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(1) (a) For eigen values, $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 10\lambda^2 + 28\lambda - 24 = 0$$

$$\Downarrow$$

$$\boxed{\lambda = 2, 2, 6}$$

(i) For $\lambda = 2$: $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ [$AX = \lambda X$]

we get $\Rightarrow$ $x_1 + x_2 + x_3 = 1$.

For linearly independence, two eigen vectors: $(-1, 0, 1)^T$, $(-1, 1, 0)^T$ Ans

(ii) For $\lambda = 6$: $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 6 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

we get:

$$\left.\begin{aligned} x_2 + x_3 &= 3x_1 \\ x_1 + x_3 &= x_2 \\ x_1 + x_2 &= 3x_3 \end{aligned}\right\} \Rightarrow \begin{pmatrix} x_1 = x_3 \\ x_2 = 2x_3 \end{pmatrix}$$

So one choice will be: $(1, 2, 1)^T$ Ans

(1) (b) $|A - \lambda I| = 0 = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} \Rightarrow (\lambda - 1)^3 = 0$

$$\Rightarrow \boxed{\lambda = 1, 1, 1}$$

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$$\Rightarrow Ax = x \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\Rightarrow$ we get : only $\boxed{x_2 = 0}$

so eigen vectors : $(1,0,0)^T, (1,1,0)^T, (0,0,1)^T$ Ans

1\. (c) $|A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 12\lambda - 16 = 0$

$\Downarrow$

$\boxed{\lambda = -2, -2, 4}$

(i) For $\lambda = -2$:-

$$Ax = -2x \Rightarrow \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

we get : only $x_1 + x_3 = x_2$,

so one choice of two eigen vectors are : $(0,1,1)^T, (1,0,-1)^T$. Ans

(ii) For $\lambda = 4$:-

$$Ax = 4x \Rightarrow \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

we get : $\boxed{x_3 = 2x_1 = 2x_2}$

so eigen vector = $(1,1,2)^T$ Ans

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1.(a) $|A-\lambda I| = 0 = \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0.$

$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \Rightarrow \boxed{\lambda = 1, 1, 3}$

(i) For $\lambda = 1$:- $AX = X$

$\Rightarrow \begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

we get: $\boxed{3x_1 + x_2 = x_3}$

so: eigen vectors: $(1, 0, 3)^T; (1, -3, 0)^T$

Ans

(ii) For $\lambda = 3$:- $AX = 3X$

$\Rightarrow \begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

we get: $x_1 = -x_2 = x_3 \Rightarrow$ eigen vector: $(1, 1, -1)^T$

Ans

2.(a) given: $AX = \lambda X$; and $|A| \neq 0$, it means A^{-1} exists.

(i) $\Rightarrow AX = \lambda X \Rightarrow X = \lambda(A^{-1}X)$ [multiply by A^{-1}].

$\Rightarrow A^{-1}X = \frac{1}{\lambda} X$

$\Rightarrow A^{-1}X = \lambda^{-1}X$

so λ^{-1} is eigen value of A^{-1}. Proved.

and corresponding eigen vector will remain same equal to x.

(ii)

$(A-kI)X_1 = \lambda_1 X_1$; λ_1 = eigen value of $(A-kI)$

$\Rightarrow AX_1 = \lambda_1 X_1 + kX_1$ and X_1 = eigen vector of $(A-kI)$.

$\Rightarrow \boxed{AX_1 = (\lambda_1 + k)X_1}$

so compare with $Ax = \lambda x$, we get

$\boxed{\lambda_1 = (\lambda - k)}$ and $\boxed{X_1 = X}$

Ans

(2) (b) characteristic equation of A :-

$\Rightarrow |A - \lambda I| = 0$

take transpose : $\Rightarrow \boxed{|A^T - \lambda I| = 0}$

so both have same eigen value, but need not have same eigen vector.

(2) (c) Let $Ax = \lambda x$, and $(P^{-1}AP)X_1 = \lambda_1 X_1$

$\Rightarrow$ characteristic polynomial for $P^{-1}AP$ is:-

$\Rightarrow |P^{-1}AP - \lambda_1 I| = 0$

$\Rightarrow |P^{-1}AP - \lambda_1 P^{-1}P| = 0$

$\Rightarrow |P^{-1}| \, |A - \lambda_1 I| \, |P| = 0 \Rightarrow (|A - \lambda_1 I| = 0)$

$\Rightarrow$ So it tells us that, eigen value of A is also $\lambda_1 \Rightarrow$ ($\lambda_1 = \lambda$) proved

(3.) (a). Let A be the hermitian matrix, then $\boxed{A = A^*}$.

$\Rightarrow$ Let $Ax = \lambda x$, $x \in C$, $\lambda \in C$.

Take transpose, conjugate :-

$$(Ax)^* = (\lambda x)^*$$

$$\Rightarrow x^* A^* = \bar{\lambda} x^* = x^* A \quad [A = A^*].$$

Post multiply by 'x' :-

$$\Rightarrow x^* A x = \bar{\lambda} x^* x$$

$$\Rightarrow x^* \lambda x = \bar{\lambda} x^* x \Rightarrow (\lambda - \bar{\lambda}) x^* x = 0$$

as $x \neq 0 \Rightarrow$ so: $\boxed{\lambda = \bar{\lambda}} \Rightarrow \boxed{\lambda \in R}$

proved

(3.) (b) Following the same as in (a) but here $\boxed{A = -A^*}$ so: (bi).

we get: $(\lambda + \bar{\lambda}) x^* x = 0 \Rightarrow \lambda =$ purely imag. or zero.

as $\boxed{b = 0}$ also

proved

(3.) (c) • $A =$ real sym. matrix.

• $\lambda_1, \lambda_2 \rightarrow$ distinct eigen values : $\lambda_1 \neq \lambda_2$

• $x_1, x_2 \rightarrow$ corresponding eigen vectors.

$$\underbrace{\boxed{Ax_1 = \lambda_1 x_1}}_{(1)} ; \quad \underbrace{\boxed{Ax_2 = \lambda_2 x_2}}_{(2)}$$

$\Rightarrow$ Multiply (1) by x_2^T :-

$$x_2^T A x_1 = \lambda_1 x_2^T x_1 \quad \text{--- (3)}$$

$\Rightarrow$ Multiply (2) by x_1^T :-

$$x_1^T A x_2 = \lambda_2 x_1^T x_2 \quad \text{--- (4.)}$$

Since $A = A^T \Rightarrow x_2^T A x_1 = x_2^T A^T x_1 = (\underbrace{A x_2}_{\lambda_2 x_2})^T x_1$

$\Rightarrow (\lambda_2 x_2)^T x_1 = \lambda_1 x_2^T x_1$

$\Rightarrow (\lambda_2 - \lambda_1) x_2^T x_1 = 0$

But $\lambda_1 \neq \lambda_2 \Rightarrow$ Hence, $x_2^T x_1 = 0$ so they are orthogonal.

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3. (d) A = unitary matrix $\Rightarrow AA^* = I = A^*A$.

$\Rightarrow Ax = \lambda x \Rightarrow |A| \cancel{|x|} = |\lambda|^n \cancel{|x|}$

$\Rightarrow \boxed{|\lambda| = 1}$ $\qquad$ $|A| = 1, -1$

modulus of (λ) $\qquad$ determinant of A.

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3. (e) We know, $A = -A^T$ [Skew symmetric]

$$|A| = (-1)^n |A|$$

as n = odd $\Rightarrow |A| = -|A| \Rightarrow |A| = 0$

proved

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3. (f) Idempotent matrix : $A^2 = I$

$\Rightarrow Ax = \lambda x \Rightarrow A^2 x = \lambda(Ax) = \lambda(\lambda x)$

$\Rightarrow \boxed{x = \lambda^2 x}$

clearly : $|\lambda = 0, 1|$ Ans

(3.) (g) Nilpotent matrix : $A^n = 0$

⇒ Let $n = 3$:- $AX = \lambda X$ Ans

$$A^3X = \lambda(A^2X) = 0 \Rightarrow \boxed{\lambda = 0}$$

as $\begin{pmatrix} A \neq 0 \\ X \neq 0 \end{pmatrix}$.

Proved

(4.) (i) $(AB)X = \lambda X$; λ = eigen value of AB

$\lambda = 0$

⇒ $\det(AB) = 0$
as $X \neq 0$
⇒ so: $\det(BA) = 0$
⇓
one of eigen value will be 0. Ans

$\lambda \neq 0$

Multiply by B:
$BABX = B\lambda X$
⇒ $(BA)(BX) = \lambda(BX)$
(BA) will have eigen value = λ and eigen vector = BX.

Proved

(ii) If given that: $|I - BA| \neq 0$.

We know : $|I - BA| = |I - AB| \neq 0$

so $(I - BA)$ is if invertible, then $(I - AB)$ is also invertible.

Proof :- $(I - BA) = (BB^{-1} - BABB^{-1}) = B(I - AB)B^{-1}$

so $|I - BA| = |B||I - AB||B^{-1}| = |I - AB|$ Ans

(5.) Let the given statement is true

so $\boxed{M = A + iB}$ where M = Hermitian matrix.

$\Rightarrow M^* = (\overline{M})^T = (\overline{A + iB})^T = (A - iB)^T$

$\Rightarrow M^* = A^T - iB^T$

and as $A = A^T$ (A = symmetric)

$B = -B^T$ (B = skew-symmetric).

so:

$$\boxed{M^* = A + iB = M}$$

so proved that it supports M is hermitian matrix.

(6.) Let $M = (I - A)(I + A)^{-1}$

$\Rightarrow$ Observation: $\boxed{A + A^* = 0}$

$\Rightarrow MM^* = \left((I - A)(I + A)^{-1}\right)\left(\overline{(I - A)(I + A)^{-1}}\right)^T$

$= (I - A)(I + A)^{-1}\left((I - \overline{A})(I + \overline{A})^{-1}\right)^T$

$= (I - A)(I + A)^{-1}\left((I + \overline{A})^T\right)^{-1}(I - \overline{A})^T$

$= (I - A)\left((I + A^*)(I + A)\right)^{-1}(I - A^*)$

$= (I - A)(2I)^{-1}(I - A^*)$

$= \frac{1}{2}(I - A)(I - A^*) = I$

so M is unitary matrix. ANS

(7)(a) Using $\lambda x = Ax$ $\Rightarrow$ λ = eigen values

x = eigen vectors

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$$

$\Rightarrow$ Combining all three eqn: we have:-

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & -2 & 4 \\ 2 & 0 & 0 \\ 0 & 2 & 4 \end{pmatrix}$$

Solving:

(i) $\left.\begin{aligned} -2x_1 + x_2 &= -4 \\ -x_1 + x_3 &= -2 \\ x_1 + x_3 &= 4 \end{aligned}\right\}$ $x_1 = 3$, $x_2 = 2$, $x_3 = 1$

(ii) $\left.\begin{aligned} -2x_4 + x_5 &= 2 \\ -x_4 + x_6 &= 0 \\ x_4 + x_6 &= 0 \end{aligned}\right\}$ $x_4 = x_6 = 0$, $x_5 = 2$

(iii) $\left.\begin{aligned} -2x_7 + x_8 &= 0 \\ -x_7 + x_9 &= 2 \\ x_7 + x_9 &= 4 \end{aligned}\right\}$ $x_7 = 1$, $x_8 = 2$, $x_9 = 3$

So: $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ Ans

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7 (b) (i) $|A-\lambda I| = 0 \Rightarrow \boxed{\lambda = 0, 4, 6}$

- $\boxed{\lambda = 0}$: $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow x_1 = x_3 = \frac{x_2}{2}$

  $\Rightarrow (1, 2, 1)^T$.

- $\boxed{\lambda = 4}$: $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow x_2 = 0$

  $x_1 + x_3 = 0$

  one possibility $\Rightarrow (-1, 0, 1)^T$.

- $\boxed{\lambda = 6}$: $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 6\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow -x_2 = x_3 = x_1$

  $\Rightarrow (1, -1, 1)^T$.

So: $P = [X_1 \quad X_2 \quad X_3]$

$= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ Ans

7 (b) (ii) $|A-\lambda I| = 0 \Rightarrow \boxed{\lambda = -1, -1, 5}$

- $\boxed{\lambda = -1}$: $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow x_1 + 2x_2 = x_3$.

So two eigenvectors $= (2, -1, 0)^T$ and $(1, 0, 1)^T$

- $\boxed{\lambda = 5}$: $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 5\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow$ $5x_1 + x_3 = 2x_2$

  $x_1 = x_3 + x_2$

  $x_1 + 2x_2 + 5x_3 = 0$

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$\Rightarrow$ From three eq^n: $x_1 = -x_3$ and $x_2 = -2x_3$
One eigen vector is $(-1,-2,1)^T$.

So $P = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ Ans

7 (b) (iii) $|A - \lambda I| = 0 \Rightarrow \boxed{\lambda = 1, 3, -2}$.

- $\Rightarrow \boxed{\lambda = 1}$:- $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow x_1 + 3x_3 = 2x_2$
$x_1 + x_3 = 0$
$x_1 + 3x_2 = 2x_3$.

One possibility: $(-1,1,1)^T$.

- $\boxed{\lambda = 3}$:- $\left.\begin{matrix} x_1 + 2x_2 = 3x_3 \\ x_1 + x_3 = 2x_2 \\ x_1 + 3x_2 = 4x_3 \end{matrix}\right\}$ $\left.\begin{matrix} x_2 = x_3 \\ x_1 = x_2 \end{matrix}\right\}$ $(1,1,1)^T$.

- $\boxed{\lambda = -2}$:- $\left.\begin{matrix} 4x_1 + 3x_3 = 2x_2 \\ x_1 + 3x_2 + x_3 = 0 \end{matrix}\right\}$ $(-11,-1,14)^T$

So: $P = \begin{pmatrix} -1 & 1 & -11 \\ 1 & 1 & -1 \\ 1 & 1 & 14 \end{pmatrix}$ Ans

7 (b) (iv) $|A - \lambda I| = 0 \Rightarrow \lambda = 1, 2, 3$.

- $\boxed{\lambda = 1} \Rightarrow A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow$ $\left.\begin{matrix} x_1 = x_3 \\ x_1 = -x_2 \\ x_3 = -x_2 \end{matrix}\right\}$ $(1,-1,1)^T$

- $\boxed{\lambda = 2} \Rightarrow A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow$ $\left.\begin{matrix} x_3 = 0 \\ 3x_1 = -2x_2 \end{matrix}\right\}$ $(-2,3,0)^T$.

• $\boxed{\lambda=3}$:- $A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 3\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \Rightarrow \left.\begin{array}{l} x_1=-x_3\\ 3x_1=-x_2\end{array}\right\} (-1,3,1)$

So $P = \begin{bmatrix}1&-2&-1\\-1&3&3\\1&0&1\end{bmatrix}$ Ans

9. (a) Characteristic equation of A:

$$|A-\lambda I|=0$$

$$\Rightarrow \lambda^3-7\lambda^2-7\lambda+1=0$$

⇒ From cayley-hamiltonian, A should satisfy this equation:-

$$A^3-7A^2-7A+I=0$$

Multiply by A^{-1}. ⇒ $\boxed{A^{-1}=-A^2+7A+7I}$ Ans

(or)

$$A^{-1}=\begin{bmatrix}7&-3&-3\\-1&0&1\\-1&1&0\end{bmatrix}$$ Ans

9. (b) ⇒ $|A-\lambda I|=0 \Rightarrow \lambda^3-17\lambda^2+62\lambda-40=0$

So: $A^3-17A^2+62A-40I=0$

$$\Rightarrow A^{-1}=\frac{A^2-17A+62I}{40}$$

$$=\begin{bmatrix}-0.2&-0.9&0.5\\0.5&1.25&-0.75\\-0.3&-0.6&0.5\end{bmatrix}$$ Ans

9.(c) $|A-\lambda I|=0 \Rightarrow \lambda^3-11\lambda^2+6\lambda-1=0$

So: $A^3-11A^2+6A-I=0$

$$\Rightarrow \boxed{A^{-1}=(A^2-11A+6I)}$$

9.(e)

$$A^{-1}=\begin{bmatrix}2 & 0 & -1\\ 5 & 1 & 0\\ 0 & 1 & 3\end{bmatrix}$$ Ans

10. (i) $|A-\lambda I|=0 \Rightarrow \lambda^3-3\lambda^2+5\lambda-11=0$

$\Rightarrow$ Cayley-Hamiltonian said that: 'A' satisfies this equation so:-

$$\boxed{A^3-3A^2+5A-11I=0} \quad \text{—— (1)}$$

To check: put $A^3=\begin{pmatrix}3 & 2 & 6\\ 11 & 3 & 1\\ 4 & 24 & 9\end{pmatrix}$; $A^2=\begin{pmatrix}-1 & 4 & 2\\ 2 & -1 & 2\\ 8 & 8 & 1\end{pmatrix}$

and $A=\begin{pmatrix}1 & 2 & 0\\ -1 & 1 & 1\\ 4 & 0 & 1\end{pmatrix}$ into eq. (1).

We will get: $RHS=LHS=0$.

So verified.

(ii) From (i):- $A^{-1}=\frac{1}{11}(A^2-3A+5I)$ Ans

$$=\frac{1}{11}\begin{pmatrix}1 & -2 & 2\\ 5 & 1 & -1\\ -4 & 8 & 3\end{pmatrix}$$ Ans

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$\Rightarrow$ Multiply by A in eqn: (1):-

$$A^4 = 3A^3 - 5A^2 + 11A$$
$$= 3(3A^2 - 5A + 11I) - 5A^2 + 11A$$
$$= 4A^2 - 4A + 33I \quad \text{Ans}$$

and

$$A^4 = \begin{bmatrix} 25 & 8 & 8 \\ 12 & 25 & 4 \\ 16 & 32 & 33 \end{bmatrix} \quad \text{Ans}$$

(11) Since A is diagonizable, it must have three linearly independent eigen vectors.

$\Rightarrow$ But the eigen-value = 3 is repeated two times and $\gamma \neq 3$: so:-

$$|A - 3I| = 0 \quad \text{i.e.} \quad (A - 3I)X = 0,$$

we should be choose 2 variables arbitarily:-

$\Rightarrow$ Two rows of $(A - 3I)$ must be zero rows or it should have rank = 1.

$$\therefore \; A - 3I = \begin{bmatrix} 1 & \alpha & -1 \\ 2 & 2 & \beta \\ 1 & 1 & \gamma - 3 \end{bmatrix} \xrightarrow{E_{12}(-2);\, E_{13}(-1)} \begin{bmatrix} 1 & \alpha & -1 \\ 0 & 2-2\alpha & \beta+2 \\ 0 & 1-\alpha & \gamma-2 \end{bmatrix}$$

$$\xrightarrow{E_{23}(-1/2)} \begin{bmatrix} 1 & \alpha & -1 \\ 0 & 2-2\alpha & \beta+2 \\ 0 & 0 & \gamma - \frac{\beta}{2} - 3 \end{bmatrix}$$

$\Rightarrow$ For rank $= 1 \Rightarrow \boxed{\beta = -2 ;\ \alpha = 1 ;\ \gamma = 2}$. Ans

The matrix A will be: $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$

$\Rightarrow$ As δ is an eigen-value, it must satisfy: $|A - \delta I| = 0$

$\Rightarrow \quad \delta^3 - 11\delta^2 + 39\delta - 45 = 0$

$\Rightarrow$ as $\delta \neq 3 \Rightarrow \boxed{\delta = 5}$. Ans

(12) (a) $|A - \lambda I| = 0 \Rightarrow \boxed{\lambda = -2, 8}$.

- For $\lambda = -2$: $\begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow x_2 = -3x_1$

$$\Downarrow$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

- For $\lambda = 8$: $\begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 8\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow x_1 = 3x_2$

$$\Downarrow$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

So $P = k\begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}$. ; $\boxed{k > 0}$

$\Rightarrow$ 'k' because to make P = orthogonal, by doing: $PP^T = I \Rightarrow k = \frac{1}{\sqrt{10}}$

So $P = \frac{1}{\sqrt{10}}\begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}$ Ans

12. (b) $|A-\lambda I| = 0 \Rightarrow \lambda^3 - 6\lambda^2 - 135\lambda - 400 = 0$

$$\Rightarrow \boxed{\lambda = -5, -5, 16}$$

• When $\lambda = -5$:- $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -5\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\Rightarrow x_3 + 4x_1 = 2x_2 \Rightarrow \begin{pmatrix} -5 \\ -8 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \sqrt{21} \\ 2\sqrt{21} \end{pmatrix}$

• When $\lambda = 16$:- $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 16\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\Rightarrow x_1 = 4x_3 \; ; \; x_2 = -2x_3 \Rightarrow \begin{pmatrix} 4\sqrt{5} \\ -2\sqrt{5} \\ \sqrt{5} \end{pmatrix}$.

$\Rightarrow$ So: $P = k\begin{pmatrix} 0 & -5 & 4\sqrt{5} \\ \sqrt{21} & -8 & -2\sqrt{5} \\ 2\sqrt{21} & 4 & \sqrt{5} \end{pmatrix}$

$\Rightarrow$ If P = orthogonal, then $PP^T = I \Rightarrow \boxed{k = \frac{1}{\sqrt{105}}}$ Ans

12. (c) $|A - \lambda I| \Rightarrow \boxed{\lambda = 0, 3}$

• $\boxed{\lambda = 0}$:- $\begin{bmatrix} 1 & (1-i) \\ (1+i) & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$\Rightarrow \left.\begin{aligned} x_1 + (1-i)x_2 &= 0 \\ (1+i)x_1 + 2x_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ (1+i) \end{pmatrix}$$

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- $\boxed{\lambda=3}$: $\left.\begin{array}{l}(1-i)x_2 = 2x_1\\(1+i)x_1 = x_2\end{array}\right\}$ $\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}\sqrt{2}\\(1+i)\sqrt{2}\end{pmatrix}$.

$\Rightarrow$ So: $P = K\begin{pmatrix}-2 & \sqrt{2}\\(1+i) & (1+i)\sqrt{2}\end{pmatrix}$.

when $PP^* = I \Rightarrow K = \frac{1}{\sqrt{6}} \Rightarrow P = \frac{1}{\sqrt{6}}\begin{pmatrix}-2 & \sqrt{2}\\(1+i) & (1+i)\sqrt{2}\end{pmatrix}$

Ans

(13.) Let $X = \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}_{5\times1}$ $\Rightarrow$ We observe:

$AX = X$

$\Rightarrow \boxed{A^{-1}X = X}$

$\Downarrow$

$\Rightarrow$ sum of elements in 1^{st} row $= 1$

so sum of all the elements of $A^{-1} = 5$

Ans

(14.) A is nilpotent, so there exist a non negative integer k such that

$A^k = 0$.

Now let $B = I - A + A^2 + \ldots\ldots (-1)^{k-1}A^{k-1}$.

$\Rightarrow$ clearly $(I+A)(B) = B + AB$

$= B + A(I - A + A^2 \cdots)$

$= I - (-1)^k A^k = I$.

and also $B(I+A) = I$ so $(I+A)^{-1} = B$.

hence $(I+A)$ is invertible. (OR) $\Rightarrow |I+A| \neq 0$.

(8.) $|A-\lambda I| = 0 \Rightarrow \boxed{\lambda = -4, -3, -2}$ = eigen values of A.

calculating eigen vectors :-

(a) $\lambda = -4$

$\Downarrow$

$x_2 = x_3 = 0$

$\Downarrow$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(b) $\lambda = -3$

$\Downarrow$

$x_3 = 0$

$x_2 = x_1$

$\Downarrow$

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(c) $\lambda = -2$

$\Downarrow$

$x_3 = x_2 = 2x_1$

$\Downarrow$

$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\Rightarrow$ so eigen vectors are linearly independent : then 'A' can be written as:

$$\boxed{A = PDP^{-1}}$$

where $P = [X_1 \ X_2 \ X_3] = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

and $D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

Calculating $P^{-1} = \dfrac{adj(P)}{|P|} = \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1/2 \end{pmatrix}$.

also $D^{50} = \begin{pmatrix} 4^{50} & 0 & 0 \\ 0 & 3^{50} & 0 \\ 0 & 0 & 2^{50} \end{pmatrix}$

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and $e^{2D} = \begin{bmatrix} e^{-8} & 0 & 0 \\ 0 & e^{-6} & 0 \\ 0 & 0 & e^{-4} \end{bmatrix}$.

(i) $A^{50} = (PDP^{-1})^{50} = PD^{50}P^{-1}$

$\Rightarrow A^{50} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4^{50} & 0 & 0 \\ 0 & 3^{50} & 0 \\ 0 & 0 & 2^{50} \end{pmatrix} \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1/2 \end{pmatrix}$.

$\Rightarrow A^{50} = \begin{bmatrix} 4^{50} & 3^{50} - 4^{50} & 1/2(4^{50} - 2\cdot 3^{50} + 2^{50}) \\ 0 & 3^{50} & 2^{50} - 3^{50} \\ 0 & 0 & 2^{50} \end{bmatrix}$

Ans

(ii) $e^{2A} = e^{2(PDP^{-1})} = Pe^{2D}P^{-1}$

$e^{2A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} e^{-8} & 0 & 0 \\ 0 & e^{-6} & 0 \\ 0 & 0 & e^{-4} \end{pmatrix} \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1/2 \end{pmatrix}$

$e^{2A} = \begin{bmatrix} e^{-8} & e^{-6} - e^{-8} & \frac{1}{2}(e^{-8} - 2e^{-6} + e^{-4}) \\ 0 & e^{-6} & e^{-4} - e^{-6} \\ 0 & 0 & e^{-4} \end{bmatrix}$

Ans

(15.) $A =$ nilpotent matrix and $UU^* = I$

$\Rightarrow$ According to schur's lemma:

$U^*AU = T$ (upper triangular)

$\Rightarrow$ to take power to 15:-

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$\Rightarrow U^* A^{15} U = T^{15} = 0$ [As $A^{15} = 0$]

Obviously if :- diagonal entries in T is zero and T is upper triangular then : $T^{15} = 0$ automatically

Hence, proved.

So: $T = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & 0 & \ddots & \vdots \\ 0 & & \ddots & \vdots \\ & & & 0 \end{bmatrix}$

Ans