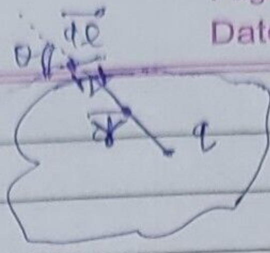
Assignment - 2



(1.) $\vec{E} = \frac{q}{4\pi\epsilon_0}\left(\frac{1-\sqrt{\alpha r}}{r^2}\right)\hat{a}_r$

$\Rightarrow$ (a) $\oint \vec{E}\cdot\vec{dl} = \oint E\cdot\underbrace{(dl\cos\theta)}_{dr} = \oint E\,dr$

$= \oint \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r^2} - \frac{\sqrt{\alpha}}{r^{3/2}}\right)dr = 0$. [Independent of contour].

(b) $\oiint \vec{E}\cdot\vec{ds} = \oiint \frac{q}{4\pi\epsilon_0}\left(\frac{1-\sqrt{\alpha r}}{\cancel{r^2}}\right)\cdot(\cancel{r^2}\sin\theta\,d\theta\,d\phi)$

$= \frac{q}{4\pi\epsilon_0}(1-\sqrt{\alpha r})\cdot(4\pi) = \frac{q}{\epsilon_0}(1-\sqrt{\alpha r})$. Ans

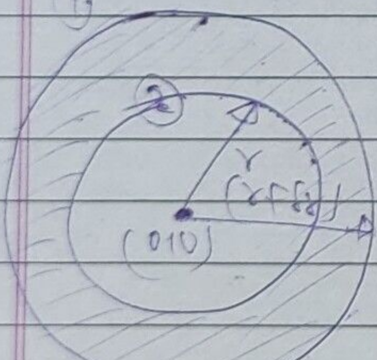
(Or)

$\oiint \vec{E}\cdot\vec{ds} = \oiint \frac{q}{4\pi\epsilon_0}\left(\frac{1-\sqrt{\alpha r}}{r^2}\right)\cdot(r^2 d\Omega) = \frac{q}{\epsilon_0}(1-\sqrt{\alpha r})$ Ans

(c) $\oiint_{S^2(r+\delta r)} \vec{E}\cdot\vec{ds} = \frac{q}{\epsilon_0}\left(1-\sqrt{\alpha r\left(1+\frac{\delta r}{r}\right)}\right) = \frac{q}{\epsilon_0}\left(1-\sqrt{\alpha r}\left(1+\frac{\delta r}{2r}\right)\right)$. Ans

$\Rightarrow$ Now: for a shell:

$\left(\oiint_{S^2(r+\delta r)} + \oiint_{S^2(r)}\right)\vec{E}\cdot\vec{ds} = \iiint_V (\vec{\nabla}\cdot\vec{E})\,dV$ [Gauss divergence theorem].



$= \left(\iiint dV\right)\cdot\left[(\vec{\nabla}\cdot\vec{E})(r)\right]$

$= (\vec{\nabla}\cdot\vec{E}(r))\cdot\frac{4\pi}{3}\left((r+\delta r)^3 - r^3\right)$.

$= (\vec{\nabla}\cdot\vec{E})(r)\cdot\frac{4\pi}{\cancel{3}}\,\frac{\delta r}{r}\times\cancel{3}\times r^3$

$= \left[(\vec{\nabla}\cdot\vec{E})(r)\right]\cdot 4\pi r^2\left[\delta(r)\right]$

{divergence of $\vec{E}$ as a function of r.} — differential radius.

$\Rightarrow \frac{q}{\epsilon_0}\left(1-\sqrt{\alpha r}\left(1+\frac{\delta r}{2r}\right) - 1 + \sqrt{\alpha r}\right) = \vec{\nabla}\cdot\vec{E}(r)\cdot 4\pi r^2\,\delta r$.

$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E}(r) = \frac{-\sqrt{\alpha}\,q}{8\pi\epsilon_0 r^{5/2}}}$ Ans

(2) (a) $\vec{\nabla}\cdot\vec{E} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(A_1h_2h_3) + \cdots\cdots\right]$

$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{Ae^{-br}}{r^2}\cdot r^2\sin\theta\right) + 0 + 0\right]$

$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(Ae^{-br}\right) = \frac{A}{r^2}\frac{d(e^{-br})}{dr} = \frac{-Abe^{-br}}{r^2} = \frac{\rho}{\epsilon_0}$.

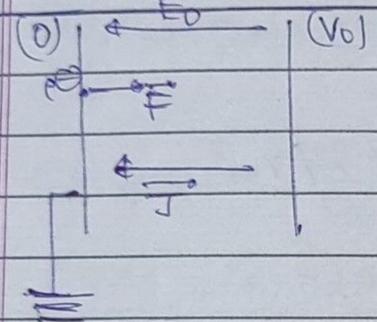
⇒ as $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ [maxwell's equation for static field]

$\epsilon_0(\vec{\nabla}\cdot\vec{E}) = \rho$ ⇒ $\boxed{\rho = \frac{-Ab\epsilon_0 e^{-br}}{r^2}}$. Ans

(b) $Q_T = \int_0^\infty \rho\, dV = \int_0^\infty \frac{-Ab\epsilon_0 e^{-br}}{\cancel{r^2}}(4\pi \cancel{r^2} dr) = \frac{-4\pi Ab\epsilon_0}{b}$

$= -4\pi A\epsilon_0$ Ans

(3)



⇒ 1-D Poisson equation:-

$\boxed{\vec{\nabla}^2(\phi) = -\rho/\epsilon_0}$

This is the combination of:

$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \\ \vec{E} = -\vec{\nabla}\phi\end{array}\right\}$ ϕ = potential

⇒ From Poisson's eqn ⇒ $\boxed{\rho = -\epsilon_0\vec{\nabla}^2(\phi)}$ where $\boxed{\phi = \phi(x)}$

⇒ we know that:

$\vec{J} = \sigma\vec{E} = -\rho\cdot\vec{v}_e$ [negative (direction consider)]

velocity of electron.

so: $\boxed{\vec{J} = +\epsilon_0 v\,\vec{\nabla}(\phi)}$ —— (1)

⇒ Energy conservation over electron: $\left(\frac{1}{2}mv^2\right) - eV_\phi = 0$ [potential at distance (x)]

[velocity at distance (x)]

⇒ So we have:- $\boxed{v = \sqrt{\frac{2e}{m}}\sqrt{\phi}}$ —— (2)

Given in Ques

⇒ Plug in two equations:

$$J = +\epsilon_0 \sqrt{\frac{2e}{m}} \sqrt{\phi}\cdot(\nabla^2\phi) = \text{constant}$$

⇒ As ϕ is only function of r: $\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial r^2}\right)$.

⇒ We have: $J = +\epsilon_0 \sqrt{\frac{2e}{m}} \sqrt{\phi}\cdot\left(\frac{\partial^2\phi}{\partial r^2}\right)$ $\left(\frac{\partial\phi}{\partial r} = t\right) = \frac{d\phi}{dr}$

$$= \epsilon_0 \sqrt{\frac{2e}{m}} \sqrt{\phi}\cdot\left(\frac{\partial t}{\partial r}\right).$$

⇒ Multiply $\frac{\partial\phi}{\partial r}$:- $\sqrt{\frac{m}{2e}}\left(\frac{J}{\epsilon_0}\right)\int_0^{\phi}\frac{d\phi}{\sqrt{\phi}} = \int_0^t t\,dt$

⇒ $\sqrt{\frac{m}{2e\epsilon_0^2}}\cdot J\, 2\sqrt{\phi} = \frac{t^2}{2}$ ⇒ $t = \frac{d\phi}{dr} = 2\sqrt{J}\left(\frac{m\phi}{2e\epsilon_0^2}\right)^{1/4}$

⇒ $\int_0^{V_0}\frac{d\phi}{\phi^{1/4}} = 2\sqrt{J}\left(\frac{m}{2e\epsilon_0^2}\right)^{1/4} d = \frac{4}{3}V_0^{3/4}$.

⇒ $$\boxed{J = \frac{4V_0^{3/2}}{9d^2}\cdot\sqrt{\frac{2e\epsilon_0^2}{m}}}$$ Ans

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(ii) Again ϕ is only function of r, so:

$$\vec{\nabla}\cdot(\vec{\nabla}\phi) = -\frac{\rho}{\epsilon_0} = \vec{\nabla}\cdot\left(\frac{d\phi}{dr}\hat{r}\right) = -\frac{K\phi}{\epsilon_0}.$$

⇒ $$\boxed{\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\cdot\frac{d\phi}{dr}\right)\right) = -\frac{K\phi}{\epsilon_0}}$$

To solve this equation for ϕ, we assume: $\boxed{\phi(r) = \frac{V(r)}{r}}$

such that: $\lim_{r\to\infty}\phi(r) = 0$ (asymptotics

⇒ So: $\frac{r}{r^2}V''(r) = -\frac{KV(r)}{r\epsilon_0}$ ⇒ $V''(r) = -\frac{K(V(r))}{\epsilon_0}$.

⇒ take $\epsilon_0 \to \epsilon$ and multiply by $V'(r)$:-

$$V'(r)\cdot d(V'(r)) = \frac{-K}{\epsilon}V(r)\cdot d(V(r))$$

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$\Rightarrow \boxed{v'(r) = \sqrt{\frac{-k}{\epsilon}}\, v(r) = \frac{d(v(r))}{dr}}$

The sign will depend on k as $\epsilon > 0$:-

$\boxed{k > 0}$

$\Rightarrow v'(r) = -\sqrt{\frac{k}{\epsilon}}\, v(r).$

$\Rightarrow \boxed{\phi(r) = \frac{v(r)}{r} = \frac{A e^{\sqrt{-k/\epsilon}\, r}}{r}}$

- as $k > 0 \Rightarrow$
  $\phi(r) = Re(\ldots).$

To calculate value of A :- we assume a surface having radius equal to just 'a' outside the given sphere

$$\oiint \vec{E}\cdot d\vec{s} = \left(\frac{Q}{\epsilon_0}\right) = \oiint -\frac{d\phi}{dr}\cdot d\vec{s}$$

$$\Rightarrow \iint \left(\frac{d\phi}{dr}\right)\cdot(a^2 \sin\theta\, d\theta\, d\phi) = \frac{-Q}{\epsilon_0}$$

as $r = a$ (const), $\frac{\partial\phi}{\partial r} = \frac{d\phi}{dr} = $ const.

$$\Rightarrow \frac{d\phi}{dr}\cdot(4\pi a^2) = \frac{-Q}{\epsilon_0} = \frac{-Q}{\epsilon}$$

where $\phi = \frac{A}{r}\cos\left(\sqrt{\frac{k}{\epsilon}}\, r\right)$

$$A = \frac{Q}{4\pi\epsilon\left(a \sin\left(\sqrt{\frac{k}{\epsilon}}\, a\right)\sqrt{\frac{k}{\epsilon}} + \cos\left(\sqrt{\frac{k}{\epsilon}}\, a\right)\right)}$$

put the value of A in

$$\phi(r) = Re\left(\frac{A}{r} e^{\sqrt{-k/\epsilon}\, r}\right) \text{ where } k > 0$$

$$= \frac{A}{r}\cos\left(\sqrt{\frac{k}{\epsilon}}\, r\right) \quad \underline{\text{Ans}}$$

$\boxed{k < 0}$

$\Rightarrow \ln|v(r)| = \sqrt{\frac{-k}{\epsilon}}\, r + c$

$\Rightarrow \boxed{v(r) = A e^{\sqrt{-k/\epsilon}\, r}}$

- as $k < 0 \Rightarrow \phi(r) = \frac{A}{r} e^{\sqrt{-k/\epsilon}\, r}$

To calculate value of A we assume a surface having radius equal to just 'a' outside the given sphere.

$$\oiint \vec{E}\cdot d\vec{r} = \oiint -\frac{d\phi}{dr}\cdot d\vec{s} = \frac{Q}{\epsilon}$$

$$\Rightarrow \frac{d\phi}{dr} = \frac{-Q}{4\pi a^2 \epsilon}$$

and $\phi = \frac{A}{r} e^{+\sqrt{-k/\epsilon}\, r}$ where $k < 0$.

solving : $A = \frac{-Q}{4\pi\epsilon\left(a\sqrt{\frac{-k}{\epsilon}} - 1\right)} e^{-\sqrt{\frac{-k}{\epsilon}}\, a}$

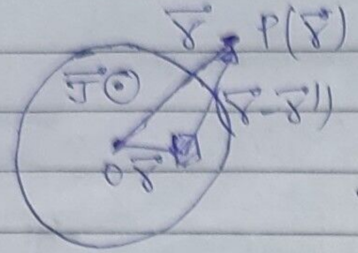
put the value of A in:

$$\phi(r) = \frac{A}{r} e^{\sqrt{k'/\epsilon}\cdot r}$$

where $k' = -k$.

Ans

(5.) let's assume that we have a cylinder and a point (P) inside or outside it, where we have to calculate $\vec{B}$ due to the current flowing inside the cylinder.



$$\Rightarrow \vec{J} = \frac{I}{\pi(a^2-b^2)}\,\hat{a}_z \qquad \text{so } B_z = 0$$

Then $\vec{B}_P = B_\rho \hat{a}_\rho + B_\phi \hat{a}_\phi$ and $B_z = 0$

Due to cylindrical symmetry, B_ρ and B_ϕ will not depend on ϕ and depend only on ρ.

- We know, $\vec{\nabla}\cdot\vec{B} = 0$ (For static fields.)

$$\Rightarrow \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho B_\rho) + \cancel{\frac{\partial}{\partial\phi}(B_\phi)}^{\,0}\right] = 0$$

as $B_\phi = B_\phi(\rho)$ ← does not depend on ϕ.

$$\Rightarrow \boxed{B_\rho = \frac{c_1}{\rho}} \quad \text{where } c_1 = \text{constant}$$

- $$\vec{B}_P(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d^3\vec{r}'$$

$$= \frac{\mu_0}{4\pi}\cdot\frac{I}{\pi(a^2-b^2)}\int \frac{\hat{a}_z\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d^3\vec{r}'$$

Assume that $\rho = 0$, $P \to 0$:-

$$\vec{B}_0(\vec{0}) = \frac{\mu_0 I}{4\pi^2(a^2-b^2)}\,\hat{a}_z \times \cancelto{0}{\int \frac{-\vec{r}'\,d^3\vec{r}'}{(r')^3}}$$

[due to cylindrical symmetry (infinite)]

and hence $\vec{B}_0(\vec{0}) = 0 \Rightarrow B_\rho\big|_{\rho=0} = 0 \Rightarrow \boxed{c_1 = 0}$

which implies that $B_\rho = 0$ (always $\forall\, \rho \in [0, a]$)

- Hence,

$$\vec{B} = B_\phi \hat{a}_\phi \Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \cancelto{0}{\frac{\partial \vec{D}}{\partial t}}\right)$$

$$\Rightarrow \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho B_\phi)\right) = \mu_0 J.$$

⇒ as ρB_ϕ is function of only ρ ⇒ $\frac{\partial}{\partial\rho} = \frac{d}{d\rho}$ and integrate.

$$\rho B_\phi = \frac{\mu_0 J \rho^2}{2} \Rightarrow \boxed{B_\phi = \frac{\mu_0 J\rho}{2}}$$

• $\vec{B}_p = \frac{\mu_0 J\rho}{2}\hat{a}_\phi = \frac{\mu_0 J\rho}{2}(\hat{a}_z\times\hat{a}_\rho) = \frac{\mu_0}{2}(J\hat{a}_z \times \rho\hat{a}_\rho)$

$\boxed{\vec{B}_p = \frac{\mu_0}{2}(\vec{J}\times\vec{r})}$ (For bigger cylinder)

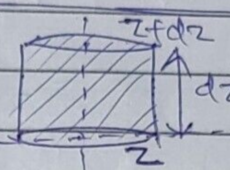
For smaller: $\boxed{\vec{B}_p' = \frac{\mu_0}{2}(\vec{J}\times\vec{r}')}$

Hence, $\vec{B}_{net_p} = \frac{\mu_0}{2}(\vec{J}\times(\vec{r}-\vec{r}')) = \frac{\mu_0}{2}(\vec{J}\times\vec{d}) = \text{constant}$

Ans

(7) We know:

$\oiint \vec{B}\cdot d\vec{s} = 0$ (Magnetostatics gauss's law)



⇒ $(B(\rho, z+dz) - B(\rho, z))\pi\rho^2 + B_\rho(dz)(2\pi\rho) = 0$

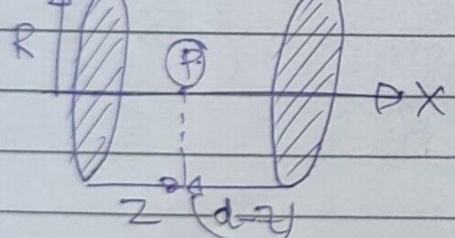
⇒ $\left(\frac{B(\rho, z+dz) - B(\rho, z)}{dz}\right)\rho = -B_\rho(z)$.

⇒ $\frac{\partial(B_z)}{\partial z}\left(\frac{-\rho}{2}\right) = B_\rho = \frac{-\rho}{2}(2\alpha z B_0) = -B_0\alpha\rho z$

For ($\rho=0$)

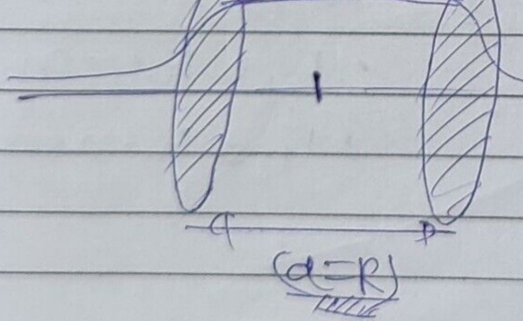
$\boxed{B_\rho = -B_0\alpha\rho z}$ Ans

(10.)



⇒ $\vec{B}_p = \frac{\mu_0(2i\pi R^2)}{4\pi}\left(\frac{1}{(R^2+x^2)^{3/2}} + \frac{1}{(R^2+(a-x)^2)^{3/2}}\right)$

⇒ $\frac{dB}{dx} = 0$ and $\frac{d^2B}{dx^2} = 0$.

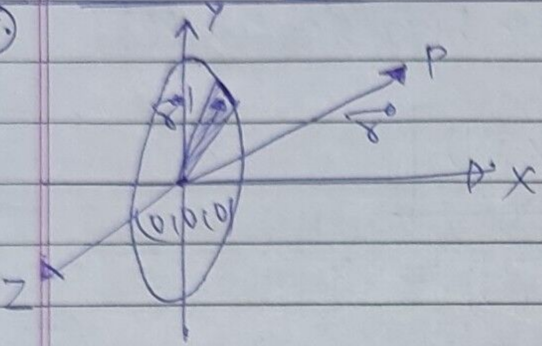


(called Helmholtz coils.)

$$\Rightarrow \frac{d^2B}{dx^2} = \frac{-3\mu_0 i R^2}{2} \frac{\left(R^2+d^2/4\right)^{3/2}}{\left(R^2+\frac{d^2}{4}\right)^{5/2}} \left(R^2-d^2\right)\times 2 = 0 \quad \left[\text{For } x=\frac{d}{2}\right]$$

$$\Rightarrow \boxed{d = a = R} \quad \text{Ans}$$

6.



$$\Rightarrow \vec{B}_p = \frac{\mu_0 I}{4\pi} \oint \left(\frac{d\vec{l}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right)$$

We know that:-

$$\boxed{\oint d\vec{l}' \circ = \iint \left(d\vec{s}' \times \vec{\nabla}'\right) \circ}$$

So:

$$\vec{B}_p = \frac{\mu_0 I}{4\pi} \iint \left(\frac{(d\vec{s}' \times \vec{\nabla}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right)$$

$$= \frac{\mu_0 I}{4\pi} \iint \left[\vec{\nabla}'\left(\frac{d\vec{s}' \cdot (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right) - \left(\frac{\vec{\nabla}' \cdot (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right) d\vec{s}'\right]$$

$$= \frac{\mu_0 I}{4\pi} \iint \left[\vec{\nabla}'\left(\frac{d\vec{s}' \cdot (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right) + \left(4\pi\delta^3(\vec{r}-\vec{r}')\right) d\vec{s}'\right]$$

$$\boxed{\vec{\nabla}^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})}$$

$$= \frac{\mu_0 I}{4\pi} \left[\vec{\nabla}' \iint d\vec{s}' \cdot \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} + \iint \underbrace{\cancel{4\pi\delta^3(\vec{r}-\vec{r}')} \cdot d\vec{s}'}_{0}\right]$$

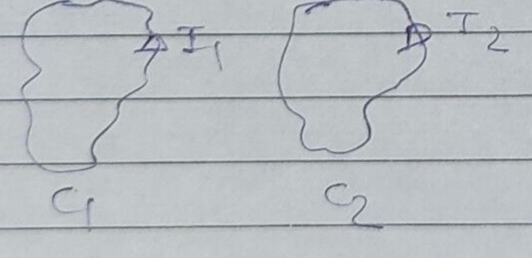
as we know

$$\left\{\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3} \text{ and } \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right) = 0. \text{ for } (r \neq 0)\right\}$$

$$= \frac{\mu_0 I}{4\pi} \vec{\nabla}' \iint \underbrace{\frac{d\vec{s} \cdot (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}_{d\Omega}$$

$$= \frac{\mu_0 I}{4\pi} \vec{\nabla}\Omega \quad \text{Ans}$$

8.



- F_{21} = Force on 2 due to 1
- F_{12} = Force on 1 due to 2

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$\Rightarrow \vec{B_1} = \frac{\mu_0 I_1}{4\pi} \oint_{c_1} \left(\frac{\vec{dl_1} \times (\vec{r_2} - \vec{r_1})}{|\vec{r_2} - \vec{r_1}|^3} \right)$

$\Rightarrow \vec{F_{21}} = \oint_{c_2} I_2 (\vec{dl_2} \times \vec{B}) = \oint_{c_2} I_2 \frac{\mu_0 I_1}{4\pi} \oint_{c_1} \vec{dl_2} \times \left(\vec{dl_1} \times \frac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} - \vec{r_1}|^3} \right)$

$\Rightarrow \vec{F_{21}} = \left(\frac{\mu_0 I_1 I_2}{4\pi} \right) \left[\oint_{c_1} \oint_{c_2} \left(\vec{dl_2} \cdot \vec{\nabla} \left(\frac{1}{|\vec{r_2} - \vec{r_1}|} \right) \right) \vec{dl_1} \right.$

$0 \quad \left. - \oint_{c_1} \oint_{c_2} \left(\vec{dl_2} \cdot \vec{dl_1} \right) \frac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} - \vec{r_1}|^3} \right]$

$= \left(\frac{\mu_0 I_1 I_2}{4\pi} \right) \left[- \oint_{c_1} \oint_{c_2} \left(\vec{dl_1} \cdot \vec{dl_2} \right) \cdot \frac{\vec{r_1} - \vec{r_2}}{|\vec{r_1} - \vec{r_2}|^3} \right]$

$= - \vec{F_{12}}$ Hence, proved