this is different from what is used in lambda calculus

Intuitionist Implicational logic (IIL)

Definition :-

Implicational formulae are built from propositional variables using the implicational connective "$\rightarrow$" : if σ and τ are formulas, then so is $(\sigma \rightarrow \tau)$. An infinite sequence of propositional variables (denoted $a, b, c, \ldots$) is assumed to be given. There are no propositional constants and no other connectives than "$\rightarrow$". "$\rightarrow$" is right-associative.

Defn : IIL has the foll. rules :-

[] indicates that it has been discharged

$$(\rightarrow E) : \frac{\sigma \rightarrow \tau \qquad \sigma}{\tau} \qquad\qquad (\rightarrow I) : \frac{\begin{matrix}[\sigma] \\ \vdots \\ \tau\end{matrix}}{\sigma \rightarrow \tau}$$

Each application of $(\rightarrow I)$ is said to discharge (cancel) some, all, or none of the occurrences of σ above τ, and must be accompanied by a discharge label that lists all the occurrences of σ that it discharges. If none are discharged then the appl^n of $\rightarrow I$ is vacuous.

Example :

1. A proof of $(a \rightarrow a \rightarrow c) \rightarrow a \rightarrow c$

$[a \rightarrow a \rightarrow c]$ (0011) $\quad [a]$ (0012) $\quad (\rightarrow E)$

$a \rightarrow c$ (001) $\quad [a]$ (002) $\quad (\rightarrow E)$

c (00) $\quad (\rightarrow I)$

$a \rightarrow c$ (0) $\quad (\rightarrow I)$ {discharging 'a' at 0012, 002}

$(a \rightarrow a \rightarrow c) \rightarrow a \rightarrow c$ $\quad \phi$ {discharging $a \rightarrow a \rightarrow c$ at 0011}

don't take the result type. Take all other than result, and derive the result somehow.

Here, result is c

2. A proof of $(a \rightarrow a \rightarrow c) \rightarrow a \rightarrow a \rightarrow c$

$[a \rightarrow a \rightarrow c]$ (00011) $\quad [a]$ (00012) $\quad (\rightarrow E)$

$a \rightarrow c$ (0001) $\quad [a]$ (0002) $\quad (\rightarrow E)$

c (000) $\quad (\rightarrow I)$

$a \rightarrow c$ (00) $\quad$ {discharging 'a' at position 00012}

$(\rightarrow I)$

$a \rightarrow a \rightarrow c$ $\quad$ {discharging 'a' at 0002}

$(a \rightarrow a \rightarrow c) \rightarrow a \rightarrow a \rightarrow c$ $\quad \phi$ {discharging $a \rightarrow a \rightarrow c$ at 00011}.

the proof of an IIL formula looks like construction tree of a term having type as IIL formula.

25.4.2025.

The Curry-Howard isomorphism :-

Defn : (C-H mapping from lamda to logic)

If Δ is a TA_λ-deduction of $\Gamma \mapsto M : \tau$, the corresponding logic deduction Δ_L is defined thus.

(i) $M \equiv x$ and $\Delta : x : \tau \mapsto x : \tau$

Δ_L : just τ

(ii) $M \equiv PQ$ and $\Gamma = \Gamma_1 \cup \Gamma_2$ and last step in Δ has the form

$$\dfrac{\begin{matrix}(\Delta_1) \\ \Gamma_1 \mapsto P : \sigma \to \tau\end{matrix} \qquad \begin{matrix}(\Delta_2) \\ \Gamma_2 \mapsto Q : \sigma\end{matrix}}{\Gamma_1 \cup \Gamma_2 \mapsto (PQ) : \tau} \ (\to E)$$

let Δ_{1L} correspond to Δ_1

Δ_{2L} " " Δ_2

$\therefore \Delta_L$ ---- " to Δ

Δ_L is obtained by apply $(\to E)$ of SIL to Δ_{1L}, Δ_{2L}.

(iii) $M \equiv \lambda x . P$, $\tau \equiv \rho \to \sigma$, $\Gamma = \Gamma' - x$ and the last step in Δ is

$$\dfrac{\begin{matrix}\Delta' \\ \Gamma' \mapsto P : \sigma\end{matrix}}{\Gamma' - x \mapsto (\lambda x . P) : \rho \to \sigma} \ (\to I)$$

Δ_L is obtained from Δ_L' by discharging all occurrences of ρ in Δ_L' whose positions are the same as the positions of the free occurrences of x in P.

and in vacous, there is no discharge.

Example:

Δ

$x : a \to a \to c \vdash x : a \to a \to c \qquad z : a \vdash z : a$ 1

$\to E$ $\quad x : a \to a \to c,\ z : a \vdash (xz) : a \to c \qquad y : a \vdash y : a$ 2

$(\to E)$ $\quad x : a \to a \to c,\ z : a,\ y : a \vdash (xz)\, y : c$ 3

$(\to I)$ $\quad x : a \to a \to c,\ y : a \vdash \lambda z.(xz)\, y : a \to c$ discharge z 4

$(\to I)$ $\quad x : a \to a \to c \vdash \lambda y.\lambda z.(xz)\, y : a \to a \to c$ discharge y 5

$(\to I)$ $\quad \vdash \lambda x.\lambda y.\lambda z.(xz)\, y : (a \to a \to c) \to a \to a \to c$. discharge x 6

Δ_L

$a \to a \to c$ [#] $\qquad a$ [*] 1

$a \to c \qquad a$ [**] 2

c discharge a at [*] 3

$a \to c$ discharge a at [**] 4

$a \to a \to c$ 5

$(a \to a \to c) \to a \to a \to c$ discharge a at [#]. 6

z is coming from [*], and hence discharge z only at that position.

it is one proof, there may be more than one.