differential Vector Calculus.

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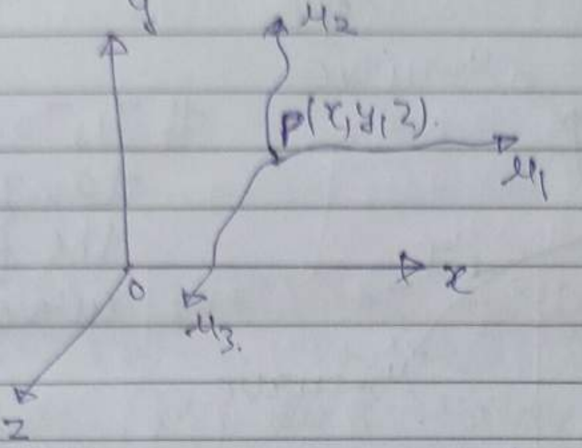
☆ Orthogonal curvilinear coordinate system:-



· From point P, assume curves are originating, and named u_1, u_2, u_3.

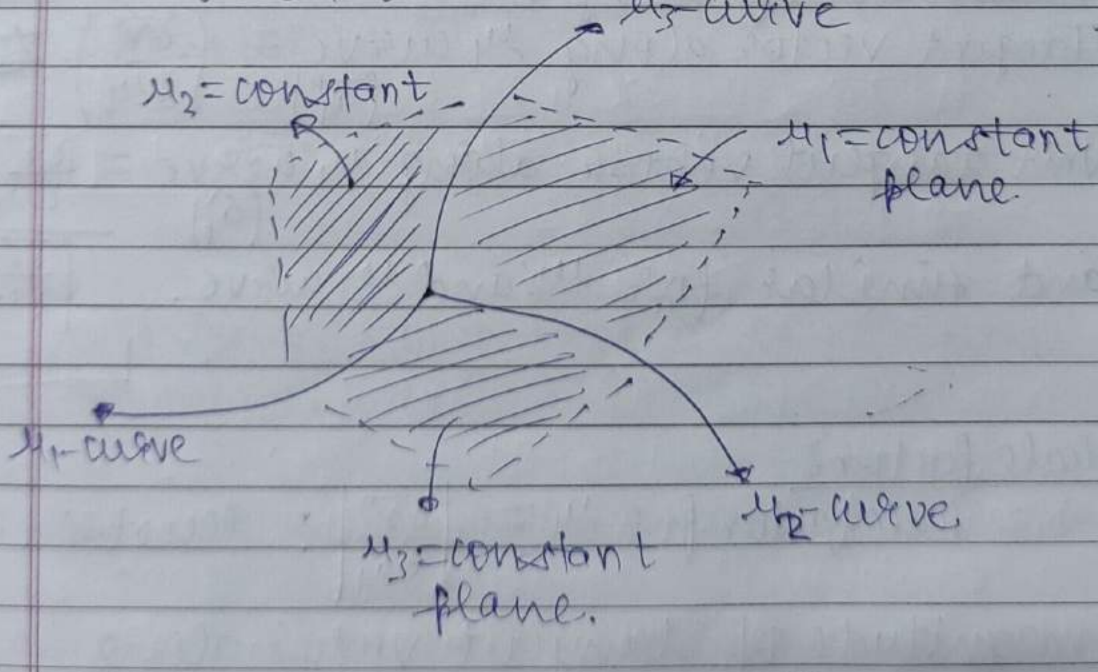
These $\{u_1, u_2, u_3 \rightarrow$ are functions of x, y, z.
$x, y, z \longrightarrow$ are also functions of u_1, u_2, u_3

matrix: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}(u_1\ u_2\ u_3)$ is invertible

of

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}(x\ y\ z).$$

- u_1, u_2, u_3 are then called curvilinear coordinates.



(a) If the three shown planes are $\perp^r$ to each other, then it is called orthogonal

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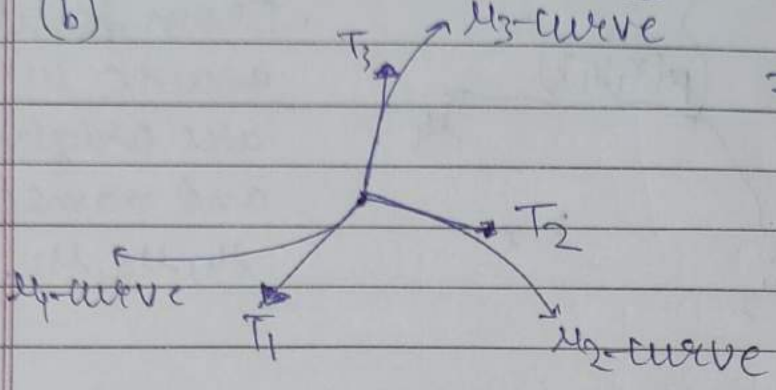
curvilinear coordinate system.

(b) 

T_1, T_2, T_3 are tangents to the following curves.

If all the three tangents are mutually $\perp^r$ to each other, then also called "orthogonal curvilinear coordinate system."

- In cartesian system, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = (x, y, z)$
but here in curvilinear system, $\boxed{\vec{r} = (u_1, u_2, u_3)}$

☆ Calculation of tangent vector :-

- Tangent vector along u_1 curve $(\vec{a}_1) = \left(\frac{\partial \vec{r}}{\partial u_1}\right)$ Ans
- Unit tangent vector along u_1 curve $(\hat{a}_1) = \dfrac{\frac{\partial \vec{r}}{\partial u_1}}{\left|\frac{\partial \vec{r}}{\partial u_1}\right|}$ Ans — ①

and similar for u_2 and u_3 curve.

Scale factor :-

- h_1 = scale factor for $u_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|$ = denotes the magnitude of tangent vector along u_1-curve.

unit tangent vector

$\therefore$ in general, $\boxed{\frac{\partial \vec{r}}{\partial u_i} = \hat{a}_i h_i}$ magnitude of tangent vector (scale factor).

tangent vector.

- Consider a particular surface where $u_i = \text{const}$, then $du_i = 0$ [But remember u_i is a $f(x,y,z)$].

$\Rightarrow$ so: $\frac{\partial u_i}{\partial x} \cdot dx + \frac{\partial u_i}{\partial y} dy + \frac{\partial u_i}{\partial z} dz = 0$

for such a surface.

Actually: $\boxed{du_i(x,y,z) = u_i(x+dx, y+dy, z+dz) - u_i(x,y,z).}$ — ①

read it as du_i as a function of x,y,z and so-on.

- For vector differentiation, we use a vector differential operator, also called dell operator denoted by $\vec{\nabla}$.

$$\boxed{\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}} = \text{dell operator.}$$

- $\boxed{\vec{\nabla} u_i \cdot d\vec{r} = 0}$ $\Rightarrow$ same as eqn: ① above.

where $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$.

$\Rightarrow$ $\vec{\nabla} u_i$ = gradient of u_i = grad u_i = Normal vector of the u_i-surface.

(unit ~~normal~~ vector) $\boxed{\hat{A}_i = \frac{\vec{\nabla} u_i}{|\vec{\nabla} u_i|}}$ Ans

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⇒ Take care $\hat{a}_i$ = unit tangent vector
$\hat{A}_i$ = unit normal vector.

- x is a $f(u_1, u_2, u_3)$ so:-

$$\boxed{dx = \sum_{i=1}^{3} \frac{\partial x}{\partial u_i} du_i} \quad \text{(1)}$$

Similarly:-

$$\boxed{dy = \sum_{i=1}^{3} \frac{\partial y}{\partial u_i} \cdot du_i} \quad \text{(2)}$$

$$\boxed{dz = \sum_{i=1}^{3} \frac{\partial z}{\partial u_i} du_i} \quad \text{(3)}$$

- Obviously, distance between two points will not change:-

$$\Rightarrow ds^2 = dx^2 + dy^2 + dz^2$$

Putting values:-

$$\# \quad \boxed{ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}\, du_i\, du_j}$$

where:-

$$\# \quad \boxed{g_{ij} = \frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_j} + \frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_j} + \frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_j}}, \quad \text{Ans}$$

$$\Rightarrow g_{ij} = \sum_{s=(x,y,z)} \frac{\partial s}{\partial u_i}\frac{\partial s}{\partial u_j}.$$

- When one has an orthogonal curvilinear coordinates system then g_{ij} can be visualised as a matrix. This can be combination of symmetric as well as skew symmetric but putting in eqⁿ, will automatically eliminate anti-sym. part, so

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- "g_{ij} will be a symmetric matrix in orthogonal curvilinear coordinate system".

when $i=j$
$\boxed{g_{ij}>0}$

when $i\neq j$
$\boxed{g_{ij}=0}$

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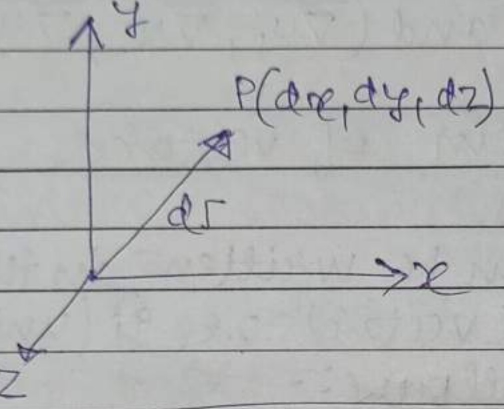
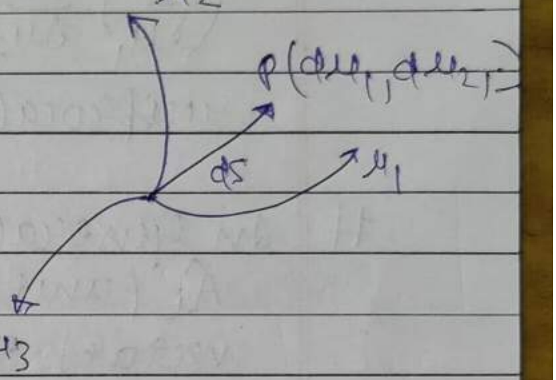
☆ General coordinate transformation :- (GCT) :-





$$\boxed{(x, y, z) \longrightarrow (u_1, u_2, u_3)}$$

- We know :-

(a) $$dx = \sum_{i=1}^{3} \frac{\partial x}{\partial u_i} \cdot du_i$$

(b) $$ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}\, du_i\, du_j$$

$\boxed{GCT}$

$\boxed{\text{Note}}$:-

Given a point P in rectangular coordinates (x, y, z) we can, from associate a unique set of coordinates (u_1, u_2, u_3) called the curvilinear coordinates of P. [Schaum's outlines].

(u_1, u_2, u_3); (x, y, z) are assumed to be have continous derivatives and single-valued so that the correspondence between

(x, y, z) and (u_1, u_2, u_3) are ~~[illegible]~~ unique.
[Schaum's outline].

$\hat{a}_1, \hat{a}_2, \hat{a}_3$ and $\hat{A}_1, \hat{A}_2, \hat{A}_3$ both sets are analogous to the $\hat{i}, \hat{j}, \hat{k}$ unit vectors in rectangular coordinates but are unlike them in that they may change directions from point to point.

Remember:
$\left(\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}\right)$ and $(\vec{\nabla} u_1, \vec{\nabla} u_2, \vec{\nabla} u_3)$ constitute reciprocal system of vectors.

In general, $\vec{A}$ can be written in terms of $\hat{A}_i$ (unit normal vector) or $\hat{a}_i$ (unit tangent vector) as follows:-

(some constant)

(i) $\vec{A} = \sum_{i=1}^{3} A_i \hat{a}_i = \sum_{i=1}^{3} \alpha_i \frac{\partial \vec{r}}{\partial u_i}$ where $\boxed{A_i = \alpha_i h_i}$

where α_i are called "contra-varient components." or components of contravarient vector.

(some constant)

(ii) $\vec{A} = \sum_{i=1}^{3} B_i \hat{A}_i = \sum_{i=1}^{3} \beta_i \vec{\nabla} u_i$ where $\boxed{B_i = |\vec{\nabla} u_i| \cdot \beta_i}$

where β_i are called "co-varient components." or components of co-varient vectors.

☆ GCT from $(u_1, u_2, u_3) \longrightarrow (\bar{u}_1, \bar{u}_2, \bar{u}_3)$:-

Now let B_i $(i = 1, 2, 3)$ are components of a contravarient vector.

Remember: $B_i(u) = B_i$ as a function $\underbrace{(u_1, u_2, u_3)}$
[combinedly (u)]

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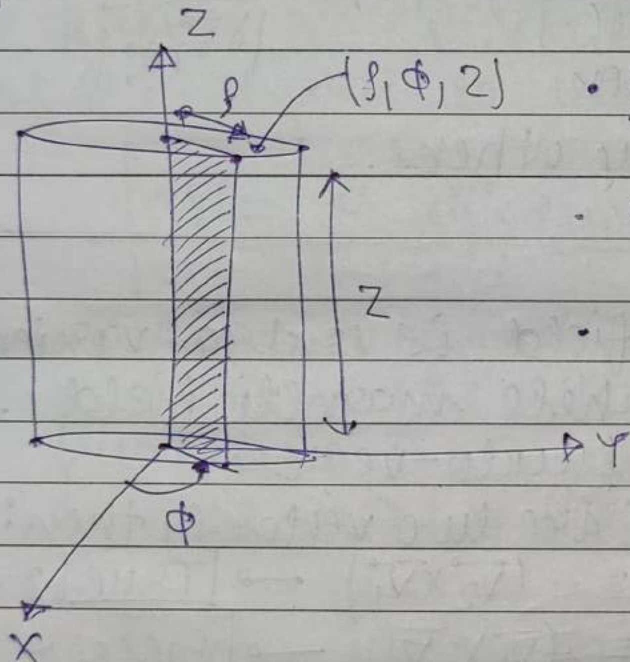
$\Rightarrow \quad B_i(u) \longrightarrow \bar{B}_i(\bar{u}) \Rightarrow \boxed{\bar{B}_i(\bar{u}) = \sum_{j=1}^{3} \frac{\partial \bar{u}_i}{\partial u_j} B_j(u).}$

$$\Rightarrow \begin{pmatrix} \bar{B}_1(\bar{u}) \\ \bar{B}_2(\bar{u}) \\ \bar{B}_3(\bar{u}) \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{u}_i}{\partial u_j} \end{pmatrix} \begin{pmatrix} B_1(u) \\ B_2(u) \\ B_3(u) \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

{column vector independent of 'B'}.

★★ # This transformation is "homogenous", i.e., any component in '$\bar{u}$' sys. will depend on all component, in u'sy of a contravarient vector.

★ Cylindrical coordinate system:-



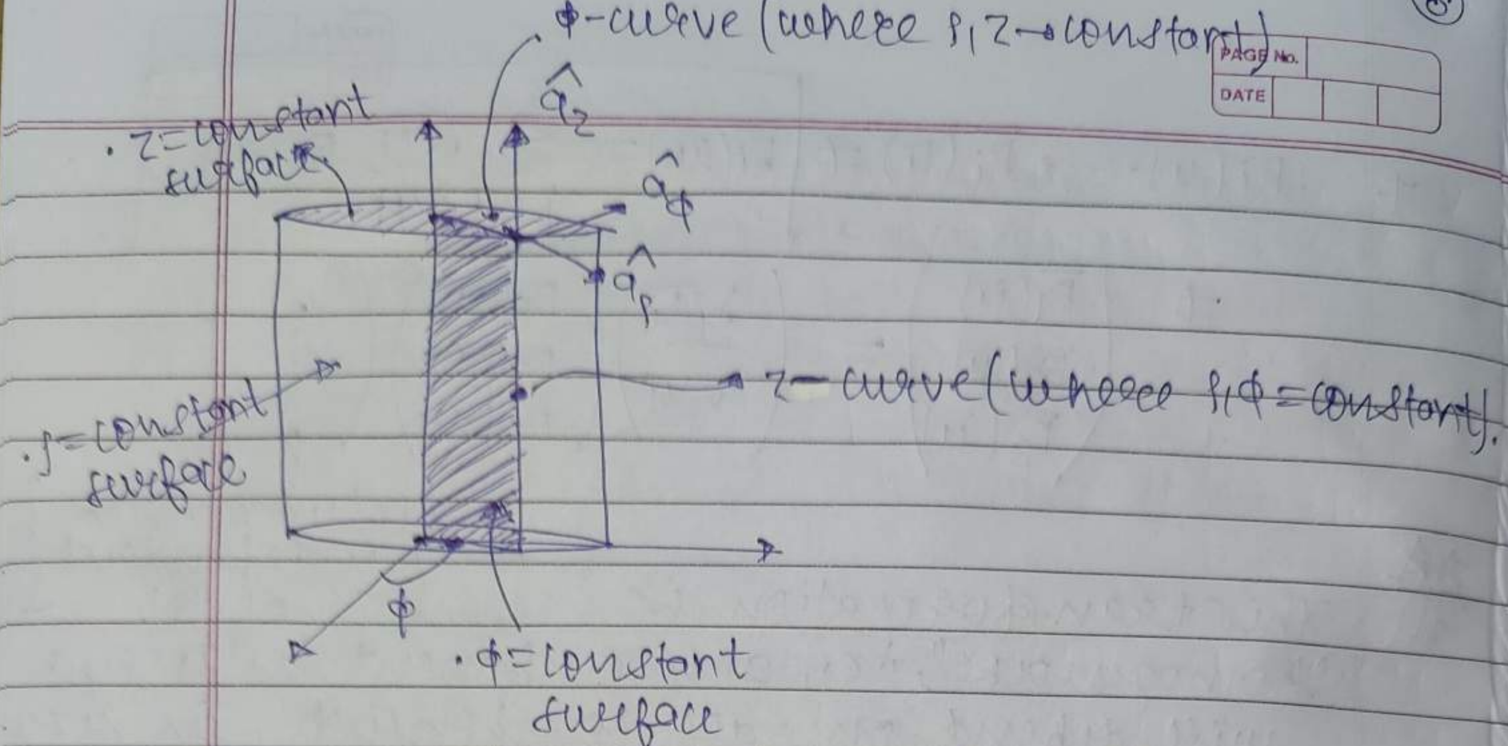
- ρ = planar, radial distance $\in [0, \infty)$
- ϕ = Azimuthal angle $\in [0, 2\pi)$
- $z \in (-\infty, \infty)$.

Here, we can say that, by seeing the ranges of ρ, ϕ, z :-

(a) $z, \rho \rightarrow$ non-compact.

(b) $\phi \rightarrow$ compact.

- $\underbrace{x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z}_{\text{relation b/w cartesian and cylindrical coordinate system.}}$



$\Rightarrow$ Here, $\vec{r} = \rho(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + z\,\hat{k}$.

- Unit tangent vector along radial direction $= \hat{a}_\rho = \dfrac{\partial\vec{r}/\partial\rho}{|\partial\vec{r}/\partial\rho|} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

and similarly others.

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Extra :- (a) Electric field is contra-varient vector while magnetic field is a axial / pseudo-vector.

(b) If $\vec{v_1}$ and $\vec{v_2}$ are two vectors, then:

(i) $\vec{v_1} \times \vec{v_2} = -(\vec{v_2} \times \vec{v_1}) \to \boxed{\text{True}}$

(ii) $\vec{\nabla} \times \vec{v_1} = +(\vec{v_1} \times \vec{\nabla}) \to \boxed{\text{False}}$?

actually :-

$$\boxed{\vec{\nabla} \times \vec{v_1} \neq -(\vec{v_1} \times \vec{\nabla}) \neq (\vec{v_1} \times \vec{\nabla})}$$ Ans

(c) Electric field is a gradient while magnetic field is a ~~[illegible]~~ curl of some vector :-

(scalar = invariant)

$\boxed{\vec{E} = -\vec{\nabla}\phi}$ = gradient of ϕ.

(contravarient vector)

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and $\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$ (Vector potential)
(Axial / pseudo vector).

(d) $\vec{E} = -\vec{\nabla}\phi$ $\qquad \vec{B} = \vec{\nabla} \times \vec{A}$
On changing dimensions, ϕ = scalar / invariant will not change, but $\vec{\nabla}$ and $\vec{A}$ will change on changing dimensions:-
so on changing dimensions,
$\vec{E}$ = will change
$\vec{B}$ = will not change / remain same.

We know $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is a function of (u_1, u_2, u_3) so we write:-

$$\boxed{d\vec{r} = \sum_{i=1}^{3} \frac{\partial \vec{r}}{\partial u_i} \cdot du_i = \sum_{i=1}(h_i \hat{a}_i) \cdot du_i}$$

where $du_i \equiv \vec{r}(u_1 + du_1, u_2 + du_2, u_3 + du_3) - \vec{r}(u_1, u_2, u_3)$.
{read it as $\vec{r}$ as a function of $u_i + du_i$ $(i = 1, 2, 3)$.}

For distance, $\boxed{ds^2 = d\vec{r} \cdot d\vec{r}}$ ——— (1)
$\Rightarrow ds^2 = |d\vec{r}|^2 = \sum_{i=1}^{3} (h_i du_i)^2$.
$\Rightarrow \boxed{ds = \sqrt{\sum_{i=1}^{3} (h_i du_i)^2}}$ Ans

☆ Calculating volume (dV) of small orthogonal curvilinear ~~[illegible]~~ cuboidal system:-

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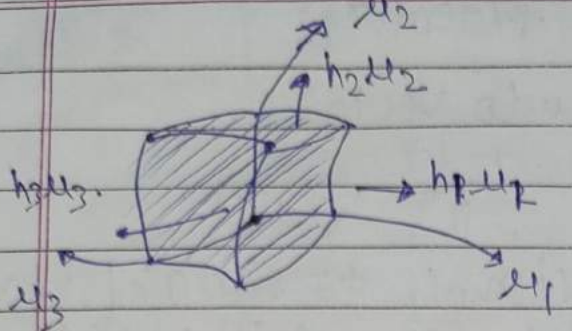


$\Rightarrow$ The volume element $= dV$

$= dV = [h_1 du_1 \hat{a}_1 \; h_2 du_2 \hat{a}_2 \; h_3 du_3 \hat{a}_3]$.

$\Rightarrow \boxed{dV = h_1 h_2 h_3 \, du_1 du_2 du_3 \, [\hat{a}_1 \; \hat{a}_2 \; \hat{a}_3]}$ 1.

☆ Cylindrical: continue:-

$\Rightarrow$ We know: $x = \rho\cos\phi$. $\Rightarrow dx = (d\rho)\cdot\cos\phi - \rho\sin\phi \, d\phi$.

$y = \rho\sin\phi \Rightarrow dy = (d\rho)\sin\phi + \rho\cos\phi \, d\phi$

$z = z \quad \Rightarrow dz = dz$.

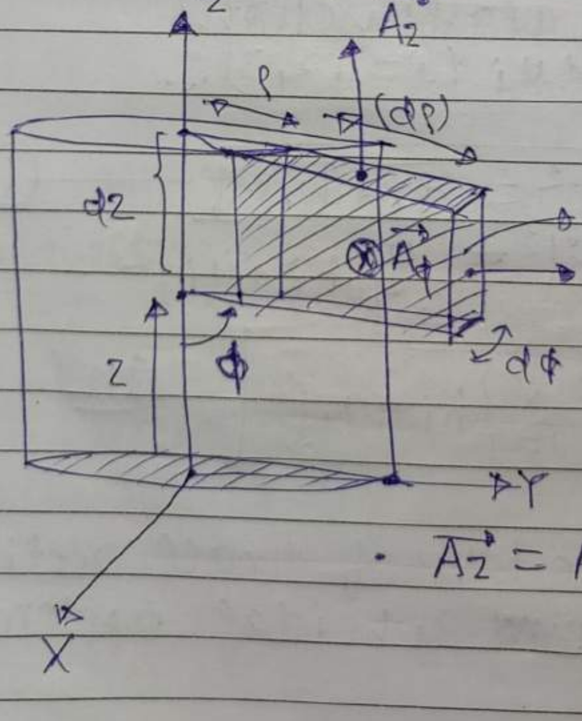
$$\Rightarrow \underbrace{h_\rho = h_z = 1 = \left|\frac{\partial \vec{r}}{\partial z}\right| = \left|\frac{\partial \vec{r}}{\partial \rho}\right| \; ; \; h_\phi = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = \rho.}_{\text{scale factor in cylindrical system.}}$$

$\Rightarrow$ Here, we can also assume that:-

$\boxed{(u_1, u_2, u_3) \longleftrightarrow (\rho, \phi, z)}$

- $ds^2 = dx^2 + dy^2 + dz^2 = dz^2 + d\rho^2 + (\rho\, d\phi)^2$

[as ϕ is dimension-less and $h_\phi = \rho$.]



Volume $= (\rho\, d\phi)(dz)(d\rho)$

Ans

- $\vec{A_z}$ = Area of that plane surface of which $\vec{A_z}$ is the normal.

and similarly $\vec{A_\phi}$ and $\vec{A_\rho}$.

(a) $\boxed{\vec{A_z} = (\rho \cdot d\phi)(d\rho) \cdot \hat{a}_z}$ —— (1.)

(b) $\boxed{\vec{A_\phi} = (dz)(d\rho) \cdot \hat{a}_\phi}$ —— (2.)

(c) $\boxed{\vec{A_\rho} = (dz)(\rho \cdot d\phi) \cdot \hat{a}_\rho}$ —— (3.)

Example:- If $(u_1, u_2, u_3) \equiv (x, y, z)$ and $(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\rho, \phi, z)$ then given (V_x, V_y, V_z), what are (V_ρ, V_ϕ, V_z) ?

Solution:-

We know: $\boxed{\bar{V}_i(\bar{u}) = \sum_{j=1}^{3} \frac{\partial \bar{u}_i}{\partial u_j} V_j(u)}$

and $\rho = \sqrt{x^2+y^2}$; $\phi = \tan^{-1}(y/x)$.

(i) $V_\rho = \left(\frac{\partial \rho}{\partial x} \cdot V_x + \frac{\partial \rho}{\partial y} \cdot V_y + \frac{\partial \rho}{\partial z} \cdot V_z\right) = \cos\phi V_x + \sin\phi V_y$ Ans

(ii) $V_\phi = \frac{\partial \phi}{\partial x} V_x + \frac{\partial \phi}{\partial y} V_y + \frac{\partial \phi}{\partial z} V_z = \frac{-\sin\phi V_x + \cos\phi V_y}{\rho}$ Ans

(iii) $V_z = V_z$. Ans

let $\boxed{V_\rho' = V_\rho}$; $\boxed{V_\phi' = \rho \cdot V_\phi}$; $\boxed{V_z' = V_z}$.

∴ So, we can write:-

$$\begin{pmatrix} V_\rho' \\ V_\phi' \\ V_z' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}.$$

Ans

($\sim$: tilde, $\wedge$: Hat, $-$: bar)

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☆ Jacobian :-

⇒ when transformation is: $(x_1, x_2, x_3) \to (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$

so for infinitesimal :-

$$(dx_1, dx_2, dx_3) \longrightarrow (d\tilde{x}_1, d\tilde{x}_2, d\tilde{x}_3)$$

but actually some factor, called the "Jacobian factor" will relate them too.

- It means :-

$$dx_1 \cdot dx_2 \cdot dx_3 \neq d\tilde{x}_1 \cdot d\tilde{x}_2 \cdot d\tilde{x}_3$$

- Actually :-

$$\boxed{dx_1 \cdot dx_2 \cdot dx_3 = J\left(\frac{x_1, x_2, x_3}{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3}\right) \cdot d\tilde{x}_1 \cdot d\tilde{x}_2 \cdot d\tilde{x}_3}$$

where, Jacobian $= J\left(\frac{x_1, x_2, x_3}{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3}\right)$ = factor multiplied when to another system.

- Value of Jacobian :-

$$\text{Jacobian} = \det \begin{pmatrix} \frac{\partial x_1}{\partial \tilde{x}_1} & \frac{\partial x_1}{\partial \tilde{x}_2} & \frac{\partial x_1}{\partial \tilde{x}_3} \\ \vdots & \vdots & \vdots \\ & & \end{pmatrix}_{3\times 3} = J\left(\frac{x_1, x_2, x_3}{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3}\right)$$

Ans

For orthogonal system (curvilinear) :-

$$\boxed{J\left(\frac{x_i}{\tilde{x}_j}\right) = \frac{1}{J\left(\frac{\tilde{x}_i}{x_j}\right)}} \Leftrightarrow \boxed{J\left(\frac{x_1, x_2, x_3}{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3}\right) = \frac{1}{J\left(\frac{\tilde{x}_1, \tilde{x}_2, \dots}{x_1, x_2, \dots}\right)}}$$

← proved by determinant of matrix given above.

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Note :- In curvilinear system,

$$\boxed{\vec{\nabla} = \sum_{i=1}^{3} \frac{\hat{a}_i}{h_i}\frac{\partial}{\partial u_i}}$$

\# ρ = is the jacobian in cylindrical system from cartesian.

(a) volume element in cartesian = $(dx)(dy)(dz)$.

(b) volume element in cylindrical = $\underbrace{(\rho)}_{\boxed{\text{Jacobian}}}(d\rho)(dz)(d\phi)$

Theorems :-

① $\left\{\frac{\partial \vec{r}}{\partial u_i}\right\}$ and $\{\vec{\nabla} u_i\}$ form a reciprocal system of vectors.

Note :- $\vec{a}, \vec{b}, \vec{c}$ and $\frac{\vec{a}\times\vec{b}}{[abc]}, \frac{\vec{b}\times\vec{c}}{[abc]}, \frac{\vec{c}\times\vec{a}}{[abc]}$ forms reciprocal system of vectors.

Proof :-

$$\Rightarrow \boxed{d\vec{r} = \sum_{i=1}^{3} \frac{\partial \vec{r}}{\partial u_i} \cdot du_i} \quad \text{(already proved).} \quad —— ①$$

We know that :-

$$\vec{\nabla} u_j \cdot d\vec{r} = du_j \quad \text{[From page no. ③]}.$$

$\Rightarrow$ From eqn. ① :-

$$du_j = \vec{\nabla} u_j \cdot \sum_{i=1}^{3} \frac{\partial \vec{r}}{\partial u_i} \cdot du_i$$

$$\Rightarrow du_j = \sum_{i=1}^{3} (\vec{\nabla} u_j) \cdot \frac{\partial \vec{r}}{\partial u_i}\, du_i .$$

Compare coefficients : $\boxed{\vec{\nabla} u_j \cdot \frac{\partial \vec{r}}{\partial u_i} = \delta_{ij}}$

$i = j \Rightarrow \delta_{ij} = 1$

$i \neq j \Rightarrow \delta_{ij} = 0$

only a symbol to represent dot pdt b/w two.

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(2.) (a) $\boxed{|\vec{\nabla} u_i| = \frac{1}{h_i}}$

- We can also prove that $\boxed{\delta_{ij} = \frac{\partial u_i}{\partial u_j}}$ $i = j \Rightarrow (1)$, $i \neq j \Rightarrow (0)$

remember (u_1, u_2, u_3) → does not depend on eachother.

(b) $\boxed{\hat{A}_i = \hat{a}_i}$ Ans

Proof :-

From 2(a).

$$\Rightarrow \hat{A}_i = \frac{\vec{\nabla} u_i}{|\vec{\nabla} u_i|} = h_i \cdot \vec{\nabla} u_i = h_i \cdot |\vec{\nabla} u_i| \cdot \hat{A}_i$$

☆ lemma's / Identities :-

(1.) $\vec{\nabla} \cdot (\phi \vec{A})$ = divergence of $(\phi \vec{A})$ where ϕ = scalar or invariant

$$= \underbrace{\left(\sum_{i=1}^{3} \hat{e}_i \cdot \frac{\partial}{\partial x_i} \right)}_{\vec{\nabla}} \cdot (\phi \vec{A})$$

where: $\begin{pmatrix} \hat{e}_1 = \hat{i} & \hat{e}_2 = \hat{j} \\ x_1 = x & \hat{e}_3 = \hat{k} \\ x_2 = y & \\ x_3 = z & \end{pmatrix}$

$$= \sum_{i=1}^{3} \hat{e}_i \left(\frac{\partial}{\partial x_i} (\phi \vec{A}) \right)$$

$$= \underbrace{\left(\sum_{i=1}^{3} \hat{e}_i \frac{\partial \phi}{\partial x_i} \right)}_{} \cdot \vec{A} + \phi \underbrace{\left(\sum_{i=1}^{3} \hat{e}_i \frac{\partial \vec{A}}{\partial x_i} \right)}_{\vec{\nabla} \cdot \vec{A}}$$

equal to $\left(\sum_{i=1}^{3} \hat{e}_i \frac{\partial}{\partial x_i} \right) \cdot \phi$ as ϕ = scalar.

$$= \vec{\nabla} \phi \cdot \vec{A} + \phi \, \vec{\nabla} \cdot \vec{A}$$

So: $\boxed{\vec{\nabla} \cdot (\phi \vec{A}) = (\vec{\nabla} \phi) \cdot \vec{A} + \phi (\vec{\nabla} \cdot \vec{A})}$ Ans

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$\therefore \hat{a}_1 \times \hat{a}_2 = \hat{a}_3$; $\hat{a}_3 \times \hat{a}_1 = \hat{a}_2$; $\hat{a}_2 \times \hat{a}_3 = \hat{a}_1$.

(2.) $$\boxed{\vec{\nabla} \times (\phi \vec{A}) = (\vec{\nabla}\phi) \times \vec{A} + \phi \underbrace{(\vec{\nabla} \times \vec{A})}_{\text{curl of } \vec{A}}}$$

(3.) $$\boxed{\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - (\vec{\nabla} \times \vec{B}) \cdot \vec{A}}$$

☆ Theorems : continue :- (From here : $\hat{a}_1, \hat{a}_2, \hat{a}_3$ are unit normal vector).

(3.) $$\boxed{\hat{a}_1 = h_2 h_3 \left((\vec{\nabla} u_2) \times (\vec{\nabla} u_3) \right)}$$

and similarly for $\hat{a}_2$ and $\hat{a}_3$. • $\hat{a}_1, \hat{a}_2, \hat{a}_3$ form $\underbrace{\text{right handed}}$ system

Proof :-

$$\therefore \left((\vec{\nabla} u_2) \times (\vec{\nabla} u_3) \right) = \left(\frac{\hat{a}_2}{h_2} \times \frac{\hat{a}_3}{h_3} \right) = \frac{(\hat{a}_2 \times \hat{a}_3)}{h_2 h_3} = \frac{\hat{a}_1}{h_2 h_3}.$$

(4.) (a) $$\boxed{\vec{\nabla} \cdot (A_1 \hat{a}_1) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (A_1 h_2 h_3)}$$

and similarly for 2 and 3 :-

Proof :-

$$\therefore \vec{\nabla} \cdot (A_1 \hat{a}_1) = \vec{\nabla} \left(A_1 h_2 h_3 \cdot (\vec{\nabla} u_2 \times \vec{\nabla} u_3) \right)$$

$$= \vec{\nabla} \cdot \left(\underbrace{(A_1 h_2 h_3)}_{\substack{\text{scalar or} \\ \text{invariant}}} \cdot \underbrace{(\vec{\nabla} u_2 \times \vec{\nabla} u_3)}_{\text{vector.}} \right)$$

$\therefore$ So here we can use : $\vec{\nabla} \cdot (\phi \vec{A})$. identity.

$$\therefore \vec{\nabla} \cdot (A_1 \hat{a}_1) = (A_1 h_2 h_3) \cdot \underbrace{\left(\vec{\nabla} \cdot (\vec{\nabla} u_2 \times \vec{\nabla} u_3) \right)}_{0}$$

$$+ \vec{\nabla} (A_1 h_2 h_3) \cdot \underbrace{(\vec{\nabla} u_2 \times \vec{\nabla} u_3)}_{\left(\frac{\hat{a}_1}{h_2 h_3} \right)}.$$

already proven

$\#$ $\boxed{\vec{\nabla} \cdot (A_1 \hat{a}_1) = \vec{\nabla} \cdot (A_1 \hat{a}_1)}$ Hence, ~~proved~~. proven

If $\vec{A} = \text{grad}\,\phi = \vec{\nabla}\phi$, then

$$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot(\vec{\nabla}\phi) = \underset{\text{(Laplacian)}}{\vec{\nabla}^2}\phi = \text{Laplacian acting on scalar field (on scalars)}$$

Result

We know: $\vec{A} = A_1\hat{a}_1 + A_2\hat{a}_2 + A_3\hat{a}_3$.

So: $$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_1h_3) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]}$$

by using 4(a).

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4. (b) $$\boxed{\vec{\nabla}\times(A_1\hat{a}_1) = \frac{\hat{a}_2}{h_3h_1}\frac{\partial}{\partial u_3}(A_1h_1) - \frac{\hat{a}_3}{h_1h_2}\frac{\partial}{\partial u_2}(A_1h_1)}$$

Proof:- $$\vec{\nabla}\times(A_1\hat{a}_1) = \vec{\nabla}\times(A_1(h_1\vec{\nabla}u_1)) = \vec{\nabla}\times(A_1h_1\vec{\nabla}u_1)$$
$$= (\vec{\nabla}(A_1h_1))\times\vec{\nabla}u_1 + (A_1h_1)(\underbrace{\vec{\nabla}\times(\vec{\nabla}u_1)}_{0})$$
$$= (\vec{\nabla}(A_1h_1))\times\frac{\hat{a}_1}{h_1}$$
$$= \left(\frac{\hat{a}_2}{h_2}\frac{\partial}{\partial u_2}(A_1h_1) + \frac{\hat{a}_3}{h_3}\frac{\partial}{\partial u_3}(A_1h_1)\right)\times\frac{\hat{a}_1}{h_1}$$

Ans

Result:-

We know: $\vec{A} = A_1\hat{a}_1 + A_2\hat{a}_2 + A_3\hat{a}_3$.

So: $$\vec{\nabla}\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{a}_1 & h_2\hat{a}_2 & h_3\hat{a}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$= \frac{1}{h_1h_2h_3}\left[h_1\hat{a}_1\left(\frac{\partial}{\partial u_2}(h_3A_3) - \frac{\partial}{\partial u_3}(h_2A_2)\right) + \ldots\right]$$

Ans

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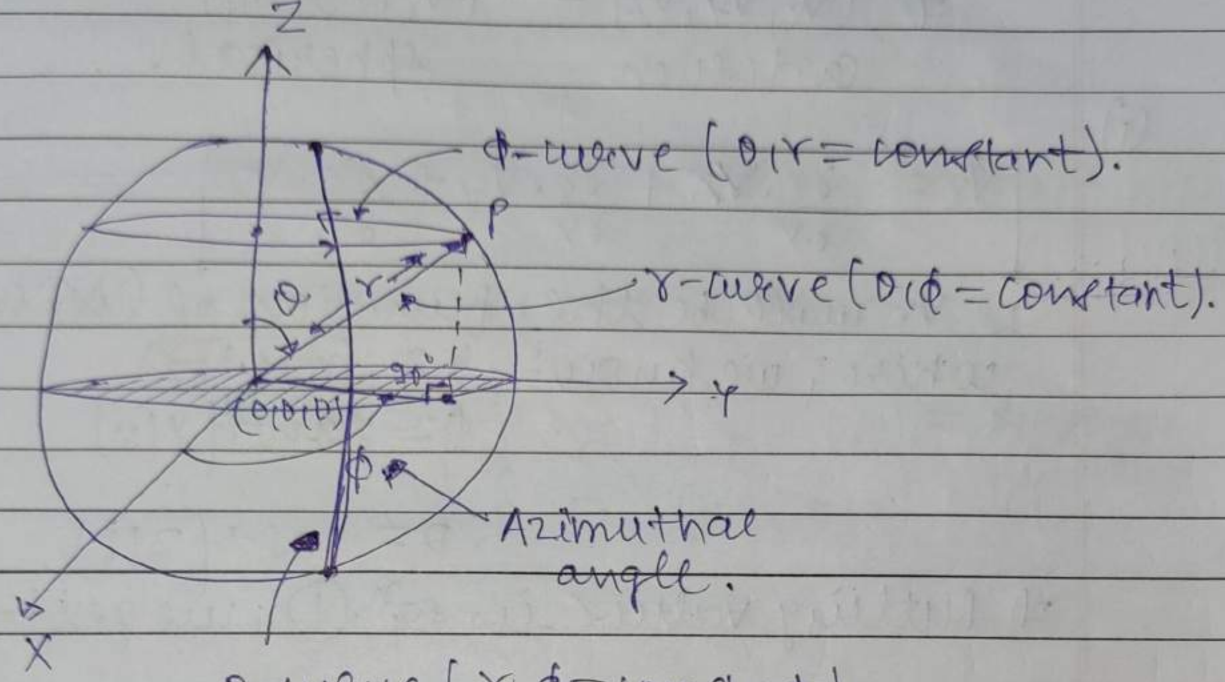
☆ Spherical (Polar) coordinates:- (r, θ, ϕ)



⇒ $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta.$

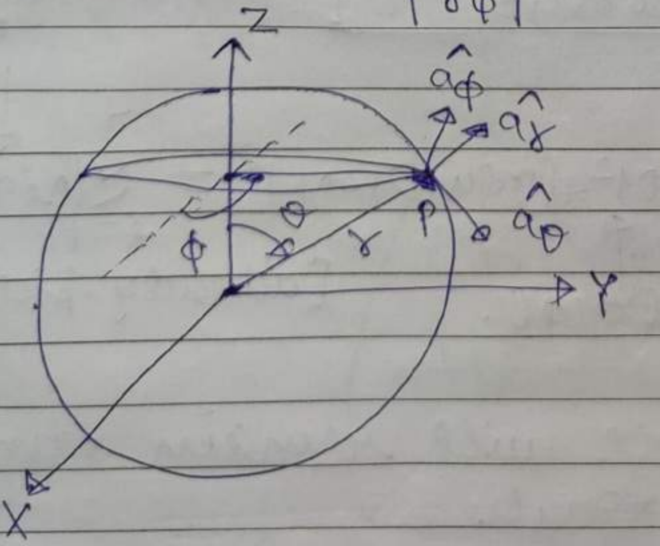
⇒ $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(a) $h_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = 1$

(b) $h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = r$

(c) $h_\phi = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = r\sin\theta$

scale factors along $\hat{a}_r, \hat{a}_\theta$ and $\hat{a}_\phi$ respectively.



(a) $\hat{a}_r = \dfrac{\frac{\partial \vec{r}}{\partial r}}{h_r} = \ldots\ldots$

(b) $\hat{a}_\theta = \dfrac{\partial \vec{r}}{\partial \theta} \times \dfrac{1}{h_\theta} = \ldots\ldots$

(c) $\hat{a}_\phi = \dfrac{\partial \vec{r}}{\partial \phi} \times \dfrac{1}{h_\phi} = \ldots$

☆ Transformation:-

$\Rightarrow (V_x, V_y, V_z) \longrightarrow (V_r, V_\theta, V_\phi)$
cartesian spherical.

(i)

$$\boxed{V_r = \frac{\partial r}{\partial x}\cdot V_x + \frac{\partial r}{\partial y}\cdot V_y + \frac{\partial r}{\partial z}\cdot V_z} \longrightarrow (1.)$$

as V_r will be the function of (V_x, V_y, V_z).
where, we know: $r = \sqrt{x^2+y^2+z^2}$

$\phi = \tan^{-1}(y/x)$

$\theta = \cos^{-1}(z/r)$.

$\Rightarrow$ Putting values in eq^n. (1), we get:-

$$\boxed{V_r = (\sin\theta\cos\phi)V_x + (\sin\theta\sin\phi)V_y + (\cos\theta)V_z.} \longrightarrow (2)$$

(ii)

$$\boxed{V_\theta = \frac{\cos\theta\cdot\cos\phi V_x + \cos\theta\sin\phi V_y - \sin\theta V_z}{r}} \longrightarrow (3.)$$

(iii)

$$\boxed{V_\phi = \frac{-\sin\phi V_x + \cos\phi V_y}{r\sin\theta}} \longrightarrow (4.)$$

(in curvilinear)

(iv) $ds^2 = dr^2 + (r\,d\theta)^2 + ((r\sin\theta)\cdot d\phi)^2 = \sum_{i=1}^{3}(h_i\,dq_i)^2$

spherical. [already proved].

obviously distance will remain same so they are equal.

matrix

$\boxed{dr^2 = (dr)^2}$

$ds^2 = dr^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}\,dx_i\,dx_j$

$= \sum_{i=1}^{3} (h_i\,dx_i)^2.$

- When $i=j=r \Rightarrow g_{rr} = 1$ (compare $(dr)^2$ coefficient)
- when $i=j=\theta \Rightarrow g_{\theta\theta} = r^2$
- when $i=j=\phi \Rightarrow g_{\phi\phi} = r^2\sin^2\theta$

$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (r\sin\theta)^2 \end{pmatrix}$ = diagonal matrix in spherical system.

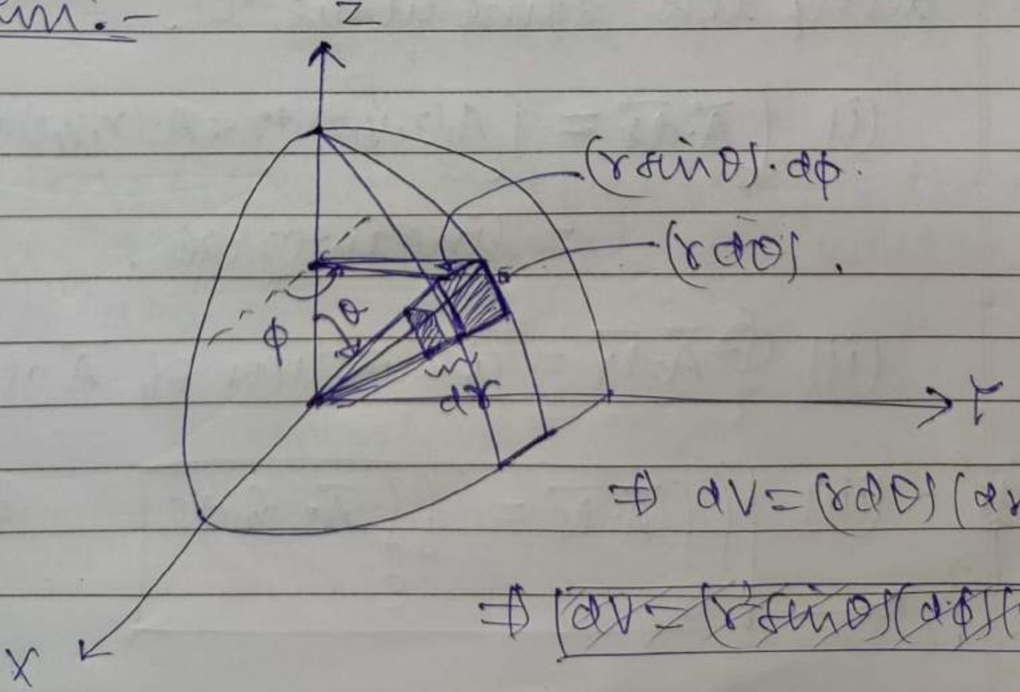
$\boxed{\text{Note}}$:- Let $V_r' = V_r = \sin\theta\cos\phi\,V_x + \sin\theta\sin\phi\,V_y + \cos\theta\,V_z$

$V_\theta' = rV_\theta = \cos\theta\cos\phi\,V_x + \cos\theta\sin\phi\,V_y - \sin\theta\,V_z$

$V_\phi' = r\sin\theta\,V_\phi = -\sin\phi\,V_x + \cos\phi\,V_y.$

actually, transformation done above is of $(V_r', V_\theta', V_\phi')$ not of (V_r, V_θ, V_ϕ). Ans

☆ Volume element (dV) in spherical polar system:-



$\Rightarrow dV = (r\,d\theta)(dr)(r\sin\theta\,d\phi)$

$\Rightarrow \boxed{dV = (r^2\sin\theta)(d\phi)(d\theta)(dr)}$

so, here Jacobian $= h_r \cdot h_\theta \cdot h_\phi = 1 \times r \times r\sin\theta$
$= r^2 \sin\theta$.
[more, less due to symmetry in spherical geometry].
Ans

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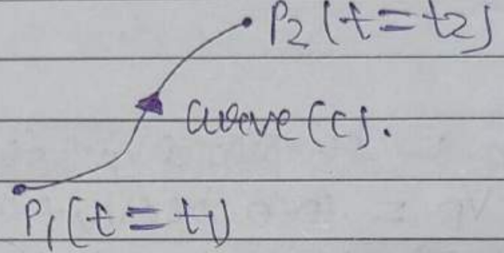
Vector Integral Calculus

- Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$
  where: $x(t)$ means 'x' as a function of time (t).



- Vector field exists:
  $\vec{A}(x,y,z) = A_1(x,y,z)\hat{i} + A_2(x,y,z)\hat{j} + A_3(x,y,z)\hat{k}$.

where $A_i (i = 1,2,3)$ = continous function of (x,y,z).

along the same curve 'C'.

(i) $\int_C \vec{A} \cdot d\vec{r} = \int_C A_1(x,y,z)\,dx + A_2(x,y,z)\,dy + A_3(x,y,z)\,dz$.
= line integral.

(ii) $\oint_C \vec{A} \cdot d\vec{r}$ = circulation of $\vec{A}$ along 'C'.

(iii) $\boxed{\iint_\varepsilon \vec{A} \cdot d\vec{s} = \iint_\varepsilon \vec{A} \cdot \hat{n}\, ds}$

$d\vec{s} = ds \cdot \hat{n}$
= Area vector of surface 'ε'.