Homework 5 . Predicate Logic: Syntax and Semantics

Submit your solution in PDF file (and Latex source file if you use Latex) to blackboard by **11:59pm Sun Nov 20**.

1. Consider the following sentence.

   Every number greater than or equal to 4 can be written as the sum of two prime numbers.

   (a) Write *a language* (as defined in Def 2.1) such that some formulas of the language can be used to represent the sentence above.

   (b) Write a formula of your language that should reflect the meaning of the sentence above.

   (c) In terms of your language, write three example terms, three example atomic formulas, and three example formulas.

2. Write a formula to represent the following information. Your formula should be as close as possible to the intended meaning of these sentences.

   (a) *There is a mother to all children.*

   (b) ALL ITEMS NOT AVAILABLE AT ALL STORES.

   Note, the sentence above is a disclaimer in the weekly flyer of specials of a grocery store chain.

3. Given the language defined in Def 2.1, which of the following are formulas defined by Def 2.5?

   (a) $f(x, c)$

   (b) $R(c, f(d, z))$

   (c) $\forall x(P(x))$

   (d) $((\exists x)(((\forall y)P(z)) \to R(x, y)))$

4. Given

$$((\exists x)(((\forall y)P(z)) \to R(x, y))),$$

(a) List all its subformulas.
(b) Draw its formation tree.

5. Which of the following terms are substitutable for x in the corresponding formulas?

   (a) $f(z, y)$ in $((\exists y)(P(y) \wedge R(x, z)))$.
   (b) $g(f(z, y), a)$ in $((\exists x)(P(x) \wedge R(x, y)))$.

6. Connect predicate calculus to the study of this course. To focus on the substance, we need to extend (informally) predicate calculus (syntax) as follows

   - You can use sets or proposition as a parameter of a predicate.
   - You can use $=$ as a predicate symbol in the normal way. For example, that two terms t_1 and t_2 are equal can be represented by $t_1 = t_2$.
   - You can use a variable to refer to a function or a set or a proposition. (e.g., $((\exists \mathcal{V})\mathcal{V}(x) = T)$ where T is constant.)

   The form of formula will be expanded accordingly to the extension above. Consider the definition of *consequence* (Def 3.7 of Part I).

   (a) Represent it as a formula. You need to introduce all predicates you need in the formula, in the way we did in L11.1 for the subset example. Intuitively, a concept name and its parameters/arguments you identify in the definition is a good candidate of a predicate. You also need to introduce any constant or function symbol you need. Note the main "if" in a definition should be understood as "if and only if." Note statement "$\forall x \in A, x \in B$" in the subset definition. We translate it to "$((\forall x)(x \in A \rightarrow x \in B))$."
   (b) Let your formula be α. What can you logically derive (some intuition/experience from your earlier study is needed here) from the formula ($\alpha\wedge$ "B is a consequence of $\{A, A \rightarrow B\}$")? Note you should translate the English in the formula using predicate(s) you introduced. Your answer to this question has to be in the form of a formula. You need to do a variable substitution (Σ in English definition would be replaced by $\{A, A \rightarrow B\}$, and σ by B).

7. 1) Let $A = \{1, 2\}$. a) List all functions from A to A in the form of sets of pair. For example, one function is $\{(1, 1), (2, 2)\}$. If we let the function be named g. Then in the example function, $g(1)$ is 1 and $g(2)$ is 2. b) List all unary relations on A. Your relations must be represented as sets.

   2) Show that $\forall x(p(x) \rightarrow q(f(x))) \wedge \forall x p(x) \wedge \exists x \neg q(x)$ is satisfiable. The format of your structure should follow the one in L12.1. In your structure, you must use the domain $A = \{1, 2\}$. You must represent the function assigned to f in the set of pairs. You must represent the relations assigned to p and q in the set forms. Note you have to figure out the constants in the language on which the formula is defined.

8. Prove that $\mathcal{A} \models \neg\exists x\varphi(x)$ if and only if $\mathcal{A} \models \forall x\neg\varphi(x)$. You have to follow the proof format we used earlier. Working backward again is a good idea. You have to be able to apply the definitions.