Itô Integrals and Processes

Itô Integrals

Probability Setup:

$$(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$$

Definition of Itô Integral:

$$I_T^{\Delta} = \int_0^T \Delta_t \, dW_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta_{t_i}^{(n)} (W_{t_{i+1}} - W_{t_i}) = I_T^{\Delta}(n)$$

Partition of the interval:

$$\pi_n = \{0 = t_0, t_1, t_2, \ldots, t_n = T\}$$

Definition of step process:

$$\Delta_{t_i}^{(n)} = \begin{cases} \Delta_0 & \text{on } [t_0, t_1) \\ \Delta_1 & \text{on } [t_1, t_2) \\ \vdots & \\ \Delta_{n-1} & \text{on } [t_{n-1}, t_n) \end{cases}$$

Existence of the Itô Integral

Limit only exists if Δ_t satisfies:

1. Adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$
2. Square-integrable in L^2 space:

$$\mathbb{E}_{\mathbb{P}}\left[\left(\int_0^T \Delta_t^2\, dt\right)^{1/2}\right] < \infty$$

$$\Delta_t \in L_T^2, \quad L_T^2 \equiv L^2([0, T] \times \Omega)$$

General Itô Processes

Let $I : [0, T] \times \Omega \to \mathbb{R}$ be a stochastic process:

$$I_t = I_0 + \int_0^t \mu_u \, du + \int_0^t \sigma_u \, dW_u$$

- $\int_0^t \mu_u \, du$: **Riemann integral (Drift)**
- $\int_0^t \sigma_u \, dW_u$: **Itô integral (Diffusion)**

$$\sigma_u \in L_T^2$$

Example: Geometric Brownian Motion

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \qquad S_t = S_0 + \int_0^t \mu S_u \, du + \int_0^t \sigma S_u \, dW_u$$

Integral with respect to Itô process:

$$Trading\ P\&L = \int_0^t \theta_u \, dS_u$$

Strategy holding θ_u units of stock for time $u \in [0, t]$

Defined Itô Process of Underlying Price S_t

- *Let's say we want to value some financial contract.*

$$C_t = f(S_t, t)$$

- *We would also like to understand the dynamics of* C_t

Using Taylor Expansion

Consider small dt and then use Taylor expansion.

$$df(S_t) = f(S_{t+dt}) - f(S_t) = f(S_t + dS_t) - f(S_t)$$

$$df(S_t) = \frac{\partial f}{\partial S}(dS_t) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS_t)^2 + \frac{1}{6}\frac{\partial^3 f}{\partial S^3}(dS_t)^3 + \dots$$

GBM dynamics: $(dS_t)^k = (\mu S_t dt + \sigma S_t dW_t)^k$

It can be shown (e.g. using $\int_0^t W_s dW_s$ and $f(x) = \frac{1}{2}x^2$):

$$\text{As } dt \to 0 : \begin{cases} dt^k, & k > 1 \\ dt \cdot dW_s & \\ (dW_s)^k, & k > 3 \end{cases} \qquad \text{All approach 0}$$

Itô's Lemma Intuition

Itô's Lemma: Identity used in stochastic calculus.

Retain terms, up to:

- First order in time increment
- Second order in Wiener process increment

As $dt \to 0$, **Discard:**

- dt^k, $k > 1$
- $dt \cdot dW_s$
- $(dW_s)^k$, $k > 3$

Keep:

- dt terms
- dW_t terms
- $(dW_t)^2 = dt$

Itô's Rule for $f : \mathbb{R} \to \mathbb{R}$

$$df(S_t) = \frac{\partial f}{\partial S} dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS_t)^2$$

Using GBM:

$$df(S_t) = \left[\mu_t \frac{\partial f}{\partial S}(S_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial S^2}(S_t)\right] dt + \sigma_t \frac{\partial f}{\partial S}(S_t) dW_t$$

Generic Itô drift-diffusion process:

$$dS_t = \mu_t dt + \sigma_t dW_t \quad \Rightarrow \quad (dS_t)^2 = \sigma_t^2 dt$$