Step-by-Step Explanation of Black-Scholes Derivation

1. Portfolio Construction (Delta Hedging)

We create a **delta-hedged portfolio** to eliminate directional risk:

$$\Pi = C - \Delta S$$

This represents a portfolio consisting of one long call option (C) and a short position of Δ shares in the underlying stock (S).

- The purpose is to offset the sensitivity of the option value to changes in the stock price.
- This portfolio is designed to be "locally" riskless over very short intervals, meaning small changes in S will not affect the value of Π.
- $\Delta = \frac{\partial C}{\partial S}$, the hedge ratio, is chosen to match the option's delta.

2. No-Arbitrage and Risk-Free Return

Since the hedged portfolio Π is theoretically riskless over an infinitesimal time, it must earn the risk-free rate r:

$$d\Pi = r\Pi dt$$

This assumes:

- A risk-free asset exists, offering a fixed rate r.
- There is no arbitrage — two assets with the same risk and payoff must have the same price.

3. Stock Price Dynamics (GBM)

The underlying stock is assumed to follow Geometric Brownian Motion:

$$dS = \mu S dt + \sigma S dW$$

- μ is the expected rate of return.
- σ is the volatility (standard deviation of returns).
- dW is a Wiener process increment, modeling randomness.
- This ensures stock prices remain positive and incorporate both drift and randomness.

4. Applying Itô's Lemma to the Option Price

Since the option price $C(S,t)$ depends on a stochastic process $S(t)$, we use Itô's Lemma:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}dt$$

- The first term represents time decay (theta).
- The second term captures price sensitivity (delta).
- The third term accounts for curvature/convexity (gamma) due to volatility.

5. Dynamics of the Hedged Portfolio

Substitute dC and dS into $d\Pi$:

$$d\Pi = dC - \Delta dS$$

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \left(\frac{\partial C}{\partial S} - \Delta\right) dS$$

To eliminate the stochastic component (dS term), we choose $\Delta = \frac{\partial C}{\partial S}$.

6. Elimination of Risk (Delta Hedging)

With $\Delta = \frac{\partial C}{\partial S}$, the portfolio becomes riskless:

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt$$

This must earn the risk-free return:

$$r\left(C - \frac{\partial C}{\partial S}S\right) dt = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt$$

7. The Black-Scholes Partial Differential Equation

Rearranging gives:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

This is the famous **Black-Scholes equation**.

8. Boundary Condition: Terminal Payoff

For a European call option:

$$C(S,T) = \max(S - K, 0)$$

This is the intrinsic value of the option at expiration.

9. Closed-Form Solution: Black-Scholes Formula

Solving the PDE with this boundary condition yields:

$$C(S,t) = S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2)$$

Where:

$$d_1 = \frac{\ln(S/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

- $\mathcal{N}(\cdot)$ is the cumulative normal distribution.
- $S\mathcal{N}(d_1)$: Present value of expected stock under risk-neutral measure.
- $Ke^{-r(T-t)}\mathcal{N}(d_2)$: Discounted strike price weighted by probability of exercise.

10. Extension to Put Options: Put-Call Parity

European put prices can be derived using put-call parity:

$$C - P = S - Ke^{-r(T-t)} \Rightarrow P = C - S + Ke^{-r(T-t)}$$

This relationship ensures consistent pricing between puts and calls with the same strike and maturity.