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PHYS 512 – Computational Physics

Question 5:

a) $$G = \sum_{x=0}^{N-1} \exp\left(\frac{-2\pi i k x}{N}\right) = \sum_{x=0}^{N-1} \left[\exp\left(\frac{-2\pi i k}{N}\right)\right]^x$$

$$\underset{\text{Geometric series}}{=} \frac{1-\exp(-2\pi i k N/N)}{1-\exp(-2\pi i k/N)} = \frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)}$$

b). As $k \to 0$, $\begin{cases} \exp(-2\pi i k) \simeq 1 - 2\pi i k \\ \exp(-2\pi i k/N) \simeq 1 - \frac{2\pi i k}{N} \end{cases}$

$$\Rightarrow G = \frac{1 - 1 + 2\pi i k}{1 - 1 + \frac{2\pi i k}{N}} = N, \text{ as } k \to 0.$$

- For any integer $k = z$:

$$\exp(-2\pi i k) = \exp(-2\pi i z) = \cos(-2\pi z) + i \underbrace{\cancel{\sin(-2\pi z)}}_{0} = 1.$$

$$\exp\left(-\frac{2\pi i k}{N}\right) = \exp\left(-2\pi i \frac{k}{N}\right) \begin{cases} \neq 0, & \text{if } k/N \notin \mathbb{Z} \\ = 0, & \text{if } k/N \in \mathbb{Z} \end{cases}$$

$$\Rightarrow G = 0, \text{ for any } k \neq z'.N, \; z' \in \mathbb{Z}$$

c) $$\sin\left(\frac{2\pi k x}{N}\right) = \frac{1}{2i} \cdot \left[\exp\left(\frac{2\pi i k x}{N}\right) - \exp\left(\frac{-2\pi i k x}{N}\right)\right]$$

$$\Rightarrow F\left[\sin\left(\frac{2\pi k x}{N}\right)\right](k') = \frac{1}{2i} \sum_{x=0}^{N-1} \left[\exp\left(\frac{2\pi i k x}{N}\right) - \exp\left(\frac{-2\pi i k x}{N}\right)\right]$$
$$\times \exp\left(\frac{-2\pi i k' x}{N}\right)\Big]$$

$$\Leftrightarrow F(k') = \frac{1}{2i} \sum_{x=0}^{N-1} \exp\left[\frac{-2\pi i (k+k')x}{N}\right] - \frac{1}{2i} \sum_{x=0}^{N-1} \exp\left[\frac{-2\pi i (k+k')x}{N}\right]$$

$$= \frac{1}{2i}\left\{ \frac{1-\exp[-2\pi i(k'-k)]}{1-\exp[-2\pi i(k'-k)/N]} - \frac{1-\exp[-2\pi i(k'+k)]}{1-\exp[-2\pi i(k'+k)/N]} \right\}$$

$$\Rightarrow \mathrm{Re}\{\mathcal{F}[\sin](k')\} = \begin{cases} \frac{1}{2}N & (k'=k) \\ -\frac{1}{2}N & (k'=-k) \\ =0 & (\text{o.w}) \end{cases}$$

e). $$f(x) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{N}\right) = \frac{1}{2} - \frac{1}{4}\left[\exp\left(2\pi i\frac{x}{N}\right) + \exp\left(\frac{-2\pi i x}{N}\right)\right]$$

$$\Rightarrow F(k) = \sum_{x=0}^{N-1}\left\{\frac{1}{2} - \frac{1}{4}\left[\exp\left(2\pi i\frac{x}{N}\right) + \exp\left(\frac{-2\pi i x}{N}\right)\right]\right\}\exp\left(\frac{-2\pi i k x}{N}\right)$$

$$= \sum_{x=0}^{N-1}\left\{\frac{1}{2}\underbrace{\exp\left(\frac{-2\pi i k x}{N}\right)}_{\substack{\text{Geometric} \\ =N,\ \text{when } k=0 \\ =0,\ \text{o.w}}} - \frac{1}{4}\underbrace{\exp\left[\frac{-2\pi i x(k-1)}{N}\right]}_{\substack{\text{Geometric} \\ =N,\ \text{when } k=1 \\ =0,\ \text{o.w}}} + \frac{1}{4}\underbrace{\exp\left[\frac{2\pi i(k+1)x}{N}\right]}_{\substack{\text{Geometric} \\ =N,\ \text{when } k=-1 \\ =0,\ \text{o.w}}}\right\}$$

$$\Rightarrow F(k) = \frac{N}{2}\delta_{k,0} + \frac{N}{4}\delta_{k,1} - \frac{N}{4}\delta_{k,-1}$$

- Let $G = \sin\left(\frac{2\pi x k}{N}\right) f(x)$

$$\Rightarrow \mathcal{F}\{G\}(k') = \mathcal{F}\left\{\sin\left(\frac{2\pi x k}{N}\right)\right\} * \mathcal{F}\{f(x)\}$$

$$= \sum_{k''} \mathcal{F}\left\{\sin\left(\frac{2\pi x k}{N}\right)\right\}(k'') \cdot \mathcal{F}\{f(x)\}(k'-k'')$$

$$= \sum_{k''} \mathcal{F}\left\{\sin\left(\frac{2\pi x k}{N}\right)\right\}(k'')\left[\frac{N}{2}\delta_{k''-k',0} + \frac{N}{4}\delta_{k''-k',1} - \frac{N}{4}\delta_{k''-k',-1}\right]$$

$$F(k') = \underbrace{\frac{N}{2} \mathcal{F}\left\{ \sin\left(\frac{2\pi x k}{N}\right)\right\}(k')}_{\text{Evaluate at } k'} + \underbrace{\frac{N}{4} \mathcal{F}\left\{ \sin\left(\frac{2\pi x k}{N}\right)\right\}(k'+1) - \frac{N}{4} \mathcal{F}\left\{ \sin\left(\frac{2\pi x k}{N}\right)\right\}(k'-1)}_{\text{Evaluate at } k' \text{ direct neighbors}}$$