Poisson Distribution

Poisson distribution was discovered by the French mathematician & physicist Simeon Denis Poisson (1781-1840) who published it in 1837. Poisson distribution is found in cases of events that occur rarely (Poisson distribution is the distribution of rare events).

The Poisson distribution is a limiting case of the binomial distribution under the following conditions:

1. n, the no of trails is indefinitely large, i.e. n → ∞.

2. p ,the constant probability for the success of each trail is indefinitely small, i.e. p → 0.

3. np = λ(say) is finite.

Poisson distribution as a limiting case of the Binomial distribution

Let $X \sim B(n, p)$
Then
$$P(X = x) = p_x = nC_x p^x q^{n-x}, \ x = 0,1,2,3,...n; \ q = 1-p$$

$$i.e\ P(X = x) = \frac{n!}{x!(n-x)!}\left(\frac{np}{n}\right)^x (1-p)^{n-x}, \ x = 0,1,2,3,...n$$

$$= \frac{(n-(x-1))(n-(x-2))(n-(x-3))...(n-1)n}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{np}{n}\right)^{n-x}$$

$$= \frac{n(1-(x-1/n)n(1-(x-2)/n)n(1-(x-3)/n)...n(1-(n-1)/n)n}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n^x(1-(x-1)/n)(1-(x-2)/n)(1-(x-3)/n)...(1-(n-1)/n)(1)}{x!}\left(\frac{np}{n}\right)^x\frac{\left(1-\frac{np}{n}\right)^n}{\left(1-\frac{np}{n}\right)^x}$$

$$i.e\ P(X = x) = \frac{(1-(x-1)/n)(1-(x-2)/n)(1-(x-3)/n)...(1-(n-1)/n)(1)}{x!}(np)^x\frac{\left(1-\frac{np}{n}\right)^n}{\left(1-\frac{np}{n}\right)^x} \quad ....(I)$$

Now let $n \to \infty$ such that $np = \lambda$ is finite. Using this in *(I)* we get

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \ x = 0,1,2,... \left(\because \lim_{n \to \infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \ \& \ \lim_{n \to \infty}\left(1-\frac{\lambda}{n}\right)^x = 1\right)$$

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Definition: A random variable X is said to follow a Poisson distribution with parameter 'λ' if it assumes only non-negative, integral values & its probability distribution is given by:

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x=0,1,2,3,... \ \&\ \lambda>0\cdot$$

We shall use the notation $X \sim P(\lambda)$ to denote that X is a Poisson variate with parameter 'λ' .

Solved Examples:

1. A transmission channel has a per-digit error probability $p=0.01$ Calculate the probability of more than 1 error in 10 received digits using (i) Binomial distribution (ii) Poisson distribution

   Solution: (i) Let X denote the number of errors in 10 received digits

   Then $X \sim B(n=10, p=0.01)$ Therefore
   $$P(\text{ more than 1 error in 10 received digits}) = P(X > 1)$$
   $$= 1 - P(X \leq 1)$$
   $$= 1 - \left[10C_0(0.01)^0(0.99)^{10} + 10C_1(0.01)^1(0.99)^9\right]$$
   $$= 0.004$$

   (ii) Let X denote the number of errors

   Then X *follows a* $P(\lambda = np = 10(0.01) = 0.1)$

   Therefore

   $$P(\text{ more than 1 error in 10 received digits}) = P(X > 1)$$
   $$= 1 - P(X \leq 1)$$

   $$= 1 - \left[\frac{e^{-0.1}(0.1)^0}{0!} + \frac{e^{-0.1}(0.1)^{01}}{1!}\right] = 0.004$$

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2. Find the probability that atmost 4 defective bulbs will be found in a box of 200 bulbs if it is known that 2% of the bulbs are defective.

   Solution: We do Poisson approximation to Binomial here.

   Let X denote number of defective bulbs in the box

   We have $\lambda = np = 200(\dfrac{2}{100}) = 4$

   Therefore $X \sim P(\lambda = 4)$

   Hence

   $P(\text{atmost 4 defective bulbs}) = P(X \leq 4)$
   $$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$
   $$= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!} + \frac{e^{-4}4^4}{4!}$$
   $$= 0.6283$$

3. If X is a Poisson Variate such that $P(X = 1) = P(X = 2)$ find $P(X = 3)$ and $E(X^2)$.

   Solution: Given: $P(X = 1) = P(X = 2)$ Since $X \square P(\lambda)$

   This implies $\dfrac{e^{-\lambda}\lambda}{1!} = \dfrac{e^{-\lambda}\lambda^2}{2!}$

   $\Rightarrow \lambda = 2$

   $E(X^2) = \lambda^2 + \lambda$
   $\qquad = 4 + 2 = 6$

   Now $P(X = 3) = \dfrac{e^{-\lambda}\lambda^3}{3!}$

   $\qquad = \dfrac{8e^{-2}}{6}$

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