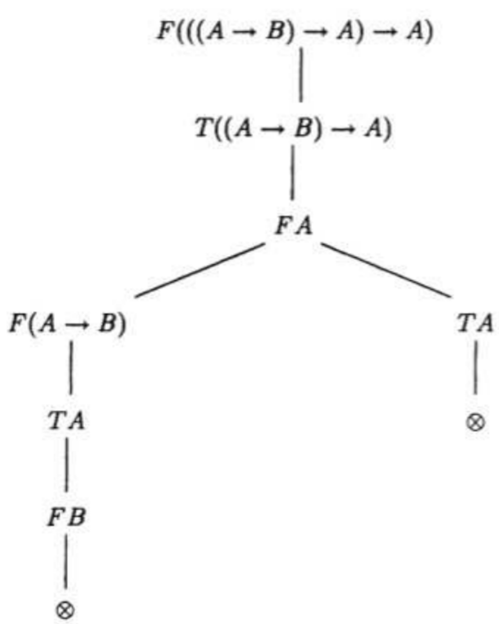
L06 Soundness and Completeness of Tableau Proofs in Propositional Logic
Chapter 5, Chapter 6

1. Motivation

Recall given a proposition α, to show it is a tautology, we construct a tableau proof. How is the tautology of the proposition related to its tableaux proof?

(purely "syntactic")	Meaning of the proposition: tautology
	$(((A \to B) \to A) \to A)$ is a tautology

We introduce two concepts: **soundness** and **completeness** for a proof. Proof here refers to a formal proof, i.e., precisely defined proof, e.g., tableaux proof. Our proof that $\Sigma \subseteq Cn(\Sigma)$ is not a formal proof because we don't formally define what that proof is.

The proof (syntactic) of sth is sound if whenever there is such a proof, this sth is "valid" (meaning).

Completeness of a proof of sth: whenever sth is valid, there is a proof (syntactic)

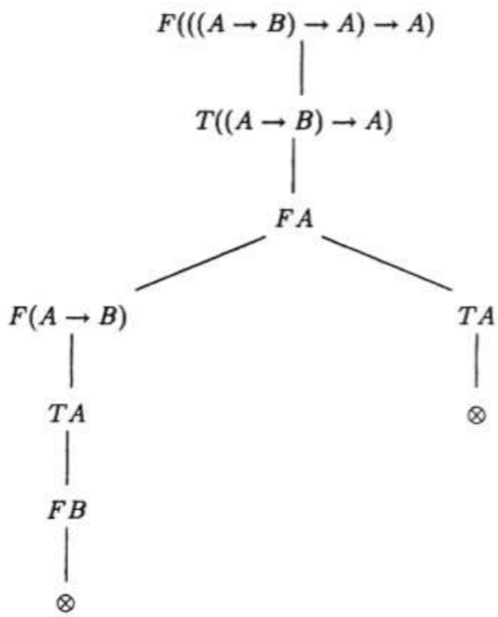
2. Overview of the results

For tableau proof, it is both *sound* and *complete*.
Write the soundness and completeness results precisely

- Soundness. For any proposition α, if there is a tableau proof of α, then α is valid or a tautology.
- Completeness. For any proposition α, if α is valid, there is a tableau proof of α.

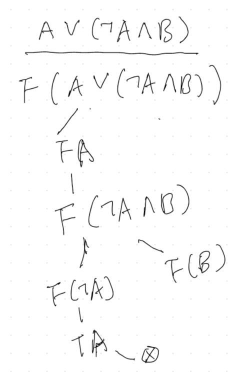
Intuition on why we have the results.

- Soundness, why?
    - In the tableau proof, we try to make the proposition F. However, we get contradiction for every possible case.

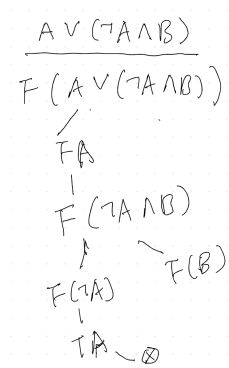


    - If there is a case (i.e., a valuation) where the proposition α is false, it means there is a path in the tableau for Fα where every entry agrees with the valuation. For example, in tableau below:

        valuation v, *v(A) = F, v(B) = F, v(-A* /\ B) = *F*, *v(A* \/ (-*A* /\ B)) = F which agrees with every entry on the non-contradictory path.



- Completeness, why?
    - If a proposition is valid, there is no way we can find an assignment to make the proposition to be false
    - If there is no tableau proof, that means that all finished tableau for F α is not contradictory, which means some path is not contradictory! Then we can construct a valuation *v* to agree with each entry of the path including the root enty F α, i.e., $v(\alpha) = F$ contradicting that α is tautology. For example, consider the tableau below, from the non-contradictory path, we can get an assignment to make the root proposition false! $v(A) = F$ and $v(B) = F$.



Study the theorems and lemmas of Chapter 5. See if you understand them. We will prove the theorems.

Informal*

Proof --- algorithm

A proof (of proposition) is sound: if for any proposition there is a proof of this proposition ⇒ this proposition is tautology.

A proof (of proposition) is complete: if for any tautology, there is a proof by that method.

For algorithms, we have similar concepts (soundness/completeness). Example using an algorithm

```
odd(x)
  if x = 3 then output odd
  else infinite loop
Sound but not complete

odd(x)
  output odd
complete but not sound
```

3. Precise statements (as theorems or lemmas) and write proofs for them.

First we write the precise statements of the soundness/completeness result. Based on the intuitive proof ideas, we write down the precise proof of the theorems.

3.1 Soundness result and proof

The soundness result is presented as

Theorem 5.1 [Soundness] If α is tableau provable, then α is valid, i.e., $\vdash \alpha \;\Rightarrow\; \models \alpha$.

The intuitive idea used in proving the theorem is presented as a lemma

Lemma 5.2 If $\mathcal{V}$ is a valuation that agrees with the root entry of a given tableau τ given as in Definition 4.1 as $\cup\tau_n$, then τ has a path P every entry of which agrees with V.

Once we have intuitive ideas of proofs, we can write a precise proof of Theorem 5.1 using Lemma 5.2.

3.2 Completeness result and proof

The completeness result is presented as

Theorem 5.3 If α is valid, then α is tableau provable, i.e., $\models \alpha \;\Rightarrow\; \vdash \alpha$.

The intuitive idea used in proving the theorem is presented as a lemma

Lemma 5.4 Let P be a noncontradictory path of a finished tableau τ.
Define a truth assignment $\mathcal{A}$ on all propositional letters A as follows:
$\mathcal{A}(A) = T$ if TA is an entry on P.
$\mathcal{A}(A) = F$ otherwise.

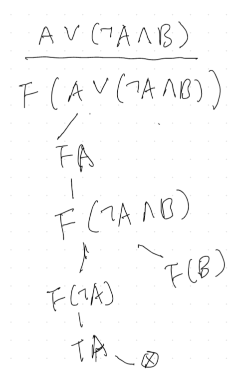
If $\mathcal{V}$ is the unique valuation (Theorem 3.3) extending the truth assignment $\mathcal{A}$, then $\mathcal{V}$ agrees with all entries of P.

Once we have intuitive ideas of proofs, we can write a precise proof of Theorem 5.3 using Lemma 5.4.

4. Prove the lemmas - Part I

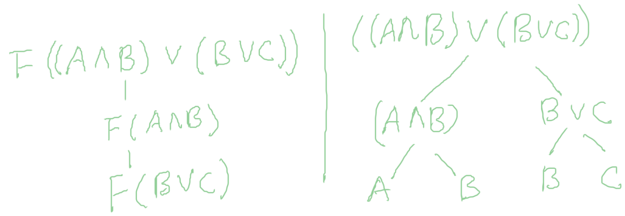
Consider Lemma 5.4 and its intuitive idea:

- For a finished tableau for F α with a non-contradictory path, we can construct a valuation *v* to agree with each entry of the path. For example, consider the tableau below, from the non-contradictory path, we can get an assignment to make root proposition false! *v(A) = F* and *v(B) = F*.



- How do we prove the statement above, i.e., the constructed valuation agrees with EVERY entry?
  - Valuation agrees with proposition letters! (that's where we start to construct the valuation)
  - Valuation agrees with proposition one logical connective (with one or two proposition letters) - by following the definition of the logical connective. For, *v(-A ∧B)* agrees with *F(-A ∧B)* (because *v(B) = F*)
  - Discuss proposition with one more logical connective, and continue …
  - …

  Alternatively, every atomic tableau (or a one-step analysis of an entry) reduces the entry by 1 level/depth/height instead of 1. See the example in diagram. The new proposition has one level less (and two less logical connectives)



So an alternative process is as follows

- Valuation agrees with proposition letters! (that's where we start to construct the valuation)
- Valuation agrees with proposition one level higher than depth of propositional letters - by following the definition of the logical connective. For, *v(-A ∧B)* agrees with *F(-A ∧B)* (because *v(B) = F*)
- Discuss proposition with one more level of depth, and continue …
- …

- The above is a perfect example where *inductive proof* helps. Its basic idea is
  - Prove the basic case (the simplest possible proposition)
  - (instead of repeating the cases, we make) inductive hypothesis that the statement is true for propositions with depth of *k*.
  - (now we just prove propositions with one level more on depth) Inductive proof.

5. Inductive Proof Method (or Prove by Induction)

Here are the details about the proof by induction method.

- Have a *precise* statement on *what* to prove.
- Figure out: prove the statement by induction on *what* (lets call it *indicator*).
- Prove the *base case* (i.e., the simplest case according to the *indicator*)
  - Formulate a precise statement on what to prove in the base case.
  - Prove the base case
- Write the Inductive hypothesis. Formulate the hypothesis precisely when *the indicator is a number k* or for all numbers at most *k*.
- The inductive step:
  - Formulate precisely the statement (with *indicator being k+1*) to prove.
  - Prove the statement.

Example for proving Lemma 5.4.

Lemma 5.4 Let P be a noncontradictory path of a finished tableau τ.
Define a truth assignment $\mathcal{A}$ on all propositional letters A as follows:
$\mathcal{A}(A) = T$ if TA is an entry on P.
$\mathcal{A}(A) = F$ otherwise.

If $\mathcal{V}$ is the unique valuation (Theorem 3.3) extending the truth assignment $\mathcal{A}$, then $\mathcal{V}$ agrees with all entries of P.

- Have a *precise* statement on *what* to prove:

  (Given the definition of P and V, the statement is) *V agrees with all entries of P.*

- Figure out: prove the statement by induction on *what* (lets call it *indicator*).

  (from intuition above, our induction will be) on *the depth of an entrie.*
- Prove the *base case* (i.e., the simplest case according to the *indicator*)
  - Formulate a precise statement on what to prove in the base case:

    (the simplest case is when the indication, i.e., depth, becomes 0) V agrees with all entries of P with depth of 0.
  - Prove the base case

    (An entry with depth of 0 is $\mathrm{X}\alpha$ where X is T or F and α is a proposition letter. By the construction of V, V agrees with the entry.)
- Write the Inductive hypothesis. Formulate the hypothesis precisely. Formulate the hypothesis precisely when *the indicator is a number k* or for all numbers at most *k*.

  V agrees with entries of P with depth at most n. (this is assumed to be true. We do NOT prove it. Instead, we use it in the inductive step.)
- The inductive step: F
  - Formulate precisely the statement (with *indicator being n+1*) to prove.

    V agrees with entries of P with depth n+1.
  - Prove the statement.

    (Proof ideas:
    - Decomposition. Since we know V agrees with entries of height at most n, we would like to decompose the entries with depth $n+1$ into entries of depth of at most n. The decomposition is natural by the formation tree (the structure) of the proposition!
    - Proof by cases. For the entry with $n+1$ depth, it has 10 cases:
      $T(\alpha_1 \vee \alpha_2), T(\alpha_1 \wedge \alpha_2), T(\alpha_1 \to \alpha_2), T(\alpha_1 \leftrightarrow \alpha_2), T(\neg\alpha_1),$
      $F(\alpha_1 \vee \alpha_2), F(\alpha_1 \wedge \alpha_2), F(\alpha_1 \to \alpha_2), F(\alpha_1 \leftrightarrow \alpha_2),$ or $F(\neg\alpha_1)$

      We need to prove each case.

)

Exercise: following the inductive proof method, fill in the details of the method (except the proof part inside the method) for proving Lemma 5.2.

6. Prove the lemmas - Part II

Proof of Lemma 5.4.

We prove this claim by induction on *the number of connectives* of the entries on P.

- Base case (the entries with 0 connectives). We will prove $\mathcal{V}$ agrees with all entries, with 0 connectives, of P.
  For every such entry E, with 0 connectives, of P,
  since it has 0 connectives, it must be of the form TA or FA.
  Case 1. $E = TA$. By the definition of
  $\mathcal{A}$, $\mathcal{V}(A) = T$ and thus

  $\mathcal{V}$ agrees with E.
  Case 2. $E = FA$. By the definition of $\mathcal{A}$,
  $\mathcal{V}(A) = F$, hence,
  $\mathcal{V}$ agrees with E.
  Therefore, $\mathcal{V}$ agrees with E, by case 1 and 2.
  $\mathcal{V}$ agrees with E.

- Inductive hypothesis (IH) (on number of connectives not more than n). We *assume* $\mathcal{V}$ agrees with all entries, *with at most n ($n \geq 0$) connectives*, of P.

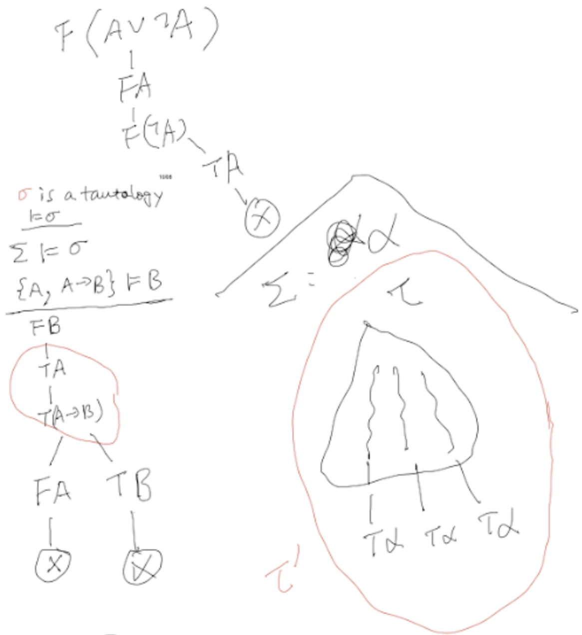
- Prove the case of entries with $n+1$ connectives, i.e., $\mathcal{V}$ agrees with all entries, *with $n+1$ connectives*, of P.
  For every such entry E, with $n+1$ connectives, of P,
  since it has $n+1$ connectives, it must be of one of the forms:
  $T(\alpha_1 \vee \alpha_2), T(\alpha_1 \wedge \alpha_2), T(\alpha_1 \rightarrow \alpha_2), T(\alpha_1 \leftrightarrow \alpha_2), T(\neg\alpha_1)$,
  $F(\alpha_1 \vee \alpha_2), F(\alpha_1 \wedge \alpha_2), F(\alpha_1 \rightarrow \alpha_2), F(\alpha_1 \leftrightarrow \alpha_2)$, or $F(\neg\alpha_1)$
  where α_1 (and α_2 respectively) has *at most n* connectives.
  We prove by cases.
  Case 1. $E = T(\alpha_1 \vee \alpha_2)$.
  Since τ is finished, P is finished and thus E is reduced.
  By definition of *reduced*, $T(\alpha_1)$ or $T(\alpha_2)$ must occur on P.
  Case 1.1 $T(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_1)$,
  and thus $\mathcal{V}(\alpha_1) = T$. Therefore,
  $\mathcal{V}(T(\alpha_1 \vee \alpha_2)) = T$, hence,
  $\mathcal{V}$ agrees with E.
  Case 1.2 $T(\alpha_2)$ occurs on P.
  We can prove, similarly to case 1.1, that
  $\mathcal{V}$ agrees with E.
  $\mathcal{V}$ agrees with E, by cases 1.1 to 1.2.
  Case 2 to 10. We can prove similarly,
  $\mathcal{V}$ agrees with E.
  Therefore, $\mathcal{V}$ agrees with E, by cases 1 to 10.
  $\mathcal{V}$ agrees with E.

QED

Exercise: prove Lemma 5.2.

7. Tableau proof for *deductions from premises*. Its soundness and completeness results, and their proofs.

- Motivation of the tableau proof from premises
  Motivation example
  {A. A->B} ⊨ B



- Write the definition of the tableau proof from premises. Compare yours with the one in the book. (Now you have all skills needed to at least fully understand the definition of tableau proof from premises)
- Formulate the soundness and completeness result about tableau proof from premises.
- What are the intuitive ideas to prove them? (very similar to Lemma 5.2 for soundness and Lemma 5.4 for completeness.)
- Write a formal proof for the soundness and completeness results.

8. Appendix

TH5.1

Assume $\models\alpha$ false

$\exists \mathcal{V}, \mathcal{V}(\alpha) = F$

for any finite table with root $F\alpha$

$\exists$ a path p agrees with $\mathcal{V}$

Let τ_1 be the ~~ctd~~ contradictory tableau for α

$\exists P_1$ in τ_1, P_1 agrees with $\mathcal{V}$

Since P1 is contradictory, $\exists T\beta, F\beta$ on P_1

$\mathcal{V}(\beta) = T \quad \mathcal{V}(\beta) = F$

contradiction that a valuation never assigns T and F to the same proposition

$\square$

Get back to chapter 5 on proving soundness and completeness result

Prove soundness result Theorem 5.1.

Prove $\vdash \sigma$ implies $\models \sigma$. (\vdash for $\vdash$)

Proof

(b2.t) Assume $\vdash \sigma$

We next prove (b2.b) by contradiction

(F3) Assume the opposite, i.e., $\models \sigma$ is false.

(F4) there exists a valuation V, $V(\sigma) = F$ (by (F3))

(F5) by (b2.t), there exists a contradictory tableau τ

(F6) (F4) implies that V agrees with $F\sigma$

(F7) by Lemma 5.2, there exists a path P_1 from τ such that *V* agrees with every entry of this path.

(F8) Since τ is contradictory (by (F5)), P_1 is contradictory.

(F9) Since P_1 is contradictory, there exists $T\beta$ and $F\beta$ on P_1.

(F10) Since V agrees with P_1, $V(\beta) = T$ and $V(\beta) = F$.

Contradicting that a valuation does not assign two values to one proposition.

Hence conclusion (b2.b) holds.

(b2.b) $\models \sigma$

(b1) $\vdash \sigma$ implies $\models \sigma$.

QED