PHN-005 (Physics) **B.Tech. I Year (CSE+ECE)** **Tutorial Sheet–I**
Autumn Semester 2022

Curvilinear Coordinate Calculus

Note: Unit tangent vectors are denoted by $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, $\mathbf{a}_z$, etc.

1. (a) Derive the algebraic equations for the coordinate curves and coordinate surfaces in (i) cylindrical and (ii) spherical polar coordinate systems.
(b) Show explicitly that (i) and (ii) are orthogonal curvilinear coordinate systems.

2. Let $\mathbf{H} = \rho \sin\phi\, \mathbf{a}_\rho - \rho z \cos\phi\, \mathbf{a}_\phi + \rho\, \mathbf{a}_z$. At point $P(\rho_0, \phi_0, z_0)$, find:

(a) a unit vector along **H**
(b) the component of **H** parallel to $\mathbf{a}_x$
(c) the component of **H** normal to $\rho = \rho_0$
the component of **H** tangential to $\phi = \phi_0$.

3. Show that a unit normal to the surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$ is given by $\boldsymbol{n} = \pm \frac{\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}}{\sqrt{EF-G^2}}$, where $E \equiv \left(\frac{\partial \boldsymbol{r}}{\partial u}\right)^2, F \equiv \left(\frac{\partial \boldsymbol{r}}{\partial v}\right)^2, G \equiv \frac{\partial \boldsymbol{r}}{\partial u} . \frac{\partial \boldsymbol{r}}{\partial v}$.

4. Let the rectangular coordinates (x, y, z) of any point be expressed as single-valued functions with continuous derivatives of (u_1, u_2, u_3): $x = x(u_1, u_2, u_3), y = y(u_1, u_2, u_3), z = z(u_1, u_2, u_3)$. Assume these can solved for u_1, u_2, u_3 in terms of *x, y, z:* $u_1 = u_1(x, y, z)$, $u_2 = u_2(x, y, z)$, $u_3 = u_3(x, y, z)$, involving single-valued functions with continuous derivatives.
(a) If a vector $\boldsymbol{A}$ is written out in a basis $\left\{\frac{\partial r}{\partial u_1}, \frac{\partial r}{\partial u_2}, \frac{\partial r}{\partial u_3}\right\}$ in the coordinate system (u_1, u_2, u_3) with components (A_1, A_2, A_3) , and in a basis $\left\{\frac{\partial r}{\partial u_1'}, \frac{\partial r}{\partial u_2'}, \frac{\partial r}{\partial u_3'}\right\}$ in the coordinate system (u_1', u_2', u_3') with components (A_1', A_2', A_3'), show that: $A_i'(u_1', u_2', u_3') = \sum_{j=1}^{3} \frac{\partial u_i'}{\partial u_j} A_j(u_1, u_2, u_3), i = 1,2,3.$
(b)) If a vector $\boldsymbol{A}$ is written out in a basis $\{\nabla u_1, \nabla u_2, \nabla u_3\}$ in the coordinate system (u_1, u_2, u_3) with components $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, and in a basis $\{\nabla u_1', \nabla u_2', \nabla u_3'\}$ in the coordinate system (u_1', u_2', u_3') with components $(\mathcal{A}_1', \mathcal{A}_2', \mathcal{A}_3')$, show that: $\mathcal{A}_i'(u_1', u_2', u_3') = \sum_{j=1}^{3} \frac{\partial u_j}{\partial u_i'} \mathcal{A}_j(u_1, u_2, u_3), i = 1,2,3.$

5. Using the differential length element, find the length of each of the following curves:

(a) ρ = constant, $\phi_1 < \phi < \phi_2$, z = constant
(b) r = constant, $\theta = \theta_0$, $\phi_1 < \phi < \phi_2$
(c) r = constant, $\theta_1 < \theta < \theta_2$, ϕ = constant

6. Calculate the areas of the following surfaces using the differential surface area ds:

(a) $z = 1,\ 1 < \rho < 3,\ 0 < \phi < \pi/4$
(b) $r = 10,\ \pi/4 < \theta < 2\pi/3,\ 0 < \phi < 2\pi$

7. Use the differential volume dv to determine the volumes of the following regions:

(a) $\rho_1 < \rho < \rho_2,\ \phi_1 < \phi < \phi_2,\ z_1 < z < z_2$

(c) $r_1 < r < r_2,\ \theta_1 < \theta < \theta_2,\ \phi_1 < \phi < \phi_2$

8. Consider the following coordinate transformation:

$x = uv\cos\phi, y = uv\sin\phi, z = \frac{1}{2}(u^2 - v^2); u \geq 0, v \geq 0, 0 \leq \phi < 2\pi.$

(a) Verify that the above describes an orthogonal curvilinear coordinate system.
(b) Obtain algebraic equations for the coordinate curves and surfaces.

9. Is $\nabla \times \nabla \boldsymbol{\varphi} = 0$ true in cylindrical and spherical polar coordinates?

10. Find a vector **A** such that $\mathbf{B} = \nabla \times \mathbf{A}$ for a constant vector **B** (in three dimensions).

11. (a) Find the components of acceleration in cylindrical coordinates parallel and perpendicular to ρ of a particle moving in the x-y plane.
(b) Calculate the derivatives of the unit tangent vectors along r, θ, ϕ with respect to r, θ, ϕ each.
(c) Using (b), verify the following identity in **spherical polar coordinates** for vectors **A** and **B**:

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B}.\nabla)\mathbf{A} - (\nabla.\mathbf{A})\mathbf{B} + (\nabla.\mathbf{B})\mathbf{A} - (\mathbf{A}.\nabla)\mathbf{B}\ .$$

12. (a) Using the standard form of Stokes theorem, prove that:

$$\oint_C d\boldsymbol{r} \circ = \iint_S (d\boldsymbol{S} \times \boldsymbol{\nabla}) \circ,$$

where S is an open surface and C its boundary.

(b) Using the standard form of Gauss' divergence theorem, prove that:

$$\oiint_\Sigma d\boldsymbol{S} \circ = \iiint_V dV\, \boldsymbol{\nabla}\circ\,,$$

where Σ is a closed surface.

In parts (a) and (b), $\circ$ implies either a scalar multiplication or a vector dot or cross product.

13. (a) If $\int_C \boldsymbol{F}.d\boldsymbol{r}$ is independent of the path C joining any two points, show that there exists a function ϕ such that $F = \nabla\phi$. Calculate ϕ for $\boldsymbol{F} = x^2\,\mathbf{a}_x + y^2\mathbf{a}_y + z^2\mathbf{a}_z$.
(b) If **F** is irrotational then prove that **F** is conservative.
(c) If the integral $\int_A^B \mathrm{F}.\, d\ell$ is regarded as the work done in moving a particle from A to B.

F = $2xy\,\mathbf{a}_x + (x^2 - z^2)\,\mathbf{a}_y - 3xz^2\,\mathbf{a}_z$**.** Find the work done by the force field on a particle that travels from
A(0, 0, 0) to B (2, 1, 3) along

(i) the segment (0, 0, 0) $\longrightarrow$ (0, 1, 0) $\longrightarrow$ (2, 1, 3)
(ii) the straight line (0, 0, 0) to (2, 1, 3)

14. If V = (x + y)z, evaluate $\oint_S V d\mathbf{S}$**,** where S is the surface of the cylindrical wedge defined by $0 < \phi < \pi/2$, $0 < z < 2$ and $d\mathbf{S}$ is normal to the surface.

15. Consider a sphere circumscribed by a cylinder of equal radii (r) centered at the origin with the height (of the cylinder) equal to the diameter of the sphere. Given a vector
$\boldsymbol{A} = A_r(r)\hat{\boldsymbol{a}}_{\boldsymbol{r}} + A_\theta(\theta)\hat{\boldsymbol{a}}_{\boldsymbol{\theta}} + A_\phi(\phi)\hat{\boldsymbol{a}}_{\boldsymbol{\phi}}$**,**
(a) verify the Gauss' divergence theorem applied to the region inside the cylinder but exterior to the sphere;
(b) verify the Stokes theorem in the annulus formed by the intersection of the $z = (0 <)b < r$ with the aforementioned sphere and the circumscribing cylinder.