Sets, Relations and Languages

Lin Chen

Email: Lin.Chen@ttu.edu

TEXAS TECH
UNIVERSITY

Ordered Pair

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- x, y are called components (of the ordered pair)

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- x, y are called components (of the ordered pair)
- Order matters
  - $(x, y) \neq (y, x)$ if $x \neq y$
  - $\{x, y\} = \{y, x\}$
- Components can be identical
  - (x, x) is a valid ordered pair
  - $(x, x) \neq x$

Cartesian Product

- $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$
- $\{a, b\}\times\{c, d\} = \{(a, c), (a, d), (b, c), (b, d)\}$
- $2^{\{a\}\times\{b\}} =?$

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- $2^{\{a\} \times \{b\}} = 2^{\{(a,b)\}} = \{\emptyset, (a, b)\}$
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- $2^{\{a\} \times \{b\}} = 2^{\{(a,b)\}} = \{\emptyset, \{(a, b)\}\}$
- $2^{\{a\}} \times 2^{\{b\}} = \big\{\{a\}, \emptyset\big\} \times \big\{\{b\}, \emptyset\big\} = \{(\{a\}, \{b\}), (\{a\}, \emptyset), (\emptyset, \{b\}), (\emptyset, \emptyset)\}$

Ordered Tuple

- $(a_1, a_2, \cdots, a_n)$ is an ordered (n-)tuple
- Each a_i is called the (i-th) component (of the ordered tuple)
- Ordered 2-tuple is ordered pair

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- Each a_i is called the (i-th) component (of the ordered tuple)
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- $A_1 \times A_2 \times A_3 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) : a_i \in A_i \text{ for all } i\}$
  - $\{a\} \times \{b, c\} \times \{d\} = \{(a, b, d), (a, c, d)\}$

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  - $\{a\} \times \{b, c\} \times \{d\} = \{(a, b, d), (a, c, d)\}$
  - The definition implies that $\big((a, b), c\big) = (a, b, c)$. It is true since we use () to indicate that objects are ordered. Both $\big((a, b), c\big)$ and (a, b, c) denote the 3 objects are placed in the order of a, b, c .

Relations

- A relation R is a set of ordered pairs
  - Attendance sheet produces a relation: $\{(Alice, X), (Bob, -), \cdots\}$
  - $<$ over natural numbers is also a relation

    { (0,1), (0,2), (0,3), ...

    (1,2), (1,3), (1,4), ...

    (2,3), (2,4), (2,5), ...

    ... }

Functions

- A function is a special kind of relation (all functions are relations, but not all relations are functions)
- A relation $R \subseteq A \times B$ is a function if:

 - For each $a \in A$, there is exactly one ordered pair in R with the first component a

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- Two relations (C: set of cities, S: set of states)
  - $R_1 = \{(x, y): x \in C, y \in S, x \text{ is a city in state } y\}$
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Functions

- A function f that is a subset of $A \times B$ is written as:
  - $f : A \mapsto B$
- $(a, b) \in f$ is written $f(a) = b$
- A is the domain of the function
- if $A' \subseteq A, f(A') = \{b : f(a) = b \text{ for some } a \in A'\}$ is the image of A'
- The range of a function is the image of its domain

Functions

- A function $f : A \mapsto B$ is:

  - one-to-one, if no two elements in A match to the same element in B

  - onto, if each element in B is mapped to by at least one element in A

  - bijection, if it is both one-to-one and onto

Inverse

- The inverse of a binary relation $R \subseteq A \times B$ is $\{(b, a) \colon (a, b) \in R\}$, and denoted as R^{-1}.

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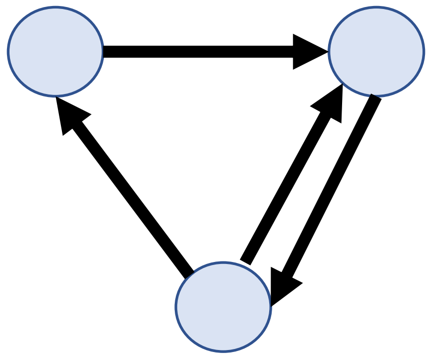
  - $R_1 = \{(x, y) \colon x \in C, y \in S, x \text{ is a city in state } y\}$
  - $R_2 = \{(y, x) \colon x \in C, y \in S, x \text{ is a city in state } y\}$
- The inverse of a relation, which is not a function, can be a function
- The inverse of a function may fail to be a function
- If a function is bijection, then its inverse is also a function

Relation Graph

- Representing a relation $R \subseteq A \times A$ using a directed graph

  - A directed graph (or digraph) is a graph that is made up of a set of vertices (nodes) connected by edges, where the edges have a direction associated with them



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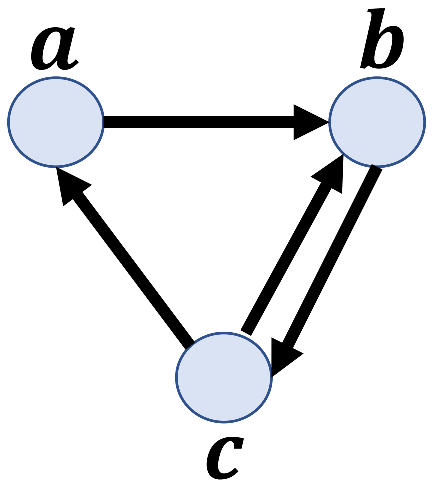
  - Use nodes to represent elements

  - Use directed edge to represent an ordered pair



$$R = \{(a,b),(b,c),(c,b),(c,a)\}$$

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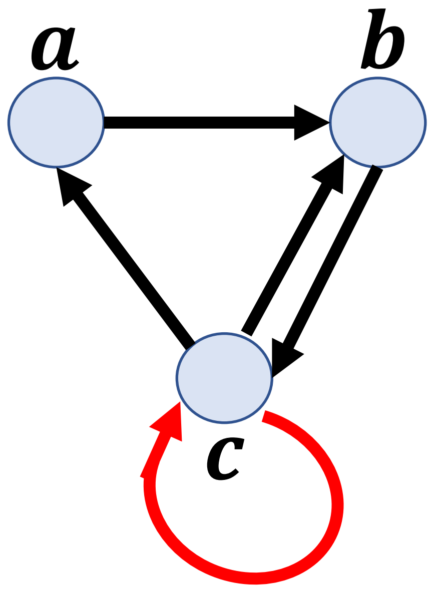
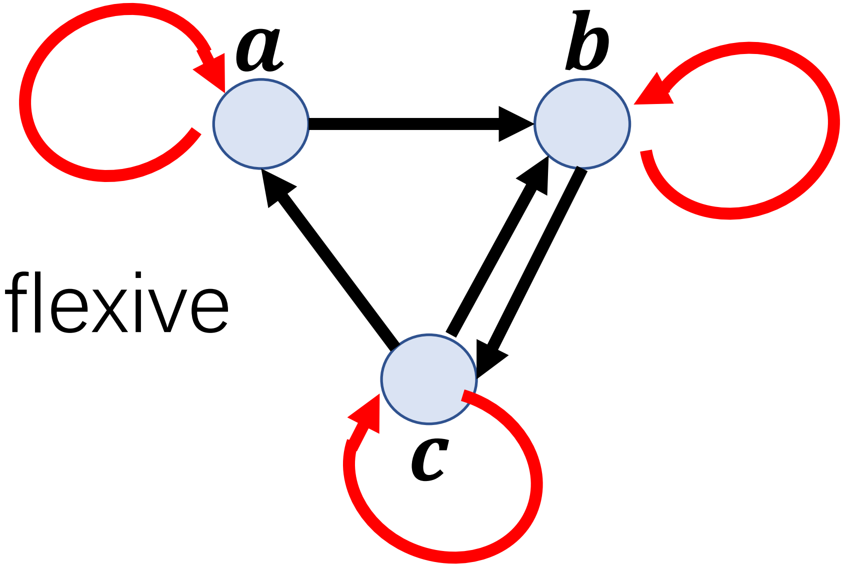
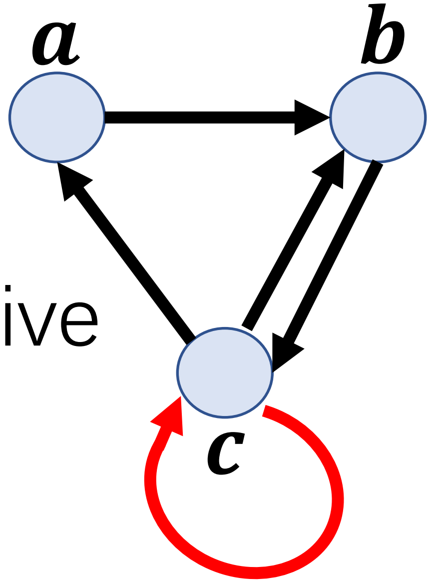


Relation Types

- A relation $R \subseteq A \times A$ is reflexive if
  - $(a, a) \in R$ for every $a \in A$
  - Equivalently, every node has a self-loop



reflexive

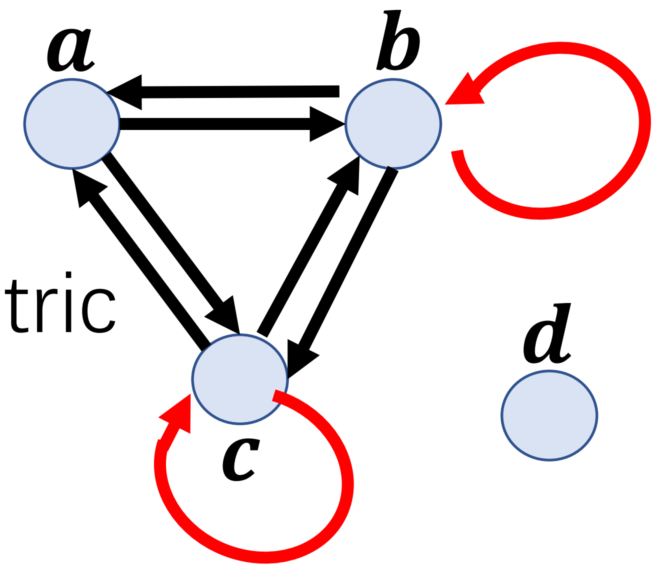


Not reflexive

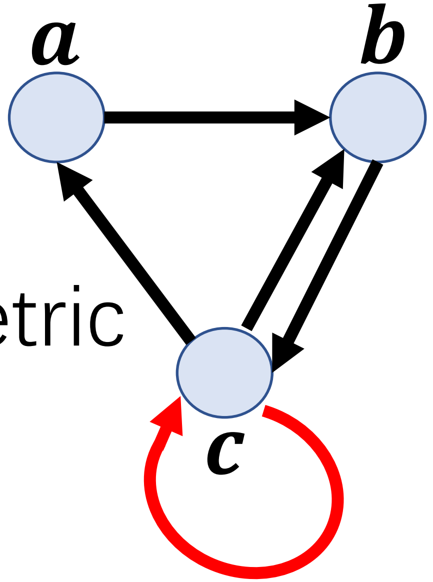
Relation Types

- A relation $R \subseteq A \times A$ is symmetric if
  - $(a, b) \in R$ if $(b, a) \in R$
  - Equivalently, every edge between two nodes is "two-ways"

symmetric



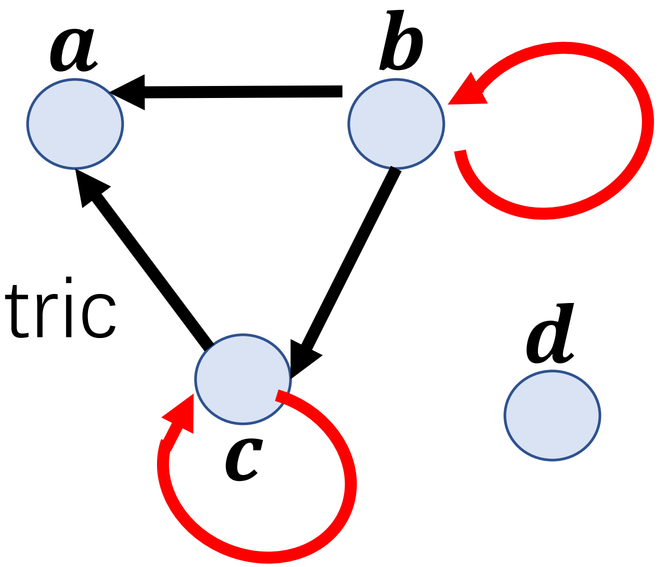
Not symmetric



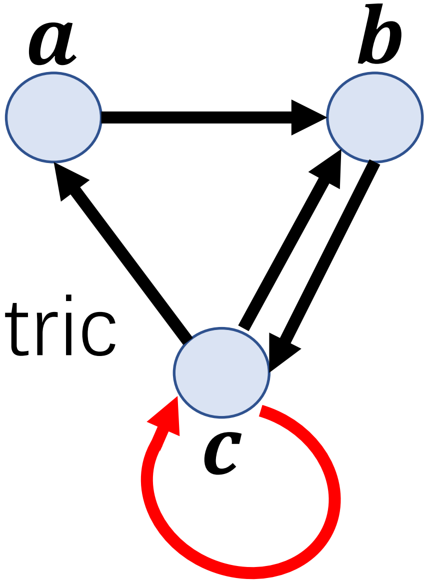
Relation Types

- A relation $R \subseteq A \times A$ is antisymmetric if
  - If $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$
  - Equivalently, every edge between two nodes is "one-way"

antisymmetric



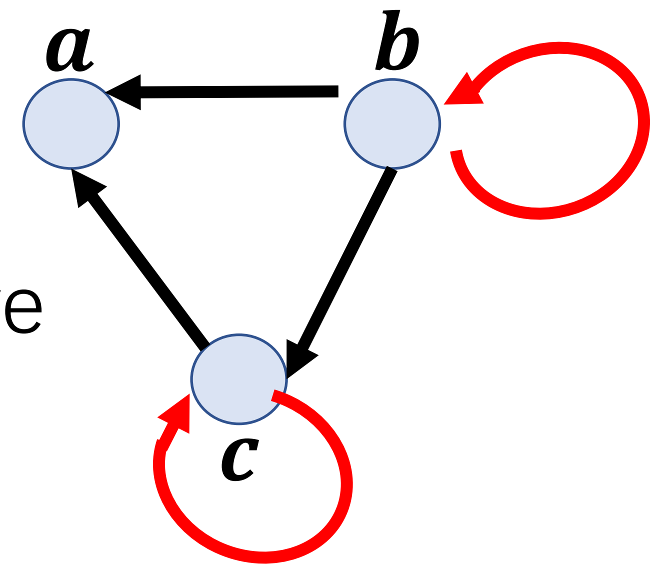
Not antisymmetric



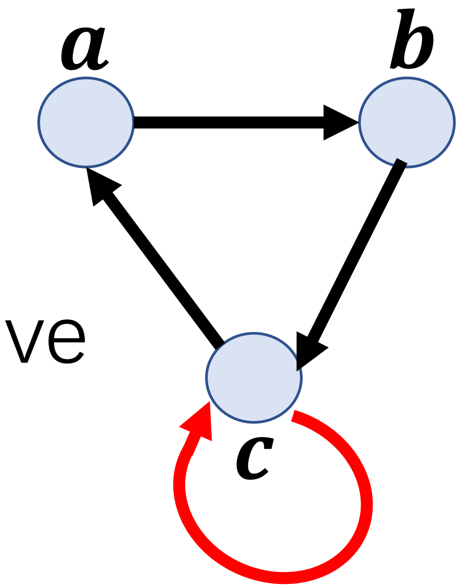
Relation Types

- A relation $R \subseteq A \times A$ is transitive if
  - If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$
  - Equivalently, you can always "shortcut"

transitive



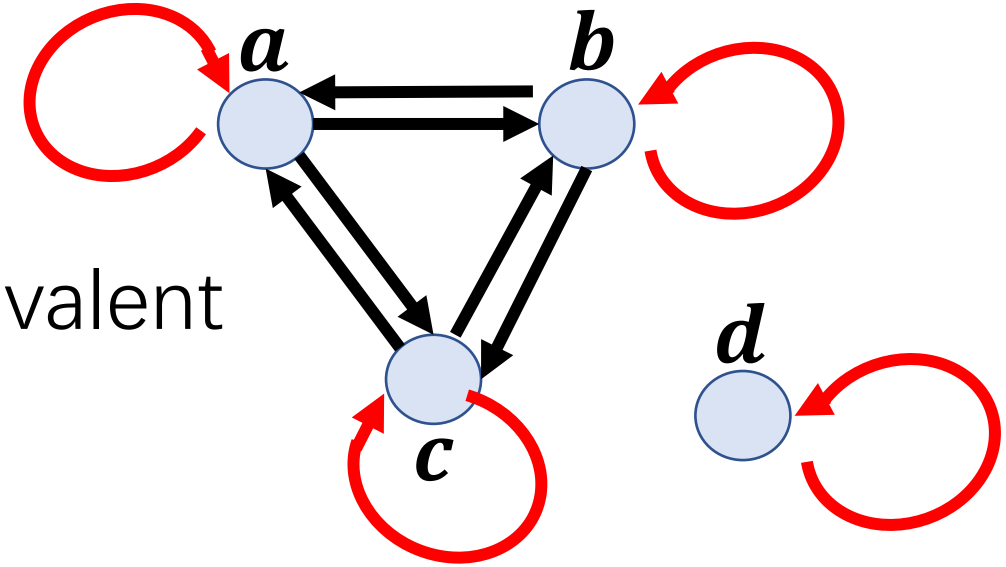
Not transitive



Relation Types

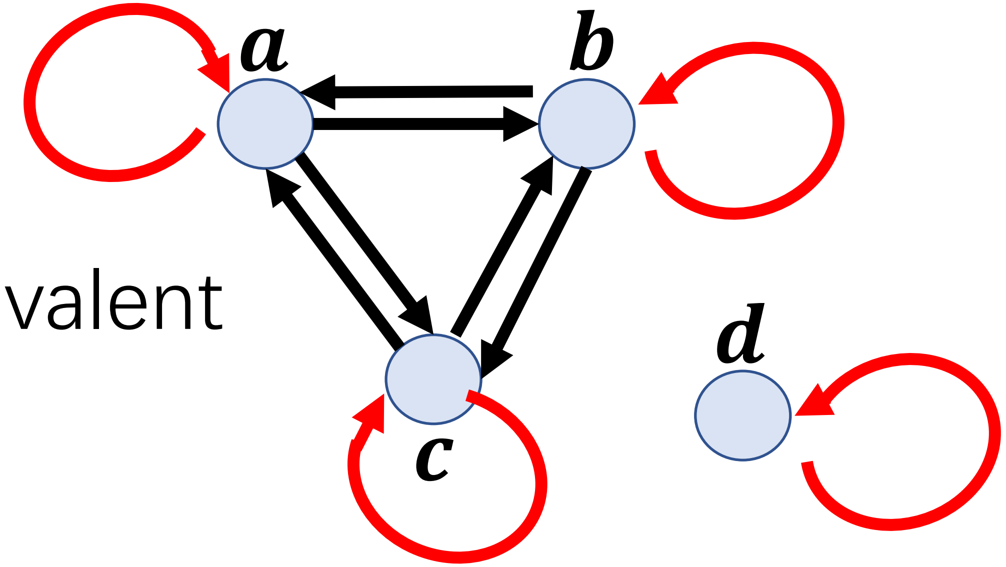
- A relation $R \subseteq A \times A$ is an equivalence relation if
  - If it is reflexive, symmetric and transitive

Equivalent



Relation Types

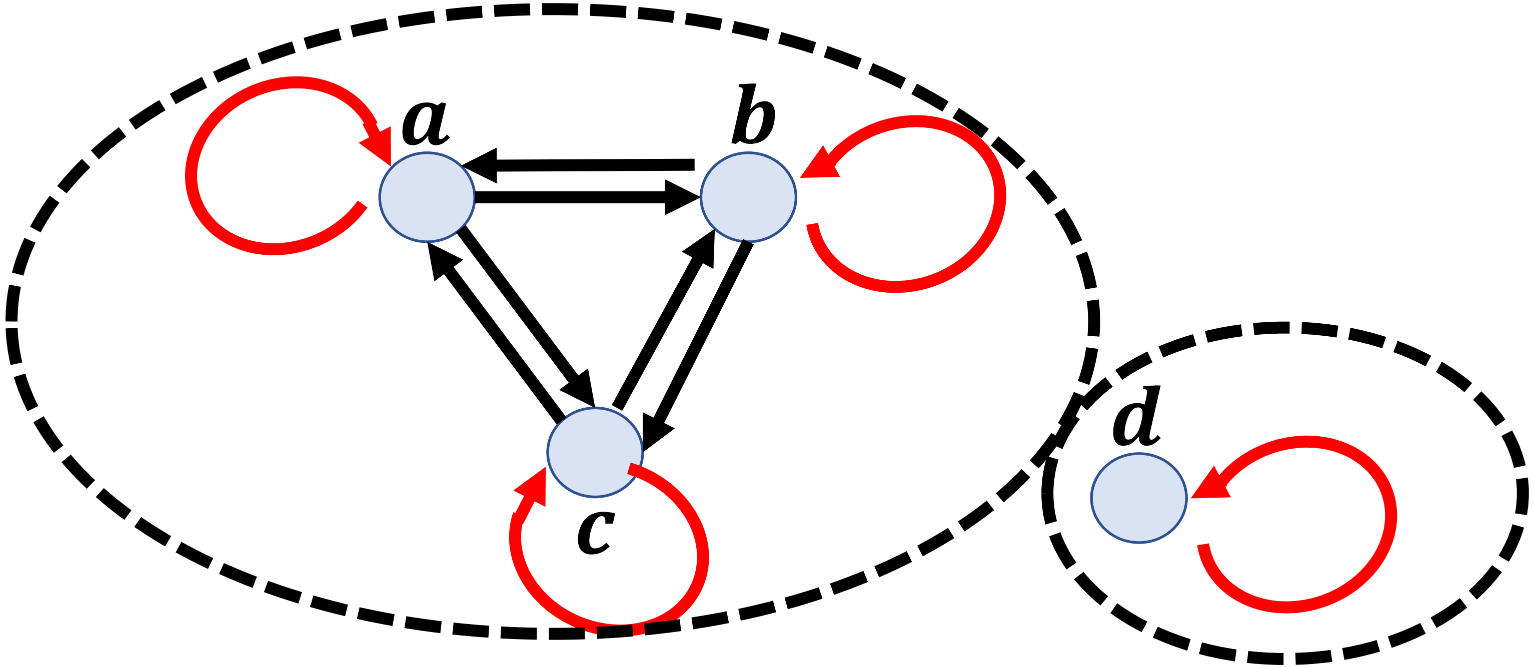
- A relation $R \subseteq A \times A$ is an equivalence relation if
  - If it is reflexive, symmetric and transitive
- Example: A (All English words), $(\mathrm{w}_1, \mathrm{w}_2) \in R$ if they start with same letter



Equivalent

Relation Types

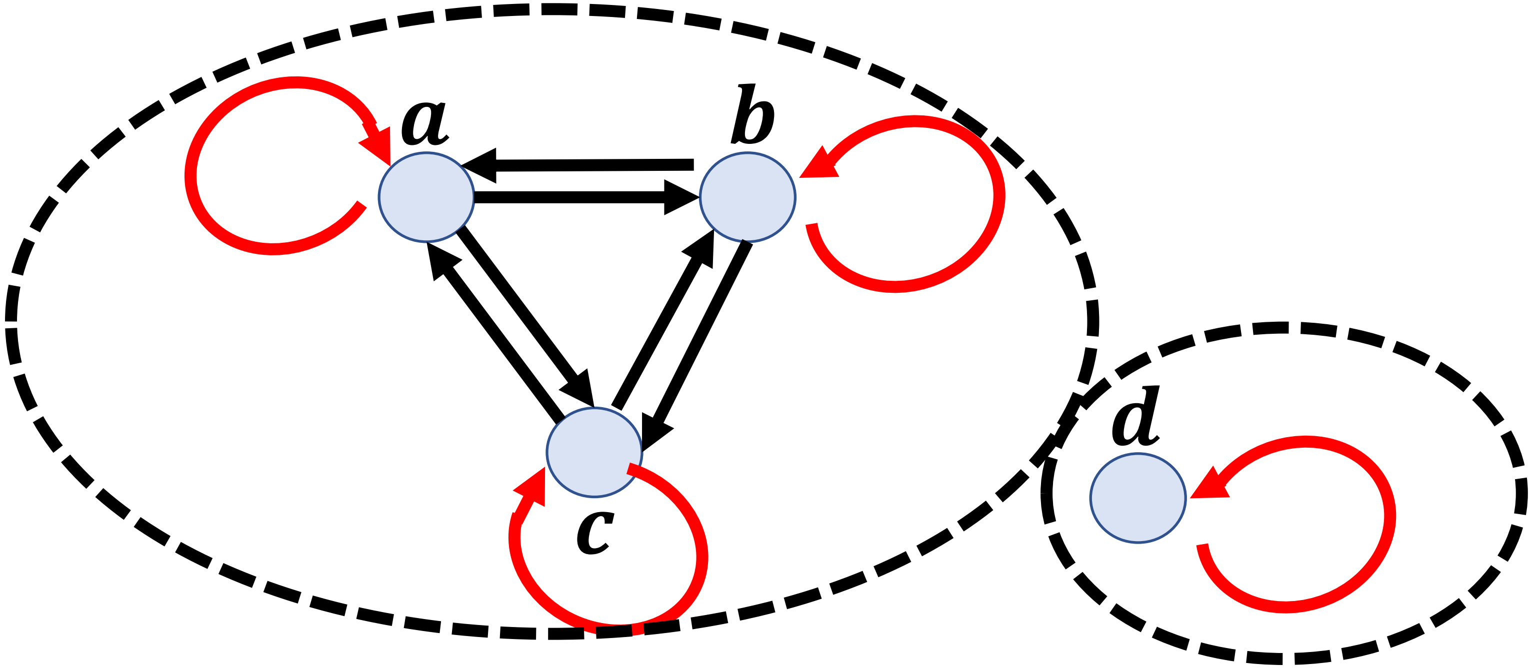
- A relation $R \subseteq A \times A$ is an equivalence relation if
  - If it is reflexive, symmetric and transitive
- Each connected component form an equivalence class



Relation Types

- A relation $R \subseteq A \times A$ is an equivalence relation if
  - If it is reflexive, symmetric and transitive
- Each connected component form an equivalence class

We can pick any element of an equivalent class to represent this class and denote it as $[a]$



Relation Types

- A relation $R \subseteq A \times A$ is a partial order if
  - If it is reflexive, antisymmetric and transitive
- Example:

1. allow a person to be considered as an ancestor of himself/herself, then

$$R = \{(a, b): a, b \text{ are persons and a is an ancestor of } b\}$$

2. $\leq$ defined on natural numbers

Relation Types

- A relation $R \subseteq A \times A$ is a total order if
  - If it is a partial order, and
  - For all $a, b \in A$, either $(a, b) \in R$ or $(b, a) \in R$
- Example:
  1. Ancestor relationship is not a total order
     $$R = \{(a, b): a, b \text{ are persons and a is an ancestor of } b\}$$
  2. $\leq$ defined on natural numbers is a total order

Closure

- A set $A \subseteq B$ is closed under a relation $R \subseteq ((B \times B) \times B)$ if:
  - $a_1, a_2 \in A$ and $((a_1, a_2), c) \in R \Rightarrow c \in A$
  - That is, if a_1, a_2 are both in A, and $((a_1, a_2), c)$ is in the relation, then c is also in A
- N is closed under addition
- N is not closed under subtraction or division

Cardinality

- How can we tell if two sets A and B have the same cardinality?
  - Calculate $|A|$ and $|B|$, see if the two numbers equal

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- What if I don't do the calculation?
  - Pick one apple from one basket, then pick one from another
  - If both of them become empty at the same step → same cardinality

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  - Create a bijection $f : A \mapsto B$

    Why it has to be a bijection?

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    Why it has to be a bijection?

    - Bijection is one-one: if I pick one apple from A, I must also pick one from B, thus no two apples in A are mapped to the same apple in B

Cardinality

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- What if I don't do the calculation?
  - Pick one apple from one basket, then pick one from another
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    Why it has to be a bijection?

    - Bijection is onto: Eventually, the basket B should be empty.

Cardinality

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  - Calculate $|A|$ and $|B|$, see if the two numbers equal
- What if I don' t do the calculation?
  - Pick one apple from one basket, then pick one from another
  - Create a bijection $f : A \mapsto B$
- If there is a bijection, we say A and B are equinumerous

Cardinality

- A set is finite if it is equinumerous with $\{1,2,\cdots,n\}$ for some n
- Otherwise, the set if infinite

Countable sets

- A set is finite if it is equinumerous with $\{1,2,\cdots,n\}$ for some n
- Otherwise, the set if infinite
- Not all infinite sets are equinumerous
  - A set is countably infinite if it is equinumerous with N
  - A set is countable if it is finite or countably infinite
  - Otherwise, it is uncountable

Countable sets

- A set is countable if it is equinumerous with N
  - Even numbers of N

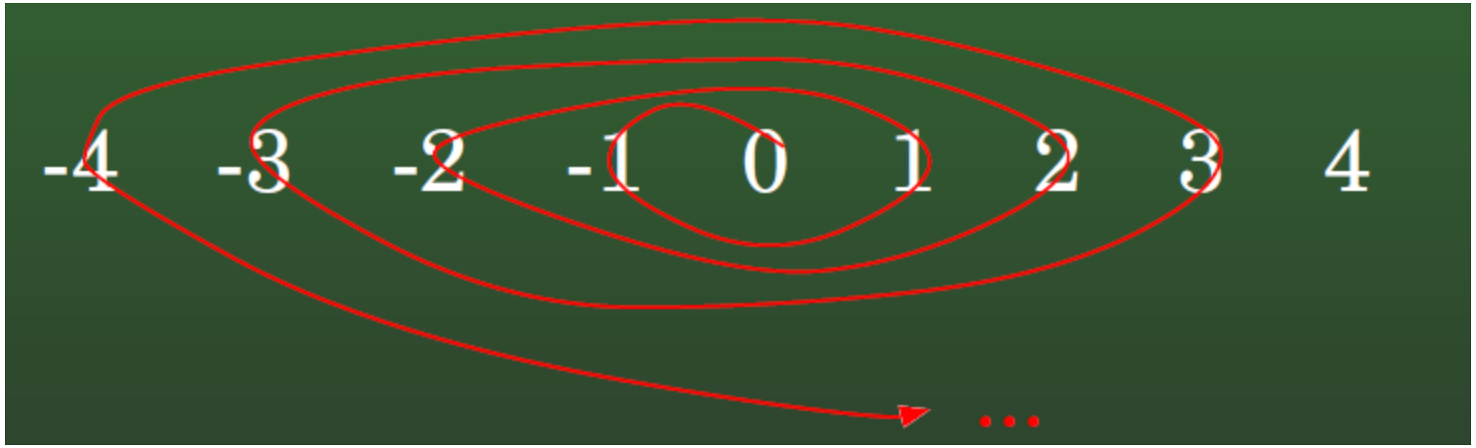
    $f(x) = 2x$

Countable sets

- A set is countable if it is equinumerous with N
  - Even numbers of N

    $f(x) = 2x$
  - All integers

    $f(x) = \left\lceil \frac{x}{2} \right\rceil * (-1)^x$



Countable sets

- A set is countable if it is equinumerous with N
  - Union of two disjoint countable sets

$$\begin{tikzcd}
a_0, \arrow[d] & a_1, \arrow[d] & a_2, \arrow[d] & a_3, \arrow[d] & a_4, \arrow[d] & \cdots \\
b_0, \arrow[ur] & b_1, \arrow[ur] & b_2, \arrow[ur] & b_3, \arrow[ur] & b_4, & \cdots
\end{tikzcd}$$

$$f(x) = \begin{cases} a_{\frac{x}{2}} & \text{if } x \text{ is even} \\ b_{\frac{x-1}{2}} & \text{if } x \text{ is odd} \end{cases}$$

Countable sets

- A set is countable if it is equinumerous with N
  - $N \times N$

$$
\begin{array}{cccccc}
(0,0), & (0,1), & (0,2), & (0,3), & (0,4), & \cdots \\
(1,0), & (1,1), & (1,2), & (1,3), & (1,4), & \cdots \\
(2,0), & (2,1), & (2,2), & (2,3), & (2,4), & \cdots \\
\dots, & \dots, & \dots, & \dots, & \dots, & \dots
\end{array}
$$

$$f(x,y) = \frac{(x+y)(x+y+1)}{2} + x$$

Uncountable sets

- A set is countable if it is equinumerous with N
  - There exist uncountable sets, e.g., the real number within $(0,1)$
  - How can we show it is uncountable?

Uncountable sets

- A set is countable if it is equinumerous with N
  - There exist uncountable sets, e.g., the real number within $(0,1)$
  - How can we show it is uncountable?
    - Proof by contradiction

      Suppose it is countable, then there exists some bijection, but given the bijection, we can find some real number which is not mapped to any element in N

Uncountable sets

- A set is countable if it is equinumerous with N
  - There exist uncountable sets, e.g., the real number within $(0,1)$

    Suppose there is a bijection:

0	$0.1\,3\,5\,7\,2\,8\,0\,0\,0\,6\,7\ \ldots$
1	$0.2\,3\,7\,5\,0\,6\,2\,1\,1\,4\,5\ \ldots$
2	$0.9\,4\,4\,8\,2\,3\,1\,8\,9\,0\,2\ \ldots$
3	$0.6\,2\,2\,6\,3\,3\,7\,7\,9\,3\,9\ \ldots$
…	…

Uncountable sets

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  - There exist uncountable sets, e.g., the real number within $(0,1)$

    Suppose there is a bijection:

$$
\begin{array}{ll}
0 & 0.\,1\;3\;5\;7\;2\;8\;0\;0\;0\;6\;7\;\ldots \\
1 & 0.\,2\;3\;7\;5\;0\;6\;2\;1\;1\;4\;5\;\ldots \\
2 & 0.\,9\;4\;4\;8\;2\;3\;1\;8\;9\;0\;2\;\ldots \\
3 & 0.\,6\;2\;2\;6\;3\;3\;7\;7\;9\;3\;9\;\ldots \\
\ldots & \qquad\qquad \ldots
\end{array}
$$

Consider $0.\,2457\cdots$ Does this number appear in the list?

Proof Techniques

- Three basic proof techniques used in this class
  - Induction
  - Diagonalization
  - Pigeonhole Principle

Induction

- Claim: Can create exact postage for any amount $\geqslant$ \$0.08 using only 3 cent and 5 cent stamps

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  - Base case:

    Can create postage for 0.08 using one 5-cent and one 3-cent stamp

Induction

- Claim: Can create exact postage for any amount $\geqslant$ \$0.08 using only 3 cent and 5 cent stamps

  -Inductive case

  To show: if we can create exact postage for \$x using only 3-cent and 5-cent stamps, we can create exact postage for \$x + \$0.01 using 3-cent and 5-cent stamps

  - Two cases:

    1. Exact postage for \$x uses at least one 5-cent stamp

    2. Exact postage for \$x uses no 5-cent stamps

Induction

- Claim: Can create exact postage for any amount ⩾ \$0.08 using only 3 cent and 5 cent stamps

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    1. Exact postage for \$x uses at least one 5-cent stamp
    2. Exact postage for \$x uses no 5-cent stamps

    Replace one 5-cent with two 3-cent stamps

Induction

- Claim: Can create exact postage for any amount ⩾ \$0.08 using only 3 cent and 5 cent stamps

  -Inductive case

  To show: if we can create exact postage for \$x using only 3-cent and 5-cent stamps, we can create exact postage for \$x + \$0.01 using 3-cent and 5-cent stamps

  - Two cases:

    1. Exact postage for \$x uses at least one 5-cent stamp
    2. Exact postage for \$x uses no 5-cent stamps

       Replace three 3-cent with two 5-cent stamps

Induction

What is wrong with the following purported proof that all horses are the same color?

Proof by induction on the number of horses:

Basis Step. There is only one horse. Then clearly all horses have the same color.

Induction Hypothesis. In any group of up to n horses, all horses have the same color.

Induction Step. Consider a group of $n + 1$ horses. Discard one horse; by the induction hypothesis, all the remaining horses have the same color. Now put that horse back and discard another; again all the remaining horses have the same color. So all the horses have the same color as the ones that were not discarded either time, and so they all have the same color.

Pigeonhole Principle

- A, B are finite sets, with $|A| > |B|$, then there is no one-to-one function from A to B
- If you have n pigeonholes, and $> n$ pigeons, and every pigeon is in a pigeonhole, there must be at least one hole with > 1 pigeon.

Pigeonhole Principle

- Claim: Let $R \subseteq A \times A$ be a relation. If there exists a path in the directed graph representation of R, then there exists a path of length at most $|A|$.

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  - Suppose on the contrary the shortest path has length $\geq |A+1|$

  - In this path $(a_1, a_2, \cdots, a_n)$ there must be some node appears at least twice (pigeonhole principle)

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  - Suppose on the contrary the shortest path has length $\geq |A + 1|$

  - In this path $(a_1, a_2, \cdots, a_n)$ there must be some node appears at least twice (pigeonhole principle)

  - $(a_1, a_2, \cdots a_i, a_{i+1}, \cdots, a_j = a_i, a_{j+1}, \cdots, a_n)$

  - $(a_1, a_2, \cdots a_i, \qquad a_{j+1}, \cdots, a_n)$ is shorter

Strings

- An alphabet Σ is a finite set of symbols
  - $\Sigma_1 = \{0,1\}$
  - $\Sigma_2 = \{a, b, c, \cdots, z\}$
- A string is a finite sequence of symbols from an alphabet
  - apple, banana are both strings over $\Sigma_2 = \{a, b, c, \cdots, z\}$
  - 100110 is a string over $\Sigma_1 = \{0,1\}$

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  - apple, banana are both strings over $\Sigma_2 = \{a, b, c, \cdots, z\}$
  - 100110 is a string over $\Sigma_1 = \{0,1\}$
- Length of a string = number of symbols in the string
  - $|\text{apple}|=5, |100110|=6$

Strings

- Empty string: e
  - $|e| = 0$

Strings

- For string w, we use $w(j)$ to denote the symbol on j-th position
  - $w = \text{apple}, w(2) = w(3) = \text{p}$
  - $\Sigma_2 = \{a, b, c, \cdots, z\}$

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- String operation: $\circ$ Contatenation
  - apple $\circ$ banana = applebanana
  - 110 $\circ$ 101=110101
- We may also drop $\circ$ if it is clear from context
  - $w_1 \circ w_2 = w_1 w_2$ for two strings w_1, w_2

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- $w \circ e = we = ew = w$

Strings

- Substring:

  - v is a substring of w if and only if $w = xvy$ for some strings x, y
- A string is a substring of itself (by taking $x = y = e$)
- e is a substring of any string

Strings

- If $w = xv$, v is a suffix of w
- If $w = xv$, x is a prefix of w

Strings

- A string can be concatenated with itself
  - $w \circ w = w\ w = w^2$
  - $w^{i+1} = w^i w$
  - $w^0 = e$
- Example: $\mathrm{apple}^2 = \mathrm{appleapple}$

Strings

- Reversal: writing backwards
  - $w = \text{apple}$, $w^R = \text{elppa}$
  - $e^R = e$

Formal language

- The set of all strings over alphabet Σ is denoted as Σ^*
  - $e \in \Sigma^*$
- Any subset of Σ^* is called a language
  - English is a subset of $\{a, b, \cdots, z\}^*$

Language operations

- Concatenation
  - $L_1, L_2 \subseteq \Sigma^*$, $L_1 \circ L_2 = L_1 L_2 = \{w \in \Sigma^*: w = xy, x \in L_1, y \in L_2\}$
  - Example: $\{a, ab\}\{b, bb\} = \{abb, ab, abbb\}$

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  - Example: $\{a, ab\}\{b, bb\} = \{abb, ab, abbb\}$
- A language can be concatenated with itself
  - $L^2 = LL = \{w^2 : w \in L\}$
  - $L^0 = \{e\}$

Language operations

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Language operations

- Complement
  - For $L \in \Sigma^*, \overline{L} = \Sigma^* - L$
- Kleene star
  - L^*: the set of all strings obtained by concatenating zero or more strings of L
  - $L^* = \{w \in \Sigma^*: w = w_1 w_2 \cdots w_k \text{ for some } k \geq 0, w_i \in L, 1 \leq i \leq k\}$
  - $\Sigma^* = \{w \in \Sigma: w = w_1 w_2 \cdots w_k \text{ for some } k \geq 0, w_i \in \Sigma, 1 \leq i \leq k\} =$ all the strings over alphabet Σ