Indian Institute of Technology Roorkee
End-term Examination, Autumn Semester, 2022-2023
Mathematics I (MAN – 001)

Time: **3 Hours** Marks: **100**

All questions are compulsory. Marks are indicated against each question.

Q.1) (a) Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ real orthogonal matrices such that $\det(\mathbf{A}) + \det(\mathbf{B}) = 0$. Then, compute $\det(\mathbf{A} + \mathbf{B})$. [6]

(b) Find the values of the constants α and β such that the system of equations

$$2x + y + 3z = 4, \qquad x + (\alpha + 1)y + 2z = 1, \qquad (\alpha - 1)x + 2y + 3z = \beta + 1$$

has (i) a unique solution, (ii) infinite number of solutions and (iii) no solution. [6]

Q.2) (a) Show that a real matrix $\mathbf{A}_{4\times4}$ is diagonalizable if and only if it has 4 linearly independent eigenvectors. [6]

(b) Verify the Cayley-Hamilton theorem for $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 4 \\ 5 & -2 & 2 \end{pmatrix}$ and hence find $\mathbf{A}^{-1}$. [6]

Q.3) (a) If $f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases}$ then discuss the continuity of $\dfrac{\partial^2 f}{\partial x \partial y}$ and the differentiability of $\dfrac{\partial f}{\partial x}$ at $(0, 0)$. [7]

(b) Let $f(x, y)$ be a function having continuous second order partial derivatives. If $x = \alpha \cosh u \cos v$ and $y = \alpha \sinh u \sin v$, where $\cosh u = \dfrac{e^u + e^{-u}}{2}$, $\sinh u = \dfrac{e^u - e^{-u}}{2}$ and α is a real constant, then show that

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = \frac{\alpha^2}{2}(\cosh(2u) - \cos(2v))\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right). \quad [7]$$

Q.4) (a) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be an invertible function and $g : D \subset \mathbb{R}^2 \mapsto \mathbb{R}$ be a homogeneous function of degree m. If $u(x, y) = f^{-1}(g(x, y))$, then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(m - 1 - m\frac{f''(u)f(u)}{f'(u)^2}\right)m\frac{f(u)}{f'(u)},$$

assuming all derivatives in the above expression exist. Hence, for $u(x, y) = \tan^{-1}\left(\dfrac{x^3 - y^3}{x^6 + y^6}\right)$, find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$. [7]

(b) Find the quadratic approximation of $f(x, y) = e^{xy}$ by using Taylor's theorem about the point $(0, 0)$ in the region $|x| \leq 0.1$ and $|y| \leq 0.1$. Also calculate, upto 5 decimal places, the maximum absolute error in the approximation. [7]

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Q.5) (a) Let D be the region cut out of the solid $S = \{(x, y, z) \in \mathbb{R}^3 : 2x^2 + y^2 + z^2 \leq 4\}$ by the elliptic cylinder $E = \{(x, y, z) \in \mathbb{R}^3 : 2x^2 + y^2 = 1\}$. Find the volume of the solid region D. [6]

(b) Let R be a region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1, xy = 4$ and the lines $y = x, y = 9x$. Find the value of

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy.$$ [6]

Q.6) (a) If m, n, α are positive integers, then find the value of $\int_0^1 \int_0^1 (y \log x)^n (x - xy^\alpha)^m dydx$ in terms of the Beta and Gamma functions. [6]

(b) Find the mass of a plate in the shape of the curve $\left(\frac{x}{2}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$, the density being given by $\rho = \mu xy$. [6]

Q.7) (a) Show that the line integral $\int_C (2x + y + z)dx + (2y + x + z^2)dy + (2z + 2yz + x)dz$ is independent of the path C. Find the value of the integral along any C joining the points $(1, 1, 0)$ and $(3, 2, 5)$. [6]

(b) Let $\frac{\partial u}{\partial \vec{v}}$ denote the directional derivative of $u(x, y)$ in the direction of the vector $\vec{v}$. Assume that $f(x, y)$ and $g(x, y)$ have continuous second order partial derivatives in a region R bounded by a piecewise smooth simple closed curve C (positively oriented) in the xy-plane. Using the Green's theorem, show that

$$\oint_C \left(f\frac{\partial g}{\partial \vec{n}} - g\frac{\partial f}{\partial \vec{n}}\right) ds = \iint_R (f\nabla^2 g - g\nabla^2 f) dxdy,$$

where $\vec{n}$ is the unit outward normal to the curve C and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. [6]

Q.8) (a) Let S be the surface of the cylinder $x^2 + y^2 = 4$ between the planes $z = 0$ and $y + z = 5$. Find the surface integrals of the vector field $\vec{F} = xz\,\vec{i} + yz\,\vec{j} + xy\,\vec{k}$ over the surface S. [6]

(b) Let S_1 be the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ lying above the xy-plane and S_2 be the plane region $x^2 + y^2 \leq 4$. For a vector field $\vec{F}$ having continuous second order derivatives, show that

$$\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS + \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = 0.$$

If $\vec{F} = (2x - y)\,\vec{i} - 2yz^2\,\vec{j} - y^2 z\,\vec{k}$, then find the value of $\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$. [6]