The Remainder and Factor Theorems

Remainder Theorem

$$p(x) = (x - a) \cdot q(x) + r$$

When you divide a polynomial $f\left(x\right)$ by a divisor $d\left(x\right)$, you get a quotient $q\left(x\right)$ and a remainder polynomial $r\left(x\right)$. This is written as $\dfrac{f\left(x\right)}{d\left(x\right)}=q\left(x\right)+\dfrac{r\left(x\right)}{d\left(x\right)}$. The degree of the remainder must be less than the degree of the divisor.

Example 1

Divide $2x^4+3x^3+5x-1$ by x^2-2x+2

$$
\begin{array}{r}
2x^2+7x+10 \\
x^2-2x+2\overline{\big)\,2x^4+3x^3+0x^2+5x-1} \\
\underline{-\left(2x^4-4x^3+4x^2\right)} \\
7x^3-4x^2+5x-1 \\
\underline{-\left(7x^3-14x^2+14x\right)} \\
10x^2-\ \ 9x-\ \ 1 \\
\underline{10x^2-20x+20} \\
11x+21
\end{array}
$$

$$\frac{2x^4+3x^3+5x-1}{x^2-2x+2} = 2x^2+7x+10+\frac{11x-21}{x^2-2x+2}$$

Remainder Theorem

If a polynomial $f(x)$ is divided by $x-k$, then the remainder is $r = f(k)$.

Synthetic Division: A process for division of a polynomial by a binomial.

Example 2: Divide

$$2x^3 - 7x^2 - 8x + 16 \text{ by } x - 4$$

Step 1: Write the terms of the polynomial in descending order.

$$2x^3 - 7x^2 - 8x + 16$$

Then write just the coefficients as shown.

$$2 \quad -7 \quad -8 \quad 16$$

Step 2: Write the constant k, of the divisor x – k to the left

$$4\,\rfloor \quad 2 \quad -7 \quad -8 \quad 16$$

Bring the first coefficient down. Multiply the first coefficient by k and write the product under the second coefficient. Add the second column. Write the sum as shown.

To write the result

$$\frac{2x^3 - 7x^2 - 8x + 16}{x - 4} = 2x^2 + x - 4$$

Example 3:

Divide:

$x^3 + 2x^2 - 6x - 9$ by $x - 2$

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -6 & -9 \\ & & 2 & 8 & 4 \\ \hline & 1 & 4 & 2 & -5 \end{array}$$

$$\frac{x^3 + 2x^2 - 6x - 9}{x - 2} = x^2 + 4x + 2 - \frac{5}{x - 2}$$

Proof of Remainder Theorem

Let us assume that $q(x)$ and 'r' are the quotient and the remainder respectively when a polynomial $p(x)$ is divided by a linear polynomial $(x - a)$. By division algorithm,

Dividend = (Divisor × Quotient) + Remainder.

Using this, $p(x) = (x - a) \cdot q(x) + r$.

Substitute $x = a$

$p(a) = (a - a) \cdot q(a) + r$

$p(a) = (0) \cdot q(a) + r$

$p(a) = r$

i.e. the remainder = $p(a)$.

Factor Theorem

$$p(x) = (x - a) \cdot q(x)$$

Factor Theorem

A polynomial $f\left(x\right)$ has a factor $x-k$ if and only if $f\left(k\right)=0$

Example 4

One zero of $f\left(x\right)=x^3-2x^2-9x+18$ is $x=2$.

Find the other zeros of the function.

x = 2

Subtract 2 from both sides

x – 2 = 0

So, x – 2 is a factor

$$\begin{array}{r|rrrr} 2 & 1 & -2 & -9 & 18 \\ & & 2 & 0 & -18 \\ \hline & 1 & 0 & -9 & 0 \end{array}$$

Because $f\left(2\right.$ -2 is a factor.

Use syntheti her factors.

Because $f(2) = 0$, you know that $x - 2$ is a factor.

Use synthetic division to find the other factors.

$$
\begin{array}{r|rrrr}
2 & 1 & -2 & -9 & 18 \\
  &   & 2 & 0 & -18 \\
\hline
  & 1 & 0 & -9 & 0
\end{array}
$$

The result gives the coefficients of the quotient.

$$f(x) = (x-2)(x^2-9)$$

$$=$$

b. x^3+2x^2-6x-9 by $x+3$

To find the value of k, rewrite the divisor in the form $x-k$. Because $x+3=x-(-3)$, $k=-3$

$$
\begin{array}{r|rrrr}
-3 & 1 & 2 & -6 & -9 \\
 & & -3 & 3 & 9 \\
\hline
 & 1 & -1 & -3 & 0
\end{array}
$$

$$\frac{x^3+2x^2-6x-9}{x+3}=x^2-x-3$$

Example 4

Factor

$f(x)=2x^3+11x^2+18x+9$ given that $f(-3)=0$

Because $f(-3)=0$, you know that $x-(-3)=0$

or $x+3=0$ is a factor of $f(x)$.

Use synthetic division to find the other factors.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

The result gives the coefficients quotient

$$2x^3+11x^2+18x+9=(x+3)(2x^2+5x+3)$$

$$=(x+3)(2x+3)(x+1)$$

gcd(a,b)
greatest common divisor of a and b

Arayabhata-Euclid's algorithm:
How to find gcd(a,b),
the greatest common divisor of a and b

Based on a single observation: if $a = b\,q + r$,
then any divisor of a and b is also a divisor of r,
and any divisor of b and r is also a divisor of a,
so $\gcd(a,b) = \gcd(b,r)$

Euclid algorithm:

Example: gcd(55,35)

55 = 35*1 + 20	so gcd(55,35) = gcd(35,20)
35 = 20*1 + 15	so gcd(35,20) = gcd(20,15)
20 = 15*1 + 5	so gcd(20,15) = gcd(15,5)
15 = 5*3 + 0	so gcd(15,5) = gcd(5,0)