L05 Tableau Proofs in Propositional Logic
Chapter 4

1. Motivation

Major concept $\Sigma \models \sigma$

One goal is to *automatically* construct a proof to show the above

First work on $\{\} \models \sigma$, i.e., σ is a tautology.

Tableaux proof idea (intuition: proof by contradiction)

(A \/ -A) is tautology
Proof (by contradiction) [informal]

Assume (A \/ -A) is F.
(working forward) by definition of \/
(1) A is F, and
(2) -A is F
(3) A is T (by (2) and definition of -)
Contradiction (1) and (3)

QED

Prove (A /\ B) is tautology
Proof (by contradiction) [**not a proof** below]

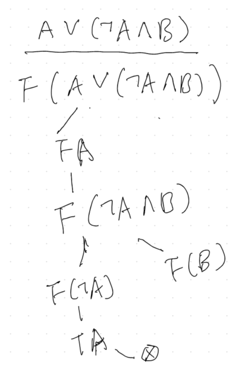
Assume (A /\ B) is F.
(working forward) by definition of /\

We consider two case	we consider 3 cases
(1) A is F …. No contradiction	
	A is F B is T
(2) **OR** B is F … No contradiction	A is F B is F
No Contradiction	A is T B is F

QED

2. Motivation of tableaux proof

Motivation of tableaux proof: proof by contradiction. A signed proposition Fα means assume α is false or we would like α to be false. Worked out atomic tableaux ourselves. Work out a few examples on reducing a signed proposition (all paths are contradiction, no path is contradiction or some contradiction and some not)



Finishing studying the finite tableaux / tableaux definition. Check what are defined? Using drawing (tree, path) to check understanding of the definition.

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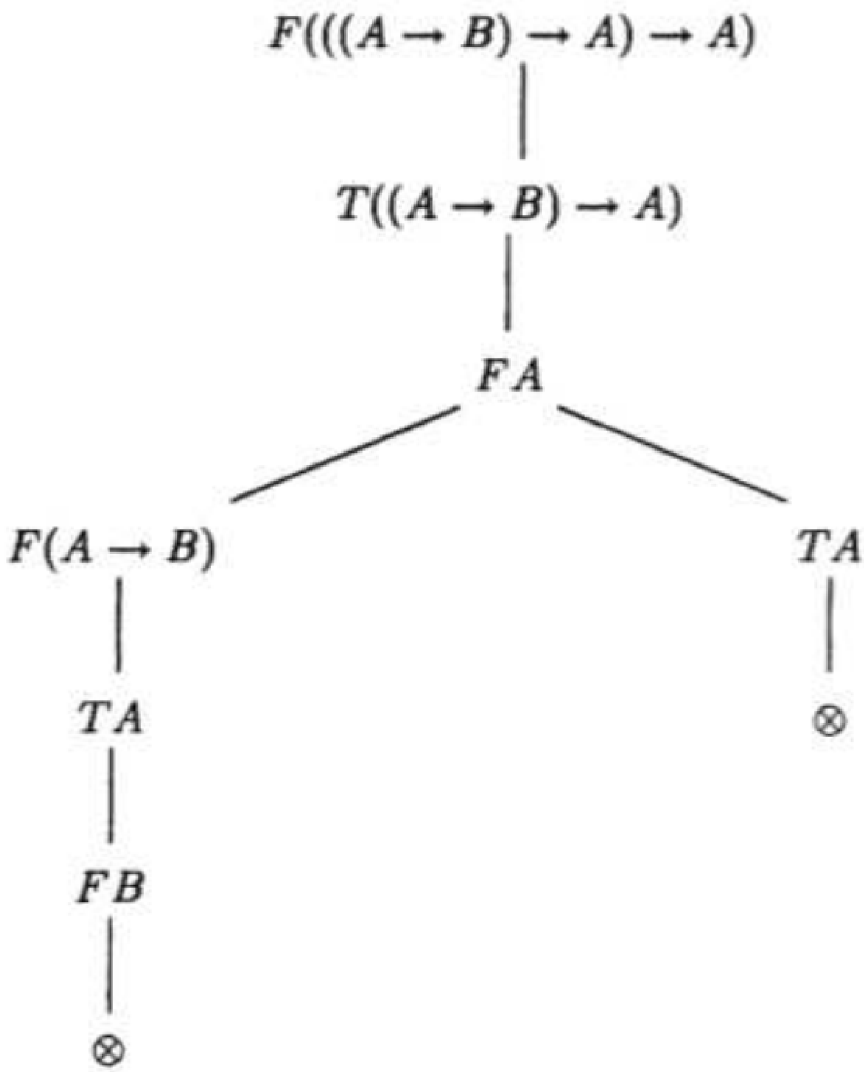
T((A \/ B) /\ (C \/ D))
 |
T(A\/ B)
|
T(C \/ D)
|                    \
T(A)                T(B)
```

A precise informal reading of signed proposition:
F(\alpha): we would like to find a way (valuation) to make \alpha to be false
T(\alpha): we would like to find a way (valuation) to make \alpha to be true
Entries on a path is understood as conjunction. For example

```
T(A \/ B)
  |       \
T(A)    T(B)
```

Read as we would like to find a valuation to make A \/ B true and to make A true.
In this case, since we have to make A to be true and thus A \/ B true, we don't need to consider an alternative case of making B to be true.

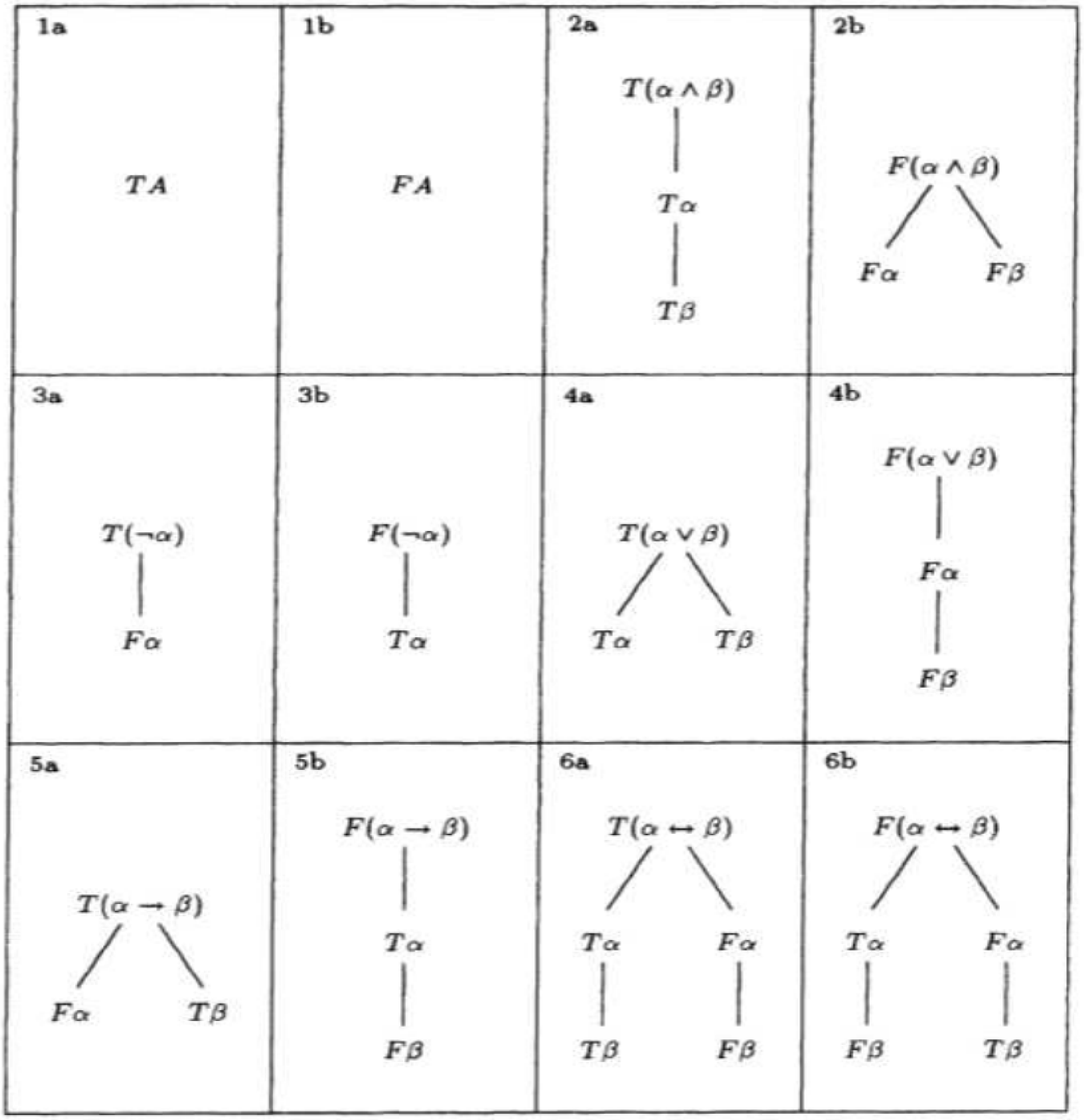
Prove $(((A \to B) \to A) \to A)$ is a tautology



3. Studying the definitions

3.1 Definitions of finite tableau and tableau

Read from tableaux def (most time of class spent on the first definition)


1a
TA
1b
FA
2a
$T(\alpha \wedge \beta)$
$T\alpha$
$T\beta$
2b
$F(\alpha \wedge \beta)$
$F\alpha$
$F\beta$
3a
$T(\neg\alpha)$
$F\alpha$
3b
$F(\neg\alpha)$
$T\alpha$
4a
$T(\alpha \vee \beta)$
$T\alpha$
$T\beta$
4b
$F(\alpha \vee \beta)$
$F\alpha$
$F\beta$
5a
$T(\alpha \rightarrow \beta)$
$F\alpha$
$T\beta$
5b
$F(\alpha \rightarrow \beta)$
$T\alpha$
$F\beta$
6a
$T(\alpha \leftrightarrow \beta)$
$T\alpha$
$F\alpha$
$T\beta$
$F\beta$
6b
$F(\alpha \leftrightarrow \beta)$
$T\alpha$
$F\alpha$
$F\beta$
$T\beta$

Definition 4.1 A finite tableau is a binary tree, labeled with signed propositions called entries, that satisfies the following inductive definition:

(i) All atomic tableaux are finite tableaux.

(ii) If τ is a finite tableau, P a path on τ, E an entry of τ occurring on P and τ' is obtained from τ by adjoining the unique atomic tableau with root entry E to τ at the end of the path P, then τ' is also a finite tableau.

If $\tau_0, \tau_1, ..., \tau_n, ...$ is a (finite or infinite) sequence of finite tableaux such that, for each $n \geq 0$, τ_{n+1} is constructed from τ_n by an application of (ii), then $\tau = \cup \tau_n$ is a tableau.

Drawing a picture helps to understand the definition.

Exercise: figure out the union of graphs, apply it to the union of two finite tableaux

G1 = (V1, E1) G2 = (V2, E2)

G1 U G2 = (V1 U V2, E1 U E2)

3.2 Definitions of tableau proof and complete systematic tableau

We would like to precisely define

- Tableau proof, and
- A systematic way to construct a tableau proof called complete systematic tableau
  - Continue to “analyze” (or reduce) each node in a tableau until each node is “reduced” (so that no cheating to say there is no tableau proof by intentionally stopping the analyze of each node)

Definition 4.2 Let τ be a tableau, P a path on τ and E an entry occurring on P.

(i) E has been reduced on P if all the entries on one path through the atomic tableau with root E occur on P. (E.g., TA and FA are reduced for every propositional letter A. $T\neg\alpha$ and $F\neg\alpha$ are reduced (on P) if $F\alpha$ and α, respectively, appear on P. $T(\alpha \vee \beta)$ is reduced if either $T\alpha$ or $T\beta$ appears on P. $F(\alpha \vee \beta)$ is reduced if both $F\alpha$ and $F\beta$ appear on P.)

(ii) P is contradictory if, for some proposition α, $T\alpha$ and $F\alpha$ are both entries on P. P is finished if it is contradictory or every entry on P is reduced on P.

(iii) τ is finished if every path through τ is finished.

(iv) τ is contradictory if every path through τ is contradictory. (It is, of course, then finished as well.)

Definition 4.3 A tableau proof of a proposition α is a contradictory tableau with root entry $F\alpha$. A proposition is tableau provable, written $\vdash \alpha$, if it has a tableau proof.

A tableau refutation for a proposition α is a contradictory tableau starting with $T\alpha$. A proposition is tableau refutable if it has a tableau refutation.

Definition 4.4 Let R be a signed proposition. We define the complete systematic tableau (CST) with root entry R by induction.

- Let τ_0 be the unique atomic tableau with R at its root.
- Assume that τ_m has been defined. Let n be the smallest level of τ_m containing an entry that is unreduced on some noncontradictory path in τ_m and let E be the leftmost such entry of level n. We now let τ_{m+l} be the tableau gotten by adjoining the unique atomic tableau with root E to the end of every noncontradictory path of τ_m on which E is unreduced. The union of the sequence τ_m is our desired complete systematic tableau.

Theorem 4.5 Every CST is finished.

Theorem 4.6 If $\tau = \cup\tau_n$ is a contradictory tableau, then for some m, τ_m is a finite contradictory tableau. Thus, in particular, if a CST is a proof, it is a finite tableau.

Definition 4.7 Define the degree of a proposition α, $d(\alpha)$ by induction:

(i) If α is a propositional letter, then $d(\alpha) = 0$.

(ii) If α is $\neg\beta$, then $d(\alpha) = d(\beta) + 1$.

(iii) If α is $\beta \vee \gamma$, $\beta \wedge \gamma$, $\beta \rightarrow \gamma$ or $\beta \leftrightarrow \gamma$, then $d(\alpha) = d(\beta) + d(\gamma) + 1$. The degree of a signed proposition $T\alpha$ or $F\alpha$ is the degree of α. If P is a path in a tableau τ, then $d(P)$ the degree of P is the sum of the signed propositions on P that are not reduced on P.

Theorem 4.8 Every CST is finite