Assignment-8

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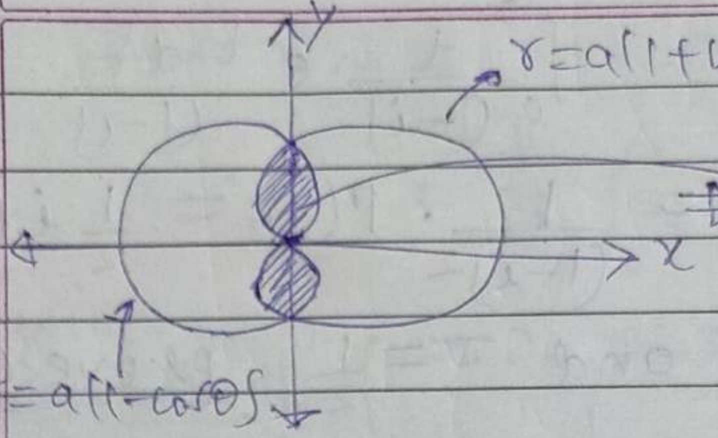
(1.)



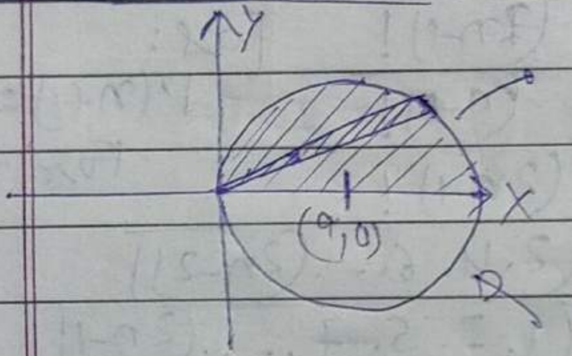
$$A = 4\times\int_0^{\pi/2}\left(\int_0^{a(1-\cos\theta)} r\,dr\right)d\theta$$

$$= 4\times\frac{a^2}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta$$

$$= 2a^2\left(\frac{3\pi}{4}-2\right).$$

$$= \frac{a^2}{2}(3\pi-8) \text{ Ans}$$

(2.)



$x^2+y^2=2ax$

$r^2=2ar\cos\theta$

$\boxed{r=2a\cos\theta}$

And for covering upper part only $\theta \to 0$ to $\pi/2$.

So: $$I = 2.\int_0^{\pi/2}\left(\int_0^{2a\cos\theta}\left(\int_{nx}^{mx} dz\right).r\,dr\right)d\theta$$

(For lower part)

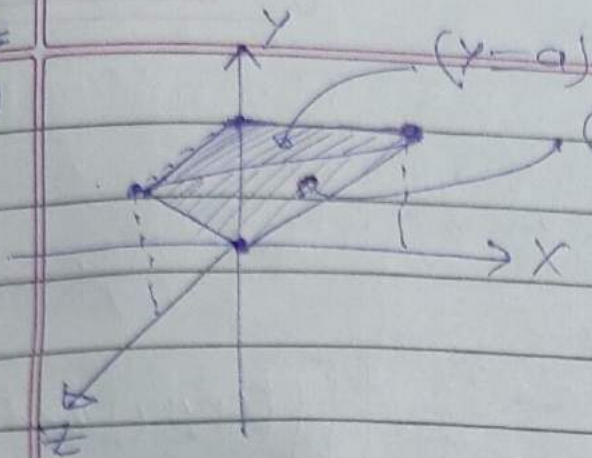
$$I = 2.\int_0^{\pi/2}(m-n)(\cos\theta).\frac{8a^3}{3}.\cos^3\theta\,d\theta$$

$$= \frac{16a^3}{3}(m-n).\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$= \frac{16a^3}{3}(m-n)\times\frac{3\pi}{16} = \pi a^3(m-n)$$

Ans

3.



Here, the region has been bounded by:

$$\begin{Bmatrix} x \geq 0 \\ z \geq 0 \\ y \leq a \\ y \geq x+z \end{Bmatrix}$$

M-I

$$I = \text{volume} = \int_0^a \left(\int_0^{a-z} \left(\int_{x+z}^{a} dy \right) dx \right) dz.$$

$$\Rightarrow \boxed{I = a^3/6} \quad \text{Ans}$$

M-II

It may be seen that the volume will be equal to: $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x+y+z \leq a$.

So applying dirichlet's integral:

$$I = a^3 \cdot \frac{(\Gamma(1))^3}{\Gamma(4)} = \frac{a^3 \cdot (1)^3}{3!} = \frac{a^3}{6} \quad \text{Ans}$$

4.

$$I = \int_0^{2\pi} \int_0^{2} \int_{\frac{x^2+y^2}{2}}^{2} (z\,dz) \cdot \frac{1}{\rho} \cdot \rho\, d\rho \cdot d\phi \quad ; \; x^2+y^2 = 4$$

$$= (2\pi) \times \left(\frac{1}{2}\right) \times \left(\frac{1}{4}\right) \times \frac{32}{5} \times 4 = \frac{32\pi}{5} \quad \text{Ans}$$

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(5.) Volume of $D = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} r^2 \sin\theta \cdot dr \cdot d\theta \cdot d\phi.$

$$= \left(\frac{8}{3}\right) \times (2) \times (2\pi) = \left(\frac{32\pi}{3}\right).$$

$\Rightarrow$ So, average value $(I_{avg}) = \frac{3}{32\pi} \iiint_D (x+y+z)\, dx\, dy\, dz.$

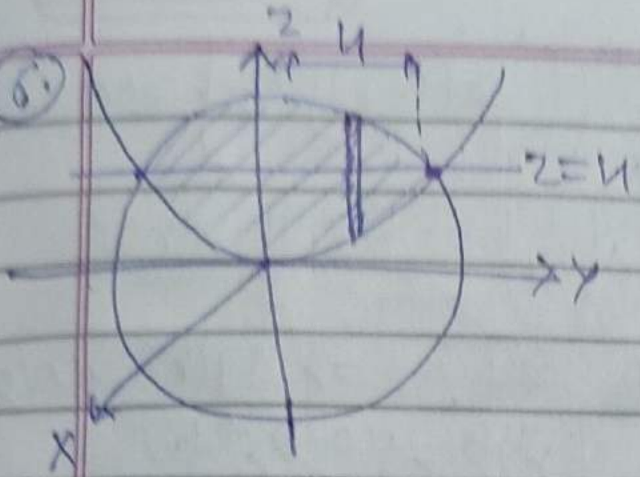
$$\Rightarrow I_{avg} = \frac{3}{32\pi} \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \big(r\sin\theta(\sin\phi + \cos\phi) + r\cos\theta\big) r^2 \sin\theta \, dr\, d\theta\, d\phi.$$

$$= \frac{3}{32\pi} \int_0^{2\pi}\int_0^{\pi} (4) \cdot \big(\sin^2\theta(\sin\phi + \cos\phi) + \sin\theta\cos\theta\big)\, d\theta\, d\phi.$$

$$= \frac{3}{8\pi} \int_0^{2\pi} (\sin\phi + \cos\phi) \cdot \frac{\pi}{2}\, d\phi.$$

$$= \left(\frac{3}{16}\right) \cdot \int_0^{2\pi} (\cos\phi + \sin\phi)\, d\phi = 0 \quad \underline{\text{Ans}}$$

(6.)



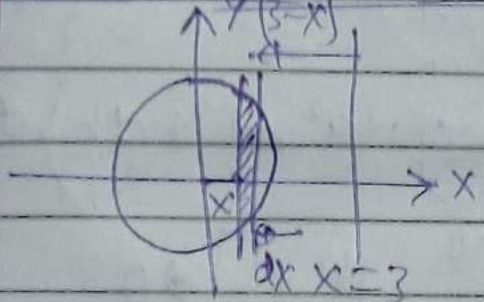
$$\Rightarrow V = \int_0^{2\pi}\int_0^{4}\left(\int_{\frac{r^2}{4}}^{\sqrt{32-r^2}} dz\right) r\,dr\,d\theta$$

$$= 2\pi \cdot \int_0^4 \left(\sqrt{32-r^2} - \frac{r^2}{4}\right) r\,dr.$$

$$= 2\pi\left(\frac{1}{3}\cdot 64(2\sqrt{2}-1) - 16\right)$$

$$= \frac{32\pi}{3}\left(8\sqrt{2}-7\right) \quad \underline{\text{Ans}}$$

(7.)



$$dV = (dx)\left[2\pi(3-x)\right] 2\cdot\sqrt{4-x^2}$$

$$V_T = \int_{-2}^{+2} 4\pi\cdot(3-x)\sqrt{4-x^2}\,dx$$

$$V_T = (4\pi)\cdot\int_0^2 6\sqrt{4-x^2}\,dx.$$

$$\Rightarrow V_T = \underline{(4\pi\times 6)}\times(2)\cdot\int_0^2\left(1-\left(\frac{x}{2}\right)^2\right)^{1/2} dx.$$

$$V_T = 48\pi\cdot \beta\left(\frac{1}{2},\frac{3}{2}\right) = 24\pi^2 \quad \underline{\text{Ans}}$$

(8.) Intersection plane will be an ellipse:

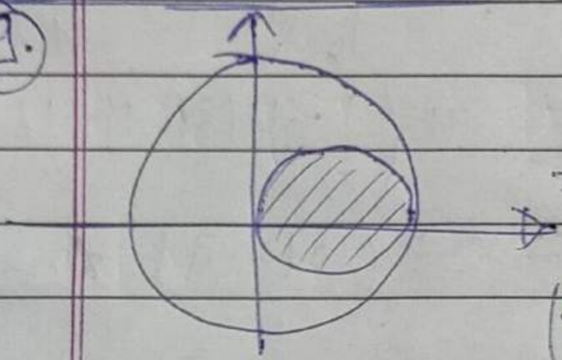
$$x^2+y^2/8 = 1 \Rightarrow x = r\cos\theta;\ y = 2\sqrt{2}\,r\sin\theta.$$

$$\Rightarrow I = \text{volume} = \int_0^{2\pi}\int_0^1\int_{3x^2+\frac{y^2}{4}}^{4-x^2-\frac{y^2}{4}} dz\cdot\left(2\sqrt{2}\,r\,dr\,d\theta\right)$$

$$I = (2\sqrt{2})(4)\cdot\int_0^{2\pi}\left(\int_0^1 r(1-r^2)\,dr\right)d\theta$$

$$= (8\sqrt{2})\cdot(2\pi)\cdot\left(\frac{1}{4}\right) = 4\sqrt{2}\pi \quad \text{Ans}$$

9.



$\therefore\ x^2+y^2 = ax \Rightarrow (r = a\cos\theta)$.

$$\therefore V = 2\cdot\int_0^{\pi/2}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{+\sqrt{a^2-r^2}} z^2\,dz\cdot r\,dr\,d\theta$$

(For other region)

$$\therefore V = 2\cdot\int_0^{\pi/2}\left(\int_0^{a\cos\theta}\left(\frac{2}{3}\right)\cdot(a^2-r^2)^{3/2}\,r\,dr\right)d\theta$$

$$= \left(\frac{4}{3}\right)\cdot\int_0^{\pi/2}\left(\int_{a\sin\theta}^{a} t^4\,dt\right)d\theta$$

$$= \left(\frac{4}{3}a^5\right)\times\left(\frac{1}{5}\right)\cdot\int_0^{\pi/2}\left(1-(\sin\theta)^5\right)d\theta$$

$$= \frac{4a^5}{15}\left(\frac{\pi}{2}-\frac{8}{15}\right) \quad \text{Ans}$$

10. Mass $= M = \iint \rho\,dx\,dy = \rho\int_{-2}^{1}\left(\int_{y^2}^{2-y} dx\right)dy$

$$\boxed{M = \frac{9\rho}{2}}$$

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- $\bar{x} = \frac{1}{M}\iint x\,dx\,dy = \left(\frac{2}{9}\right)\cdot\int_{-2}^{1}\left(\int_{y^2}^{2-y} x\,dx\right)dy.$

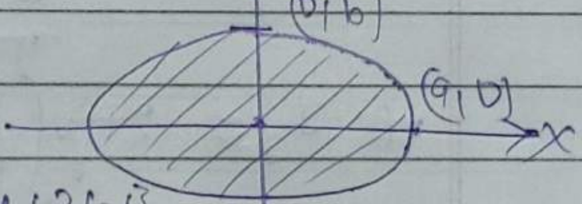
$$\boxed{\bar{x} = 8/5}$$

- $\bar{y} = \frac{2}{9}\left[\cancel{\int_0^1\left(\int_{-\sqrt{x}}^{+\sqrt{x}} y\,dy\right)}\,dx + \int_1^4\left(\int_{-\sqrt{x}}^{2-x} y\,dy\right)dx\right]$

$$\boxed{\bar{y} = -\frac{1}{2}}$$

$\Rightarrow$ for:- COM $= \left(\frac{8}{5}, -\frac{1}{2}\right)$ Ans

(11) For $\frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{b^{2/3}} = 1$ $\Rightarrow$ Mass $= \left(\frac{\mu a^2 b^2}{20}\right)$.



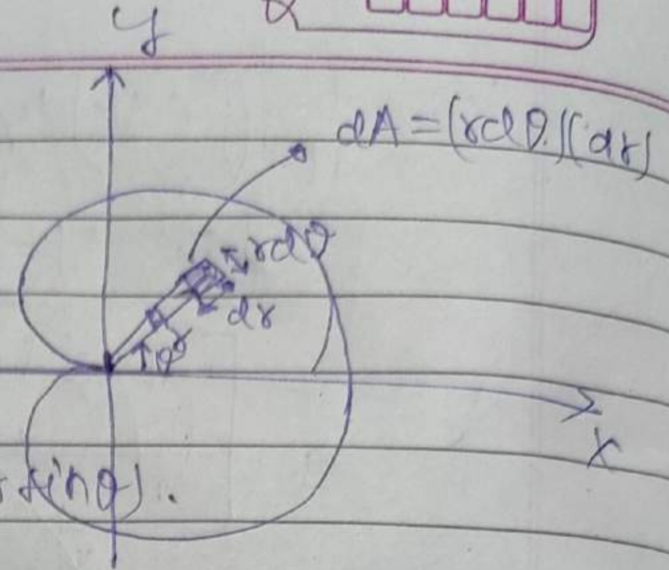
$\Rightarrow M = 4\times\int_0^b\left(\int_0^{a\cdot\left(1-\frac{y^{2/3}}{b^{2/3}}\right)^{3/2}} \rho\cdot dx\right)dy$

$= 4\times\int_0^b\left(\frac{\mu}{2}\right)\cdot a^2\cdot\left(1-\left(\frac{y}{b}\right)^{2/3}\right)^3\cdot y\,dy.$

$= (2\mu a^2)\cdot\int_0^b\left(y - \frac{y^3}{b^2} + \frac{3y^{5/3}}{b^{2/3}} + 3\cdot\frac{y^{7/3}}{b^{4/3}}\right)dy$

$\boxed{M = \left(\frac{\mu a^2 b^2}{20}\right)}$ Ans

(12.) Take a small area element on cardiod, and revolve it about $y=0$.



$$\Rightarrow dV = \left[(r\,d\theta)(dr)\right] \times (2\pi r \sin\theta).$$

$$\Rightarrow \boxed{dV = (2\pi r^2 dr)\cdot(\sin\theta\, d\theta)}. \Rightarrow \boxed{dm = \rho\, dV}$$

- We know that M.I. of ring about one of the diameter will be $\left(\frac{1}{2} M r^2\right)$.

Hence,

$$dI_y = \frac{1}{2}\cdot(dm)\cdot(r\sin\theta)^2 + (dm)(r\cos\theta)^2$$

$$= (dm)(r^2)\left(\frac{\sin^2\theta}{2} + \cos^2\theta\right).$$

$$\Rightarrow I_y = 2\pi\rho \int_0^{\pi} \left(\int_0^{a(1+\cos\theta)} r^4\, dr\right) \sin\theta \left(\frac{\sin^2\theta}{2} + \cos^2\theta\right) d\theta$$

$$= \frac{2\pi a^5}{5}\rho \int_0^{\pi} (1+\cos\theta)^5 \cdot \frac{1}{2}(1+\cos^2\theta)\sin\theta\, d\theta$$

$$= \frac{\pi a^5}{5}\rho \int_{+1}^{-1} (1+t)^5 \cdot (1+t^2)(-dt).$$

$$= \frac{\pi a^5}{5}\rho \int_{-1}^{1} (1+t)^5 (1+t^2)\, dt.$$

$$I_y = \frac{\pi a^5 \rho}{5} \int_0^1 (1+t^2)\left[(1+t)^5+(1-t)^5\right] dt$$

$$= \frac{\pi a^5 \rho}{5} \int_0^1 (1+t^2)(2+20t^2+10t^4)\, dt$$

$$= \frac{2\pi a^5 \rho}{5} \int_0^1 (1+t^2)(1+10t^2+5t^4)\, dt$$

$$= \frac{2\pi a^5 \rho}{5} \times \frac{176}{21} = \frac{352}{105}\pi \rho a^5 \text{ Ans}$$