**Indian Institute of Technology Roorkee**

**MAN-001(Mathematics-1), Autumn Semester: 2020-21**

**Assignment-2: Matrix Algebra II**

(1) For each matrix, find all eigenvalues and the corresponding linearly independent eigenvectors;

(a) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

(2) (a) Let $\lambda$ be an eigenvalue of a nonsingular square matrix $A$ of order $n$ and $\mathbf{x}$ be the corresponding eigenvector. Show that $\lambda^{-1}$ is an eigenvalue of $A^{-1}$ and identify the corresponding eigenvector. Also, identify the eigenvalue and eigenvector of $(A-kI)$, where $I$ is the identity matrix and $k$ is a scalar.

(b) Let $A$ be a square matrix of size $n$. Show that $A$ and $A^T$ have same eigenvalues. Are their eigenvectors also same?

(c) If $A$ and $P$ are square matrices of order $n$, and $P$ is nonsingular, then prove that $A$ and $P^{-1}AP$ have the same eigenvalues.

(3) Prove that

(a) all eigenvalues of a Hermitian matrix are real.

(b) eigenvalues of a skew Hermitian matrix are purely imaginary or zero.

(c) eigenvectors corresponding to two distinct eigenvalues of a real symmetric matrix are orthogonal.

(d) eigenvalues of a unitary matrix have unit modulus.

(e) any skew-symmetric matrix of odd order has zero determinant.

(f) the eigenvalues of an idempotent matrix are either 0 or 1.

(g) all eigenvalues of a nilpotent matrix are 0.

(4) Let $A$ and $B$ be square matrices of order $n$. Show that if $\lambda$ be an eigenvalue of $AB$ then it will be an eigenvalue of $BA$. Hence, prove that $I-AB$ is invertible iff $I-BA$ is invertible.

(5) Prove that every Hermitian matrix can be written as $A+iB$, where $A$ is a real symmetric matrix, and $B$ is a real skew-symmetric matrix.

(6) Given that $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I-A)(I+A)^{-1}$ is a unitary matrix.

(7) (a) The eigenvalues of a $3\times 3$ matrix $A$ are $2, 2, 4$ and the corresponding eigenvectors are $(-2,1,0)^T$, $(-1,0,1)^T$ and $(1,0,1)^T$. Find A.

(b) Find a matrix $P$ such that $P^{-1}AP$ is a diagonal matrix, where $A$ is

(i) $\begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 0 & -1 \\ 3 & 4 & 0 \\ 3 & 2 & 0 \end{bmatrix}$.

(8) Find $e^{2A}$ and $A^{50}$ when (a) $A = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ (b) $A = \begin{bmatrix} -2 & 4 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(9) Using Cayley-Hamilton theorem, find the inverse of the following matrices:

(a) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 6 & 8 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

(10) Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix}$. Hence, find $A^{-1}$ and $A^4$.

(11) Let $A = \begin{pmatrix} 4 & \alpha & -1 \\ 2 & 5 & \beta \\ 1 & 1 & \gamma \end{pmatrix}$. Given that the eigenvalues of the matrix $A$ are $3, 3, \delta$ (where $\delta \neq 3$) and $A$ is diagonalizable, find the values of constants $\alpha, \beta, \gamma, \delta$.

(12) Find an orthogonal or unitary matrix $P$ such that $P^*AP$ is diagonal where $A$ is given by the following:

(a) $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$

(13) Let $A$ be a $5 \times 5$ invertible matrix with row sums 1. That is $\sum_{j=1}^{5} a_{ij} = 1$ for $1 \leq i \leq 5$. Then, prove that the sum of all the entries of $A^{-1}$ is 5.

(14) Let $A$ be a nilpotent matrix. Show that $I + A$ is invertible.

(15) Suppose that $A^{15} = 0$. Show that there exists a unitary matrix $U$ such that $U^*AU$ is $5 \times 5$ upper triangular with diagonal entries 0.

**ANSWERS**

1. (a) $2, 2, 6, (-1, 0, 1)^T, (-1, 1, 0)^T, (1, 2, 1)^T$ (b) $1, 1, 1, (1, 0, 1)^T, (1, 0, 0)^T$
(c) $-2, -2, 4$ $(0, 1, 1)^T, (1, 0, -1)^T, (1, 1, 2)^T$ (d) $1, 1, 3, (1, 0, 3)^T, (1, -3, 0)^T, (-1, 1, -1)^T$

7. (a) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

(b) (i) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & -11 & 1 \\ 1 & -1 & 1 \\ 1 & 14 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \\ 1 & 0 & 1 \end{bmatrix}$

8. (a) $\begin{bmatrix} \alpha & \beta-\alpha & (\alpha-2\beta+\gamma)/2 \\ 0 & \beta & \gamma-\beta \\ 0 & 0 & \gamma \end{bmatrix}$ where $\alpha = e^{-8}, \beta = e^{-6}, \gamma = e^{-4}$ for $e^{2A}$ and
$\alpha = 4^{50}, \beta = 3^{50}, \gamma = 2^{50}$ for $A^{50}$

(b) $\frac{1}{6}\begin{bmatrix} 6\alpha & 4(\beta-\alpha) & 3(\beta-\alpha) \\ 0 & 6\beta & 0 \\ 0 & 0 & 6\beta \end{bmatrix}$ where $\alpha = e^{-4}, \beta = e^{8}$ for $e^{2A}$ and
$\alpha = 2^{50}, \beta = 4^{50}$ for $A^{50}$

9. (a) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -0.20 & -0.90 & 0.50 \\ 0.50 & 1.25 & -0.75 \\ -0.30 & -0.60 & 0.50 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

10. $\frac{1}{11}\begin{bmatrix} 1 & -2 & 2 \\ 5 & 1 & -1 \\ -4 & 8 & 3 \end{bmatrix}$, $\begin{bmatrix} 25 & 8 & 8 \\ 12 & 25 & 4 \\ 16 & 32 & 33 \end{bmatrix}$

11. $\alpha = 1, \beta = -2, \gamma = 2, \delta = 5$.

12. (a) $\frac{1}{\sqrt{10}}\begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -5/\sqrt{105} & 4/\sqrt{21} \\ 1/\sqrt{5} & -8/\sqrt{105} & -2/\sqrt{21} \\ 2/\sqrt{5} & 4/\sqrt{105} & 1/\sqrt{21} \end{bmatrix}$ (c) $\begin{bmatrix} -2/\sqrt{6} & 1/\sqrt{3} \\ (1+i)/\sqrt{6} & (1+i)/\sqrt{3} \end{bmatrix}$