L04 Propositional Logic - Semantics - 1
Chapter 2: Def 2.6 -- proof of 2.8
Chapter 3

Recall: Propositional Logic (a language)

- Syntax (of the language by which we can *represent* statements, called *propositions*)
- Semantics (of the language by which we can *reason* with the propositions)

Semantics

- Define the *meaning* of each propositional letter (intuitively, either T or F)
- Define the *meaning* of each logical connectives (following intuition of logical connectives)
- Define the *meaning* of a proposition

Section 1: Motivating example using arithmetic expression

Recall: we have defined the syntax of arithmetic expressions.

What is the meaning of an arithmetic expression: e.g., 1+2?

Intuitively, the meaning of an arithmetic expression is a value or a number: e.g., 1+2 is 3.

How to define the meaning? (problem decomposition - we decompose the syntactic structure of an expression as follows)

- Value/meaning of a number 5: 5 (this 5 is in a different language from arithmetic expression)
- Value/meaning of a variable x: we can really **assign** any value to it. This assignment gives meaning to a variable
- Value of \alpha (an expression) + \beta (an expression): the value of \alpha plus (different from syntax +) value of \beta

What is the meaning of 5 - 5 - 5? (could be two structure (5-5)-5 and 5-(5-5))

Once defining the meaning, we can talk about how the meaning of 1+2 is related to the meaning of 2+1. The communicative law

$$x + y = y + x$$

Really means that for any arithmetic expression (syntactic thing) *x* and expression *y*, the meaning of x + y (syntactic thing) is the same as that of y + x (syntactic thing).

Note that we do need meaning of the functions +, -, *, / …
You can imagine that these functions are defined by table (normally infinite). E.g.,

X	Y	+
1	1	2
1	2	3
…		

In reality, we just memorize part of the table and memorize an algorithm for adding/multiplying/dividing numbers.

Now are you clear what you have learned in elementary/secondary school is syntax and what are semantics?

Section 2: Semantics of Propositional Logic

What is the meaning of a proposition? Note that different from arithmetic expressions, we do not have constants in a proposition (syntax). The meaning of a proposition is True or False (**T** or **F** for short)!

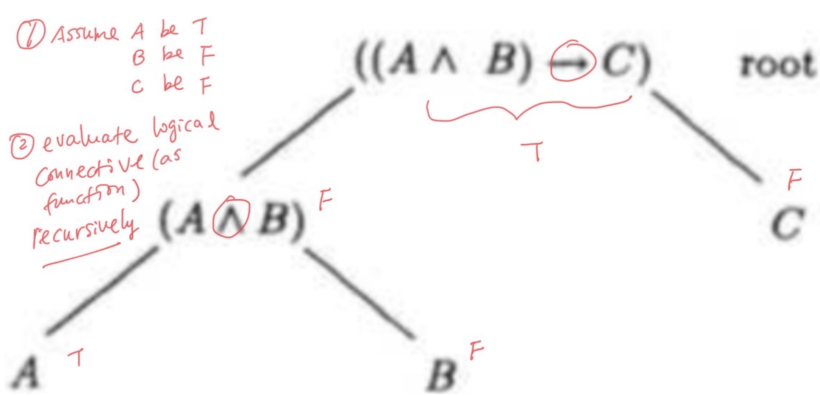
How do we define the meaning of propositions? Similarly to arithmetic expression, we decompose a proposition and assign meaning to the components

- Most basic proposition: propositional letter or variable: we have to **assign** it a value. For example, what is the meaning of P? We can only obtain the meaning of it by assigning a value. So, the base of semantic of propositions is to assign variables in a proposition values.
- Logical connectives (meaning - in fact each connective is a function meaning wise)
  - \alpha and \beta (- binary function that takes two input and gives one output)
  - \alpha or \beta
  - …
- Once we have the above we can evaluate a proposition! Example: meaning of ((A and B) → C)?

A	B	C	$(A \wedge B)$	$((A \wedge B) \rightarrow C)$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

FIGURE 4.

In summary, intuitively, given the (1) *truth values of propositional letters* of a proposition, by using the definitions of (2) *connectives*, we can know the (3) *truth value of the components of the proposition (including itself)* according to logical connectives and *the value* of its children (recursively). See below for a value (meaning) of the proposition $((A \wedge B) \rightarrow C)$.



We will give formal definitions below. Some of them may superficially seem to be different from intuition (but they do agree if you think more carefully and understand set theory: sets, functions, relations).

2.1 Definition of "meaning" of propositional variables/letters

Corresponding to (1), we have def 3.1

Definition 3.1: A **truth assignment** $\mathcal{A}$ is a function that assigns to each propositional letter A a unique truth value $\mathcal{A}(A) \in \{T, F\}$.

Domain of $\mathcal{A}$: propositional letterS

Codomain of $\mathcal{A}$: {T, F}

Example of a truth assignment

- $\mathcal{A}_1$ = { (A1, T), (A2, F), ….} // enumeration of a function as a set of pairs.
- …
-

Intuition (2) and (3) leads to

Definition 3.2: A **truth valuation** $\mathcal{V}$ is a function that assigns to each proposition α a unique truth value $\mathcal{V}(\alpha)$ so that its value on a compound proposition (that is, one with a connective) is determined in accordance with the appropriate truth tables. Thus, for example, $\mathcal{V}((\neg\alpha)) = T$ iff $\mathcal{V}(\alpha) = F$ and $\mathcal{V}((\alpha \vee \beta)) = T$ iff $\mathcal{V}(\alpha) = T$ or $\mathcal{V}(\beta) = T$. We say that $\mathcal{V}$ **makes** α **true** if $\mathcal{V}(\alpha) = T$.

A precise def on the condition of the "function" $\mathcal{V}$ in the def 3.2

For any compound proposition ($\alpha \wedge \beta$) where α and β are proposition,
$\mathcal{V}(\alpha \wedge \beta)$ = T iff $\mathcal{V}(\alpha)$ = T and $\mathcal{V}(\beta)$ = T

Recall the following definitions in discrete math (in latex)

Set theory definitions

- set, membership (not defined but can use)
- subset (defined based on set, membership and simple logic)

A \textit{relation} on A and B is a subset of $A \times B$.

Here, relation is the concept being defined and A,B are the parameters. You can tell these are parameters because if you remove these, the concept doesn't make much sense.

$A \times B$ = $\{(x, y) \mid x \in A, y \in B\}$

{a, b} X {1, 2} = {(a, 1), (a, 2), (b, 1), (b,2)}

A \textit{function} f from A to B is a relation on A and B such that $\forall x \in A, \exists$ a unique $y \in B$ such that $(x, y) \in f$. *A* is called the **domain** of *f* and *B* is called the **range (codomain)** of f.

/\ a function from {T, F} and {T, F} to {T, F}. /\ = {(T, T, T), (T, F, F), …} (see def of /\ for a table representation of a function)

Example:

Let A={1}, B={2,3} and $\mathcal{R}$ = {(1,2),(1,3)}

$\mathcal{R}$ is a relation on A and B, but not a function from A to B.

Th3.3 For any truth assignment $\mathcal{A}$, there is a unique truth valuation $\mathcal{V}$ such that $\mathcal{A} \subseteq \mathcal{V}$.

A(P1) = T V(P1) = T
A(P2) = F V(P2) = T
V((P1 /\ P2)) = T
…. ...

Corollary 3.4 If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two valuations that agree on the *support* of α, the finite set of propositional letters used in the construction of the proposition α, then $\mathcal{V}_1(\alpha) = \mathcal{V}_2(\alpha)$.

(A1 /\ A2)

V1(A1) = V2(A1) = T
V1(A2) = V2(A2) = F
V1(A3) = T, V2(A3) = F
……

V1((P1 /\ P2)) must be the same as V2((P1 /\ P2))

Example of tautology
$(A(\neg A))(\alpha(\neg\alpha))$

A is a subset of B if for any $x \in A, x \in B$.
A is a subset of B if for any $x, (x \in A) => (x \in B)$.

To prove a statement: *sth* implies *sthElse*
Proof.
Assume sth.
….

sthElse
QED

2.2 Definition of logical connectives

Meaning of logical connectives (definition technique: enumeration, we call it *truth table* here)

Definition 2.6 (Truth tables):

α	β	$(\alpha \vee \beta)$
T	T	T
T	F	T
F	T	T
F	F	F

α	β	$(\alpha \wedge \beta)$
T	T	T
T	F	F
F	T	F
F	F	F

α	β	$(\alpha \rightarrow \beta)$
T	T	T
T	F	F
F	T	T
F	F	T

α	β	$(\alpha \leftrightarrow \beta)$
T	T	T
T	F	F
F	T	F
F	F	T

α	$\neg\alpha$
T	F
F	T

The definition above might not always follow our daily life understanding of the logical connectives

- Or: This will be done by you **or** me. (Usually exclusive: not (you AND me))

We can define the meaning of propositions formally later. But **intuitively**, by definition, we can have the meaning of any proposition *informally*. Example:

A	B	C	$(A \wedge B)$	$((A \wedge B) \rightarrow C)$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

FIGURE 4.

Are the logical connectives we used sufficient to represent all possible logical connectives we may use (in daily life)?

- n-ary connective (**truth functional connectives**)
  - Syntax: $\sigma(A_1, \ldots, A_n)$: it involves *n* parameters
  - Semantics:
    - $\sigma(A_1, \ldots, A_n)$ is a *function* from $D_1, \ldots, D_n$ to D where $D_1 = \ldots = D_n = D = \{T, F\}$.
    - It can be defined by truth table, e.g.,

	A	B	C	?
1	T	T	T	T
2	T	T	F	F
3	T	F	T	F
4	T	F	F	F
5	F	T	T	T
6	F	T	F	F
7	F	F	T	F
8	F	F	F	T

FIGURE 7.

- Example of majority of three proposition
  - Syntax maj(A1, A2, A3)
  - Semantics
    - Intuitive meaning: maj(A1, A2, A3) takes the value of the majority of A1, A2, A3.
    - Formal definition (using truth table)
- **Adequate**: (see def 2.7)

  Definition 2.7: A set S of truth functional connectives is *adequate* if, given any truth functional connective σ, we can find a proposition built up from the connectives in S with the same abbreviated truth table as σ.

  (if you realize that an *abbreviated truth table* could be sth precisely defined, you will read the book to find its meaning. This shows how important you realized what may already have been defined in a definition.)
- Answer to our question

  Theorem 2.8 (Adequacy): $\{\neg, \wedge, \vee\}$ *is adequate.*

- Proof of TH 2.8
  - Intuitive ideas

	A	B	C	?
1	T	T	T	T
2	T	T	F	F
3	T	F	T	F
4	T	F	F	F
5	F	T	T	T
6	F	T	F	F
7	F	F	T	F
8	F	F	F	T

FIGURE 7.

(A=T and B=T and C= T) or (A=F and B=T and C=T) or (A=F and B = F and C=F)
(A /\ (B /\ C)) \/ ((-A) /\ B /\ C) \/ ((-A) /\ (-B) /\ (-C))

- Go through the proof:
  - read and understand definitions!
  - Proof method (also problem solving methodology)

Theorem 2.8 (Adequacy): $\{\neg, \wedge, \vee\}$ *is adequate.*

Proof: Let $A_1, \ldots, A_k$ be distinct propositional letters and let a_{ij} denote the entry (T or F) corresponding to the i^{th} row and j^{th} column of the truth table for $\sigma(A_1, \ldots, A_k)$ as in Figure 6. Suppose that at least one T appears in the last column.

A_1	$\cdots$	A_j	$\cdots$	A_k	$\cdots$	$\sigma(A_1, \ldots, A_k)$
						b_1
						b_2
						.
						.
		a_{ij}				b_i

FIGURE 6.

For any proposition α, let α^T be α and α^F be $(\neg\alpha)$. For the i^{th} row denote the *conjunction* $(A_1^{a_{i1}} \wedge \ldots \wedge A_k^{a_{ik}})$ by a_i. Let $i_1, \ldots, i_m$ be the rows with a T in the last column. The desired proposition is the *disjunction* $(a_{i_1} \vee \ldots \vee a_{i_m})$. The proof that this proposition has the given truth table is left as Exercise 14. (Note that we abused our notation by leaving out a lot of parentheses in the interest of readability. The convention is that of *right associativity*, that is, $A \wedge B \wedge C$ is an abbreviation for $(A \wedge (B \wedge C))$.) We also indicate a disjunction over a set of propositions with the usual set-theoretic terminology. Thus, the disjunction just constructed would be written as $\bigvee\{a_i | \ b_i = T\}$. $\square$

2.3. Formal definition of Consequence

Intuition: a set of propositions $\Sigma = \{A \to B, A\}$, and a proposition $\sigma = B.$ We say\sigma $isaconsequenceof$\Sigma$.Reason: wheneverevery propositionofΣistrue, then$ \sigma$istrue.Whatdoesitmeanby"whener"?Anassignmentofthepropositionvariablesin$ \Sigma$!So, wehavethefollo$ng whenevers

- A=T, B=T: every proposition of Σ is T! And σ is T!

The following is not "whenever":

- A = T, B=F: $A \to B$ is not T!
- A = F, B = T: A is not T!
- A = F, B = F: A is not T!

Now we have the following formal definition:

Definition 3.7 Let Σ be a (possibly infinite) set of propositions. We say that σ is a consequence of Σ (and write $\Sigma \models \sigma$) if, for any valuation $\mathcal{V}$,

$$(\mathcal{V}(\tau) = T \text{ for all } \tau \in \Sigma) \Rightarrow \mathcal{V}(\sigma) = T.$$

Note that, if Σ is empty, $\Sigma \models \sigma$ (or just $\models \sigma$) iff σ is valid. We also write this as $\models \sigma$. This definition gives a semantic notion of consequence.

Example 1

- Intuitive one
  - Σ = { john is father or john is mother of peter $\to$ john is parent of peter, john is father of peter}.
  - σ is "john is parent of peter"
  - $\Sigma \models \sigma$
- Formal one
  - Σ = { $f \vee m \to p$, f}
  - σ: p
  - $\Sigma \models \sigma$

Example 2

- Σ = {a, b}
- $a \wedge b$ **a** consequence of Σ? ($Cn(\Sigma)$: all consequences of Σ.

Let

(p1) Σ be a set of propositions and

(P2) $Cn(\Sigma)$ be the set of consequences of Σ.

Prove $\Sigma \subseteq Cn(\Sigma)$.

Proof

(B3.t) $\forall x$,

(B4.t) Assume $x \in \Sigma$

(B7.t) for any valuation $\mathcal{V}$,

(B8.t) Assume $\mathcal{V}$ is a model of Σ

(F9) for any $y \in Sigma$, $\mathcal{V}(y) = T$. by (B8.t) and (A3)

(B8.b) $\mathcal{V}(x) = T$.

by (B4.t), (F9) and $\forall$ (and substitution)

(B7.b) if $\mathcal{V}$ is a model of Σ then $\mathcal{V}(x) = T$.

(B6) for any valuation $\mathcal{V}$, if $\mathcal{V}$ is a model of Σ then $\mathcal{V}(x) = T$.

(B5) $\Sigma \models x$ by (B6) and (A2)

(B4.b) $x \in Cn(\Sigma)$ by (p2) and (B5)

(B3.b) $x \in \Sigma \Longrightarrow x \in Cn(\Sigma)$.

(B2) $\forall x\, x \in \Sigma \Longrightarrow x \in Cn(\Sigma)$

(B1) $\Sigma \subseteq Cn(\Sigma)$ by (A1) and (B2)

QED

Definition of concepts and apply the definition to a statement

1. When working backward from (B1), its main concept is subset (with two parameters)
   - Definition:
     A is a **subset** of B if $\forall x \quad x \in A \Longrightarrow x \in B$.
   - Substitution
     A: Σ
     B: $Cn(\Sigma)$
   - Result: the definition instance for main concept with parameters at (B1)
     (A1) Σ is a **subset** of $Cn(\Sigma)$ if $\forall x \quad x \in \Sigma \Longrightarrow x \in Cn(\Sigma)$.
2. When working backward from (B5), its main concept is consequence
   - Definition

     Definition 3.7 Let Σ be a (possibly infinite) set of propositions. We say that σ is a consequence of Σ (and write $\Sigma \models \sigma$) if, for any valuation $\mathcal{V}$,

     $(\mathcal{V}(\tau) = T$ for all $\tau \in \Sigma) \Rightarrow \mathcal{V}(\sigma) = T$.

     Note that, if Σ is empty, $\Sigma \models \sigma$ (or just $\models \sigma$) iff σ is valid. We also write this as $\models \sigma$. This definition gives a semantic notion of consequence.
   - Substitution:
     Σ: Σ
     σ: x
   - The resulting **definition instance** is
     (A2) $\Sigma \models x$ **if** for any valuation $\mathcal{V}$, if $\mathcal{V}$ is a model of Σ then $\mathcal{V}(x) = T$.

3. When to obtain (F9)

- Definition

  Definition 3.8 A valuation $\mathcal{V}$ is a model of Σ if $\mathcal{V}(\sigma) = T$ for every $\sigma \in \Sigma$. We denote by $\mathcal{M}(\Sigma)$ the set of all models of Σ.

- Substitution:

  V: V
  Σ: Σ

- The resulting of applying the definition to concept at (8.t)

  (A3) V is a model of Σ if (iff) for every $\sigma \in \Sigma$, $V(\sigma) = T$

  note in general A → B doesn't mean B → A
  note: A <-> B if and only if A->B and B->A.
  Note: main "if" is a definition means "if and only if"

Exercise: continue the proof above and finish. Write the final version.

2.4. Self-learn concepts in this and almost every subject

By reading the text, for any concept, you are expected to be able to know/create

- Motivation of the concept
- Intuitive meaning of the concept
- Definition of the concept
- Examples of the concept
- Proofs

Example of the self-learning of "consequence" concept, see above.

2.5 Application

An application of proposition: how to represent 2-queen problem using a set of propositions, and how the solution to 2-queen problem is related to the model(s) of the propositions.

P(i,j): propositional variable for cell (i,j), when it is true, it means there is a queen on cell (i,j)

p(1,1) -> -p(1,2) [p(1,1) → -p2?]
p(1,1) -> -p(2,1)
p(1,1) -> -p(2,2) // diagonal constraints
p(1,2) -> …
…

p(1,1) V p(1,2)
p(2,1) V p(2,2)

=== a more systematic way ===
For each row, there is one and only one queen:
  // There is one
p(1,1) \/ p(1,2)
  // only one?
p(1,j) -> (for all i \in 1..2 and $$i \neq j) -p(1, i) // note this is pseudocode