

**Indian Institute of Technology Roorkee**  
**MAN-001(Mathematics-I), Autumn Semester: 2022-23**  
**Assignment-5: (Taylor's theorem, Maxima-Minima)**

1. Let  $f(x) = \sin x$ . If  $f(h) = f(0) + hf'(\theta h)$ ,  $0 < \theta < 1$  then show that  $\lim_{h \rightarrow 0} \theta = \frac{1}{\sqrt{3}}$ .
2. Using Taylor's theorem, show that:
  - (a)  $1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2} e^x$ ,  $x > 0$
  - (b)  $x - \frac{x^3}{3!} < \sin x < x$ ,  $x > 0$
  - (c)  $1 + \frac{x}{2} - \frac{x^3}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$ ,  $x > 0$ .
3. Obtain the Taylor's series expansion of the maximum order for the function  $f(x, y) = x^2y + 3y - 2$  about the point  $(1, -2)$ .
4. Obtain Taylor's expansion of  $\tan^{-1} \frac{y}{x}$  about  $(1, 1)$  up to second degree term. Hence compute  $f(1.1, 0.9)$ .
5. Find the linear approximation of the following functions at point  $P_0$ . Also, find the maximum absolute error in this approximation.
  - (a)  $f(x, y) = 2x^2 - xy + y^2 + 3x - 4y + 1$  at  $P_0 = (-1, 1)$  and  $R: |x + 1| < 0.1, |y - 1| < 0.1$ .
  - (b)  $f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$  at  $P_0 = (3, 2)$  and  $R: |x - 3| < 0.1, |y - 2| < 0.1$ .
6. Use Taylor's formula to find a quadratic approximation of  $f(x, y) = \sin x \sin y$  at the origin. How accurate is the approximation if  $|x| \leq 0.1$  and  $|y| \leq 0.1$ ?
7. Prove that the function  $\left(\frac{1}{x}\right)^x$ ,  $x > 0$  has a maximum at  $x = e^{-1}$ .
8. Examine the following functions for local extrema and saddle points:
  - (a)  $xy - x^2 - y^2 - 2x - 2y + 4$
  - (b)  $x^3 + y^3 - 3axy$
  - (c)  $x^2y^2 - 5x^2 - 8xy - 5y^2$
  - (d)  $2(x - y)^2 - x^4 - y^4$
  - (e)  $y \sin x$
9. Find the absolute maximum and minimum values of the following functions  $f(x, y)$  in the closed region  $R$ :
  - (a)  $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ ,  $R$ : triangular plate in the first quadrant bounded by the lines  $x = 0, y = 0, y = 9 - x$ .
  - (b)  $f(x, y) = 3x^2 + y^2 - x$ ,  $R: 2x^2 + y^2 \leq 1$ .
10. Find the extrema of the following functions using the method of Lagrange multipliers:
  - (a)  $f(x, y) = 3x + 4y$  subject to the condition  $x^2 + y^2 = 1$ .
  - (b)  $f(x, y, z) = x^m y^n z^p$  subject to the condition  $x + y + z = a$ .
  - (c)  $f(x, y) = xy$  subject to the condition  $\frac{x^2}{8} + \frac{y^2}{2} = 1$ .
11. Prove that if the perimeter of a triangle is constant, its area is maximum when the triangle is equilateral.

12. Find the shortest distance from the point  $(1, 2, -1)$  to the sphere  $x^2 + y^2 + z^2 = 24$ .
13. The plane  $x + y + z = 1$  cuts the cylinder  $x^2 + y^2 = 1$  in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.
14. Find the quadratic approximation of  $f(x, y) = \sqrt{1 + 4x^2 + y^2}$  at  $(1, 2)$  and use it to compute the approximate value of  $f(1.1, 2.05)$ .
15. Find the maximum and minimum of  $f(x, y) = (x+y)e^{-x^2-y^2}$  on the region defined by  $x^2 + y^2 \leq 1$ .
16. Consider the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $f(0, 0) = 0$  and  $f(x, y) = e^{-\frac{1}{x^2+y^2}}$  when  $(x, y) \neq (0, 0)$ . Show that the  $n$ th Taylor polynomial of  $f$  around  $(0, 0)$  is the zero polynomial for every non-negative integer  $n$ .

## Answers

3.  $f(x, y) = -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2) + (x - 1)^2(y + 2)$ .
4.  $f(x, y) = \frac{\pi}{4} - \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1) + \frac{1}{4}(x - 1)^2 - \frac{1}{4}(y - 1)^2$ , 0.685.
5. (a)  $f(x, y) = -2 - 2(x + 1) - (y - 1)$ ,  $E(x, y) \leq 0.08$ .  
 (b)  $f(x, y) = 8 + 4(x - 3) - (y - 2)$ ,  $E(x, y) \leq 0.04$ .
6.  $f(x, y) = xy$ ,  $E(x, y) \leq 0.0013$ .
8. (a) Maximum value 8 at  $(-2, -2)$ .  
 (b) Maximum at  $(a, a)$  if  $a < 0$ , minimum at  $(a, a)$  if  $a > 0$ , and saddle point at  $(0, 0)$ .  
 (c) Maximum value 0 at  $(0, 0)$ , saddle point at  $(\pm 1, \mp 1)$ ,  $(\pm 3, \pm 3)$ .  
 (d) Maximum value 8 at  $(\pm\sqrt{2}, \mp\sqrt{2})$  and saddle point at  $(0, 0)$ .  
 (e) Saddle point at  $(n\pi, 0)$  for  $n \in I$ .
9. (a) Absolute maximum value 4 at  $(1, 1)$  and absolute minimum value -61 at  $(0, 9)$  and  $(9, 0)$ .  
 (b) Absolute maximum value  $\frac{3+\sqrt{2}}{2}$  at  $(-1/\sqrt{2}, 0)$  and absolute minimum value -1/12 at  $(1/6, 0)$ .
10. (a) Maximum value 5 at  $(3/5, 4/5)$  and minimum value -5 at  $(-3/5, -4/5)$ .  
 (b)  $\frac{m^m n^n p^p a^{m+n+p}}{(m+n+p)^{m+n+p}}$ .  
 (c) Maximum value 2 at  $(\pm 2, \pm 1)$  and minimum value -2 at  $(\mp 2, \pm 1)$ .
12.  $\sqrt{6}$ .
13.  $(-1/\sqrt{2}, -1/\sqrt{2}, 1 + \sqrt{2})$  is farthest from the origin,  $(1, 0, 0)$  and  $(0, 1, 0)$  are closest to the origin.
14. 3.1691
15. Maximum value  $\frac{1}{\sqrt{e}}$  at  $(1/2, 1/2)$  and minimum value  $-\frac{1}{\sqrt{e}}$  at  $(-1/2, -1/2)$ .