Quicksort

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Sorting Review

- Insertion Sort
  - $T(n) = Q(n^2)$
  - In-place
- Merge Sort
  - $T(n) = Q(n \lg(n))$
  - Not in-place
- Heap Sort
  - $T(n) = Q(n \lg(n))$
  - In-place

Sorting

- Assumptions
  1. No knowledge of the keys or numbers we are sorting on.
  2. Each key supports a comparison interface or operator.
  3. Sorting entire records, as opposed to numbers, is an implementation detail.
  4. Each key is unique (just for convenience).

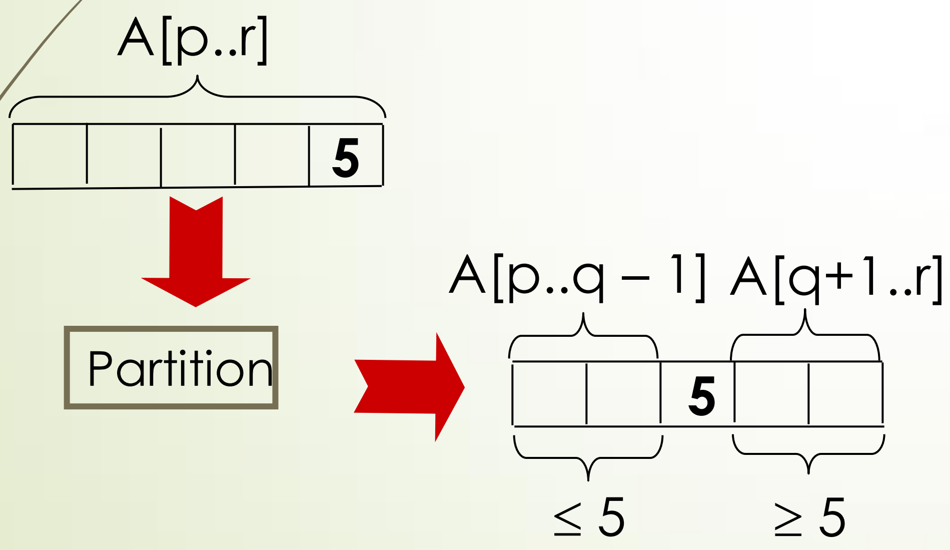
Comparison Sorting

QuickSort Design

- Follows the **divide-and-conquer** paradigm.
- ***Divide:*** Partition (separate) the array $A[p..r]$ into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$.
  - Each element in $A[p..q-1] < A[q]$.
  - $A[q] <$ each element in $A[q+1..r]$.
  - Index q is computed as part of the partitioning procedure.
- ***Conquer:*** Sort the two subarrays by recursive calls to quicksort.
- ***Combine:*** The subarrays are sorted in place – no work is needed to combine them.

Pseudocode

```
Quicksort(A, p, r)
    if p < r then
        q := Partition(A, p, r);
        Quicksort(A, p, q – 1);
        Quicksort(A, q + 1, r)
```




```
Partition(A, p, r)
    i = p – 1, x, := A[r];
    for j := p to r – 1 do
        if A[j] ≤ x then
            i := i + 1;
    A[i + 1] ↔ A[r];
    return i + 1
```

Example

initially: p 2 5 8 3 9 4 1 7 10 **6** r — i j

note: pivot (x) = 6

next iteration: 2 5 8 3 9 4 1 7 10 **6** — i at 2, j at 5

next iteration: 2 5 8 3 9 4 1 7 10 **6** — i at 5, j at 8

next iteration: 2 5 8 3 9 4 1 7 10 **6** — i at 5, j at 3

next iteration: 2 5 3 8 9 4 1 7 10 **6** — i at 3, j at 9

```
Partition(A, p, r)
    x, i := A[r], p – 1;
    for j := p to r – 1 do
        if A[j] ≤ x then
            i := i + 1;
            A[i] ↔ A[j]
    A[i + 1] ↔ A[r];
    return i + 1
```

Example (Continued)

next iteration:	2	5	3	8	9	4	1	7	10	**6**
			i		j					
next iteration:	2	5	3	8	9	4	1	7	10	**6**
			i			j				
next iteration:	2	5	3	4	9	8	1	7	10	**6**
				i			j			
next iteration:	2	5	3	4	1	8	9	7	10	**6**
					i			j		
next iteration:	2	5	3	4	1	8	9	7	10	**6**
					i				j	
next iteration:	2	5	3	4	1	8	9	7	10	**6**
					i					j
after final swap:	2	5	3	4	1	**6**	9	7	10	8
					i					j

Partition(A, p, r)
    x, i := A[r], p – 1;
    for j := p **to** r – 1 **do**
        if A[j] $\leq$ x **then**
            i := i + 1;
            A[i] $\leftrightarrow$ A[j]
    A[i + 1] $\leftrightarrow$ A[r];
    return i + 1

Partitioning

- Select the last element $A[r]$ in the subarray $A[p..r]$ as the *pivot* – the element around which to partition.
- As the procedure executes, the array is partitioned into four (possibly empty) regions.

1. $A[p..i]$ — All entries in this region are **< *pivot***.
2. $A[i+1..j-1]$ — All entries in this region are **> *pivot***.
3. $A[r]$ = *pivot*.
4. $A[j..r-1]$ — Not known how they compare to *pivot*.

- The above hold before each iteration of the *for* loop, and constitute a *loop invariant*. (4 is not part of the loopi.)

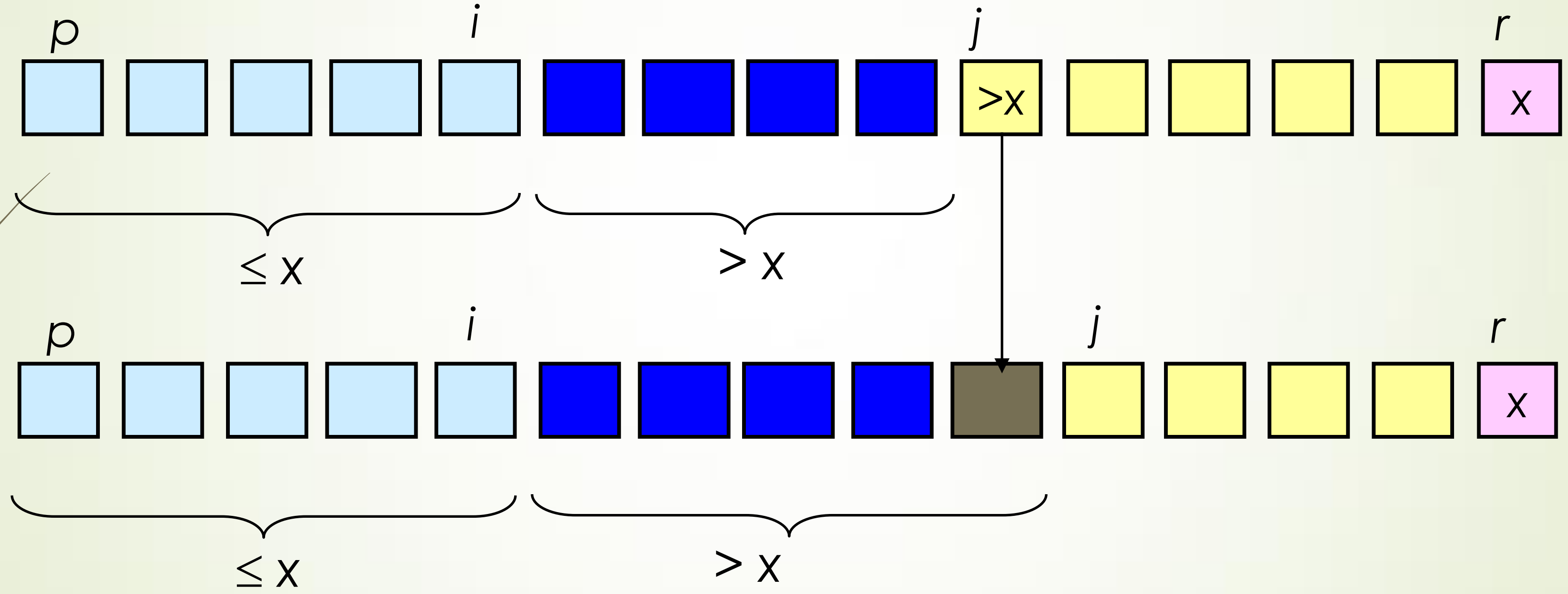
Correctness of Partition

- Use loop invariant.
- **Initialization:**
  - Before first iteration
    - $A[p..i]$ and $A[i+1..j-1]$ are empty – Conds. 1 and 2 are satisfied (trivially).
    - r is the index of the *pivot*
      - Cond. 3 is satisfied.
- **Maintenance:**
  - **Case 1:** $A[j] > x$
    - Increment j only.
    - Loop Invariant is maintained.

```
Partition(A, p, r)
    x, i := A[r], p – 1;
    for j := p to r – 1 do
        if A[j] ≤ x then
            i := i + 1;
            A[i] ↔ A[j]
    A[i + 1] ↔ A[r];
    return i + 1
```

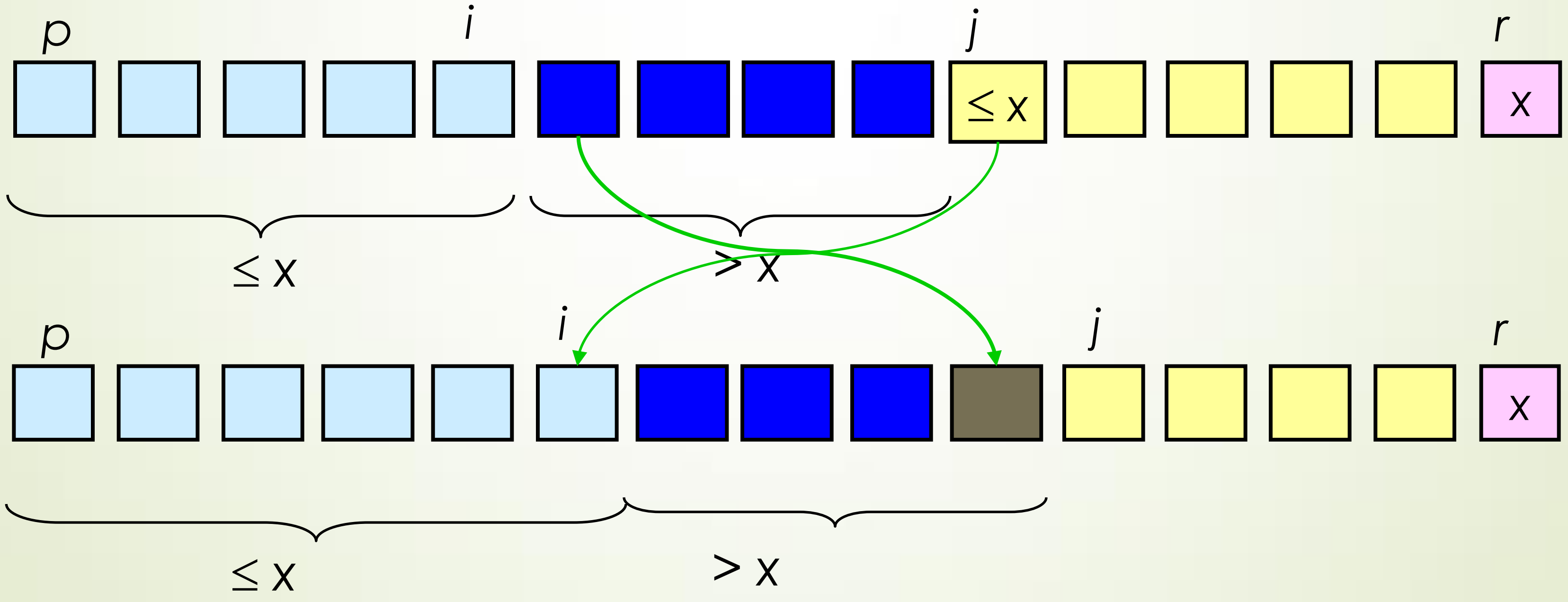
Correctness of Partition

Case 1:



Correctness of Partition

- **Case 2:** $A[j] \le x$
  - Increment i
  - Swap $A[i]$ and $A[j]$
    - Condition 1 is maintained.
- Increment j
  - Condition 2 is maintained.
- $A[r]$ is unaltered.
  - Condition 3 is maintained.



Correctness of Partition

- **Termination:**
  - When the loop terminates, $j = r$, so all elements in A are partitioned into one of the three cases:
    - $A[p..i] \le$ ***pivot***
    - $A[i+1..j-1] >$ ***pivot***
    - $A[r] =$ ***pivot***
- The last two lines swap $A[i+1]$ and $A[r]$.
  - *Pivot* moves from the end of the array to between the two subarrays.
  - Thus, procedure *partition* correctly performs the divide step.

Complexity of Partition

- PartitionTime(n) is given by the number of iterations in the *for* loop.
- $\Theta(n)$: $n = r - p + 1$.

<u>Partition(A, p, r)</u>

x, i := A[r], p – 1;
for j := p **to** r – 1 **do**
  if A[j] $\leq$ x **then**
    i := i + 1;
    A[i] $\leftrightarrow$ A[j]
A[i + 1] $\leftrightarrow$ A[r];
return i + 1

Quicksort Overview

- To sort a[left...right]:

1. if left < right:

   1.1. Partition a[left...right] such that:

   all a[left...p-1] are less than a[p], and

   all a[p+1...right] are >= a[p]

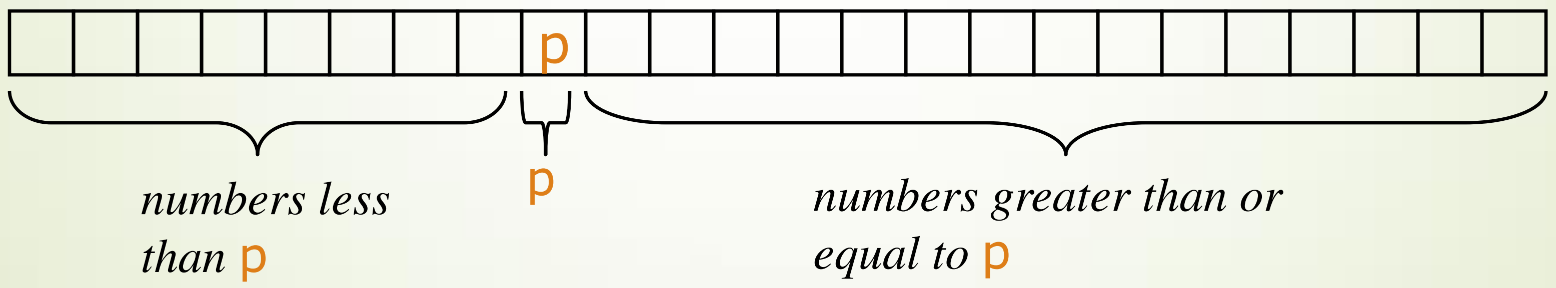
   1.2. Quicksort a[left...p-1]

   1.3. Quicksort a[p+1...right]

2. Terminate

Partitioning in Quicksort

- A key step in the Quicksort algorithm is partitioning the array
  - We choose some (any) number p in the array to use as a pivot
  - We partition the array into three parts:



Alternative Partitioning

- Choose an array value (say, the first) to use as the pivot
- Starting from the left end, find the first element that is greater than or equal to the pivot
- Searching backward from the right end, find the first element that is less than the pivot
- Interchange (swap) these two elements
- Repeat, searching from where we left off, until done

Alternative Partitioning

- To partition a[left...right]:

1. Set pivot = a[left], l = left + 1, r = right;
2. while l < r, do
   2.1. while l < right & a[l] < pivot , set l = l + 1
   2.2. while r > left & a[r] >= pivot , set r = r - 1
   2.3. if l < r, swap a[l] and a[r]
3. Set a[left] = a[r], a[r] = pivot
4. Terminate

Example of partitioning

- choose pivot: 4 3 6 9 2 4 3 1 2 1 8 9 3 5 6
- search: 4 3 6 9 2 4 3 1 2 1 8 9 3 5 6
- swap: 4 3 3 9 2 4 3 1 2 1 8 9 6 5 6
- search: 4 3 3 9 2 4 3 1 2 1 8 9 6 5 6
- swap: 4 3 3 1 2 4 3 1 2 9 8 9 6 5 6
- search: 4 3 3 1 2 4 3 1 2 9 8 9 6 5 6
- swap: 4 3 3 1 2 2 3 1 4 9 8 9 6 5 6
- search: 4 3 3 1 2 2 3 1 4 9 8 9 6 5 6
- swap with pivot: 1 3 3 1 2 2 3 4 4 9 8 9 6 5 6

Partition Implementation

```
int Partition(int[] a, int left, int right)
{
    int p = a[left], l = left + 1, r = right;
    while (l < r)
    {
        while (l < right && a[l] < p)   l++;
        while (r > left && a[r] >= p)   r--;
        if (l < r)
        {
            int temp = a[l]; a[l] = a[r]; a[r] = temp;
        }
    }
    a[left] = a[r];
    a[r] = p;
    return r;
}
```

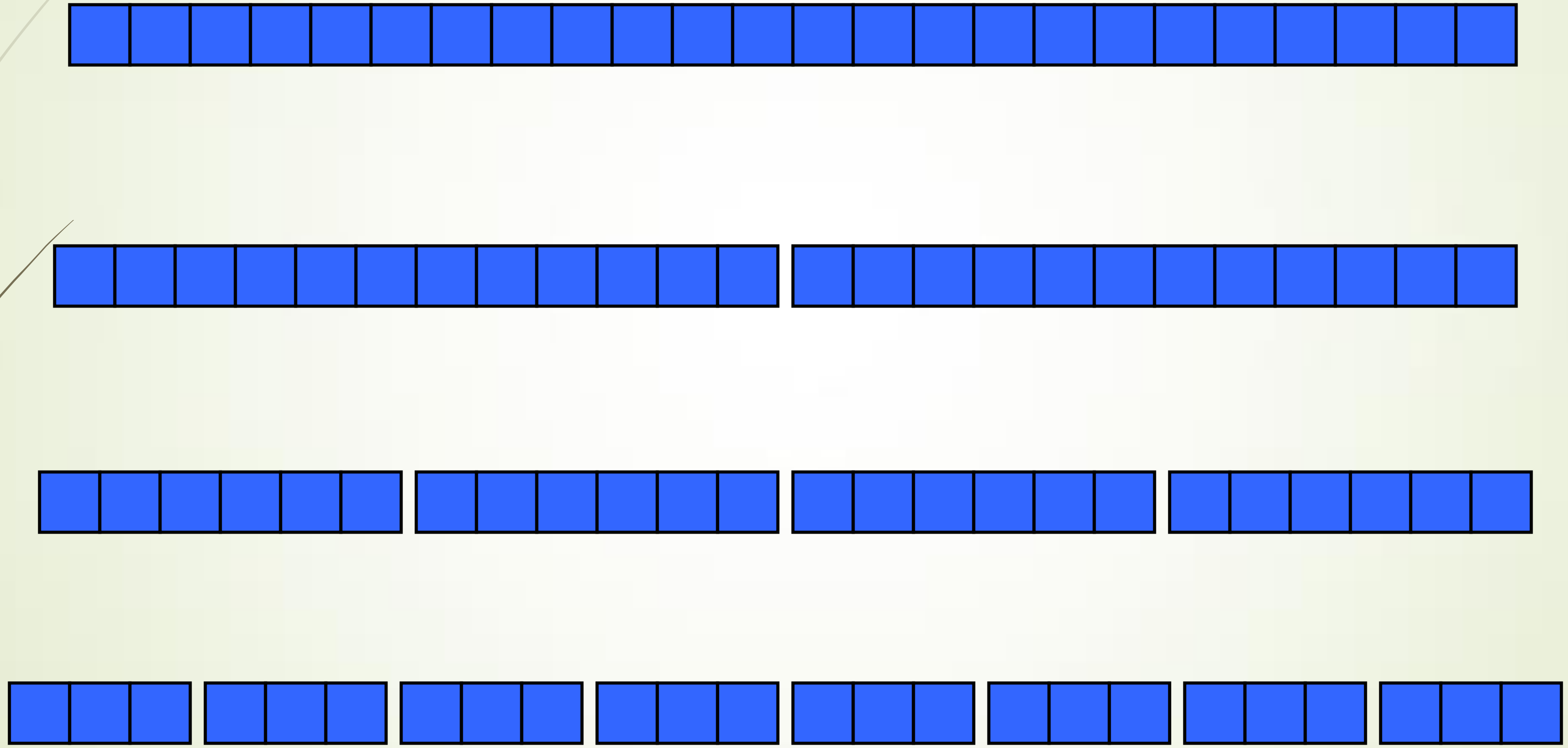
Quicksort Implementation

```
void Quicksort(int[] array, int left, int right)
{
    if (left < right)
    {
        int p = Partition(array, left, right);
        Quicksort(array, left, p - 1);
        Quicksort(array, p + 1, right);
    }
}
```

Analysis of quicksort—best case

- Suppose each partition operation divides the array almost exactly in half
- Then the depth of the recursion in $\log_2 n$
  - Because that's how many times we can halve n

Partitioning at various levels



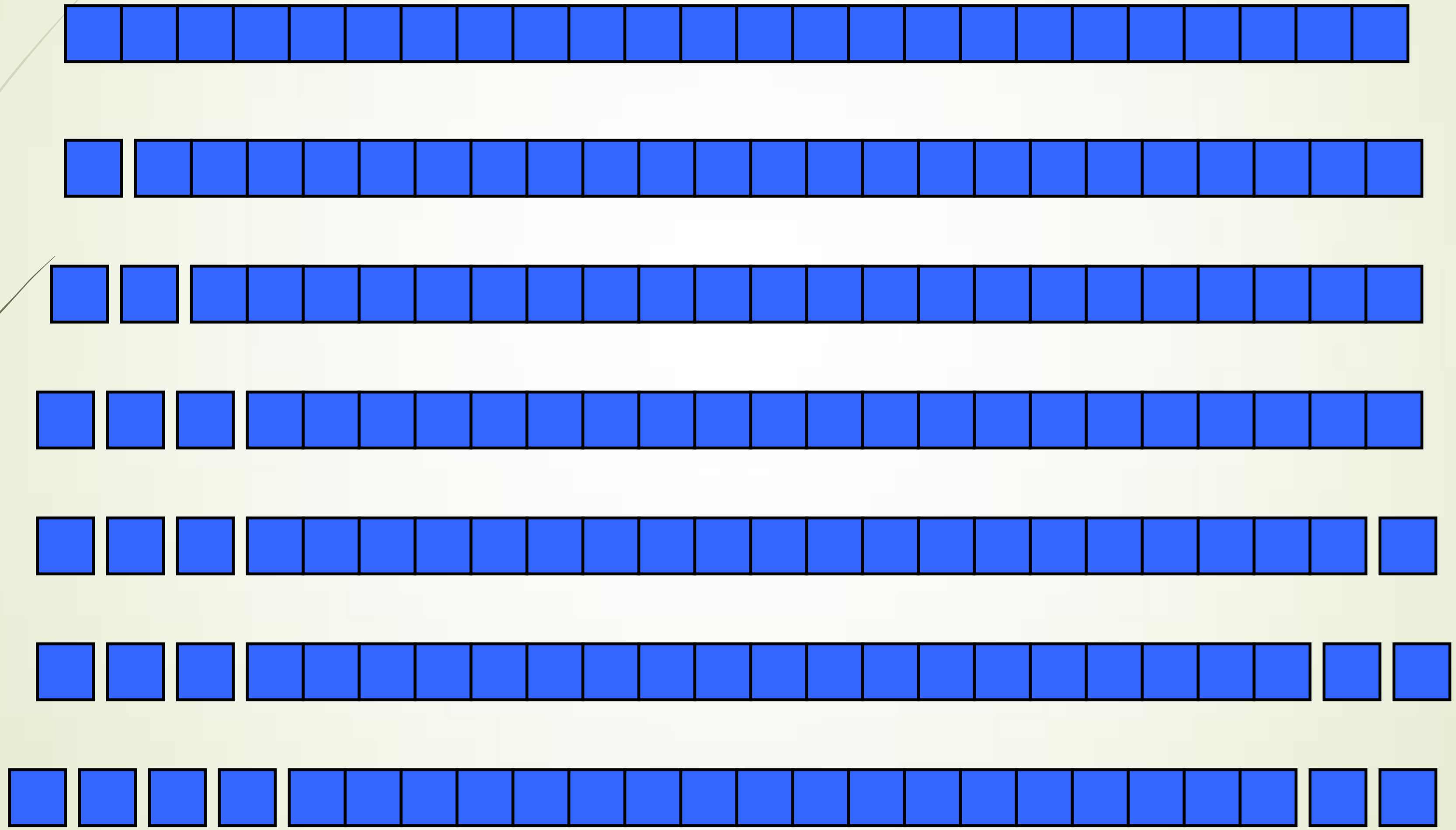
Best Case Analysis

- We cut the array size in half each time
- So the depth of the recursion in $\log_2 n$
- At each level of the recursion, all the partitions at that level do work that is linear in n
- $O(\log_2 n) * O(n) = O(n \log_2 n)$
- Hence in the best case, quicksort has time complexity $O(n \log_2 n)$
- What about the worst case?

Worst case

- In the worst case, partitioning always divides the size n array into these three parts:
  - A length one part, containing the pivot itself
  - A length zero part, and
  - A length n-1 part, containing everything else
- We don't recur on the zero-length part
- Recurring on the length n-1 part requires (in the worst case) recurring to depth n-1

Worst case partitioning



Worst case for quicksort

- In the worst case, recursion may be n levels deep (for an array of size n)
- But the partitioning work done at each level is still n
- $O(n) * O(n) = O(n^2)$
- So worst case for Quicksort is $O(n^2)$
- When does this happen?
  - There are many arrangements that *could* make this happen
  - Here are two common cases:
    - When the array is already sorted
    - When the array is *inversely* sorted (sorted in the opposite order)

Typical case for quicksort

- If the array is sorted to begin with, Quicksort is terrible: $O(n^2)$
- It is possible to construct other bad cases
- However, Quicksort is *usually* $O(n \log_2 n)$
- The constants are so good that Quicksort is generally the faster algorithm.
- Most real-world sorting is done by Quicksort

Picking a better pivot

- Before, we picked the *first* element of the subarray to use as a pivot
  - If the array is already sorted, this results in $O(n^2)$ behavior
  - It's no better if we pick the *last* element
- We could do an *optimal* quicksort (guaranteed $O(n \log n)$) if we always picked a pivot value that exactly cuts the array in half
  - Such a value is called a **median**: half of the values in the array are larger, half are smaller
  - The easiest way to find the median is to *sort* the array and pick the value in the middle (!)

Quicksort for Small Arrays

- For very small arrays (N<= 20), quicksort does not perform as well as insertion sort
- A good cutoff range is N=10
- Switching to insertion sort for small arrays can save about 15% in the running time

Mergesort vs Quicksort

- Both run in O(n lg$n$)
  - Mergesort – always.
  - Quicksort – on average
- Compared with Quicksort, Mergesort has less number of comparisons but larger number of moving elements
- In Java, an element comparison is expensive but moving elements is cheap. Therefore, Mergesort is used in the standard Java library for generic sorting

Mergesort vs Quicksort

In C++, copying objects can be expensive while comparing objects often is relatively cheap. Therefore, quicksort is the sorting routine commonly used in C++ libraries

Note these last two ***rules*** are not really language specific, but rather how the language is ***typically*** used.

Summary

- Discussed stable and in-place sorting technique
- It’s part of standard C library
- Has numerous applications in medical monitoring, defence and mission critical applications