Homework – 7

1. Use the pumping theorem to show that the following languages are not context-free

   a). $\{www: w \in \{a, b\}^*\}$

   (c) Assume $L = \{www : w \in \Sigma^*\}$ were context-free. Then there is a number $k > 0$ such that for any $w \in L$ such that $|w| \geq k$ there exist $u, v, x, y, z \in \Sigma^*$ such that $w = uvxyz$, $|vxy| \leq k$, $vy \neq e$, and $uv^n xy^n z \in L$ for all $n \geq 0$. Consider the string $w = a^k b a^k b a^k b$. This string is in L and satisfies $|w| \geq k$. By our assumption, u, v, x, y, and z exist as above. Neither v nor y can contain more than one b. This follows from the fact that $|vxy| \leq k$, so in particular $|v|, |y| \leq k$ so each cannot contain more than one b. In fact, neither v nor y can contain any instance of b at all. Suppose, without loss of generality, that v contained a b. Then uv^2xy^2z contains four occurrences of b and hence certainly cannot be in L (as four is not divisible by three). Similarly, if v and y each contained a b, the string uv^2xy^2z would have five instances of b and by the same reasoning could not be in L. So the only case remaining is $v, y \in L(a^*)$. Suppose $v = a^p$ and $y = a^q$, where $p, q \leq k$. Without loss of generality, let us condier the case when v is in the first set of a's and y is in the second set of a's. Then $uv^2xy^2z = a^{k+p} b a^{k+q} a^k b$, which cannot be in L, since at least one of p and q must be nonzero. Having exhausted all possible cases, we conclude that the context-free pumping fails on w and hence L cannot be context-free.

   b). $\{w \in \{a, b, c\}^*: w \text{ has equal number of } a's, b's \text{ and } c's\}$

   The intersection of context-free language and regular language is context-free. So if it is context-free, we intersect L with $a^*b^*c^*$ and get $\{a^n b^n c^n: n \geq 0\}$ which should be context-free, but $\{a^n b^n c^n: n \geq 0\}$ not context-free, a contradiction.

2. Decide whether the following language is context-free or not, and state your reason:

   a). $\{a^m b^n c^p: m = n \ or \ n = p \ or \ m = p\}$

   Context-free. Hint: This is the union of $\{a^m b^m c^p: m \geq 0, p \geq 0\}$, $\{a^m b^p c^p: m \geq 0, p \geq 0\}$, $\{a^m b^p c^m: m \geq 0, p \geq 0\}$, each of which is essentially like $\{a^n b^n: n \geq 0\}$, which can be generated by similar context-free grammar, e.g., $\{a^m b^m c^p: m \geq 0, p \geq 0\}$ is the concatenation of $\{a^n b^n: n \geq 0\}$ and c^*, where one can use $S \to S_1 S_2$, while $S_1 \to aS_1 b | e$ is the one for $\{a^n b^n: n \geq 0\}$, and $S_2 \to cS_2 | e$ for c^*. Or you can modify the PDA in our slide for it.

   b). $\{a^m b^n c^p: m \neq n \ or \ n \neq p \ or \ m \neq p\}$

   Context-free. Hint: This is the union of $\{a^m b^n c^p: m \neq n\}$, $\{a^m b^n c^p: n \neq p\}$, $\{a^m b^n c^p: m \neq p\}$. Each of them, say, $\{a^m b^n c^p: m \neq p\}$, is essentially the same as $\{a^m c^p: m \neq p\}$, which you can use the material in slides for showing the **complement** of $\{a^n b^n: n \geq 0\}$ is context-free.

   c). $\{a^m b^n c^p: m = n \ and \ n = p \ and \ m = p\}$

   Not context-free. This is $\{a^n b^n c^n: n \geq 0\}$