Property

The subject-construction theorem: 9.4.2025.

Let Δ be a TA_λ-deduction of a formula $\Gamma \mapsto M : \tau$.

(i) If we remove from each formula in Δ everything except its subject, Δ changes to a tree of terms which is exactly the construction-tree for M.

(ii) If M is an atom, $M \equiv x$, then $\Gamma = \{x : \tau\}$ and Δ contains only one formula, namely the axiom

$$x : \tau \mapsto x : \tau$$

(iii) If $M \equiv PQ$ the last step in Δ must be an application of $(\to E)$ to two formulae with form

$$\Gamma \restriction P \mapsto P : \sigma \to \tau \qquad \Gamma \restriction Q \mapsto Q : \sigma \quad \text{for some } \sigma.$$

restriction subjects$(\Gamma) = FV(P)$

(iv) If $M \equiv \lambda x . P$ then τ must have form $\rho \to \sigma$

if $x \in FV(P)$ the last step in Δ must be an application of $(\to I)_{main}$ to

$$\Gamma, x : \rho \mapsto P : \sigma$$

if $x \notin FV(P)$ the last step in Δ must be an application of $(\to I)_{vac}$ to

$$\Gamma \mapsto P : \sigma$$

Deductions in TA_λ may not be unique

Example :-

$$\Delta_M \quad \dfrac{\dfrac{\dfrac{y : a \mapsto y : a}{\mapsto (\lambda y . y) : a \to a}(\to I)}{\mapsto (\lambda x . \lambda y . y) : (\sigma \to \sigma) \to (a \to a)}(\to I) \qquad \dfrac{z : \sigma \mapsto z : \sigma}{\mapsto (\lambda z . z) : \sigma \to \sigma}(\to I)}{\mapsto (\lambda x . \lambda y . y)(\lambda z . z) : a \to a}(\to E)$$

here $M \equiv (\lambda x . \lambda y . y)(\lambda z . z)$ $\tau \equiv a \to a$ $\Gamma = \phi$

here σ can be anything and this makes the Δ not unique.

(Property)

Uniqueness of deductions for normal forms.

Let M be a β-nf and Δ a TA_λ-deduction of $\Gamma \mapsto M:\tau$.

Then (i) every type in Δ has an occurrence in τ or in a type in Γ,

(ii) Δ is unique, i.e., if Δ' is also a deduction of $\Gamma \mapsto M:\tau$ then $\Delta' \equiv \Delta$.

Subject reduction and expansion (Property)

If P has type τ we can think of P as being in some sense "safe". If P represents a stage in some computation which continues by β-reducing P then all later stages in the computation are also "safe". (Unsafe means mismatch of types.)

Subject-reduction theorem :-

If $\Gamma \vdash_\lambda P:\tau$ and $P \triangleright_\beta Q$ then $\Gamma \vdash_\lambda Q:\tau$

$\vdash_\lambda$ means there is a deduction of $\langle \Gamma, P, \tau \rangle$ in TA_λ.

~~Exact~~ Proof :-

$P \equiv (\lambda x.M)N \qquad Q \equiv M[N/x]$

Let $x \in FV(M)$, then by the subject-construction theorem the lower steps of Δ must have the form

$$\cfrac{\cfrac{\Gamma_1,\ x:\sigma \mapsto M:\tau}{\Gamma_1 \mapsto (\lambda x.M):\sigma\to\tau}\,(\to I)\text{ main} \qquad \Gamma_2 \mapsto N:\sigma}{\Gamma_1 \cup \Gamma_2 \mapsto \underbrace{((\lambda x.M)N)}_{Q} : \tau} \quad Q:\tau$$

Now $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\text{Subjects}(\Gamma) = FV(P)$

so we have a deduction for $\Gamma \mapsto P:\tau$

but $(\lambda x.M)N \triangleright_\beta Q$. i.e $P \triangleright_\beta Q$.

So we also have a deduction for $\Gamma \mapsto Q:\tau$. $\square$.

Subject ~~expansion~~ theorem :-

If $\Gamma \vdash_\lambda Q:\tau$ and $P \triangleright_\beta Q$ [*] then $\Gamma \vdash_\lambda P:\tau$.

[*] by non-duplicating and non-cancelling contractions.

The above condition [*] is very important. Removing it will make the conclusion false.

1. Let M be a λ-term. Let $CT(M)$ be ~~a~~ the construction tree for M.
Let $S_{p\text{-}\ell}^{M}$ be the set of all the pairs of labels-position in the $CT(M)$.

S_{ℓ}^{M} be the set obtained from $S_{p\text{-}\ell}^{M}$ by removing the labels.
i.e. S_{ℓ}^{M} contains only the positions.

Problem :- 1.1. Given $S_{incomplete} \subsetneq S_{\ell}^{M}$. Construct a unique M corresponding to S_{ℓ}^{M}.

1.2. Find a minimum sized set $S_{incomplete}$ so that a unique M can be constructed from the set.

2. Let $\Sigma = \{0, 1, 2\}$

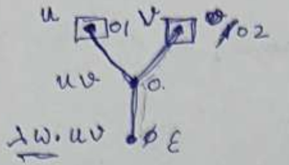
2.1. ~~Let~~ given a regular expression R_{Σ} obtain the structure of λ-term M.

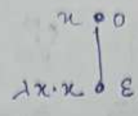
2.2. Suggest types of regular expressions that are meaningful w.r.t. λ-terms.

2.3. Suggest types of regular expressions that are not meaningful w.r.t. λ-terms.

- Extension of Construction-Tree
- Problems related to Construction tree.

4.3.2025.





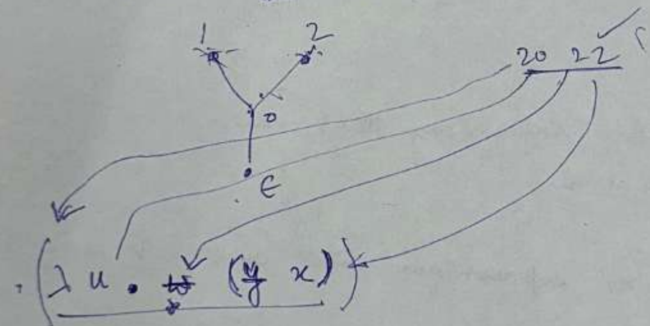
u : 01   v : 02

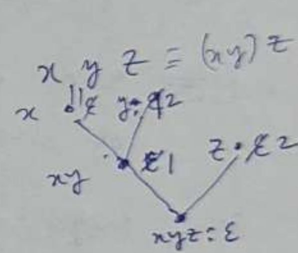
λw. (uv)

0   1 2

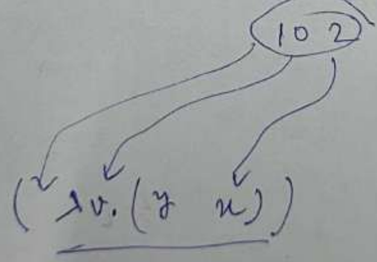
{ε, 0, 01, 02}

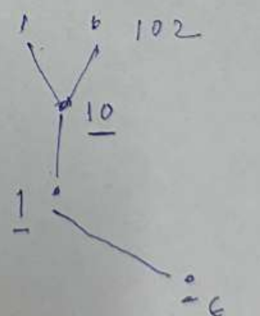
{ε, —, 01, —}





0+





x • ε

λx. λy. x