Indian Institute of Technology Roorkee
Mid-term Examination 2022

Course Title: Mathematics I | Total Time: 90 Minutes
Course Code: MAN – 001 | Total Marks: 50

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Tutorial Batch: 01 **Branch:** Btech-CSE **Year:** 1st

All questions are compulsory. Marks are indicated against each question. Full mark will be awarded only for the complete answer.

Q.1) Let $\mathbf{A} = \begin{pmatrix} 3 & -3 & 3 & 3 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 2 & 2 & 2 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ -4 \\ 4 \\ 0 \end{pmatrix}$. Using the elementary row operations, find the inverse of $\mathbf{A}$ and hence solve the system $\mathbf{Ax} = \mathbf{b}$. [6]

Q.2) (a) Investigate for what values of λ and μ, the following system of equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$3x + 6y + 3\lambda z = 3\mu$$

has (i) no solution, (ii) a unique solution, and (iii) infinitely many solutions. Moreover, if $\lambda = 2$ and $\mu = 9$, then find the solution, if it exists. [3]

(b) Show that $\mathbf{A} = \begin{pmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 2 & 2 & 6 \end{pmatrix}$ is diagonalizable. Find a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. [4]

Q.3) (a) Let $\mathbf{A}$ be a 3×3 real skew-symmetric matrix. Then, prove the following,

(i) $i\mathbf{A}$ is a hermitian matrix,

(ii) $\mathbf{x}^T\mathbf{Ax} = 0$ for any $\mathbf{x} = (x_1, x_2, x_3)^T$, and

(iii) $\mathbf{P}^T\mathbf{AP}$ is a skew-symmetric matrix for any matrix $\mathbf{P}$ of order $3 \times m$. [3]

(b) Let $\mathbf{P}$ be a 3×3 orthogonal matrix whose first two rows are multiples of $(1, 3, x)$ and $(1, 0, -1)$. Find the value of x and the matrix $\mathbf{P}$. [3]

Q.4) Using the Cayley–Hamilton theorem, find $\mathbf{A}^{100}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$ [6]

Please turn over...

Q.5) (a) For natural numbers m and n, define a function

$$f_{m,n}(x,y)=\begin{cases}\dfrac{x^m y^n}{x^{2m}+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

(i) Find the values of m and n for which f is continuous at $(0,0)$. Justify your answer.

(ii) Find the values of m and n for which the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(0,0)$ and also compute them. Justify your answer. [4]

(b) Check the differentiability of the following function at $(0,0)$,

$$f(x,y)=\begin{cases}\dfrac{4x^3y}{x^4+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

[3]

Q.6) (a) Let

$$f(x,y)=(1+xy)(\sin(\tan^{-1}x)-y\cos(\tan^{-1}x))$$

and

$$g(x,y)=(x-y)(\cos(\tan^{-1}x)+y\sin(\tan^{-1}x)).$$

Using the Jacobian method, prove or disprove that f and g are functionally independent. Find the relationship between f and g if they are functionally dependent. [3]

(b) If $z=x^2e^{-y}-y^3e^x$ where x and y are implicit functions of t defined by $xe^x+t^2x-te^x-t=0$ and $e^ty^3-2ey^2+y-2t=0$, then find $\dfrac{dz}{dt}$ at $t=1$. [3]

Q.7) (a) Let $u(x,y)=\ln\left(\dfrac{x^3+y^3}{x+y}\right)+\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$. Then, using Euler's theorem, evaluate $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$. [3]

(b) Let $f(x,y)=\begin{cases}y\sin\left(\dfrac{1}{x}\right), & x\neq0,\\ 0, & x=0\end{cases}$, $g(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^4}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$ Examine the existence of the limit and the iterated/repeated limits of f and g at $(0,0)$. [3]

Q.8) Let $w=f(u,v)$ satisfies the equation $w_{uu}+w_{vv}+w_v=0$. If $u=\dfrac{x^2-y^2}{2}$ and $v=xy$, then show that w also satisfies the equation $w_{xx}+w_{yy}+yw_x+xw_y=0$. [6]