Defn: logic to lambda mapping. $\Delta \to \Delta_\lambda$

(i) Δ : single formula τ any term variable x.

Then Δ_λ : $x:\tau \mapsto x:\tau$

(ii) Last step in Δ is $(\to E)$ applied to the conclusions of deductions Δ' and Δ''.

let $\Delta'_\lambda : \Gamma' \mapsto M : \sigma \to \tau$ $\quad \Delta''_\lambda : \Gamma'' \mapsto N : \sigma$

Then replace all term variables in Δ''_λ by distinct new ones so that there is no common term variable in Δ'_λ and Δ''_λ.

Now apply $\to E$ to obtain Δ_λ.

(iii) if the last step in Δ is an occurrence of $(\to I)$ with form

$$\begin{array}{l} [\rho] \\ \vdots \\ \underline{\sigma} \\ \rho \to \sigma \end{array} \left.\right\} \text{a deduction } \Delta'$$

{ discharge $K \geq 0$ occurrences of $\underline{\rho_1}, \dots, \underline{\rho_K}$ of ρ }.

$\underline{K \geq 1}$ let $\Delta^{*\prime}_\lambda$: $\Gamma, v_1:\rho, \dots v_K:\rho \mapsto P:\sigma$

(v_i's are distinct)

replace all v_i's by a new term variable x to obtain

$\Gamma, x:\rho \mapsto P^*:\sigma$ $\quad P^* \equiv P[x/v_1 \dots x/v_K]$

Apply $\to$ Imain to obtain $\boxed{\Gamma \mapsto (\lambda x. P^*) : \rho \to \sigma}$: Δ_λ

$\underline{K=0}$ $\Delta^{*\prime}_\lambda$: $\Gamma \mapsto P : \sigma$

Choose a new variable x not in $\Delta^{*\prime}_\lambda$.

and apply $\to$ I vac to obtain.

Δ_λ : $\boxed{\Gamma \mapsto (\lambda x . P) : \rho \to \sigma}$

Proof of $a \to a \to a$:-

#1: Δ_1

$$\dfrac{\dfrac{[a]\ (00)}{a \to a\ (0)}\ (\to I)\ \{\text{disch. } a \text{ vac.}\}}{a \to a \to a\ \phi}\ (\to I)\ \{\text{disch. } a \text{ at } 00\}$$

$(\Delta_1)_\lambda$:

$$\dfrac{\dfrac{x : a \mapsto x : a}{x : a \mapsto (\lambda y . x) : a \to a}\ (\to I)\text{vac}\ \ y : a}{\mapsto \underline{(\lambda x . \lambda y . x)} : a \to a \to a}\ (\to I)\text{main}$$

#2. Δ_1

$$\dfrac{\dfrac{[a]\ (00)}{a \to a\ (0)}\ (\to I)\ \{\text{disch. } a \text{ at } 00\}}{a \to a \to a}\ (\to I)\ \{\text{disch. } a \text{ vac.}\}.$$

$(\Delta_2)_\lambda$:

$$\dfrac{\dfrac{x : a \mapsto x : a}{\mapsto (\lambda x . x) : a \to a}\ (\to I)\text{main}}{\mapsto \underline{(\lambda y . \lambda x . x)} : a \to a \to a}\ (\to I)\text{vac.}$$

the terms are different.

All the assumptions must be discharged.

$$\sigma \equiv \left[\dfrac{\dfrac{\sigma}{\sigma}}{\sigma \to \sigma}\right]\ \{\text{disch. } \sigma \text{ vac.}\}$$

$$\sigma \equiv \left[\dfrac{\dfrac{\sigma\ (00)}{\sigma}}{\sigma \to \sigma}\right]\ \{\text{disch. } \sigma \text{ at } 00\}$$

$$\dfrac{\begin{matrix}\sigma\ (00) \\ \vdots \\ \tau\end{matrix}}{\sigma \to \tau}\ \{\text{disch } \sigma \text{ at } 00\}$$

Proof of $a \to a$

#. Δ_1 : $\dfrac{[a]\ (00)}{a \to a}\ (\to I)\ \{\text{disch } a \text{ vac.}\}$ $\longleftrightarrow$ $(\Delta_1)_\lambda$: $\dfrac{x : a \mapsto x : a}{x : a \mapsto (\lambda y . x) : a \to a}\ (\to I)\text{vac}\ \ y : a$

Δ_2 : $\dfrac{[a]\ (00)}{a \to a}\ (\to I)\ \{\text{disch } a \text{ at } 00\}$ $\longleftrightarrow$ $(\Delta_2)_\lambda$: $\dfrac{x : a \mapsto x : a}{\mapsto (\lambda x . x) : a \to a}\ (\to I)\text{main}$

Disch. occurrences at top must be marked as [ ].

Observations :-

1. There can be more than one (different) λ-terms that have same type.
   i.e, $M_1 : \tau$ and $M_2 : \tau$ is possible.

2. ~~There cannot be two distinct deductions of a λ-term.~~
   So ~~$M : \tau$ and $M : \tau'$~~ is not possible. (i.e. $\tau = \tau'$.)

3. ~~There cannot be two distinct proofs of a type.~~

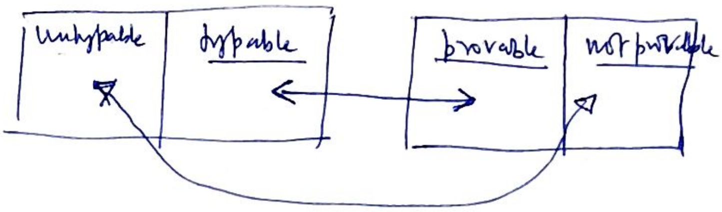
4. From a proof of τ we can obtain only one λ-term M
   with $M : \tau$

5. If τ is not provable then there is no λ-term M
   ~~for~~ with $M : \tau$.
   i.e τ cannot be assigned to any λ-term.

6. M is typable $\Leftrightarrow$ τ is provable
   (with type τ)

C-H isomorphism

1. provable formulae $\longleftrightarrow$ types of closed terms.
2. logic proofs $\longleftrightarrow$ TA_λ - proofs
3. logic deductions $\longleftrightarrow$ TA_λ - deductions.



$\tau = ((a \to b) \to a) \to a$ is not provable in IIL.
(Pierce's law)

Example 1 logic to lambda

Δ :

$$\dfrac{\dfrac{a \to a \to c \quad a}{a \to c}(\to E) \quad a}{c}(\to E)$$

Δ_λ :

$$\dfrac{\dfrac{x : a \to a \to c \mapsto x : a \to a \to c \quad \dfrac{}{y : a \mapsto y : a}\text{(axiom)}}{x : a \to a \to c, y : a \mapsto (xy) : a \to c}(\to E) \quad z : a \mapsto z : a}{x : a \to a \to c, y : a, z : a \mapsto xyz : c}(\to E)$$

Example 2 logic to lambda $\quad \rho = (a \to a \to c) \to a \to a \to c$

Δ :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a \to a \to c]\ (00011) \quad [a]\ (00012)}{a \to c\ (0001)}(\to E) \quad [a]\ (0002)}{c\ (000)}(\to E)}{a \to c\ (00)}(\to I)\ \{\text{discharging } a \text{ at } 00012\}}{a \to a \to c\ (0)}\ \{\text{discharg } a \text{ at } 0002\}}{(a \to a \to c) \to a \to a \to c\ \ \phi}\ \{\text{dischg } [a \to a \to c] \text{ at } (0011)\}.$$

Δ_L :

$$\dfrac{x : a \to a \to c \mapsto x : a \to a \to c \quad \dfrac{}{y : a \mapsto y : a}}{x : a \to a \to c, y : a \mapsto xy : a \to c}(\to E) \quad \dfrac{}{z : a \mapsto z : a}(\to E)$$

$$\dfrac{\dfrac{\dfrac{x : a \to a \to c, y : a, z : a \mapsto xyz : c}{x : a \to a \to c, z : a \mapsto \lambda y . xyz : a \to c}(\to I)\text{main}}{x : a \to a \to c \mapsto \lambda z . \lambda y . xyz : a \to a \to c}(\to I)\text{main}}{\mapsto \lambda x . \cancel{z} . \lambda y . xyz : (a \to a \to c) \to a \to a \to c.}(\to I\ \text{main})$$

$\boxed{\text{proof structure same but discharge labels different, so the } \lambda\text{-terms } \text{~~became~~ are different.}}$

Example 3 logic to lambda. ρ is same as in Example 2.

Δ :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a \to a \to c]\ (00011) \quad [a]\ (00012)}{a \to c\ (0001)}\to E \quad [a]\ (0002)}{c\ (000)}\to E}{a \to c\ (00)}(\to I)\ \{\text{discharging } a \text{ at } 00012, 0002\}}{a \to a \to c\ (0)}(\to I)\ \{\text{dischg } a \text{ vac.}\}}{(a \to a \to c) \to a \to a \to c\ \ \phi}(\to I)\ \{\text{dischg. } a \to a \to c \text{ at } 00011\}$$

Δ_L :

$$\dfrac{x : a \to a \to c \mapsto x : a \to a \to c \quad y : a \mapsto y : a}{x : a \to a \to c, y : a \mapsto xy : a \to c}(\to E) \quad z : a \mapsto z : a\ (\to E)$$

$(z' : a \mapsto z' : a)$

$(z' : a \mapsto z' : a)$

$[z'/z, z'/y]$

$$\dfrac{\dfrac{\dfrac{x : a \to a \to c, y : a, z : a \mapsto xyz : c \quad (z' : a,\ xz'z' : c)}{x : a \to a \to c \mapsto \lambda z' . xz'z' : a \to c}(\to I)\text{main}}{x : a \to a \to c \mapsto \lambda y' . \lambda z' . xz'z' : a \to a \to c}(\to I)\text{vac}}{\mapsto \lambda x . \lambda y' . \lambda z' . xz'z' : (a \to a \to c) \to a \to a \to c.}\to In.$$

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