Homework 2. Propositional Logic. Tableau Proof.

Submit your home work to blackboard by **11:59pm Thur Sept 29.** .

1. (6) (Attendance and grading issues). Divya Mannava and Vamsi Krishna Pagadala are our graders.

   - If you have any issues/requests about attendance, please contact Vamsi directly and he will take notes and answer your questions. His email is:

     vpagadal@ttu.edu

   - If you have any issues/requests about grading, please write to either Divya or Vamsi whose emails are *dmannava@ttu.edu* and *vpagadal@ttu.edu* respectively. I do work out a rubric with Divya and Vamsi for grading each homework, and we go through a sample set of submissions together on how to grade. Whom do you need to contact if you cannot attend a class or have an attendance issue? What email do you use for that contact? Whom do you contact if you have doubts on the grading of your homework? What email do you use for that contact?

   Answer:

   For attendance issues, I will contact Vamsi Krishna Pagadala

   Email id used to contact vamshi is vpagadal@ttu.edu

   For doubts on grading of the homework, I will contact Divya or Vamsi

   Email id of Divya is dmannava@ttu.edu and email id of Vamsi is vpagadal@ttu.edu

2. (14) Study the proof of $\Sigma \subseteq Cn(\Sigma)$ in Section 2.3 of L04 and the note after the proof to learn how to work backwards step by step. A key in the one-step backwards is the application of the definition of a concept to a use of the concept.

   (a) Based on the working backwards method, write a final proof for the following statement: for any proposition α, α is a consequence of $\{\alpha\}$.
   Answer:
   The main concept is consequence.
   A1) Let α be a propositional letter.
   A2) α is a proposition [a propositional letter is a proposition]

A3) $\{\alpha\}$ is a set of proposition. [α is an element in a set]
A4) Valuation $\mathcal{V}$ is a model of Σ if $\mathcal{V}(\sigma)$ = T [By definition of model of set of propositions]
A5) $\mathcal{V}(\alpha)$ = T from A4
A6) $\{\alpha\} \models \alpha$ [from A4 and A5]

(b) Let Σ_1 and Σ_2 be sets of propositions. Using the working backwards method, prove $\Sigma_1 \subseteq \Sigma_2$ implies $Cn(\Sigma_1) \subseteq Cn(\Sigma_2)$.
Answer:
The main concept is subset.
A1) Σ_1 and Σ_2 be the set of propositions.
A2) $Cn(\Sigma_1)$ and $Cn(\Sigma_2)$ be the set of consequences of the sets Σ_1 and Σ_2 respectively.
A3) $\forall$ x $\in \Sigma_1 \Rightarrow$ x $\in Cn(\Sigma_1)$ [By the proof of $\Sigma \subseteq Cn(\Sigma)$]
A4) $\forall$ x $\in \Sigma_2 \Rightarrow$ x $\in Cn(\Sigma_2)$ [By the proof of $\Sigma \subseteq Cn(\Sigma)$]
A5) If we assume $\Sigma_1 \subseteq \Sigma_2$ then x $\in \Sigma_1 \Rightarrow$ x $\in \Sigma_2$ [By the definition of subset]
A6) If x $\in \Sigma_1 \Rightarrow$ x $\in \Sigma_2$ then x $\in Cn(\Sigma_1) \Rightarrow$ x $\in Cn(\Sigma_2)$ [By the proof of $\Sigma \subseteq Cn(\Sigma)$]
A7) Therefore $Cn(\Sigma_1) \subseteq Cn(\Sigma_2)$ [From A6]

Remember to write the reason for each statement in your proof. Your proof should be in the final form.

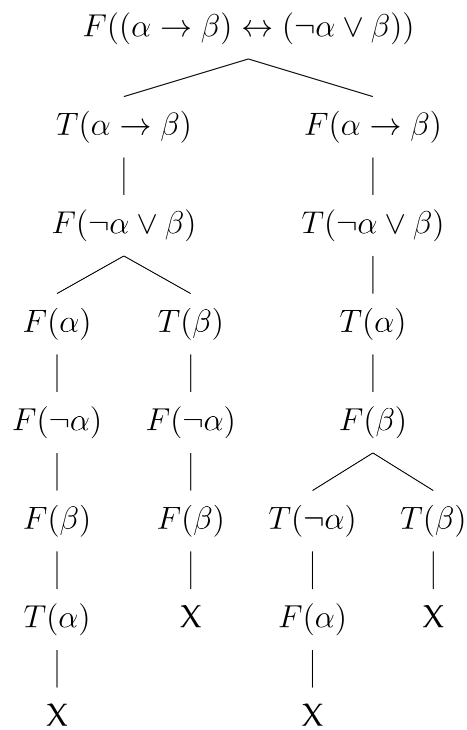
3. (10) Find the definition of a *model of a set of propositions* and definition of *a proposition is a consequence of a set of propositions* from the textbook. Rewrite each of the definitions using the concept of *a valuation makes a proposition true* (Definition 3.2) where appropriate. In your new definition, you are NOT allowed to directly apply a valuation $\mathcal{V}$ to a proposition σ in the form $\mathcal{V}(\sigma)$. You are NOT allowed to use T directly in your definitions.

Answer:
Model of set of propositions:
i) Σ is a set of propositions
ii) Truth valuation of a proposition is assigning a truth value to the proposition.
iii) Truth Valuation $\mathcal{V}$ is a model of set of propositions Σ if it assigns truth values to all the propositions in Σ.
Truth valuation of $(A \wedge B)$ is $\mathcal{V}(A)$=T and $\mathcal{V}(B)$=T
Truth valuation of $(A \vee B)$ is $\mathcal{V}(A)$=T and $\mathcal{V}(B)$=F or $\mathcal{V}(A)$=F and $\mathcal{V}(B)$=T
Truth valuation of $(A \rightarrow B)$ is $\mathcal{V}(A)$=F and $\mathcal{V}(B)$=T
Truth valuation of $(A \leftrightarrow B)$ is $\mathcal{V}(A)$=F and $\mathcal{V}(B)$=F
Truth valuation of $(\neg A)$ is $\mathcal{V}(A)$=F

Proposition is a consequence of set of propositions:
i) α is a proposition
ii) Σ is a set of propositions
iii) For $\tau \in \Sigma$, and $\forall\ \tau$ in Σ if valuation makes a propositions τ true and α true, then α is the consequence of Σ

4. (15) Give a tableau proof of $((\alpha \to \beta) \leftrightarrow (\neg\alpha \vee \beta))$.

Answer:



5. (10) Which entries of the tableaux in Figure 1 are *reduced*? Which are not?

T((A ∨ B) ∧ (A ∧ B))

T(A ∨ B)

T(A ∧ B)

TA

TB

Figure 1: A tableau

Answer:

Entries $T((A \vee B) \wedge (A \wedge B))$ and $T(A \wedge B)$ are reduced.

Entry $T(A \vee B)$ is not reduced.

6. (15) Draw the CST of $T((A \vee B) \wedge (C \vee D))$.

Answer:

$$T((A \vee B) \wedge (C \vee D))$$

$$T(A \vee B)$$

$$T(C \vee D)$$

$$TA \qquad\qquad TB$$

$$TC \quad TD \quad TC \quad TD$$

7. (10) (Write definition) Recall the *language* of propositional logic in L02. We now expand it with a new connective *majority*. While most of the original connectives in the language such as $\wedge$ are used in an *infix form* to form a proposition. For example, if α and β are propositions, then $(\alpha \wedge \beta)$ is a proposition. With the

expanded language, we can write new propositions. For the new connective *majority*, it allows exactly three parameters, and a prefix form has to be used for it to form a new proposition. For example, for propositional letters A_1, A_2, A_3, *majority(*A_1, A_2, A_3*)* is a proposition. In fact, we can nest these connectives. For example, *majority(majority(*A_1, A_2, A_3*)*, $(A_1 \wedge A_2), (A_3 \vee B)$*)* is a new *propositions*, and so is *(majority(*$A_1, A_2, A_3) \wedge A_1)$

Write a definition of the new *proposition*. You can refer to the definition of original proposition from the book/L02. Clearly, an inductive (recursive) definition is needed here.

Answer:

i) Propositional letters are propositions.

ii) If A,B and C are propositions, then majority(A,B,C) is also a proposition. If A and majority(A,B,C) are propositions, then (A $\wedge$ majority(A,B,C)), (A $\vee$ majority(A,B,C)),(A $\rightarrow$ majority(A,B,C)),(A $\leftrightarrow$ majority(A,B,C)), ($\neg$ majority(A,B,C)) are also propositions.

iii) A string of symbols is a proposition if and only if it can be obtained by starting with propositional letters (i) and repeatedly applying (ii).

8. (10) Study carefully the proofs in L06. Prove the completeness result of the tableaux proof, i.e., Theorem 5.3. Follow the proof of soundness result in L06. Do not skip steps in your proof. Your proof should be in the final form (e.g., all labels for statements will be without prefix b or F). You may use lemma 5.4 directly.

   Answer:

   Completeness result is presented as if σ is valid, then σ is tableau provable, i.e., $\models \sigma \Rightarrow \vdash \sigma$
   We can prove this by contradiction
   A1) Assume that σ is not provable. [non contradictory path with root entry F(σ)]
   A2) Also assume $\models \sigma$ [σ is valid]
   A3) If there is a non contradictory path from finite tableau, then $\mathcal{V}$ agrees with all the entries present in the path. [lemma 5.4]
   A4) implies $\mathcal{V}(\sigma)$ = F
   A5) σ is valid(i.e., tautology) implies $\mathcal{V}(\sigma)$ = T
   A6) Since $\mathcal{V}(\sigma)$ = T and $\mathcal{V}(\sigma)$ = F, it is a contradiction. [Valuation does not assign two values to one proposition]
   Hence our assumption is wrong and given statement is correct
   $\models \sigma \Rightarrow \vdash \sigma$

9. (10) Study carefully the proofs in L06. Prove lemma 5.2. You have to follow the methods we studied in L06.

Answer:
We prove this claim by induction on the depth of the entries on P.

- Base case (the entries with depth 0). We will prove $\mathcal{V}$ agrees with all entries, with depth 0, of P.
  For every such entry E, with depth 0, of P,
  since the depth is 0, it must be of the form TA or FA.
  Case 1. $E = TA$. By the definition of
  atomic tableau, $\mathcal{V}(A) = T$ and thus
  $\mathcal{V}$ agrees with E.
  Case 2. $E = FA$. By the definition of atomic tableau,
  $\mathcal{V}(A) = F$, hence,
  $\mathcal{V}$ agrees with E.
  Therefore, $\mathcal{V}$ agrees with E, by case 1 and 2.
  $\mathcal{V}$ agrees with E.

- Inductive hypothesis (IH) (on depth not more than n). We *assume* $\mathcal{V}$ agrees with all entries of P with depth at most n(n≥0).

- Prove the case of entries with depth of $n+1$, i.e., $\mathcal{V}$ agrees with all entries with depth n+1, of P.
  For every such entry E, with depth $n+1$, of P,
  since it has depth of $n+1$, it must be of one of the forms:
  $T(\alpha_1 \vee \alpha_2), T(\alpha_1 \wedge \alpha_2), T(\alpha_1 \rightarrow \alpha_2), T(\alpha_1 \leftrightarrow \alpha_2), T(\neg\alpha_1)$,
  $F(\alpha_1 \vee \alpha_2), F(\alpha_1 \wedge \alpha_2), F(\alpha_1 \rightarrow \alpha_2), F(\alpha_1 \leftrightarrow \alpha_2)$, or $F(\neg\alpha_1)$
  where α_1 (and α_2 respectively) has *at most depth of* n.
  We prove by cases.
  Case 1. $E = T(\alpha_1 \vee \alpha_2)$.
  Since τ is finite tableau, P a path on τ, E an entry on τ occurring on path P and τ' is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then τ' is also finite
  By definition of atomic tableau, $T(\alpha_1)$ or $T(\alpha_2)$ must occur on P.
  Case 1.1 $T(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_1)$,
  and thus $\mathcal{V}(\alpha_1) = T$. Therefore,
  $\mathcal{V}(\alpha_1 \vee \alpha_2) = T$, hence,
  $\mathcal{V}$ agrees with E.
  Case 1.2 $T(\alpha_2)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_2)$,
  and thus $\mathcal{V}(\alpha_2) = T$. Therefore,
  $\mathcal{V}(\alpha_1 \vee \alpha_2) = T$, hence,
  $\mathcal{V}$ agrees with E.
  $\mathcal{V}$ agrees with E, by cases 1.1 to 1.2.

  Case 2. $E = F(\alpha_1 \vee \alpha_2)$.
  Since τ is finite tableau, P a path on τ, E an entry on τ occurring on path P and τ' is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then τ' is also finite
  By definition of atomic tableau, $F(\alpha_1)$ and $F(\alpha_2)$ must occur on P.

Case 2.1 $F(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $F(\alpha_1)$,
and thus $\mathcal{V}(\alpha_1) = F$.
Case 2.2 $F(\alpha_2)$ occurs on P. By IH, $\mathcal{V}$ agrees with $F(\alpha_2)$,
and thus $\mathcal{V}(\alpha_2) = F$. Therefore,
$\mathcal{V}(\alpha_1 \vee \alpha_2) = F$, hence,
$\mathcal{V}$ agrees with E.
$\mathcal{V}$ agrees with E, by cases 2.1 to 2.2.

Case 3. $E = T(\alpha_1 \wedge \alpha_2)$.
Since τ is finite tableau, P a path on τ, E an entry on τ occurring on path P and $\tau^{'}$ is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then $\tau^{'}$ is also finite
By definition of atomic tableau, $T(\alpha_1)$ and $T(\alpha_2)$ must occur on P.
Case 3.1 $T(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_1)$,
and thus $\mathcal{V}(\alpha_1) = T$.
Case 3.2 $T(\alpha_2)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_2)$,
and thus $\mathcal{V}(\alpha_2) = T$. Therefore,
$\mathcal{V}(\alpha_1 \wedge \alpha_2) = T$, hence,
$\mathcal{V}$ agrees with E.
$\mathcal{V}$ agrees with E, by cases 3.1 to 3.2.

Case 4. $E = F(\alpha_1 \wedge \alpha_2)$.
Since τ is finite tableau, P a path on τ, E an entry on τ occurring on path P and $\tau^{'}$ is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then $\tau^{'}$ is also finite
By definition of atomic tableau, $F(\alpha_1)$ or $F(\alpha_2)$ must occur on P.
Case 4.1 $F(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $F(\alpha_1)$,
and thus $\mathcal{V}(\alpha_1) = F$. Therefore,
$\mathcal{V}(\alpha_1 \wedge \alpha_2) = F$, hence,
$\mathcal{V}$ agrees with E.
Case 4.2 $F(\alpha_2)$ occurs on P. By IH, $\mathcal{V}$ agrees with $F(\alpha_2)$,
and thus $\mathcal{V}(\alpha_2) = F$. Therefore,
$\mathcal{V}(\alpha_1 \wedge \alpha_2) =$, hence,
$\mathcal{V}$ agrees with E.
$\mathcal{V}$ agrees with E, by cases 4.1 to 4.2.

Case 5. $E = T(\alpha_1 \rightarrow \alpha_2)$.
Since τ is finite tableau, P a path on τ, E an entry on τ occurring on path P and $\tau^{'}$ is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then $\tau^{'}$ is also finite
By definition of atomic tableau, $F(\alpha_1)$ or $T(\alpha_2)$ must occur on P.
Case 5.1 $F(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $F(\alpha_1)$,
and thus $\mathcal{V}(\alpha_1) = F$. Therefore,
$\mathcal{V}(\alpha_1 \rightarrow \alpha_2) = T$, hence,

$\mathcal{V}$ agrees with E.
Case 5.2 $T(\alpha_2)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_2)$,
and thus $\mathcal{V}(\alpha_2) = T$. Therefore,
$\mathcal{V}(\alpha_1 \to \alpha_2) = T$, hence,
$\mathcal{V}$ agrees with E.
$\mathcal{V}$ agrees with E, by cases 5.1 to 5.2.

Case 6. $E = F(\alpha_1 \to \alpha_2)$.
Since τ is finite tableau, P a path on τ, E an entry on τ occurring on path P and τ' is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then τ' is also finite
By definition of atomic tableau, $T(\alpha_1)$ and $F(\alpha_2)$ must occur on P.
Case 6.1 $T(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_1)$,
and thus $\mathcal{V}(\alpha_1) = T$.
Case 6.2 $F(\alpha_2)$ occurs on P. By IH, $\mathcal{V}$ agrees with $F(\alpha_2)$,
and thus $\mathcal{V}(\alpha_2) = F$. Therefore,
$\mathcal{V}(\alpha_1 \to \alpha_2) = F$, hence,
$\mathcal{V}$ agrees with E.
$\mathcal{V}$ agrees with E, by cases 6.1 to 6.2.

Case 7. $E = T(\alpha_1 \leftrightarrow \alpha_2)$.
Since τ is finite tableau, P a path on τ, E an entry on τ occurring on path P and τ' is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then τ' is also finite
By definition of atomic tableau, $T(\alpha_1)$ and $T(\alpha_2)$ or $F(\alpha_1)$ and $F(\alpha_2)$ must occur on P.
Case 7.1 $T(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_1)$,
and thus $\mathcal{V}(\alpha_1) = T$.
Case 7.2 $T(\alpha_2)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_2)$,
and thus $\mathcal{V}(\alpha_2) = T$. Therefore,
$\mathcal{V}(\alpha_1 \leftrightarrow \alpha_2) = T$, hence,
$\mathcal{V}$ agrees with E.
Case 7.3 $F(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $F(\alpha_1)$,
and thus $\mathcal{V}(\alpha_1) = F$.
Case 7.4 $F(\alpha_2)$ occurs on P. By IH, $\mathcal{V}$ agrees with $F(\alpha_2)$,
and thus $\mathcal{V}(\alpha_2) = F$. Therefore,
$\mathcal{V}(\alpha_1 \leftrightarrow \alpha_2) = T$, hence,
$\mathcal{V}$ agrees with E.
$\mathcal{V}$ agrees with E, by cases 7.1 to 7.4.

Case 8. $E = F(\alpha_1 \leftrightarrow \alpha_2)$.
Since τ is finite tableau, P a path on τ, E an entry on τ occurring on path P and τ' is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then τ' is also finite
By definition of atomic tableau, $T(\alpha_1)$ and $F(\alpha_2)$ or $F(\alpha_1)$ and $T(\alpha_2)$ must occur on P.

Case 8.1 $T(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_1)$,
and thus $\mathcal{V}(\alpha_1) = T$.
Case 8.2 $F(\alpha_2)$ occurs on P. By IH, $\mathcal{V}$ agrees with $F(\alpha_2)$,
and thus $\mathcal{V}(\alpha_2) = F$. Therefore,
$\mathcal{V}(\alpha_1 \leftrightarrow \alpha_2) = F$, hence,
$\mathcal{V}$ agrees with E.
Case 8.3 $F(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $F(\alpha_1)$,
and thus $\mathcal{V}(\alpha_1) = F$.
Case 8.4 $T(\alpha_2)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_2)$,
and thus $\mathcal{V}(\alpha_2) = T$. Therefore,
$\mathcal{V}(\alpha_1 \leftrightarrow \alpha_2) = F$, hence,
$\mathcal{V}$ agrees with E.
$\mathcal{V}$ agrees with E, by cases 8.1 to 8.4.

Case 9. $E = T(\neg\alpha_1)$.

Since τ is finite tableau, P a path on τ, E an entry on τ occurring on path P and $\tau^{'}$ is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then $\tau^{'}$ is also finite

By definition of atomic tableau, $F(\alpha_1)$ must occur on P.
$F(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $F(\alpha_1)$,
and thus $\mathcal{V}(\alpha_1) = F$. Therefore,
$\mathcal{V}(\neg\alpha_1) = T$, hence,
$\mathcal{V}$ agrees with E.

Case 10. $E = F(\neg\alpha_1)$.

Since τ is finite tableau, P a path on τ, E an entry on τ occurring on path P and $\tau^{'}$ is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then $\tau^{'}$ is also finite

By definition of atomic tableau, $T(\alpha_1)$ must occur on P.
$T(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_1)$,
and thus $\mathcal{V}(\alpha_1) = T$. Therefore,
$\mathcal{V}(\neg\alpha_1) = F$, hence,
$\mathcal{V}$ agrees with E.

$\mathcal{V}$ agrees with E.
Therefore, $\mathcal{V}$ agrees with E, by cases 1 to 10.
$\mathcal{V}$ agrees with E.

Appendix. A proof (see latex source for the latex code for this proof).

Proof. In this proof, the *number of connectives* of an entry of a path in a tableau, is defined as the number of connectives of the proposition of this entry.

We prove this claim by induction on the number of connectives of the entries on P.

- Base case (the entries with 0 connectives). We will prove $\mathcal{V}$ agrees with all entries, with 0 connectives, of P.
  For every such entry E, with 0 connectives, of P,
  since it has 0 connectives, it must be of the form TA or FA.
  Case 1. $E = TA$. By the definition of
  $\mathcal{A}$, $\mathcal{V}(A) = T$ and thus
  $\mathcal{V}$ agrees with E.
  Case 2. $E = FA$. By the definition of $\mathcal{A}$,
  $\mathcal{V}(A) = F$, hence,
  $\mathcal{V}$ agrees with E.
  Therefore, $\mathcal{V}$ agrees with E, by case 1 and 2.
  $\mathcal{V}$ agrees with E.

- Inductive hypothesis (IH) (on number of connectives not more than n). We *assume* $\mathcal{V}$ agrees with all entries, *with at most* n *(*$n \geq 0$*) connectives*, of P.

- Prove the case of entries with $n + 1$ connectives, i.e., $\mathcal{V}$ agrees with all entries, *with* $n + 1$ *connectives*, of P.
  For every such entry E, with $n + 1$ connectives, of P,
  since it has $n + 1$ connectives, it must be of one of the forms:
  $T(\alpha_1 \vee \alpha_2), T(\alpha_1 \wedge \alpha_2), T(\alpha_1 \rightarrow \alpha_2), T(\alpha_1 \leftrightarrow \alpha_2), T(\neg\alpha_1)$,
  $F(\alpha_1 \vee \alpha_2), F(\alpha_1 \wedge \alpha_2), F(\alpha_1 \rightarrow \alpha_2), F(\alpha_1 \leftrightarrow \alpha_2)$, or $F(\neg\alpha_1)$
  where α_1 (and α_2 respectively) has *at most* n connectives.
  We prove by cases.
  Case 1. $E = T(\alpha_1 \vee \alpha_2)$.
  Since τ is finished, P is finished and thus E is reduced.
  By definition of *reduced*, $T(\alpha_1)$ or $T(\alpha_2)$ must occur on P.
  Case 1.1 $T(\alpha_1)$ occurs on P. By IH, $\mathcal{V}$ agrees with $T(\alpha_1)$,
  and thus $\mathcal{V}(\alpha_1) = T$. Therefore,
  $\mathcal{V}(\alpha_1 \vee \alpha_2) = T$, hence,
  $\mathcal{V}$ agrees with E.
  Case 1.2 $T(\alpha_2)$ occurs on P.
  We can prove, similarly to case 1.1, that
  $\mathcal{V}$ agrees with E.
  $\mathcal{V}$ agrees with E, by cases 1.1 to 1.2.
  Case 2 to 10. We can prove similarly,
  $\mathcal{V}$ agrees with E.

Therefore, $\mathcal{V}$ agrees with E, by cases 1 to 10.
$\mathcal{V}$ agrees with E.

QED