So, here Jacobian $= h_r \cdot h_\theta \cdot h_\phi = 1 \times r \times r\sin\theta$
$= r^2\sin\theta$.
[more, less due to symmetry in spherical geometry]. Ans

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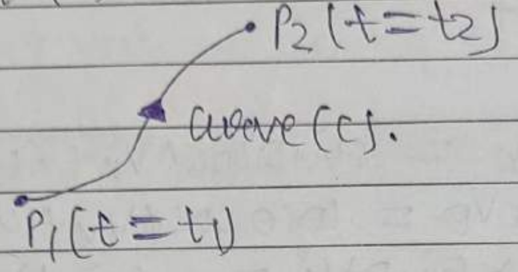
Vector Integral Calculus

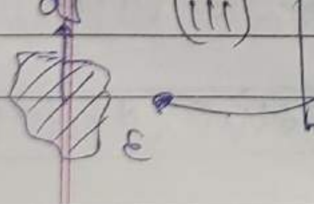
- Let $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$
where: $x(t)$ means 'x' as a function of time (t).



- Vector field exists:
$$\vec{A}(x,y,z) = A_1(x,y,z)\,\hat{i} + A_2(x,y,z)\,\hat{j} + A_3(x,y,z)\,\hat{k}.$$
where $A_i\,(i = 1,2,3)$ = continous function of (x,y,z).
along the same curve 'C'.

(i) $\int_C \vec{A}\cdot\vec{dr} = \int_C A_1(x,y,z)\,dx + A_2(x,y,z)\,dy + A_3(x,y,z)\,dz$.
= line integral over 'C'.

(ii) $\oint_C \vec{A}\cdot\vec{dr}$ = circulation of $\vec{A}$ along 'C'.



(iii) $\boxed{\iint_\varepsilon \vec{A}\cdot\vec{ds} = \iint_\varepsilon \vec{A}\cdot\hat{n}\,ds}$

$\vec{ds} = ds\cdot\hat{n}$
= Area vector of surface 'ε'.

"surface integral"
$\equiv$
integral over a closed surface

Note: $\iint$ by default means $\oint$.

(b) surface taken will be "orientable surface". It means outward and inward are distinguished properly.

(iv) $\iint \overline{ds}\times\vec{A}$ or $\iint \vec{ds}\times\vec{A}$; both are equal.

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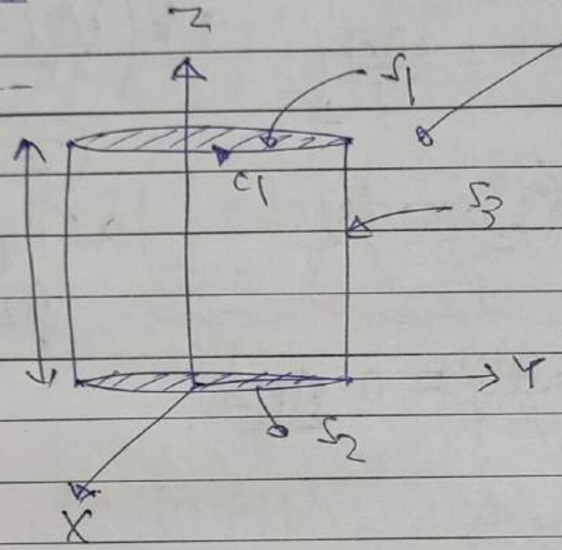
Example:-



Given, vector field $\vec{A} = \dots$
$\vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$.
Calculate surface integral of $\vec{A}$ over the given cylinder.

Solution:-

(a) $$\oint_{C_1} \vec{A}\cdot\vec{dr} = \oint_{C_1}\left(A_x dx + A_y dy + A_z \cancel{dz}^{\,0}\right)$$

$$= \oint_{C_1}\left(x^2 dx + y^2 dy\right) = \oint_{C_1}\frac{1}{3}d(x^3+y^3).$$

$$= 0.$$

{as there is (no) multi valued function, so its in closed loop, is trivial and will be equal to zero.}

(x^3+y^3) = single-valued function.
= Polynomial function.

Similarly: $\boxed{\oint_{C_2}\vec{A}\cdot\vec{dr} = 0 = \oint_{C_1}\vec{A}\cdot\vec{dr} = 0}$

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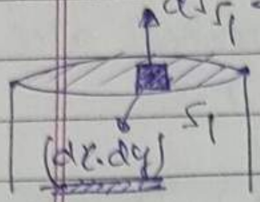
(2.) Surface integral over $S_1 = \iint_{S_1} (\vec{\nabla}\times\vec{A})\cdot\vec{dS} = S_1$

$\vec{dS}_{S_1} = \hat{a}_z (dx)(dy)$

$$= \iint_{S_1} (\vec{\nabla}\times\vec{A})_z \, dx\, dy$$



(x, y will be zero in dot pdt.)   {only (z) comp. responsible}.

⇒ Let's calculate: $\vec{\nabla}\times\vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) + \hat{k}\underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)}_{\text{(z-component)}}$

{Making use of cylindrical symmetry: $\left(\frac{\partial A_y}{\partial x} = 0 = \frac{\partial A_x}{\partial y}\right)$}

as $\left(A_y = y^2,\ A_z = z^2\right)$.

⇒ Back to $S_1 = \iint_{S_1} \cancel{(\vec{\nabla}\times\vec{A})_z}^{\,0}\, dx\cdot dy$.

$$= \iint_{S_1} 0 \cdot dx \cdot dy = 0 \ \underline{\text{Ans}}.$$

Note: (a) $\iint_{S_1} (\vec{\nabla}\times\vec{A})\cdot\vec{dS}$ = flux of vector potential $\vec{A}$ through S_1 surface.

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(3.) Now we will calculate the volume integral over the whole cylinder by two different ways and find that both are equal.

(a)

Volume integral $= \iiint (\vec{\nabla}\cdot\vec{A})\, dV$ (when we integrate then they become zero)

$$= 2 \cdot \int_{z=0}^{b} \int_{x=-a}^{x=a} \int_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}} (\cancel{x}^{\,0} + \cancel{y}^{\,0} + z)\, \underbrace{dx.dy.dz}_{(dV)}$$

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$$= 2 \cdot \int_{z=0}^{b} \int_{x=-a}^{x=+a} \int_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}} (z \cdot dz)(dx \cdot dy).$$

Actually, this integration will split into two parts, just as we write the volume = $\underbrace{\text{(base area)}}_{\text{I.}} \times \underbrace{\text{(height)}}_{\text{II.}}$.

But the condition for this to happen will be :-
"Integrand and limits of the integral should be factorisable".

$$= 2 \cdot \underbrace{\left(\int_{z=0}^{b} z\,dz\right)}_{= b^2/2} \cdot \underbrace{\left(\iint_{S_1} dx \cdot dy\right)}_{\substack{\text{area of} \\ \text{'}S_1\text{' surface}}}$$

area of infinitesimally small element on 'S_1' surface

$$= (\pi a^2) b^2 .\quad \text{Ans.}$$

We can also
Doing the same thing in cylindrical coordinates :-

$$= \int_{z=0}^{b} \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} z\, \underbrace{(d\rho)(\rho\, d\phi)\, dz}.$$

But how we directly write the thing for cylindrical coordinates.

as $dz = dz$ [z will not change]
to make dimensionful we multiple 'ρ' to $d\phi$.

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(b) We will calculate $\underbrace{\oint\!\!\oint \vec{A}\cdot\vec{ds}}$ $= \{ S_1 \cup S_2 \cup S_3$.
(For whole cylinder).

$$\Rightarrow \oint\!\!\oint \vec{A}\cdot\vec{ds} = \underbrace{\iint_{S_1} \vec{A}\cdot\vec{ds}}_{K_1} + \underbrace{\iint_{S_2} \vec{A}\cdot\vec{ds}}_{K_2} + \underbrace{\iint_{S_3} \vec{A}\cdot\vec{ds}}_{K_3}.$$

(i) $K_1 = \iint_{S_1} \vec{A}\cdot\vec{ds}$, $\vec{ds} = (dx)(dy)\cdot\hat{k}$ $= \iint_{S_1} z^2 \underbrace{(dx)(dy)}_{\text{area of } S_1 \text{ surface [infinitesimally small]}}$.

{only z-component will remain after dot pdt} as $z = b$

$$= b^2 \underbrace{\iint_{S_1} dx\cdot dy}_{\text{area of surface } S_1}$$

$$\boxed{K_1 = \pi a^2 b^2}.$$

(ii) $K_2 = \iint_{S_2} \vec{A}\cdot\vec{ds} = 0$ [as $z = 0$].

(iii) we will use (ρ, ϕ, z) here:-

$K_3 = \iint_{S_3} \vec{A}\cdot\vec{ds} = 0.$ [see proof on next page]. $\vec{ds} = \hat{a}_\rho\cdot(\rho\, d\phi)(dz)$.

So: $\boxed{\oint\!\!\oint \vec{A}\cdot\vec{ds} = \pi a^2 b^2 = \iiint_V (\vec{\nabla}\cdot\vec{A})\, dV.}$

{From both the methods, the answer is same.}

Ans

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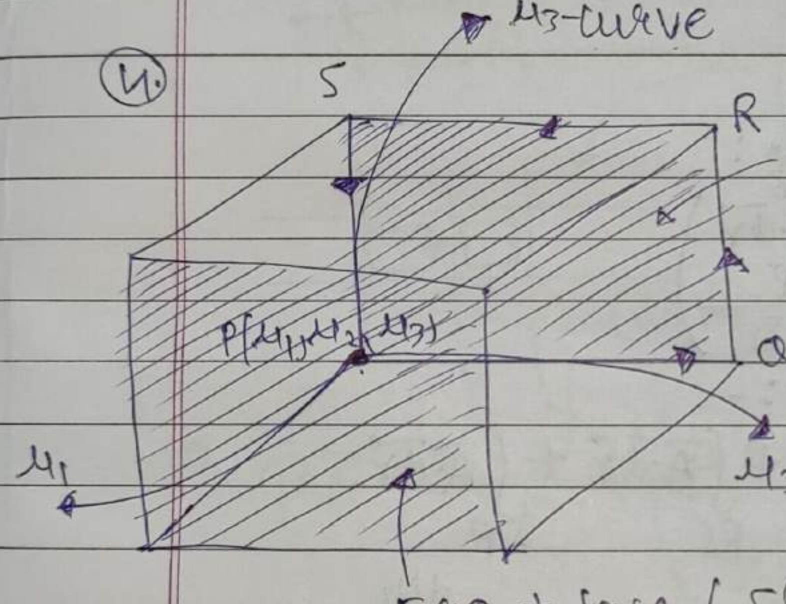
Ques:- why $k_3 = 0$?

Solution:- Write $\vec{A}$ in terms of $\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ and then take out dot pdt, it will turn out to be zero $\Rightarrow$ $\boxed{k_3 = 0}$

[Take components of $\vec{A}$ along $\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ and use:
$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$
]

(4.)



back surface $(S_1^{(2)})$.

Front face $(S_1^{(1)})$.

We know, that:

$\Rightarrow$ ΔV or $dV = h_1 h_2 h_3\, du_1\, du_2\, du_3$. [For orthogonal].

(only notation):

(a) Firstly, we will calculate: $\iint_{S_1^{(1)} + S_1^{(2)}} \vec{A} \cdot \vec{ds} \equiv T$ $\quad (\hat{n} \cdot ds)$

$\Rightarrow T = \Sigma\; S_1^{(2)} \cup S_1^{(1)}$.

$= -A_1(u_1, u_2, u_3) \cdot (h_2\, du_2)(h_3\, du_3)$.

[-ve because $\hat{n}$ in $(-)$ direction]

For $S_1^{(2)}$

$+ A_1(u_1 + du_1, u_2, u_3) \cdot (h_2\, du_2) \cdot (h_3\, du_3)$.

For $S_1^{(1)}$

Taylor series:-

$$\boxed{f(x+a) = f(x) + a f'(x) + \frac{a^2}{2} f''(x) + \ldots}$$

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Remember, A_1, h_2, h_3 are all functions of (u_1, u_2, u_3).

$$T = \iint_{S_1^{(1)} + S_1^{(2)}} \vec{A} \cdot \vec{dS} = (du_2 \cdot du_3) \left[(A_1 h_2 h_3)(u_1 + du_1, u_2, u_3) - A_1 h_2 h_3 (u_1, u_2, u_3) \right]$$

[A_1, h_2, h_3 are function of u_1, u_2, u_3]

$$\# \boxed{T = \frac{\partial}{\partial u_1} \overbrace{(A_1 h_2 h_3)}_{(u_1, u_2, u_3)} \cdot du_1 \cdot du_2 \cdot du_3}$$

(b) Now divide T by ΔV:-

$$\boxed{\frac{T}{\Delta V} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (A_1 h_2 h_3)_{(u_1, u_2, u_3)}}$$

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(5.) We will be calculating:

$$\lim_{\Delta S_1 \to 0} \left(\frac{\int_{PQRSP} \vec{A} \cdot \vec{dr}}{\Delta S_1} \right)$$

Let numerator be: N:-

$$N = \int_{PQ} \vec{A} \cdot \vec{dr} + \int_{QR} \vec{A} \cdot \vec{dr} + \int_{RS} \vec{A} \cdot \vec{dr} + \int_{SP} \vec{A} \cdot \vec{dr}.$$

$$= A_2 h_2 du_2 + A_3 h_3 (u_1, u_2 + du_2, u_3)\, du_3 - A_2 h_2 (u_1, u_2, u_3 + du_3)\, du_2 - A_3 du_3 h_3.$$

Taylor expansion:-

$$\boxed{N = \left(\frac{\partial}{\partial u_2} (A_3 h_3) - \frac{\partial}{\partial u_3} (A_2 h_2) \right) du_2 du_3.} \quad \text{--- (1.)}$$

We know: $\Delta S_1 = (h_2 h_3 du_2 du_3)$:-

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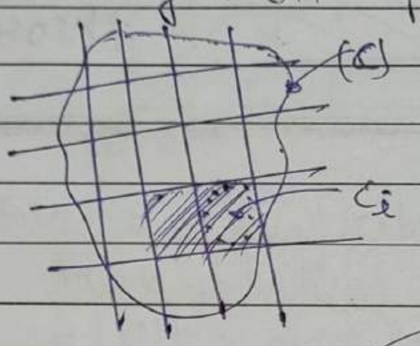
So: $\boxed{\dfrac{N}{\Delta S_1} = \dfrac{1}{h_2 h_3}\left(\dfrac{\partial}{\partial u_2}(A_3 h_3) - \dfrac{\partial}{\partial u_3}(A_2 h_2)\right)}$

$\Rightarrow$ let curve PQRSP $= \gamma_1$:-

$$\lim_{\Delta S_1 \to 0}\left(\frac{N}{\Delta S_1}\right) = \left(\frac{\oint_{\gamma_1} \vec{A}\cdot d\vec{r}}{\Delta S}\right) = (\vec{\nabla}\times\vec{A})\cdot\hat{q}_1 \quad \text{Ans}$$

☆ Stokes theorem :-

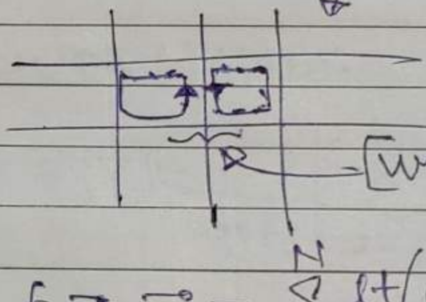
(a) Let any arbitrary curve C :-



• divided into infinite small area elements.

So: we can write :-

$$\boxed{\oint_C \vec{A}\cdot d\vec{r} = \sum_{i=1}^{\infty} \oint_{C_i} \vec{A}\cdot d\vec{r}} \quad \text{--- (1)}$$



[would both cancel eachother]

$$\Rightarrow \oint_C \vec{A}\cdot d\vec{r} = \sum_{i=1}^{N} \underbrace{\lim_{\Delta S_i \to 0}\left(\frac{\oint_{C_i} \vec{A}\cdot d\vec{r}}{\Delta S_i}\right)}_{((\vec{\nabla}\times\vec{A})\cdot\hat{n}_i)} \Delta S_i \qquad \begin{bmatrix} \Delta S_i \to 0 \\ \Rightarrow N \to \infty \end{bmatrix}$$

$$\boxed{\oint_C \vec{A}\cdot d\vec{r} = (\vec{\nabla}\times\vec{A})\cdot\hat{n}_i (dS)} \quad \text{--- (2)}$$

and

$$\boxed{\oint_C \vec{A}\cdot d\vec{r} = \iint_{\Sigma} (\vec{\nabla}\times\vec{A})\cdot d\vec{S}} \quad \text{--- (3)} \quad \left\{\text{Stokes theorem}\right\}$$

(not partial diff. operator).

$\oint$ and $\partial \Sigma = C$ = Boundary of the open surface Σ is C.

☆ Gauss-divergence Theorem:-

Similarly:-

$$\oiint_S \vec{A}\cdot\vec{ds} = \sum_{i=1}^{N} \underbrace{\text{limit}_{\Delta V_i \to 0} \left(\frac{\oiint_{S_i} \vec{A}\cdot\vec{ds}}{\Delta V_i} \right)}_{(\vec{\nabla}\cdot\vec{A})} \Delta V_i$$

$$\boxed{\oiint_S \vec{A}\cdot\vec{ds} = \iiint_V (\vec{\nabla}\cdot\vec{A})\, dV}$$ (Gauss-divergence theorem).