1. Consider the grammar $G=(V,\Sigma,R,S)$, where

$$V=\{a,b,S,A\},$$
$$\Sigma=\{a,b\},$$
$$R=\{S\to AA,$$
$$A\to AAA,.$$
$$A\to a,$$
$$A\to bA,$$
$$A\to Ab\}.$$

Write all the strings of $L(G)$ whose length is at most 4.

2. Consider the context free grammar $G=(V,\Sigma,R,S)$, where

$$V=\{a,b,S,A,B\},$$
$$\Sigma=\{a,b\},$$
$$R=\{S\to aB,$$
$$S\to bA,$$
$$A\to a,$$
$$A\to aS,$$
$$A\to BAA,$$
$$B\to b,$$
$$B\to bS,$$
$$B\to ABB\}.$$

Show that $ababba\in L(G)$.

$$S\Rightarrow aB\Rightarrow|\Rightarrow|abaB\Rightarrow ababS\Rightarrow ababbA\Rightarrow ababba$$

3. Construct context-free grammars that generate each of these language

a). $\{ww^R : w\in\{a,b\}^{¿}\}$

b). $\{w\in\{a,b\}^{¿} : w=w^R\}$

A: $a¿.\{ww^R : w\in\{a,b\}^{¿}\}$

$$S\to aSa$$
$$S\to bSb$$
$$S\to e$$

b). $\{w\in\{a,b\}^{¿} : w=w^R\}$

$S\to aSa$
$S\to bSb$
$S\to a$

$S \rightarrow b$
$S \rightarrow e$

4.

Let $G = (V, \Sigma, R, S)$, where $V = \{a, b, S\}$, $\Sigma = \{a, b\}$, and $R = \{S \rightarrow aSb, S \rightarrow aSa, S \rightarrow bSa, S \rightarrow bSb, S \rightarrow e\}$. Show that $L(G)$ is regular.

A: We first prove that $L(G)=M=\{w \in \{a,b\}^{¿} : |w| is\, even\}$.
To show $L(G)=M$, we show $L(G) \subseteq M$ and $M \subseteq L(G)$.

Step 1. We show that $M \subseteq L(G)$.
Proof by induction on the string length:
Base case: Obviously $e \in L(G)$.
Induction hypothesis: Suppose any string of length $2k$, $k \geq 0$, is contained in $L(G)$.
Consider $w \in \{a,b\}^{¿}$, $|w|=2k+2$.
Depending on the first and last symbols, there are three possibilities: $w=aua, w=bub$or aub, where $|u|=2k$.According to the induction hypothesis, we have $S \Rightarrow^{¿} u$
i) If $w=aua$, then $S \Rightarrow aSa \Rightarrow^{¿} aua=w$
ii) If $w=bub$, then $S \Rightarrow bSb \Rightarrow^{¿} bub=w$
iii) If $w=aub$, then $S \Rightarrow aSb \Rightarrow aub=w$
Therefore, $w \in L(G)$. Hence, $M \subseteq L(G)$

Step 2. We show that $L(G) \subseteq M$.
Proof by induction on the derivation length (i.e., number of ⇒'s in the derivation):
Base case: If the derivation length is 1, then the only string that can be derived is $S \Rightarrow e$. $|e|=0$, which is even.
Induction hypothesis: Suppose any derivation with length at most k generates a string of even length, we consider a derivation with length $k+1$. Consider the first derivation. It can be $S \Rightarrow aSa$, $S \Rightarrow bSb, S \Rightarrow aSb, S \Rightarrow bSb$. According to the hypothesis, with at most k derivations we always have $S \Rightarrow u$ for some u of even length. Hence if the first derivation is aSa, bSb, aSb, bSa, then with additional k more derivations we get aua, bub, aub, bua, respectively, whose length is even in all cases. Thus, $L(G) \subseteq M$.

Step 3. We have proved so far that $L(G)=M$. Since $M=((a+b)(a+b))^{¿}$,

$L(G)$ is regular.

5. Show that the following languages are context-free by exhibiting contextfree grammars generating each:

i) $\{a^n b^m c^{m+n} : n, m \geq 0\}$.

ii) $\{a^m b^n c^p d^q : m+n = p+q\}$

iii) $\{uawb : u, w \in \{a,b\}^{¿}, |u| = ¿ w \vee\}$

A: i) $\{a^n b^m c^{m+n} : n, m \geq 0\}$.

The language $G = (V, \Sigma, R, S)$ where $V = \{a, b, c, A, B, S\}$, terminals $\Sigma = \{a, b, c\}$, and rules $R = \{S \to A, A \to aAc, A \to B, A \to e, B \to bBc, B \to e\}$.

ii) $\{a^m b^n c^p d^q : m+n = p+q\}$.

The language $G = (V, \Sigma, R, S)$ where $V = \{a, b, c, d, A, B, S\}$, terminals $\Sigma = \{a, b, c, d\}$, and rules

$$\{S \to S_1 \vee S_2$$

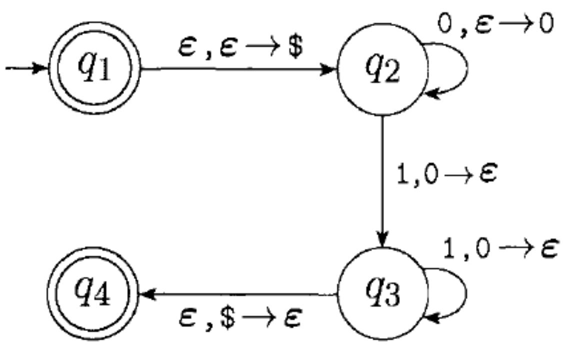
$$S_1 \to a S_1 d \,|\, A_1, A_1 \to b A_1 d \,|\, B_1, B_1 \to b B_1 c \vee e$$

$$S_2 \to a S_2 d \,|\, A_2, A_2 \to a A_2 c \,|\, B_2, B_2 \to b B_2 c \vee e\}$$

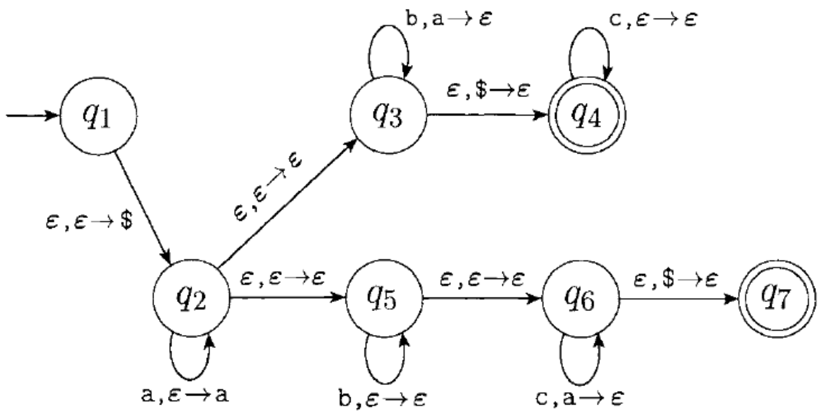
iii) $\{uawb : u, w \in \{a,b\}^{¿}, |u| = ¿ w \vee\}$

The language $G = (V, \Sigma, R, S)$ where $V = \{a, b, T, S\}$, terminals $\Sigma = \{a, b\}$, and rules $R = \{S \to Tb, T \to aTa, T \to bTb, T \to aTb, T \to bTa, T \to a\}$.

6. What does the following PDA's accept?



A: $\{0^n 1^n : n \geq 0\}$



A: $\{a^i b^j c^k : i=j \vee i=k\}$

(Additional question: can you modify the PDA for $\{a^i b^j c^{i+j} : i,j \geq 0\}$?)

7. (*) What does the following PDA accept?

$$K=\{q\}, \Sigma=\{0,1\}, \Gamma=\{a,b,0,1\}, F=\{q\},$$

$$\Delta=\{((q,0,e),(q,a)),$$

$$((q,1,e),(q,1))$$

$$((q,0,1),(q,0))$$

$$((q,1,a),(q,0))$$

$$((q,1,0),(q,e))$$

$$((q,1,1),(q,b))$$

$$((q,0,b),(q,e))\}$$

8. Use Pumping theorem to show the followings are not context-free:
   a). $\{a^n b^n c^n : n \geq 0\}$
   b). $\{a^p : p \, is \, prime\}$
   c). $\{a^{n^2} : n \geq 0\}$
   d). $\{a^n b^n a^n b^n : n \geq 0\}$
   e). $\{ww : w \in \{a,b\}^{¿}\}$

   A:

a). Suppose on the contrary that $L=\{a^n b^n c^n : n\geq 0\}$ is CFG, then

there exists some sufficiently large number N, for any $n\geq N$, we have $a^n b^n c^n = uvxyz$ such that $|vy|>0$, $|vxy|\leq N$, and $u v^i x y^i z\in L$ for any $i\geq 0$.
Pick $n=N$ and consider $a^N b^N c^N = uvxyz$. $|vxy|\leq N$, so there are 5 different possibilities.
i). $vxy=a\cdots a, \vee b\cdots b, \vee c\cdots c$, i.e., it only consists one symbol
We show the case of $vxy=a\cdots a$,the other two cases are the same.
Since $|vy|>0$, we know $v^2 x y^2$ contains exactly ¿$vy\vee$¿ more a's than vxy. That is, $u v^2 x y^2 z$ will contain $N+|vy|>N$ copies of a, i.e., $u v^2 x y^2 z = a^{N+¿vy\vee¿} b^N c^N \notin L,¿$ contradicting that $u v^i x y^i z\in L$ for any $i\geq 0$.

ii).$vxy=a\cdots ab\cdots b$ or $vxy=b\cdots bc\cdots c$, i.e., vxy contains both a,b or b,c .We show that case of $vxy=a\cdots ab\cdots b$, the other case is the same.
Since $|vy|>0$, we assume $vy=a^\alpha b^\beta$ for some $\alpha,\beta\geq 0$ and $\alpha+\beta>0$. Now we have $u v^2 x y^2 z = a^{N+\alpha} b^{N+\beta} c^N \notin L$, contradicting that $u v^i x y^i z\in L$ for any $i\geq 0$

Note that since $|vxy|\leq N$, it is impossible for vxy to contain all a,b,c.
Thus we have exhausted all the possibilities.

b). Proof essentially the same as that in slide for non-regularity

c) Suppose on the contrary that $L=\{a^{n^2} : n\geq 0\}$ is CFG, then there

exists some sufficiently large number N, for any $n\geq N$, we have $a^{n^2}=uvxyz$ such that $|vy|>0$, $|vxy|\leq N$, and $u v^i x y^i z\in L$ for any $i\geq 0$.

Pick $n=N$ and consider $a^{N^2}=uvxyz$. Let $vxy=a^\beta$ for some $1\leq\beta\leq N$.

Then $u v^2 x y^2 z = a^{N^2+\beta}\in L$. Hence, there exists some integer N_1 such

that $N^2+\beta=N_1^2$. Obviously $N_1>N$, i.e., $N_1\geq N+1$. However,

$N_1^2\geq (N+1)^2 > N^2+N$, implying that $\beta>N$, contradicting that $\beta\leq N$.

Hence, L is not CFG.

d). Suppose on the contrary that $L=\{a^n b^n a^n b^n : n\geq 0\}$ is CFG, then

there exists some sufficiently large number N, for any $n\geq N$, we have $a^n b^n a^n b^n = uvxyz$ such that $|vy|>0$, $|vxy| \leq N$, and $u v^i x y^i z \in L$ for any $i \geq 0$.

Pick $n=N$ and consider $a^N b^N a^N b^N = uvxyz$. $|vxy| \leq N$. We divide $a^N b^N a^N b^N$ into 4 substrings of equal length, and let them be w_1, w_2, w_3, w_4 where $w_1 = w_3 = a^N$, $w_2 = w_4 = b^N$. There are 3 different possibilities.

i). vxy is a substring of w_1 or w_2 or w_3 or w_4.

We show the case that vxy is a substring of w_1, the other 3 cases are the same. Since $|vy|>0$, we know $v^2 x y^2$ contains exactly $¿ vy \vee ¿$ more a's than vxy. That is, $u v^2 x y^2 z$ will contain $N + |vy| > N$ copies of a, i.e., $u v^2 x y^2 z = a^{N + ¿vy \vee ¿} b^N a^N b^N \notin L, ¿$ contradicting that $u v^i x y^i z \in L$ for any $i \geq 0$.

ii). $vxy = a \cdots ab \cdots b$, and is a substring of $w_1 w_2$ or $w_3 w_4$. We show the case that vxy is a substring of $w_1 w_2$, the other case is the same. Since $|vy|>0$, we assume $vy = a^\alpha b^\beta$ for some $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$. Now we have $u v^2 x y^2 z = a^{N+\alpha} b^{N+\beta} a^N b^N \notin L$, contradicting that $u v^i x y^i z \in L$ for any $i \geq 0$.

iii). $vxy = b \cdots ba \cdots a$, and is a substring of $w_2 w_3$. Since $|vy|>0$, we assume $vy = b^\beta a^\alpha$ for some $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$. Now we have $u v^2 x y^2 z = a^N b^{N+\beta} a^{N+\alpha} b^N \notin L$, contradicting that $u v^i x y^i z \in L$ for any $i \geq 0$.

e) Apply pumping theorem on $a^n b^n a^n b^n$, show that the resulted string cannot be expressed as ww

9. Determine whether the following statement is correct or wrong, and state your reason.

a). Language $\{a^{6n} b^{3m} c^{p+10} : n \geq 0, m \geq 0, p \geq 0\}$ is regular.

True

b). Let $\boldsymbol{L}_1, \boldsymbol{L}_2 \ldots \boldsymbol{L}_i \ldots$ be regular languages, then $\cup_{i=1}^{\infty} L_i$ is also regular.

False, consider $L_i = \{a^i b^i\}$.

c). $\boldsymbol{A}$ and $\boldsymbol{B}$ are two context-free languages, so is $\boldsymbol{A} \oplus \boldsymbol{B}$, where

$\boldsymbol{A \oplus B} = (\boldsymbol{A - B}) \cup (\boldsymbol{B - A})$.

False, consider $B \subseteq A = \{a, b\}^{¿}$, then it is essentially the complement of B, which is not necessarily CFG.

Q: what if I replace context-free with regular?

d). Language $\{a^m b^n c^l : m, n, l \in Z_{\geq 0}, m + n > 3l\}$ is context free.

True.

e). Language $\{a^m (bc)^n : m, n \geq 0\}$ is not regular.

False. It is the same as $a^{¿} (bc)^{¿}$

f). Language $\{a^n b^m : m \equiv n \bmod 2\}$ is regular

True. It is the same as $\{(aa)^{¿} (bb)^{¿}\} \cup \{a (aa)^{¿} b (bb)^{¿}\}$

g). The concatenation of a regular language and non-regular language is non-regular.

False. $a^{¿}$ concatenate with $\{a^m b^n : m \leq n\}$ is $a^{¿} b^{¿}$

10.

(a) Give a context-free grammar for the language

$$L_3 = \{xy \mid x, y \in \{a, b\}^*, |x| = |y| \text{ and } x \text{ and } y^R \text{ differ in one positions }\}.$$

For example, $abbbbaba, abbbbbbb \in L_3$, but $aababb \notin L_3$.

(b) Design a PDA $M = (K, \Sigma, \Gamma, \Delta, s, F)$ accepting the language L_3.

Solution: (a) We can construct the context-free grammar $G = (V, \Sigma, R, S)$ for language L_3, where

$V = \{a, b, S, A, B\}; \Sigma = \{a, b\}$; and

$$R = \{S \to aSa, S \to bSb, S \to aAb, A \to aAa, A \to bAb, A \to e,$$

$$S \to bBa, B \to aBa, B \to bBb, B \to e\}$$

(b) The PDA $M = (K, \Sigma, \Gamma, \Delta, s, F)$ is defined below:

	(q, σ, β)	(p, γ)
	(p, e, e)	(q, S)
$K = \underline{\{p, q\}}$	(q, e, S)	(q, aSa)
	(q, e, S)	(q, bSb)
$\Sigma = \{a, b\}$	(q, e, S)	(q, aAb)
	(q, e, A)	(q, aAa)
$\Gamma = \underline{\{a, b, S, A, B\}}$	(q, e, A)	(q, bAb)
	(q, e, A)	(q, e)
$s = \underline{p}$	(q, e, S)	(q, bBa)
	(q, e, B)	(q, aBa)
$F = \underline{\{q\}}$	(q, e, B)	(q, bBb)
	(q, e, B)	(q, e)
	(q, a, a)	(q, e)
	(q, b, b)	(q, e)

11.

Give a context-free grammar for language:

$$L_1 = \{a^m b^n c w w^R | m, n \in \mathbb{N}, n \le m \le 2n, w \in \{a, b\}^*\}$$

A:

$$G = (V, \Sigma, R, S)$$
$$V = \{a, b, S, S_1, S_2\}$$
$$\Sigma = \{a, b\}$$
$$R = \{$$
$$S \to S_1 c S_2,$$
$$S_1 \to a S_1 b,$$
$$S_1 \to aa S_1 b,$$
$$S1 \to e,$$
$$S_2 \to a S_2 a,$$
$$S_2 \to b S_2 b,$$
$$S_2 \to e$$
$$\}$$

12.

(20%) Let $\Sigma = \{a, b, c\}$. Let $L_3 = \{w|w \in \{a, b, c\}^*, \#_b(w) = \#_c(w)\}$. Where $\#_z(w)$ is the number of appearances of the character z in w. For example, the string $x = baccabcbcb \in L_3$, since $\#_b(x) = \#_c(x) = 4$. Similarly, the string $x = abcaba \notin L_3$, since $\#_b(x) = 2$ and $\#_c(x) = 1$.

(a) Construct a context-free grammar that generates the language L_3.

(b) Construct a pushdown automata that accepts L_3.

A:

$R = \{S \to bSc|cSb|SS|AS|e, A \to aA|e\}.$