Indian Institute of Technology Roorkee
MAN-001(Mathematics-1)
Autumn Semester: 2022-23
Assignment-7: (Gamma and Beta Functions)

(1) Evaluate (i) $\Gamma(7)$ (ii) $\Gamma(\frac{7}{2})$.

(2) Show that (i) $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{2}{\sqrt{3}}\sqrt{\pi}$

(ii) $\Gamma(m+\frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(2m+1)}{2^{2m}\Gamma(m+1)}$, where $m \in \mathbb{Z}$.

(iii) $2^{2m-1}\Gamma(m)\Gamma(m+\frac{1}{2}) = \sqrt{\pi}\Gamma(2m)$, where $m \in \mathbb{Z}$.

(3) For $s > 0$, $p > 0$, show that

(i) $\int_0^\infty x^{p-1}e^{-sx}dx = \Gamma(p)/s^p$ (ii) $\int_0^\infty e^{-s^2x^2}dx = \sqrt{\pi}/2s$.

(4) Show that $\Gamma(p) = \int_0^1 (\ln(\frac{1}{y}))^{p-1}dy$; $p > 0$; using this evaluate $\int_0^1 (\ln(\frac{1}{y}))^{-1/2}dy$.

(5) Show that for integer $m > -1, n > 0$

$$\int_0^1 x^m(\ln x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}.$$

(6) Show that for $c > 1$, ,

$$\int_0^\infty \frac{x^c}{c^x}dx = \frac{\Gamma(c+1)}{(\ln(c))^{c+1}}.$$

(7) Show that for $r > -1$,

$$\int_0^\infty x^r e^{-s^2x^2}dx = \frac{1}{2s^{r+1}}\Gamma(\frac{r+1}{2}).$$

(8) Using refection property show that $\int_0^{\pi/2} \tan^n\theta d\theta = \frac{\pi}{2}\sec\frac{n\pi}{2}$.

(9) Prove the following:

(i) $\beta(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta d\theta$.
(ii) $\beta(x,y) = \int_0^{infty} \frac{t^{x-1}}{(1+t)^{x+y}}dt$.
(iii) $\beta(x,y) = \beta(x+1,y) + \beta(x,y+1)$.
(iv) $\frac{1}{x+y}\beta(x,y) = \frac{1}{x}\beta(x+1,y) = \frac{1}{y}\beta(x,y+1)$.

(v) $\int_0^1 t^{m-1}(1-t^2)^{n-1}dt = \frac{1}{2}\beta(\frac{m}{2}, n)$.
(vi) $\int_0^1 (1-t^6)^{-1/6}dt = \frac{\pi}{3}$.

(10) Show that for any $m \in \mathbb{N}$,

$$\beta(m,m) = \frac{\sqrt{\pi}\Gamma(m)}{2^{2m-1}\Gamma(m+1/2)}.$$

(11) Evaluate the following integrals in terms of Gamma and Beta functions:
(i) $\int_0^\infty e^{-x^4}dx$ (ii) $\int_0^\infty x^{-7/4}e^{-\sqrt{x}}dx$ (iii) $\int_0^a x^9(a^6-x^6)^{\frac{1}{3}}dx$.

(12) Prove that $\int_0^\infty xe^{-x}\cos x dx = 0$.

(13) Compute $\int_0^\infty xe^{-x}\sin x dx = \frac{1}{2}$.

(14) Prove that

$$\Gamma(n+\frac{1}{2}) = \sqrt{\pi}\prod_{k=1}^{n}\frac{2k-1}{2} \text{ for } n \in \mathbb{N}.$$

Answers.
(1) (i) 720 (ii) $\frac{15}{8}\sqrt{\pi}$ (4) $\sqrt{\pi}$ (11) (i) $\Gamma(\frac{5}{4})$ (ii)$\frac{8}{3}\sqrt{\pi}$ (iii) $\frac{a^6}{6}\beta(\frac{5}{3},\frac{4}{3})$.