Context-Free Grammars

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TEXAS TECH

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Context-Free Grammars

- Recall our goal
  - we want to study languages of interest
  - we start with "artificial" languages "easier" than, e.g., English
    * Regular language

      DFA as a checker

      Regular expression as a generator

To communicate with a machine in English, it should be able to verify whether I have inputted a valid English sentence, and be able to generate a valid English sentence as a response

Context-Free Grammars

- Regular language is not powerful enough:
  - $\{0^n 1^n : n \geq 0\}$
- How do we characterize such languages?
- Context free grammars

Context-Free Grammars

- Regular language is not powerful enough:
  - $\{0^n 1^n : n \geq 0\}$
- How do we characterize such languages?
- Context free grammars
  - In a regular language, we start with basic symbols, and use union, concatenation and Kleene star to generate longer strings
  - In a context free language, we start with a single symbol, and iteratively replace it with other symbols to generate longer strings

Context-Free Grammars

- Examples:

$S \rightarrow aS$
$S \rightarrow Bb$
$B \rightarrow cB$
$B \rightarrow e$

Generating a string:
S - replace S with aS
aS - replace S wtih Bb
aBb - replace B wtih cB
$acBb$ - replace B wtih e
acb - Final String

Context-Free Grammars

- Example:
  - $S \rightarrow aMb$
  - $M \rightarrow A$
  - $M \rightarrow B$
  - $A \rightarrow e$
  - $A \rightarrow aA$
  - $B \rightarrow e$
  - $B \rightarrow bB$

Generating a string:

S - replace S with aMb

aMb - replace M wtih B

aBb - replace B wtih bB

$abBb$ - replace B wtih e

abb - Final String

The replacement does not depend on the context

Context-Free Grammars

- A context-free grammar G is a quadruple (V, Σ, R, S), where
  - V is an alphabet
  - Σ (set of terminals) is a subset of V
  - $V - \Sigma$ (set of nonterminals or variables)
  - R (set of rules) is a finite subset of $(V - \Sigma) \times V^*$

    Each rule: Nonterminal → Terminals and/or Nonterminals
  - S (the start symbol) is an element of $V - \Sigma$

Context-Free Grammars

- Start with the initial symbol
- Repeat:
  - Pick any non-terminal in the string
  - Replace that nonterminal with the right-hand side of some rule that has that nonterminal as a left-hand side
- Until all elements in the string are terminals

Context-Free Grammars

- More example:
- $\{a^n b^n : \geq 0\}$
  - $S \rightarrow aSb$
  - $S \rightarrow e$

Equivalently, - $S \rightarrow aSb$ | e

$S \Rightarrow_G aSb \Rightarrow aaSbb \Rightarrow aabb$

Directly yields

$S \Rightarrow_G^* aabb$

Yields

Context-Free Grammars

- More example:

⟨SENTENCE⟩ → ⟨NOUN-PHRASE⟩⟨VERB-PHRASE⟩
⟨NOUN-PHRASE⟩ → ⟨CMPLX-NOUN⟩ | ⟨CMPLX-NOUN⟩⟨PREP-PHRASE⟩
⟨VERB-PHRASE⟩ → ⟨CMPLX-VERB⟩ | ⟨CMPLX-VERB⟩⟨PREP-PHRASE⟩
⟨PREP-PHRASE⟩ → ⟨PREP⟩⟨CMPLX-NOUN⟩
⟨CMPLX-NOUN⟩ → ⟨ARTICLE⟩⟨NOUN⟩
⟨CMPLX-VERB⟩ → ⟨VERB⟩ | ⟨VERB⟩⟨NOUN-PHRASE⟩
⟨ARTICLE⟩ → `a` | `the`
⟨NOUN⟩ → `boy` | `girl` | `flower`
⟨VERB⟩ → `touches` | `likes` | `sees`
⟨PREP⟩ → `with`

Context-Free Grammars

- More example:

⟨SENTENCE⟩ → ⟨NOUN-PHRASE⟩⟨VERB-PHRASE⟩
⟨NOUN-PHRASE⟩ → ⟨CMPLX-NOUN⟩ | ⟨CMPLX-NOUN⟩⟨PREP-PHRASE⟩
⟨VERB-PHRASE⟩ → ⟨CMPLX-VERB⟩ | ⟨CMPLX-VERB⟩⟨PREP-PHRASE⟩
⟨PREP-PHRASE⟩ → ⟨PREP⟩⟨CMPLX-NOUN⟩
⟨CMPLX-NOUN⟩ → ⟨ARTICLE⟩⟨NOUN⟩
⟨CMPLX-VERB⟩ → ⟨VERB⟩ | ⟨VERB⟩⟨NOUN-PHRASE⟩
⟨ARTICLE⟩ → a | the
⟨NOUN⟩ → boy | girl | flower
⟨VERB⟩ → touches | likes | sees
⟨PREP⟩ → with

⟨SENTENCE⟩ ⇒ ⟨NOUN-PHRASE⟩⟨VERB-PHRASE⟩
⇒ ⟨CMPLX-NOUN⟩⟨VERB-PHRASE⟩
⇒ ⟨ARTICLE⟩⟨NOUN⟩⟨VERB-PHRASE⟩
⇒ a ⟨NOUN⟩⟨VERB-PHRASE⟩
⇒ a boy ⟨VERB-PHRASE⟩
⇒ a boy ⟨CMPLX-VERB⟩
⇒ a boy ⟨VERB⟩
⇒ a boy sees

Context-Free Grammars

- More example:

⟨SENTENCE⟩ → ⟨NOUN-PHRASE⟩⟨VERB-PHRASE⟩
⟨NOUN-PHRASE⟩ → ⟨CMPLX-NOUN⟩ | ⟨CMPLX-NOUN⟩⟨PREP-PHRASE⟩
⟨VERB-PHRASE⟩ → ⟨CMPLX-VERB⟩ | ⟨CMPLX-VERB⟩⟨PREP-PHRASE⟩
⟨PREP-PHRASE⟩ → ⟨PREP⟩⟨CMPLX-NOUN⟩
⟨CMPLX-NOUN⟩ → ⟨ARTICLE⟩⟨NOUN⟩
⟨CMPLX-VERB⟩ → ⟨VERB⟩ | ⟨VERB⟩⟨NOUN-PHRASE⟩
⟨ARTICLE⟩ → a | the
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⟨SENTENCE⟩ ⇒ ⟨NOUN-PHRASE⟩⟨VERB-PHRASE⟩
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⇒ a ⟨NOUN⟩⟨VERB-PHRASE⟩
⇒ a boy ⟨VERB-PHRASE⟩
⇒ a boy ⟨CMPLX-VERB⟩
⇒ a boy ⟨VERB⟩
⇒ a boy sees

the boy sees a flower
a girl with a flower likes the boy

Context-Free Grammars

- More examples:

$$W = \{S, A, N, V, P\} \cup \Sigma,$$
$$\Sigma = \{\text{Jim}, \text{big}, \text{green}, \text{cheese}, \text{ate}\},$$
$$R = \{P \to N,$$
$$P \to AP,$$
$$S \to PVP,$$
$$A \to \text{big},$$
$$A \to \text{green},$$
$$N \to \text{cheese},$$
$$N \to \text{Jim},$$
$$V \to \text{ate}\}$$

S: sentence
A: adjective
N: noun
V: verb
P: phrase

Context-Free Grammars

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S: sentence
A: adjective
N: noun
V: verb
P: phrase

$S \Rightarrow PVP$
$\Rightarrow NVN$
$\Rightarrow$ Jim ate cheese

Context-Free Grammars

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$$R = \{P \to N,$$
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$$A \to \text{big},$$
$$A \to \text{green},$$
$$N \to \text{cheese},$$
$$N \to \text{Jim},$$
$$V \to \text{ate}\}$$

S: sentence
A: adjective
N: noun
V: verb
P: phrase

$S \Rightarrow PVP$
$\Rightarrow APVAP$
$\Rightarrow ANVAN$
$\Rightarrow$ big Jim ate green cheese

Context-Free Grammars

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$$W = \{S, A, N, V, P\} \cup \Sigma,$$
$$\Sigma = \{\text{Jim, big, green, cheese, ate}\},$$
$$R = \{P \to N,$$
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$S \Rightarrow PVP$
$\Rightarrow APVP$
$\Rightarrow ANVN$
$\Rightarrow$ big cheese ate Jim

Context-Free Grammars

- More examples:

$$W = \{S, A, N, V, P\} \cup \Sigma,$$
$$\Sigma = \{\text{Jim, big, green, cheese, ate}\},$$
$$R = \{P \to N,$$
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$$A \to \text{big},$$
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S: sentence
A: adjective
N: noun
V: verb
P: phrase

$$S \Rightarrow PVP$$
$$\Rightarrow APVAP$$
$$\Rightarrow AAPVAAP$$
$$\Rightarrow AAAPVAAAP$$
$$\cdots$$
$$\Rightarrow AA \cdots ANVAA \cdots AN$$

Context-Free Grammars

- More example:
- all strings of properly balanced left and right

$$
\begin{aligned}
V &= \{S, (,)\}, \\
\Sigma &= \{(,)\}, \\
R &= \{S \to e, \\
&\qquad S \to SS, \\
&\qquad S \to (S)\}.
\end{aligned}
$$

$$S \Rightarrow SS \Rightarrow (S)S \Rightarrow ()S \Rightarrow ()(S) \Rightarrow ()(())$$

Context-Free Grammars

- More example:
- Some non-regular language can be generated by CFG

$$\{0^n 1^n : n \geq 0\}$$

- $S \rightarrow 0S1$
- $S \rightarrow e$

Context-Free Grammars

- More example:
- The complement of some non-regular language can be generated by CFG

  $\bar{L}$, where $L = \{0^n 1^n : n \geq 0\}$

Context-Free Grammars

- More example:
- The complement of some non-regular language can be generated by CFG

$\bar{L}$, where $L = \{0^n 1^n : n \geq 0\}$

There are three types of strings in $\bar{L}$:

1. $w = 0^m 1^n,\ m < n$
2. $w = 0^m 1^n,\ n < m$
3. w contains 10 as a substring.

Context-Free Grammars

- More example:

  $\bar{L}$, where $L = \{0^n 1^n : \geq 0\}$

  - type 1: $0^m 1^n$ with more 1s than 0s

Context-Free Grammars

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  - type 1: $0^m 1^n$ with more 1s than 0s

    Any string of type 1 falls into one of the followings:
    * the string 1,
    * consists of a string of type 1, followed by one 1, or
    * consists of one 0, followed by an arbitrary string of type 1, followed by one 1.

Context-Free Grammars

- More example:

  $\bar{L}$, where $L = \{0^n 1^n : \geq 0\}$

  - type 1: $0^m 1^n$ with more 1s than 0s

    Any string of type 1 falls into one of the followings:

    * the string 1,
    * consists of a string of type 1, followed by one 1, or
    * consists of one 0, followed by an arbitrary string of type 1, followed by one 1.

      - $S_1 \rightarrow 1$
      - $S_1 \rightarrow S_1 1$
      - $S_1 \rightarrow 0 S_1 1$

Context-Free Grammars

- More example:

  $\bar{L}$, where $L = \{0^n 1^n : \geq 0\}$

  - type 2: $0^m 1^n$ with more 0s than 1s

    Any string of type 2 falls into one of the followings:
    * the string 0,
    * starts with 0, followed by a string of type 2, or
    * consists of one 0, followed by an arbitrary string of type 2, followed by one 1.

Context-Free Grammars

- More example:

  $\bar{L}$, where $L = \{0^n 1^n : \geq 0\}$

  - type 2: $0^m 1^n$ with more 0s than 1s

    Any string of type 2 falls into one of the followings:

    * the string 0,
    * starts with 0, followed by a string of type 2, or
    * consists of one 0, followed by an arbitrary string of type 2, followed by one 1.

      - $S_2 \rightarrow 0$
      - $S_2 \rightarrow 0S_2$
      - $S_2 \rightarrow 0S_2 1$

Context-Free Grammars

- More example:

  $\bar{L}$, where $L = \{0^n 1^n : \geq 0\}$

  - type 3: contains 10

    Any string of type 3 is as follows:
    * Start with an arbitrary binary string, followed by 10, then followed by arbitrary binary string

Context-Free Grammars

- More example:

  $\bar{L}$, where $L = \{0^n 1^n : \geq 0\}$

  - type 3: contains 10

    Any string of type 3 is as follows:

    * Start with an arbitrary binary string, followed by 10, then followed by arbitrary binary string

      - $X \rightarrow e$
      - $X \rightarrow 0X$
      - $X \rightarrow 1X$
      - $S_3 \rightarrow X10X$

Context-Free Grammars

- More example:

  $\bar{L}$, where $L = \{0^n 1^n : \geq 0\}$

  - $S \rightarrow S_1$
  - $S \rightarrow S_2$
  - $S \rightarrow S_3$

  - $S_1 \rightarrow 1$
  - $S_1 \rightarrow S_1 1$
  - $S_1 \rightarrow 0S_1 1$

  - $S_2 \rightarrow 0$
  - $S_2 \rightarrow 0S_2$
  - $S_2 \rightarrow 0S_2 1$

  - $X \rightarrow e$
  - $X \rightarrow 0X$
  - $X \rightarrow 1X$
  - $S_3 \rightarrow X10X$

Context-Free Grammars

- More example:

  $\bar{L}$, where $L = \{0^n 1^n : \geq 0\}$

  - $S \rightarrow S_1$
  - $S \rightarrow S_2$
  - $S \rightarrow S_3$

$$
\begin{aligned}
S &\rightarrow S_1 | S_2 | S_3 \\
S_1 &\rightarrow 1 | S_1 1 | 0 S_1 1 \\
S_2 &\rightarrow 0 | 0 S_2 | 0 S_2 1 \\
S_3 &\rightarrow X 10 X \\
X &\rightarrow e | 0X | 1X
\end{aligned}
$$

- $S_1 \rightarrow 1$
- $S_1 \rightarrow S_1 1$
- $S_1 \rightarrow 0 S_1 1$

- $S_2 \rightarrow 0$
- $S_2 \rightarrow 0 S_2$
- $S_2 \rightarrow 0 S_2 1$

- $X \rightarrow e$
- $X \rightarrow 0X$
- $X \rightarrow 1X$
- $S_3 \rightarrow X10X$

Context-Free Grammars

- More example:
- A CFG that verifies addition: $L = \{a^n b^m c^{m+n} : n \geq 0, m \geq 0\}$

Context-Free Grammars

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- A CFG that verifies addition: $L = \{a^n b^m c^{m+n} : n \geq 0, m \geq 0\}$
  - 1. we construct $a^n c^n$: every time we add one a, we also add one c
  - 2. b is inserted between a^n and c^n: add one b, we also add one c

Context-Free Grammars

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- A CFG that verifies addition: $L = \{a^n b^m c^{m+n} : n \geq 0, m \geq 0\}$
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  - 2. b is inserted between a^n and c^n: add one b, we also add one c

$S \rightarrow A$

$A \rightarrow e | aAc$

Context-Free Grammars

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$$S \rightarrow A$$
$$A \rightarrow e|aAc|B$$

Context-Free Grammars

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- A CFG that verifies addition: $L = \{a^n b^m c^{m+n} : n \geq 0, m \geq 0\}$
  - 1. we construct $a^n c^n$: every time we add one a, we also add one c
  - 2. b is inserted between a^n and c^n: add one b, we also add one c

$S \rightarrow A$

$A \rightarrow e|aAc|B$

$B \rightarrow e|bBc$