

Sets, Relations and Languages

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TEXAS TECH
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Sets – Definition

- A set is an unordered collection of objects
- $S = \{a, b, c\}$
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An abstraction of an object, can be numbers, points, lines, graphs, vectors, matrices...

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- $\{\text{red}, \text{blue}, \text{red}\} = \{\text{red}, \text{blue}\}$
- $\{\text{red}, \text{blue}\} = \{\text{blue}, \text{red}\}$

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- $\{\text{red}, \text{blue}\} = \{\text{blue}, \text{red}\}$
- Element may also be a set: $\{\text{red}, \{\text{red}\}\} \neq \{\text{red}\}$
- Empty set: $\{\}$ or \emptyset
- $\{\{\}\}$ is **not** empty

Sets – Cardinality

- Cardinality of a set is the number of elements in the set
- $|\{a, b, c\}| = 3$
 $|\{\text{red}, \text{blue}\}| = 2$
 $|\{\{a, b\}, c\}| = 2$

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 $|\{\text{red}, \text{blue}\}| = 2$
 $|\{\{a, b\}, c\}| = 2$
- $|\{\}| = 0$

Sets – Singleton

- Set with one element is called Singleton

$$|\{\text{red}\}|=1$$

$$|\{a\}| = 1$$

Sets – Singleton

- Set with one element is called Singleton

$$|\{\text{red}\}|=1$$

$$|\{a\}| = 1$$

- $|\{\{\}\}| = 1$

Sets – Membership

- Element a contained in a set $S = \{a, b, c\}$: $a \in S$
 - $3 \in \{1, 3, 5\}$
 - $a \notin \{b, c, d\}$
 - $a \in \{a, \{b, c\}\}$
 - $b \notin \{a, \{b, c\}\}$

Sets – Describing

- Different ways of writing a set
 - List all members: $S = \{a, b, c\}$, $S = \{2, 3, 4, 5 \dots\}$
 - $S = \{x : x \text{ has a certain property}\}$
 - $S = \{x \mid x \text{ has a certain property}\}$
 - $S = \{x : x \in N \text{ and } x < 10\}$
 - $S = \{x : x \text{ is prime}\}$

Sets – Subset

- A is a subset of B , denoted as $A \subseteq B$ if:
 - Any element of A is also an element of B
 - Any $(\forall) x \in A$ implies $(\Rightarrow) x \in B$
 - If $x \in A$, then $x \in B$

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 - If $x \in A$, then $x \in B$
- A is a proper subset of B , denoted as $A \subset B$ if:
 - $A \subseteq B$ and B contains at least one element not in A

Sets – Subset (examples)

- A is a subset of B :
 - $\{a, b, c\} \subseteq \{a, b, c, d\}$
 - $\{1, 3, 5, 7, \dots\} \subseteq \{1, 2, 3, 4, 5, \dots\}$
 - $\emptyset \subseteq A$ for any set A
 - $A \subseteq A$
- A is a proper subset of B :
 - $\{a, b, c\} \subset \{a, b, c, d\}$
 - $\{1, 3, 5, 7, \dots\} \subset \{1, 2, 3, 4, 5, \dots\}$
 - $\emptyset \subset A$ for any set $A \neq \emptyset$
 - $A \not\subset A$

Sets – Operations

- Union
 - $A \cup B = \{x: x \in A \text{ or } x \in B\}$
 - $\{1,3,9\} \cup \{3,5,7\} = \{1,3,5,7,9\}$
 - $\{\text{red, blue}\} \cup \{\text{red, \{red, blue\}}\} = \{\text{red, blue, \{red, blue\}}\}$

Sets – Operations

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 - $A \cup B = \{x: x \in A \text{ or } x \in B\}$
 - $\{1,3,9\} \cup \{3,5,7\} = \{1,3,5,7,9\}$
 - $\{\text{red}, \text{blue}\} \cup \{\text{red}, \{\text{red}, \text{blue}\}\} = \{\text{red}, \text{blue}, \{\text{red}, \text{blue}\}\}$
- Intersection
 - $A \cap B = \{x: x \in A \text{ and } x \in B\}$
 - $\{1,3,9\} \cap \{3,5,7\} = \{3\}$
 - $\{\text{red}, \text{blue}\} \cap \{\text{red}, \{\text{red}, \text{blue}\}\} = \{\text{red}\}$
 - $\{\text{red}, \text{blue}\} \cap \{\{\text{red}, \text{blue}\}\} = \{\}$

Sets – Operations

- Difference
 - $A - B = \{x: x \in A \text{ and } x \notin B\}$
 - $\{1,3,9\} - \{3,5,7\} = \{1,9\}$
 - $\{\text{red, blue}\} - \{\text{red, \{red, blue\}}\} = \{\text{blue}\}$

Sets – Power Set

- Power set: Set of all subsets
 - $2^S = \{x : x \subseteq S\}$
 - $2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
 - $2^\emptyset = \{\emptyset\}$
 - $|2^S| = 2^{|S|}$

$A = \{a, b, c\}$
The power set $P(A) = \{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$
Now, the Power Set has $2^3 = 8$ elements.

Sets – Partition

- Π is a partition of S if:
 - $\Pi \subseteq 2^S$
 - $\emptyset \notin \Pi$
 - $A \cap B = \emptyset$ for any $A, B \in \Pi, A \neq B$
 - $\cup \Pi = S$

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 - $\cup \Pi = S$
- $\{\{a\}, \{b\}, \{c\}\}$ is a partition of $\{a, b, c\}$

Sets – Proof Format

- Q: For two arbitrary sets A , B , show
 $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

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- What does it mean by $X = Y$?
 - Any element of X is also an element of Y , and every element of Y is also an element of X .

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$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
- What does it mean by $X = Y$?
 - Any element of X is also an element of Y , and every element of Y is also an element of X .
 - Equivalently, $X \subseteq Y$ and $Y \subseteq X$.

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- Q: For two arbitrary sets A, B , show
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
- We first show: $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$
 - We aim to show any $x \in (A \cap B) \cup C, x \in (A \cup C) \cap (B \cup C)$

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 - By definition of Union, $x \in (A \cap B) \cup C$ means $x \in (A \cap B)$ or $x \in C$

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 - By definition of Union, $x \in (A \cap B) \cup C$ means $x \in (A \cap B)$ or $x \in C$
 - Thus, we aim to show
 - i) $x \in (A \cap B)$, then $x \in (A \cup C) \cap (B \cup C)$, and
 - ii) $x \in C$, then $x \in (A \cup C) \cap (B \cup C)$

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 - Thus, we aim to show
 - i) $x \in (A \cap B)$, then $x \in (A \cup C) \cap (B \cup C)$, and
 1. If $x \in (A \cap B)$, then by definition of intersection, $x \in A$ and $x \in B$;

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 2. Because $x \in A$, $x \in (A \cup C)$ (by the definition of union)

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$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
- We first show: $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$
 - We aim to show any $x \in (A \cap B) \cup C$, $x \in (A \cup C) \cap (B \cup C)$
 - By definition of Union, $x \in (A \cap B) \cup C$ means $x \in (A \cap B)$ or $x \in C$
 - Thus, we aim to show
 - i) $x \in (A \cap B)$, then $x \in (A \cup C) \cap (B \cup C)$, and
 1. If $x \in (A \cap B)$, then by definition of intersection, $x \in A$ and $x \in B$;
 2. Because $x \in A$, $x \in (A \cup C)$ (by the definition of union)
 3. Because $x \in B$, $x \in (B \cup C)$

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- Q: For two arbitrary sets A, B , show
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
- We first show: $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$
 - We aim to show any $x \in (A \cap B) \cup C$, $x \in (A \cup C) \cap (B \cup C)$
 - By definition of Union, $x \in (A \cap B) \cup C$ means $x \in (A \cap B)$ or $x \in C$
 - Thus, we aim to show
 - i) $x \in (A \cap B)$, then $x \in (A \cup C) \cap (B \cup C)$, and
 1. If $x \in (A \cap B)$, then by definition of intersection, $x \in A$ and $x \in B$;
 2. Because $x \in A$, $x \in (A \cup C)$ (by the definition of union)
 3. Because $x \in B$, $x \in (B \cup C)$
 4. Hence, $x \in (A \cup C) \cap (B \cup C)$ (by the definition of intersection)

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- Q: For two arbitrary sets A, B , show
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
- We first show: $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$
 - We aim to show any $x \in (A \cap B) \cup C$, $x \in (A \cup C) \cap (B \cup C)$
 - By definition of Union, $x \in (A \cap B) \cup C$ means $x \in (A \cap B)$ or $x \in C$
 - Thus, we aim to show
 - ii) $x \in C$, then $x \in (A \cup C) \cap (B \cup C)$
 1. Because $x \in C$, $x \in (A \cup C)$ (why?)
 2. Because $x \in C$, then $x \in (B \cup C)$ (why?)
 3. Because $x \in (A \cup C)$ and $x \in (B \cup C)$, $x \in (A \cup C) \cap (B \cup C)$ (why?)

Sets – Proof Format

- Q: For two arbitrary sets A, B , show
 $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- We first show: $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$

$x \in (A \cap B) \cup C$, then $x \in (A \cup C) \cap (B \cup C)$

$x \in (A \cap B)$ or $x \in C$, then $x \in (A \cup C) \cap (B \cup C)$

i) $x \in (A \cap B)$, then $x \in (A \cup C) \cap (B \cup C)$, and
ii) $x \in C$, then $x \in (A \cup C) \cap (B \cup C)$

Summarize and rewrite in the logic flow

Sets – Proof Format

- Q: For two arbitrary sets A, B , show
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
- We next show: $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$

Homework