Assignment: 3

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(1) let $|f-L_1| < \varepsilon_1$ where $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$
$|g-L_2| < \varepsilon_2$ $\delta_1, \delta_2, \delta_3 > 0$
$|h-L_3| < \varepsilon_3$
when $||(x,y)-(a,b)|| < \delta_i$ $(i=1,2,3)$.

(a) $|(f-g)-(L_1-L_2)| = |(f-L_1)-(g-L_2)|$
$\leq |f-L_1|+|g-L_2| < \varepsilon_1+\varepsilon_2 = \varepsilon_0 > 0$
so when $||(x,y)-(a,b)|| < \delta_0$ then $\varepsilon_0 > 0$ exists.
$\Rightarrow$ function is continous.

(b) Let's prove f^2 is continous.
we already know $(f+g)$ is continous as $(f-g)$ continuity holds.

$\Rightarrow$ $|f^2-L_1^2| = |f+L_1|\cdot|f-L_1| < \varepsilon_1\cdot C = \varepsilon' > 0$
as $\varepsilon_1 > 0$ and $C = |f+L_1| \geq 0$.
so (ε', δ') pair exist $\Rightarrow$ f^2 is continous.

$\Rightarrow$ Now, $f\cdot g = \frac{1}{2}\left[\underbrace{(f+g)^2}_{\text{continous}} - \underbrace{(f^2+g^2)}_{\text{continous}}\right]$

and we have proved in (a) part, that sum/difference of continous functions is continous so $(f\cdot g)$ is also continous.

1. (c) We know:

$$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2}.$$

and we now that $(f+g)$ and $(f-g)$ or $(g-f)$ both are continous so the max. of (f,g) also continous.

(d) $$\min(f,g,h) = \frac{\left(\frac{(f+g) - |(f-g)|}{2} + h\right) - \left(\frac{(f+g) - |f-g|}{2} - h\right)}{2}$$

$\Rightarrow$ obviously, formed sum/difference of continous functions, so it will be continous.

2. (a) $\frac{xy^3}{x+y}$ $\Rightarrow$ no indeterminancy at $(-1,2)$

so: limit $= \frac{(-1)(2)^3}{-1+2} = -8$ Ans

(b) Let path taken be $y = mx^3$

$$\lim_{(x,y)\to(0,0)} \frac{x^3(mx^3)}{x^6 + m^2x^6} = \left(\frac{m}{1+m^2}\right) = \text{depend on } 'm'$$

so limit is not unique

it means $\Rightarrow$ does not exist.

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(c) take path be $y = mx+1$ which is passing through $(0,1)$

$$\Rightarrow \lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{mx+1}{x}\right) = \lim_{(x,y)\to(0,1)} \tan^{-1}\left(m+\frac{1}{x}\right).$$

$$\Rightarrow \text{for } \left.\begin{array}{l} x\to 0^{+} \Rightarrow L = \pi/2 \\ x\to 0^{-} \Rightarrow L = -\pi/2 \end{array}\right\} \text{does not exist.}$$

(d) take path as $y = mx$:

$$\lim_{(x,y)\to(0,0)} \left(\frac{x\cdot(mx)}{\sqrt{x^2(1+m^2)}} = \frac{mx}{\sqrt{1+m^2}}\right) = 0 \ \text{Ans}$$

(e) take $y = mx$:

$$L = \lim_{x\to 0} \frac{\sin^2((m+1)x)}{(1+|m|)|x|} = \lim_{x\to 0} \frac{\sin^2((m+1)x)\cdot((m+1)x)^2}{((m+1)x)^2\,(1+|m|)|x|}$$

$$= 0 \ \text{Ans}$$

(f) take path as $y = x$:-

then: $f(x,x) = \begin{cases} 1 & x \geq 1 \\ -1 & x < 1 \end{cases}$

so obviously does not exist.

(g) take $y = mx$:- $L = \lim_{x\to 0} \dfrac{2mx^3}{(1+m^2)x^2} = 0$ Ans

3. (a) (i) $\lim_{y\to 0} f(x,y) = 1$

(ii) $\lim_{x\to 0} f(x,y) = -1$

} proved the first line.

(iii) $\lim_{(x,y)\to(0,0)} \left(\frac{x+y}{x-y}\right)$.

take $y = mx$:- $\lim_{x\to 0}\left(\frac{1+m}{1-m}\right)$ ⇒ depends on 'm'

so does not exist.

(b) $f(x,y) = \begin{cases} x\sin(1/y) & y \neq 0 \\ 0 & y = 0 \end{cases}$

(i) take observation:-

$\lim_{(x,y)\to(0,0)} \underbrace{x}_{\text{zero approaching}} \cdot \underbrace{\sin(1/y)}_{\text{some no. b/w } [-1,1]} = 0.$ Ans

(ii) actually, the limit inside brackets does not exist : as when we are $y\to 0$ $\sin(\ldots) \in [-1,1]$ ⇒ which is not fixed. and then we take x close to zero but not zero

(iii) $x\to 0$. will make the whole equal to zero. Ans

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(3) (c) (i) $\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = \lim_{x\to 0}\left(\frac{0}{x^2}\right) = 0$. Ans

(ii) $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = \lim_{y\to 0}\left(\frac{0}{y^2}\right) = 0$ Ans

→ both denominators are "tending zero".

(iii) $\lim_{(x,y)\to(0,0)} (f(x,y)) = \lim_{x\to 0} \frac{m^2x^4}{m^2x^4+(1-m)^2x^2}$

$\Rightarrow L = \lim_{x\to 0}\left[\frac{m^2x^2}{m^2x^2+(1-m)^2}\right]$

$\boxed{m=1}$ $(L=1)$ $\boxed{m\neq 1}$ $(L=0)$

So the limit does not exist. Ans

(4.) It is obvious: if we take circle of radius 'δ' around (a,b); it contains both rational as well as irrational points.

- Suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L$ (exists).

$\therefore$ By ϵ-δ definition, for any given $\epsilon>0$, there must exist a $\delta>0$ satisfying $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$, s.t.

$|f(x,y) - L| < \varepsilon$ —— (1.)

- If any rational value of 'x' satisfies the $\sqrt{(x-a)^2+(y-b)^2} < \delta$ inequality then:- eqn. (1) gives:-
  $$|0-L| < \varepsilon \Rightarrow \boxed{L \in (-\varepsilon, \varepsilon)} \text{ —— (2.)}$$

- If any irrational value satisfies:- then: $|1-L| < \varepsilon \Rightarrow \boxed{L \in (1-\varepsilon, 1+\varepsilon)}$ —— (3.)

So solving (2) and (3): for $\varepsilon = 1/2$ will give intersection equal to ϕ. So our assumption is wrong.
Limit does not exist.

(5.) (a) $L = \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}$

$$= \lim_{\substack{x\to 0 \\ y\to 0}} \frac{\sin^{-1}(x+2y)}{(x+2y)} \cdot \underbrace{\frac{(x+2y)}{(2x+4y)}}_{(1/2)} \cdot \underbrace{\frac{2x+4y}{\tan^{-1}(2x+4y)}}_{1}$$

$= (1/2) = f(0,0)$ So function is continous at $(0,0)$.

(i) $\Rightarrow f_x(0,0) = \dfrac{\partial}{\partial x} f(x,y)\Big|_{(0,0)} = \dfrac{d}{dx}(f(x,0))\Big|_{x=0}$

$$= \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2\sin^{-1}(h)-\tan^{-1}(2h)}{(2h)\cdot\tan^{-1}(2h)}$$

$$= \lim_{h\to 0} \underbrace{\frac{2\sin^{-1}(h)-\tan^{-1}(2h)}{(4h^2)}}_{\text{(L-Hopital)}} \cdot \cancel{\frac{2h}{\tan^{-1}(2h)}}^{1}$$

$$= \lim_{h\to 0} \frac{(1+4h^2)-\sqrt{1-h^2}}{(1+4h^2)\sqrt{1-h^2}\,(4h)}$$

apply L-Hopital on remaining:-

$$\Rightarrow f_x(0,0) = L = \lim_{h\to 0} \left(8-\frac{1}{\sqrt{1-h^2}}\right)\cdot\frac{h}{4} = 0. \text{ (exists)}.$$

(ii) $f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k}$

and same as (i) it also exists. and $f_y(0,0)=0$. Ans

(5) (b) • $\lim_{(x,y)\to(0,0)} f(x,y) = 0+0 = 0 = f(0,0).$

$\Rightarrow$ function is continous.

as discussed in 3(b).

• $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$

$= \lim_{h\to 0} \underbrace{\sin(1/h)}_{\text{some number in } [-1,1]}$ = does not exist

Similarly $f_y(0,0)$ does not exist.

(5.) (c) • $x = r\cos\theta$ and $y = r\sin\theta$:-

$$\lim_{(x,y)\to 0} f(x,y) = \lim_{r\to 0} r^2 \sin\theta\cos\theta \log(r^2).$$

$$= \lim_{r\to 0} (\sin(2\theta)) \cdot \left[\frac{\log(r)}{\underbrace{(1/r^2)}_{\text{apply L-Hopital.}}}\right].$$

$$= 0. = f(0,0)$$

So the function is continous.

• $f_x(0,0) = \lim_{x\to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$

• $f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = 0.$

extra 'x' and 'y' outside logarithm make them zero.

So partiable exist.

(5.) (d) • $\lim_{\substack{x\to y \\ (y+\delta = x)}} f(x,y) = \lim_{\delta\to 0} \frac{(y+\delta)^3 + y^3}{(\delta)}$

$$= \lim_{\delta\to 0} [3(\delta+y)^2 . (1)] = 3y^2$$

depends on variable, so limit does not exist.

• $f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(h,0) - f(0,0)}{h}$ $[h = \Delta x]$

$\Rightarrow f_x(0,0) = \lim_{(\Delta x) \to 0} \frac{(\Delta x)^2 - 0}{(\Delta x)} = 0$ (exists)

Ans

• $f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{(\Delta y)}$

$= \lim_{\Delta y \to 0} \frac{-(\Delta y)^2 - 0}{(\Delta y)} = 0$ (exists)

Ans

(6.) (a) $f_x(0,y)$:-

$f_x(0,y) = \frac{\partial}{\partial x} f(x,y)\Big|_{(0,y)} = \lim_{h \to 0} \frac{f(0+h, y) - f(0,y)}{h}$

$= \lim_{h \to 0} \left[\frac{\frac{y(h^2-y^2)}{(h^2+y^2)} + y}{h} \right]$

$= \lim_{h \to 0} \frac{y}{h} \cdot \frac{2h^2}{(h^2+y^2)} = 0$ **Ans**

(b) $f_y(x,0)$:-

$f_y(x,0) = \frac{\partial}{\partial y} f(x,y)\Big|_{(x,0)} = \lim_{k \to 0} \frac{f(x, 0+k) - f(x,0)}{k}$

$= \lim_{k \to 0} \frac{\frac{k(x^2-k^2)}{(x^2+k^2)} - 0}{(k)} = 1$ **Ans**

(0)

(c) $f_x(0,0) = \lim_{\substack{\Delta x \to 0 \\ \text{or} \\ h \to 0}} \frac{f(h,0) - f(0,0)}{h} = 0$ **Ans**

(-k)

(d) $f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = -1$ **Ans**

7. (i) $f_{xy}(0,0) = \frac{\partial}{\partial y}(f_x) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$

$$= \lim_{k \to 0} \frac{\lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} - \lim_{h \to 0} \frac{f(h,0) - }{h}}{k}$$

$$= \lim_{k \to 0} \frac{\frac{-hk - 0}{h} - 0}{k} = -1$$

(ii) $f_{yx}(0,0) = \frac{\partial}{\partial x}(f_y) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$

$$= \lim_{h \to 0} \frac{\lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k} - \lim_{k \to 0} \frac{f(0,}{k}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{hk}{k} - 0}{h} = 1.$$

So both exists, but $f_{yx}(0,0) \neq f_{xy}(0,0)$

proved.

8. # $\lim_{(x,y) \to (0,0)} (|x| + |y|) = 0 \Rightarrow$ (no indeterminancy here)

$\Rightarrow$ can also be proved by (ϵ, δ) pair :-

let for $\delta > 0 \ni \|(x,y) - (0,0)\| < \delta$:-

then we know:

$$\big||x| + |y|\big| < x^2 + y^2 < \delta^2 = \varepsilon$$

so $\varepsilon > 0$ exists for $\delta > 0$ it means function is continous.

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- $\mathcal{E}(H) = \dfrac{f(P+H) - f(P) - A\cdot H}{\sqrt{h^2+k^2}}$ $\begin{bmatrix} h \\ k \end{bmatrix}$

$\Rightarrow \mathcal{E}(H) = \dfrac{f(0+h, 0+k) - f(0,0) - \left[\frac{\partial f}{\partial x}\Big|_{(0,0)} \; \frac{\partial f}{\partial y}\Big|_{(0,0)}\right]}{\sqrt{h^2+k^2}}$

$\Rightarrow \boxed{\mathcal{E}(H) = \dfrac{|h|+|k|}{\sqrt{h^2+k^2}}}$

- So: $\lim\limits_{\substack{H\to 0 \\ \left(\substack{h\to 0 \\ k\to 0}\right)}} \mathcal{E}(H) = \lim\limits_{(h,k)\to(0,0)} \dfrac{|h|+|k|}{\sqrt{h^2+k^2}} = \dfrac{(1+|m|)}{\sqrt{1+m^2}}$
$(k = mh)$

So the function is not differentiable at $(0,0)$ as $\mathcal{E}(H) \not\to 0$ as $H\to 0$.

(9.) (i) $f(h,0) = h$; $f(0,k) = k$ where $h,k \neq 0$.

$\Rightarrow f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = 1$

$f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k} = 1$ } ~~proved~~

(ii) let $H = \begin{pmatrix} h \\ k \end{pmatrix}$; $A = \left[\frac{\partial f}{\partial x}\Big|_{(0,0)} \; \frac{\partial f}{\partial y}\Big|_{(0,0)}\right] = [1 \; 1]$

$\mathcal{E}(H) = \dfrac{f(H) - 0 - [1 \; 1]\begin{bmatrix} h \\ k \end{bmatrix}}{\sqrt{h^2+k^2}} = \dfrac{(h+k)(hk)}{(h^2+k^2)}$

$\Rightarrow \lim\limits_{(h,k)\to(0,0)} \dfrac{(h+k)(hk)}{(h^2+k^2)} = \lim\limits_{r\to 0} \dfrac{r^2 \sin\theta\cos\theta \cdot r(\cos\theta+\sin\theta)}{r^2} = 0.$
(exists)

$\begin{pmatrix} \Delta x = h = r\sin\theta \\ \Delta y = k = r\cos\theta \end{pmatrix}$.

so $\varepsilon(H) \to 0$ as $H \to 0$ $\Rightarrow$ function is differentiable.

(10) (a)
- $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} h^2 \sin\left(\frac{1}{h^2}\right) = 0.$
- $\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} k^2 \sin\left(\frac{1}{k^2}\right) = 0$
- We know that:

$$f(p+H) - f(p) = \overset{0}{\cancel{A}}H + \varepsilon_1 h + \varepsilon_2 k$$

$\Rightarrow$ $h^3 \sin\left(\frac{1}{h^2}\right) + k^3 \sin\left(\frac{1}{k^3}\right) = h \cdot \varepsilon_1 + k \cdot \varepsilon_2.$

$\Rightarrow$ compare: $\varepsilon_1 = h^2 \sin(1/h^2)$; $\varepsilon_2 = k^2 \sin(1/k^2)$

- And as $(h,k) \to (0,0)$, $\begin{pmatrix} \varepsilon_1 \to 0 \\ \varepsilon_2 \to 0 \end{pmatrix}$ $\Rightarrow$ function is differentiable.

(b) $\frac{\partial f}{\partial x} = f_x(x,y) = \begin{cases} -2\cos\left(\frac{1}{x^2}\right) + 3x^2 \sin\left(\frac{1}{x^2}\right), & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$

$\Rightarrow$ $\lim_{(x,y) \to (0,0)} f_x(x,y) =$ does not exist.

(1st term will oscillate).

(c) $\frac{\partial f}{\partial y} = f_y(x,y) = \begin{cases} 3y^2 \sin\left(\frac{1}{y^2}\right) - 2\cos\left(\frac{1}{y^2}\right) & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$

here, 2nd term oscillates when $(x,y) \to (0,0)$ so not continous.

(11.) Given: $f(x,y) = \begin{cases} |xy|^p & xy \neq 0 \\ 0 & xy = 0 \end{cases}$

(a) $\lim_{(x,y)\to(0,0)} |xy|^p = 0$ (for continuity).
$= L$.
when $p = 0 \Rightarrow L = 1$
when $p > 0 \Rightarrow L = 0$ $\Rightarrow$ Ans
when $p < 0 \Rightarrow L =$ does not exist

(b) $\epsilon(H) = \dfrac{|xy|^p}{\sqrt{x^2+y^2}}$; $\left\{ \dfrac{\partial f}{\partial x}\Big|_{(0,0)} = \dfrac{\partial f}{\partial y}\Big|_{(0,0)} = 0 \right\}$.

$\Rightarrow$ For differentiability: $L_1 = \lim_{(x,y)\to(0,0)} \epsilon(H) = 0$.

$\Rightarrow$ take $y = mx$; path $\Rightarrow L = \lim_{x\to 0} \dfrac{|m|^p \cdot (x^{2p})}{\sqrt{1+m^2}\cdot|x|}$.

So it has become (a) part like,
so $2p - 1 > 0 \Rightarrow \boxed{p > 1/2}$ Ans.

(12.) (a) $\dfrac{\partial f}{\partial x}\Big|_{(0,0)} = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} h\cdot\cos\left(\dfrac{1}{h}\right) = 0$.

$\dfrac{\partial f}{\partial y}\Big|_{(0,0)} = \lim_{k\to 0} k\cdot\cos\left(\dfrac{1}{k}\right) = 0$.

$\Rightarrow \epsilon(H) = \sqrt{(h^2+k^2)}\cos\left(\dfrac{1}{\sqrt{h^2+k^2}}\right) = 0$. when $(h,k)\to(0,0)$

So function is differentiable at $(0,0)$

(b) $\frac{\partial f}{\partial x} = f_x(x,y) = \begin{cases} 2x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) + \frac{x}{\sqrt{x^2+y^2}}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y)\neq 0 \\ 0 & (x,y)=0. \end{cases}$

$\Rightarrow \lim_{(x,y)\to(0,0)} f_x(x,y) =$ doesnot exist (2^{nd} term oscillates).

(look $y=mx$ path and take a look).

- Similarly $f_y(x,y)$ is also discontinous.

(13.) (b) take $f(x,y) = \frac{x^2y}{x^2+y^2}$.

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{mx^3}{(1+m^2)x^2} = 0$ (exists).

function is continous.

$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0 = \left.\frac{\partial f}{\partial y}\right|_{(0,0)}$.

$\Rightarrow \varepsilon(H) = \frac{h^2k}{(h^2+k^2)^{3/2}}$ $\Rightarrow$ take path as $k=mx$; $\Rightarrow$ will depend on path or value of m

So function $\to$ not differentiable.

(14.) (a) $f(0,0)=0$; $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{\sqrt{h^2}}{h} =$ doesnot exist.

So not differentiable.

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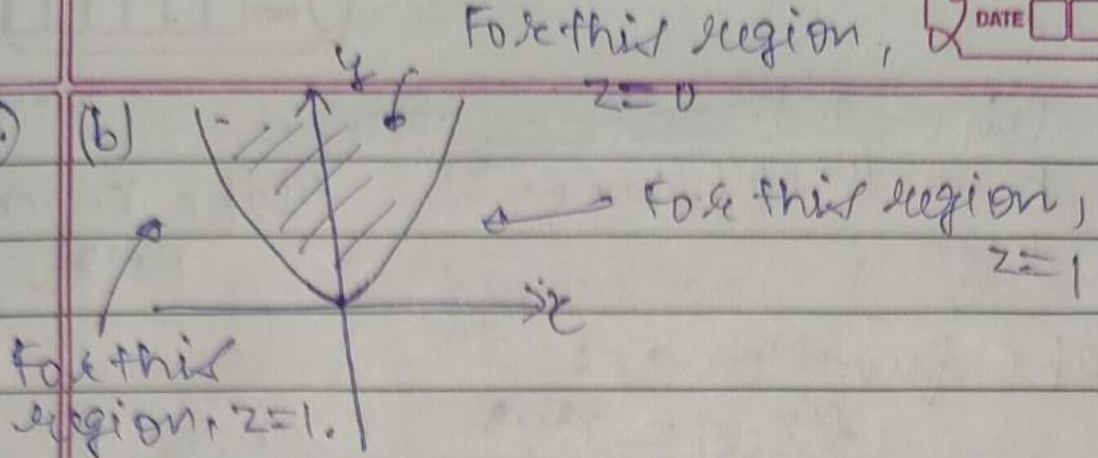
(14.) (b)



$\Rightarrow$ Visualise in 3d, For all the points on boundary conditions of two regions, there is discontinuity as well as non-differentiability

(14.) (c) $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} [\ldots] = 0 = \left.\frac{\partial f}{\partial y}\right|_{(0,0)}$

$\Rightarrow \varepsilon(H) = \frac{h^2 K}{(h^2+K^2)^{3/2}}$ $\Rightarrow$ same as (13) (b).

Non-differentiable ~~Ans~~

(14.) (d) $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0 = \left.\frac{\partial f}{\partial y}\right|_{(0,0)}$. and $f(0,0) = 0$

$\varepsilon(H) = \frac{(x^2+y^2)}{\sqrt{x^2+y^2}} \Rightarrow H \to 0 \Rightarrow \varepsilon(H) \to 0.$

So it is differentiable ~~Ans~~

(14.) (e) Clearly: $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$

$\Rightarrow \varepsilon(H) = \frac{h^2 K^2}{(h^4+K^4)\sqrt{h^2+K^2}}$.

$\therefore$ $L = \lim_{\substack{h\to 0\\ k\to 0}} \frac{h^2k^2}{(h^4+k^2)\sqrt{h^2+k^2}} = 0.$ (take $k=mx$ path).

(13.) (a) take function as:

$f(x,y) = \frac{xy}{x^4+y^2}$ (given in (11)(e)).

This is differentiable and partials also exist.

but is discontinous at $(0,0)$.