Theory of Automata

Home work-1

1) Show that $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$

Ans: To prove this, we aim to show any $x \in (A \cup C) \cap (B \cup C)$, $x \in (A \cap B) \cup C$

→ By the definition of intersection, $x \in (A \cup C) \cap (B \cup C)$ means $x \in (A \cup C)$ and $x \in (B \cup C)$

→ Thus we aim to show

i) $x \in (A \cup C)$, then $x \in (A \cap B) \cup C$

ii) $x \in (B \cup C)$, then $x \in (A \cap B) \cup C$

(a) If $x \in (A \cup C)$ then $x \in A$ (or) $x \in C$

case i) → Because $x \in C$, $x \in (A \cap B) \cup C$ [By definition of union] (1)

case ii) Now let us consider $x \in A$

Also According to statement (ii) $x \in (B \cup C)$ which means $x \in B$ (or) $x \in C$

Case (ii-a): If $x \in A$ and $x \in B$ then $x \in (A \cap B)$ [By defintion of intersection]

→ If $x \in (A \cap B)$ then $x \in (A \cap B) \cup C$ [By definition of union] (2)

case (ii-b): If $x \in A$ and $x \in C$ then $x \in (A \cap B) \cup C$ [By definition of Union] (3)

From (1) (2) and (3) we can say $x \in (A \cap B) \cup C$

$\Rightarrow (A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$

Hence Proved

2) Write the following explicitly,

Ans: a) $\emptyset \times \{1, 2\}$

The cartision product of a null set with any set is a null set.

So $\emptyset \times \{1,2\} = \emptyset$

b) $2^{\{1,2\}} \times \{1,2\}$

$2^{\{1,2\}}$ is the power set of $\{1,2\}$

$\therefore 2^{\{1,2\}} = \{\emptyset, \{1\} \{2\}, \{1,2\}\}$

Now $2^{\{1,2\}} \times \{1,2\} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \times \{1,2\}$

$2^{\{1,2\}} \times \{1,2\} = \{(\emptyset,1) (\emptyset,2) (\{1\},1) (\{1\},2) (\{2\},1) (\{2\},2) (\{1,2\},1)$
$(\{1,2\},2)\}$

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3) Let $f: A \mapsto B$. Show that following relation is an equivalence relation on A: $(a,b) \in R$ iff $f(a) = f(b)$

Ans: Given condition is for a relation R on A

$(a,b) \in R$ iff $f(a) = f(b)$

To prove that the relation R is an equivalence relation, we need to show R is reflexive, symmetric and transitive

i) To check R is reflexive, we need to see if $(a,a) \in R \Rightarrow f(a) = f(a)$ which is true.

So R is reflexive ——— (1)

ii) According to the given condition
if $(a,b) \in R$ then $f(a) = f(b)$
then if $(b,a) \in R$ then $f(b) = f(a)$ which is true
So $(a,b) \in R$ and $(b,a) \in R$ so
R is symmetric —— (2)

iii) If $(a,b) \in R$ then $f(a) = f(b)$
If $(b,c) \in R$ then $f(b) = f(c)$
From the above two statements and if
$(a,c) \in R$ then $f(a) = f(c)$ $[f(a) = (f(b) = f(c)]$
Hence R is transitive —— (3)
From (1) (2) and (3) we can say that the relation R on A is an equivalence relation.

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4) Let R_1 and R_2 be any two partial orders on the same set A. Show $R_1 \cap R_2$ is partial order.

Ans: A relation R is a partial order if R is reflexive, anti-symmetric and transitive.

i) To prove $R_1 \cap R_2$ is reflexive
Let us consider $(a,a) \in R_1$ and $(a,a) \in R_2$ since they are reflexive.
So $(a,a) \in R_1 \cap R_2$ [By the definition of intersection]
$\therefore$ $R_1 \cap R_2$ is reflexive —— (1)

ii) To prove $(R_1 \cap R_2)$ is anti-symmetric, let us go with the contradiction method which means we consider $R_1 \cap R_2$ as non anti symmetric and if there is any error then our assumption is wrong.

Since R_1 and R_2 are anti-symmetric

$(a, b) \in R_1$ and $(a,b) \in R_2$ but $(b,a) \notin R_1$ and $(b,a) \notin R_2$

But as we considered $R_1 \cap R_2$ as non anti-symmetric, if $(a,b) \in R_1$ & $(a,b) \in R_2$ then $(a,b) \in R_1 \cap R_2$ and also $(b,a) \in R_1 \cap R_2$

$\Rightarrow (b,a) \in R_1$ & $(b,a) \in R_2$

which is a contradiction. So our assumption is wrong and $R_1 \cap R_2$ is antisymmetric — (2)

iii) Consider (a,b) and $(b,c) \in (R_1 \cap R_2)$ then

$(a,b), (b,c) \in R_1$ and $(a,b), (b,c) \in R_2$ [By intersection]

$\rightarrow$ Since R_1 and R_2 are transitive, $(a,c) \in R_1$ and $(a,c) \in R$.

$\rightarrow (a,c) \in R_1 \cap R_2$ [By definition of intersection]

Thus $R_1 \cap R_2$ is transitive — (3)

From (1) (2) & (3) $R_1 \cap R_2$ is also a Partial Order

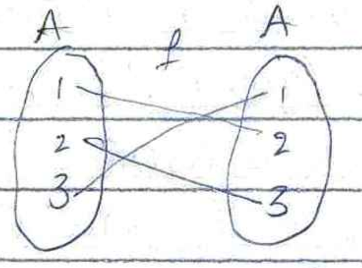
5) Show that any function from a finite set to itself contains a cycle.

Ans: A set is said to be finite if it contains finite numbers of elements in it.

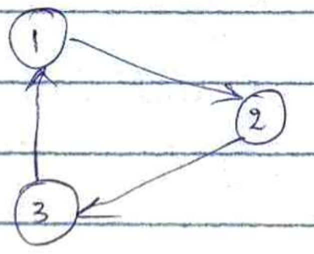
Let us consider a finite set

$A = \{1, 2, 3\}$

A function from a finite set to itself can be $\{(1,2),(2,3)(3,1\}$



So we can show this as



Thus we can say in a finite set, a function goes from first element and goes to last and last element points to first one making it a cycle.

So we can say that a function from from a finite set to itself contains a cycle.