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CSN-373: Probability theory

(1.) Fair coin $\Rightarrow$ P(head) = P(tail) = $\frac{1}{2}$

$\Rightarrow$ Sample space, $S = \{ T^n H \mid n \geq 0, n$ is an integer $\}$
where T = tail and H = Head.

$\Rightarrow$ P(to get first head in 7 tosses) = $P(T^6 H)$

$= (1/2)^6 \cdot (1/2) = (1/2)^7$

$= 0.781\,\%$ Ans

$\Rightarrow$ If $P(H) = P(T) + 0.025$ and we have $P(H) + P(T) = 1$

$\Rightarrow$ we get: $P(H) = 0.5125$

$P(T) = 0.4875$

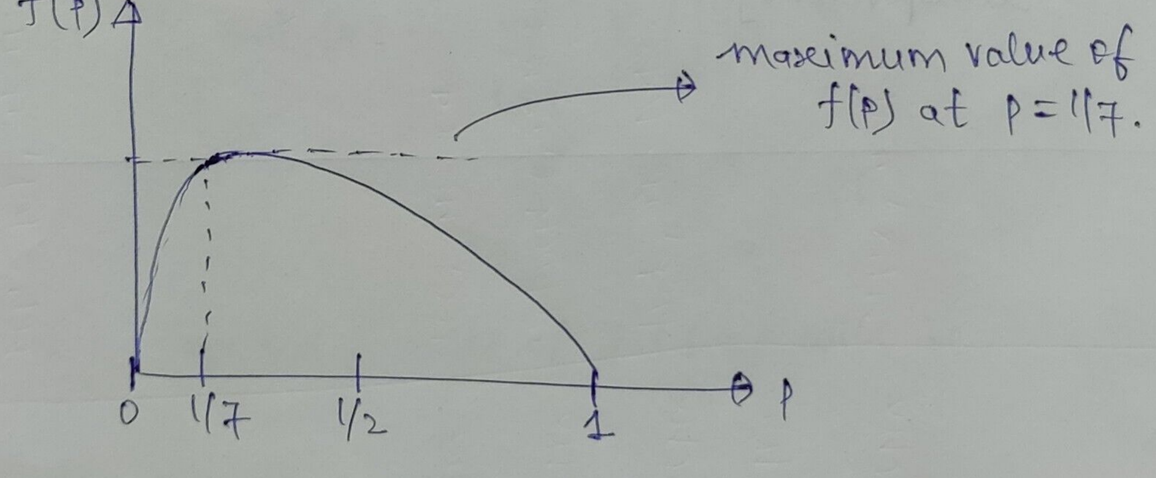
then:

$P(T^6 H) = (0.4875)^6 \cdot (0.5125) = 0.6879\,\%$ Ans

Hence, the probability that head occurs in 7 tosses decreases.

$\Rightarrow$ If P(head) = p, P(tail) = 1-p.

then: $P(T^6 H) = (1-p)^6 p = f(p)$



If P(head) $\to$ 1, then $P(T^6 H) \to 0$ Ans

② A : Getting a head on first toss } two events.
B : Getting a head on second toss.

⇒ Given that: $I(A|B) = I(A)$

say, $C = A$ given B (another event)

⇒ We have: $I(C) = I(A)$

$$-\log(P(C)) = -\log(P(A))$$

$$P(C) = P(A)$$

$$\boxed{P(A|B) = P(A)} \quad \ldots\ldots\ldots (i)$$

⇒ We know: $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = P(A)$

$$\boxed{P(A\cap B) = P(A)\cdot P(B)} \quad \ldots\ldots (ii)$$

From (i) and (ii), we have that A and B are independent events.

Hence proved

{ to prove that all the outcomes that one can get from the two tosses are indeed independent, we have to prove that both the tosses are independent }

) Let X = Number of tosses / guesses alice has to make
Y = number that bob guesses.

$$\Rightarrow P_Y(y) = \frac{1}{20} \quad \text{(uniform dist.)} \quad \forall\, y \in \{1,2,3 \ldots, 20\}$$

$$\Rightarrow P_X(x) = \sum_{i=1}^{20} P_Y(i) \cdot P_{X|Y}(x|i)$$

$$= \frac{1}{20}\left[10\times\left(\frac{37}{40}\right)^{x-1} \cdot \frac{3}{40} + 5\times\left(\frac{59}{60}\right)^{x-1} \cdot \frac{1}{60} + 5\times\left(\frac{29}{30}\right)^{x-1} \cdot \frac{1}{30}\right]$$

$$\Rightarrow E[X] = \sum_{x=1}^{\infty} x \cdot P_X(x) = \sum_{x=1}^{\infty}\left[\frac{3}{80}\, x\left(\frac{37}{40}\right)^{x-1} + \frac{x}{240}\left(\frac{59}{60}\right)^{x-1} + \frac{1}{120}\, x\left(\frac{29}{30}\right)^{x-1}\right]$$

$$= \frac{3}{80} \cdot \left(\frac{40}{3}\right)^2 + \frac{1}{240}(60)^2 + \frac{1}{120}(30)^2$$

$$= 20/3 + 15 + 15/2 = 29.167 \ \underline{\text{Ans}}$$

Hence Expected number of guesses will be ~ 30 Ans

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③ (b) $$P_X(x) = \sum_{i=1}^{20} P_Y(i) \cdot P_{X|Y}(x|i)$$

$$= \frac{1}{20} \times 20 \times \left(\frac{19}{20}\right)^{x-1} \cdot \left(\frac{1}{20}\right) = \frac{1}{20} \cdot \left(\frac{19}{20}\right)^{x-1}$$

$$\Rightarrow E[X] = \sum_{x=1}^{\infty} x \cdot P_X(x) = \sum_{x=1}^{20} \frac{x}{20}\left(\frac{19}{20}\right)^{x-1}$$

$$= \frac{1}{20} \times (20)^2 = 20$$

Hence expected number of guesses will be 20 Ans.

④ (i) • $P(X <= 9) = F(9) = \frac{\sqrt{9}}{100} = \frac{3}{100} = 0.03$

where X = wait time

• $P(X > 9) = 1 - P(X <= 9) = 0.97$ Ans

(ii) $P(X > 19 \mid X > 9) = \frac{P(X > 19)}{P(X > 9)} = \frac{1 - F(19)}{1 - F(9)} = \frac{1 - 1/20}{1 - 3/100}$

$= \frac{95}{97}$ Ans

(iii) $P(X = 16) = P(X <= 16) - P(X < 16)$

$= F(16) - P(X \leq (16 - \delta))$ where $\delta \to 0^+$

$= 3/5 - \lim_{\delta \to 0^+} \left(\frac{\sqrt{16 - \delta}}{100} \right)$

$= 3/5 - 4/100 = 56/100 = 0.56$ Ans.

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⑤. • Let X be discrete random variable. If i show that total probability is equal to 1, then X will be indeed discrete.

• If $\max(X_A, X_B) = x$, then three situations are possible:

(i) $X_A = x$, $X_B = x$

(ii) $X_A = x$, $X_B < x$

(iii) $X_A < x$, $X_B = x$

$\to$ hence $\sum_x P_X(x) = \sum_{x=1}^{20} P_{X_A}(X_A = x) \cdot P_{X_B}(X_B < x) + \sum_{i=1}^{20} P_{X_A}(X_A < x) \cdot P_{X_B}(X_B = x)$

$+ \sum_{i=1}^{20} P_{X_A}(X_A = x) \cdot P_{X_B}(X_B = x)$

total probability

$= \sum_{x=1}^{20} P_{X_A}(X_A = x) \cdot \sum_{x=1}^{20} P_{X_B}(X_B = x) = 1$

$\Rightarrow$ Hence total probability is 1 $\Rightarrow$ X is indeed a discrete random variable. Ans

Calculating PMF(X) :-

$\Rightarrow P_X(x) = P_{X_A}(X_A = x) \cdot P_{X_B}(X_B = x) + P_{X_A}(X_A = x) \cdot P_{X_B}(X_B < x)$

$+ P_{X_A}(X_A < x) \cdot P_{X_B}(X_B = x)$

$= \frac{1}{20} \cdot \frac{1}{20} + \frac{1}{20} \cdot \frac{x-1}{20} + \frac{1}{20} \cdot \frac{x-1}{20}$

$= \left(\frac{2x-1}{400}\right)$ { assuming X_A and X_B follows uniform distribution }

$\Rightarrow \boxed{P_X(x) = \frac{2x-1}{400}} \quad \forall\ x \in \{1, 2, 3 \ldots 20\}$

Ans.

$\Rightarrow$ Calculating PDF(X) :-

$\Rightarrow F_X(x) = \sum_{i=1}^{x} \left(\frac{2i-1}{400}\right) = \left(\frac{1}{400}\right)\left[2x \frac{x(x+1)}{2} - x\right] = \frac{x^2}{400}$

$\Rightarrow \boxed{F_X(x) = \frac{x^2}{400}} \quad \forall\ x \in \{1, 2, 3 \ldots 20\}$ or $x \in [1, 20]$

↳ distribution function

Ans.