Logic!

Logic

- Crucial for mathematical reasoning
- Used for designing electronic circuitry

- Logic is a system based on propositions.
- A proposition is a statement that is either true or false (not both).
- We say that the truth value of a proposition is either true (T) or false (F).

- Corresponds to 1 and 0 in digital circuits

The Statement/Proposition Game

"Elephants are bigger than mice."

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? true

The Statement/Proposition Game

"520 < 111"

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? false

The Statement/Proposition Game

"$y > 5$"

Is this a statement? yes

Is this a proposition? no

Its truth value depends on the value of y, but this value is not specified.

We call this type of statement a **propositional function** or **open sentence**.

The Statement/Proposition Game

"Today is January 1 and $99 < 5$."

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? false

The Statement/Proposition Game

"Please do not fall asleep."

Is this a statement? no

It's a request.

Is this a proposition? no

Only statements can be propositions.

The Statement/Proposition Game

"If elephants were red,
they could hide in cherry trees."

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? probably false

The Statement/Proposition Game

"$x < y$ if and only if $y > x$."

Is this a statement? yes

Is this a proposition? yes

… because its truth value does not depend on specific values of x and y.

What is the truth value of the proposition? true

Combining Propositions

As we have seen in the previous examples, one or more propositions can be combined to form a single compound proposition.

We formalize this by denoting propositions with letters such as p, q, r, s, and introducing several logical operators.

Logical Operators (Connectives)

We will examine the following logical operators:

• Negation	(NOT)
• Conjunction	(AND)
• Disjunction	(OR)
• Exclusive or	(XOR)
• Implication	(if – then)
• Biconditional	(if and only if)

Truth tables can be used to show how these operators can combine propositions to compound propositions.

Negation (NOT)

Unary Operator, Symbol: $\neg$

P	$\neg P$
true (T)	false (F)
false (F)	true (T)

Conjunction (AND)

Binary Operator, Symbol: $\wedge$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (OR)

Binary Operator, Symbol: $\vee$

P	Q	P$\vee$Q
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or (XOR)

Binary Operator, Symbol: ⊕

P	Q	P⊕Q
T	T	F
T	F	T
F	T	T
F	F	F

Implication (if - then)

Binary Operator, Symbol: $\rightarrow$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional (if and only if)

Binary Operator, Symbol: $\leftrightarrow$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Statements and Operators

Statements and operators can be combined in any way to form new statements.

P	Q	$\neg P$	$\neg Q$	$(\neg P)\vee(\neg Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Statements and Operations

Statements and operators can be combined in any way to form new statements.

P	Q	$P \wedge Q$	$\neg\,(P \wedge Q)$	$(\neg P) \vee (\neg Q)$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Equivalent Statements

P	Q	$\neg(P\wedge Q)$	$(\neg P)\vee(\neg Q)$	$\neg(P\wedge Q)\leftrightarrow(\neg P)\vee(\neg Q)$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

The statements $\neg(P\wedge Q)$ and $(\neg P) \vee (\neg Q)$ are logically equivalent, since $\neg(P\wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$ is always true.

Tautologies and Contradictions

A tautology is a statement that is always true.

Examples:

- $R \vee (\neg R)$
- $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

If $S \rightarrow T$ is a tautology, we write $S \Rightarrow T$. (A formula *S* is said to **tautologically imply** a formula *T* if every valuation that causes *S* to be true also causes *T* to be true.)

If $S \leftrightarrow T$ is a tautology, we write $S \Leftrightarrow T$.

Tautologies and Contradictions

A contradiction is a statement that is always false.

Examples:

- $R \wedge (\neg R)$
- $\neg(\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q))$

The negation of any tautology is a contra-diction, and the negation of any contradiction is a tautology.

Exercises

We already know the following tautology:

$\neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$

exercise:

Show that $\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$.

Propositional Functions

Propositional function (open sentence):
statement involving one or more variables,

e.g.: $x-3 > 5$.

Let us call this propositional function $P(x)$, where P is the predicate and x is the variable.

What is the truth value of $P(2)$? false

What is the truth value of $P(8)$? false

What is the truth value of $P(9)$? true

Propositional Functions

Let us consider the propositional function $Q(x, y, z)$ defined as:

$x + y = z$.

Here, Q is the predicate and x, y, and z are the variables. (A predicate is an expression of one or more variables defined on some specific domain. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.)

What is the truth value of $Q(2, 3, 5)$? true

What is the truth value of $Q(0, 1, 2)$? false

What is the truth value of $Q(9, -9, 0)$? true

Universal Quantification

Let P(x) be a propositional function.

Universally quantified sentence:

For all x in the universe of discourse P(x) is true.

Using the universal quantifier $\forall$:

$\forall x\ P(x)$ "for all x P(x)" or "for every x P(x)"

(Note: $\forall x\ P(x)$ is either true or false, so it is a proposition, not a propositional function.)

Universal Quantification

Example:

S(x): x is a IITR student.

G(x): x is a genius.

What does $\forall x\ (S(x) \rightarrow G(x))$ mean ?

"If x is a IITR student, then x is a genius."

or

"All IITR students are geniuses."

Existential Quantification

Existentially quantified sentence:

There exists an x in the universe of discourse for which P(x) is true.

Using the existential quantifier $\exists$:

$\exists x\, P(x)$ "There is an x such that P(x)."

"There is at least one x such that P(x)."

(Note: $\exists x\, P(x)$ is either true or false, so it is a proposition, but no propositional function.)

Existential Quantification

Example:

P(x): x is a MIT professor.

G(x): x is a genius.

What does $\exists x\ (P(x) \wedge G(x))$ mean ?

"There is an x such that x is a MIT professor and x is a genius."

or

"At least one MIT professor is a genius."

Quantification

Another example:

Let the universe of discourse be the real numbers.

What does $\forall x \exists y\ (x + y = 320)$ mean ?

"For every x there exists a y so that $x + y = 320$."

Is it true? yes

Is it true for the natural numbers? no

Disproof by Counterexample

A counterexample to $\forall x\, P(x)$ is an object c so that $P(c)$ is false.

Statements such as $\forall x\, (P(x) \rightarrow Q(x))$ can be disproved by simply providing a counterexample.

Statement: "All birds can fly."

Disproved by counterexample: Penguin.

Negation

$\neg(\forall x\, P(x))$ is logically equivalent to $\exists x\, (\neg P(x))$.

$\neg(\exists x\, P(x))$ is logically equivalent to $\forall x\, (\neg P(x))$.

De Morgan's laws for quantifiers