Intuitionist Implicational logic (IIL)

Definition:-

Implicational formulae are built from propositional variables using the implicational connective "$\to$" : if σ and τ are formulas, then so is $(\sigma \to \tau)$. An infinite sequence of propositional variables (denoted $a, b, c, \ldots$) is assumed to be given. There are no propositional constants and no other connectives than "$\to$". "$\to$" is right-associative.

Defn : IIL has the foll. rules :-

$$(\to E) : \frac{\sigma \to \tau \qquad \sigma}{\tau} \qquad\qquad (\to I) : \frac{\begin{array}{c}[\sigma]\\ \vdots \\ \tau\end{array}}{\sigma \to \tau}$$

Each application of $(\to I)$ is said to discharge (cancel) some, all, or none of the occurrences of σ above τ, and must be accompanied by a discharge label that lists all the occurrences of σ that it discharges. If none are discharged then the appl^n of $\to I$ is vacuous.

Example :

1. A proof of $(a \to a \to c) \to a \to c$

$$\frac{[a \to a \to c]\ (0011) \qquad [a]\ (0012)}{a \to c \quad (001)}\ (\to E)$$

$$\frac{a \to c \quad (001) \qquad [a]\ (002)}{c \quad (00)}\ (\to E)$$

$$\frac{c \quad (00)}{a \to c \quad (0)}\ (\to I)\ \{\text{discharging 'a' at } 0012, 002\}$$

$$\frac{a \to c \quad (0)}{(a \to a \to c) \to a \to c \quad \phi}\ (\to I)\ \{\text{discharging } a \to a \to c \text{ at } 0011\}$$

2. A proof of $(a \to a \to c) \to a \to a \to c$

$$\frac{[a \to a \to c]\ (00011) \qquad [a]\ (00012)}{a \to c \quad (0001)}\ (\to E)$$

$$\frac{a \to c \quad (0001) \qquad [a]\ (0002)}{c \quad (000}\ (\to E)$$

$$\frac{c \quad (000}{a \to c \quad (00)}\ (\to I)\ \{\text{discharging 'a' at position } 00012\}$$

$$\frac{a \to c \quad (00)}{a \to a \to c}\ (\to I)\ \{\text{discharging 'a' at } 0002\}$$

$$\frac{a \to a \to c}{(a \to a \to c) \to a \to a \to c \quad \phi}\ \{\text{discharging } a \to a \to c \text{ at } 00011\}.$$

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The Curry-Howard isomorphism :-

Defn : (C-H mapping from Lamda to logic)

If Δ is a TA_λ-deduction of $\Gamma \mapsto M : \tau$, the corresponding logic deduction Δ_L is defined thus.

(i) $M \equiv x$ and $\Delta : x : \tau \mapsto x : \tau$

Δ_L : just τ

(ii) $M \equiv P\,Q$ and $\Gamma = \Gamma_1 \cup \Gamma_2$ and last step in Δ has the form

$$\dfrac{\begin{matrix} (\Delta_1) & & (\Delta_2) \\ \Gamma_1^{\phi} \mapsto P : \sigma \to \tau & & \Gamma_2 \mapsto Q : \sigma \end{matrix}}{\Gamma_1 \cup \Gamma_2 \mapsto (PQ) : \tau} \; (\to E)$$

let Δ_{1L} correspond to Δ_1
Δ_{2L} " " Δ_2
$\therefore \Delta_L$ ---- " to Δ

Δ_L is obtained by apply $(\to E)$ of IL to Δ_{1L}, Δ_{2L}.

(iii) $M \equiv \lambda x. P$, $\tau \equiv \rho \to \sigma$, $\Gamma = \Gamma' - x$ and the last step in Δ is

$$\dfrac{\begin{matrix} \Delta' \\ \Gamma' \mapsto P : \sigma \end{matrix}}{\Gamma' - x \mapsto (\lambda x. P) : \rho \to \sigma} \; (\to I)$$

Δ_L is obtained from Δ_L' by discharging all occurrences of ρ in Δ_L' whose positions are the same as the positions of the free occurrences of x in P.

Example:

Δ

$$x : a \to a \to c \vdash x : a \to a \to c \qquad z : a \vdash z : a \qquad 1$$

$(\to E)$

$$x : a \to a \to c,\ z : a \vdash (xz) : a \to c \qquad y : a \vdash y : a \qquad 2$$

$(\to E)$

$$x : a \to a \to c,\ z : a,\ y : a \vdash (xz)\,y : c \qquad 3$$

$(\to I)$

$$x : a \to a \to c,\ y : a \vdash \lambda z.(xz)\,y : a \to c$$

dischg z 4

$(\to I)$

$$x : a \to a \to c \vdash \lambda y.\lambda z.(xz)\,y : a \to a \to c$$

dischg y 5

$(\to I)$

$$\vdash \lambda x.\lambda y.\lambda z.(xz)\,y : (a \to a \to c) \to a \to a \to c.$$

dischg x 6

Δ_L

$$a \to a \to c\ [\#] \qquad a\ [*] \qquad 1$$

$$a \to c \qquad a\ [**] \qquad 2$$

$$c$$

dischge a at [*] 3

$$a \to c$$

dischg a at [**] 4

$$a \to a \to c \qquad 5$$

$$(a \to a \to c) \to a \to a \to c$$

dischg a at [#]. 6