Turing Machine

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Turing Machine

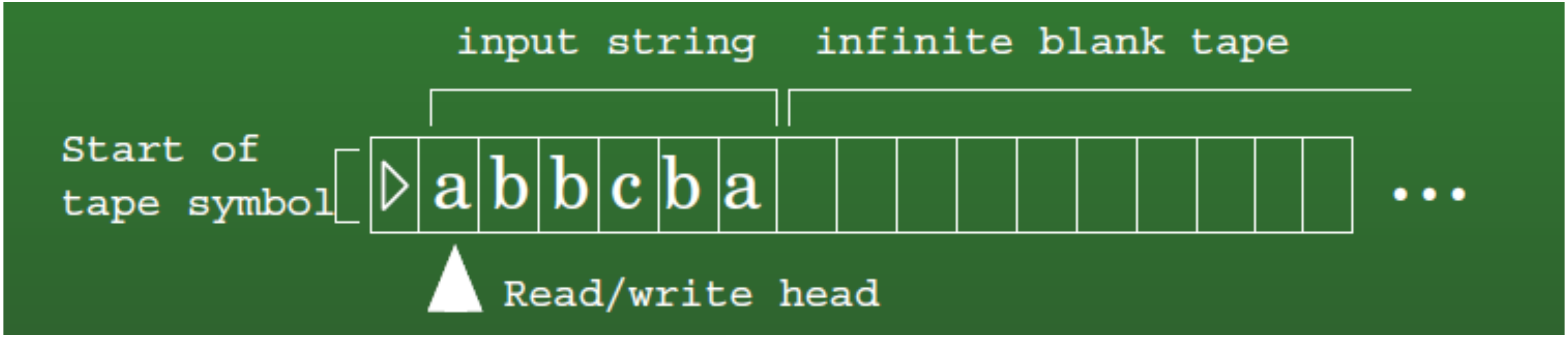
Handicapped machines

- **DFA limitations**
  - Tape head moves only one direction
  - Tape is read-only
  - Tape length is a constant
- **PDA limitations**
  - Tape head moves only one direction
  - Tape is read-only, but stack is writable
  - Stack has only LIFO(last-in, first-out) access
  - Tape length is constant, but stack is not bounded.

Turing Machine

- **What about**
  - Writable, 2-way tape?
  - Random-access ‘stack?

Turing Machine



- Head can both read and write, and move in both directions
- Tape has unbounded length.
- ⊔ is blank symbol. In practice, all but a finite number of tape squares are blank.

Turing Machine

A ***Turing machine*** is a 7-tuple, $(K, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where K, Σ, Γ are all finite sets and

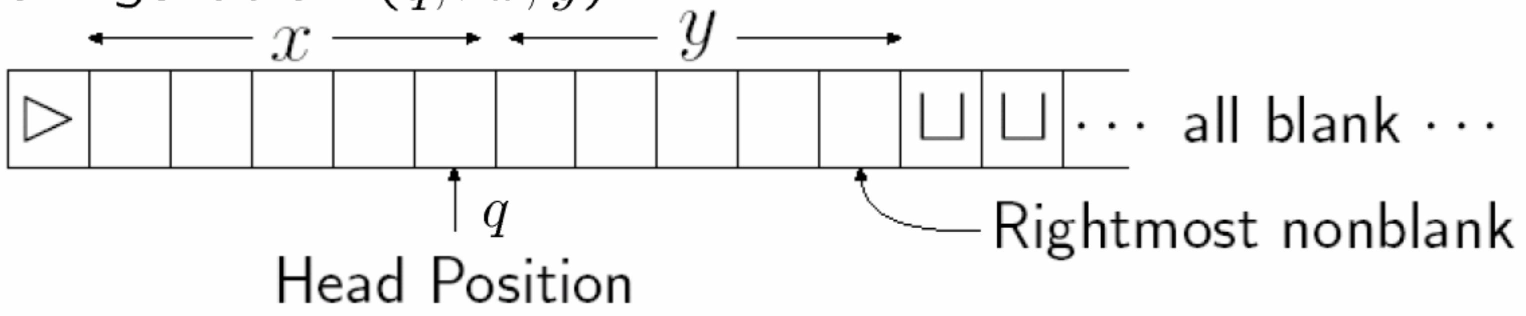
1. K is the set of states,
2. Σ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta : K \times \Gamma \longrightarrow K \times \Gamma \times \{L, R\}$ is the transition function,
5. $q_0 \in K$ is the start state,
6. $q_{accept} \in K$ is the accept state, and
7. $q_{reject} \in K$ is the reject state, where $q_{reject} \neq q_{accept}$.

Turing Machine

Turing Machines Configuration

Definition: A configuration of a TM $M = (K, \Sigma, \delta, s, H)$ is a member of $K \times \triangleright\Sigma^* \times (\Sigma^*(\Sigma - \{\sqcup\}) \cup \{e\})$.

Configuration $(q, \triangleright x, y)$:



Remark:

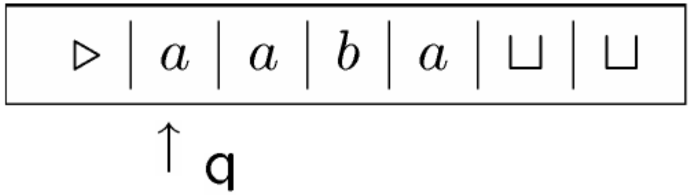
- A configuration whose state component is in H will be called halted configuration.

Turing Machine

- A simplified notation of configuration:

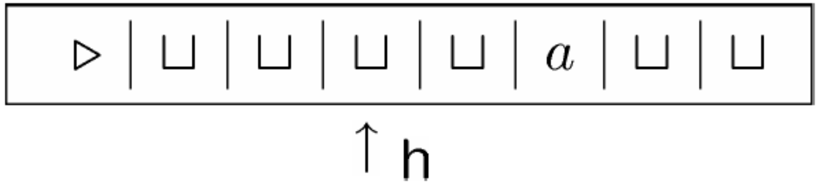
$$(q, wa, u) \Rightarrow (q, w\underline{a}u)$$

Example of configuration:



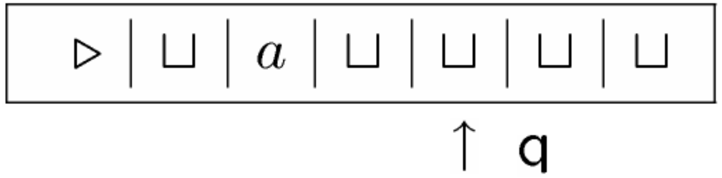
$(q, \rhd a, aba)$
or $(q, \rhd \underline{a}aba)$

$\rhd\ qaaba$



$(h, \rhd \sqcup \sqcup\sqcup, \sqcup a)$
or $(h, \rhd \sqcup \sqcup\underline{\sqcup} \sqcup a)$

$\rhd\sqcup\sqcup\ h\ \sqcup\sqcup\ a$



$(q, \rhd \sqcup a \sqcup \sqcup, e)$
or $(q, \rhd \sqcup a \sqcup \underline{\sqcup} e)$

$\rhd\sqcup\ a \sqcup q \sqcup$

Turing Machine

Remark:

- For any Turing Machine M, let $\vdash_M^*$ be the Reflexive, transitive closure of $\vdash_M$.
Configuration C_1 yields configuration C_2 if $C_1 \vdash_M^* C_2$.

- A **computation** by M is a sequence of configuration C_0, C_1, $\cdots$, C_n, for some $n \geq 0$ such that

$$C_0 \vdash_M C_1 \vdash_M \cdots \vdash_M C_n$$

we say that the computation is of length n or that it has n steps, denoted by $C_0 \vdash_M^n C_n$.

Turing Machine

Run a Turing machine on an input, the Turing machine may:

- Accept (enter q_{accept})
- Reject (enter q_{reject})
- Loop (running forever)

Turing Machine

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- Reject (enter q_{reject})
- Loop (running forever)

• M **accepts** $w \in (\sum - \{\sqcup, \triangleright\})^*$ if $(s, \triangleright\underline{\sqcup}w)$ yields an accepting configuration; M **rejects** w if $(s, \triangleright\underline{\sqcup}w)$ yields an rejecting configuration.

• Let $\sum_0 \subseteq (\sum - \{\sqcup, \triangleright\}$ be a alphabet — input alphabet of M.

M **decides** $L \subseteq \sum_0^*$ if $\forall w \in \sum_0^*$ the following is true:

- ☐ $w \in L$ iff M accepts w;
- ☐ $w \notin L$ iff M rejects w.

Turing Machine

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A language is Turing-decidable or simply decidable if some Turing machine decides it.

The collection of the strings that a Turing machine accepts is the language recognized by the machine.

A language is Turing-recognizable (or semi-decidable) if some Turing machine recognizes it.

Can we use Turing machine as checker for the language it recognizes/decides?

Turing Machine-example

Turing machine (TM) M_1 that decides $B = \{w\#w \mid w \in \{0,1\}^*\}$

M_1 = "On input string w:

1. Zig-zag across the tape to corresponding positions on either side of the # symbol to check whether these positions contain the same symbol. If they do not, or if no # is found, *reject*. Cross off symbols as they are checked to keep track of which symbols correspond.
2. When all symbols to the left of the # have been crossed off, check for any remaining symbols to the right of the #. If any symbols remain, *reject*; otherwise, *accept*."

Turing Machine-example

Turing machine (TM) M_1 that decides $B = \{w\#w \mid w \in \{0,1\}^*\}$

0 1 1 0 0 0 # 0 1 1 0 0 0 $\sqcup$ ...

x 1 1 0 0 0 # 0 1 1 0 0 0 $\sqcup$ ...

x 1 1 0 0 0 # x 1 1 0 0 0 $\sqcup$ ...

x 1 1 0 0 0 # x 1 1 0 0 0 $\sqcup$ ...

x x 1 0 0 0 # x 1 1 0 0 0 $\sqcup$ ...

x x x x x x # x x x x x x $\sqcup$ ...

accept

Turing Machine-example

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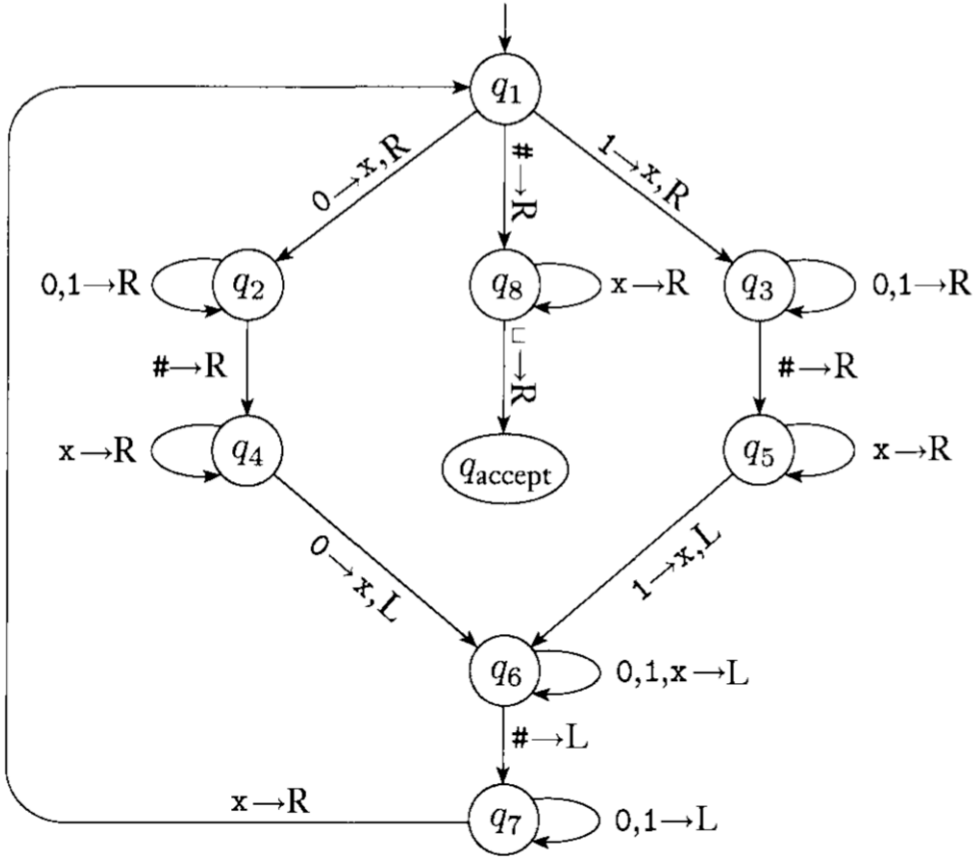


Incomplete, move to a reject state once lacking out-edge

Turing Machine-example

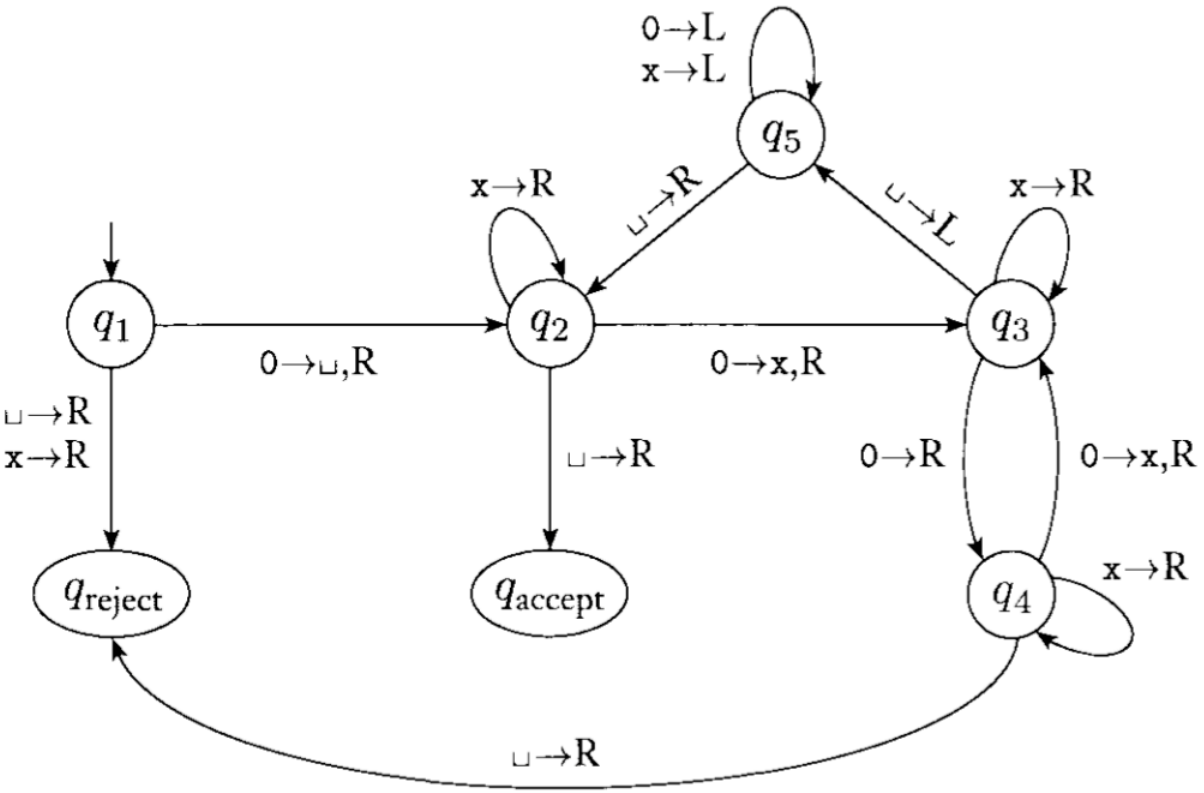
Turing machine (TM) M_2 that decides $A = \{0^{2^n} \mid n \geq 0\}$

M_2 = "On input string w:

1. Sweep left to right across the tape, crossing off every other 0.
2. If in stage 1 the tape contained a single 0, *accept*.
3. If in stage 1 the tape contained more than a single 0 and the number of 0s was odd, *reject*.
4. Return the head to the left-hand end of the tape.
5. Go to stage 1."

Turing Machine-example

Turing machine (TM) M_2 that decides $A = \{0^{2^n} \mid n \geq 0\}$



Turing Machine-example

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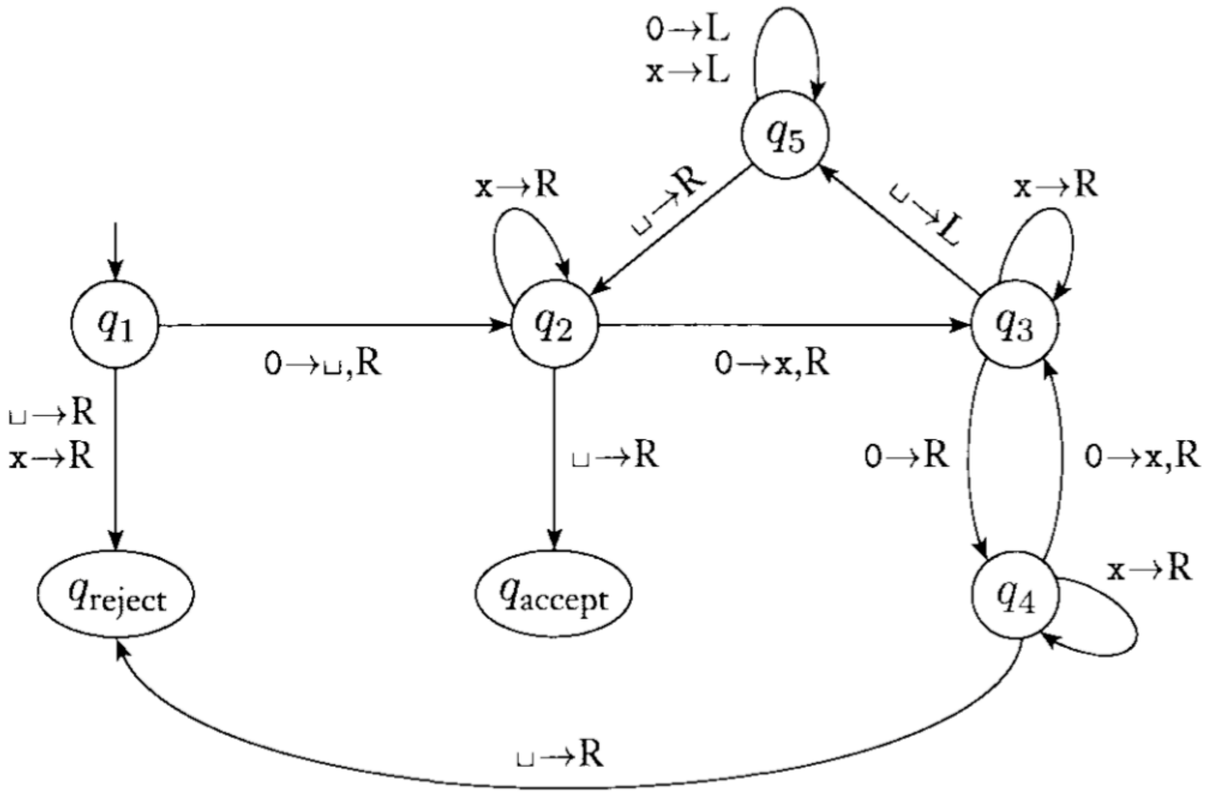

$q_1 0000$
$\sqcup q_2 000$
$\sqcup \mathrm{x} q_3 00$
$\sqcup \mathrm{x} 0 q_4 0$
$\sqcup \mathrm{x} 0 \mathrm{x} q_3 \sqcup$
$\sqcup \mathrm{x} 0 q_5 \mathrm{x} \sqcup$
$\sqcup \mathrm{x} q_5 0 \mathrm{x} \sqcup$

$\sqcup q_5 \mathrm{x} 0 \mathrm{x} \sqcup$
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$\sqcup q_2 \mathrm{x} 0 \mathrm{x} \sqcup$
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$\sqcup \mathrm{xx} q_3 \mathrm{x} \sqcup$
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$q_5 \sqcup \mathrm{xxx} \sqcup$
$\sqcup q_2 \mathrm{xxx} \sqcup$
$\sqcup \mathrm{x} q_2 \mathrm{xx} \sqcup$
$\sqcup \mathrm{xx} q_2 \mathrm{x} \sqcup$
$\sqcup \mathrm{xxx} q_2 \sqcup$
$\sqcup \mathrm{xxx} \sqcup q_{\text{accept}}$

Variants of Turing Machine

Can we strengthen a Turing machine by equipping it with more "resources"?

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Multi-tape Turing machine: Turing machine has only one read/write tape, what if we allow multiple tapes?

$$\delta\colon Q \times \Gamma^k \longrightarrow Q \times \Gamma^k \times \{\mathrm{L}, \mathrm{R}, \mathrm{S}\}^k$$

$$\delta(q_i, a_1, \ldots, a_k) = (q_j, b_1, \ldots, b_k, \mathrm{L}, \mathrm{R}, \ldots, \mathrm{L})$$

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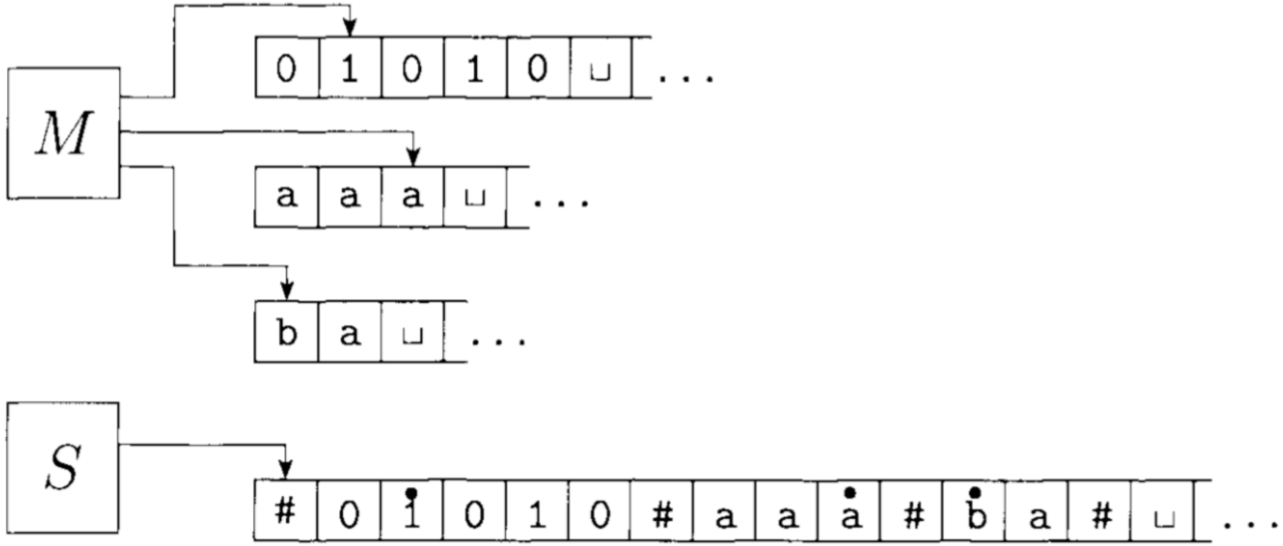
Can multi-tape Turing machine decides/recognizes more languages than Turing machine?

Variants of Turing Machine

Every multitape Turing machine has an equivalent single-tape Turing machine.

We can simulate a multi-tape Turing machine using a single-tape Turing machine.



Variants of Turing Machine

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We can simulate a multi-tape Turing machine using a single-tape Turing machine.

$S =$ "On input $w = w_1 \cdots w_n$:

1. First S puts its tape into the format that represents all k tapes of M. The formatted tape contains
$$\#\dot{w}_1 w_2 \cdots w_n \#\dot{\sqcup}\#\dot{\sqcup}\# \cdots \#$$
2. To simulate a single move, S scans its tape from the first #, which marks the left-hand end, to the $(k+1)$st #, which marks the right-hand end, in order to determine the symbols under the virtual heads. Then S makes a second pass to update the tapes according to the way that M's transition function dictates.
3. If at any point S moves one of the virtual heads to the right onto a #, this action signifies that M has moved the corresponding head onto the previously unread blank portion of that tape. So S writes a blank symbol on this tape cell and shifts the tape contents, from this cell until the rightmost #, one unit to the right. Then it continues the simulation as before."

Variants of Turing Machine

Every multitape Turing machine has an equivalent single-tape Turing machine.

A language is Turing-recognizable if and only if some multitape Turing machine recognizes it.

Variants of Turing Machine

Can we strengthen a Turing machine by equipping it with non-determinism?

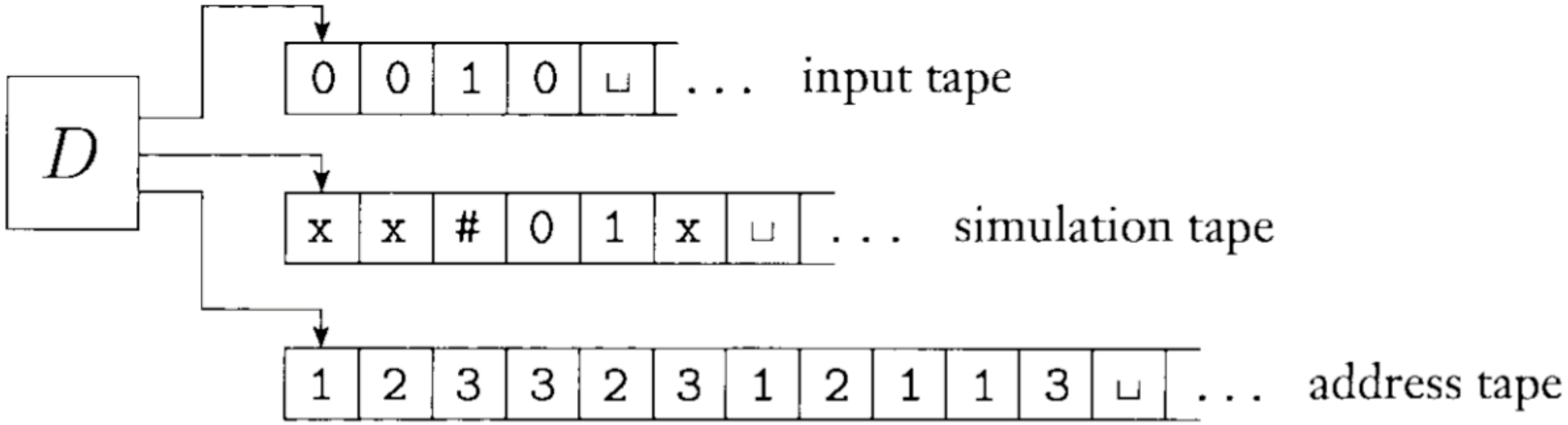
Nondeterministic Turing machine: from transition function to transition relation.

$$\delta : Q \times \Gamma \longrightarrow \mathcal{P}(Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\})$$

Variants of Turing Machine

Every nondeterministic Turing machine has an equivalent deterministic Turing machine.

We can simulate a non-deterministic Turing machine using a deterministic Turing machine.



Variants of Turing Machine

Every nondeterministic Turing machine has an equivalent deterministic Turing machine.

We can simulate a non-deterministic Turing machine using a deterministic Turing machine.

1. Initially tape 1 contains the input w, and tapes 2 and 3 are empty.
2. Copy tape 1 to tape 2.
3. Use tape 2 to simulate N with input w on one branch of its nondeterministic computation. Before each step of N consult the next symbol on tape 3 to determine which choice to make among those allowed by N's transition function. If no more symbols remain on tape 3 or if this nondeterministic choice is invalid, abort this branch by going to stage 4. Also go to stage 4 if a rejecting configuration is encountered. If an accepting configuration is encountered, *accept* the input.
4. Replace the string on tape 3 with the lexicographically next string. Simulate the next branch of N's computation by going to stage 2.

Variants of Turing Machine

Every nondeterministic Turing machine has an equivalent deterministic Turing machine.

A language is Turing-recognizable if and only if some nondeterministic Turing machine recognizes it.