Theory of Automata – Home Work 1

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1. **Show that $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$**

Sol : We aim to show any $x \in (A \cap B) \cup C$, $x \in (A \cup C) \cap (B \cup C)$
By definition of Union, $x \in (A \cap B) \cup C$ means $x \in (A \cap B)$ or $x \in C$
Thus, we aim to show i) $x \in (A \cap B)$, then $x \in (A \cup C) \cap (B \cup C)$ and
1. If $x \in (A \cap B)$, then by definition of intersection, $x \in A$ and $x \in B$;
2. Because $x \in A$, $x \in (A \cup C)$ (by the definition of union)
3. Because $x \in B$, $x \in (B \cup C)$ (by the definition of union)
4. Hence, $x \in (A \cup C) \cap (B \cup C)$ (by the definition of intersection)

Also, ii) $x \in C$, then $x \in (A \cup C) \cap (B \cup C)$
1. Because $x \in C$, $x \in (A \cup C)$ (by definition of union)
2. Because $x \in C$, then $x \in (B \cup C)$ (by definition of union)
3. Because $x \in (A \cup C)$ and $x \in (B \cup C)$, $x \in (A \cup C) \cap (B \cup C)$ (by the definition of intersection)

Hence, $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup$ C

2. **Write each of the followings explicitly**
   a). $\emptyset \times \{1,2\}$ = $\{(\emptyset,1), (\emptyset, 2)\}$
   b).$2^{\{1,2\}} \times \{1,2\}$=$\{\emptyset,\{(1,2)\}\}\times\{1,2\}$=$\{(\emptyset, 1),(\emptyset, 2),(\{1,2\},1),(\{1,2\},2)\}$

3. **Let $f: A \mapsto B$. Show that the following relation R is an equivalence relation on A: $(a, b) \in R$ if and only if $f(a) = f(b)$.**

Sol : To show that a relation is an equivalence relation, we must show that it is a) reflexive, b) symmetric, and c) transitive.

To show a relation is equivalence, we must show that it is reflexive, symmetric and transitive.

a) To do this, we must show that f(a) = f(a). This is true, since equality is **reflexive**.
b) Given that f(a) = f(b), we must show that f(b) = f(a). This is true, since equality is **symmetric**.
c) Given that f(a) = f(b) and that f(b) = f(c), we can conclude that f(a) = f(c),

since equality is **transitive**.
Hence, the given relation is equivalence relation.

4. **Let R_1 and R_2 be any two partial orders on the same set A. Show that $R_1 \cap R_2$ is a partial order.**

Sol : $R1$ and R1 and $R2$ and R2 are by definition subsets of $S \times S$ which are reflexive, antisymmetric, and transitive. Now we need to check that $R1 \cap R2$ is also reflexive, antisymmetric, and transitive.

- **Reflexive:** Since (a,a) must be in both $R1$ and $R2$ for any a∈S, (a,a) will also be in $R1 \cap R2$, so it is reflexive.
- **Antisymmetric:** Now, this is a conditional property. If (a,b)∈R1∩R2 and (b,a)∈R1∩R2, it must be the case that a=b. Since we know that this property is satisfied for both R1 and R2, it must also hold for R1∩R2.
- **Transitivity:** Likewise, since we know that the transitive property holds for both R1 and R2, there can be no two elements (a,b)∈R1∩R2 and (b,c)∈R1∩R2 without it also being the case that (a,c)∈R1∩R2.

Hence, **$R_1 \cap R_2$ is a partial order.**

5. Show that any function from a finite set to itself contains a cycle.
Sol : To prove that any function from a finite set to itself contains a cycle

Suppose there are n nodes i.e., $(a_1, a_2, \ldots a_n)$, there must be at least one node that would be appear twice. (**Pigeonhole Principle**)

- $(a_1, a_2, \ldots ai, a_{i+1}, \ldots a_j = a_i, a_{j+1}, \ldots a_n)$.

When there are n+1 nodes, that creates a cycle, due to repetition.

Hence, **function from a finite set to itself contains a cycle.ßß**