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(1.)(a) For $\vec{u}$ to be constant, u_1, u_2, u_3 are constant:

$$\frac{du_i}{dt} = 0 \quad (i = 1, 2, 3).$$

$$\Rightarrow \frac{d\vec{u}}{dt} = \frac{du_1}{dt}\hat{i} + \frac{du_2}{dt}\hat{j} + \frac{du_3}{dt}\hat{k} = 0.$$

(b) $|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \text{constant (say } K)$

$\Rightarrow u_1^2 + u_2^2 + u_3^2 = K^2$

differentiate w.r.t to t:-

$$u_1\frac{du_1}{dt} + u_2\frac{du_2}{dt} + u_3\frac{du_3}{dt} = 0.$$

$$\Rightarrow \boxed{\vec{u} \cdot \frac{d\vec{u}}{dt} = 0} \quad \text{Proved}$$

(c) For vector function to not change the direction, its tangent should be $\parallel$ to it:-

$$\boxed{\vec{u} \times \frac{d\vec{u}}{dt} = 0} \quad \text{Ans}$$

(2.)(a) hyperbolic functions:

$$\vec{r} = \sinh(t)\vec{a} + \cosh(t)\vec{b}$$

$$\frac{d\vec{r}}{dt} = \cosh(t)\vec{a} + \sinh(t)\vec{b}$$

$$\frac{d^2\vec{r}}{dt^2} = \sinh(t)\vec{a} + \cosh(t)\vec{b} = \vec{r} \quad \text{Ans}$$

(2) (b) $\frac{d\vec{r}}{dt} = ne^{nt}\vec{a} - ne^{-nt}\vec{b}$

$\frac{d^2\vec{r}}{dt^2} = n^2e^{nt}\vec{a} + n^2e^{-nt}\vec{b} = n^2\vec{r}$. Ans

(c) $\frac{d\vec{r}}{dt} = -n\sin(nt)\hat{i} + n\cos(nt)\hat{j}$

$\Rightarrow \vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(nt) & \sin(nt) & 0 \\ -n\sin(nt) & n\cos(nt) & 0 \end{vmatrix} = n\hat{k}$

Ans

(3.) $\frac{d\vec{r}}{dt} = \vec{v} = -\sin(t-1)\hat{i} + \cosh(t-1)\hat{j} + 3\alpha t^2\hat{k}$

$\Rightarrow \vec{a} = \frac{d^2\vec{r}}{dt^2} = -\cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + 6\alpha t\hat{k}$

$\Rightarrow$ so at $t=1$: $\vec{r}\cdot\vec{a} = 0$

$\Rightarrow -\cos^2(0) + \sin^2(0) + (\alpha t^3 \cdot 6\alpha t) = 0$

$\Rightarrow 6\alpha^2 = 1$

$\boxed{\alpha = \pm\frac{1}{\sqrt{6}}}$ Ans

(4.) (i) $\vec{r}\cdot\vec{r} = a^2\cos^2(t) + a^2\sin^2(t) + b^2t^2$

$= a^2 + b^2t^2 = |\vec{r}|^2$ Proved

(ii) We know that:-

$|\vec{a}\times\vec{b}|^2 = \left(\sqrt{a^2b^2 - (\vec{a}\cdot\vec{b})^2}\right)^2 = a^2b^2 - (\vec{a}\cdot\vec{b})^2$

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$$\Rightarrow |\vec{r}' \times \vec{r}''|^2 = \left(\sqrt{(a^2+b^2)(a^2) - 0}\right)^2$$

$$= a^2(a^2+b^2) \quad \underline{Ans}$$

4. (iii) $$I = (\vec{r}' \times \vec{r}'') \cdot \vec{r}'''$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} \cdot (a\sin t\,\hat{i} - a\cos t\,\hat{j})$$

$$= \begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

$$= b(a)^2 \quad \underline{Ans}$$

5. $$I = \frac{d}{dt}\left([f\ f'\ f'']\right) = \frac{d}{dt}\begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$\Rightarrow I = \underbrace{\begin{vmatrix} f_1' & f_2' & f_3' \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} f_1 & f_2 & f_3 \\ f_1'' & f_2'' & f_3'' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}}_{0} + \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}$$

$$I = [f\ f'\ f'''] \quad \underline{Ans}$$

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(6.) (i) $\vec{\nabla}\phi = (2z^4 - 2xy)\hat{i} - x^2\hat{j} + 8xz^3\hat{k}$
$= 10\hat{i} - 4\hat{j} + 16\hat{k}$
$|\vec{\nabla}\phi| = \sqrt{100+16+256} = \sqrt{372}$ Ans

(ii) $\frac{\partial\phi}{\partial x} = (y+y^2+z^2) \Rightarrow \phi = (y+y^2+z^2)x + c_1(y,z)$

where c_1 is the function of y and z.
$\Rightarrow \frac{\partial\phi}{\partial y} = x(1+2y) + \frac{\partial c_1}{\partial y} = x+z+2xy$

$\Rightarrow \frac{\partial c_1}{\partial y} = z \Rightarrow \boxed{c_1 = yz + c_2(z).}$

where c_2 is the function of z.
$\Rightarrow \frac{\partial\phi}{\partial z} = x(2z) + y + \frac{\partial c_2}{\partial z} = y + 2zx$

$\Rightarrow \frac{\partial c_2}{\partial z} \Rightarrow \boxed{c_2 = K\ (\text{constant}).}$

So: $\phi = (y+y^2+z^2)x + yz + K.$
and $\phi(1,1,1) = 3 = K+1+3 \Rightarrow \boxed{K=-1}$
hence:
$\phi = xy + yz + x(y^2+z^2) - 1$ Ans

(7.) (i) $\vec{\nabla}\left((x^2+y^2+z^2)^{n/2}\right) = n.(x^2+y^2+z^2)^{\frac{n-2}{2}}.(x\hat{i}+y\hat{j}+z\hat{k})$
$= n.r^{n-2}.\vec{r}$ Ans

(7) (ii) $\vec{\nabla}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-1}{2(x^2+y^2+z^2)^{3/2}} \cdot 2\cdot(x\hat{i}+y\hat{j}+z\hat{k})$

$= \frac{-\vec{r}}{r^3}$ Ans

(iii) $\vec{\nabla}(f(r)) = \frac{\partial f(r)}{\partial x}\hat{i} + \frac{\partial f(r)}{\partial y}\hat{j} + \frac{\partial f(r)}{\partial z}\hat{k}$

$= f'(r)\cdot\frac{\partial r}{\partial x}\hat{i} + f'(r)\cdot\frac{\partial r}{\partial y}\hat{j} + f'(r)\cdot\frac{\partial r}{\partial z}\hat{k}$

$= f'(r)\cdot\left(\frac{1}{2r}\right) 2(x\hat{i}+y\hat{j}+z\hat{k})$

$= \frac{f'(r)}{r}\cdot\vec{r}$ Ans

(iv) We know:

$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}.$

Put $\vec{A} = \vec{a}\times\vec{b}$ and $\vec{B} = \vec{r}$,

$\vec{\nabla}\times\vec{A} = 0, \quad (\vec{B}\cdot\vec{\nabla})\vec{A} = 0.$

so:

$\vec{\nabla}([r \; a \; b]) = (\vec{a}\times\vec{b})(\underbrace{\vec{\nabla}\times\vec{r}}_{0}) + (\vec{A}\cdot\vec{\nabla})\vec{B}$

$= (\vec{A}\cdot\vec{\nabla})\vec{B}$

Let $\vec{a}\times\vec{b} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

where $A_1, A_2,$ and $A_3 =$ constants.

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then: $\vec{\nabla}([\vec{r}\ \vec{a}\ \vec{b}]) = A_1 \frac{\partial x}{\partial x}\hat{i} + A_2 \frac{\partial y}{\partial y}\hat{j} + A_3 \frac{\partial z}{\partial z}\hat{k}$;

$= \vec{A} = (\vec{a} \times \vec{b})$ Ans

8. (i) $|\vec{v_1}| = (\vec{\nabla}\phi \cdot \hat{n}) = (2\hat{i} - 4\hat{j} - 8\hat{k}) \cdot \frac{(2\hat{i} + 2\hat{j} + \hat{k})}{3}$

$= \frac{4-8-8}{3} = \frac{-12}{3} = -4$ Ans

(ii) $v_2 = (\vec{\nabla}\phi \cdot \hat{n}) = (2\hat{i} + \hat{j} + \hat{k}) \cdot \frac{(2\hat{i} - \hat{j} + 2\hat{k})}{3}$

$= \frac{4-1+2}{3} = 5/3$ Ans

(iii) $v_3 = \vec{\nabla}\phi \cdot \hat{PQ} = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$

$= \frac{8+8+12}{\sqrt{21}} = \frac{28 \times \sqrt{21}}{21}$

$= \frac{4\sqrt{21}}{3}$ Ans

(iv) the directional derivative will be greatest in the direction of $\vec{\nabla}\phi$ at that point:

$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{4\hat{i} + 3\hat{j} + 5\hat{k}}{\sqrt{16+9+25}} = \frac{4\hat{i} + 3\hat{j} + 5\hat{k}}{\sqrt{50}}$

and $(\text{direction derivative})_{max} = |\vec{\nabla}\phi| = 5\sqrt{2}$

Ans

9. (i) $\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2\hat{i}+5\hat{j}-2\hat{k}}{\sqrt{8+25}} = \frac{2\hat{i}+5\hat{j}-2\hat{k}}{\sqrt{33}}$ Ans

(ii) Angle b/w the surfaces = Angle b/w normal to the surfaces.

$\Rightarrow \hat{n}_1 = \frac{4\hat{i}-2\hat{j}+4\hat{k}}{6}$; $\hat{n}_2 = \frac{4\hat{i}-2\hat{j}-\hat{k}}{\sqrt{21}}$

$\Rightarrow \cos\theta = \frac{16+4-4}{6\sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$ Ans

10. (i) $\vec{a} = a\hat{i}+b\hat{j}+c\hat{k}$; $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

$I_1 = \vec{\nabla}\cdot(\vec{r}\times\vec{a}) = \vec{\nabla}\cdot\left[(yc-zb)\hat{i}+(az-cx)\hat{j} + (bx-ay)\hat{k}\right].$

$= 0$ proved

(ii) $$\vec{\nabla}\times(\vec{r}\times\vec{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yc-zb & az-cx & bx-ay \end{vmatrix}$$

$= -2(a\hat{i}+b\hat{j}+c\hat{k}) = -2\vec{a}$

Hence, proved

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(10) (iii) $\vec{\nabla}(ax+by+cz) = (a\hat{i}+b\hat{j}+c\hat{k}) = \vec{a}$

Proved

(iv) $\vec{\nabla}\cdot\left((x^2+y^2+z^2)(a\hat{i}+b\hat{j}+c\hat{k})\right) = 2xa\hat{i}+2yb\hat{j}+2zc\hat{k}$

$= 2\vec{a}\cdot\vec{r}$ Ans

Hence, proved

(11) (i) $\vec{F}$ is solenoidal $\Leftrightarrow (\vec{\nabla}\cdot\vec{F}=0)$

$\Rightarrow 0+0+a=0$

$\boxed{a=0}$ Ans

(ii) $\vec{F}$ is irrotational $\Leftrightarrow (\vec{\nabla}\times\vec{F}=0)$

$\Rightarrow (2y-2y)\hat{i}+(-2z+az)\hat{j}+(-\,-)\hat{k}=0$

Clearly: $2z = az \Rightarrow a=2$ Ans.

(iii) $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (6xy+z^3) & (3x^2-z) & (3xz^2-y) \end{vmatrix}$$

$= 0\hat{i}+0\hat{j}+0\hat{k} = 0.$

Hence, $\vec{F}$ is conservative, means there exists a scalar function, ϕ, such that

$$\vec{F} = \vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right).$$

$\Rightarrow \frac{\partial\phi}{\partial x} = (6xy+z^3) \Rightarrow \phi = xz^3+3x^2y+c_1(y,z).$

where c_1 is function of y and z.

$\Rightarrow \dfrac{\partial \phi}{\partial y} = 3x^2 + \dfrac{\partial c_1}{\partial y} = 3x^2 - z \Rightarrow \boxed{c_1 = -yz + c_2(z)}$.

where c_2 is the function of z.

$\Rightarrow \dfrac{\partial \phi}{\partial z} = 3xz^2 - y + \dfrac{\partial c_2}{\partial z} = 3xz^2 - y \Rightarrow c_2 = k$
$= \text{constant}$

and hence:

$$\boxed{\phi = xz^3 + 3x^2y - yz + k}$$

(12.)

(i) $\vec{\nabla}\cdot(\phi\vec{F}) = \vec{\nabla}\phi\cdot\vec{F} + \phi(\vec{\nabla}\cdot\vec{F})$. (To prove)

$\Rightarrow$ let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$:-

$$\vec{\nabla}\cdot(\phi F_1\hat{i} + \phi F_2\hat{j} + \phi F_3\hat{k}) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\phi F_i)\cdot\hat{e_i}$$

$$= \sum_{i=1}^{3}\left(\frac{\partial \phi}{\partial x_i}\cdot F_i + \frac{\partial F_i}{\partial x_i}\cdot\phi\right)\hat{e_i}$$

$$= \vec{F}\cdot(\vec{\nabla}\phi) + (\vec{\nabla}\cdot\vec{F})\phi \quad \underline{\text{Ans}}$$

Hence, <u>proved</u>

(ii) $\vec{\nabla}\times(\phi\vec{F}) = (\vec{\nabla}\phi)\times\vec{F} + \phi(\vec{\nabla}\times\vec{F})$ (To prove).

$\Rightarrow$ R.H.S $=$
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi F_x & \phi F_y & \phi F_z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(\phi F_z) - \frac{\partial}{\partial z}(\phi F_y)\right)\hat{i} + \left(\frac{\partial}{\partial z}(\phi F_x) - \frac{\partial}{\partial x}(\phi F_z)\right)\hat{j}$$
$$+ \left(\frac{\partial}{\partial x}(\phi F_y) - \frac{\partial}{\partial y}(\phi F_x)\right)\hat{k}$$

$$\Rightarrow \underbrace{\left(\left(\frac{\partial \phi}{\partial y} F_z - \frac{\partial \phi}{\partial z} F_y\right)\hat{i} + \left(\frac{\partial \phi}{\partial z} F_x - \frac{\partial \phi}{\partial x} F_z\right)\hat{j} + \left(\frac{\partial \phi}{\partial x} F_y - \frac{\partial \phi}{\partial y} F_x\right)\hat{k}\right)}^{(\vec{\nabla}\phi \times \vec{F})}$$

$$+ \phi \underbrace{\left(\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}\right)}_{(\vec{\nabla} \times \vec{F})}$$

$$\Rightarrow \vec{\nabla}\phi \times \vec{F} + \phi \cdot (\vec{\nabla} \times \vec{F})$$ proved

(13) (i) We know:

$$\vec{\nabla}(\phi \vec{A}) = \vec{\nabla}\phi \cdot \vec{A} + \phi(\vec{\nabla} \cdot \vec{A})$$

$$\Rightarrow \vec{\nabla}\left(\frac{1}{r^3} \cdot \vec{r}\right) = \vec{\nabla}\left(\frac{1}{r^3}\right) \cdot \vec{r} + \frac{1}{r^3}(\vec{\nabla} \cdot \vec{r})$$

$$= \frac{-3}{r^5}(\cancel{\vec{r} \cdot \vec{r}}^{\,r^2}) + \frac{1}{r^3}(3)$$

$$= -3/r^3 + 3/r^3 = 0$$ Ans.

(ii) $\vec{\nabla}(r^3 \cdot \vec{r}) = \vec{\nabla}(r^3) \cdot \vec{r} + r^3 \cdot (\vec{\nabla} \cdot \vec{r})$

$$= 3 \cdot r\, r^2 + r^3 \cdot 3 = 6r^3$$ Ans

(iii) $\vec{\nabla} \cdot \left(r \cdot \frac{-3}{r^5} \vec{r}\right) = -3 \cdot \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^4}\right)$.

$$= -3 \cdot \left(\frac{-4}{r^6} \cdot r^2 + \frac{1}{r^4} \times 3\right)$$

$$= 3/r^4$$ Ans.

Hence, proved all

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13 (IV) $\vec{\nabla}\cdot(r^n(\vec{a}\times\vec{r})) = \vec{\nabla}(r^n)\cdot(\vec{a}\times\vec{r}) + r^n(\vec{\nabla}\cdot(\vec{a}\times\vec{r}))$

$= n\cdot r^{n-2}[\vec{r}\cdot(\vec{a}\times\vec{r})]$

0 [both are ⊥ʳ vectors]

∴ Ans: 0 Ans

14 (b) $\vec{\nabla}\cdot(f(r)\cdot\vec{r}) = [\vec{\nabla}\cdot(f(r))]\cdot\vec{r} + (\vec{\nabla}\cdot\vec{r})\,f(r)$.

$= c\cdot\frac{(-3)}{r^5}r^2 + 3f(r). = 0$

when $f(r) = \frac{c}{r^3}$.

(a) $\vec{\nabla}\cdot((f(r)\cdot\vec{r})) = (\vec{\nabla} f(r))\cdot\vec{r} + (\vec{\nabla}\cdot\vec{r})\cdot f(r)$

$= \left(\frac{f'(r)\vec{r}}{r}\right)\cdot\vec{r} + 3f(r)$

$= r\cdot f'(r) + 3\cdot f(r)$

Hence, proved

when: $r\cdot f'(r) + 3f(r) = 0$

then: $\frac{dy}{dx} = -\frac{3}{x}y$

$\ln|y| = -3\ln|x| + c$

∴ $\boxed{f(r) = \frac{+c_1}{r^3}}$ hence, proved

15. (i) $\vec{\nabla}\times(\underbrace{r^n}_{\phi}\underbrace{\vec{r}}_{\vec{A}}) = (\vec{\nabla}(r^n))\times\vec{r} + (\vec{\nabla}\times\vec{r})r^n$

$= \underbrace{(nr^{n-2}\vec{r})\times(\vec{r})}_{\text{parallel vectors}} + 0 = 0 + 0$

$= 0$

Ans

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- So for all n, curl of $r^n\hat{r}$ is zero, hence it will be always irrotational.
- $\vec{\nabla}\cdot(r^n\vec{r}) = \vec{\nabla}(r^n)\cdot\vec{r} + (\vec{\nabla}\cdot\vec{r})\,r^n$

$$= n\cdot \frac{r^{n-2}}{r^2} + 3r^n = 0. \quad \text{[For solenoidal]}$$

$$\Rightarrow n = \frac{-3r^{n-2}}{r^{n-2}} = -3. \ \underline{\text{Ans}}$$

It means, $r^n\hat{r}$ will be solenoidal only for $n=-3$ $\underline{\text{Ans}}$

(15) (ii) For irrotation, curl should be zero.

Hence, $\vec{\nabla}\times(f(r)\cdot\vec{r}) = 0$

$$\Rightarrow \left[\vec{\nabla}(f(r))\right]\times\vec{r} + f(r)\cdot\underbrace{(\vec{\nabla}\times\vec{r})}_{0} = 0$$

$$\Rightarrow \underbrace{\left(\frac{f'(r)}{r}\cdot\vec{r}\right)\times\vec{r}}_{\text{parallel vectors}} = 0 \quad \Rightarrow 0=0$$

Hence $\underline{\text{proved}}$

(16.) $$I = \vec{\nabla}\times\left(\frac{1}{r^3}(\vec{a}\times\vec{r})\right) = \vec{\nabla}\left(\frac{1}{r^3}\right)\times(\vec{a}\times\vec{r}) + \frac{(\vec{\nabla}\times(\vec{a}\times\vec{r}))}{r^3}$$

$$= \underbrace{\left(\frac{-3}{r^5}\vec{r}\right)\times(\vec{a}\times\vec{r})}_{I_1} + \underbrace{\frac{\vec{\nabla}\times(\vec{a}\times\vec{r})}{r^3}}_{I_2}$$

and $I = (I_1 + I_2)$.

$\Rightarrow I_1 = \frac{-3}{r^5}\left(\vec{r}\times(\vec{a}\times\vec{r})\right) = \frac{-3}{r^5}\left(r^2(\vec{a}) - (\vec{a}\cdot\vec{r})\vec{r}\right)$

$\Rightarrow I_1 = \frac{-3}{r^3}\vec{a} + \frac{3\vec{r}}{r^5}(\vec{a}\cdot\vec{r})$ — (1)

$\Rightarrow I_2 = \frac{1}{r^3}\left(\vec{\nabla}\times\left((bz-cy)\hat{i} + (cx-az)\hat{j} + (ay-bx)\hat{k}\right)\right)$

$= \frac{1}{r^3}(2\vec{a}) = \frac{2\vec{a}}{r^3}$ — (2)

So: $I = I_1 + I_2 = \frac{-\vec{a}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{a}\cdot\vec{r})$ Ans

(17) (i) $\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}\times\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

$= \hat{i}\left(\frac{\partial^2 F_y}{\partial y\,\partial x} - \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_x}{\partial z^2} + \frac{\partial^2 F_z}{\partial z\,\partial x}\right)$
$+ \hat{j}\left(\frac{\partial^2 F_z}{\partial z\,\partial y} - \frac{\partial^2 F_y}{\partial z^2} - \frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_x}{\partial x\,\partial y}\right)$
$+ \hat{k}\left(\frac{\partial^2 F_x}{\partial x\,\partial z} - \frac{\partial^2 F_z}{\partial x^2} - \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_y}{\partial y\,\partial z}\right)$.

Arrange +ve and -ve seperately and adding and subtracting : $\frac{\partial^2 F_x}{\partial x^2}, \frac{\partial^2 F_y}{\partial y^2}, \frac{\partial^2 F_z}{\partial z^2}$ $\Rightarrow$ it becomes equal to:

$\vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \vec{\nabla}^2\vec{F}$.

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(17.) (ii) Take curl of eqn. in (i):

$$\text{curl}(\text{curl}(\text{curl}\vec{F})) = \vec{\nabla}\times(\vec{\nabla}g) - \vec{\nabla}\times(\vec{\nabla}^2\vec{F})$$
$$= \underbrace{\vec{\nabla}\times(\vec{\nabla}g)}_{0} - \underbrace{\vec{\nabla}\times(\vec{\nabla}\cdot\vec{h})}_{0}$$
$$= 0$$

Hence the answer is 0. Ans

(18.) (i) $\vec{\nabla}\cdot(\vec{\nabla}(r^{-1})) = \vec{\nabla}\cdot((-1)(r^{-3})\cdot\vec{r})$

$$= \vec{\nabla}\cdot\left(-\frac{\vec{r}}{r^3}\right)$$

and we have proved that: $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = 0$

hence: $\boxed{\vec{\nabla}^2\left(\frac{1}{r}\right) = 0}$ proved

(ii) $\vec{\nabla}^2(r^n\vec{r}) = \vec{\nabla}\cdot(\vec{\nabla}(r^n\vec{r}))$

We know that:

$$\vec{\nabla}^2(\vec{F}) = \text{grad}(\text{div}\vec{F}) - \text{curl}(\text{curl}(\vec{F}))$$

$$\Rightarrow \vec{\nabla}^2(r^n\vec{r}) = \text{grad}(\text{div}(r^n\vec{r})) - \vec{\nabla}\times(\underbrace{\vec{\nabla}\times r^n\vec{r}}_{0})$$
$$= \vec{\nabla}(\vec{\nabla}(r^n)\cdot\vec{r} + (\vec{\nabla}\cdot\vec{r})r^n)$$
$$= \vec{\nabla}((n+3)r^n)$$
$$= n(n+3)r^{n-2}\vec{r}$$

as $\vec{\nabla}\times\vec{r} = 0$ and $r^n\vec{r}$ is along $\vec{r}$.

Hence, proved

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(18.) (iii) $\vec{\nabla}^2(f(r)) = \vec{\nabla}\cdot(\vec{\nabla}f(r)) = \vec{\nabla}\cdot\left(\frac{f'(r)}{r}\cdot\vec{r}\right)$

$= \vec{\nabla}\left(\frac{f'(r)}{r}\right)\cdot\vec{r} + (\vec{\nabla}\cdot\vec{r})\cdot\frac{f'(r)}{r}$

$= 3\cdot\frac{f'(r)}{r} + \frac{\partial}{\partial r}\left(\frac{f'(r)}{r}\right)\cdot\underbrace{\left(\sum\left(\frac{\partial r}{\partial x}\hat{i}\right)\cdot\vec{r}\right)}_{r}$

$= 3\cdot\frac{f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$

$= f''(r) + 2\cdot\frac{f'(r)}{r}$ Proved

(19.) In 18, we have:

$\vec{\nabla}^2(f(r)) = f''(r) + 2\cdot\frac{f'(r)}{r}$.

$\Rightarrow$ So: $f''(r) + 2\cdot\frac{f'(r)}{r} = 0$

Let $f'(r) = t \Rightarrow \frac{dt}{dr} + \frac{2t}{r} = 0$

$\Rightarrow \frac{dt}{t} = -2\frac{dr}{r}$

$\Rightarrow \ln|t| = -2\ln|r| + C$

$\Rightarrow t = \frac{a}{r^2} = f'(r) = \frac{d[f(r)]}{dr}$

Let $f(r) = y$:- $\boxed{f(r) = \left(a + \frac{b}{r}\right)}$ Ans

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(20.) (I) $\vec{\nabla}\cdot(\vec{\nabla}\phi) = \vec{\nabla}\cdot(2x\hat{i} + (-2y)\hat{j}) = 2-2 = 0$

(II) $\vec{\nabla}^2\left(\vec{\nabla}\cdot\left(\frac{1}{r^2}\vec{r}\right)\right) = \vec{\nabla}^2\left(\vec{\nabla}\left(\frac{1}{r^2}\right)\cdot\vec{r} + \frac{\vec{\nabla}\cdot\vec{r}}{r^2}\right)$

$= \vec{\nabla}^2\left(\frac{-2}{r^2} + \frac{3}{r^2}\right) = \vec{\nabla}^2\left(\frac{1}{r^2}\right)$

$= \vec{\nabla}\cdot\left(\frac{-2}{r^4}\vec{r}\right) = -2\left[\vec{\nabla}\left(\frac{1}{r^4}\right)\cdot\vec{r} + \frac{\vec{\nabla}\cdot\vec{r}}{r^4}\right]$

$= 2/r^4$ Ans

(III) We know:

$\vec{\nabla}^2(\phi) = \text{div}(\text{grad}(\phi))$

$= \text{div}\left(\vec{\nabla}\left(\frac{1}{r^3}\right)\cdot x + \vec{\nabla}(x)\cdot\frac{1}{r^3}\right)$

$= \text{div}\left(\frac{-3x\vec{r}}{r^5} + \frac{\hat{i}}{r^3}\right)$

$= -3\left(\vec{\nabla}\left(\frac{x}{r^5}\right)\cdot\vec{r} + (\vec{\nabla}\cdot\vec{r})\frac{x}{r^5}\right) + \left(\vec{\nabla}\left(\frac{1}{r^3}\right)\cdot\hat{i} + (\vec{\nabla}\cdot\hat{i})\frac{1}{r^3}\right)$

$\vec{\nabla}^2\phi = -3\left(\frac{3x}{r^5} + \left(\frac{-5}{r^7}\vec{r}\cdot x + \frac{\hat{i}}{r^5}\right)\cdot\vec{r}\right) + \frac{-3}{r^5}x.$

$= \frac{-12x}{r^5} - 3\left(\frac{-5x}{r^5} + \frac{x}{r^5}\right) = 0$ Ans