Sets, Relations and Languages

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TEXAS TECH
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Sets – Definition

- A set is an unordered collection of objects
- $S = \{a, b, c\}$
  - a, b, c are elements or members of the set S

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An abstraction of an object, can be numbers, points, lines, graphs, vectors, matrices⋯

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- {red, blue} = {blue, red}

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- {red, blue, red} = {red, blue}
- {red, blue} = {blue, red}
- Element may also be a set: {red, {red}} ≠ {red}
- Empty set: {} or ∅
- {{}} is **not** empty

Sets – Cardinality

- Cardinality of a set is the number of elements in the set
- $|\{a, b, c\}| = 3$

  |{red, blue}|=2

  $|\{\{a, b\}, c\}| = 2$

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  $|\{\{a, b\}, c\}| = 2$
- $|\{\}| = 0$

Sets – Singleton

- Set with one element is called Singleton

  |{red}|=1

  $|\{a\}| = 1$

Sets – Singleton

- Set with one element is called Singleton

  |{red}|=1

  $|\{a\}| = 1$
- $|\{\{\}\}| = 1$

Sets – Membership

- Element a contained in a set $S = \{a, b, c\}$: $a \in S$
  - $3 \in \{1,3,5\}$
  - $a \notin \{b, c, d\}$
  - $a \in \{a, \{b, c\}\}$
  - $b \notin \{a, \{b, c\}\}$

Sets – Describing

- Different ways of writing a set
  - List all members: $S = \{a, b, c\}$, $S = \{2,3,4,5 \ldots\}$
  - $S = \{x : x$ has a certain property$\}$
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    $S = \{x : x \in N \text{ and } x < 10\}$

    $S = \{x : x \text{ is prime}\}$

Sets – Subset

- A is a subset of B, denoted as $A \subseteq B$ if:
  - Any element of A is also an element of B
  - Any ($\forall$) $x \in A$ implies ($\Rightarrow$) $x \in B$
  - If $x \in A$, then $x \in B$

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- A is a proper subset of B, denoted as $A \subset B$ if:
  - $A \subseteq B$ and B contains at least one element not in A

Sets – Subset (examples)

- A is a subset of B:
  - $\{a, b, c\} \subseteq \{a, b, c, d\}$
  - $\{1,3,5,7, \ldots\} \subseteq \{1,2,3,4,5, \ldots\}$
  - $\emptyset \subseteq A$ for any set A
  - $A \subseteq A$
- A is a proper subset of B:
  - - $\{a, b, c\} \subset \{a, b, c, d\}$
  - $\{1,3,5,7, \ldots\} \subset \{1,2,3,4,5, \ldots\}$
  - $\emptyset \subset A$ for any set $A \neq \emptyset$
  - $A \not\subset A$

Sets – Operations

- Union
  - $A \cup B = \{x: x \in A \text{ or } x \in B\}$
  - $\{1,3,9\} \cup \{3,5,7\} = \{1,3,5,7,9\}$
  - {red, blue} ∪ {red, {red, blue}} = {red, blue, {red, blue}}

Sets – Operations

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  - {red, blue} ∪ {red, {red, blue}} = {red, blue, {red, blue}}
- Intersection
  - $A \cap B = \{x: x \in A \text{ and } x \in B\}$
  - $\{1,3,9\} \cap \{3,5,7\} = \{3\}$
  - {red, blue} ∩ {red, {red, blue}} = {red}
  - {red, blue} ∩ {{red, blue}} = {}

Sets – Operations

- Difference
  - $A - B = \{x : x \in A \text{ and } x \notin B\}$
  - $\{1,3,9\} - \{3,5,7\} = \{1,9\}$
  - {red, blue} − {red, {red, blue}} = {blue}

Sets – Power Set

- Power set: Set of all subsets
  - $2^S = \{x : x \subseteq S\}$
  - $2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
  - $2^{\emptyset} = \{\emptyset\}$
  - $|2^S| = 2^{|S|}$

Sets – Partition

- Π is a partition of S if:
  - $\Pi \subseteq 2^S$
  - $\emptyset \notin \Pi$
  - $A \cap B = \emptyset$ for any $A, B \in \Pi, \mathrm{A} \neq B$
  - $\cup\, \Pi = S$

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  - $A \cap B = \emptyset$ for any $A, B \in \Pi, \mathrm{A} \neq B$
  - $\cup\, \Pi = S$
- $\{\{a\}, \{b\}, \{c\}\}$ is a partition of $\{a, b, c\}$

Sets – Proof Format

- Q: For two arbitrary sets A, B, show $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

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  \- Any element of X is also an element of Y, and every element of Y is also an element of X.

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- What does it mean by $X = Y$?

  - Any element of X is also an element of Y, and every element of Y is also an element of X.

  - Equivalently, $X \subseteq Y$ and $Y \subseteq X$.

Sets – Proof Format

- Q: For two arbitrary sets A, B, show $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- We first show: $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$
  - We aim to show any $x \in (A \cap B) \cup C$, $x \in (A \cup C) \cap (B \cup C)$

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  - By definition of Union, $x \in (A \cap B) \cup C$ means $x \in (A \cap B)$ or $x \in C$

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  - By definition of Union, $x \in (A \cap B) \cup C$ means $x \in (A \cap B)$ or $x \in C$
  - Thus, we aim to show
    i) $x \in (A \cap B)$, then $x \in (A \cup C) \cap (B \cup C)$, and
    ii) $x \in C$, then $x \in (A \cup C) \cap (B \cup C)$

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  - Thus, we aim to show

    i) $x \in (A \cap B)$, then $x \in (A \cup C) \cap (B \cup C)$, and

    1. If $x \in (A \cap B)$, then by definition of intersection, $x \in A$ and $x \in B$;

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  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
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    1. If $x \in (A \cap B)$, then by definition of intersection, $x \in A$ and $x \in B$;
    2. Because $x \in A$, $x \in (A \cup C)$ (by the definition of union)

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  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- We first show: $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$
  - We aim to show any $x \in (A \cap B) \cup C$, $x \in (A \cup C) \cap (B \cup C)$
  - By definition of Union, $x \in (A \cap B) \cup C$ means $x \in (A \cap B)$ or $x \in C$
  - Thus, we aim to show

    i) $x \in (A \cap B)$, then $x \in (A \cup C) \cap (B \cup C)$, and

    1. If $x \in (A \cap B)$, then by definition of intersection, $x \in A$ and $x \in B$;
    2. Because $x \in A$, $x \in (A \cup C)$ (by the definition of union)
    3. Because $x \in B$, $x \in (B \cup C)$

Sets – Proof Format

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  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- We first show: $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$
  - We aim to show any $x \in (A \cap B) \cup C$, $x \in (A \cup C) \cap (B \cup C)$
  - By definition of Union, $x \in (A \cap B) \cup C$ means $x \in (A \cap B)$ or $x \in C$
  - Thus, we aim to show

    i) $x \in (A \cap B)$, then $x \in (A \cup C) \cap (B \cup C)$, and

    1. If $x \in (A \cap B)$, then by definition of intersection, $x \in A$ and $x \in B$;
    2. Because $x \in A$, $x \in (A \cup C)$ (by the definition of union)
    3. Because $x \in B$, $x \in (B \cup C)$
    4. Hence, $x \in (A \cup C) \cap (B \cup C)$ (by the definition of intersection)

Sets – Proof Format

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  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- We first show: $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$
  - We aim to show any $x \in (A \cap B) \cup C$, $x \in (A \cup C) \cap (B \cup C)$
  - By definition of Union, $x \in (A \cap B) \cup C$ means $x \in (A \cap B)$ or $x \in C$
  - Thus, we aim to show

    ii) $x \in C$, then $x \in (A \cup C) \cap (B \cup C)$

    1. Because $x \in C$, $x \in (A \cup C)$ (why?)
    2. Because $x \in C$, then $x \in (B \cup C)$ (why?)
    3. Because $x \in (A \cup C)$ and $x \in (B \cup C)$, $x \in (A \cup C) \cap (B \cup C)$ (why?)

Sets – Proof Format

- Q: For two arbitrary sets A, B, show

  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- We first show: $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$

$x \in (A \cap B) \cup C$, then $x \in (A \cup C) \cap (B \cup C)$

$x \in (A \cap B)$ or $x \in C$, then $x \in (A \cup C) \cap (B \cup C)$

i) $x \in (A \cap B)$, then $x \in (A \cup C) \cap (B \cup C)$, and
ii) $x \in C$, then $x \in (A \cup C) \cap (B \cup C)$

Summarize and rewrite in the logic flow

Sets – Proof Format

- Q: For two arbitrary sets A, B, show $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- We next show: $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$

Homework