Indian Institute of Technology Roorkee

MAN-001(Mathematics-1), Autumn Semester: 2022-23

Assignment-4: (Euler's theorem, Chain Rule, Jacobian)

1. Let $u = \cos^{-1}\left(\dfrac{x^2+y^2}{\sqrt{x}-\sqrt{y}}\right)$. Then, by using Euler's theorem, prove that

   (i) $2x\dfrac{\partial u}{\partial x} + 2y\dfrac{\partial u}{\partial y} + 3\cot u = 0$. (ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + y^2\dfrac{\partial^2 u}{\partial y^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{3}{2}(\csc^2 u + \cot u)$

2. If $z = x^m f\left(\frac{y}{x}\right) + x^n g\left(\frac{y}{x}\right)$, then show that

$$x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

3. If $f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log_e x - \log_e y}{x^2+y^2}$, prove that, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$.

4. Use the chain rule to compute $\frac{du}{dt}$ if

   (i) $u = \sin(x^2+y^2),\ x = t^2+3,\ y = t^3$.

   (ii) $u = \tan^{-1}\left(\frac{y}{x}\right),\ x = e^t - e^{-t},\ y = e^t + e^{-t}$.

   (iii) $u = x^2+y^2+z^2$ and $x = e^{2t},\ y = e^{2t}\cos 3t,\ z = e^{2t}\sin 3t$.

5. Use the chain rule to compute $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ for $z = \cos(x^3+y^3),\ x = st,\ y = t^2+s^2$.

6. Find the first order partial derivatives of z with respect to x and y if $xy+yz+xz = 1$.

7. If $z = e^x\sin y + e^y\cos x$, where x and y are implicit functions of t defined by $x^3+x+e^t+t^2+t-1 = 0$ and $yt^3+y^3t+t+y = 0$, then find $\frac{dz}{dt}$ at $t = 0$.

8. Find the values of n so that the function $v = r^n(3\cos^2\theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial v}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial \theta}\right) = 0$$

9. If $v = v(r)$, where $r^2 = \sum_{i=1}^{n} x_i^2$, show that

$$\sum_{i=1}^{n}\frac{\partial^2 v}{\partial x_i^2} = \frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r}\frac{\partial v}{\partial r}$$

10. Let $w = f(u,v)$ satisfy the Laplace equation $w_{uu} + w_{vv} = 0$. If $u = \frac{x^2-y^2}{2}$ and $v = xy$, then show that w also satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

11. Compute the Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$, where $x = \rho\sin\theta\cos\phi,\ y = \rho\sin\theta\sin\phi,\ z = \rho\cos\theta$.

12. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ where $x = \sqrt{2}u - \sqrt{\frac{2}{3}}v,\ y = \sqrt{2}u + \sqrt{\frac{2}{3}}v$

13. Check whether the functions are functionally dependent or not? If yes, then find a relation between them.

    (i) $f(x,y) = \log x - \log y,\ g(x) = \frac{x^2+3y^2}{2xy}$ (ii) $f(x,y) = \frac{y}{x},\ g(x) = \frac{x-y}{x+y}$

14. Show that the following functions satisfy the necessary condition for functional dependence

$$u = x + y + z, \quad v = x^2 + y^2 + z^2, \quad w = x^3 + y^3 + z^3 - 3xyz$$

Also find a relation among u, v, w.

15. Let $f(x, y) = e^x \cos y$ and $g(x, y) = x + \ln(\cos y)$ for $x \in \mathbb{R}$ and $0 < y < \pi$. Then, find the Jacobian $\dfrac{\partial(f, g)}{\partial(x, y)}$, and use it to check whether the functions are functionally dependent or not. If yes, then find a relation between them.

16. Find the Jacobian $\dfrac{\partial(u, v)}{\partial(s, t)}$ where $u = 3x + 2y$, $v = x - 2y$, $x = st$ and $y = s^2 + t^2$.

17. If $x_1 = u_1(1 - u_2),\ x_2 = u_1u_2(1 - u_3),\ x_3 = u_1u_2u_3(1 - u_4),\ x_4 = u_1u_2u_3u_4$, then prove that

$$\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_1, u_2, u_3, u_4)} = u_1^3u_2^2u_3$$

18. If the roots of equation $(t - x)^3 + (t - y)^3 + (t - z)^3 = 0$ in t are u, v, w, then show that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(x - y)(y - z)(z - x)}{(u - v)(v - w)(w - u)}$$

19. If $x = r\cos\theta, y = r\sin\theta$, prove that

(i) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$ (ii) $\left(\frac{\partial^2 r}{\partial x^2}\right) \cdot \left(\frac{\partial^2 r}{\partial y^2}\right) = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2$

(iii) $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$

20. If $u = \log_e(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x + y + z)^2}$$

Answers

4. (i) $4xt\cos(x^2 + y^2) + 6yt^2\cos(x^2 + y^2)$, (ii) $-2/(e^{2t} + e^{-2t})$, (iii) $8e^{4t}$.

5. $-3(tx^2 + 2y^2s)\sin(x^3 + y^3)$, $-3(sx^2 + 2y^2t)\sin(x^3 + y^3)$.

6. $\frac{\partial z}{\partial x} = -\frac{y+z}{x+y}$, $\frac{\partial z}{\partial y} = -\frac{x+z}{x+y}$.

7. -2

8. $n = 2, -3$

11. $\rho^2 \sin\theta$

12. $\frac{4}{\sqrt{3}}$

13. (i) dependent. $f(x, y) = \log(g(x, y)) + \sqrt{(g(x, y))^2 - 3})$

(ii) dependent. $f(x, y) = \frac{1-g(x,y)}{1+g(x,y)}$

14. $w = \frac{u(3v-u^2)}{2}$

15. 0, dependent, $\ln(f(x, y)) - g(x, y) = 0$

16. $16s^2 - 16t^2$