Indian Institute of Technology Roorkee
MAN-001(Mathematics-1)
Autumn Semester: 2022-23
Assignment-10: Vector Calculus II (Line and surface integrals, Greens, Gauss and Stokes' theorem and their applications)

(1) Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = x^2\hat{i} - xz\hat{j} + y^2\hat{k}$ along the path C joining the points $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1) \to (0,0,1)$ via straight lines.

(2) Show that $\vec{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$ is a conservative vector field and find a function ϕ such that $\vec{F} = \nabla\phi$. Also, find the work done by a moving particle from $(0,1,-1)$ to $(\pi/2,-1,2)$.

(3) If $\vec{F} = \dfrac{x}{x^2+y^2}\hat{j} - \dfrac{y}{x^2+y^2}\hat{i}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ for the various curves C from $(0,1)$ to $(1,0)$ along

(i) the arc of $x^2+y^2 = 1$ lying in the second, third and fourth quadrant.

(ii) $x + y = 1$.

(iii) the arc of $x^2 + y^2 = 1$ lying in the first quadrant.

Is the vector field $\vec{F}$ conservative? If so, find ϕ such that $\vec{F} = \nabla\phi$. Why is the line integral not path independent?

(4) Evaluate the surface integral $\iint_S \vec{F}\cdot\hat{n}dS$, if

(i) $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is the surface of $x^2+y^2+z^2 = 1$ in the first octant.

(ii) $\vec{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of $x^2+y^2 = 16$ in the first octant between $z = 0$ and $z = 5$.

(iii) $\vec{F} = \dfrac{\vec{r}}{r^3}$ and S is the surface of $x^2+y^2+z^2 = a^2$.

(5) If $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$, evaluate the volume integral $\iiint_V \nabla\cdot\vec{F}dV$ over the entire surface of the region above the xy- plane bounded by the cone $z^2 = x^2+y^2$ and the plane $z = 4$.

(6) Evaluate $\iiint_V \phi dV$, where $\phi = 45x^2y$ and V is the closed region bounded by the planes $4x + 2y + z = 8, x = 0, y = 0$ and $z = 0$.

(7) Evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$, where $\vec{F} = y^2\hat{\imath} + y\hat{\jmath} - xz\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above xy- plane.

(8) Verify Greens theorem for

(i) $\oint_C [(xy^2 - 2xy)dx + (x^2y + 3)dy]$ around the boundary curve C of the region enclosed by $y = 8x$ and $x = 2$.

(ii) $\oint_C [(xy + y^2)dx + x^2dy]$, C bounds the region enclosed by $y = x$ and $y = x^2$.

(iii) $\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ and C bounds the region enclosed by $x = 0, y = 0$ and $x + y = 1$.

(9) By converting into the line integral, evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$, where $\vec{F} = (x - z)\hat{\imath} + (x^3 + yz)\hat{\jmath} - 3xy^2\hat{k}$ and S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above xy-plane.

(10) Verify Stokes' theorem for $\vec{F} = (x^2+y^2)\hat{\imath} - 2xy\hat{\jmath}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0$ and $y = b$.

(11) Verify Gauss' divergence theorem for

(i) $\vec{F} = (2x - z)\hat{\imath} - x^2y\hat{\jmath} + 4xz^2\hat{k}$ taken over the region bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$.

(ii) $\vec{F} = 2x^2y\hat{\imath} - y^2\hat{\jmath} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ and $x = 2$.

(12) Evaluate $\iint_S \vec{F} \cdot \vec{n} ds$, where $\vec{F} = (x^2 - yz)\hat{\imath} + (y^2 - zx)\hat{\jmath} + (z^2 - xy)\hat{k}$ and S is a rectangular parallelepiped $0 \le x \le a, 0 \le y \le b$ and $0 \le z \le c$.

Answers.

(1) $\frac{3}{2}$ (2) $\phi = y^2 \sin x + xz^3 - 4y + 2z, 4\pi + 15$ (3) $\frac{3\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}, \phi = \tan^{-1}\left(\frac{y}{x}\right)$

(4)(i) $\frac{3}{8}$ (ii) 90 (iii) 4π (5) 320π (6) 128 (7) 0 (9) 12π (12) $abc(a+b+c)$.