The Ducci Game

On Thursday, March 20, 2014, the annual Sulski Lecture held at Holy Cross welcomed Joseph Silverman of Brown University as this year's speaker. His presentation, titled "Dynamical Systems from a Number Theorist's Perspective," discussed the study of arithmetic dynamics, an area of dynamical systems in the field of number theory. In his lecture, he states that for a dynamicist, the periodic points of a given function, f, are the numbers satisfying the equation $f(z) = z$ for $n = 1, 2, 3, \ldots$ A number theorist then asks himself, "What sorts of numbers may appear as periodic?" In response, Silverman stated Northcott's Theorem, which says that a rational function has only finitely many rational periodic points. More specifically, the Ducci Map is one of the most studied dynamical systems in mathematics, which has a variety of possible periodic points that behave in different ways. This paper studies the dynamics of the Ducci Map and, more specifically the types of points that are periodic points as well as the fate of orbits for differing cases.

Enrico Ducci, an Italian mathematician born in 1864 and died in 1940, first discovered the map. After graduating in Naples in 1887, he taught in a number of secondary schools, as well as the Military College of Naples. He is also known for being the author of several works that are of a didactic nature.

The Ducci Game is defined by the map:

$$(a_1, a_2, \ldots, a_n) \to (|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_n - a_1|).$$

where a_i exists in the integers for $i = 1, 2, 3, \ldots, n$.

In other words, given a sequence of integers, the Ducci function maps the sequence to another sequence by taking the absolute differences of neighboring entries in the original sequence.

For example, let S be a sequence of integers such that S = {5,7, 12, 23, 9}.

Then, D(5, 7, 12, 23, 9) = (|5-7|, |7-12|, |12-23|, |23-9|, |9-5|).

= (2, 5, 11, 14, 4).

The above output is the first iteration of the Ducci function.

The second iterate of the function is D^2(5, 7, 12, 23, 9) = D(2, 5, 11, 14, 4).

= (|2-5|, |5-11|, |11-14|, |14-4|, |4-2|).

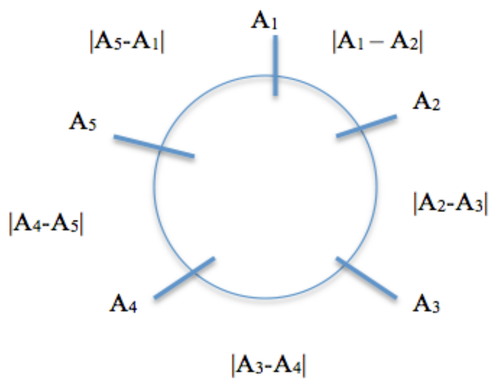
= (3, 6, 3, 10, 2).

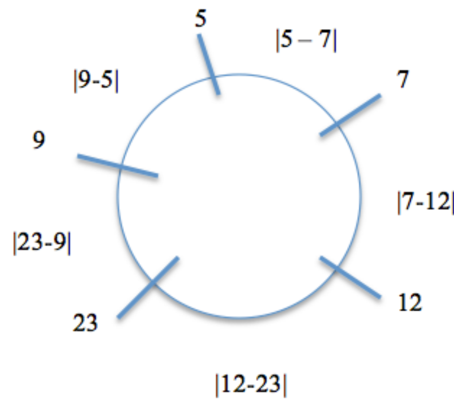
It is important to note that, with each iteration, the numbers of the sequence are decreasing. It will be shown later in the paper that this is a crucial factor for determining the periodicity of a given sequence.

By the above example, one can see that the first number of the sequence appears in the first and last entry of the new sequence generated by the Ducci function.

$$(a_1, a_2, ..., a_n) \to (|a_1 - a_2|, |a_2 - a_3|, ..., |a_n - a_1|).$$

As a result, the function can easily be represented visually using a circle. Using the same example from above, arrange the five integers in a circle.





A new circle represents each subsequent iterate by subtracting the absolute value of the neighboring points on the circle.

After examining the basics of the function, one must look at the behavior of its orbits. An important focus is finding what sequences are periodic. To begin, take a simple case and let n = 2. Therefore, the sequence has two integers. Furthermore, let these integers be 4 and 9. Then, D(4, 9) = (5, 5). Then, D(5, 5) = (0, 0). One can see that this sequence quickly converges to 0. Now, suppose n = 3. Thus, the sequence has three integers and let these integers be 4, 9, and 12. Then, (4, 9, 12) → (5, 3, 8) → (2, 5, 3) → (3, 2, 1) → (1, 1, 2) → (0, 1, 1) → (1, 0, 1) → (1, 1, 0) → (0, 1, 1). This sequence is eventually periodic of period three.

Now, let n = 4 with the integers being 4, 9, 12, and 7. Then, (4, 9, 12, 7) → (5, 3, 5, 3) → (2, 2, 2, 2) → (0, 0, 0, 0). Like the case when n = 2, this sequence also converges to 0.

From these basic examples, it seems that when n is even, the sequence under the Ducci Map converges to 0. On the other hand, when n is odd, the sequence is eventually periodic. Unfortunately, the behavior of the Ducci Map is not this simple. Take, for instance, n = 6 and let the six integers be 4, 9, 12, 7, 13, and 2. Then, (4, 9, 12, 7, 13, 2) → (5, 3, 5, 6, 11, 2) → (2, 2, 1, 5, 9, 3) → (0, 1, 4, 4, 6, 1) → (1, 3, 0, 2, 5, 1) → (2, 3, 2, 3, 4, 0) → (1, 1, 1, 1, 4, 2) → (0, 0, 0, 3, 2, 1) → (0, 0, 3, 1, 1, 1) → (0, 3, 2, 0, 0, 1) → (3, 1, 2, 0, 1, 1) → (2, 1, 2, 1, 0, 2) → (1, 1, 1, 1, 2, 0) → (0, 0, 0, 1, 2, 1) → (0, 0, 1, 1, 1, 1) → (0, 1, 0, 0, 0, 1) → (1, 1, 0, 0, 1, 1) → (0, 1, 0, 1, 0, 0) → (1, 1, 1, 1, 0, 0) → (0, 0, 0, 1, 0, 1) → (0, 0, 1, 1, 1, 1). Although 6 is an even number, the sequence does not converge to 0, but rather turns out to be eventually periodic of period 6.