# Introduction to feedback control

ME 416 - Prof. Tron

Friday 3rd May, 2019

We start with a dynamical system modeled by the differential equation $\dot{x} = f(x, u)$, where $x$ is the state of the system and $u$ is the input to the system.

The goal of **feedback control** is to design a vector field $u(x)$ which, given the current state $x$, provides the input $u$, so that the **closed-loop system** $\dot{x} = f(x, u(x))$ exhibits a desired behavior (e.g., converge to a prescribed point).

Note that the closed-loop equation $\dot{x} = f(x, u(x))$ is an ODE.

TODO: put link to page on Dynamical Systems.

DEF: a point $x_0$ is an **equilibrium** of an ODE $\dot{x} = f(x)$ if $f(x_0) = 0$.

Intuitively, if the system starts at $x(0) = x_0$, then $\dot{x} = 0$, and the state will remain fixed to that equilibrium point.

EXAMPLE: in a pendulum, there are two equilibria: the "down" position (with the pendulum's weight below the fulcrum) and the "up" position (with the weight above the fulcrum)

TODO: put drawings of down and up equilibria in the pendulum.

Given any system $\dot{x} = f(x)$, finding the set of equilibria is equivalent to finding the set of solutions to the equation $f(x) = 0$.

Despite the fact that trajectories at an equilibrium remain at that equilibrium, trajectories that pass near that equilibrium might converge or diverge. This is captured by the notion of **stability**.

DEF: an equilibrium $x_0$ is said to be **Lyapunov stable** (or simply **stable**) if there exist positive constants $\delta, \epsilon$ such that, if $x(0)$ is in a ball of radius $\delta$, centered at $x_0$, then $x(t)$ remains in a ball of radius $\epsilon$, also centered at $x_0$, for all $t > 0$. In symbols: $\exists \delta, \epsilon > 0 : x(0) \in B_\delta(x_0) \implies x(t) \in B_\epsilon(x_0) \forall t > 0$.

Intuitively, the definition means that a trajectory starting near a Lyapunov stable equilibrium, stays around that equilibrium.

EXAMPLE: The "down" equilibrium of a friction-less pendulum is Lyapunov stable. For any trajectory starting near this position, the pendulum will start oscillating but without ever going beyond the initial angle.

DEF: an equilibrium $x_0$ is **unstable** if it not stable.

EXAMPLE: The "up" equilibrium of a pendulum is unstable.

Intuitively, an equilibrium is unstable if any minimal perturbation makes the trajectory go away from the that equilibrium.

DEF: an equilibrium $x_0$ is **asymptotically stable** if there exists a positive constant $\delta$ such that, if $x(0)$ is in a ball of radius $\delta$ centered at $x_0$, the trajectory converges, in the limit, to $x_0$.

EXAMPLE: The "down" equilibrium of a pendulum *with friction* is asymptotically stable. Any trajectory that does not start from the "up" equilibrium, will converge to that equilibrium.

In symbols: $\exists \delta > 0 : x(0) \in B_\delta(x_0) \implies \lim_{t\to\infty} x(t) = x_0$.

DEF: an equilibrium $x_0$ is **globally asymptotically stable** if, in the definition, we can take $\delta = \infty$.

Level **: TODO: exponentially stable, uniformly * stable.