Property

9.4.2025.

The subject-construction theorem :

Let Δ be a TA_λ-deduction of a formula $\Gamma \mapsto M : \tau$.

(i) If we remove from each formula in Δ everything except its subject, Δ changes to a tree of terms which is exactly the construction-tree for M.

(ii) If M is an atom, $M \equiv x$, then $\Gamma = \{x : \tau\}$ and Δ contains only one formula, namely the axiom

$$x : \tau \mapsto x : \tau$$

(iii) If $M \equiv PQ$ the last step in Δ must be an application of $(\to E)$ to two formulae with form

$$\Gamma \upharpoonright P \mapsto P : \sigma \to \tau \qquad \Gamma \upharpoonright Q \mapsto Q : \sigma \quad \text{for some } \sigma.$$

restriction subjects$(\Gamma) = FV(P)$

(iv) If $M \equiv \lambda x . P$ then τ must have form $\rho \to \sigma$

if $x \in FV(P)$ the last step in Δ must be an application of $(\to I)_{main}$ to

$$\Gamma, x : \rho \mapsto P : \sigma$$

if $x \notin FV(P)$ the last step in Δ must be an application of $(\to I)_{vac}$ to

$$\Gamma \mapsto P : \sigma$$

Deductions in TA_λ may not be unique

Example :-

$$\Delta_M \quad \dfrac{\dfrac{\dfrac{y : a \mapsto y : a}{\mapsto (\lambda y . y) : a \to a}\,(\to I)}{\mapsto (\lambda x . \lambda y . y) : (\sigma \to \sigma) \to (a \to a)}\,(\to I) \qquad \dfrac{z : \sigma \mapsto z : \sigma}{\mapsto (\lambda z . z) : \sigma \to \sigma}\,(\to I)}{\mapsto (\lambda x . \lambda y . y)(\lambda z . z) : a \to a}\,(\to E)$$

think of ->Ivac

Here $M \equiv (\lambda x . \lambda y . y)(\lambda z . z)$ $\quad \tau \equiv a \to a$ $\quad \Gamma = \phi$

here σ can be anything and this makes the Δ_λ not unique.

Deductions in TA may not be unique but it will always be unique if M in beta-normal form.

(Property)

Uniqueness of deductions for normal forms.

let M be a β-nf and Δ a TA_λ-deduction of $\Gamma \mapsto M:\tau$.

Then (i) every type in Δ has an occurrence in τ or in a type in Γ,

(ii) Δ is unique, i.e., if Δ' is also a deduction of $\Gamma \mapsto M:\tau$ then $\Delta' \equiv \Delta$.

Subject reduction and expansion (Property)

If P has type τ we can think of P as being in some sense "safe".

If P represents a stage in some computation which continues by β-reducing P then all later stages in the computation are also "safe". (Unsafe means mismatch of types.)

Subject-reduction theorem :-

If $\Gamma \vdash_\lambda P:\tau$ and $P \rhd_\beta Q$ then $\Gamma \vdash_\lambda Q:\tau$

$\vdash_\lambda$ means there is a deduction of $\langle \Gamma, P, \tau \rangle$ in TA_λ.

~~Goal~~ Proof :-

$P \equiv (\lambda x.M)N$ $\quad Q \equiv M[N/x]$

let $x \in FV(M)$, then by the subject-construction theorem the lower steps of Δ must have the form

subject construction theorem used for getting the lower or last steps in deduction tree.

$$\dfrac{\dfrac{\Gamma_1, x:\sigma \mapsto M:\tau}{\Gamma_1 \mapsto (\lambda x.M):\sigma\to\tau}(\to I)\,\text{main} \qquad \Gamma_2 \mapsto N:\sigma}{\Gamma_1 \cup \Gamma_2 \mapsto \underbrace{((\lambda x.M)N)}_{Q}:\tau} \leftarrow Q:\tau$$

Now $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\text{Subjects}(\Gamma) = FV(P)$

so we have a deduction for $\Gamma \mapsto P:\tau$

but $(\lambda x.M)N \rhd_\beta Q$. i.e $P \rhd_\beta Q$.

so we also have a deduction for $\Gamma \mapsto Q:\tau$. $\square$

subject ~~expansion~~ theorem :-

If $\Gamma \vdash_\lambda Q:\tau$ and $P \rhd_\beta Q$ [*] then $\Gamma \vdash_\lambda P:\tau$.

[*] by non-duplicating and non-cancelling contractions.

the above condition [*] is very important. Removing it will make the conclusion false.

1. Let M be a λ-term. Let $CT(M)$ be ~~a~~ the construction tree for M.
Let S^M_{p-l} be the set of all the pairs of label-position in the $CT(M)$.
S^M_l be the set obtained from S^M_{p-l} by removing the labels.
i.e. S^M_l contains only the positions.

Problem :- 1.1. Given $S_{incomplete} \subsetneq S^M_l$. Construct a unique M corresponding to S^M_l. No, we can't create a unique lambda term.

1.2. Find a minimum sized set $S_{incomplete}$ so that a unique M can be constructed from the set.

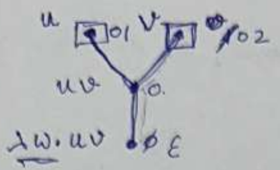
2. Let $\Sigma = \{0, 1, 2\}$

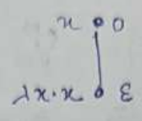
2.1. ~~Let~~ given a regular expression R_Σ obtain the structure of λ-term M.

2.2. Suggest types of regular expressions that are meaningful w.r.t. λ-terms.

2.3. Suggest types of regular expressions that are not meaningful w.r.t. λ-terms.

- Extension of Contraction-Tree 4.3.2025.
- Problems related to Contraction tree.



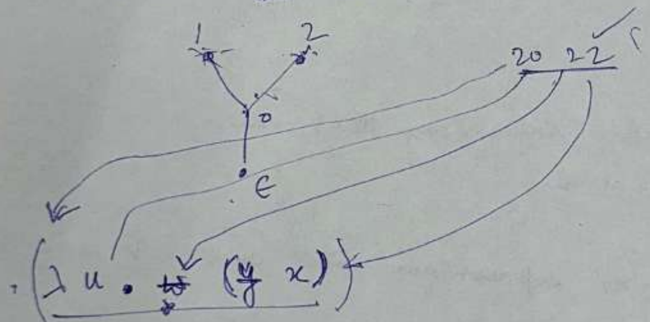


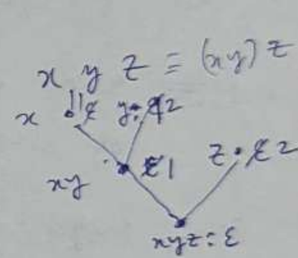
u : 01 v : 02

$$\lambda w. (uv)$$
$$\downarrow \quad \downarrow\downarrow$$
$$0 \quad 1 \; 2$$

$\{\varepsilon, 0, 01, 02\}$

$\{\varepsilon, \_, \underline{01}, \_\}$





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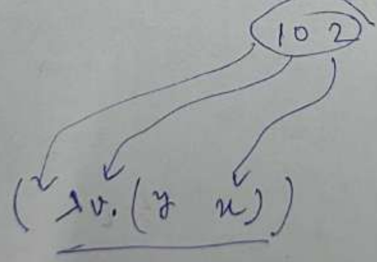

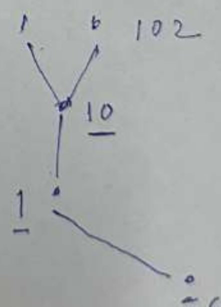

x • ε

λx. λy. x