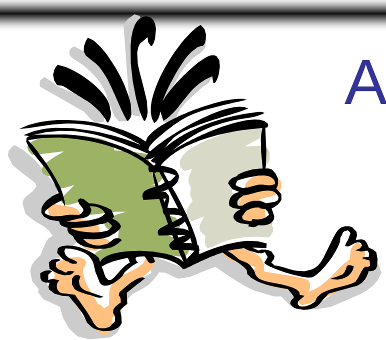
# Hashing



Acknowledge to : George Bebis

# The Search Problem

- Find items with keys matching a given search key
  - Given an array A, containing n keys, and a search key x, find the index i such as x=A[i]
  - As in the case of sorting, a key could be part of a large record.

example of a record

| | |
|---|---|
| **Key** | **other data** |

# Applications

- Keeping track of customer account information at a bank
  - Search through records to check balances and perform transactions
- Keep track of reservations on flights
  - Search to find empty seats, cancel/modify reservations
- Search engine
  - Looks for all documents containing a given word

# Special Case: Dictionaries

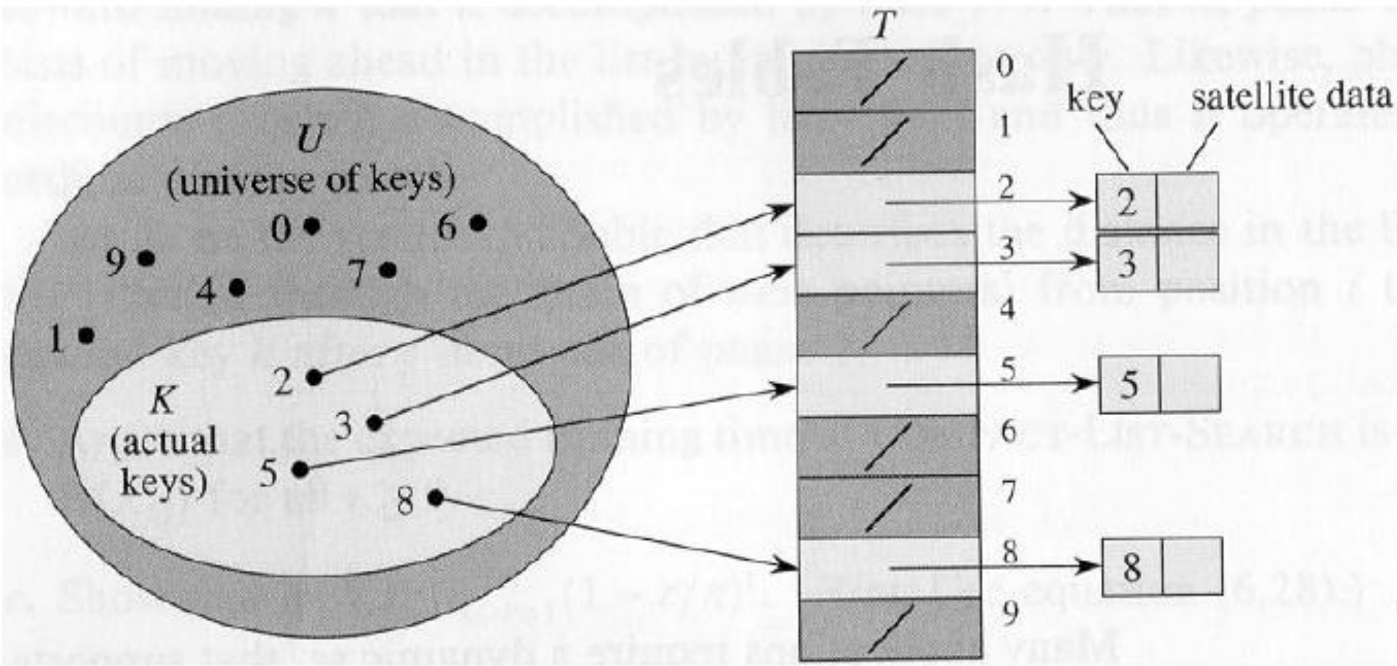
- **Dictionary** = data structure that supports mainly two basic operations: *insert* a new item and *return an item with a given key*
- Queries: return information about the set S:
  - Search (S, k)
  - Minimum (S), Maximum (S)
  - Successor (S, x), Predecessor (S, x)
- Modifying operations: change the set
  - Insert (S, k)
  - Delete (S, k) – not very often

# Direct Addressing

- Assumptions:
  - Key values are distinct
  - Each key is drawn from a universe U = {0, 1, . . . , m - 1}
- Idea:
  - Store the items in an array, indexed by keys

- **Direct-address table** representation:
  - An array T[0 . . . m - 1]
  - Each **slot**, or position, in T corresponds to a key in U
  - For an element x with key k, a pointer to x (or x itself) will be placed in location T[k]
  - If there are no elements with key k in the set, T[k] is empty, represented by NIL

# Direct Addressing (cont'd)




(insert/delete in O(1) time)

# Operations

*Alg.:* DIRECT-ADDRESS-SEARCH($T$, $k$)

**return** $T[k]$

*Alg.:* DIRECT-ADDRESS-INSERT($T$, $x$)

$T[key[x]] \leftarrow x$

*Alg.:* DIRECT-ADDRESS-DELETE($T$, $x$)

$T[key[x]] \leftarrow NIL$

- Running time for these operations: $O(1)$

# Comparing Different Implementations

- Implementing dictionaries using:
  - Direct addressing
  - Ordered/unordered arrays
  - Ordered/unordered linked lists

| | Insert | Search |
|---|---|---|
| direct addressing | O(1) | O(1) |
| ordered array | O(N) | O(lgN) |
| ordered list | O(N) | O(N) |
| unordered array | O(1) | O(N) |
| unordered list | O(1) | O(N) |

# Examples Using Direct Addressing

## Example 1:

(i) Suppose that the keys are integers from 1 to 100 and that there are about 100 records

(ii) Create an array $A$ of 100 items and store the record whose key is equal to $i$ in $A[i]$

## Example 2:

(i) Suppose that the keys are nine-digit social security numbers

(ii) We can use the same strategy as before but it very inefficient now: an array of 1 billion items is needed to store 100 records !!

- $|U|$ can be very large
- $|K|$ can be much smaller than $|U|$

# Hash Tables

- When $K$ is much smaller than $U$, a **hash table** requires much less space than a **direct-address table**
  - Can reduce storage requirements to $|K|$
  - Can still get $O(1)$ search time, but on the average case, not the worst case

# Hash Tables

**Idea:**

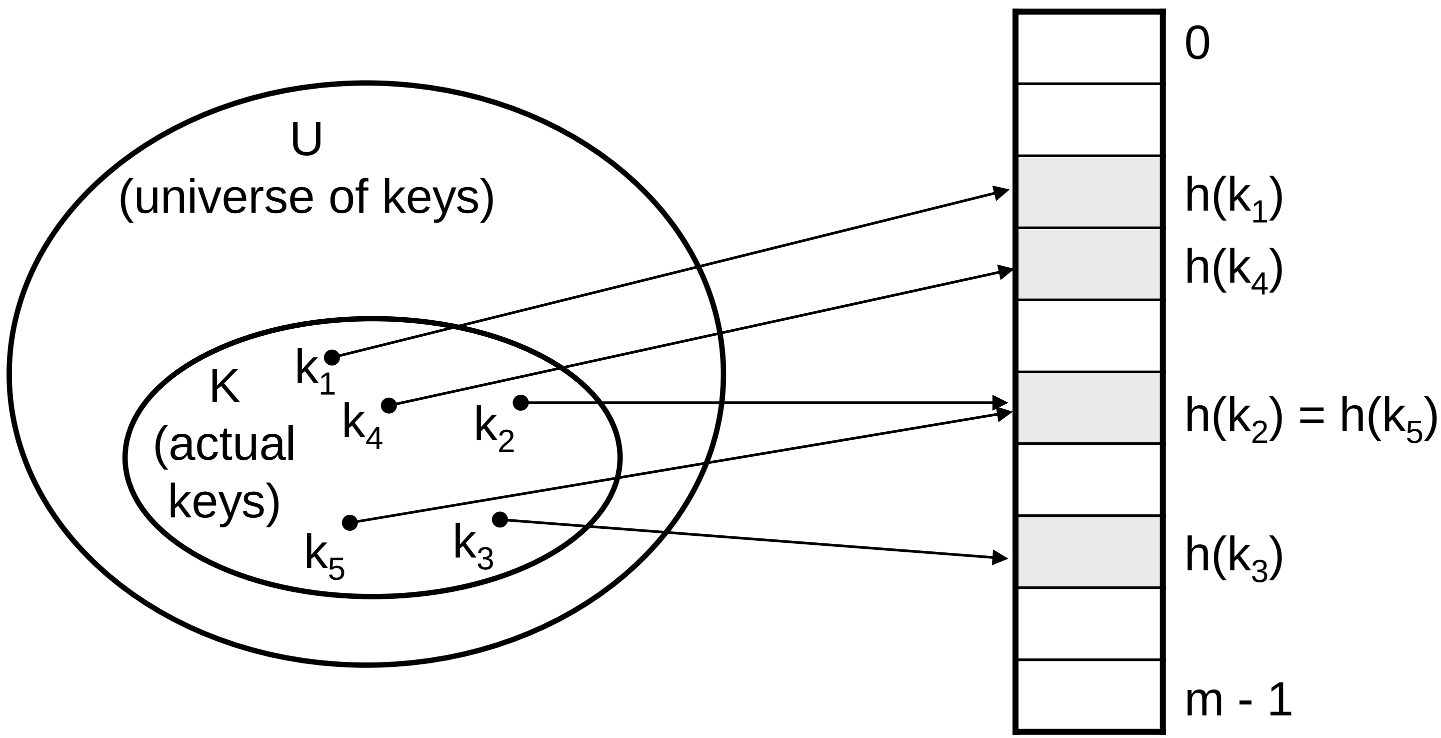
- Use a function $h$ to compute the slot for each key
- Store the element in slot $h(k)$

- A **hash function** $h$ transforms a key into an index in a hash table $T[0...m-1]$:

$$h : U \rightarrow \{0, 1, . . . , m - 1\}$$

- We say that $k$ **hashes** to slot $h(k)$
- Advantages:
  - Reduce the range of array indices handled: $m$ instead of $|U|$
  - Storage is also reduced

# Example: HASH TABLES



# Revisit Example 2

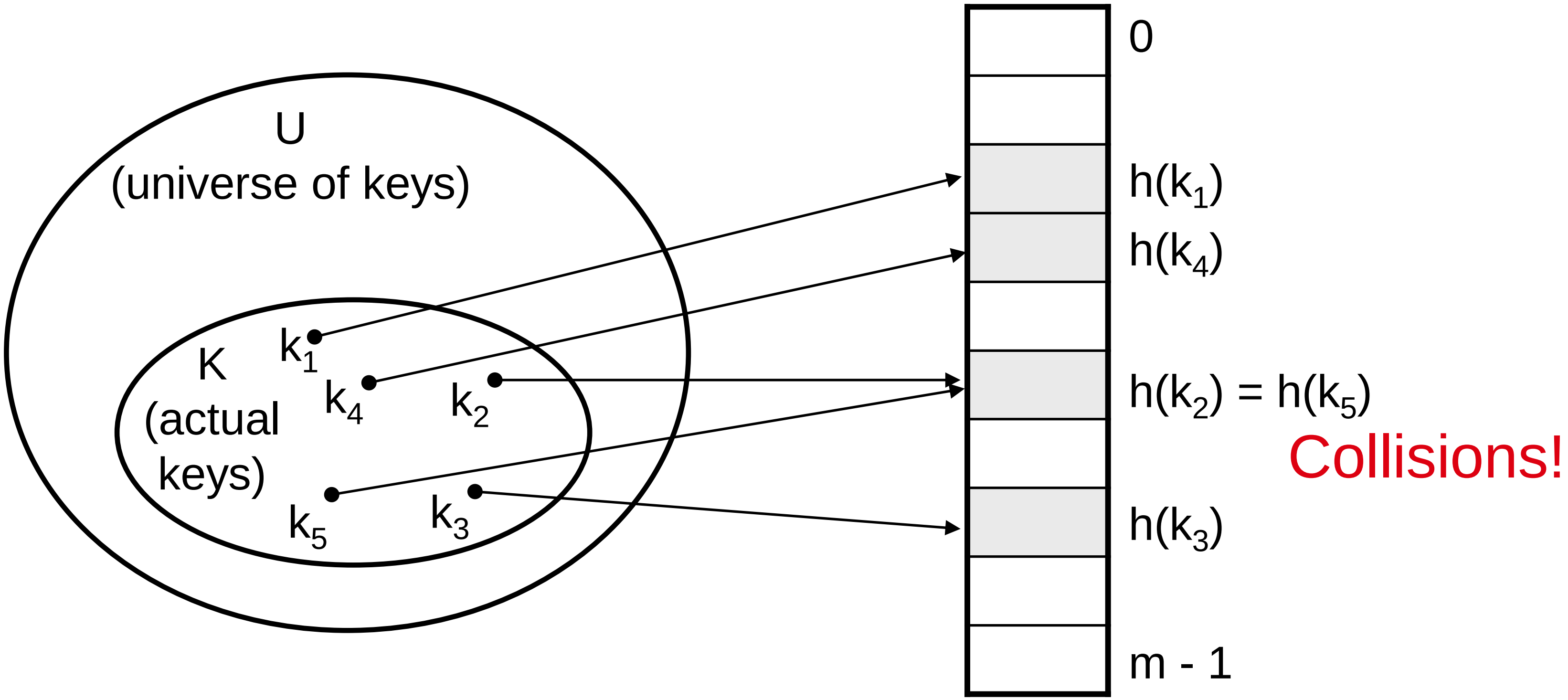
Suppose that the keys are nine-digit social security numbers

Possible hash function

$h(ssn) = sss \bmod 100$ (last 2 digits of ssn)

e.g., if $ssn = 10123411$ then $h(10123411) = 11$)

# Do you see any problems with this approach?



# Collisions

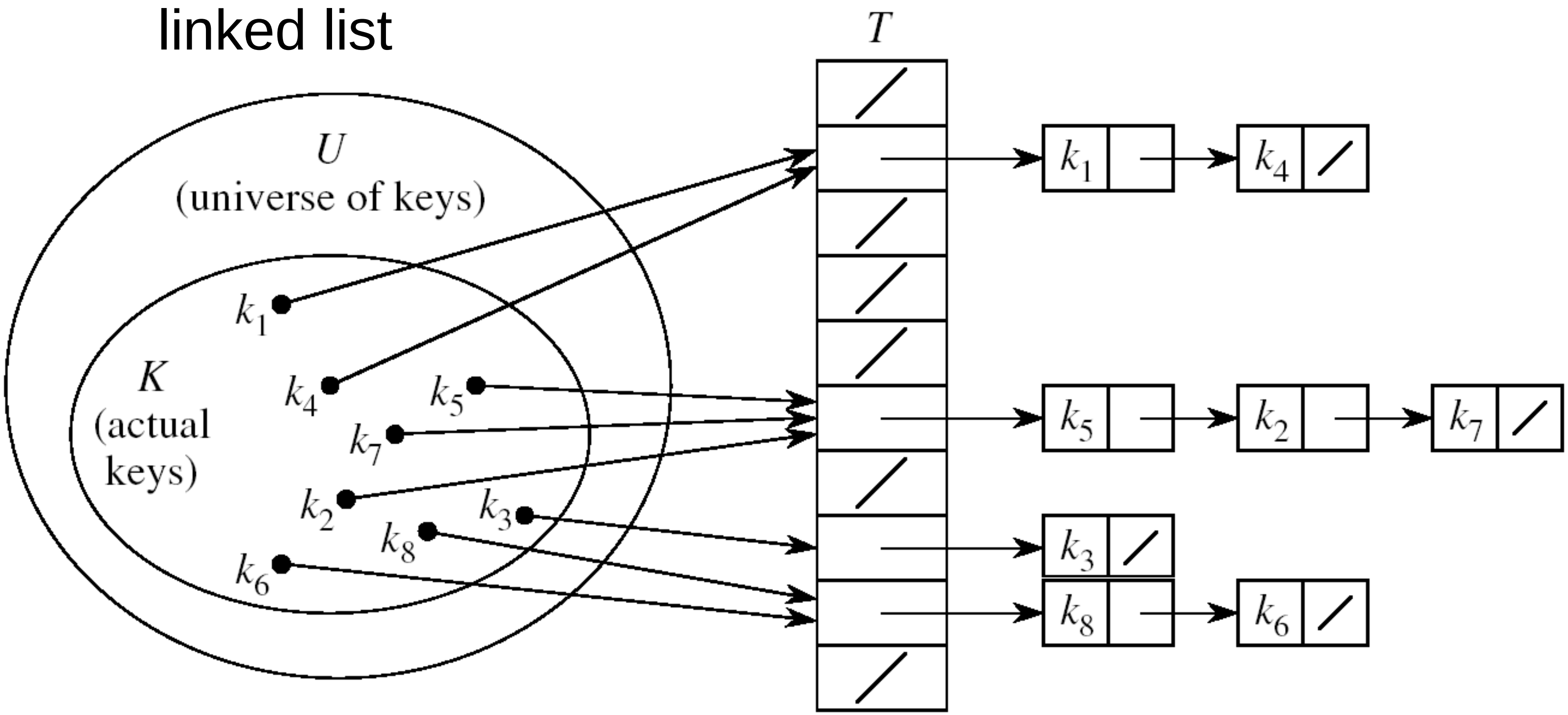
- Two or more keys hash to the same slot!!
- For a given set $K$ of keys
  - If $|K| \leq m$, collisions may or may not happen, depending on the hash function
  - If $|K| > m$, collisions will definitely happen (i.e., there must be at least two keys that have the same hash value)
- Avoiding collisions completely is hard, even with a good hash function

# Handling Collisions

- We will review the following methods:
  - Chaining
  - Open addressing
    - Linear probing
    - Quadratic probing
    - Double hashing
- We will discuss chaining first, and ways to build “good” functions.

# Handling Collisions Using Chaining

- **Idea**:
  - Put all elements that hash to the same slot into a linked list




  - Slot j contains a pointer to the head of the list of all elements that hash to j

# Collision with Chaining - Discussion

- Choosing the size of the table
  - Small enough not to waste space
  - Large enough such that lists remain short
  - Typically 1/5 or 1/10 of the total number of elements
- How should we keep the lists: ordered or not?
  - Not ordered!
    - Insert is fast
    - Can easily remove the most recently inserted elements

# Insertion in Hash Tables

*Alg.:* CHAINED-HASH-INSERT($T$, $x$)

insert $x$ at the head of list $T[h(key[x])]$

- Worst-case running time is $O(1)$
- Assumes that the element being inserted isn't already in the list
- It would take an additional search to check if it was already inserted

# Deletion in Hash Tables

*Alg.:* CHAINED-HASH-DELETE(T, x)

delete x from the list T[h(key[x])]

- Need to find the element to be deleted.
- Worst-case running time:
  - Deletion depends on searching the corresponding list

# Searching in Hash Tables

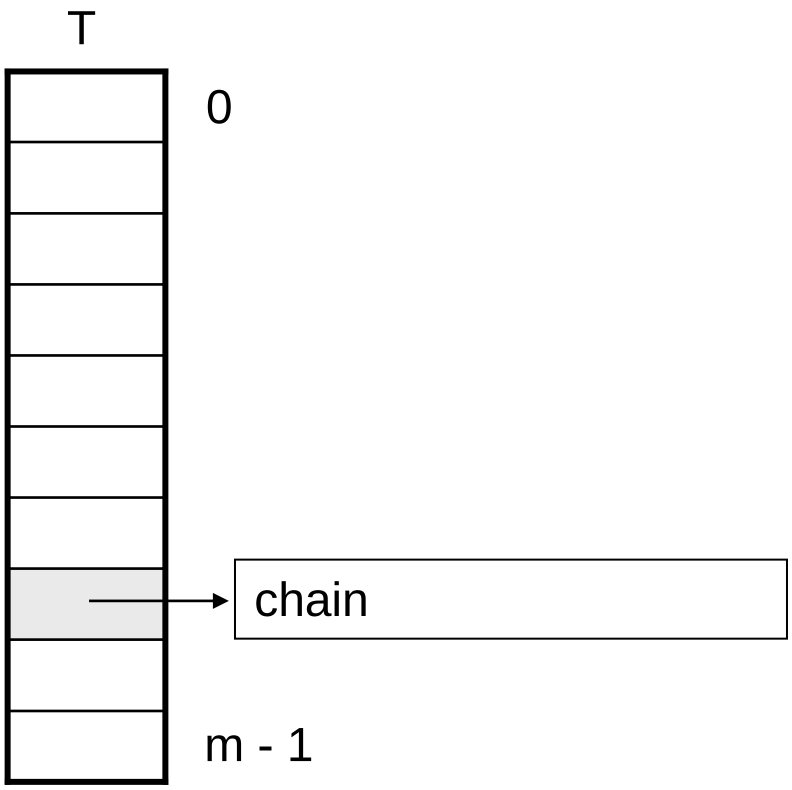
*Alg.:* CHAINED-HASH-SEARCH(T, k)

search for an element with key $k$ in list $T[h(k)]$

- Running time is proportional to the length of the list of elements in slot $h(k)$

# Analysis of Hashing with Chaining: Worst Case

- How long does it take to search for an element with a given key?
- Worst case:
  - All $n$ keys hash to the same slot
  - Worst-case time to search is $\Theta(n)$, plus time to compute the hash function



# Analysis of Hashing with Chaining: Average Case

- Average case
  - depends on how well the hash function distributes the $n$ keys among the $m$ slots
- **Simple uniform hashing** assumption:
  - Any given element is equally likely to hash into any of the $m$ slots (i.e., probability of collision Pr(h(x)=h(y)), is 1/m)
- Length of a list:

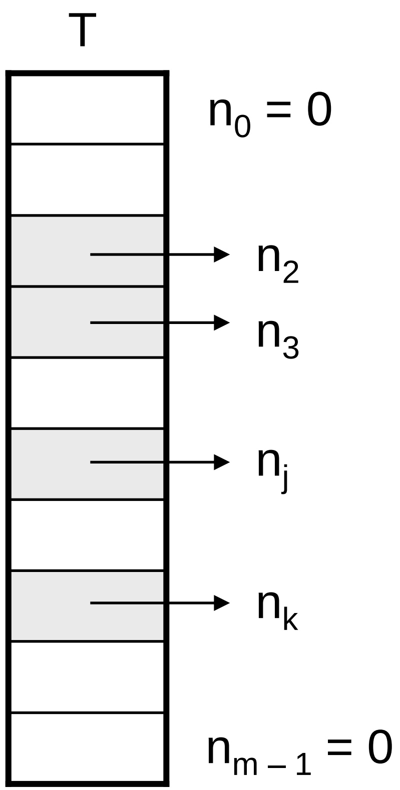
$$T[j] = n_j, \qquad j = 0, 1, \ldots, m-1$$

- Number of keys in the table:

$$n = n_0 + n_1 + \cdots + n_{m-1}$$

- Average value of $n_j$:

$$E[n_j] = \alpha = n/m$$

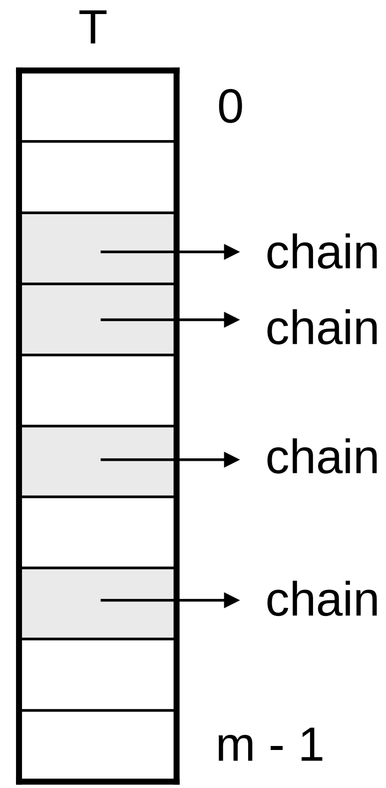


# Load Factor of a Hash Table

- Load factor of a hash table T:

$$\alpha = n/m$$

  - $n$ = # of elements stored in the table
  - $m$ = # of slots in the table = # of linked lists
- $\alpha$ encodes the average number of elements stored in a chain
- $\alpha$ can be <, =, > 1



# Case 1: Unsuccessful Search (i.e., item not stored in the table)

**Theorem**

An unsuccessful search in a hash table takes expected time $\Theta(1+\alpha)$ under the assumption of simple uniform hashing

(i.e., probability of collision Pr(h(x)=h(y)), is 1/m)

**Proof**

- Searching unsuccessfully for any key k
  - need to search to the end of the list T[h(k)]
- Expected length of the list:
  - $E[n_{h(k)}] = \alpha = n/m$
- Expected number of elements examined in an unsuccessful search is α
- Total time required is:
  - O(1) (for computing the hash function) + α → $\Theta(1+\alpha)$

# Case 2: Successful Search

Successful search: $\Theta(1+\frac{a}{2})=\Theta(1+a)$ time on the average

(search half of a list of length $a$ plus $O(1)$ time to compute $h(k)$)

# Analysis of Search in Hash Tables

- If m (# of slots) is proportional to n (# of elements in the table):
- $n = O(m)$
- $\alpha = n/m = O(m)/m = O(1)$

$\Rightarrow$ Searching takes constant time on average

# Hash Functions

- A hash function transforms a key into a table address
- **What makes a good hash function?**

  (1) Easy to compute

  (2) Approximates a random function: for every input, every output is equally likely (simple uniform hashing)

- In practice, it is very hard to satisfy the simple uniform hashing property
  - i.e., we don’t know in advance the probability distribution that keys are drawn from

# Good Approaches for Hash Functions

- Minimize the chance that closely related keys hash to the same slot
  - Strings such as pt and pts should hash to different slots
- **Derive a hash value that is independent from any patterns that may exist in the distribution of the keys**

# The Division Method

- **Idea:**
  - Map a key $k$ into one of the $m$ slots by taking the remainder of $k$ divided by $m$

$$h(k) = k \bmod m$$

- **Advantage**:
  - fast, requires only one operation
- **Disadvantage**:
  - Certain values of $m$ are bad, e.g.,
    - power of 2
    - non-prime numbers

# Example - The Division Method

- If $m = 2^p$, then $h(k)$ is just the least significant $p$ bits of $k$
  - $p = 1 \Rightarrow m = 2$
    $\Rightarrow h(k) = \{0, 1\}$ , least significant 1 bit of $k$
  - $p = 2 \Rightarrow m = 4$
    $\Rightarrow h(k) = \{0, 1, 2, 3\}$, least significant 2 bits of $k$
- Choose $m$ to be a prime, not close to a power of 2
  - Column 2: k mod 97
  - Column 3: k mod 100

| | m 97 | m 100 |
|---|---|---|
| 16838 | 57 | 38 |
| 5758 | 35 | 58 |
| 10113 | 25 | 13 |
| 17515 | 55 | 15 |
| 31051 | 11 | 51 |
| 5627 | 1 | 27 |
| 23010 | 21 | 10 |
| 7419 | 47 | 19 |
| 16212 | 13 | 12 |
| 4086 | 12 | 86 |
| 2749 | 33 | 49 |
| 12767 | 60 | 67 |
| 9084 | 63 | 84 |
| 12060 | 32 | 60 |
| 32225 | 21 | 25 |
| 17543 | 83 | 43 |
| 25089 | 63 | 89 |
| 21183 | 37 | 83 |
| 25137 | 14 | 37 |
| 25566 | 55 | 66 |
| 26966 | 0 | 66 |
| 4978 | 31 | 78 |
| 20495 | 28 | 95 |
| 10311 | 29 | 11 |
| 11367 | 18 | 67 |

# The Multiplication Method

**Idea:**

- Multiply key $k$ by a constant $A$, where $0 < A < 1$
- Extract the fractional part of $kA$
- Multiply the fractional part by $m$
- Take the floor of the result

$$h(k) = \lfloor m(kA - \lfloor kA \rfloor) \rfloor = \lfloor m\ \underbrace{(k\ A \bmod 1)}_{\text{fractional part of } kA = kA - \lfloor kA \rfloor} \rfloor$$

- **Disadvantage:** Slower than division method
- **Advantage:** Value of $m$ is not critical, e.g., typically $2^p$

# Example – Multiplication Method

- The value of $m$ is not critical now (e.g., $m = 2^p$)

assume $m = 2^3$

```
      .101101 (A)
       110101 (k)
---------------
1001010.0110011 (kA)
```

discard: 1001010

shift .0110011 by 3 bits to the left

011.0011

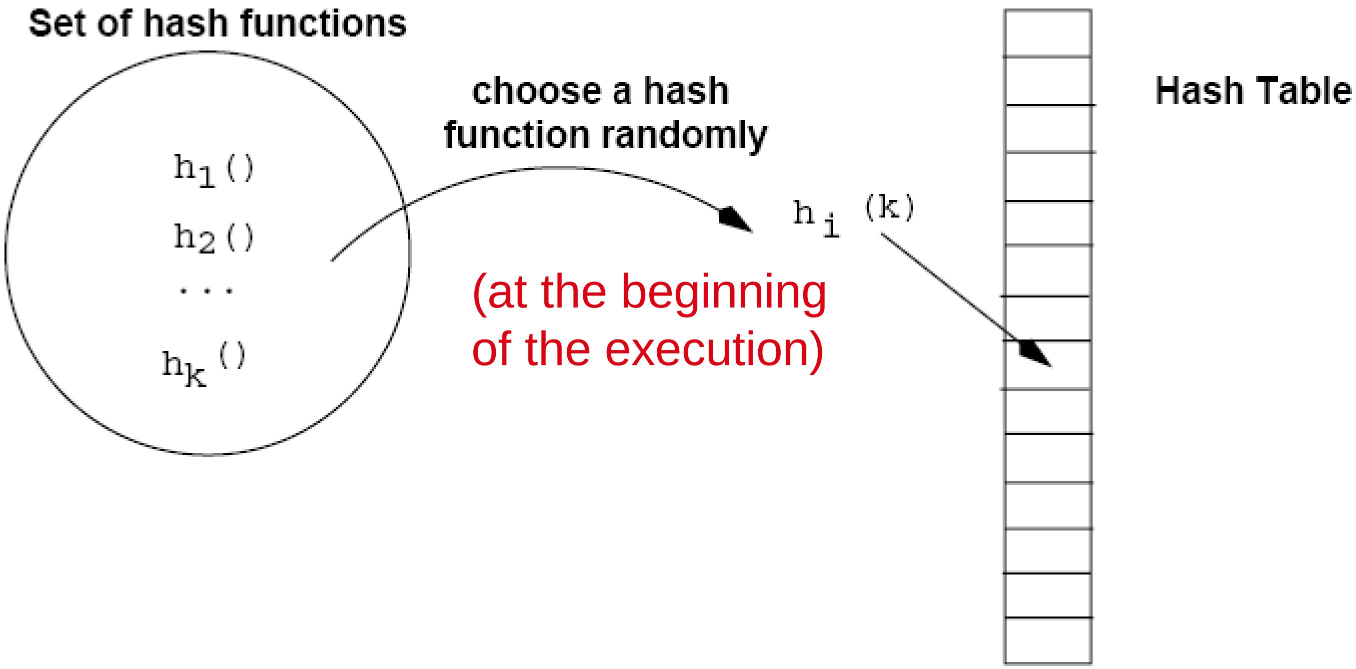
take integer part: 011

thus, h(110101)=011

# Universal Hashing

- In practice, keys are not randomly distributed
- Any fixed hash function might yield $\Theta(n)$ time
- Goal: hash functions that produce random table indices irrespective of the keys
- Idea:
  - Select a hash function at random, from a designed class of functions at the beginning of the execution

# Universal Hashing




(at the beginning
of the execution)

# Definition of Universal Hash Functions

H={h(k): U→(0,1,..,m-1)}

$H$ is said to be universal if

for $x \neq y$, **|(h() $\in$ H: h(x)=h(y)|=|H|/m**

(notation: |H|: number of elements in H - cardinality of H)

# How is this property useful?

- What is the probability of collision in this case ?

It is equal to the probability of choosing a function $h \in U$ such that $x \neq y$ --> $h(x) = h(y)$ which is

$$\Pr(h(x)=h(y)) = \frac{|H|/m}{|H|} = \frac{1}{m}$$

# Universal Hashing – Main Result

With universal hashing the chance of collision between distinct keys $k$ and $l$ is no more than the $1/m$ chance of collision if locations $h(k)$ and $h(l)$ were randomly and independently chosen from the set $\{0, 1, \ldots, m - 1\}$

# Designing a Universal Class of Hash Functions

- Choose a prime number $p$ large enough so that every possible key $k$ is in the range $[0 \ldots p - 1]$

  $Z_p = \{0, 1, \ldots, p - 1\}$ and $Z_p^* = \{1, \ldots, p - 1\}$

- Define the following hash function

  $h_{a,b}(k) = ((ak + b) \bmod p) \bmod m,$

  $\forall\ a \in Z_p^*$ and $b \in Z_p$

- The family of all such hash functions is

  $\mathcal{H}_{p,m} = \{h_{a,b} : a \in Z_p^* \text{ and } b \in Z_p\}$

The class $\mathcal{H}_{p,m}$ of hash functions is universal

- $a$ , $b$: chosen randomly at the beginning of execution

# Example: Universal Hash Functions

*E.g.:* $p = 17$, $m = 6$

$$h_{a,b}(k) = ((ak + b) \bmod p) \bmod m$$

$$h_{3,4}(8) = ((3 \cdot 8 + 4) \bmod 17) \bmod 6$$

$$= (28 \bmod 17) \bmod 6$$

$$= 11 \bmod 6$$

$$= 5$$

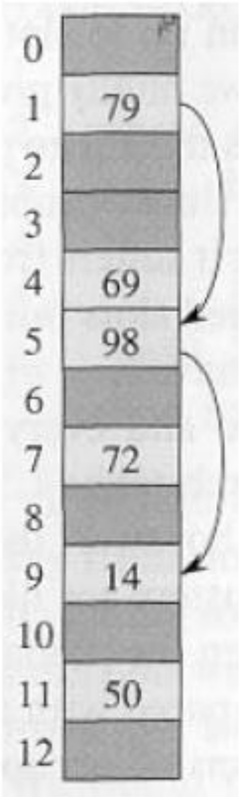
# Advantages of Universal Hashing

- Universal hashing provides good results on average, independently of the keys to be stored
- Guarantees that no input will always elicit the worst-case behavior
- Poor performance occurs only when the random choice returns an inefficient hash function – this has small probability

# Open Addressing

- If we have enough contiguous memory to store all the keys ($m > N$) $\Rightarrow$ store the keys in the table itself
- No need to use linked lists anymore
- Basic idea:
  - <u>Insertion:</u> if a slot is full, try another one, until you find an empty one
  - <u>Search:</u> follow the same sequence of probes
  - <u>Deletion:</u> more difficult ... (we'll see why)
- Search time depends on the length of the probe sequence!

e.g., insert 14



# Generalize hash function notation:

- A hash function contains two arguments now:

  (i) Key value, and (ii) Probe number

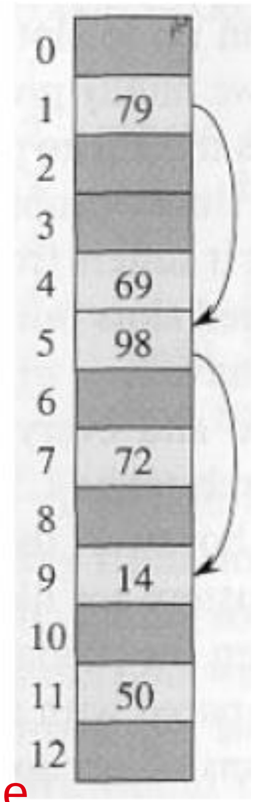
  $h(k,p), \quad p=0,1,\ldots,m-1$

- Probe sequences

  <h(k,0), h(k,1), ..., h(k,m-1)>

  - Must be a permutation of <0,1,...,m-1>
  - There are $m!$ possible permutations
  - Good hash functions should be able to produce all $m!$ probe sequences

insert 14




Example

<1, 5, 9>

# Common Open Addressing Methods

- Linear probing
- Quadratic probing
- Double hashing

- **Note:** None of these methods can generate more than $m^2$ different probing sequences!

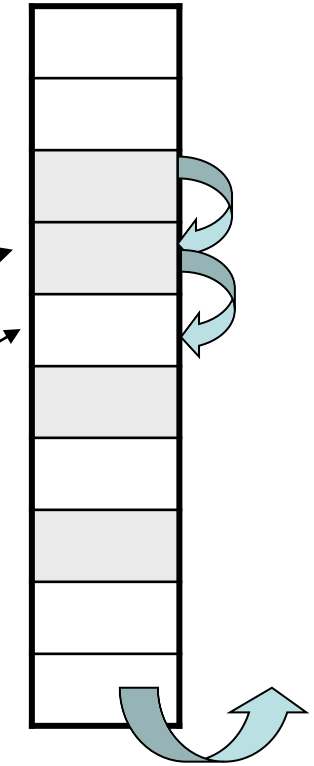
# Linear probing: Inserting a key

- Idea: when there is a collision, check the next available position in the table (i.e., probing)

$$h(k,i) = (h_1(k) + i) \bmod m$$

$$i = 0, 1, 2, \ldots$$

- First slot probed: $h_1(k)$
- Second slot probed: $h_1(k) + 1$
- Third slot probed: $h_1(k)+2$, and so on



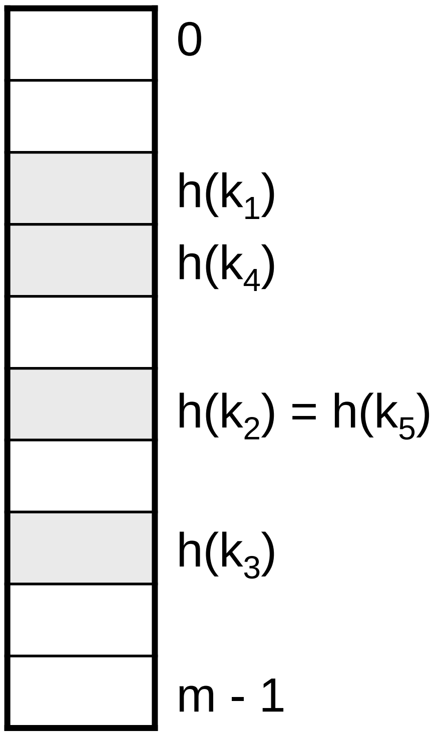
probe sequence: < h1(k), h1(k)+1 , h1(k)+2 , ....>

- Can generate $m$ probe sequences maximum, why?

wrap around

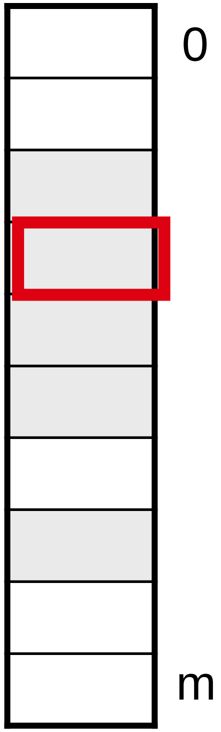
# Linear probing: Searching for a key

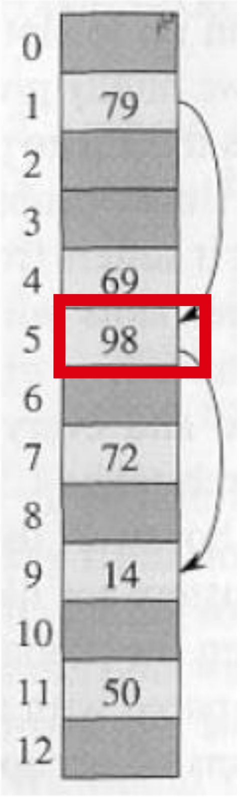
- Three cases:
  - (1) Position in table is occupied with an element of equal key
  - (2) Position in table is empty
  - (3) Position in table occupied with a different element
- Case 2: probe the next higher index until the element is found or an empty position is found
- The process wraps around to the beginning of the table



# Linear probing: Deleting a key

- Problems
  - Cannot mark the slot as empty
  - Impossible to retrieve keys inserted after that slot was occupied
- Solution
  - Mark the slot with a sentinel value DELETED
- The deleted slot can later be used for insertion
- Searching will be able to find all the keys



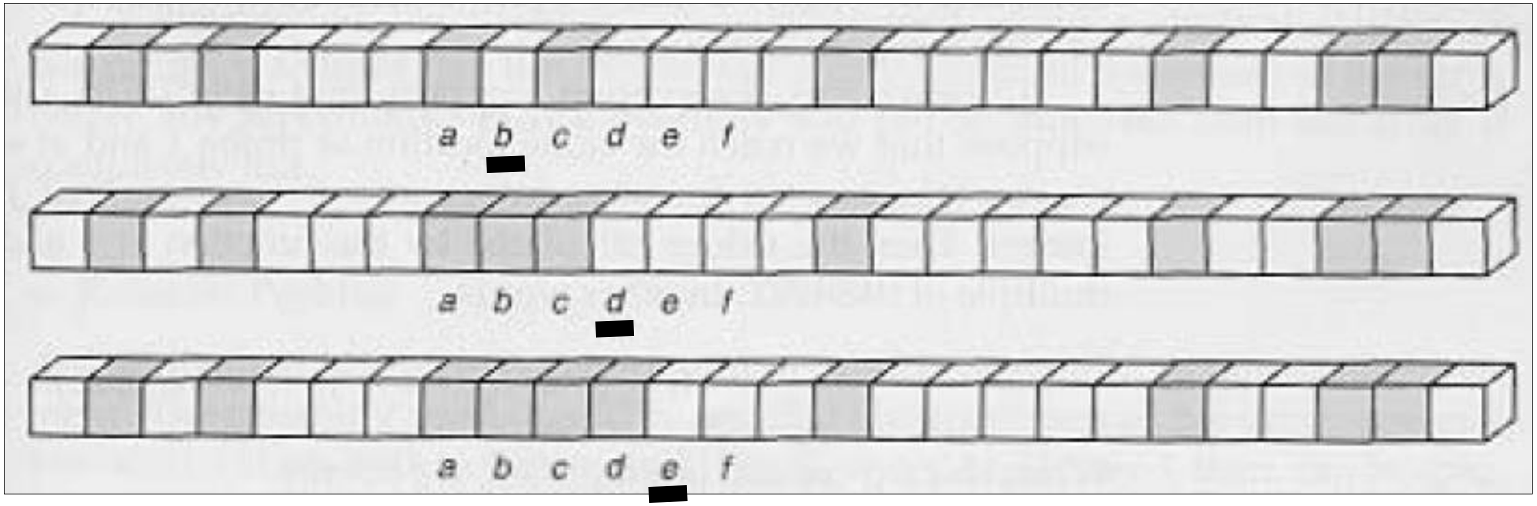



# Primary Clustering Problem

- Some slots become more likely than others
- Long chunks of occupied slots are created

$\Rightarrow$ search time increases!!

initially, all slots have probability 1/m




Slot b: 2/m

Slot d: 4/m

Slot e: 5/m

# Quadratic probing

$h(k,i) = (h'(k) + c_1 i + c_2 i^2) \bmod m$, where $h' : U --> (0, 1, \ldots, m-1)$

i=0,1,2,...

- Clustering problem is less serious but still an issue (*secondary clustering*)

- How many probe sequences quadratic probing generate ? $m$

(the initial probe position determines the probe sequence)

# Double Hashing

(1) Use one hash function to determine the first slot

(2) Use a second hash function to determine the increment for the probe sequence

$$h(k,i) = (h_1(k) + i\, h_2(k)) \bmod m, \quad i=0,1,...$$

- Initial probe: $h_1(k)$
- Second probe is offset by $h_2(k) \bmod m$, so on ...
- Advantage: avoids clustering
- Disadvantage: harder to delete an element
- Can generate $m^2$ probe sequences maximum

# Double Hashing: Example

$h_1(k) = k \bmod 13$

$h_2(k) = 1 + (k \bmod 11)$

$h(k,i) = (h_1(k) + i\, h_2(k)) \bmod 13$

- Insert key 14:

$h_1(14,0) = 14 \bmod 13 = 1$

$h(14,1) = (h_1(14) + h_2(14)) \bmod 13$

$= (1 + 4) \bmod 13 = 5$

$h(14,2) = (h_1(14) + 2\, h_2(14)) \bmod 13$

$= (1 + 8) \bmod 13 = 9$

| | |
|---|---|
| 0 | |
| 1 | 79 |
| 2 | |
| 3 | |
| 4 | 69 |
| 5 | 98 |
| 6 | |
| 7 | 72 |
| 8 | |
| 9 | 14 |
| 10 | |
| 11 | 50 |
| 12 | |

# Analysis of Open Addressing

- Ignore the problem of clustering and assume that all probe sequences are equally likely

<u>Unsuccessful retrieval:</u>

Prob(*probe hits an occupied cell*)= $a$ (load factor)

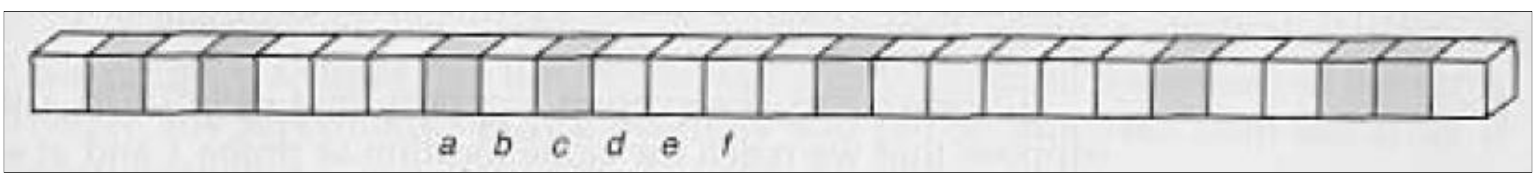
Prob(*probe hits an empty cell*)= $1-a$

probability that a probe terminates in 2 steps: $a(1-a)$

probability that a probe terminates in k steps: $a^{k-1}(1-a)$

What is the average number of steps in a probe ?

$$E(\#steps) = \sum_{k=1}^{m} ka^{k-1}(1-a) \leq \sum_{k=0}^{\infty} ka^{k-1}(1-a) = (1-a)\frac{1}{(1-a)^2} = \frac{1}{1-a}$$



# Analysis of Open Addressing (cont'd)

Successful retrieval:

$$E(\#steps) = \frac{1}{a}\, ln(\frac{1}{1-a})$$

Example (similar to **Exercise 11.4-4, page 244**)

Unsuccessful retrieval:

a=0.5 E(#steps) = 2

a=0.9 E(#steps) = 10

Successful retrieval:

a=0.5 E(#steps) = 3.387

a=0.9 E(#steps) = 3.670