Orthogonal curvilinear coordinate system

(summary)

1. $$ds^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}\, du_i\, du_j = \sum_{i=1}^{3} (h_i\, du_i)^2 .$$

where : g_{ij} = metric = diagonal matrix

$$= \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}_{3\times 3}.$$

$\Rightarrow$ For $i=j \Rightarrow \boxed{g_{ii} = h_i^2} > 0$ ← compare the above eqn.

$\Rightarrow$ For $i \neq j \Rightarrow \boxed{g_{ij} = 0}$

2. $$\{A_i(u_i)\} \xrightarrow{u \to \bar{u}} (\bar{A}_i(\bar{u}_i))$$

$\Rightarrow$ where $A_i(u)$ means $A_i(u_1, u_2, u_3)$ means A_i as a function of u_1, u_2, u_3.
where A_1, A_2, A_3 are components of a vector in curvilinear coordinates.

Remember this transformation is "homogenous".
It means that any component in $\bar{u}$ system will depend on all component, in u system, of a vector $\vec{A}$.

$$\boxed{\bar{A}_i(\bar{u}) = \sum_{j=1}^{3} \frac{\partial \bar{u}_i}{\partial u_j} A_j(u_1, u_2, u_3)}$$

3. $$\vec{\nabla} = \frac{\hat{a}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{a}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{a}_3}{h_3}\frac{\partial}{\partial u_3} = \sum_{i=1}^{3} \frac{\hat{a}_i}{h_i}\frac{\partial}{\partial u_i}.$$

$$\Rightarrow \boxed{\vec{\nabla} = \underbrace{\sum_{i=1}^{3} \frac{\hat{a}_i}{h_i}\frac{\partial}{\partial u_i}}_{\text{(curvilinear)}} = \underbrace{\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}_{\text{(cartesian)}}}.$$

(a) $\vec{\nabla}\phi = \sum_{i=1}^{3} \frac{\hat{q}_i}{h_i} \frac{\partial\phi}{\partial x_i}$ = grad ϕ = gradient of ϕ.

where ϕ = scalar, invariant.

(b) $\vec{\nabla}\cdot\vec{A}$ = Divergence of $\vec{A}$ = div $\vec{A}$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(A_1 h_2 h_3) + \frac{\partial}{\partial x_2}(A_2 h_1 h_3) + \frac{\partial}{\partial x_3}(A_3 h_1 h_2)\right].$$

\# If $\vec{A} = \vec{\nabla}\phi$ = grad ϕ = gradient of ϕ :-

\# $\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{\nabla}\phi = \underset{|||}{\vec{\nabla}^2}\phi$. = Laplacian on scalar field.

(Laplacian operator)

\# So: we are doing $\left(A_i \to \frac{1}{h_i}\frac{\partial\phi}{\partial x_i}\right)$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial x_3}\right)\right]$$

Ans

(c) $\nabla\times\vec{A} = \vec{\nabla}\times\vec{A}$ = curl of $\vec{A}$

$$= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{q}_1 & h_2\hat{q}_2 & h_3\hat{q}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Ans