End-Term Examination, Spring Semester 2022-23

PHN - 006: Quantum Mechanics and Statistical Mechanics

Duration: 180 minutes   Max. Marks: 70   Weightage: 50%

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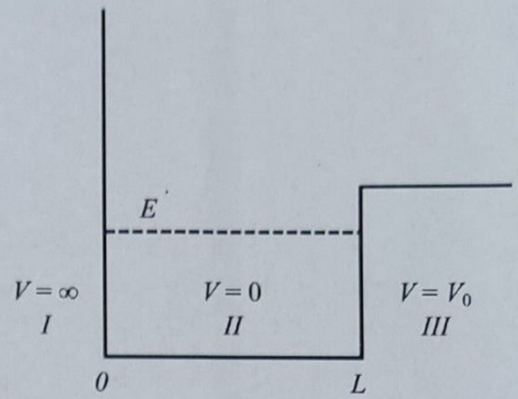
1. (a) The angular frequency of the surface waves in a liquid is given by $\omega = \sqrt{gk + \frac{Tk^3}{\rho}}$, where g is the acceleration due to gravity, k is the wavenumber, ρ is the density of the liquid, and T is the surface tension. Find the phase and group velocities for the limiting cases when the surface waves have: (i) very large wavelengths and (ii) very small wavelengths. [5]

   (b) Show that the zero-point energy of a quantum linear harmonic oscillator is a manifestation of the uncertainty principle. [5]

2. (a) A particle of mass m moves in a one-dimensional potential given by,

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ 0 & \text{for } 0 \le x \le L \\ V_0 & \text{for } x > L. \end{cases}$$

   (i) Obtain the appropriate wave functions in the three regions. (ii) Using the boundary conditions, find the transcendental equation satisfied by the energy eigenvalue $E < V_0$ [3+4]



   (b) A particle of mass m is subjected to the potential given by,

$$V(x) = \begin{cases} \infty & \text{for } x \le 0 \\ \frac{1}{2}m\omega^2 x^2 & \text{for } x > 0. \end{cases}$$

   Find the energy eigenvalues. [5]

   (c) How many electrons can an f subshell occupy? [3]

3. (a) Find the probability that an electron in the ground state of a hydrogen atom can be found beyond the Bohr radius. [5]

   (b) Draw a vector diagram showing an electron's spin angular momentum, $\boldsymbol{S}$, and its possible z-components along a chosen z-axis. What angles can the vector $\boldsymbol{S}$ make with the z-axis? [3]

   (c) Consider the motion of electrons in a one-dimensional periodic potential of a metal crystal. Using the Kronig-Penny model equation given by, $\frac{P \sin Ka}{Ka} + \cos Ka = \cos ka$, where $K = \sqrt{\frac{2mE}{\hbar^2}}$, discuss the origin of energy bands and forbidden bands in solids, qualitatively. Show that for $P << 1$, the energy of the lowest energy band is $E = \frac{\hbar^2 P}{ma^2}$. [4+3]

4. A system consists of a one-dimensional simple harmonic quantum oscillator. Consider an ensemble of such systems in contact with a heat reservoir at temperature T.

   (a) Show that the partition function Z of the system can be expressed as $Z = \left(2\sinh\frac{\hbar\omega}{2kT}\right)^{-1}$. [6]

   (b) Show that the average energy of the oscillator is $\frac{\hbar\omega}{2}\coth\frac{\hbar\omega}{2kT}$. [4]

Note that the hyperbolic functions are defined as $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, and $\coth x = \cosh x/\sinh x$.

5. (a) The density of the metallic sodium is $9.7 \times 10^2\,\mathrm{kg/m^3}$. Compute the average energy of free electrons at $T = 0\,\mathrm{K}$. The atomic weight of sodium is 23. [4]

(b) Using the distribution function $f(\epsilon)$ for identical and indistinguishable particles of odd half integral spins, show that [4]

$$\int_0^{\epsilon_F} f(\epsilon)d\epsilon = kT \ln\left[\frac{1 + e^{\epsilon_F/kT}}{2}\right].$$

(c) Show further that [2]

$$\int_0^{\infty} f(\epsilon)d\epsilon = kT \ln 2 + \int_0^{\epsilon_F} f(\epsilon)d\epsilon.$$

6. (a) A system consisting of seven identical particles has ten accessible energy states. Calculate the entropy of the system if the particles follow (i) the Maxwell-Boltzmann distribution, (ii) the Fermi-Dirac distribution, and (iii) the Bose-Einstein distribution. [2+2+2]

(b) Consider an assembly of N_1 atoms in state 1 and N_2 atoms in state 2, all in thermal equilibrium at the temperature T with radiation of frequency ν and energy density $u(\nu)$. Obtain the relation between Einstein's A and B coefficients of spontaneous and stimulated emission of radiation, respectively. [4]