L09 Refine Resolution
Chapter 9

1. Motivation

Resolution is an improvement of systems in (chapter 7). It can be further improved (efficiency purpose).

Under resolution, we are interested in **refutation** instead of proof (they are "equivalent")

- A tautology cannot never be a "reason" leading to a contradiction (an empty clause)
  - A clause can be taken as a constraint over its literals. E.g., (A \/ B) (A \/ -A \/ B) {{A}, {-A}}
  - A tautology allows its literals to take any value

- Given an assignment, require one parent to be "false" (unsatisfied) under the given assignment.

- Ordered resolution: order propositional letters. E.g., { {D, -B}, {-B, A}, {A, B, C} } Assume order A, B,C,D. Although we can resolve the last clause with any other ones, but it is not necessary, which can be understood in two ways
  - Since C has a higher order than B which was resolved on, the resolvent (with any of the other clauses) still contains C and thus we can not get empty clause by resolving the resolvent and other clauses on literals with lower order unless we resolve that resolvent with a clause on literal C.
  - Since the literal in C to be resolved on is not the highest, we can simply "set" the highest one, i.e., C, to be "true" and the whole clause is satisfied without affecting the ability to obtain empty clause using clauses with literals having lower order.

  {D, -B} {A, -D, C}

Every restriction of the resolution refutation methods are still sound. Each of them is complete.

2. Study the definitions and results in the textbook

Todo for next class: write the needed definitions / study them in the book, and the soundness and completeness result of the improved resolution. Understand the intuition behind ordered resolution.

Definition 9.1 T-resolutions are resolutions in which neither of the parent clauses is a tautology. $\mathcal{R}^T(S)$ is the closure of S under T-resolutions.

Lemma 9.2 Any restriction of a sound method, i.e., one that allows fewer deductions than the sound method, is itself sound. In particular, as resolution is sound, so is $\mathcal{R}^T$, i.e., if $\Box \in \mathcal{R}^T(S)$, S is unsatisfiable.

Theorem 9.3 If S is unsatisfiable, then $\Box \in \mathcal{R}^T(S)$.

Definition 9.4 Let $\mathcal{A}$ be an assignment. An $\mathcal{A}$-resolution is a resolution in which at least one of the parents is false in $\mathcal{A}$. $\mathcal{R}^{\mathcal{A}}$ is the closure of S under $\mathcal{A}$-resolutions. This procedure is often called semantic resolution.

Theorem 9.5 For any $\mathcal{A}$ and S, if $S \in \mathtt{UNSAT}$, then $\Box \in \mathcal{R}^{\mathcal{A}}(S)$.

Ordered resolution

Definition 9.6 Assume that we have indexed all the propositional letters. We define $\mathcal{R}^<(S)$, for ordered resolution, as usual except that we only allow resolutions of $C_1 \sqcup \{p\}$ and $C_2 \sqcup \{\bar{p}\}$ when p has a higher index than any propositional letter in C_1 or C_2.

Theorem 9.8 [Completeness of ordered resolution] If S is unsatisfiable, then there is an ordered resolution refutation of S, i.e., $\Box \in \mathcal{R}^<(S)$.

Definition 8.7 $\mathcal{R}(S)$ is the closure of S under resolution, i.e., the set determined by the following inductive definition:

1. If $C \in S, C \in \mathcal{R}(S)$.
2. If $C_1, C_2 \in \mathcal{R}(S)$ and C is a resolvent of C_1 and C_2, then $C \in \mathcal{R}(S)$.

Let $S = \{\{p\}, \{\neg q\}, \{\neg p, \neg q\}\}$. The analysis in which we eliminate first p and then q can be represented below: