in the world frame.

As seen in the interpretation rotations, the expression ${}^{\mathcal{W}}R_{\mathcal{B}} \begin{bmatrix}1\\0\end{bmatrix}$ produced the first column of ${}^{\mathcal{W}}R_{\mathcal{B}}$, which represents the direction of the x-axis of $\mathcal{B}$ in $\mathcal{W}$. As a result, the linear velocity of the robot will always be "forward" (x-axis of $\mathcal{B}$), and the robot cannot move sideways, no matter what speeds are given to the two motors (this makes sense, given the no-slip conditions of the wheels).

Angular velocity Subtracting (**??**) from (**??**) we instead get:

$$k(s_{LW} - s_{RW})\, {}^{\mathcal{W}}R_{\mathcal{B}} \begin{bmatrix}1\\0\end{bmatrix} = 2\omega S\, {}^{\mathcal{W}}R_{\mathcal{B}} \begin{bmatrix}0\\d\end{bmatrix}, \tag{14}$$

where we expanded ${}^{\mathcal{W}}\dot{R}_{\mathcal{B}} = \omega S\, {}^{\mathcal{W}}R_{\mathcal{B}}$. Now, we use a couple of "tricks": we recall that S happens to be a valid 2-D rotation, and 2-D rotations commute. Hence, we switch the order of S and ${}^{\mathcal{W}}R_{\mathcal{B}}$:

$$k(s_{LW} - s_{RW})\, {}^{\mathcal{W}}R_{\mathcal{B}} \begin{bmatrix}1\\0\end{bmatrix} = 2\omega\, {}^{\mathcal{W}}R_{\mathcal{B}} S \begin{bmatrix}0\\d\end{bmatrix}. \tag{15}$$

Multiplying on both sides by ${}^{\mathcal{W}}R_{\mathcal{B}}^{\mathrm{T}}$, recalling that, for rotation matrices, ${}^{\mathcal{W}}R_{\mathcal{B}}^{\mathrm{T}}\, {}^{\mathcal{W}}R_{\mathcal{B}} = I$, and carrying out the last multiplication on the right hand side, we arrive to the equation:

$$k(s_{LW} - s_{RW}) \begin{bmatrix}1\\0\end{bmatrix} = -2d\omega \begin{bmatrix}1\\0\end{bmatrix}. \tag{16}$$

In this system of two equations, the second one is trivial ($0 = 0$), while the other gives us an expression for the angular velocity:

$$\omega = \frac{k}{2d}(s_{RW} - s_{LW}). \tag{17}$$

Together with (**??**), this equation gives all the quantities we were looking for.