Assignment-10

(1.) $$I = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 x^2\,dx + \int_0^1 (-\cancel{z})\,dy + \int_0^1 dz + \int_{C_4} \vec{F}\cdot d\vec{r}.$$

$\Rightarrow$ For curve C_4:

$$\frac{z-1}{0} = \frac{x}{1} = \frac{y}{1} = t \Rightarrow \vec{r} = t\hat{i} + t\hat{j} + \hat{k}$$

$$\frac{d\vec{r}}{dt} = (\hat{i} + \hat{j}).$$

Hence,

$$\Rightarrow I = \frac{1}{3} + 0 + 1 + \int_1^0 (x^2 - xz)\,dt$$

$$= \frac{4}{3} + \int_0^1 (t - t^2)\,dt = \frac{4}{3} + \frac{1}{6} = \frac{9}{6}$$

Ans

(2.) $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2\cos x + z^3) & (2y\sin x - 4) & (3xz^2 + 2) \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(3z^2 - 3z^2) + \hat{k}(2y\cos x - 2y\cos x)$$

$$= 0$$

$\Rightarrow$ Hence, $\vec{F}$ is irrotational and conservative vector field

(2.)

$\Rightarrow$ Let $\vec{F} = \vec{\nabla}\phi \Rightarrow \frac{\partial\phi}{\partial x} = (y^2\cos x + z^3)$

then: $\phi = y^2\sin x + z^3x + c(y,z)$

where c is the function of y and z

$\Rightarrow \frac{\partial\phi}{\partial y} = 2y\sin x + \frac{\partial c}{\partial y} = 2y\sin x - 4$

$\Rightarrow \boxed{c = -4y + c_1(z)}$

where c_1 is the function of z.

$\Rightarrow \frac{\partial\phi}{\partial z} = 3xz^2 + \frac{\partial c_1}{\partial z} = 3xz^2 + 2 \Rightarrow c_1 = 2z + c_2$

where c_2 = constant.

Hence,

$\boxed{\phi = y^2\sin x + xz^3 - 4y + 2z + K}$. Ans

Work done $= \phi(\text{final}) - \phi(\text{initial})$

$= (1 + 4\pi + 4 + 4 + K) - (-4 - 2 + C)$

$= (4\pi + 15)$ Ans

(3.) (i) $\vec{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \Rightarrow \frac{d\vec{r}}{d\theta} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$.

$\Rightarrow I_1 = \int_{\pi/2}^{2\pi}\left(\vec{F}\cdot\frac{d\vec{r}}{d\theta}\right)d\theta = \int_{\pi/2}^{2\pi} 1\cdot d\theta = \frac{3\pi}{2}$ Ans

(3.) (ii) $\vec{r} = t\hat{i} + (1-t)\hat{j} \Rightarrow \frac{d\vec{r}}{dt} = (\hat{i} - \hat{j})$

$\Rightarrow I_2 = \int_0^1 \frac{-1}{2t^2 - 2t + 1}dt = \left(\frac{-1}{2}\right)\frac{1}{(1/2)}\left(\tan^{-1}\left(\frac{2t-1}{1}\right)\right)_0^1$

$= -\pi/2$ Ans

(1)(iii) $\vec{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \Rightarrow \frac{d\vec{r}}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\Rightarrow I_3 = \int_{\pi/2}^{0} 1 \cdot d\theta = -\frac{\pi}{2}$ Ans

(3.) $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \hat{k}\left(\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} + \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\right)$$

$$= 0$$

Hence $\vec{F}$ is conservative.

$\Rightarrow \vec{F} = \vec{\nabla}\phi \Rightarrow \frac{\partial\phi}{\partial x} = \frac{-y}{x^2+y^2} \Rightarrow \phi = \frac{+y}{y}\tan^{-1}\left(\frac{y}{x}\right) + c(y,z).$

where c is the function of y and z.

$\Rightarrow \frac{\partial\phi}{\partial y} = \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(\frac{1}{x}\right) + \frac{\partial c}{\partial y} = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial c}{\partial y} = 0.$

means c is some constant. $= k$

$\Rightarrow \frac{\partial\phi}{\partial z} = 0 \Rightarrow \frac{\partial c}{\partial z} = 0.$

Hence:

$$\boxed{\phi = \tan^{-1}\left(\frac{y}{x}\right) + k}$$ Ans

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4. (i) $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{2\sqrt{x^2+y^2+z^2}} = (x\hat{i}+y\hat{j}+z\hat{k})$

$\Rightarrow I = \iint xyz\,|\hat{i}+\hat{j}+\hat{k}|^2 ds = \iint (3xyz)\, ds$

$I = \iint (3xyz) \frac{dy\,dz}{(\hat{n}\cdot\hat{i})} = \iint 3yz\, dy\, dz$

[take projection y-z plane].

$\Rightarrow I = \int_0^{\pi/2}\int_0^1 3r^3 \sin\theta\cos\theta \frac{r\,dr\,d\theta}{\cancel{r}}$

$= 3\times\left(\frac{1}{4}\right)\times\left(\frac{1}{2}\right)\frac{(2)}{2} = 3/8$ Ans

(ii) $\hat{n} = \frac{2(x\hat{i}+y\hat{j})}{2\sqrt{x^2+y^2}} = \left(\frac{x\hat{i}+y\hat{j}}{4}\right)$.

$\Rightarrow I = \iint \left(\frac{zx}{4}+\frac{xy}{4}\right)\frac{dy\,dz}{(x/4)} = \iint (y+z)\,dy\,dz.$

Take projection on y-z plane, become a rectangle: $z\in[0,5]$ and $y\in[0,4]$.

So:

$I = \int_0^5\int_0^4 y\,dy\,dz + \int_0^5\int_0^4 z\,dy\,dz$

$= 8\times5 + \frac{4\times25}{2} = 50+40 = 90$

Ans

(4.) (III) $\hat{n} = \dfrac{2(x\hat{i}+y\hat{j}+z\hat{k})}{2\sqrt{x^2+y^2+z^2}} = \left(\dfrac{\vec{r}}{a}\right)$.

$\Rightarrow I = \iint \left(\dfrac{1}{ar}\right) \dfrac{dy\,dz}{(x/a)} = \iint \left(\dfrac{dy\,dz}{rx}\right)$.

$\Rightarrow$ Convert to polar coordinates:

$$I = \int_0^{2\pi}\int_0^{a} \frac{r_1 dr_1 d\theta}{r\cdot(r_1\cos\theta)}$$

But better approach is this:-

$$I = \iint \frac{1}{ar}\,ds = \iint \frac{1}{ar}\cdot a^2 d\Omega$$ (solid angle)

at $r=a$: $I = \iint d\Omega = 4\pi$ **Ans**

(5.) $I = \iiint (4z + xz^2 + 3)\,dz\,\underbrace{dx\,dy}_{r\,dr\,d\theta}$.

$$= \int_0^{2\pi}\left(\int_0^{4}\left(\int_{\sqrt{x^2+y^2}}^{4} (4z+xz^2+3)\,dz\right) r\,dr\right) d\theta$$

$$= \int_0^{2\pi}\left(160 + \frac{4^6}{18}\cos\theta\right) d\theta$$

$= 320\pi$ **Ans**

(6.) We know that:

$$\iiint_D x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz = \frac{a^l b^m c^n}{\alpha\beta\gamma}\,\frac{\Gamma\left(\frac{l}{\alpha}\right)\cdot\Gamma\left(\frac{m}{\beta}\right)\cdot\Gamma\left(\frac{n}{\gamma}\right)}{\left(\Gamma\left(\frac{l}{\alpha}\right)+1+\Gamma\left(\frac{m}{\beta}\right)+\Gamma\left(\frac{n}{\gamma}\right)\right)}$$

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where $D \equiv \left[(x) \left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} + \left(\frac{z}{c}\right)^{\gamma} \leq 1 \right]$.

$\therefore$ here $\alpha = \beta = \gamma = 1$

$l = 3,\ m = 2,\ n = 1$

$a = 2,\ b = 4,\ c = 8.$

$\therefore\ I = 45 \times \frac{8 \times 16 \times 8}{(1)} \cdot \frac{\Gamma(3) \cdot \Gamma(2) \cdot \Gamma(1)}{\Gamma(3+2+1+1)}$

$= 45 \times 64 \times 16 \cdot \frac{2 \times 1 \times 1}{6!}$

$= \left(\frac{45 \times 64 \times 16}{60 \times 6}\right) = 128$ Ans

(7.) According to stokes theorem:

$I = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{s} = \oint_C \vec{F} \cdot d\vec{r}$

[as here surface is closed, we may apply it.]

where C is: $(x^2 + y^2 = a^2)$.

$\therefore\ \vec{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j}$ $\therefore\ \frac{d\vec{r}}{d\theta} = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$

$\therefore\ I = \int_0^{2\pi} \left(\vec{F} \cdot \frac{d\vec{r}}{d\theta}\right) d\theta = \int_0^{2\pi} (-a^3 \sin^3\theta + a^2 \sin\theta\cos\theta)\, d\theta$

$= a^2 \cdot \int_0^{2\pi} \frac{\sin(2\theta)}{2} d\theta - a \int_0^{2\pi} \sin^3\theta\, d\theta$

$0 \qquad 0$

$= 0$ Ans

(8)

(9) $\iint (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} = I$

where C is $x^2+y^2=4;\ z=0$

so: $\vec{r} = 2\cos\theta\,\hat{i} + 2\sin\theta\,\hat{j}$

$\therefore\ I = \int_0^{2\pi} (-2x\sin\theta + 2x^3\cos\theta)\,d\theta$

$= 2\cdot\left(\int_0^{2\pi} 8\cos^4\theta\,d\theta - 2\int_0^{2\pi} \cancel{\sin\theta\cos\theta\,d\theta}^{\,0}\right)$

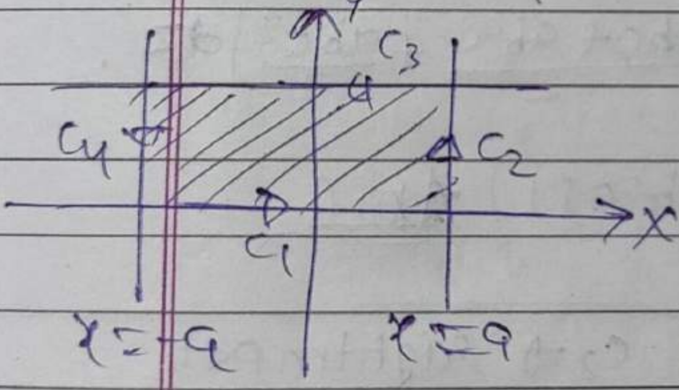
$= 16\cdot\times 4\int_0^{\pi/2} \cos^4\theta\,d\theta = (16\times 4)\times\left(\frac{3\pi}{16}\right)$

$= 12\pi$ Ans

(10) Stokes theorem:-

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS}$$

$\therefore$ L.H.S $= \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r}$



$= \int_{-a}^{+a} x^2\,dx + \int_0^b -2ay\,dy + \int_a^{-a} (x^2+b^2)\,dx + \int_b^0 2ay\,dy$

$= \frac{2a^3}{3} - \frac{2ab^2}{2} + \left(-\frac{2a^3}{3}\right) + b^2(-2a) + 2a\left(-\frac{b^2}{2}\right)$

$\therefore\ \boxed{\text{L.H.S} = -4ab^2}$ —— (1)

$$R.H.S = \int_0^b \int_{-a}^{a} (\vec{\nabla} \times \vec{F}) \cdot \hat{k} \, dx \, dy = \int_0^b \int_{-a}^{a} -4y \, dx \, dy$$

$$= -4ab^2 \quad \text{Ans}$$

Hence proved

(12.) For a closed surface S:-

$$I = \iint_S (\vec{F} \cdot \vec{n}) \, ds = \iiint_V (\vec{\nabla} \cdot \vec{F}) \, dV.$$

$$\Rightarrow \text{So: } I = \int_0^c \int_0^b \int_0^a 2(x+y+z) \, dx \, dy \, dz$$

$$= 2 \cdot \int_0^c \int_0^b \left(\frac{a^2}{2} + (y+z)a \right) dy \, dz.$$

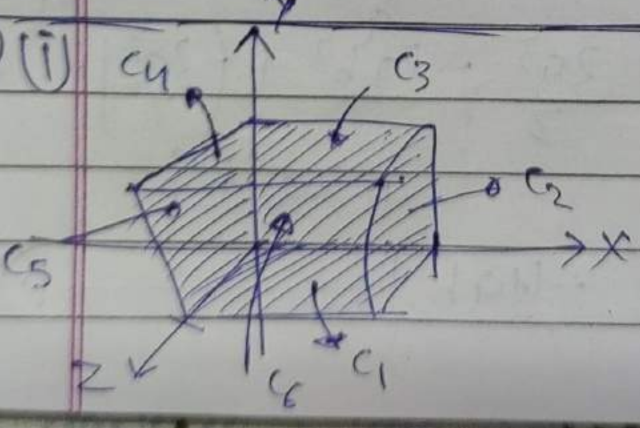
$$= 2 \cdot \int_0^c \left[\frac{a^2 b}{2} + a\left(\frac{b^2}{2} + bz \right) \right] dz.$$

$$= 2 \cdot \int_0^c \left(\left(\frac{a^2 b}{2} + \frac{ab^2}{2} \right) + (abz) \right) dz$$

$$= 2 \cdot \left(\frac{a^2bc + ab^2c}{2} + \frac{abc^2}{2} \right)$$

$$= abc(a+b+c) \quad \text{Ans}$$

(14) (i)



$C_2 \Rightarrow$ rightmost
$C_3 \Rightarrow$ backward (xy-pl)
$C_4 \Rightarrow$ upward plane
$C_5 \Rightarrow$ leftmost
$C_6 \Rightarrow$ front face
$C_1 =$ bottommost.

→ Gauss-divergence theorem:

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,d\vec{s} = \iiint_V (\vec{\nabla}\cdot\vec{F})\,dV.}$$

→ Calculating R.H.S:

$$R.H.S = \int_0^1\int_0^1\int_0^1 (2-x^2+8xz)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 (2-x^2+8xz)\,dx\,dz$$

$$= \int_0^1 \left(2-\frac{1}{3}+\frac{8z}{2}(1)\right) dz$$

$$= \left(\frac{5}{3}+2\right) = \frac{11}{3}. \quad \text{——— (1)}$$

→ $$L.H.S = \left(\iint_{C_1} + \iint_{C_2} + \dots \iint_{C_6}\right)(\vec{F}\cdot d\vec{s})$$

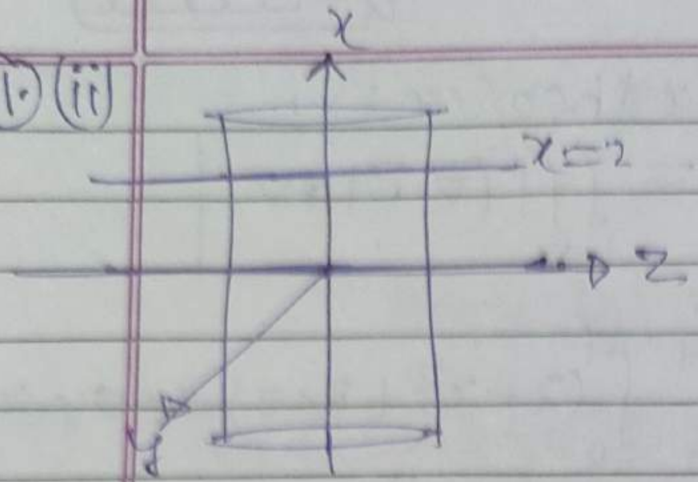
$$= \underbrace{\int_0^1\int_0^1 (x^2y)\,dx\,dz}_{(C_1)} + \underbrace{\int_0^1\int_0^1 (2x-z)\,dy\,dz}_{(C_2)} + \underbrace{\int_0^1\int_0^1 (-x^2y)\,dx\,dz}_{(C_4)}$$

$$+ \underbrace{\int_0^1\int_0^1 (z-2x)\,dy\,dz}_{(C_5)} + \underbrace{\int_0^1\int_0^1 (-4xz^2)\,dy\,dx}_{(C_3)} + \underbrace{\int_0^1\int_0^1 (4xz^2)\,dy\,dx}_{(C_6)}$$

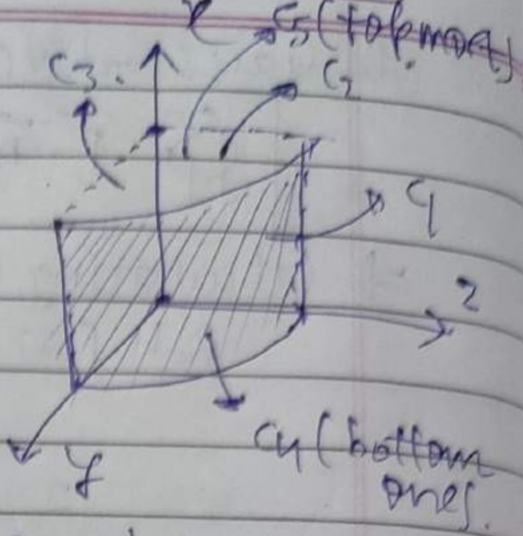
$$= 0 + \frac{3}{2} + \left(\frac{-1}{3}\right) + \left(\frac{1}{2}\right) + 0 + 2 = 4 - \frac{1}{3} = \frac{11}{3}$$

Hence, proved

11. (ii)



$\equiv$



$$\Rightarrow R.H.S = \iiint (4xy - 2y + 8xz)\, dx\, dy\, dz$$

$$= \int_0^{3}\int_0^{\sqrt{9-z^2}}\left(\int_0^{2} (4xy - 2y + 8xz)\, dx\right) dy \cdot dz.$$

$$= \int_0^{\pi/2}\int_0^{3} (8r\cos\theta - 4r\cos\theta + 16 r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{3} r^2 dr \cdot (4\cos\theta + 16\sin\theta)\, d\theta$$

$$= (9\times 4)\cdot\int_0^{\pi/2} (\cos\theta + 4\sin\theta)\, d\theta$$

$$= 36\times(1+4) = 180. \quad \text{——— (1)}$$

$$\Rightarrow L.H.S = \left(\oiint_{C_1} + \cdots \oiint_{C_5}\right) \vec{F}\cdot d\vec{S}$$

$$= (I_1 + I_2 + I_3 + I_4 + I_5)$$

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⇒ $I_2 = 0 = I_4 = I_3$

⇒ $I_5 = \iint (2x^2 y)\, dy\, dz = 8 \cdot \int_0^3 \int_0^{\sqrt{9-z^2}} y\, dy\, dz$

$= \left(\frac{8}{2}\right) \cdot \int_0^3 (9 - z^2)\, dz$

$= 4 \cdot (27 - 9) = 72$

⇒ $I_1 = \iint \vec{F} \cdot \vec{ds}$, convert this into polar form (cylinder)

⇒ $\left.\begin{aligned} y &= \rho \cos\theta \\ z &= \rho \sin\theta \\ x &= x \end{aligned}\right\}$ $I_1 = \int \int \vec{F} \cdot \left[(\rho\, d\theta\, dx) \cdot \hat{a}_\rho \right]$

as $\rho = 3 =$ constant and $\hat{a}_\rho = \frac{1}{3}(3\cos\theta\, \hat{j} + 3\sin\theta\, \hat{k})$

$I_1 = \int_0^2 \int_0^{\pi/2} (-9\cos^3\theta + 36x\sin^3\theta)(3\, d\theta\, dx)$

$= \int_0^2 27 \cdot \left(\int_0^{\pi/2} (4\sin^3\theta\, x - \cos^3\theta)\, d\theta \right) dx$

$= 27 \cdot \int_0^2 \left((4x)\left(\frac{2}{3}\right) - \left(\frac{2}{3}\right) \right) dx$

$= (18 \times 6) = 108.$

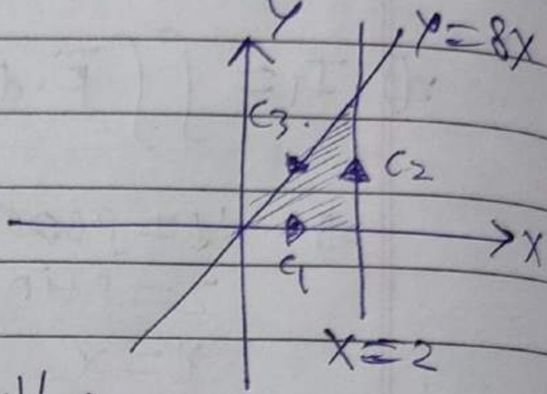
So. $\sum_{i=1}^{5} I_i = 108 + 72 + 0 = 180$ Ans

Hence, proved

(8). We know that: Green's theorem is:

$$\oint_C (f\,dx + g\,dy) = \iint_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

where S is the closed surface.



(I) $\text{L.H.S} = \left(\oint_{C_1} + \int_{C_2} + \int_{C_3}\right)$

$$= 0 + \int_0^{16} (4y+3)\,dy$$

$$+ \int_2^0 \left(x(64x^2) - 2x(8x)\right)dx + \left(x^2(8x)+3\right)(8\,dx).$$

$$= 0 + (2(256) + 48) + \int_2^0 (128x^3 - 16x^2 + 24)\,dx$$

$$= 512 + 48 + \frac{128}{4}(-16) - \frac{16}{3}(-8) + 24(-2)$$

$$= 512 - 512 + \left(\frac{128}{3}\right) = \left(\frac{128}{3}\right).$$

$$\text{R.H.S} = \iint \left(2xy - (2xy - 2x)\right) dx\,dy$$

$$= \int_0^{16}\int_{y/8}^{2} (2x\,dx)\,dy$$

$$= \int_0^{16} \left(4 - \frac{y^2}{64}\right) dy = 64 - \frac{1}{64} \times \frac{1}{3}\, 16^3$$

$$= 128/3$$

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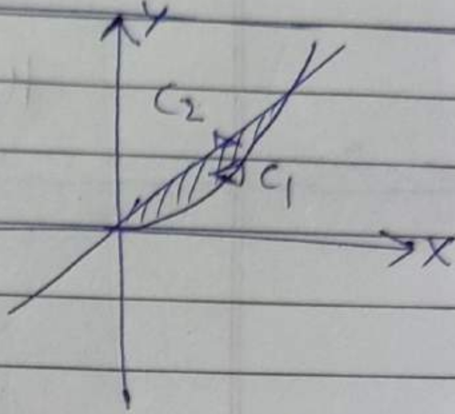
⇒ So R.H.S = L.H.S, hence, proved.

(8) (II) L.H.S $= \left(\int_{C_1} + \int_{C_2}\right)$



$= \int_0^1 [(x^3+x^4)dx + x^2(2x\,dx)]$

$+ \int_1^0 ((x^2+x^2)dx + x^2dx)$

$= \int_0^1 (3x^3+x^4)dx + \int_1^0 (3x^2dx) = -\frac{1}{20}$ — (i)

(II) R.H.S $= \int_0^1\int_{x^2}^{x} (2x-(x+2y))(dy\,dx)$

$= \int_0^1\int_{x^2}^{x} (x-2y)\,dy\,dx = \int_0^1 x(x-x^2)-(x^2-x^4)\,dx$

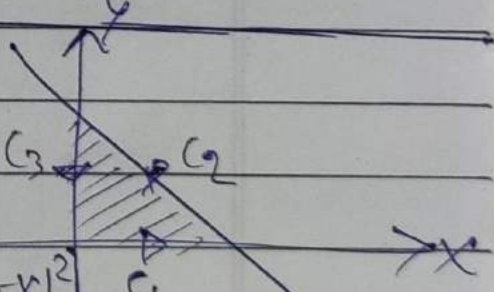
$= \int_0^1 (-x^3+x^4)\,dx$

Hence, R.H.S = L.H.S $= -\frac{1}{4}+\frac{1}{5} = -\frac{1}{20}$

proved

8. (III)

⇒ L.H.S $= \int_{C_1} + \int_{C_3} + \int_{C_2}$



$= \int_0^1 3x^2dx + \int_1^0 4y\,dy + \int_1^0 [(3x^2-8(1-x)^2)\,dx + (4(1-x)-6x(1-x))(-dx)]$

$= 1-2+8/3 = 5/3$ — (i)

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8 (III) $R.H.S = \int_0^1 \int_0^{1-x} (10xy)\,dy\,dx = 5/3.$

Hence, proved