Defn: <u>logic to lambda mapping</u>. $\Delta \to \Delta_\lambda$

(i) Δ : single formula τ any term variable x.

Then Δ_λ : $x:\tau \mapsto x:\tau$

(ii) Last step in Δ is $(\to E)$ applied to the conclusions of deductions Δ' and Δ''.

let $\Delta'_\lambda : \Gamma' \mapsto M : \sigma \to \tau$ $\quad \Delta''_\lambda : \Gamma'' \mapsto N : \sigma$

Then replace all term variables in Δ''_λ by distinct new ones so that there is no common term variable in Δ'_λ and Δ''_λ.

Now apply $\to E$ to obtain Δ_λ.

(iii) if the last step in Δ is an occurrence of $(\to I)$ with form

$$\dfrac{\left.\begin{array}{c}[\rho]\\ \vdots \\ \sigma\end{array}\right\} \text{ a deduction } \Delta'}{\rho \to \sigma} \quad \{\text{discharge } K \geq 0 \text{ occurrences of } \underline{\rho_1}, \ldots, \underline{\rho_K} \text{ of } \rho\}.$$

<u>$K \geq 1$</u> let $\Delta^{*\prime}_\lambda$: $\Gamma, v_1:\rho_1 \ldots v_K:\rho_K \mapsto P:\sigma$

(v_i's are distinct)

replace all v_i's by a new term variable x to obtain

$\Gamma, x:\rho \mapsto P^*:\sigma \qquad P^* \equiv P[x/v_1 \ldots x/v_K]$

Apply $\to I$main to obtain $\boxed{\Gamma_\lambda \mapsto \underline{(\lambda x. P^*)} : \rho \to \sigma}$: Δ_λ

<u>$K=0$</u> $\Delta^{*\prime}_\lambda$: $\Gamma \mapsto P : \sigma$

Choose a new variable x not in $\Delta^{*\prime}_\lambda$.

and apply $\to I$vac to obtain.

Δ_λ : $\boxed{\Gamma \mapsto (\lambda x . P) : \rho \to \sigma}$

Proof of $a \to a \to a$:-

#1: Δ_1

$$\dfrac{\dfrac{[a]\ (00)}{a \to a\ (0)}\ (\to I)\ \{\text{disch. } a \text{ vac.}\}}{a \to a \to a\ \phi}\ (\to I)\ \{\text{disch. } a \text{ at } 00\}$$

$(\Delta_1)_\lambda$:

$$\dfrac{\dfrac{x : a \mapsto x : a}{x : a \mapsto (\lambda y . x) : a \to a}\ (\to I)\text{vac}\ \ y : a}{\mapsto (\lambda x . \lambda y . x) : a \to a \to a}\ (\to I)\text{main}$$

#2. Δ_1

$$\dfrac{\dfrac{[a]\ (00)}{a \to a\ (0)}\ (\to I)\ \{\text{disch } a \text{ at } 00\}}{a \to a \to a}\ (\to I)\ \{\text{disch } a \text{ vac}\}.$$

$(\Delta_2)_\lambda$:

$$\dfrac{\dfrac{x : a \mapsto x : a}{\mapsto (\lambda x . x) : a \to a}\ (\to I)\text{main}}{\mapsto (\lambda y . \lambda x . x) : a \to a \to a}\ (\to I)\text{vac.}$$

the terms are different.

All the assumptions must be discharged.

$$\sigma \equiv \dfrac{\left[\ \dfrac{\sigma}{\sigma}\right.}{\sigma \to \sigma}\ \{\text{disch. } \sigma \text{ vac.}\}$$

$$\sigma \equiv \dfrac{\left[\ \dfrac{\sigma\ (00)}{\sigma}\right.}{\sigma \to \sigma}\ \{\text{disch. } \sigma \text{ at } 00\}$$

$$\dfrac{\dfrac{\dfrac{\sigma\ (00)}{\vdots}}{\tau}}{\sigma \to \tau}\ \{\text{disch } \sigma \text{ at } 00\}$$

There are two proofs for a->a and a->a->a

proof of $a \to a$

#. Δ_1 :

$$\dfrac{[a]\ (00)}{a \to a}\ (\to I)\ \{\text{disch } a \text{ vac.}\}$$

$\longleftrightarrow$ $(\Delta_1)_\lambda$:

$$\dfrac{x : a \mapsto x : a}{x : a \mapsto (\lambda y . x) : a \to a}\ (\to I)\text{vac}\ \ y : a$$

Δ_2 :

$$\dfrac{[a]\ (00)}{a \to a}\ (\to I)\ \{\text{disch } a \text{ at } 00\}$$

$\longleftrightarrow$ $(\Delta_2)_\lambda$:

$$\dfrac{x : a \mapsto x : a}{\mapsto (\lambda x . x) : a \to a}\ (\to I)\text{main}$$

Disch. occurrences at top must be marked as [ ].

Observations :-

1. There can be more than one (different) λ-terms that have same type.
   i.e, $M_1 : \tau$ and $M_2 : \tau$ is possible.

2. ~~There cannot be two distinct deductions of a λ-term.~~
   so ~~$M : \tau$ and $M : \tau'$~~ is not possible. (i.e. $\tau = \tau'$.)

3. ~~There cannot be two distinct proofs of a type.~~

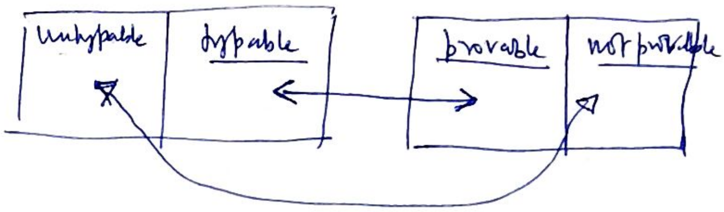
4. From a proof of τ we can obtain only one λ-term M with $M : \tau$

5. If τ is not provable then there is no λ-term M ~~for~~ with $M : \tau$.
   i.e τ cannot be assigned to any λ-term.

6. M is typable $\Leftrightarrow$ τ is provable
   (with type τ)

C-H isomorphism

1. provable formulae $\longleftrightarrow$ types of closed terms.
2. logic proofs $\longleftrightarrow$ TA_λ - proofs
3. logic deductions $\longleftrightarrow$ TA_λ - deductions.



if a term is typable in TA-lambda, then it is provable in IIL and vice-versa.

If a term is untype-able in TA-lambda, then it is not provable in IIL and vice-versa.

$\tau = ((a \to b) \to a) \to a$ is not provable in IIL.
(Pierce's law)

Example 1 logic to lambda

Δ :

$a \to a \to c \quad a \ (\to E)$

$a \to c \quad a \ (\to E)$

c

Δ_λ :

$x : a \to a \to c \mapsto x : a \to a \to c \qquad y : a \mapsto y : a$ (axiom) $(\to E)$

$x : a \to a \to c,\ y : a \mapsto (xy) : a \to c \qquad z : a \mapsto z : a \ (\to E)$

$x : a \to a \to c,\ y : a,\ z : a \mapsto xyz : c$

Example 2 logic to lambda $\quad \rho = (a \to a \to c) \to a \to a \to c$

Δ :

$[a \to a \to c]\ (00011) \qquad [a]\ (00012) \ (\to E)$

$a \to c\ (0001) \qquad [a]\ (0002) \ (\to E)$

$c\ (000)$ $(\to I)$ {discharging a at 00012}

$a \to c\ (00)$ {discharg ~~at~~ at 0002}

$a \to a \to c\ (0)$ {disch $[a \to a \to c]$ at (0011)}.

$(a \to a \to c) \to a \to a \to c \quad \phi$

Δ_L :

$x : a \to a \to c \mapsto x : a \to a \to c \qquad y : a \mapsto y : a \ (\to E)$

$x : a \to a \to c,\ y : a \mapsto xy : a \to c \qquad z : a \mapsto z : a \ (\to E)$

$x : a \to a \to c,\ y : a,\ z : a \mapsto xyz : c$ $(\to I)$ main.

$x : a \to a \to c,\ y : a,\ z : a \mapsto \lambda y.\ xyz : a \to c$ $(\to I)$ main

$x : a \to a \to c,\ z : a \mapsto \lambda z.\lambda y.\ xyz : a \to a \to c$ $(\to I$ main$)$

$x : a \to a \to c \mapsto \lambda x.\cancel{z}.\lambda y.\ xyz : (a \to a \to c) \to a \to a \to c.$

$\boxed{\text{proof structure same but discharge labels different, so the } \lambda\text{-terms ~~become~~ are different.}}$

Example 3 logic to lambda. ρ is same as in Example 2.

Δ :

$[a \to a \to c]\ (00011) \qquad [a]\ (00012) \to E$

$a \to c\ (0001) \qquad [a]\ (0002) \to E$

$c\ (000)$ $(\to I)$ {discharging a at 00012, 0002}

$a \to c\ (00)$ $(\to I)$ {disch a vac.}

$a \to a \to c\ (0)$ $(\to I)$ {disch. $a \to a \to c$ at 00011}

$(a \to a \to c) \to a \to a \to c \quad \phi$

Δ_L :

$x : a \to a \to c \mapsto x : a \to a \to c \qquad y : a \mapsto y : a \ (\to E)$ $\quad (z' : a \mapsto z' : a)$

$x : a \to a \to c,\ y : a \mapsto xy : a \to c \qquad z : a \mapsto z : a \ (\to E)$ $\quad (z' : a \mapsto z' : a)$

$[z'/z,\ z'/y]$

$x : a \to a \to c,\ \underbrace{y : a,\ z : a}_{z' : a} \mapsto \underbrace{xyz : c}_{xz'z' : c}$ $(\to I)$ main

$x : a \to a \to c \mapsto \lambda z'.\ xz'z' : a \to c$ $(\to I)$ vac

$x : a \to a \to c \mapsto \lambda y'.\lambda z'.\ xz'z' : a \to a \to c \longrightarrow \to I_n.$

$\mapsto \lambda x.\lambda y'.\lambda z'.\ xz'z' : (a \to a \to c) \to a \to a \to c.$