Indian Institute of Technology Roorkee
MAN-001(Mathematics-1), Autumn Semester: 2022-23
Assignment-3: Differential Calculus

(1) Suppose $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. Show that each of the following functions of $(x, y) \in \mathbb{R}^2$ are continuous:
  (a) $f - g$
  (b) fg
  (c) $\max\{f, g\}$
  (d) $\min\{f, g, h\}$

(2) Find the following limits, if they exist

(a) $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{xy^3}{x+y}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$ (c) $\displaystyle\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right)$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ (e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin^2(x+y)}{|x|+|y|}$ (f) $\displaystyle\lim_{(x,y)\to(1,1)} f(x, y)$ where

$f(x, y) = \begin{cases} 1, & \text{if } x + y \geq 2 \\ -1, & \text{if } x + y < 2 \end{cases}$ (g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^2 + y^2}$

(3) (a) Consider the function $f(x, y) = \dfrac{x+y}{x-y}$ for $(x, y) \in \mathbb{R}^2$ with $x - y \neq 0$. Show that $\lim\limits_{x\to 0}[\lim\limits_{y\to 0} f(x, y)] = 1$, but $\lim\limits_{y\to 0}[\lim\limits_{x\to 0} f(x, y)] = -1$. What can you say about the existence of $\lim\limits_{(x,y)\to(0,0)} f(x, y)$?

(b) Let $f(x, y) = 0$ if $y = 0$, and $f(x, y) = x \sin\left(\dfrac{1}{y}\right)$, if $y \neq 0$. Compute $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ and iterated limits $\lim\limits_{x\to 0}[\lim\limits_{y\to 0} f(x, y)]$ and $\lim\limits_{y\to 0}[\lim\limits_{x\to 0} f(x, y)]$ if they exist.

(c) Let $f(x, y) = \dfrac{x^2 y^2}{x^2 y^2 + (x-y)^2}$ if $x^2 y^2 + (x - y)^2 \neq 0$. Show that $\lim\limits_{x\to 0}[\lim\limits_{y\to 0} f(x, y)]$ and $\lim\limits_{y\to 0}[\lim\limits_{x\to 0} f(x, y)] = 0$. But $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

(4) Let $f(x, y) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$. Show that for any point (a, b), $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ does not exist.

(5) Examine the continuity of the function $f(x, y)$ at $(0, 0)$ in each of the following cases. Also check the existence of $f_x(0, 0)$ and $f_y(0, 0)$.

(a) $f(x, y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x, y) \neq (0, 0) \\ \dfrac{1}{2}, & (x, y) = (0, 0) \end{cases}$

(b) $f(x, y) = \begin{cases} x \sin\frac{1}{x} + y \sin\frac{1}{y}, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$

(c) $f(x, y) = \begin{cases} xy \log(x^2 + y^2), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(d) $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y. \end{cases}$

(6) For the function $f(x,y) = \begin{cases} \dfrac{y(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$.

Compute $f_x(0,y), f_y(x,0), f_x(0,0)$ and $f_y(0,0)$, if they exist.

(7) Show that for the function $f(x,y) = \begin{cases} -xy, & |y| \geq |x| \\ xy, & |y| < |x| \end{cases}$, $f_{xy}(0,0)$ and $f_{yx}(0,0)$ both exist and are unequal.

(8) Prove that $|x| + |y|$ is continuous, but not differentiable at $(0,0)$.

(9) Prove that $f(x,y) = \begin{cases} \frac{(x+y)\{\sqrt{(x^2+y^2)}+xy\}}{\sqrt{(x^2+y^2)}}, & \text{when } x^2+y^2 \neq 0 \\ 0, & \text{when } x^2+y^2 = 0, \end{cases}$ is differentiable at $(0,0)$. Hence, deduce that $f_x(0,0) = f_y(0,0) = 1$.

(10) Show that the function

$$f(x,y) = \begin{cases} x^3 \sin\frac{1}{x^2} + y^3 \sin\frac{1}{y^2}, & \text{when } xy \neq 0 \\ x^3 \sin\frac{1}{x^2}, & \text{when } x \neq 0 \text{ and } y = 0 \\ y^3 \sin\frac{1}{y^2}, & \text{when } x = 0 \text{ and } y \neq 0 \\ 0, & \text{when } x = y = 0 \end{cases}$$ is differentiable at $(0,0)$,

whereas none of f_x, f_y is continuous at $(0,0)$.

(11) Determine the values of p for which $f(x,y) = |xy|^p,\ xy \neq 0$, and $f(x,y) = 0,\ xy = 0$, is continuous and differential at $(0,0)$.

(12) Show that the function $f(x,y) = \begin{cases} (x^2+y^2)\cos\left(\dfrac{1}{\sqrt{x^2+y^2}}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ is differentiable at $(0,0)$ and that f_x, f_y are not continuous at $(0,0)$.

(13) Let $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$ be differentiable. Suppose that all partial derivatives of f exists at origin.

(a) Can f be extended to a continuous map from $\mathbb{R}^2 \to \mathbb{R}$?

(b) Suppose f is continuous at origin. Is f differentiable from $\mathbb{R}^2 \to \mathbb{R}$?

(14) Check differentiablility of the following functions:

(a) $f(x,y) = \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$.

(b) $f(x,y) = 1$ if $0 \leq y \leq x^2$ and $f(x,y) = 0$ otherwise.

(c) $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$ for $(x,y) \neq 0$, and $f(0,0) = 0$.

(d) $f(x,y) = x^2 + y^2$ for $(x,y) \in \mathbb{R}^2$.

(e) $f(x,y) = \dfrac{x^2 y^2}{x^4+y^2}$ for $(x,y) \neq 0$, and $f(0,0) = 0$.

Answers:

2. (a) -8 (b) does not exist (c) does not exist (d) 0 (e) 0 (f) does not exist (g) 0
3. (a) does not exist (b) 0, does not exist, 0
5. (a) continuous, 0,0 (b) continuous, does not exists (c) continuous, 0,0
   (d) discontinuous, 0,0
6. 0, 1, 0, -1.
11. For continuity $p > 0$ and for differentiability $p > 1/2$.
13. Not in general for both cases.
14. No, No, No, Yes, Yes.