Indian Institute of Technology Roorkee
MAN-001(Mathematics I)
Autumn Semester 2022–23
Assignment 6: (Multiple Integrals)

1. Sketch the region R in the xy-plane bounded by the curves $y^2 = 2x$ and $y = x$, and find its area.

2. Evaluate the following integrals by interchanging the order of integration:
   (a) $\int_0^1 \int_{4y}^4 e^{x^2} dxdy.$ (b) $\int_0^1 \int_{y^2}^1 (ye^{x^2}) dxdy.$ (c) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dydx.$
   (d) $\int_0^8 \int_{y^{\frac{1}{3}}}^2 \sqrt{(x^4+1)} dxdy.$ (e) $\int_0^3 \int_1^{\sqrt{4-y}} (x+y) dxdy.$ (f) $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2x^2}} dydx.$

3. Evaluate:
   (a) $\int\int_D (4x+2) dA$, where D is a region enclosed by the curves $y = x^2$ and $y = 2x$.
   (b) $\int\int_R [x+y] dA$, over the rectangle formed by the coordinate axes and the lines $x = 1$, $y = 2$.

4. Evaluate the following double integrals:
   (a) $\int\int_R (x^2+y^2) dA$, where R is the region of the plane given by $x^2+y^2 \leq a^2$.
   (b) $\int_0^1 \int_{\sqrt{3}\,y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2} dxdy.$

5. Show the followings by changing the order of integration:
   (a) $\int_0^{\pi/2} \int_0^{2a\cos\theta} f(r,\theta) drd\theta = \int_0^{2a} \int_0^{\cos^{-1}(r/2a)} f(r,\theta) d\theta dr.$
   (b) $\int_0^{\pi/3} \int_{a\sec^2(\theta/2)}^{(8a/3)\cos\theta} f(r,\theta) drd\theta = \left[\int_a^{4a/3} \int_0^{2\cos^{-1}(\sqrt{a/r})} + \int_{4a/3}^{8a/3} \int_0^{\cos^{-1}(3r/8a)} \right] f(r,\theta) d\theta dr.$

6. Prove that
   (a) $\int_0^a \int_0^x \frac{f'(y)dydx}{\sqrt{(a-x)(x-y)}} = \pi(f(a) - f(0)).$
   (b) $\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{bdydx}{(x^2+y^2+b^2)^{3/2}(x^2+y^2+a^2)^{1/2}} = \frac{2\pi}{a+b}.$ (By changing into polar coordinates.)

7. Evaluate the following triple integrals:
   (a) $\int\int\int_E 2x dV$, where E is the region under the plane $2x+3y+z = 6$ that lies in the first octant.
   (b) $\int\int\int_E \sqrt{3x^2+3z^2} dV$, where E is the solid bounded by $y = 2x^2+2z^2$ and the plane $y = 8$.
   (c) $\int\int\int_E xyz dV$, where E is the solid bounded by the sphere of radius 2 in the first octant.
   (d) $\int\int\int_E dV$, where E is the solid bounded by the cylinder $x^2+y^2 = 9$ and the planes $z = 1$ and $x+z = 5$.

8. Find the volume of the region bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the paraboloid $z = 4x^2 + 4y^2$.

9. Evaluate the following integrals by changing the variables into cylindrical coordinates:
   (a) $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{16-x^2-y^2} \sqrt{x^2+y^2}dzdydx$.
   (b) $\int\int\int_E \sqrt{(x^2+y^2)}dV$, where E is the region lying above the xy-plane and below the cone $z = 4 - \sqrt{x^2+y^2}$.
   (c) $\int\int\int_E dV$, where E is the region bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the plane $z = b$, where $a > b > 0$.

10. By using spherical coordinates evaluate the following triple integrals:
    (a) $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} dzdydx$.
    (b) $\int\int\int_E (x^2+y^2+z^2)^{\frac{1}{2}} dV$, where E is the region bounded by the plane $z = 3$ and the cone $z = \sqrt{x^2+y^2}$.
    (c) $\int\int\int_E (x^2+y^2+z^2)^{-\frac{3}{2}} dV$, where E is the region bounded by the spheres of radius 2 and 3.

11. Evaluate $\int\int_R (\frac{x-y}{x+y+2})^2 dxdy$, where R is the region bounded by the lines $x+y = \pm 1,\ x-y = \pm 1$. (Use the transformation $u = x+y,\ v = x-y$ and integrate over an appropriate region in $uv-$plane.)

12. Evaluate $\int\int_R (3x^2+14xy+8y^2)dxdy$, where R is the region in the first quadrant bounded by the lines $y = -\frac{3}{2}x+1$, $y = -\frac{3}{2}x+3$, $y = -\frac{1}{4}x$ and $y = -\frac{1}{4}x+1$, using the transformation $u = 3x+2y$ and $v = x+4y$.

13. Evaluate $\int\int_R e^{x^2-y^2} dA$, where R is the region in the first quadrant bounded by $x^2 - y^2 = 1,\ x^2 - y^2 = 4, y = 0$ and $y = (3/5)x$, by using the transformation $u = x^2 - y^2$ and $v = x + y$.

14. Evaluate $\int_0^3 \int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} (\frac{2x-y}{2} + \frac{z}{3})dxdydz$ by applying the transformation $u = \frac{2x-y}{2},\ v = \frac{y}{2},\ w = \frac{z}{3}$, and integrating over an appropriate region in $uvw-$plane.

Answers:

1. $\frac{2}{3}$. **2.** (a) $\frac{1}{8}(e^{16}-1)$. (b) $\frac{1}{4}(e-1)$. (c) 1. (d) $\frac{1}{6}(17^{\frac{3}{2}}-1)$. (e) $\frac{241}{60}$. (f) $\frac{\pi a^2}{6}$.
3. (a) 8. (b) 2 . **4.** (a) $\frac{\pi a^4}{2}$. (d) $\frac{4\pi}{9}$.

7. (a) 9. (b) $\frac{256\sqrt{3}\ \pi}{15}$. (c) $\frac{4}{3}$. (d) 36π. **8.** $\frac{5\pi}{2}$.

9. (a) $\frac{1024(\pi)}{15}$. (b) $\frac{64}{3}(2\pi)$. (c) $2\pi(\frac{a^3}{3} - \frac{a^b}{2} + \frac{b^3}{6})$.

10. (a) $\frac{\pi}{3}$, (b) $\frac{27\pi}{2}(2\sqrt{2}-1)$, (c) $4\pi\log(\frac{3}{2})$.

11. $\frac{2}{9}$. **12.** $\frac{64}{5}$. **13.** $\frac{\log 2}{2}(e^4 - e)$. **14.** 12.