Contents

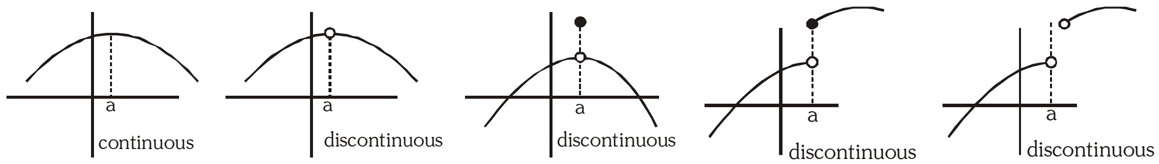
ALLEN CAREER INSTITUTE KOTA (RAJASTHAN) Path to success

MATHEMATICS

CONTINUITY

1.0 CONTINUOUS FUNCTIONS

A function for which a small change in the independent variable causes only a small change and not a sudden jump in the dependent variable are called continuous functions. Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function around that point without lifting the pen from the plane of the paper.



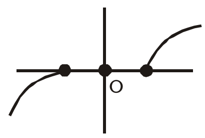
Continuity of a function at a point

A function f(x) is said to be continuous at x = a, if $\lim\limits_{x \to a} f(x) = f(a)$. Symbolically f is continuous at x = a if

$\lim\limits_{h \to 0} f(a-h) = \lim\limits_{h \to 0} f(a+h) = f(a)$, h > 0

It should be noted that continuity of a function at x = a can be discussed only if the function is defined in the immediate neighbourhood of x = a, not necessarily at x = a.

Ex. Continuity at x = 0 for the curve can not be discussed.



Illustrations

Illustration 1. If $f(x) = \begin{cases} \sin\dfrac{\pi x}{2}, & x < 1 \\ [x] & x \geq 1 \end{cases}$ then find whether f(x) is continuous or not at x = 1, where [ ] denotes greatest integer function.

Solution $f(x) = \begin{cases} \sin\dfrac{\pi x}{2}, & x < 1 \\ [x] & , x \geq 1 \end{cases}$

For continuity at x = 1, we determine, f(1), $\lim\limits_{x \to 1^-} f(x)$ and $\lim\limits_{x \to 1^+} f(x)$.

Now, f(1) = [1] = 1

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sin\frac{\pi x}{2} = \sin\frac{\pi}{2} = 1 \text{ and } \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} [x] = 1$$

so $f(1) = \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$

$\therefore$ f(x) is continuous at x = 1

Illustration 2. Consider $f(x) = \begin{cases} \dfrac{8^x - 4^x - 2^x + 1}{x^2}, & x > 0 \\ e^x \sin x + \pi x + k\ell n4, & x < 0 \end{cases}$ Define the function at x = 0 if possible, so that f(x) becomes continuous at x = 0.

Solution

$$f(0^+) = \lim_{h \to 0} \frac{8^h - 4^h - 2^h + 1}{h^2} = \lim_{h \to 0} \frac{4^h(2^h - 1) - (2^h - 1)}{h^2}$$

$$= \lim_{h \to 0} \frac{(4^h - 1)}{h} \frac{(2^h - 1)}{h} = \ell n4 \,.\, \ell n2$$

$$f(0^-) = \lim_{x \to 0^-} \left(e^x \sin x + \pi x + k\ell n4\right) = k\ell n4$$

f(x) is continuous at x = 0,

$\Rightarrow$ $f(0^+) = f(0^-) = f(0) \Rightarrow$ $\ell n4 . \ell n2 = k\ell n4$ $\Rightarrow$ $k = \ell n2$ $\Rightarrow$ $f(0) = (\ell n4)(\ell n2)$

Illustration 3. Let $f(x) = \begin{cases} \dfrac{a(1 - x\sin x) + b\cos x + 5}{x^2} & x < 0 \\ 3 & x = 0 \\ \left(1 + \left(\dfrac{cx + dx^3}{x^2}\right)\right)^{\frac{1}{x}} & x > 0 \end{cases}$

If f is continuous at x = 0, then find out the values of a, b, c and d.

Solution Since f(x) is continuous at x = 0, so at x = 0, both left and right limits must exist and both must be equal to 3.

Now

$$\lim_{x \to 0^-} \frac{a(1 - x\sin x) + b\cos x + 5}{x^2} = \lim_{x \to 0^-} \frac{(a + b + 5) + \left(-a - \frac{b}{2}\right)x^2 + \ldots}{x^2} = 3$$

(By the expansions of sinx and cosx)

If $\lim\limits_{x \to 0^-} f(x)$ exists then $a + b + 5 = 0$ and $-a - \dfrac{b}{2} = 3 \Rightarrow a = -1$ and $b = -4$

since $\lim\limits_{x \to 0^+} \left(1 + \left(\dfrac{cx + dx^3}{x^2}\right)\right)^{\frac{1}{x}}$ exists $\Rightarrow \lim\limits_{x \to 0^+} \dfrac{cx + dx^3}{x^2} = 0 \Rightarrow c = 0$

Now $\lim\limits_{x \to 0^+} (1 + dx)^{\frac{1}{x}} = \lim\limits_{x \to 0^+} \left[(1 + dx)^{\frac{1}{dx}}\right]^d = e^d$

So $e^d = 3 \Rightarrow d = \ell n\, 3$,

Hence $a = -1$, $b = -4$, $c = 0$ and $d = \ell n\, 3$.

BEGINNER'S BOX-1

1. If $f(x) = \begin{cases} \cos x; x \ge 0 \\ x + k;\ x < 0 \end{cases}$ find the value of k if $f(x)$ is continuous at x = 0.

2. If $f(x) = \begin{cases} \dfrac{|x + 2|}{\tan^{-1}(x + 2)} & ;\ x \ne -2 \\ 2 & ;\ x = -2 \end{cases}$ then discuss the continuity of $f(x)$ at x=–2

3. Let $f(x) = [x]$ & $g(x) = \text{sgn}(x)$ (where [.] denotes greatest integer function) , then discuss the continuity of $f(x) \pm g(x)$, $f(x).g(x)$ & $\dfrac{f(x)}{g(x)}$ at x= 0.

4. If $f(x) = \begin{cases} \dfrac{36^x - 9^x - 4^x + 1}{\sqrt{2} - \sqrt{1 + \cos x}} & , x \ne 0 \\ k & , x = 0 \end{cases}$ is continuous at x = 0, then k equals

(A) $16\ell n2\ \ell n3$ (B) $16\sqrt{2}\ \ell n\ 6$ (C) $16\sqrt{2}\ \ell n2\ \ell n3$ (D) None of these

5. If $f(x) = \begin{cases} \dfrac{\ell n(1 + ax) - \ell n(1 - bx)}{x} & , x \ne 0 \\ k & , x = 0 \end{cases}$ and f(x) is continuous at x = 0, then the value of k is

(A) a – b (B) a + b (C) $\ell na + \ell nb$ (D) None of these

6. Let f(x) = $\begin{cases} \dfrac{x-4}{|x-4|}+a, & x<4 \\ a+b, & x=4 \\ \dfrac{x-4}{|x-4|}+b, & x>4 \end{cases}$. Then f(x) is continuous at x = 4 when

(A) a = 0, b = 0 (B) a = 1, b = 1 (C) a = –1, b = 1 (D) a = 1, b = –1

7. If f(x) = $\begin{cases} \dfrac{\sin(a+1)x+\sin x}{x}, & x<0 \\ c, & x=0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx\sqrt{x}}, & x>0 \end{cases}$ is continuous at x = 0, then

(A) $a=-\dfrac{3}{2}, b=0, c=\dfrac{1}{2}$ (B) $a=-\dfrac{3}{2}, b=1, c=-\dfrac{1}{2}$

(C) $a=-\dfrac{3}{2}, b\in R-\{0\}, c=\dfrac{1}{2}$ (D) None of these

8. If f(x) = $\begin{cases} mx+1, & x\le\dfrac{\pi}{2} \\ \sin x+n, & x>\dfrac{\pi}{2} \end{cases}$ is continuous at x = $\dfrac{\pi}{2}$, then

(A) m = 1, n = 0 (B) m = $\dfrac{n\pi}{2}+1$ (C) n = m$\dfrac{\pi}{2}$ (D) m = n = $\dfrac{\pi}{2}$.

9. For the function f(x) = $\begin{cases} x-1, & x<0 \\ 1/4, & x=0 \\ x^2, & x>0 \end{cases}$. $\lim\limits_{x\to0^+}$ f(x) and $\lim\limits_{x\to0^-}$ f(x) are

(A) 0, 1 (B) 0, –1 (C) 1, –1 (D) None of these

10. If f(x) = $\begin{cases} \dfrac{1-\sin^2 x}{3\cos^2 x}, & x<\dfrac{\pi}{2} \\ a, & x=\dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2}, & x>\dfrac{\pi}{2} \end{cases}$. Then f(x) is continuous at x = $\dfrac{\pi}{2}$, if

(A) $a=\dfrac{1}{3}, b=2$ (B) $a=\dfrac{1}{3}, b=\dfrac{8}{3}$ (C) $a=\dfrac{2}{3}, b=\dfrac{8}{3}$ (D) None of these

11. If f(x) = $\begin{cases} \dfrac{1-\cos 10x}{x^2}, & x<0 \\ a, & x=0 \\ \dfrac{\sqrt{x}}{\sqrt{625+\sqrt{x}}-25}, & x>0 \end{cases}$, then the value of a so that f(x) may be continuous at x = 0 is

(A) 25 (B) 50 (C) –25 (D) None of these

12. If f(x) = $\begin{cases} ax^2+b, & 0 \le x < 1 \\ x+3, & 1 < x \le 2 \\ 4, & x=1 \end{cases}$, then the value of (a, b) for which f(x) cannot be continuous at x = 1 is

(A) (2, 2) (B) (3, 1) (C) (4, 0) (D) (5, 2)

13. If the function f(x) = $\begin{cases} Ax-B, & x \le 1 \\ 3x, & 1<x<2 \\ Bx^2-A, & x \ge 2 \end{cases}$ be continuous at x = 1 and discontinuous at x = 2, then

(A) A = 3 + B, B $\neq$ 3 (B) A = 3 + B, B = 3 (C) A = 3 + B, B $\in$ R (D) None of these

14. If f(x) = $\begin{cases} \dfrac{1-\sin x}{(\pi-2x)^2}, & when\, x \neq \dfrac{\pi}{2} \\ \lambda, & x=\dfrac{\pi}{2} \end{cases}$, then f(x) will be a continuous function at x = $\dfrac{\pi}{2}$ when λ is

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{1}{8}$ (D) None of these

2.0 CONTINUITY OF THE FUNCTION IN AN INTERVAL

(a) A function is said to be continuous in (a,b) if f is continuous at each & every point belonging to (a, b).

(b) A function is said to be continuous in a closed interval [a,b] if :

(i) f is continuous in the open interval (a,b)

(ii) f is right continuous at 'a' i.e. $\lim\limits_{x\to a^+} f(x) = f(a)$ = a finite quantity

(iii) f is left continuous at 'b' i.e. $\lim\limits_{x\to b^-} f(x) = f(b)$ = a finite quantity

NOTE

(i) Obseve that $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to b^+} f(x)$ do not make sense. As a consequence of this definition, if $f(x)$ is defined only at one point, it is continuous there, i.e., if the domain of $f(x)$ is a singleton, $f(x)$ is a continuous function.

Example : Consider $f(x) = \sqrt{a-x}+\sqrt{x-a}$.

$f(x)$ is a singleton function defined only at x = a. Hence $f(x)$ is a continuous function.

(ii) If f (x) & g(x) are two functions that are continuous at x = c then the function defined by :

$F_1(x) = f(x) \pm g(x)$; $F_2(x) = K\, f(x)$, where K is any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x = c.

Further, if g(c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x = c.

(iii) Some continuous functions

Function f(x)	**Interval in which f(x) is continuous**
Constant function	$(-\infty, \infty)$
x^n, n is an integer ≥ 0	$(-\infty, \infty)$
x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
$\|x - a\|$	$(-\infty, \infty)$
$p(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$	$(-\infty, \infty)$
$\frac{p(x)}{q(x)}$, where p(x) and q(x) are polynomial in x	$(-\infty, \infty) - \{x : q(x) = 0\}$
sinx, cosx, e^x	$(-\infty, \infty)$
tanx, secx	$(-\infty, \infty) - \{(2n + 1)\pi/2 : n \in I\}$
cotx, cosecx	$(-\infty, \infty) - \{n\pi : n \in I\}$
ℓnx	$(0, \infty)$

(iv) Some Discontinuous Functions

Functions	**Points of discontinuity**
[x], {x}	Every Integer
tanx, secx	$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...$
cotx, cosecx	$x = 0, \pm\pi, \pm 2\pi, ...$
$\sin\frac{1}{x}$, $\cos\frac{1}{x}$, $\frac{1}{x}$, $e^{1/x}$	x = 0

Illustrations

Illustration 4. Discuss the continuity of $f(x) = \begin{cases} |x+1| & , \quad x < -2 \\ 2x+3 & , \quad -2 \leq x < 0 \\ x^2+3 & , \quad 0 \leq x < 3 \\ x^3-15 & , \quad x \geq 3 \end{cases}$

Solution We write f(x) as $f(x) = \begin{cases} -x-1 & , \quad x < -2 \\ 2x+3 & , \quad -2 \leq x < 0 \\ x^2+3 & , \quad 0 \leq x < 3 \\ x^3-15 & , \quad x \geq 3 \end{cases}$

As we can see, f(x) is defined as a polynomial function in each of intervals $(-\infty, -2)$, $(-2,0)$, $(0,3)$ and $(3,\infty)$. Therefore, it is continuous in each of these four open intervals. Thus we check the continuity at x = –2,0,3.

At the point x = –2

$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} (-x-1) = +2-1 = 1$

$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} (2x+3) = 2.(-2)+3 = -1$

Therefore, $\lim_{x\to-2} f(x)$ does not exist and hence f(x) is discontinuous at x = –2.

At the point x = 0

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} (2x+3) = 3$

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} (x^2+3) = 3$

$f(0) = 0^2 + 3 = 3$

Therefore f(x) is continuous at x = 0.

At the point x = 3

$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} (x^2+3) = 3^2+3 = 12$

$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} (x^3-15) = 3^3-15 = 12$

$f(3) = 3^3 - 15 = 12$

Therefore, f(x) is continuous at x = 3.

We find that f(x) is continuous at all points in R except at x = – 2

3.0 REASONS OF DISCONTINUITY

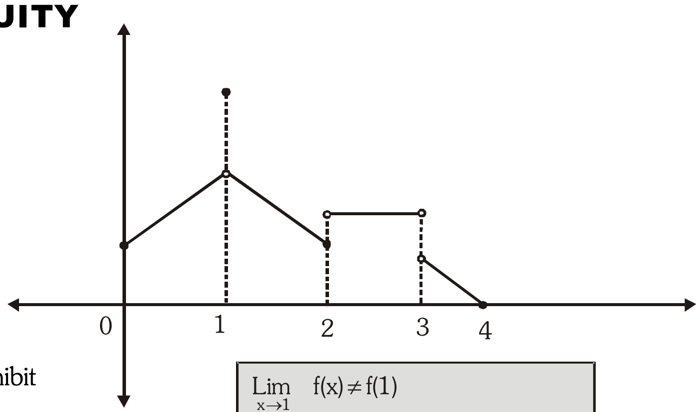
(a) Limit does not exist

i.e. $\operatorname{Lim}_{x\to a^-} f(x) \neq \operatorname{Lim}_{x\to a^+} f(x)$

(b) f(x) is not defined at x = a

(c) $\operatorname{Lim}_{x\to a} f(x) \neq f(a)$

Geometrically, the graph of the function will exhibit a break at x = a, if the function is discontinuous at x = a. The graph as shown is discontinuous at x = 1 , 2 and 3.



$\operatorname{Lim}_{x\to1} f(x) \neq f(1)$

$\operatorname{Lim}_{x\to2} f(x)$ does not exist

f(x) is not defined at x = 3

4.0 TYPES OF DISCONTINUITIES

Type-1 – (Removable type of discontinuities) – In case $\operatorname{Lim}_{x\to a} f(x)$ exists but is not equal to f(a) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\operatorname{Lim}_{x\to a} f(x) = f(a)$ & make it continuous at x = a. Removable type of discontinuity can be further classified as:

(a) **Missing point discontinuity –**

Where $\operatorname{Lim}_{x\to a} f(x)$ exists but f(a) is not defined.

(b) **Isolated point discontinuity –**

Where $\operatorname{Lim}_{x\to a} f(x)$ exists & f(a) also exists but; $\operatorname{Lim}_{x\to a} f(x) \neq f(a)$.

Illustrations

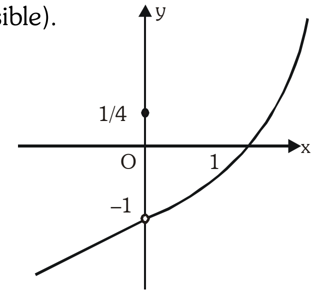
Illustration 5. Examine the function , $f(x) = \begin{cases} x-1 & , \; x<0 \\ 1/4 & , \; x=0 \\ x^2-1 & , \; x>0 \end{cases}$. Discuss the continuity, and if discontinuous remove the discontinuity by redefining the function (if possible).

Solution Graph of f(x) is shown, from graph it is seen that

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = -1$, but $f(0) = 1/4$

Thus, f(x) has removable discontinuity and f(x) could be made continuous by taking $f(0) = -1$

$$\Rightarrow \quad f(x) = \begin{cases} x-1 & , \; x<0 \\ -1 & , \; x=0 \\ x^2-1 & , \; x>0 \end{cases}$$



y = f(x) before redefining

Type-2 – (Non-Removable type of discontinuities)

In case $\lim\limits_{x\to a} f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such a discontinuity is known as non-removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

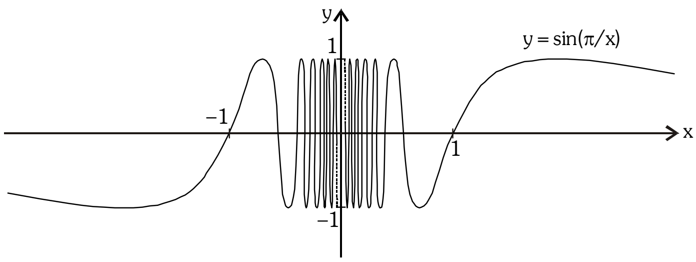
(a) **Finite type discontinuity** – In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

(b) **Infinite type discontinuity –** In such type of discontinuity atleast one of the limit viz. LHL and RHL is tending to infinity.

(c) **Oscillatory type discontinuity** –

e.g. $f(x) = \sin\dfrac{\pi}{x}$ at $x = 0$

$$f(x) = \sin\frac{\pi}{x}$$



f(x) has non removable oscillatory type discontinuity at x = 0

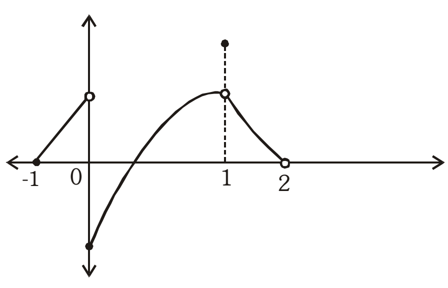
Example – From the adjacent graph note that

(i) f is continuous at x = –1

(ii) f has isolated discontinuity at x = 1

(iii) f has missing point discontinuity at x = 2

(iv) f has non removable (finite type) discontinuity at the origin.



Illustrations

Illustration 6. Show that the function, $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & ;\ \text{when } x \neq 0 \\ 0, & ;\ \text{when } x = 0 \end{cases}$ has non-removable discontinuity at $x = 0$.

Solution We have,

$$f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & ;\ \text{when } x \neq 0 \\ 0, & ;\ \text{when } x = 0 \end{cases}$$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}$$

$$= \lim_{h\to 0} \frac{1-\dfrac{1}{e^{1/h}}}{1+\dfrac{1}{e^{1/h}}} = 1 \qquad [\because\ e^{1/h} \to \infty]$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{h\to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1} = \frac{0-1}{0+1}$$

$$= -1\ [\because\ h \to 0\ ;\ e^{-1/h} \to 0]$$

$$\lim_{x\to 0^-} f(x) = -1$$

$\Rightarrow \lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0^-} f(x)$. Thus f(x) has non-removable discontinuity.

Illustration 7. $f(x) = \begin{cases} \cos^{-1}\{\cot x\} & x < \dfrac{\pi}{2} \\ \pi[x]-1 & x \geq \dfrac{\pi}{2} \end{cases}$; find jump of discontinuity,

where [ ] denotes greatest integer & { } denotes fractional part function.

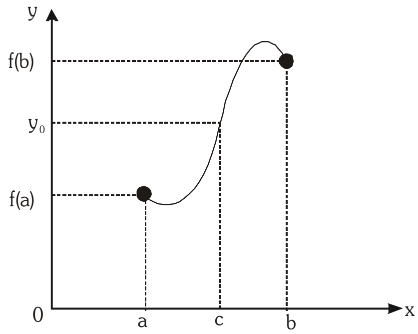
Solution $f(x) = \begin{cases} \cos^{-1}\{\cot x\} & x < \dfrac{\pi}{2} \\ \pi[x]-1 & x \geq \dfrac{\pi}{2} \end{cases}$

$$\lim_{x\to \frac{\pi^-}{2}} f(x) = \lim_{x\to \frac{\pi^-}{2}} \cos^{-1}\{\cot x\} = \lim_{h\to 0} \cos^{-1}\left\{\cot\left(\frac{\pi}{2}-h\right)\right\} = \lim_{h\to 0}\cos^{-1}\{\tan h\} = \frac{\pi}{2}$$

$$= \lim_{x\to \frac{\pi^+}{2}} f(x) = \lim_{x\to \frac{\pi^+}{2}} \pi[x]-1 = \lim_{h\to 0} \pi\left[\frac{\pi}{2}+h\right]-1 = \pi-1$$

$\therefore$ jump of discontinuity $= \pi - 1 - \dfrac{\pi}{2} = \dfrac{\pi}{2} - 1$

5.0 THE INTERMEDIATE VALUE THEOREM



The function f, being continuous on [a,b] takes on every value between f(a) and f(b)

Suppose f(x) is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b), there exists a number c between a and b such that $f(c) = y_0$

Note that a function f which is continuous in [a,b] possesses the following properties :

(a) If f(a) & f(b) posses opposite signs, then there exists atleast one root of the equation f(x) = 0 in the open interval (a,b).

(b) If K is any real number between f(a) & f(b), then there exists atleast one root of the equation f(x) = K in the open interval (a,b).

Note – In above cases the number of roots is always odd.

Illustrations

Illustration 8. Show that the function, $f(x)=(x-a)^2(x-b)^2+x$, takes the value $\frac{a+b}{2}$ for some $x_0 \in (a, b)$

Solution
$f(x) = (x-a)^2(x-b)^2 + x$
$f(a) = a$
$f(b) = b$
& $\frac{a+b}{2} \in (f(a), f(b))$

$\therefore$ By intermediate value theorem, there is atleast one $x_0 \in (a, b)$ such that $f(x_0) = \frac{a+b}{2}$.

Illustration 9. Let $f : [0, 1] \xrightarrow{\text{onto}} [0, 1]$ be a continuous function, then prove that f(x) = x for atleast one $x \in [0, 1]$

Solution
Consider $g(x) = f(x) - x$

$g(0) = f(0) - 0 = f(0) \ge 0$ $\quad \{\because \; 0 \le f(x) \le 1\}$

$g(1) = f(1) - 1 \le 0$

$\Rightarrow$ $g(0) \,.\, g(1) \le 0$

$\Rightarrow$ g(x) = 0 has atleast one root in [0, 1]

$\Rightarrow$ f(x) = x for atleast one $x \in [0, 1]$

6.0 SOME IMPORTANT POINTS

(a) If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\phi(x) = f(x).g(x)$ will **not necessarily be discontinuous at x = a**, e.g.

$$f(x) = x \;\&\; g(x) = \begin{bmatrix} \sin\frac{\pi}{x} & x \ne 0 \\ 0 & x = 0 \end{bmatrix}$$

f(x) is continuous at x = 0 & g(x) is discontinuous at x = 0, but f(x).g(x) is continuous at x = 0.

(b) If f (x) and g (x) both are discontinuous at x = a then the product function $\phi(x) = f(x).g(x)$ **is not necessarily be discontinuous at x = a**, e.g.

$$f(x) = -g(x) = \begin{bmatrix} 1 & x \ge 0 \\ -1 & x < 0 \end{bmatrix}$$

f(x) & g(x) both are discontinuous at x = 0 but the product function f.g(x) is still continuous at x = 0

(c) If f (x) and g (x) both are discontinuous at x = a then f(x) ± g(x) is not necessarily be discontinuous at x = a

(d) A continuous function whose domain is closed must have a range also in closed interval.

(e) If f is continuous at x = a & g is continuous at x = f (a) then the composite g[f(x)] is continuous at x = a. eg. $f(x)=\frac{x\sin x}{x^2+2}$ & $g(x)=|x|$ are continuous at x =0, hence the composite $(gof)(x)=\left|\frac{x\sin x}{x^2+2}\right|$ will also be continuous at x = 0

Illustrations

Illustration 10. If $f(x) = \frac{x+1}{x-1}$ and $g(x)=\frac{1}{x-2}$, then discuss the continuity of f(x), g(x) and fog (x) in R.

Solution $f(x) = \frac{x+1}{x-1}$

f(x) is a rational function it must be continuous in its domain and f is not defined at x = 1.

$\therefore$ f is discontinuous at x = 1

$g(x) = \frac{1}{x-2}$

g(x) is also a rational function. It must be continuous in its domain and g is not defined at x = 2.

$\therefore$ g is discontinuous at x = 2

Now fog(x) will be discontinuous at x = 2 (point of discontinuity of g(x))

Consider g(x) = 1 (when g(x) = point of discontinuity of f(x))

$\frac{1}{x-2} = 1 \quad \Rightarrow x = 3$

$\therefore$ fog(x) is discontinuous at x = 2 & x = 3.

7.0 SINGLE POINT CONTINUITY

Functions which are continuous only at one point are said to exhibit single point continuity

Illustration 11. If $f(x)=\begin{cases} x & \text{if } x\in Q \\ -x & \text{if } x\notin Q\end{cases}$, find the points where $f(x)$ is continuous

Solution Let x = a be the point at which $f(x)$ is continuous.

$\Rightarrow \lim\limits_{\substack{x\to a \\ \text{through rational}}} f(x) = \lim\limits_{\substack{x\to a \\ \text{through irrational}}} f(x)$

$\Rightarrow$ a = –a

$\Rightarrow$ a = 0 $\Rightarrow$ function is continuous at x = 0.

GOLDEN KEY POINTS

- All polynomials, trigonometrical functions, exponential & logarithmic functions are continuous in their domains.
- In case of non-removable (finite type) discontinuity the non-negative difference between the value of the RHL at x = a & LHL at x = a is called **the jump of discontinuity**. A function having a finite number of jumps in a given interval I is called a **piece wise continuous or sectionally continuous** function in this interval.

BEGINNER'S BOX-2

1. If $f(x) = \begin{cases} \dfrac{1}{x-1} & ; \quad 0 \le x < 2\ ;\ x \ne 1 \\ x^2 - 3 & ; \quad 2 \le x < 4 \\ 5 & ; \quad x = 4 \\ 14 - \dfrac{x^{1/2}}{2} & ; \quad x > 4 \end{cases}$, then discuss the types of discontinuity for the function.

2. Discuss the type of discontinuity for $f(x) = \begin{cases} -1 & ; \quad x \le -1 \\ |x| & ; \quad -1 < x < 1 \\ (x+1) & ; \quad x \ge 1 \end{cases}$

3. If $f(x) = \sin|x|$ & $g(x) = \tan|x|$ then discuss the continuity of $f(x) \pm g(x)$; $\dfrac{f(x)}{g(x)}$ & $f(x)\,g(x)$

4. If $g(x) = \begin{bmatrix} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{bmatrix}$, then find the points where function is continuous.

5. If $f(x) = \begin{cases} x^2 & ; \quad x \in Q \\ 1 - x^2 & ; \quad x \notin Q \end{cases}$, then find the points where function is continuous.

6. If $f(x) = \begin{cases} \dfrac{x^2}{a} & ; \quad 0 \le x < 1 \\ -1 & ; \quad 1 \le x < \sqrt{2} \\ \dfrac{2b^2 - 4b}{x^2} & ; \quad \sqrt{2} \le x < \infty \end{cases}$ then find the value of a & b if $f(x)$ is continuous in $[0,\infty)$

7. Discuss the continuity of $f(x) = \begin{cases} |x-3| & ; \quad 0 \le x < 1 \\ \sin x & ; \quad 1 \le x \le \dfrac{\pi}{2} \\ \log_{\frac{\pi}{2}} x & ; \quad \dfrac{\pi}{2} < x < 3 \end{cases}$ in $[0,3)$

8. The function $f(x) = \dfrac{4 - x^2}{4x - x^3}$

(A) Discontinuous at only one point
(B) Discontinuous exactly at two points
(C) Discontinuous exactly at three points
(D) None of these

9. If f(x) defined by $f(x) = \begin{cases} \dfrac{|x^2 - x|}{x^2 - x} & x \ne 0,1 \\ 1 & , x = 0 \\ -1 & , x = 1 \end{cases}$, then f(x) is continuous for all

(A) x
(B) x except at x = 0
(C) x except at x = 1
(D) x except at x = 0 and x = 1.

10. Let f(x) = $\begin{cases} \dfrac{x^4-5x^2+4}{|(x-1)(x-2)|}, & x \neq 1, 2 \\ 6, & x = 1 \\ 12, & x = 2 \end{cases}$. Then f(x) is continuous on the set

(A) R (B) R – {1} (C) R – {2} (D) R – {1, 2}

11. The value of f(0), so that the function $f(x) = \dfrac{\sqrt{a^2-ax+x^2}-\sqrt{a^2+ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$ becomes continuous for all x in its domain, given by (a > 0)

(A) $a^{3/2}$ (B) $a^{1/2}$ (C) $-a^{1/2}$ (D) $-a^{3/2}$.

12. The value of f(0), so that the function f(x) = $\dfrac{(27-2x)^{1/3}-3}{9-3(243+5x)^{1/5}} (x \neq 0)$ is continuous, is given by

(A) $\dfrac{2}{3}$ (B) 6 (C) 2 (D) 4

13. The points (s) of discontinuity of the function f(x) = $\begin{cases} \dfrac{1}{5}(2x^2+3), & x \leq 1 \\ 6-5x, & 1 < x < 3 \\ x-3, & x \geq 3 \end{cases}$ is (are)

(A) x = 1 (B) x = 3

(C) x = 1, 3 (D) None of these

14. Let f(x) = $\dfrac{1-\tan x}{4x-\pi}$, $x \neq \dfrac{\pi}{4}, x \in \left[0, \dfrac{\pi}{2}\right)$, f(x) is continuous in $\left[0, \dfrac{\pi}{2}\right)$, then f$\left(\dfrac{\pi}{4}\right)$ is

(A) $-\dfrac{1}{2}$ (B) $\dfrac{1}{2}$ (C) 1 (D) –1

15. Let f(x) = $\begin{cases} \dfrac{\sqrt{1+px}-\sqrt{1-px}}{x}, & -1 \leq x < 0 \\ \dfrac{2x+1}{x-2}, & 0 \leq x \leq 1 \end{cases}$. If f(x) is continuous in the interval [–1, 1], then p equals

(A) $\dfrac{1}{2}$ (B) $-\dfrac{1}{2}$ (C) –1 (D) 1

EXERCISE – 1 SCQ/MCQ

1. If $f(x)=\begin{cases} x+2 & , \text{ when } x<1 \\ 4x-1 & , \text{ when } 1 \le x \le 3 \\ x^2+5 & , \text{ when } x>3 \end{cases}$, then correct statement is -

(A) $\lim_{x\to 1} f(x)=\lim_{x\to 3} f(x)$
(B) f(x) is continuous at x = 3
(C) f(x) is continuous at x = 1
(D) f(x) is continuous at x = 1 and 3

2. If $f(x)=\begin{cases} [x]+[-x], & x\ne 2 \\ \lambda, & x=2 \end{cases}$, f is continuous at x = 2 then λ is (where [.] denotes greatest integer) -

(A) –1 (B) 0 (C) 1 (D) 2

3. Function f(x) = $\dfrac{1}{\log |x|}$ is discontinuous at -

(A) one point
(B) two points
(C) three points
(D) infinite number of points

4. Which of the following functions has finite number of points of discontinuity in R (where [.] denotes greatest integer)

(A) tan x (B) |x| / x (C) x + [x] (D) sin $[\pi x]$

5. If $f(x)=\dfrac{x-e^x+\cos 2x}{x^2}$, $x\ne 0$ is continuous at x = 0, then -

(A) $f(0)=\dfrac{5}{2}$ (B) [f(0)] = –2 (C) {f(0)} = –0.5 (D) [f(0)].{f(0)} = –1.5

where [x] and {x} denotes greatest integer and fractional part function.

6. Let $f(x)=\dfrac{x(1+a\cos x)-b\sin x}{x^3}$, $x\ne 0$ and f(0) = 1. The value of a and b so that f is a continuous function are -

(A) 5/2, 3/2 (B) 5/2, –3/2 (C) –5/2, –3/2 (D) none of these

7. 'f' is a continuous function on the real line. Given that $x^2+(f(x)-2)x-\sqrt{3}\,.f(x)+2\sqrt{3}-3=0$. Then the value of $f(\sqrt{3})$ is -

(A) $\dfrac{2(\sqrt{3}-2)}{\sqrt{3}}$ (B) $2(1-\sqrt{3})$ (C) zero (D) cannot be determined

8. If the function $f(x)=\begin{cases} x+a\sqrt{2}\sin x & , 0\le x\le \frac{\pi}{4} \\ 2x\cot x+b & , \frac{\pi}{4}<x\le \frac{\pi}{2} \\ a\cos 2x-b\sin x, & \frac{\pi}{2}<x\le \pi \end{cases}$ is continuous in the interval $[0, \pi]$, then

(A) $a=\dfrac{\pi}{6}, b=\dfrac{\pi}{12}$ (B) $a=-\dfrac{\pi}{6}, b=\dfrac{\pi}{12}$ (C) $a=-\dfrac{\pi}{6}, b=\dfrac{-\pi}{12}$ (D) $a=\dfrac{\pi}{6}, b=-\dfrac{\pi}{12}$

9. Let f(x) = $\lim_{n\to\infty}\dfrac{\ln(2+x)-x^{2n}\sin x}{1+x^{2n}}$, then f (x) is discontinuous at in its domain

(A) x = 1 only (B) x = –1 only (C) x = –1, 1 only (D) no point

10. Let f(x) be defined as follows : $\begin{cases}(\cos x-\sin x)^{\operatorname{cosec} x}, & -\dfrac{\pi}{2}<x<0\\ a, & x=0\\ \dfrac{e^{1/x}+e^{2/x}+e^{3/x}}{ae^{2/x}+be^{3/x}}, & 0<x<\dfrac{\pi}{2}\end{cases}$. If f(x) is continuous at x = 0, then (a, b) =

(A) $\left(e,\dfrac{1}{e}\right)$ (B) $\left(\dfrac{1}{e},e\right)$ (C) (e, e^2) (D) None of these

11. Consider the function f(x) = $x-|x-x^2|,\ -1\le x\le 2$. The points of discontinuity of f(x) for $x\in[-1,2]$ are

(A) x = 0 , 1 (B) x = 1, 2 (C) x = 0 , $\dfrac{1}{2}$,1 (D) None of these

12. In $x\in\left[0,\dfrac{\pi}{2}\right]$, let f (x) = $\lim_{n\to\infty}\dfrac{2^x-x^n\sin x}{1+x^n}$, then

(A) f(x) is a constant function (B) f(x) is continuous at x = 1

(C) f(x) is discontinuous at x = 1 (D) none of these

13. Given f(x) = $\dfrac{[\{|x|\}]\, e^{x^2}\{[x+\{x\}]\}}{\left(e^{\frac{1}{x^2}}-1\right)\operatorname{sgn}(\sin x)}$ for $x\neq 0$

= 0 for x = 0

where {x} is the fractional part function; [x] is the step up function and sgn(x) is the signum function of x then, f(x)

(A) is continuous at x = 0 (B) is discontinuous at x = 0

(C) has a removable discontinuity at x = 0 (D) has an irremovable discontinuity at x = 0

14. Consider f(x) = $\begin{cases} x[x]^2\log_{(1+x)}2 & \text{for } -1<x<0\\ \dfrac{\ln\left(e^{x^2}+2\sqrt{\{x\}}\right)}{\tan\sqrt{x}} & \text{for } 0<x<1\end{cases}$

where [ ] & {} are the greatest integer function & fractional part function respectively, then

(A) f(0) = $\ln 2\Rightarrow$ f is continuous at x = 0 (B) f(0) = 2 $\Rightarrow$ f is continuous at x = 0

(C) f(0) = $e^2\Rightarrow$ f is continuous at x = 0 (D) f has an irremovable discontinuity at x = 0

15. Consider $f(x) = \dfrac{\sqrt{1+x} - \sqrt{1-x}}{\{x\}} \quad x \neq 0$

$g(x) = \cos 2x \qquad -\dfrac{\pi}{4} < x < 0$

$$h(x) = \begin{cases} \dfrac{1}{\sqrt{2}} f(g(x)) & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ f(x) & \text{for } x > 0 \end{cases}$$

then, which of the following holds good.
where $\{x\}$ denotes fractional part function.

(A) 'h' is continuous at $x = 0$

(B) 'h' is discontinuous at $x = 0$

(C) $f(g(x))$ is an even function

(D) $f(x)$ is an even function

MORE THAN ONE OPTION CORRECT

16. Indicate all correct alternatives if, $f(x)=\frac{x}{2}-1$, then on the interval $[0,\pi]$

(A) tan (f (x)) & $\frac{1}{f(x)}$ are both continuous

(B) tan (f(x)) & $\frac{1}{f(x)}$ are both discontinuous

(C) tan (f (x))& $f^{-1}(x)$ are both continuous

(D) tan (f(x)) is continuous but $\frac{1}{f(x)}$ is not

17. $f(x)=\frac{2\cos x-\sin 2x}{(\pi-2x)^2}$; $g(x)=\frac{e^{-\cos x}-1}{8x-4\pi}$

$h(x) = f(x)$ for $x<\pi/2$

$= g(x)$ for $x>\pi/2$

then which of the followings does not holds ?

(A) h is continuous at $x = \pi/2$

(B) h has an irremovable discontinuity at $x=\pi/2$

(C) h has a removable discontinuity at $x = \pi/2$

(D) $f\left(\frac{\pi^+}{2}\right)=g\left(\frac{\pi^-}{2}\right)$

18. Given $f(x)=\begin{bmatrix} 3-\left[\cot^{-1}\left(\frac{2x^3-3}{x^2}\right)\right] & \text{for} & x>0 \\ \{x^2\}\cos\left(e^{1/x}\right) & \text{for} & x<0 \end{bmatrix}$ where { } & [ ] denotes the fractional part and the integral part functions respectively, then which of the following statement does not hold good -

(A) $f(0^-) = 0$

(B) $f(0^+)=3$

(C) $f(0)=0 \Rightarrow$ continuity of f at x = 0

(D) irremovable discontinuity of f at x = 0

19. Given $f(x) = b([x]^2 + [x]) + 1$ for $x \geq -1$

$= \sin(\pi(x+a))$ for $x < -1$

where [x] denotes the integral part of x, then for what values of a, b the function is continuous at $x = -1$?

(A) $a=2n+(3/2)$; $b\in R$; $n\in I$

(B) $a=4n+2$; $b\in R$; $n\in I$

(C) $a=4n+(3/2)$; $b\in R^+$; $n\in I$

(D) $a=4n+1$; $b\in R^+$; $n\in I$

20. If $f(x)=\frac{1}{x^2-17x+66}$, then $f\left(\frac{2}{x-2}\right)$ is discontinuous at x =

(A) 2

(B) $\frac{7}{3}$

(C) $\frac{24}{11}$

(D) 6,11

21. Let $f(x) = [x]$ & $g(x)=\begin{cases} 0; & x\in Z \\ x^2; & x\in R-Z \end{cases}$, then (where [.] denotes greatest integer function) -

(A) $\lim\limits_{x\to 1} g(x)$ exists, but g(x) is not continuous at x= 1.

(B) $\lim\limits_{x\to 1} f(x)$ does not exist and $f(x)$ is not continuous at x=1.

(C) gof is continuous for all x.

(D) fog is continuous for all x.

22. Let $f(x)=\frac{[\tan^2 x]-1}{\tan^2 x-1}$ at $x\neq n\pi\pm\frac{\pi}{4}$,

f(x) = 0 at $x=n\pi\pm\frac{\pi}{4}$. Then $f(x)$ is (where [x] denotes the greatest integer less than or equal to x)

(A) continuous at all x

(B) continuous at $x=\frac{\pi}{4}$

(C) discontinuous at $x=\frac{\pi}{4}$

(D) discontinuous at infinite number of points

EXERCISE - 2 MISCELLANEOUS

Note : (* one or more then one option is / are correct)

Comprehension Based Questions

Comprehension – 1

If $S_n(x) = \frac{x}{x+1} + \frac{x^2}{(x+1)(x^2+1)} + \ldots\ldots + \frac{x^{2^n}}{(x+1)(x^2+1)\ldots..(x^{2^n}+1)}$ and $x > 1$

$\lim\limits_{n\to\infty} S_n(x) = \ell$

$g(x) = \begin{cases} \frac{\sqrt{ax+b}-1}{x} & , \ x \neq 0 \\ 1 & , \ x = 0 \end{cases}$, g(x) is contineous at x = 0

$h : R \to R \quad h(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 7$

On the basis of above information, answer the following questions

1. If g(x) is continuous at x = 0 then a + b is equal to -

(A) 0 (B) 1 (C) 2 (D) 3

2. Identify the incorrect option -

(A) h(x) is surjective

(B) domain of g(x) is $[-1/2, \infty)$

(C) h(x) is bounded

(D) $\ell = 1$

Comprehension – 2

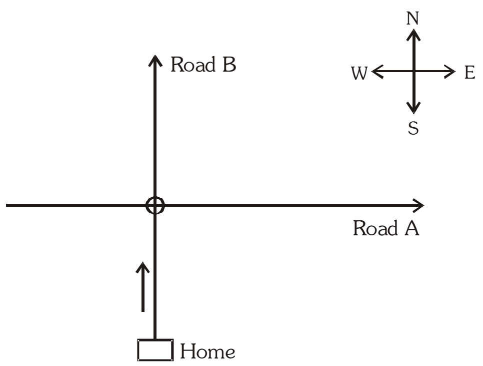
A man leaves his home early in the morning to have a walk. He arrives at a junction of road A & road B as shown in figure. He takes the following steps in later journey :

(a) 1 km in north direction

(b) changes direction & moves in north-east direction for $2\sqrt{2}$ kms.

(c) changes direction & moves southwards for distance of 2 km.

(d) finally he changes the direction & moves in south-east direction to reach road A again.



Visible/Invisible path – The path traced by the man in the direction parallel to road A & road B is called invisible path, the remaining path traced is visible.

Visible points – The points about which the man changes direction are called visible points except the point from where he changes direction last time

Now if road A & road B are taken as x-axis & y-axis then visible path & visible point represents the graph of y = f(x).

On the basis of above information, answer the following questions

3. The value of x at which the function is discontinuous -

(A) 2 (B) 0 (C) 1 (D) 3

4*. The value of x at which fof(x) is discontinuous -

(A) 0 (B) 1 (C) 2 (D) 3

5. If f(x) is periodic with period 3, then f(19) is -

(A) 2 (B) 3 (C) 19 (D) none of these

Match the column

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

6*.

	Column-I		**Column-II**
(A)	If $f(x) = \begin{cases} \sin\{x\}; & x < 1 \\ \cos x + a; & x \geq 1 \end{cases}$ where {.} denotes the fractional part function, such that f(x) is continuous at x=1. If $\lvert k \rvert = \dfrac{a}{\sqrt{2}\sin\dfrac{(4-\pi)}{4}}$ then k is	(p)	1
(B)	If the function $f(x) = \dfrac{(1-\cos(\sin x))}{x^2}$ is continuous at x = 0, then f(0) is	(q)	0
(C)	$f(x) = \begin{bmatrix} x & , x \in Q \\ 1-x, & x \notin Q \end{bmatrix}$, then the values of x at which f(x) is continuous	(r)	–1
(D)	If f(x) = x + {–x} + [x], where [x] and {x} represents integral and fractional part of x, then the values of x at which f(x) is discontinuous	(s)	$\dfrac{1}{2}$

7*. **Column-I** **Column-II**

(A) If f(x) = 1/(1–x), then the points at which the function fofof(x) is discontinuous — (p) $\frac{1}{2}$

(B) $f(u) = \frac{1}{u^2+u-2}$, where $u = \frac{1}{x-1}$. The values of x at which 'f' is discontinuous — (q) 0

(C) $f(x) = u^2$, where $u = \begin{cases} x-1, x \ge 0 \\ x+1, x < 0 \end{cases}$ The number of values of x at which 'f' is discontinuous — (r) 2

(D) The number of value of x at which the function $f(x) = \frac{2x^5 - 8x^2 + 11}{x^4 + 4x^3 + 8x^2 + 8x + 4}$ is discontinuous — (s) 1

8. **Column-I** **Column-II**

(A) If the function $f(x) = \begin{cases} \frac{\sin 3x}{x}, & x \neq 0 \\ \frac{K}{2}, & x = 0 \end{cases}$ is continuous at x = 0, then K = — (p) 6

(B) If $f(x) = \frac{x^2 - 10x + 25}{x^2 - 7x + 10}$ for $x \neq 5$ and it is continuous at x = 5 then f(5) = — (q) $2\ell n|a|$

(C) If f : R $\to$ R defined by $f(x) = \begin{cases} a^2\cos^2 x + b^2 \sin^2 x, & x \le 0 \\ e^{ax+b}, & x > 0 \end{cases}$ is continuous function then b = — (r) 3

(s) $\ell n|a|$

(t) 0

INTEGER / SUBJECTIVE TYPE

9. If $f(x)=\begin{cases}-x^2, & \text{when } x\le 0\\ 5x-4, & \text{when } 0<x\le 1\\ 4x^2-3x, & \text{when } 1<x<2\\ 3x+4, & \text{when } x\ge 2\end{cases}$, discuss the continuity of f(x) in R.

10. Let $f(x)=\begin{bmatrix}-2\sin x & \text{for} & -\pi\le x\le -\frac{\pi}{2}\\ a\sin x+b & \text{for} & -\frac{\pi}{2}<x<\frac{\pi}{2}\\ \cos x & \text{for} & \frac{\pi}{2}\le x\le \pi\end{bmatrix}$. If f is continuous on $[-\pi,\pi]$ then find the values of a + b.

11. Suppose that $f(x)=x^3-3x^2-4x+12$ and $h(x)=\begin{bmatrix}\dfrac{f(x)}{x-3} & , x\ne 3\\ K & x=3\end{bmatrix}$ then

(a) find all zeros of 'f'
(b) find the value of K that makes 'h' continuous at x =3
(c) using the value of K found in (b) determine whether 'h' is an even function.

12. If $f(x)=\dfrac{\sin 3x+A\sin 2x+B\sin x}{x^5}(x\ne 0)$ is continuous at x = 0, then find the value of A + B + f(0).

13. Examine the continuity at x= 0 of the sum function of the infinite series :

$$\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\ldots\ldots\ldots\infty$$

14. Given $f(x)=\sum_{r=1}^{n}\tan\left(\dfrac{x}{2^r}\right)\sec\left(\dfrac{x}{2^{r-1}}\right)$; $r,n\in N$

$$g(x)=\begin{bmatrix}\lim\limits_{n\to\infty}\dfrac{\ln\left(f(x)+\tan\frac{x}{2^n}\right)-\left(f(x)+\tan\frac{x}{2^n}\right)^n\cdot\left[\sin\left(\tan\frac{x}{2}\right)\right]}{1+\left(f(x)+\tan\frac{x}{2^n}\right)^n} & ; & x\ne\pi/4\\ K & ; & x=\pi/4\end{bmatrix}$$

where [ ] denotes the greatest integer function and the domain of g(x) is $\left(0,\frac{\pi}{2}\right)$. Find the value of k, if possible, so that g(x) is continuous at $x=\pi/4$. Also state the points of discontinuity of g(x) in $(0,\pi/4)$, if any.

15. Let $f(x)=\begin{cases}1+x^3, & x<0\\ x^2-1, & x\ge 0\end{cases}$; $g(x)=\begin{cases}(x-1)^{1/3}, & x<0\\ (x+1)^{1/2}, & x\ge 0\end{cases}$ Discuss the continuity of g(f(x)).

16. Consider the function $g(x)=\begin{bmatrix}\dfrac{1-a^x+xa^x\ell n a}{a^x x^2} & \text{for } x<0\\ \dfrac{2^x a^x-x\ell n2-x\ell n a-1}{x^2} & \text{for } x>0\end{bmatrix}$ where a > 0.

Find the value of [a] + [g(0)] so that the function g(x) is continuous at x = 0 , (Where [.] is G.I.F.).

17. Let $f(x) = \begin{cases} \dfrac{\left(\dfrac{\pi}{2} - \sin^{-1}\left(1-\{x\}^2\right)\right).\sin^{-1}(1-\{x\})}{\sqrt{2}\left(\{x\}-\{x\}^3\right)} & \text{for } x \neq 0 \\ \dfrac{\pi}{2} & \text{for } x = 0 \end{cases}$

where $\{x\}$ is the fractional part of x.

Consider another function g(x); such that

$g(x) = \begin{cases} f(x) & \text{for } x \geq 0 \\ 2\sqrt{2}\ f(x) & \text{for } x < 0 \end{cases}$

Discuss the continuity of the functions f(x) & g(x) at x = 0.

18. $f(x) = \dfrac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}$

for $x > 0$

$= \dfrac{\ell n(1+x+x^2) + \ell n(1-x+x^2)}{\sec x - \cos x}$

for $x < 0$, if 'f' is continuous at $x = 0$, find 'a'

now if $g(x) = \ell n\left(2 - \dfrac{x}{a}\right).\cot(x-a)$ for $x \neq a$, $a \neq 0$, $a > 0$. If 'g' is continuous at $x = a$ then show that $g(e^{-1}) = -e$

19. Let [x] denote the greatest integer function & f(x) be defined in a neighbourhood of 2 by

$f(x) = \begin{cases} \dfrac{\left(\exp\{(x+2)\ell n4\}\right)^{\frac{[x+1]}{4}} - 16}{4^x - 16}, & x < 2 \\ A\dfrac{1-\cos(x-2)}{(x-2)\tan(x-2)}, & x > 2 \end{cases}$

Find the value of A & f(2) in order that f(x) may be continuous at x = 2.

20. Let $y_n(x) = x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{\left(1+x^2\right)^2} + \ldots\ldots\ldots \dfrac{x^2}{\left(1+x^2\right)^{n-1}}$ and $y(x) = \lim\limits_{n\to\infty} y_n(x)$. Discuss the continuity of

$y_n(x)\ (n = 1,2,3\ldots..n)$ and y (x) at x = 0

NCERT CORNER

Very Short Answer

1. Show that the function f(x) given by $f(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$ is continuous at x = 0.

2. Show that the function f(x) given by $f(x)=\begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1}, & \text{when } x\neq 0 \\ 0\ , & \text{when } x=0 \end{cases}$ is discontinuous at x = 0.

3. Show that the function f(x) = 2x – |x| is continuous at x = 0.

4. Discuss the continuity of the function f(x) = $\begin{cases} \frac{|x|}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$

Short Answer

5. Discuss the continuity of the function *f*, where *f* is defined by $f(x)=\begin{cases} 2x, & \text{if } x<0 \\ 0\ , & \text{if } 0\leq x\leq 1 \\ 4x, & \text{if } x>1 \end{cases}$

6. Discuss the continuity of the function of given by f(x) = |x – 1| + |x – 2| at x = 1 and x = 2.

7. If the function f(x) defined by $f(x)=\begin{cases} \frac{\ln(1+ax)-\ln(1-bx)}{x}, & \text{if} x\neq 0 \\ k & ,\text{if } x=0 \end{cases}$ is continuous at x = 0 find k.

Long Answer

8. If the function f(x) given by $f(x)=\begin{cases} 3ax+b, & \text{if } x>1 \\ 11, & \text{if } x=1 \\ 5ax-2b, & \text{if } x<1 \end{cases}$ is continuous at x = 1, find the values of a and b.

9. Let f(x) = $\begin{cases} \frac{1-\cos 4x}{x^2}, & \text{if } x<0 \\ a, & \text{if } x=0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{if } x>0 \end{cases}$ Determine the value of a so f(x) is continuous at x = 0.

10. Show that the function f defined by f(x) = |1 – x + | x | | is everywhere continuous.

ANSWER KEY

BEGINNER'S BOX-1

1. 1 **2.** discontinuous at $x = -2$

3. All are discontinuous at $x = 0$.

4. (C) **5.** (B) **6.** (D) **7.** (C) **8.** (C) **9.** (B)
10. (B) **11.** (B) **12.** (D) **13.** (A) **14.** (C)

BEGINNER'S BOX-2

1. Non removable discontinuity at $x = 1$, isolated point removable discontinuity at $x = 4$.

2. Finite type non-removable discontinuity at $x=-1,1$

3. $f(x)\,g(x)$ & $f(x) \pm g(x)$ are discontinuous at $x = (2n+1)\frac{\pi}{2}$; $n \in I$

$\frac{f(x)}{g(x)}$ is discontinuous at $x = \frac{n\pi}{2}$; $n \in I$

4. $x = 0$

5. $x = \pm\frac{1}{\sqrt{2}}$

6. $a=-1$ & $b=1$

7. Discontinuous at $x = 1$ & continuous at $x = \frac{\pi}{2}$

8. (C) **9.** (D) **10.** (D) **11.** (C) **12.** (C)
13. (B) **14.** (A) **15.** (B)

EXERCISE-1 (SINGLE CORRECT & MORE THAN ONE OPTION CORRECT)

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	A	C	B	D	C	B	D	C	B
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	D	C	A	D	A	CD	ACD	BD	AC	ABC
Que.	21	22								
Ans.	ABC	CD								

EXERCISE-2 (MISCELLANEOUS)

- **Comprehension Based Questions**

Comprehension – 1 **1.** D **2.** C

Comprehension – 2 **3.** A **4.** B,C **5.** A

- **Match the Column**

6. (A) $\rightarrow$ (p, r); (B) $\rightarrow$ (s); (C) $\rightarrow$ (s); (D) $\rightarrow$ (p, q, r)

7. (A) $\rightarrow$ (q, s); (B) $\rightarrow$ (p, r, s); (C) $\rightarrow$ (q); (D) $\rightarrow$ (q)

8. (A) $\rightarrow$ (p); (B) $\rightarrow$ (t) ; (C) $\rightarrow$ (q)

- **Integer/Subjective Type Questions**

9. continuous every where except at $x = 0$

10. 00

11. (a) $-2, 2, 3$ (b) $K= 5$ (c) even

12. 02

13. discontinuous at $x = 0$

14. $k = 0$; $g(x)=\begin{cases} ln(\tan x) & \text{if } 0 < x < \frac{\pi}{4} \\ 0 & \text{if } \frac{\pi}{4} \le x < \frac{\pi}{2} \end{cases}$. Hence g(x) is continuous everywhere.

15. gof is discontinuous at $x = 0, 1$ and -1

16. 00, where $a = \frac{1}{\sqrt{2}}$, $g(0)=\frac{(\ell n2)^2}{8}$

17. $f(0^+) = \frac{\pi}{2}$; $f(0^-) = \frac{\pi}{4\sqrt{2}}$ $\Rightarrow$ ' f ' is dicontinuous at $x = 0$; $g(0^+) = g(0^-) = g(0) = \frac{\pi}{2}$

$\Rightarrow$ 'g' is continuous at $x = 0$

18. $a = e^{-1}$

19. $A = 1$; $f(2) = 1/2$

20. $y_n(x)$ is continuous at $x = 0$ for all n and $y(x)$ is discontinuous at $x = 0$

NCERT CORNER

4. f(x) is continuous at each point, except at $x = 0$.

5. f is continuous at all points except 1.

6. f(x) is continuous at $x = 1$ and $x = 2$.

7. $k = a + b$

8. $a = 3$ and $b = 2$

9. (8)

DIFFERENTIABILITY

1.0 MEANING OF DERIVATIVE

The instantaneous rate of change of a function with respect to the dependent variable is called derivative. Let 'f' be a given function of one variable and let Δx denote a number (positive or negative) to be added to the number x. Let Δf denote the corresponding change of 'f' then $\Delta f = f(x + \Delta x) - f(x)$

$$\Rightarrow \quad \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

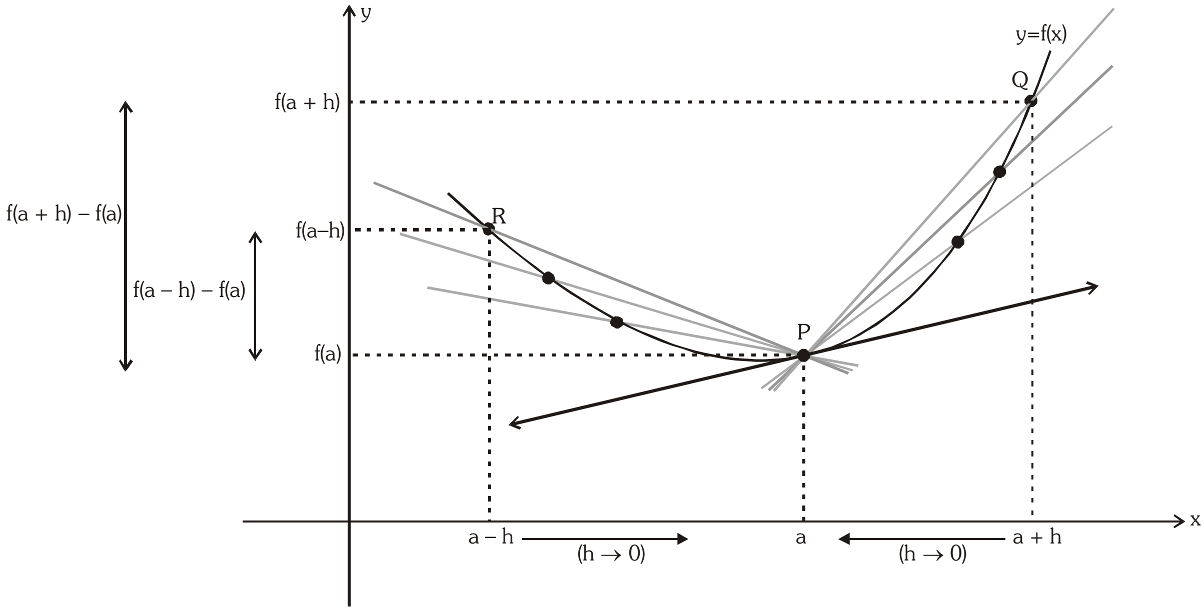
If $\Delta f/\Delta x$ approaches a limit as Δx approaches zero, this limit is the derivative of 'f' at the point x. The derivative of a function 'f' is a function ; this function is denoted by symbols such as

$$f'(x),\ \frac{df}{dx},\ \frac{d}{dx}f(x) \text{ or } \frac{df(x)}{dx}$$

$$\Rightarrow \quad \frac{df}{dx} = \lim_{\Delta x \to 0}\frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The derivative evaluated at a point a, is written, $f'(a),\ \left.\frac{df(x)}{dx}\right|_{x=a},\ f'(x)_{x=a}$, etc.

2.0 EXISTENCE OF DERIVATIVE AT x = a



(a) Right hand derivative –

The right hand derivative of f(x) at x = a denoted by $f'(a^+)$ is defined as :

$$f'(a^+) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$, provided the limit exists & is finite. $(h > 0)$

(b) Left hand derivative –

The left hand derivative of f(x) at x = a denoted by $f'(a^-)$ is defined as :

$$f'(a^-) = \lim_{h\to 0}\frac{f(a-h) - f(a)}{-h}$$, provided the limit exists & is finite. $(h > 0)$

Hence f(x) is said to be **derivable or differentiable at x = a**. $\boxed{\text{If } f'(a^+) = f'(a^-) = \text{finite quantity}}$ and it is denoted by $f'(a)$; where $f'(a) = f'(a^-) = f'(a^+)$ & it is called derivative or differential coefficient of f(x) at x = a.

3.0 DIFFERENTIABILITY & CONTINUITY

Theorem – If a function f(x) is derivable at x = a, then f(x) is continuous at x = a.

Proof : $f'(a) = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ exists.

Also $f(a+h)-f(a) = \dfrac{f(a+h)-f(a)}{h}.h \quad [h \neq 0]$

$\therefore \quad \lim\limits_{h\to 0}[f(a+h)-f(a)] = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}.h = f'(a).0 = 0$

$\Rightarrow \quad \lim\limits_{h\to 0}[f(a+h)-f(a)] = 0 \Rightarrow \lim\limits_{h\to 0}f(a+h) = f(a) \Rightarrow f(x)$ is continuous at x = a.

NOTE

(i) Differentiable $\Rightarrow$ Continuous; Continuity $\not\Rightarrow$ Differentiable; Not Differentiable $\not\Rightarrow$ Not Continuous but Not Continuous $\Rightarrow$ Not Differentiable

(ii) All polynomial, trigonometric, logarithmic and exponential function are continuous and differentiable in their domains.

(iii) If f(x) & g(x) are differentiable at x = a then the functions f(x) + g(x), f(x) – g(x), f(x). g(x) will also be differentiable at x =a & if g(a) ≠ 0 then the function f(x)/g(x) will also be differentiable at x = a.

Illustrations

Illustration 1. Let $f(x) = \begin{cases} \operatorname{sgn}(x)+x; & -\infty < x < 0 \\ -1+\sin x; & 0 \le x < \dfrac{\pi}{2} \\ \cos x; & \dfrac{\pi}{2} \le x < \infty \end{cases}$. Discuss the continuity & differentiability at x = 0 & $\dfrac{\pi}{2}$.

Solution $f(x) = \begin{cases} -1+x; & -\infty < x < 0 \\ -1+\sin x; & 0 \le x < \dfrac{\pi}{2} \\ \cos x; & \dfrac{\pi}{2} \le x < \infty \end{cases}$

To check the differentiability at x = 0

$$\text{LHD} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{-1+0-h-(-1)}{-h} = 1$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{-1+\sin h+1}{h} = 1$$

$\because$ LHD = RHD

$\therefore$ Differentiable at x = 0.

$\Rightarrow$ Continuous at x = 0.

To check the continuity at x = $\dfrac{\pi}{2}$

LHL $\lim\limits_{x\to\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to\frac{\pi}{2}^-}(-1+\sin x) = 0$

RHL $\lim\limits_{x\to\frac{\pi}{2}^+} f(x) = \lim\limits_{x\to\frac{\pi}{2}^+}\cos x = 0$

$\because$ LHL = RHL = $f\left(\dfrac{\pi}{2}\right) = 0$

$\therefore$ Continuous at $x = \frac{\pi}{2}$.

To check the differentiability at $x = \frac{\pi}{2}$

$$\text{LHD} = \lim_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h} = \lim_{h\to 0}\frac{-1+\cos h-0}{-h} = 0$$

$$\text{RHD} = \lim_{h\to 0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h} = \lim_{h\to 0}\frac{-\sin h-0}{h} = -1$$

$\because$ LHD $\neq$ RHD

$\therefore$ not differentiable at $x = \frac{\pi}{2}$.

Illustration 2. Let $f(x) = \begin{cases}\frac{1}{|x|}, & |x|\geq 1\\ ax^2+b, & |x|<1\end{cases}$ be continuous and differentiable everywhere. Then a and b are -

(A) $-\frac{1}{2}, \frac{3}{2}$ (B) $\frac{1}{2}, -\frac{3}{2}$ (C) $\frac{1}{2}, \frac{3}{2}$ (D) none of these

Solution

$$f(x) = \begin{cases}-\frac{1}{x}, & x\leq -1\\ ax^2+b, & -1<x<1\\ \frac{1}{x}, & x\geq 1\end{cases}$$

Since function is continuous everywhere

$\therefore$ LHL = RHL

at $x = -1$

$$\text{LHL} = \lim_{h\to 0} f(-1-h) = \lim_{h\to 0}\frac{-1}{(-1-h)} = 1$$

$$\text{RHL} = \lim_{h\to 0} f(-1+h) = \lim_{h\to 0} a(-1+h)^2+b = a+b$$

$\Rightarrow$ $a + b = 1$ .....(A)

Again, function is differentiable at everywhere.

$\therefore$ LHD = RHD

at $x = -1$

$$\text{LHD} = \lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h} = \lim_{h\to 0}\frac{\frac{-1}{-1-h}-\frac{1}{1}}{-h} = 1$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{+h} = \lim_{h\to 0}\frac{a(-1+h)^2+b-1}{h} = \lim_{h\to 0}\frac{a(1+h^2-2h)+b-1}{h}$$

$$= \lim_{h\to 0}\frac{a+b-1+ah^2-2ah}{h} = -2a \qquad [\because a + b = 1 \text{ from A}]$$

$\Rightarrow$ $-2a = 1$ ......(B)

$\Rightarrow$ $a = \frac{-1}{2}$ & $b = \frac{3}{2}$ (using (A) & (B))

Illustration 3. If $f(x)=\begin{cases} A+Bx^2 & x<1 \\ 3Ax-B+2 & x\ge 1 \end{cases}$

then find A and B so that f(x) become differentiable at x = 1.

Solution $f'(1^+) = \lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0}\dfrac{3A(1+h)-B+2-3A+B-2}{h} = \lim\limits_{h\to 0}\dfrac{3Ah}{h} = 3A$

$f'(1^-) = \lim\limits_{h\to 0}\dfrac{f(1-h)-f(1)}{-h} = \lim\limits_{h\to 0}\dfrac{A+B(1-h)^2-3A+B-2}{-h} = \lim\limits_{h\to 0}\dfrac{(-2A+2B-2)+Bh^2-2Bh}{-h}$

hence for this limit to be defined

$-2A + 2B - 2 = 0$

$B = A + 1$

$f'(1^-) = \lim\limits_{h\to 0} -(Bh - 2B) = 2B$

$\therefore \quad f'(1^-) = f'(1^+)$

$3A = 2B = 2(A + 1)$

$A = 2,\ B = 3$ **Ans.**

Illustration 4. If $f(x) = \begin{cases} |x-1|([x]-x) & , \ x\ne 1 \\ 0 & , \ x=1 \end{cases}$

Test the differentiability at x = 1, where [.] denotes the greatest integer function.

Solution Check the differentiability at x = 1

$Rf'(1) = \lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{h} \quad (\because x > 1)$

$= \lim\limits_{h\to 0}\dfrac{|1+h-1|([1+h]-(1+h))-0}{h} = \lim\limits_{h\to 0}\dfrac{h(1-1-h)}{h} = \lim\limits_{h\to 0}\dfrac{h(-h)}{h} = 0$

$L(f'(1)) = \lim\limits_{h\to 0}\dfrac{f(1-h)-f(1)}{-h}$

$= \lim\limits_{h\to 0}\dfrac{|1-h-1|([1-h]-(1-h))-0}{-h} = \lim\limits_{h\to 0}\dfrac{h(0-1+h)}{-h} = 1$

$Lf'(1) \ne Rf'(1)$

Hence f(x) is not differentiable at x = 1. **Ans.**

Illustration 5. $f(x) = \begin{cases} [\cos \pi x] & x\le 1 \\ 2\{x\}-1 & x>1 \end{cases}$ comment on the derivability at x = 1, where [ ] denotes greatest integer function & { } denotes fractional part function.

Solution $f'(1^-) = \lim\limits_{h\to 0}\dfrac{f(1-h)-f(1)}{-h} = \lim\limits_{h\to 0}\dfrac{[\cos(\pi-\pi h)]+1}{-h} = \lim\limits_{h\to 0}\dfrac{-1+1}{-h} = 0$

$f'(1^+) = \lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0}\dfrac{2\{1+h\}-1+1}{h} = \lim\limits_{h\to 0}\dfrac{2h}{h} = 2$

Hence f(x) is not differentiable at x = 1. **Ans.**

4.0 IMPORTANT NOTE

(a) Let $f'(a^+) = p$ & $f'(a^-) = q$ where p & q are finite then –

(i) $p = q \Rightarrow$ f is differentiable at $x = a$ $\Rightarrow$ f is continuous at $x = a$

(ii) $p \neq q \Rightarrow$ f is not differentiable at $x = a$, but f is continuous at $x = a$.

Illustrations

Illustration 6. Determine the values of x for which the following functions fails to be continuous or differentiable

$$f(x) = \begin{cases} (1-x), & x < 1 \\ (1-x)(2-x), & 1 \le x \le 2 \\ (3-x), & x > 2 \end{cases}, \text{ Justify your answer.}$$

Solution By the given definition it is clear that the function f is continuous and differentiable at all points except possibily at $x = 1$ and $x = 2$.

Check the differentiability at $x = 1$

$$q = \text{LHD} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{1-(1-h)-0}{-h} = -1$$

$$p = \text{RHD} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\{1-(1+h)\}\{2-(1+h)\}-0}{h} = -1$$

$\because$ $q = p$ $\quad\therefore$ Differentiable at $x = 1$. $\Rightarrow$ Continuous at $x = 1$.

Check the differentiability at $x = 2$

$$q = \text{LHD} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0}\frac{(1-2+h)(2-2+h)-0}{-h} = 1 = \text{finite}$$

$$p = \text{RHD} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(3-2-h)-0}{h} \to \infty \quad \text{(not finite)}$$

$\because$ $q \neq p$ $\quad\therefore$ not differentiable at $x = 2$.

Now we have to check the continuity at $x = 2$

$$\text{LHL} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(1-x)(2-x) = \lim_{h\to 0}(1-(2-h))(2-(2-h)) = 0$$

$$\text{RHL} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3-x) = \lim_{h\to 0}(3-(2+h)) = 1$$

$\therefore$ LHL $\neq$ RHL

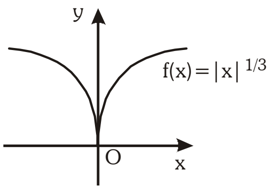
$\Rightarrow$ not continuous at $x = 2$.

(b) **Geometrical interpretation of differentiability –**

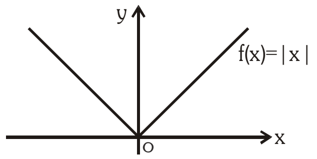
(i) If the function $y = f(x)$ is differentiable at $x = a$, then a unique tangent can be drawn to the curve $y = f(x)$ at the point $P(a, f(a))$ & $f'(a)$ represent the slope of the tangent at point P.

(ii) If a function $f(x)$ does not have a unique tangent ($p \neq q$) but is continuous at $x = a$, it geometrically implies a sharp corner at $x = a$. Note that p and q may not be finite, where $p = f'(a^+)$ and $q = f'(a^-)$

e.g. (1) $f(x) = |x|$ and $|x|^{1/3}$ is continuous but not differentiable at $x = 0$ & there is sharp corner at $x = 0$.

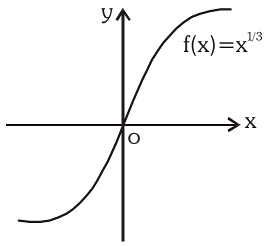


(does not have unique tangent) $\begin{cases} p \to +\infty \\ q \to -\infty \end{cases}$



(does not have unique tangent) p = 1, q = –1

(2) $f(x) = x^{1/3}$ is continuous but not differentiable at x = 0 because $f'(0^+) \to \infty$ and $f'(0^-) \to \infty$.



$p \to +\infty$, $q \to +\infty$

(have a unique tangent but does not have sharp corner)

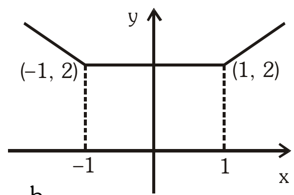
Note : $\boxed{\text{sharp corner} \Rightarrow \text{non differentiable}}$ $\boxed{\text{non differentiable} \not\Rightarrow \text{sharp corner}}$

Illustrations

Illustration 7. Let f(x) = max {(1 + x), (1 – x), 2}. Find the number of points where it is not differentiable.

Solution

$$f(x) = \begin{cases} 1-x; & x < -1 \\ 2; & -1 \le x \le 1 \\ 1+x; & x > 1 \end{cases}$$



at x = –1

$$q = \text{LHD} = f'(-1^-) = \lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h} = \lim_{h\to 0}\frac{1-(-1-h)-2}{-h} = \lim_{h\to 0}\frac{h}{-h} = -1$$

$$p = \text{RHD} = f'(-1^+) = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0}\frac{2-2}{h} = 0$$

$\because$ $q \ne p$

$\therefore$ not differentiable but continuous at x = –1 and having sharp corner.

Now, at x = 1

$$q = \text{LHD} = f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{2-2}{-h} = 0$$

$$p = \text{RHD} = f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{1+(1+h)-2}{h} = 1$$

$\because$ $q \ne p$

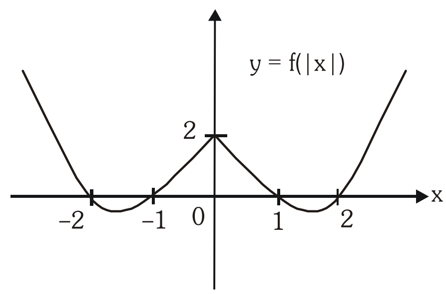
$\therefore$ not differentiable but continuous at x = 1 and having sharp corner.

$\Rightarrow$ f(x) is not differentiable at x = ±1. **Ans.**

Illustration 8. If $f(x) = \begin{cases} x-3 & x < 0 \\ x^2-3x+2 & x \ge 0 \end{cases}$. Draw the graph of the function & discuss the continuity and differentiability of $f(|x|)$ and $|f(x)|$.

Solution

$$f(|x|) = \begin{cases} |x|-3; & |x| < 0 \to \text{not possible} \\ |x|^2-3|x|+2; & |x| \ge 0 \end{cases}$$



$$f(|x|) = \begin{cases} x^2+3x+2, & x < 0 \\ x^2-3x+2, & x \ge 0 \end{cases}$$

at x = 0

$$q = LHD = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{-h} = 3$$

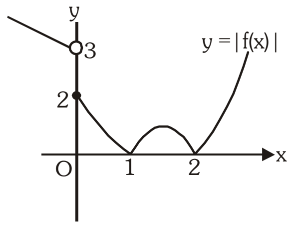
$$p = RHD = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{h} = -3$$

$\because$ $q \neq p$

$\therefore$ not differentiable at x = 0. but p & q are both are finite

$\Rightarrow$ continuous at x = 0

$$\text{Now, } |f(x)| = \begin{cases} 3-x & , \quad x<0 \\ (x^2-3x+2) & , \quad 0\le x<1 \\ -(x^2-3x+2) & , \quad 1\le x\le 2 \\ (x^2-3x+2) & , \quad x>2 \end{cases}$$



To check differentiability at x = 0

q = LHD =

$$\left.\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{3+h-2}{-h} = \lim_{h\to 0}\frac{(1+h)}{-h} \to -\infty\right\} \Rightarrow \text{not differentiable at } x = 0.$$

$$p = RHD = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{h} = -3$$

Now to check continuity at x = 0

$$\left.\begin{aligned} LHL &= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3-x = 3 \\ RHL &= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2-3x+2 = 2 \end{aligned}\right\} \Rightarrow \text{not continuous at } x = 0.$$

To check differentiability at x = 1

$$q = LHD = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h)^2-3(1-h)+2-0}{-h} = \lim_{h\to 0}\frac{h^2+h}{-h} = -1$$

$$p = RHD = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{-(h^2+2h+1-3+3h+2)-0}{h} = \lim_{h\to 0}\frac{-(h^2-h)}{h} = 1$$

$\Rightarrow$ not differentiable at x = 1.

but $|f(x)|$ is continous at x = 1, because $p \neq q$ and both are finite.

To check differentiability at x = 2

$$q = LHD = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0}\frac{-(4+h^2-4h-6+3h+2)-0}{-h} = \lim_{h\to 0}\frac{h^2-h}{h} = -1$$

$$p = RHD = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(h^2+4h+4-6-3h+2)-0}{h} = \lim_{h\to 0}\frac{(h^2+h)}{h} = 1$$

$\Rightarrow$ not differentiable at x = 2.

but $|f(x)|$ is continous at x = 2, because $p \neq q$ and both are finite.

BEGINNER'S BOX-1

1. A function is defined as follows :

$f(x) = \begin{cases} x^3 \ ; & x^2 < 1 \\ x \ ; & x^2 \geq 1 \end{cases}$ discuss the continuity and differentiability at x = 1.

2. Let $f(x) = \begin{cases} (x-1)|x-1|, & x \neq 1 \\ 0, & x = 1 \end{cases}$. Discuss the continuity and differentiability of f(x) at x = 1.

3. Let $f(x) = \begin{cases} -4 \ ; & -4 < x < 0 \\ x^2 - 4 \ ; & 0 \leq x < 4 \end{cases}$

Discuss the continuity and differentiablity of g(x) = |f(x)|.

4. The function $f(x) = \begin{cases} \dfrac{x(e^{1/x} - e^{-1/x})}{e^{1/x} + e^{-1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is

(A) Continuous everywhere but not differentiable at x = 0
(B) Continuous and differentiable everywhere
(C) Not continuous at x = 0
(D) None of these

5. If a function f(x) is defined as $f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ x^2 - x + 1, & x > 1 \end{cases}$, then

(A) f(x) is differentiable at = 0 & x = 1
(B) f(x) is differentiable at x = 0 but not at x = 1
(C) f(x) is differentiable at x = 1 but not at x = 0
(D) f(x) is not differentiable at x = 0 and x = 1

6. The value of the derivative of | x – 1 | + | x – 3 | at x = 2 is
(A) –2 (B) 0 (C) 2 (D) Does not exist

7. If f(x) = | x – 2 | and g(x) = f[f(x)], then g'(x) for x > 20 is
(A) 1 (B) 2 (C) – 1 (D) None of these

8. If $f(x) = \begin{cases} 3^x, & -1 \leq x \leq 1 \\ 4 - x, & 1 < x \leq 4 \end{cases}$, then

(A) f(x) is continuous as well as differentiable at x = 1
(B) f(x) is continuous but not differentiable at x = 1
(C) f(x) is differentiable but not continuous at x = 1
(D) None of these

9. The function $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$ is

(A) Continuous at x = 1 but not derivable at x = 1
(B) Continuous and derivable at x = 1
(C) Not derivable at x = 1
(D) Not continuous at x = 1

10. Let f(x) = $\begin{cases} ax^2+1, & x>1 \\ x+a, & x\le 1 \end{cases}$. Then f(x) is derivable at x = 1, if

(A) a = 2 (B) a = 1 (C) a = 0 (D) a = $\frac{1}{2}$

11. Let f(x) = $\begin{cases} 1, & x\le -1 \\ |x|, & -1<x<1 \\ 0, & x\ge 1 \end{cases}$, then

(A) f is continuous at x = –1 (B) f is differentiable at x = –1
(C) f is continuous everywhere (D) f is differentiable for all x.

12. If f(x) = $\begin{cases} \frac{1}{1+e^{1/x}}, & x\ne 0 \\ 0, & x=0 \end{cases}$, then f(x) is

(A) Continuous as well as differentiable at x = 0 (B) Continuous but not differentiable at x = 0
(C) Differentiable but not continuous at x = 0 (D) None of these

13. If f(x) = x^5 sgn x, where sgn x = $\begin{Bmatrix} \frac{|x|}{x}, & x\ne 0 \\ 0, & x=0 \end{Bmatrix}$, then f(x) is

(A) Differentiable as well as continuous at x = 0 (B) Continuous but not differentiable at x = 0

(C) Differentiable but not continuous at x = 0 (D) Neither differentiable nor continuous at x = 0

14. If f(x) = $\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\ne 0 \\ 0, & x=0 \end{cases}$, then f(x) is

(A) Continuous as well as differentiable for all x

(B) Continuous for all x but not differentiable at x = 0

(C) Differentiable for all x but not continuous at x = 0

(D) None of these.

15. If f(x) = $\begin{cases} \frac{[x]-1}{x-1}, & x\ne 1 \\ 0, & x=1 \end{cases}$, then f(x) is

(A) Continuous as well as differentiable at x = 1
(B) Differentiable but not continuous at x = 1
(C) Continuous but not differentiable at x = 1
(D) Neither continuous nor differentiable at x = 1

5.0 DIFFERENTIABILITY OVER AN INTERVAL

(a) f(x) is said to be differentiable over an open interval (a, b) if it is differentiable at each & every point of the open interval (a, b).

(b) f(x) is said to be differentiable over the closed interval [a, b] if :

(i) f(x) is differentiable in (a, b) &

(ii) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist.

Illustrations

Illustration 9. If $f(x)=\begin{cases} e^{-|x|}, & -5<x<0 \\ -e^{-|x-1|}+e^{-1}+1, & 0\le x<2 \\ e^{-|x-2|}, & 2\le x<4 \end{cases}$

Discuss the continuity and differentiability of f(x) in the interval (–5, 4).

Solution Check the differentiability at x = 0

$$\text{LHD}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{e^{-|-h|}-1}{-h}=\lim_{h\to 0}\frac{e^{-h}-1}{-h}=1$$

$$\text{RHD}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{-e^{-|h-1|}+e^{-1}+1-1}{h}=\lim_{h\to 0}\frac{e^{-1}(1-e^{h})}{h}=-e^{-1}$$

$\therefore$ LHD $\neq$ RHD

$\Rightarrow$ not differentiable at x = 0.

but f(x) is continous at x = 0, because p ≠ q and both are finite.

check the differentiability at x = 2

$$\text{LHD}=\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}=\lim_{h\to 0}\frac{e^{-|1-h|}+e^{-1}+1-1}{-h}=\lim_{h\to 0}\frac{e^{-1}(1-e^{h})}{-h}=e^{-1}$$

$$\text{RHD}=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{e^{-|h|}-1}{h}=\lim_{h\to 0}\frac{(e^{-h}-1)}{h}=-1$$

$\therefore$ LHD $\neq$ RHD

$\Rightarrow$ not differentiable at x = 2.

but f(x) is continous at x = 2, because p ≠ q and both are finite.

similar checking at x = 1 indicates that function is non-differentiable at x = 1

Illustration 10. $f(x) = \begin{cases} \left\{x+\frac{1}{3}\right\}[\sin \pi x] & , \quad 0 \le x < 1 \\ [2x]\operatorname{sgn}\left(x-\frac{4}{3}\right) & , \quad 1 \le x \le 2 \end{cases}$; find that points at which continuity and differentiability should be checked. Also check the continuity and differentiability of f(x) at x = 1, where [ ] denotes greatest integer function & { } denotes fractional part function.

Solution

$$f(x) = \begin{cases} \left\{x+\frac{1}{3}\right\}[\sin x\pi]; & 0 \le x < 1 \\ [2x]\operatorname{sgn}\left(x-\frac{4}{3}\right); & 1 \le x \le 2 \end{cases}$$

$$f(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ \frac{5}{6}, & x = \frac{1}{2} \\ 0, & \frac{1}{2} < x < 1 \\ -2, & 1 \le x < \frac{4}{3} \\ 0, & x = \frac{4}{3} \\ 2, & \frac{4}{3} < x < \frac{3}{2} \\ 3, & \frac{3}{2} \le x < 2 \\ 4, & x = 2 \end{cases}$$

Hence function is discontinuous & non-derivable at $x = \frac{1}{2}, 1, \frac{4}{3}, \frac{3}{2}$ & 2

6.0 DETERMINATION OF FUNCTION WHICH SATISFIES THE GIVEN FUNCTIONAL RULE

Illustrations

Illustration 11. Let f be a differentiable function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y) \ \forall \ x, y > 0$.

If f'(1) = 1. Find f(x).

Solution Put x = y = 1 in given rule $\Rightarrow$ f(1) = f(1) – f(1) = 0

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{h} \qquad \{\text{from given functional rule}\}$$

$$\therefore \quad f'(x) = \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right) - f(1)}{\frac{h}{x}} \times \frac{1}{x} = \frac{f'(1)}{x}$$

$$\therefore \quad f'(x) = \frac{1}{x} \qquad \{\because \quad f'(1) = 1\}$$

Integrating both sides $\Rightarrow$ $f(x) = \ell nx + c$

putting x = 1 we get c = 0 $\Rightarrow$ $f(x) = \ell nx$

Illustration 12. Let $f(x + y) = f(x) + f(y) - 2xy - 1$ for all x and y. If $f'(0)$ exists and $f'(0) = -\sin\alpha$, then find $f\{f'(0)\}$.

Solution

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\{f(x)+f(h)-2xh-1\}-f(x)}{h} \qquad \text{(Using the given relation)}$$

$$= \lim_{h\to 0}-2x+\lim_{h\to 0}\frac{f(h)-1}{h} = -2x+\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

[Putting $x = 0 = y$ in the given relation we find $f(0) = f(0) + f(0) + 0 - 1 \Rightarrow f(0) = 1$]

$\therefore \quad f'(x) = -2x + f'(0) = -2x - \sin\alpha$

$\Rightarrow \quad f(x) = -x^2 - (\sin\alpha).\,x + c$

$f(0) = -0 - 0 + c \Rightarrow c = 1$

$\therefore \quad f(x) = -x^2 - (\sin\alpha).\,x + 1$

So, $f\{f'(0)\} = f(-\sin\alpha) = -\sin^2\alpha + \sin^2\alpha + 1$

$\therefore \quad f\{f'(0)\} = 1.$

GOLDEN KEY POINTS

- If f(x) is differentiable at x = a & g(x) is not differentiable at x = a, then the product function F(x)=f(x).g(x) can still be differentiable at x = a.

  e.g. Consider $f(x) = x$ & $g(x) = |x|$. f is differentiable at x = 0 & g is non-differentiable at x = 0, but f(x).g(x) is still differentiable at x = 0.

- If f(x) & g(x) both are not differentiable at x = a then the product function; F(x)=f(x).g(x) can still be differentiable at x = a.

  e.g. Consider $f(x) = |x|$ & $g(x) = -|x|$. f & g are both non differentiable at x = 0, but f(x).g(x) still differentiable at x = 0.

- If f(x) & g(x) both are non-differentiable at x=a then the sum function F(x)=f(x)+g(x) may be a differentiable function.

  e.g. $f(x)=|x|$ & $g(x)=-|x|$. f & g are both non differentiable at x = 0, but (f+g)(x) still differentiable at x = 0.

- If f(x) is differentiable at x = a $\not\Rightarrow$ f'(x) is continuous at x = a.

  e.g. $f(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

- f(x) is continuous at x = a and g(x) is differentiable at x = a also g(a) = 0 than f(x) × g(x) is differentiable at x = a.

- f(x), g(x) are two differentiable function then |f(x)|, |g(x)|, max {f(x), g(x)} and min {f(x), g(x)} are continuous but may or may not be differentiable.

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. Discuss the continuity and differentiability of the function $y = f(x)$ defined parametrically; $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$.

Solution Here $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$.

Now when $t < 0$;

$x = 2t - \{-(t-1)\} = 3t - 1$ and $y = 2t^2 - t^2 = t^2 \Rightarrow y = \frac{1}{9}(x+1)^2$

when $0 \le t < 1$

$x = 2t - (-(t-1)) = 3t - 1$ and $y = 2t^2 + t^2 = 3t^2 \Rightarrow y = \frac{1}{3}(x+1)^2$

when $t \ge 1$;

$x = 2t - (t-1) = t + 1$ and $y = 2t^2 + t^2 = 3t^2 \Rightarrow y = 3(x-1)^2$

$$\text{Thus, } y = f(x) = \begin{cases} \frac{1}{9}(x+1)^2, & x < -1 \\ \frac{1}{3}(x+1)^2, & -1 \le x < 2 \\ 3(x-1)^2, & x \ge 2 \end{cases}$$

We have to check differentiability at $x = -1$ and 2.

Differentiability at $x = -1$;

$$\text{LHD} = f'(-1^-) = \lim_{h\to 0} \frac{f(-1-h) - f(-1)}{-h} = \lim_{h\to 0} \frac{\frac{1}{9}(-1-h+1)^2 - 0}{-h} = 0$$

$$\text{RHD} = f'(-1^+) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\frac{1}{3}(-1+h+1)^2 - 0}{-h} = 0$$

Hence $f(x)$ is differentiable at $x = -1$.

$\Rightarrow$ continuous at $x = -1$.

To check differentiability at $x = 2$;

$$\text{LHD} = f'(2^-) = \lim_{h\to 0} \frac{\frac{1}{3}(2-h+1)^2 - 3}{-h} = 2 \ \& \ \text{RHD} = f'(2^+) = \lim_{h\to 0} \frac{3(2+h-1)^2 - 3}{h} = 6$$

Hence $f(x)$ is not differentiable at $x = 2$.

But continuous at $x = 2$, because LHD & RHD both are finite.

$\therefore$ $f(x)$ is continuous for all x and differentiable for all x, except $x = 2$.

Illustration 2. Let $f(x) = 1 + 4x - x^2, \forall x \in R$

$$g(x) = \begin{cases} \max. \{f(t);\ x \le t \le (x+1);\ 0 \le x < 3\} \\ \min. \{(x+3);\ 3 \le x \le 5\} \end{cases}$$

Discuss the continuity and differentiability of $g(x)$ for all $x \in [0, 5]$.

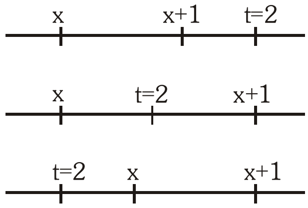
Solution Here, $f(t) = 1 + 4t - t^2$.

$f'(t) = 4 - 2t$, when $f'(t) = 0 \Rightarrow t = 2$

at $t = 2$, $f(x)$ has a maxima.

Since, $g(x)$ max. $\{f(t)$ for $t \in [x, x+1], 0 \le x < 3\}$

$$\therefore \quad g(x) = \begin{cases} f(x+1), & \text{if } \; t = 2 \text{ is on right side of } [x, x+1] \\ f(2), & \text{if } \; t = 2 \text{ is inside } [x, x+1] \\ f(x), & \text{if } \; t = 2 \text{ is on left side of } [x, x+1] \end{cases}$$



$$\therefore \quad g(x) = \begin{cases} 4 + 2x - x^2, & \text{if } \; 0 \le x < 1 \\ 5, & \text{if } \; 1 \le x \le 2 \\ 1 + 4x - x^2, & \text{if } \; 2 < x < 3 \\ 6, & \text{if } \; 3 \le x \le 5 \end{cases}$$

Which is clearly continuous for all $x \in [0, 5]$ except $x = 3$.

to check differentiability at $x = 1, 2, 3$

at $x = 1$

$$\text{LHD} = f'(1^-) = \lim_{h \to 0} \frac{f(-h) - f(1)}{-h} = \lim_{h \to 0} \frac{-(1-h)^2 + 2(1-h) + 4 - 5}{-h} = 0$$

$$\text{RHD} = f'(1^+) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{5-5}{h} = 0$$

$\therefore$ differentiable at $x = 1$

at $x = 2$

$$\text{LHD} = f'(2^-) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \frac{5-5}{-h} = 0$$

$$\text{RHD} = f'(2^+) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{1 + 4(2+h) - (2+h)^2 - 5}{h} = \lim_{h \to 0} \frac{-h^2}{h} = 0$$

$\therefore$ differentiable at $x = 2$

Function g(x) is discontinous at $x = 3 \Rightarrow$ not differentiable at $x = 3$.

BEGINNER'S BOX-2

1. Let $f(x) = \min \{|x-1|, |x+1|, 1\}$. Find the number of points where it is not differentiable.

2. Let $f(x) = \max \{\sin x, 1/2\}$, where $0 \le x \le \frac{5\pi}{2}$. Find the number of points where it is not differentiable.

3. Let $f(x) = \begin{cases} [x] & ; \; 0 < x \le 2 \\ 2x - 2 ; & 2 < x < 3 \end{cases}$, where [.] denotes greatest integer function.

(a) Find that points at which continuity and differentiability should be checked.

(b) Discuss the continuity & differentiability of f(x) in the interval (0, 3).

4. If $f(x) = \begin{cases} ax^3 + b, & \text{for } \; 0 \le x \le 1 \\ 2\cos \pi x + \tan^{-1} x, & \text{for } \; 1 < x \le 2 \end{cases}$ be the differentiable function in [0, 2], then find a and b.

5. Let $f : R \to R$ be function defined by $f(x) = \max \{x, x^3\}$. The set of all points where f(x) is not differentiable is

(A) $\{-1, 1\}$ (B) $\{-1, 0\}$ (C) $\{0, 1\}$ (D) $\{-1, 0, 1\}$

6. Let $f(x) = [x^3 - 3]$, where [.] denotes the greatest integer function. Then the number of points in the interval (1, 2) where the function is discontinuous, is

(A) 4 (B) 2 (C) 6 (D) None of these

7. The function $f(x) = \max.\{(1-x), (1+x), 2\}$, $x \in (-\infty, \infty)$, is

(A) Continuous at all points

(B) Differentiable at all points

(C) Differentiable at all points except at $x = 1$ and $x = -1$.

(D) Continuous at all points except at $x = 1$ and $x = -1$, where it is discontinuous.

***8.** Let $h(x) = \min\{x, x^2\}$, for every real number x, then

(A) h is continuous for all x (B) Continuous at all non-zero x

(C) $h'(x) = 1$, for all $x > 1$ (D) h is not differentiable at two values of x

***9.** The points where the function $f(x) = [x] + |1 - x|$, $-1 \le x \le 3$, where [.] denotes the greatest integer function, is not differentiable, are

(A) $x = -1, 0, 1, 2, 3$ (B) $x = -1, 0, 2$ (C) $x = 0, 1, 2, 3$ (D) $x = -1, 0, 1, 2$

10. If $f(x) = \lim\limits_{n\to\infty} \dfrac{(1+\cos \pi x)^n + 1}{(1+\cos \pi x)^n - 1}$, then

(A) $f(1^+) = 1$ (B) $f(1^-) = 2$

(C) f(x) is continuous at $x = 1$ (D) f(x) is not continuous at $x = 1$

11. If $f(x) = 2x + |x - x^2|$, $-1 \le x \le 1$, then f(x) is

(A) Continuous but not differentiable in [–1, 1] (B) Continuous as well as differentiable in [–1, 1]

(C) Differentiable but not continuous in [–1, 1] (D) Neither differentiable nor continuous in [–1, 1]

12. The values of constants a and b so as to make the function $f(x) = \begin{cases} \dfrac{1}{|x|}, & |x| \ge 1 \\ ax^2 + b, & |x| < 1 \end{cases}$ continuous as well as differentiable for all x, are

(A) $a = \dfrac{-1}{2}, b = \dfrac{3}{2}$ (B) $a = \dfrac{1}{2}, b = \dfrac{3}{2}$ (C) $a = \dfrac{-1}{2}, b = \dfrac{-3}{2}$ (D) None of these

13. The set of points where the function $f(x) = |x - 2| \cos x$ is differentiable, is

(A) $(-\infty, \infty)$ (B) $(-\infty, \infty) - \{2\}$ (C) $(0, \infty)$ (D) None of these

14. If $4x + 3|y| = 5y$, then y as a function of x is

(A) Differentiable at $x = 0$ (B) Continuous at $x = 0$ (C) $\dfrac{dy}{dx} = 2$ for all x (D) None of these

15. A function $f : R \to R$ satisfies the equation $f(x + y) = f(x).f(y)$ for all $x, y \in R$, $f(x) \ne 0$. Suppose that the function is differentiable everywhere and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

16. Let $f(x + y) = f(x) f(y)$ for all x, y where $f(0) \ne 0$. If $f'(0) = 2$, then f(x) is equal to (f is differentiable function)

(A) e^x (B) e^{2x} (C) 2x (D) None of these

17. If a function $f : R \to R$ be such that $f(x + y) = f(x) . f(y)$ for all $x, y \in R$ where $f(x) = 1 + x\phi(x)$ and $\lim\limits_{x\to 0} \phi(x) = 1$, then

(A) $f'(x)$ does not exist (B) $f'(x) = 2f(x)$ for all x (C) $f'(x) = f(x)$ for all x (D) None of these

EXERCISE – 1 SCQ/MCQ

SINGLE CORRECT

1. If a function f(x) is defined as $f(x)=\begin{cases}-x & , \quad x<0 \\ x^2 & , \quad 0 \le x \le 1 \\ x^2-x+1 & , \quad x>1\end{cases}$, then -

(A) f(x) is differentiable at x = 0 and x = 1
(B) f(x) is differentiable at x = 0 but not at x = 1
(C) f(x) is differentiable at x = 1 but not at x = 0
(D) f(x) is not differentiable at x = 0 and x = 1

2. If $f(x) = x^3 \operatorname{sgn} x$, then -

(A) f is differentiable at x = 0
(B) f is continous but not differentiable at x = 0
(C) $f'(0^-) = 1$
(D) none of these

3. Which one of the following functions is continuous everywhere in its domain but has atleast one point where it is not differentiable ?

(A) $f(x) = x^{1/3}$
(B) $f(x) = \frac{|x|}{x}$
(C) $f(x) = e^{-x}$
(D) f(x) = tan x

4. If $f(x)=\begin{bmatrix} x+\{x\}+x\sin\{x\} & \text{for} \quad x \ne 0 \\ 0 & \text{for} \quad x=0\end{bmatrix}$ where {x} denotes the fractional part function, then -

(A) 'f' is continuous & differentiable at x = 0
(B) 'f' is continuous but not differentiable at x = 0
(C) 'f' is continuous & differentiable at x = 2
(D) none of these

5. Let $f(x) = x^3$ and $g(x) = |x|$. Then at x = 0, the composite functions -

(A) gof is derivable but fog is not
(B) fog is derivable but gof is not
(C) gof and fog both are derivable
(D) neither gof nor fog is derivable

6. If $f(x)=\begin{cases}\dfrac{x-1}{2x^2-7x+5}, & x \ne 1 \\ \dfrac{-1}{3}, & x=1\end{cases}$, then f'(1) equals -

(A) $\frac{2}{9}$
(B) $-\frac{2}{9}$
(C) 0
(D) does not exist

7. Function $f(x) = \dfrac{x}{1+|x|}$ is differentiable in the set -

(A) $(-\infty, \infty)$
(B) $(-\infty, 0)$
(C) $(-\infty, 0) \cup (0, \infty)$
(D) $(0, \infty)$

8. Let $f(x) = \begin{bmatrix} 4x^2+2[x]x & \text{if} \quad -\frac{1}{2} \le x < 0 \\ ax^2-bx & \text{if} \quad 0 \le x < \frac{1}{2}\end{bmatrix}$ where [x] denotes the greatest integer function. Then -

(A) f(x) is continuous and differentiable in $\left(-\frac{1}{2},\frac{1}{2}\right)$ for all a, provided b = 2

(B) f(x) is continuous and differentiable in $\left(-\frac{1}{2},\frac{1}{2}\right)$ if a = 4, b = 2

(C) f(x) is continuous and differentiable in $\left(-\frac{1}{2},\frac{1}{2}\right)$ if a = 4 and b = 0

(D) for no choice of a and b, f(x) is differentiable in $\left(-\frac{1}{2},\frac{1}{2}\right)$

9. Let the function f, g and h be defined as follows -

$$f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right) & \text{for } -1\le x\le 1 \text{ and } x\ne 0 \\ 0 & \text{for } x=0\end{cases},\quad g(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{for } -1\le x\le 1 \text{ and } x\ne 0 \\ 0 & \text{for } x=0\end{cases}$$

and $h(x) = |x|^3$ for $-1 \le x \le 1$

Which of these functions are differentiable at $x = 0$?

(A) f and g only (B) f and h only (C) g and h only (D) none

10. Identify the correct statement -

(A) If f(x) is derivable at $x = a$, $|f(x)|$ will also be derivable at $x = a$.

(B) If f(x) is continuous at $x = a$, $|f(x)|$ too will be continuous at $x=a$.

(C) If f(x) is discontinuous at $x = a$, $|f(x)|$ will also be discontinuous at $x = a$.

(D) If $|f(x)|$ is continuous at $x = a$, f(x) too will be continuous at $x=a$.

11. A function f defined as $f(x) = x[x]$ for $-1 \le x \le 3$ where [x] defines the greatest integer $\le x$ is -

(A) continuous at all points in the domain of f but non-derivable at a finite number of points

(B) discontinuous at all points & hence non-derivable at all points in the domain of f

(C) discontinuous at a finite number of points but not derivable at all points in the domain of f

(D) discontinuous & also non-derivable at a finite number of points of f

12. Consider $f(x)=\begin{cases}\left[\dfrac{2(\sin x-\sin^3 x)+|\sin x-\sin^3 x|}{2(\sin x-\sin^3 x)-|\sin x-\sin^3 x|}\right], x\ne\dfrac{\pi}{2} & \text{for } x\in(0,\pi) \\ 3 & x=\dfrac{\pi}{2}\end{cases}$; where [ ] denotes the greatest integer function, then -

(A) f is continuous & differentiable at $x = \pi/2$

(B) f is continuous but not differentiable at $x = \pi/2$

(C) f is neither continuous not differentiable at $x = \pi/2$

(D) none of these

13. If $f(x + y) = f(x) f(y)$ $\forall x, y$ and $f(5) = 2$, $f'(0) = 3$; and function is differentiable then $f'(5)$ is equal to-

(A) 2 (B) 4 (C) 6 (D) 8

14. If f is a real-valued differentiable function satisfying $|f(x) - f(y)| \le (x - y)^2$, $x, y \in R$ and $f(0) = 0$, then $f(1)$ equals

(A) 1 (B) 2 (C) 0 (D) –1

MORE THAN ONE OPTION CORRECT

15. If $f(x)\begin{cases}\dfrac{x.\ell n(\cos x)}{\ell n(1+x^2)} & x\neq 0\\ 0 & x=0\end{cases}$ then -

(A) f is continuous at $x = 0$

(B) f is continuous at $x = 0$ but not differentiable at $x = 0$

(C) f is differentiable at $x = 0$

(D) f is not continuous at $x = 0$

16. Which one of the following statements is not correct ?

(A) The derivative of a differentiable periodic function is a periodic function with the same period.

(B) If f(x) and g(x) both are defined on the entire number line and are aperiodic then the function F(x) =f(x).g(x) can not be periodic.

(C) Derivative of an even differentiable function is an odd function and derivative of an odd differentiable function is an even function.

(D) Every function f(x) can be represented as the sum of an even and an odd function.

17. Let $f(x) = \begin{bmatrix}(x-e)2^{-2^{\frac{1}{e-x}}}, & x\neq e\\ 0, & x=e\end{bmatrix}$, then -

(A) f is continuous and differentiable at $x = e$

(B) f is continuous but not differentiable at $x = e$

(C) f is neither continuous nor differentiable at $x = e$

(D) geometrically f has sharp corner at $x = e$

18. Let [x] be the greatest integer function $f(x) = \dfrac{\sin\frac{1}{4}\pi[x]}{[x]}$ is -

(A) not continuous at any point

(B) continuous at $\frac{3}{2}$

(C) discontinuous at 2

(D) differentiable at $\frac{4}{3}$

19. $f(x) = (\sin^{-1}x)^2.\cos(1/x)$ if $x \neq 0$, $f(0) = 0$, f(x) is :

(A) continuous no where in $-1 \le x \le 1$

(B) continuous every where in $-1 \le x \le 1$

(C) differentiable no where in $-1 \le x \le 1$

(D) differentiable everywhere $-1 < x < 1$

20. f(x) = 1 + [cosx] x in $0 < x \le \pi/2$, where [ ] denotes greatest integer function then -

(A) it is continuous in $0 < x < \pi/2$
(B) it is differentiable in $0 < x < \pi/2$
(C) its maximum value is 2
(D) it is not differentiable in $0 < x < \pi/2$

21. Let f(x) = cos x & $H(x) = \begin{cases} \text{Min } [f(t)/0 \le t \le x] & \text{for } 0 \le x \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2} - x & \text{for } \dfrac{\pi}{2} < x \le 3 \end{cases}$, then -

(A) H(x) is continuous & derivable in [0, 3]
(B) H(x) is continuous but not derivable at x = $\pi/2$
(C) H(x) is neither continuous nor derivable at x = $\pi/2$
(D) Maximum value of H(x) in [0,3] is 1

22. If f(x) = 3 $(2x+3)^{2/3}$ + 2x + 3 then -
(A) f(x) is continuous but not differentiable at x = – 3/2
(B) f(x) is differentiable at x = 0
(C) f(x) is continuous at x = 0
(D) f(x) is differentiable but not continuous at x = – 3/2

23. Given that the derivative $f'(a)$ exists. Indicate which of the following statement (s) is/are always true -

(A) $f'(a) = \operatorname*{Limit}_{h\to a} \dfrac{f(h) - f(a)}{h - a}$
(B) $f'(a) = \operatorname*{Limit}_{h\to 0} \dfrac{f(a) - f(a-h)}{h}$
(C) $f'(a) = \operatorname*{Limit}_{t\to 0} \dfrac{f(a+2t) - f(a)}{t}$
(D) $f'(a) = \operatorname*{Limit}_{t\to 0} \dfrac{f(a+2t) - f(a+t)}{2t}$

24. Let $f(x) = \begin{cases} 2x+3; & -3 \le x < -2 \\ x+1; & -2 \le x < 0 \\ x+2; & 0 \le x \le 1 \end{cases}$. At what points the function is/are not differentiable in the interval [–3, 1]

(A) –2 (B) 0 (C) 1 (D) 1/2

25. If f(x) = cos $\pi(|x| + [x])$, then f(x) is/are (where [.] denotes greatest integer function)

(A) continuous at x = $\dfrac{1}{2}$
(B) continuous at x = 0
(C) differentiable in (2, 4)
(D) differentiable in (0, 1)

26. If f(x) = |x + 1|(|x| + |x – 1|) then at what points the function is/are not differentiable at in the interval [–2, 2]
(A) –1 (B) 0 (C) 1 (D) 1/2

EXERCISE - 2 MISCELLANEOUS

Comprehension Based Questions

Comprehension – 1

$$f(x) = \begin{cases} 2+(x-1)^2 & \text{if } x<1 \\ 2 & \text{if } x \in [1,3] \\ 2-(x-3)^2 & \text{if } x>3 \end{cases}$$

$$g(x) = \begin{cases} 2+\sqrt{-x} & \text{if } x<0 \\ x+2 & \text{if } x \in [0, 4] \\ 3x-6 & \text{if } x \in (4, \infty) \end{cases}$$

$$h(x) = \begin{cases} 4+ae^x & \text{if } x<0 \\ x+2 & \text{if } x \in [0,3] \\ b^2-7b+18-\dfrac{3}{x} & \text{if } x>3 \end{cases}$$

$$k(x) = \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)(x+4)}}}, \ x > 0$$

On the basis of above information, answer the following questions

1. Which of the following is continuous at each point of its domain -

(A) $f(x)$ (B) $g(x)$ (C) $k(x)$ (D) all three f, g, k

2. Value of (a, b) for which h(x) is continuous $\forall\ x \in R$:

(A) (4, 3) (B) (–2, 3) (C) (3, 4) (D) none of these

3. Which of the following function is not differentiable at exactly two points of its domain -

(A) $f(x)$ (B) $g(x)$ (C) $k(x)$ (D) none of these

Comprehension – 2

Let 'f' be a function that is differentiable every where and that has the following properties :

(i) $f(x) > 0$ (ii) $f'(0) = -1$ (iii) $f(-x) = \dfrac{1}{f(x)}$ & $f(x + h) = f(x).f(h)$

A standard result : $\displaystyle\int \frac{f'(x)}{f(x)}dx = \ell n|f(x)| + c$

On the basis of above information, answer the following questions

4. Range of f(x) is-

(A) R (B) R – {0} (C) R^+ (D) (0, e)

5. The range of the function $\Delta = f(|x|)$ is -

(A) [0, 1] (B) [0, 1) (C) (0, 1] (D) none of these

6. The function y = f(x) is -

(A) odd (B) even (C) increasing (D) decreasing

7. If h(x) = f'(x). then h(x) is given by -

(A) –f(x) (B) $\dfrac{1}{f(x)}$ (C) f(x) (D) $e^{f(x)}$

Match the Column

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

8.

	Column - I	**Column - II**
(A)	The number of the values of x in $(0, 2\pi)$, where the function $f(x)=\frac{\tan x+\cot x}{2}-\left\|\frac{\tan x-\cot x}{2}\right\|$ is continuous but not differentiable is	(p) 2
(B)	The number of points where the function $f(x) = \min\{1, 1 + x^3, x^2 - 3x + 3\}$ is non-derivable	(q) 0
(C)	The number of points where $f(x) = (x + 4)^{1/3}$ is non-differentiable is	(r) 4
(D)	Consider $f(x)=\begin{cases}-\frac{\pi}{2}\ell n\left(\frac{x.2}{\pi}\right)+\frac{\pi}{2}, & 0<x\le\frac{\pi}{2}\\ \sin^{-1}\sin x, & \frac{\pi}{2}<x<\frac{3\pi}{2}\end{cases}$. Number of points in $\left(0,\frac{3\pi}{2}\right)$, where $f(x)$ is non-differentiable is	(s) 1

9.

	Column - I	**Column - II**
(A)	Number of points where the function $f(x)=\begin{cases}1+\left[\cos\frac{\pi x}{2}\right] & 1<x\le 2\\ 1-\{x\} & 0\le x<1\\ \|\sin \pi x\| & -1\le x<0\end{cases}$ and $f(1) = 0$ is continuous but non-differentiable where [ ] denote greatest integer and { } denote fractional part function	(p) 0
(B)	$f(x)=\begin{cases}x^2e^{1/x}, & x\ne 0\\ 0, & x=0\end{cases}$, then $f'(0^-)$ is equal to	(q) 1
(C)	The number of points at which $g(x)=\frac{1}{1+\frac{2}{f(x)}}$ is not differentiable where $f(x)=\frac{1}{1+\frac{1}{x}}$, is	(r) 2
(D)	Number of points where tangent does not exist for the curve $y = \operatorname{sgn}(x^2 - 1)$	(s) 3

INTEGER / SUBJECTIVE TYPE QUESTIONS

10. A function f is defined as follows : $f(x)=\begin{cases}1 & \text{for } -\infty<x<0\\ 1+\sin x & \text{for } 0\le x<\frac{\pi}{2}\\ 2+\left(x-\frac{\pi}{2}\right)^2 & \text{for } \frac{\pi}{2}\le x<+\infty\end{cases}$

Discuss the continuity & differentiability at $x = 0$ & $x = \pi/2$.

11. Examine the continuity and differentiability of $f(x) = |x| + |x-1| + |x-2|$ $x \in R$. Also draw the graph of f(x).

12. Discuss the continuity & differentiability of the function $f(x) = \sin x + \sin|x|$, $x \in R$. Draw a rough sketch of the graph of f(x).

13. Examine the origin for continuity & derrivability in case of the function f defined by $f(x) = x\tan^{-1}(1/x)$, $x \neq 0$ and $f(0) = 0$.

14. Find the values of 2a+b for which the function $f(x)=\begin{cases} ax+b, & x \le -1 \\ ax^3+x+2b, & x > -1 \end{cases}$ be differentiable for all $x \in R$.

15. If $f(x) = \begin{bmatrix} ax^2 - b & \text{if } |x| < 1 \\ -\dfrac{1}{|x|} & \text{if } |x| \ge 1 \end{bmatrix}$ is derivable at $x = 1$. Find the values of a + b

16. Discuss the continuity & the derivability in [0, 2] of $f(x) = \begin{bmatrix} |2x-3|[x] & \text{for } x \ge 1 \\ \sin\dfrac{\pi x}{2} & \text{for } x < 1 \end{bmatrix}$

where [ ] denote greatest integer function.

17. Find the set of values of m for which $f(x) = \begin{bmatrix} x^m \sin\left(\dfrac{1}{x}\right) & x > 0 \\ 0 & x = 0 \end{bmatrix}$

(a) is discontinuous at x = 0

(b) is continuous but not derivable at x = 0

18. Let f(0) = 0 and f'(0) = 1. For a positive integer k, show that

$$\lim_{x\to 0}\frac{1}{x}\left(f(x)+f\left(\frac{x}{2}\right)+\ldots\ldots f\left(\frac{x}{k}\right)\right)=1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots\ldots+\frac{1}{k}$$

19. If $f(x) = -1+|x-1|, -1 \le x \le 3$; $g(x) = 2-|x+1|, -2 \le x \le 2$, then calculate (fog) (x) & (gof)(x). Draw their graph. Discuss the continuity of (fog) (x) at x = –1 & the differentiability of (gof)(x) at x = 1.

NCERT CORNER

Short Answer

1. Show that f(x) = |x| is not differentiable at x = 0.

2. Discuss the differentiability of $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0, & x=0 \end{cases}$ at x = 0.

3. If f(2) = 4 and f′(2) = 1, then find $\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2}$.

4. Discuss the differentiability of f(x) = x|x| at x = 0.

5. Show that the function $f(x) = \begin{cases} x\sin\frac{1}{x}, & \text{where } x \neq 0 \\ 0, & \text{when } x=0 \end{cases}$ is continuous but not differentiable at x = 0.

6. If f(x) = x^2+2x+7 find f′(3).

7. Find f′(2) and f′(5) when f(x) = x^2+7x+4.

Long Answer

8. For the function f given by f(x) = x^2-6x+8, prove that f′(5) – 3 f′ (2) = f′ (8).

9. Discuss the continuity and differentiability of $f(x)=\begin{cases} 1-x, & x<1 \\ (1-x)(2-x), & 1\le x\le 2 \\ 3-x, & x>2 \end{cases}$.

10. Discuss the differentiability of f(x) = |x – 1| + |x –2|.

11. Discuss the differentiability of f(x) = $|\log_e x|$ for x > 0.

ANSWER KEY

BEGINNER'S BOX-1

1. Continuous but not differentiable at x = 1

2. Continuous & differentiable at x = 1

3. Continuous everywhere but not differentiable at x = 2 only

4. (A) **5.** (D) **6.** (B) **7.** (A) **8.** (B) **9.** (B) **10.** (D) **11.** (A)

12. (D) **13.** (A) **14.** (B) **15.** (D)

BEGINNER'S BOX-2

1. 5 **2.** 3 **3.** **(a)** 1 & 2

(b) Not continuous at x = 1 & 2 and not differentiable at x = 1 & 2.

4. $a = \frac{1}{6},\ b = \frac{\pi}{4} - \frac{13}{6}$ **5.** (D) **6.** (C) **7.** (AC) **8.** (ABCD) **9.** (C)

10. (C) **11.** (A) **12.** (A) **13.** (B) **14.** (B) **16.** (B) **17.** (C)

EXERCISE-1

(SINGLE CORRECT & MORE THEN ONE OPTION CORRECT)

Que.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Ans.	D	A	A	D	C	B	A	A	C	B
Que.	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Ans.	D	A	C	C	AC	BD	BD	BCD	BD	AB
Que.	**21**	**22**	**23**	**24**	**25**	**26**				
Ans.	AD	ABC	AB	AB	AD	ABC				

EXERCISE-2 (MISCELLANEOUS)

- **Comprehension Based Questions**

  Comprehension – 1 **1.** D **2.** B **3.** B

  Comprehension – 2 **4.** C **5.** C **6.** D **7.** A

- **Match the Column** **8.** (A) → (r), (B) → (p), (C) → (s), (D) → (q)

  9. (A) → (q); (B) → (p); (C) → (s); (D) → (r)

- **Integer/Subjective Type Questions**

10. continuous but not differentiable at x = 0; differentiable & continuous at x = π/2

11. continuous ∀ x ∈ R, not differentiable at x = 0, 1 & 2

12. f(x) is continuous but not derivable at x = 0

13. continuous but not differentiable at x = 0

14. 0, where a = –1/2 b = 1

15. 2, where a = 1/2, b = 3/2

16. f is discontinuous at x =2 , f is not differentiable at x = 1, 3/2, 2

17. (a) $m \in (-\infty, 0]$ (b) $m \in (0, 1]$

19. $(fog)(x) = x+1$ for $-2 \le x \le -1$, $-(x+1)$ for $-1 < x \le 0$ & $x-1$ for $0 < x \le 2$. $(fog)(x)$ is continuous at $x = -1$, $(gof)(x) = x+1$ for $-1 \le x \le 1$ & $3-x$ for $1 < x \le 3$. $(gof)(x)$ is not differentiable at $x = 1$

EXERCISE-3(A)

Que.	1	2	3	4	5	6	7
Ans.	A	D	D	C	C	B	D

EXERCISE-3(B)

1. C **2.** B,C **3.** (A,B,C,D)

4. B **5.** 3 **6.** D **7.** A,D **8.** (BC) **9.** (D)

NCERT CORNER

2. $f(x)$ is not differentiable at $x = 0$.

3. (2)

4. $f(x)$ is differentiable at $x = 0$.

6. (8)

7. $f'(2) = 11$ and $f'(5) = 17$

9. $f(x)$ is not continuous at $x = 2$. So, it is not differentiable at $x = 2$.

10. $f(x)$ is not differentiable at $x = 2$.

11. $f(x)$ is not differentiable at $x = 1$.

* * * * *

METHODS OF DIFFERENTIATION

1.0 DEFINITION

We had defined the derivative of a real function as follows.

Suppose f is a real function and c is a point in its domain. The derivative of f at c can be given by $\lim\limits_{h\to 0}\dfrac{f(c+h)-f(c)}{h}$ provided this limit exists. Derivative of f at x = c denoted by f'(c) or $\dfrac{d}{dx}(f(x))_{x=c}$. The derivative of function defined by $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ wherever the limit exists. The derivative of f is denoted by f '(x) or $\dfrac{d}{dx}(f(x))$ or if y = f (x) by $\dfrac{dy}{dx}$ or y' or y_1. The process of finding derivative of a function is called differentiation.

2.0 DERIVATIVE OF f(x) FROM THE FIRST PRINCIPLE

Obtaining the derivative using the definition $\lim\limits_{\delta x\to 0}\dfrac{\delta y}{\delta x}=\lim\limits_{\delta x\to 0}\dfrac{f(x+\delta x)-f(x)}{\delta x}=f'(x)=\dfrac{dy}{dx}$ is called calculating derivative using first principle or ab initio or delta method.

Illustrations

Illustration 1. Differentiate each of following functions by first principle :

(i) $f(x) = \tan x$ (ii) $f(x) = e^{\sin x}$

Solution (i) $f'(x)=\lim\limits_{h\to 0}\dfrac{\tan(x+h)-\tan x}{h}=\lim\limits_{h\to 0}\dfrac{\tan(x+h-x)[1+\tan x\tan(x+h)]}{h}$

$$=\lim_{h\to 0}\frac{\tanh}{h}\,.\,(1+\tan^2x)=\sec^2x.$$ **Ans.**

(ii) $f'(x)=\lim\limits_{h\to 0}\dfrac{e^{\sin(x+h)}-e^{\sin x}}{h}=\lim\limits_{h\to 0}e^{\sin x}\dfrac{[e^{\sin(x+h)-\sin x}-1]}{\sin(x+h)-\sin x}\left(\dfrac{\sin(x+h)-\sin x}{h}\right)$

$$=e^{\sin x}\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=e^{\sin x}\cos x$$ **Ans.**

3.0 DERIVATIVE OF STANDARD FUNCTIONS

	f(x)	f'(x)		f(x)	f'(x)
(i)	x^n	nx^{n-1}	(ii)	e^x	e^x
(iii)	a^x	$a^x\ell na,\ a>0$	(iv)	ℓnx	1/x
(v)	$\log_a x$	$(1/x)\log_a e,\ a>0,\ a\neq 1$	(vi)	sinx	cosx
(vii)	cosx	– sinx	(viii)	tanx	$\sec^2x$
(ix)	secx	secx tanx	(x)	cosecx	– cosecx . cotx
(xi)	cotx	$-\text{cosec}^2x$	(xii)	constant	0
(xiii)	$\sin^{-1}x$	$\dfrac{1}{\sqrt{1-x^2}},\ -1<x<1$	(xiv)	$\cos^{-1}x$	$\dfrac{-1}{\sqrt{1-x^2}},\ -1<x<1$
(xv)	$\tan^{-1}x$	$\dfrac{1}{1+x^2},\ x\in R$	(xvi)	$\sec^{-1}x$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert>1$
(xvii)	$\text{cosec}^{-1}x$	$\dfrac{-1}{\lvert x\rvert\sqrt{x^2-1}},\ \lvert x\rvert>1$	(xviii)	$\cot^{-1}x$	$\dfrac{-1}{1+x^2},\ x\in R$

4.0 FUNDAMENTAL THEOREMS

If f and g are derivable functions of x, then,

(a) $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$ (b) $\frac{d}{dx}(cf) = c\frac{df}{dx}$, where c is any constant

(c) "PRODUCT RULE" :

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\frac{d}{dx}(fgh)(x) = \frac{d}{d(x)}f(x) \times g(x).h(x) + f(x)\frac{d}{dx}g(x).h(x) + f(x).g(x).\frac{d}{dx}h(x)$$

(d) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\left(\frac{df}{dx}\right) - f\left(\frac{dg}{dx}\right)}{g^2}$ where $g \neq 0$ known as "QUOTIENT RULE"

(e) If y = f(u) & u = g (x) then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ known as "CHAIN RULE"

Note – In general if y = f(u) then $\frac{dy}{dx} = f'(u).\frac{du}{dx}$.

Illustrations

Illustration 2. If $y = e^x \tan x + x\log_e x$, find $\frac{dy}{dx}$.

Solution $y = e^x.\tan x + x \cdot \log_e x$

On differentiating we get,

$$\frac{dy}{dx} = e^x \cdot \tan x + e^x \cdot \sec^2 x + 1 \cdot \log x + x \cdot \frac{1}{x}$$

Hence, $\frac{dy}{dx} = e^x(\tan x + \sec^2 x) + (\log x + 1)$ **Ans.**

Illustration 3. If $y = \frac{\log x}{x} + e^x \sin 2x + \log_5 x$, find $\frac{dy}{dx}$.

Solution On differentiating we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log x}{x}\right) + \frac{d}{dx}(e^x \sin 2x) + \frac{d}{dx}(\log_5 x)$$

$$= \frac{\frac{1}{x}\cdot x - \log x\,.\,1}{x^2} + e^x \sin 2x + 2e^x\,.\cos 2x + \frac{1}{x\log_e 5}$$

Hence, $\frac{dy}{dx} = \left(\frac{1 - \log x}{x^2}\right) + e^x(\sin 2x + 2\cos 2x) + \frac{1}{x\log_e 5}$ **Ans.**

Illustration 4. If $x = \exp\left(\tan^{-1}\left(\frac{y - x^2}{x^2}\right)\right)$, then $\frac{dy}{dx}$ equals -

(A) $x[1 + \tan(\log x) + \sec^2 x]$ (B) $2x[1 + \tan(\log x)] + \sec^2 x$

(C) $2x[1 + \tan(\log x)] + \sec x$ (D) $2x + x[1 + \tan(\log x)]^2$

Solution Taking log on both sides, we get

$$\log x = \tan^{-1}\left(\frac{y - x^2}{x^2}\right) \Rightarrow \tan(\log x) = (y - x^2)/x^2$$

$\Rightarrow$ $y = x^2 + x^2 \tan(\log x)$

On differentiating, we get

$\therefore \quad \frac{dy}{dx} = 2x + 2x\tan(\log x) + x\sec^2(\log x) \Rightarrow 2x[1 + \tan(\log x)] + x\sec^2(\log x)$

$= \quad 2x + x[1 + \tan(\log x)]^2$ **Ans. (D)**

Illustration 5. If $y = \log_e(\tan^{-1}\sqrt{1+x^2})$, find $\frac{dy}{dx}$.

Solution $y = \log_e(\tan^{-1}\sqrt{1+x^2})$

On differentiating we get,

$$= \frac{1}{\tan^{-1}\sqrt{1+x^2}}\cdot\frac{1}{1+(\sqrt{1+x^2})^2}\cdot\frac{1}{2\sqrt{1+x^2}}.2x$$

$$= \frac{x}{\left(\tan^{-1}\sqrt{1+x^2}\right)\left\{1+\left(\sqrt{1+x^2}\right)^2\right\}\sqrt{1+x^2}} = \frac{x}{\left(\tan^{-1}\sqrt{1+x^2}\right)(2+x^2)\sqrt{1+x^2}}$$ **Ans.**

5.0 LOGARITHMIC DIFFERENTIATION

To find the derivative of a function

(A) which is the product or quotient of a number of functions or

(B) of the form $[f(x)]^{g(x)}$ where f & g are both derivable.

It is convenient to take the logarithm of the function first & then differentiate.

Illustrations

Illustration 6. If $y = (\sin x)^{\ell n\, x}$, find $\frac{dy}{dx}$

Solution $\ell n\, y = \ell n\, x . \ell n(\sin x)$

On differentiating we get,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ell n(\sin x) + \ell n\, x.\frac{\cos x}{\sin x} \quad \Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\ell n\, x}\left[\frac{\ell n(\sin x)}{x} + \cot x\ \ell n\, x\right]$$

Ans.

Illustration 7. If $y = \frac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}$ find $\frac{dy}{dx}$

Solution Taking log of both sides

$$\log y = \log(x)^{1/2} + \log(1-2x)^{2/3} - \log(2-3x)^{3/4} - \log(3-4x)^{4/5}$$

$$\ell n\, y = \frac{1}{2}\ell n\, x + \frac{2}{3}\ell n(1-2x) - \frac{3}{4}\ell n(2-3x) - \frac{4}{5}\ell n(3-4x)$$

On differentiating we get,

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}$$

$$\frac{dy}{dx} = y\left(\frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}\right)$$ **Ans.**

6.0 PARAMETRIC DIFFERENTIATION

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

Illustrations

Illustration 8. If y = a cos t and x = a(t – sint) find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$

Solution $\frac{dy}{dx} = \frac{-a\sin t}{a(1-\cos t)} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -1$ **Ans.**

Illustration 9. Prove that the function represented parametrically by the equations. $x = \frac{1+t}{t^3}$; $y = \frac{3}{2t^2} + \frac{2}{t}$

satisfies the relationship : $x(y')^3 = 1 + y'$ (where $y' = \frac{dy}{dx}$)

Solution Here $x = \frac{1+t}{t^3} = \frac{1}{t^3} + \frac{1}{t^2}$

Differentiating w.r. to t

$$\frac{dx}{dt} = -\frac{3}{t^4} - \frac{2}{t^3}$$

$$y = \frac{3}{2t^2} + \frac{2}{t}$$

Differentiating w.r. to t

$$\frac{dy}{dt} = -\frac{3}{t^3} - \frac{2}{t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = t = y'$$

Since $x = \frac{1+t}{t^3} \Rightarrow x = \frac{1+y'}{(y')^3}$ or $x(y')^3 = 1 + y'$ **Ans.**

BEGINNER'S BOX–1

1. Find $\frac{dy}{dx}$ if -

y = (x + 1) (x + 2) (x + 3)

2. If $f(x) = \left(\frac{x^a}{x^b}\right)^{a+b} \cdot \left(\frac{x^b}{x^c}\right)^{b+c} \cdot \left(\frac{x^c}{x^a}\right)^{c+a}$, then f′(x) is equal to

(A) 1 (B) 0 (C) x^{a+b+c} (D) None of these

3. If $f(x) = \left(\frac{\sin^m x}{\sin^n x}\right)^{m+n} \cdot \left(\frac{\sin^n x}{\sin^p x}\right)^{n+p} \cdot \left(\frac{\sin^p x}{\sin^m x}\right)^{p+m}$, then f′(x) is equal to

(A) 0 (B) 1 (C) $\cos^{m+n+p}x$ (D) None of these

4. If $y = (1 + x^{1/4})(1 + x^{1/2})(1 - x^{1/4})$, then $\frac{dy}{dx} =$

(A) 1 (B) –1 (C) x (D) $\sqrt{x}$

5. If f(x) = | cos x|, then $f'\left(\frac{3\pi}{4}\right)$ is equal to

(A) $-\frac{1}{\sqrt{2}}$ (B) $\frac{1}{\sqrt{2}}$ (C) 1 (D) None of these

6. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

(A) 1 (B) –1 (C) 0 (D) None of these

7. If $y = e^{5x}\tan(x^2 + 2)$, then find $\frac{dy}{dx}$

8. If $y = x^3 e^{x^2}\sin 2x$ then find $\frac{dy}{dx}$

9. If $f(x) = \log_x(\ell n\, x)$, then $f'(x)$ at $x = e$ is

(A) e (B) $\frac{1}{e}$ (C) $\frac{2}{e}$ (D) 0

10. If $f(x) = \ell n\,|2x|$, $x \neq 0$, then $f'(x)$ is equal to

(A) $\frac{1}{x}$ (B) $-\frac{1}{x}$ (C) $\frac{1}{|x|}$ (D) None of these

11. If $f(x) = \cos x \cos 2x \cos 4x \cos 8x$, then $f'\left(\frac{\pi}{4}\right)$ is

(A) –1 (B) 2 (C) $\sqrt{2}$ (D) None of these

12. If $y = f\left(\frac{3x+4}{5x+6}\right)$ & $f'(x) = \tan x^2$ then $\frac{dy}{dx} =$

(A) $\tan x^3$

(B) $-2\tan\left[\frac{3x+4}{5x+6}\right]^2 \frac{1}{(5x+6)^2}$

(C) $f\left(\frac{3\tan x^2+4}{5\tan x^2+6}\right)\tan x^2$

(D) none

13. If $y = x^x$ then find $\frac{dy}{dx}$

14. If $y = (\sin x)^{\cos x}$ then find $\frac{dy}{dx}$

15. If $y = e^x . e^{x^2} . e^{x^3} . e^{x^4}$ then find $\frac{dy}{dx}$

16. If $y = \frac{(x+1)^3 \sqrt[4]{x-2}}{\sqrt[5]{(x-3)^2}}$ then find $\frac{dy}{dx}$

17. If $y = x^{x^x}$ then find $\frac{dy}{dx}$

18. Find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ if $y = \cos^4 t$ & $x = \sin^4 t$.

19. If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, $\frac{dy}{dx} =$

(A) $\cos\theta$ (B) $\tan\theta$ (C) $\sec\theta$ (D) $\operatorname{cosec}\theta$

20. If $x = a\left(\cos t + \ell n \tan\frac{t}{2}\right)$ and $y = a\sin t$, then $\frac{dy}{dx}$ is equal to

(A) $-\tan t$ (B) $\tan t$ (C) $\cot t$ (D) None of these

7.0 DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION

If f(x) is to be differentiated with respect to g(x).

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

8.0 DIFFERENTIATION OF IMPLICIT FUNCTIONS $\phi(x, y) = 0$

(A) To find dy/dx of implicit functions, we differentiate each term w.r.t. x regarding y as a function of x & then collect terms with dy/dx together on one side.

(B) In the case of implicit functions, generally, both x & y are present in answers of dy/dx.

Illustrations

Illustration 10. If $x^y + y^x = 2$, then find $\frac{dy}{dx}$.

Solution Let $u = x^y$ and $v = y^x$

$u + v = 2 \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0$

Now $u = x^y$ and $v = y^x$

$\Rightarrow \ell n\, u = y\, \ell n x$ and $\ell n\, v = x\, \ell n\, y$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \ell n x\frac{dy}{dx}$ and $\frac{1}{v}\frac{dv}{dx} = \ell n\, y + \frac{x}{y}\frac{dy}{dx}$

$\Rightarrow \frac{du}{dx} = x^y\left(\frac{y}{x} + \ell n x\frac{dy}{dx}\right)$ and $\frac{dv}{dx} = y^x\left(\ell n y + \frac{x}{y}\frac{dy}{dx}\right)$

$\Rightarrow x^y\left(\frac{y}{x} + \ell n x\frac{dy}{dx}\right) + y^x\left(\ell n y + \frac{x}{y}\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = -\frac{\left(y^x \ell n y + x^y.\frac{y}{x}\right)}{\left(x^y \ell n x + y^x.\frac{x}{y}\right)}$ **Ans.**

Illustration 11. If $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \cos x.....}}}$, prove that $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}$.

Solution Given function is $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1+y}} = \frac{(1+y)\sin x}{1 + y + \cos x}$

or $y + y^2 + y\cos x = (1 + y)\sin x$

Differentiate both sides with respect to x,

$$\frac{dy}{dx}+2y\frac{dy}{dx}+\frac{dy}{dx}\cos x-y\sin x = (1+y)\cos x+\frac{dy}{dx}\sin x$$

$$\frac{dy}{dx}(1+2y+\cos x-\sin x)=(1+y)\cos x+y\sin x$$

or $$\frac{dy}{dx}=\frac{(1+y)\cos x+y\sin x}{1+2y+\cos x-\sin x}$$ **Ans.**

9.0 DERIVATIVE OF A FUNCTION AND ITS INVERSE FUNCTION

If g is inverse of f, then

(A) $g\{f(x)\}=x$ (B) $f\{g(x)\}=x$

$g'\{f(x)\}f'(x)=1$ $f'\{g(x)\}g'(x)=1$

Illustrations

Illustration 12. Differentiate $\log_e(\tan x)$ with respect to $\sin^{-1}(e^x)$.

Solution Let by $=\log_e(\tan x)$ & $z=\sin^{-1}(e^x)$.

$$\frac{dy}{dz}=\frac{d(\log_e\tan x)}{d(\sin^{-1}(e^x))}=\frac{\frac{d}{dx}(\log_e\tan x)}{\frac{d}{dx}\sin^{-1}(e^x)}=\frac{\cot x.\sec^2 x}{e^x.1/\sqrt{1-e^{2x}}}=\frac{e^{-x}\sqrt{1-e^{2x}}}{\sin x\cos x}$$ **Ans.**

Illustration 13. If g is inverse of f and $f'(x)=\frac{1}{1+x^n}$, then $g'(x)$ equals :-

(A) $1+x^n$ (B) $1+[f(x)]^n$ (C) $1+[g(x)]^n$ (D) none of these

Solution Since g is the inverse of f. Therefore

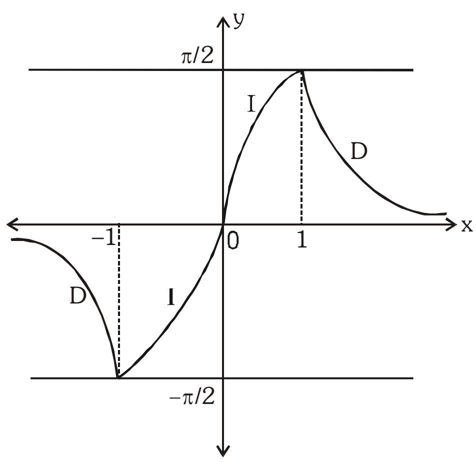
$f(g(x))=x$ for all x

$\Rightarrow \frac{d}{dx}f(g(x))=1$ for all x

$\Rightarrow f'(g(x))\,g'(x)=1 \Rightarrow g'(x)=\frac{1}{f'(g(x))}=1+(g(x))^n$ **Ans. (C)**

10.0 ANALYSIS AND GRAPHS OF SOME INVERSE TRIGONOMETRIC FUNCTIONS

(A) $$y=f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\begin{cases}2\tan^{-1}x & |x|\le 1\\ \pi-2\tan^{-1}x & x>1\\ -(\pi+2\tan^{-1}x) & x<-1\end{cases}$$



Important points –

(i) Domain is $x\in R$ & range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(ii) f is continuous for all x but not differentiable at $x = 1, -1$

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(iii) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for} \quad |x|<1 \\ \text{non existent} & \text{for} \quad |x|=1 \\ \frac{-2}{1+x^2} & \text{for} \quad |x|>1 \end{cases}$

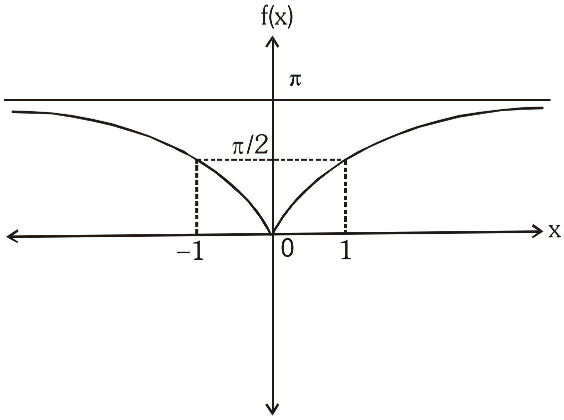
(iv) Increasing in $(-1, 1)$ & Decreasing in $(-\infty, -1) \cup (1, \infty)$

(B) Consider $y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if} \quad x \ge 0 \\ -2\tan^{-1}x & \text{if} \quad x < 0 \end{cases}$

Important points –

(i) Domain is $x \in R$ & range is $[0, \pi)$

(ii) Continuous for all x but not differentiable at $x = 0$



(iii) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for} \quad x>0 \\ \text{non existent} & \text{for} \quad x=0 \\ -\frac{2}{1+x^2} & \text{for} \quad x<0 \end{cases}$

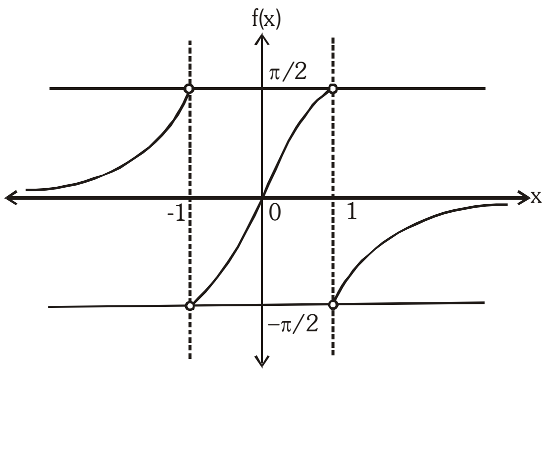
(iv) Increasing in $(0, \infty)$ & Decreasing in $(-\infty, 0)$

(C) $y = f(x) = \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & |x|<1 \\ \pi + 2\tan^{-1}x & x<-1 \\ -(\pi - 2\tan^{-1}x) & x>1 \end{cases}$

Important points –

(i) Domain is $R - \{1, -1\}$ & range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(ii) It is neither continuous nor differentiable at $x = 1, -1$



(iii) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & |x| \ne 1 \\ \text{non existent} & |x| = 1 \end{cases}$

(iv) Increasing $\forall$ x in its domain

(v) It is bounded for all x

(D) $y = f(x) = \sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x) & \text{if} \quad -1 \le x < -\frac{1}{2} \\ 3\sin^{-1}x & \text{if} \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if} \quad \frac{1}{2} < x \le 1 \end{cases}$

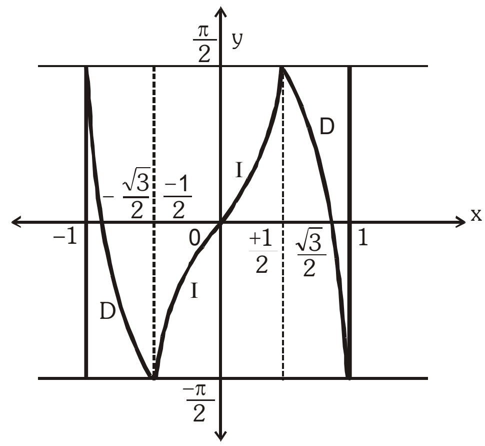
Important points –

(i) Domain is $x \in [-1,\ 1]$ & range is $\left[-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right]$

(ii) Continuous everywhere in its domain

(iii) Not derivable at $x = -\dfrac{1}{2},\ \dfrac{1}{2}$

(iv) $$\frac{dy}{dx} = \begin{cases} \dfrac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right) \\ -\dfrac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in \left(-1, -\dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right) \end{cases}$$



(v) Increasing in $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ and Decreasing in $\left[-1, -\dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right]$

(E) $$y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if} \quad -1 \le x < -\dfrac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if} \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1}x & \text{if} \quad \dfrac{1}{2} < x \le 1 \end{cases}$$

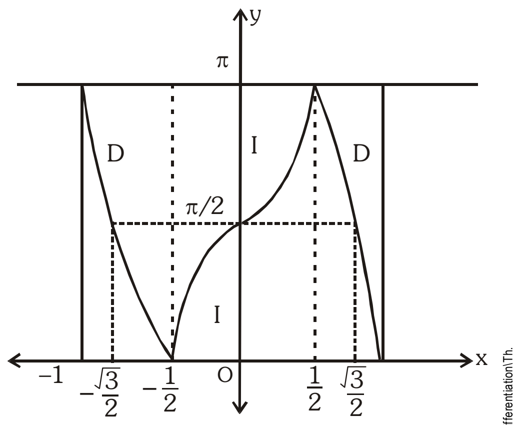
Important points –

(i) Domain is $x \in [-1,\ 1]$ & range is $[0,\ \pi]$

(ii) Continuous everywhere in its domain

(iii) Not derivable at $x = -\dfrac{1}{2},\ \dfrac{1}{2}$

(iv) $$\frac{dy}{dx} = \begin{cases} \dfrac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right) \\ -\dfrac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in \left(-1, -\dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right) \end{cases}$$



(v) Increasing in $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ & Decreasing in $\left[-1, -\dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right]$

GENERAL NOTE

Concavity is decided by the sign of 2^{nd} derivative as :

$\dfrac{d^2y}{dx^2} > 0 \Rightarrow$ Concave upwards ; $\dfrac{d^2y}{dx^2} < 0 \Rightarrow$ Concave downwards

Illustrations

Illustration 14. $\frac{d}{dx}\left\{\sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)\right\} =$

(A) $-\frac{1}{2}$ (B) 0 (C) $\frac{1}{2}$ (D) -1

Solution Let $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$. Put $x = \cos 2\theta$ $\theta \in \left(0, \frac{\pi}{2}\right]$

$\therefore$ $y = \sin^2 \cot^{-1}\left(\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\right) = \sin^2 \cot^{-1}(\cot\theta)$

$\therefore$ $y = \sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1-x}{2} = \frac{1}{2}-\frac{x}{2}$

$\therefore$ $\frac{dy}{dx} = -\frac{1}{2}$. **Ans (A)**

Illustration 15. If $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ then find

(i) $f'(2)$ (ii) $f'\left(\frac{1}{2}\right)$ (iii) $f'(1)$

Solution $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\Rightarrow$ $y = \sin^{-1}(\sin 2\theta)$

$$y = \begin{cases} \pi - 2\theta & \frac{\pi}{2} < 2\theta < \pi \\ 2\theta & \frac{-\pi}{2} \le 2\theta \le \frac{\pi}{2} \\ -(\pi + 2\theta) & -\pi < 2\theta < -\frac{\pi}{2} \end{cases} \Rightarrow f(x) = \begin{cases} \pi - 2\tan^{-1}x & x > 1 \\ 2\tan^{-1}x & -1 \le x \le 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -\frac{2}{1+x^2} & x > 1 \\ \frac{2}{1+x^2} & -1 < x < 1 \\ \frac{-2}{1+x^2} & x < -1 \end{cases}$$

(i) $f'(2) = -\frac{2}{5}$ (ii) $f'\left(\frac{1}{2}\right) = \frac{8}{5}$ (iii) $f'(1^+) = -1$ and $f'(1^-) = +1$ $\Rightarrow f'(1)$ does not exist

Ans.

BEGINNER'S BOX–2

1. Differentiate $x^{\ell nx}$ with respect to ℓnx.

2. Differentiate $(ln\,x)^{\tan x}$ with respect to $\sin(m\cos^{-1}x)$.

3. The derivative of f(tan x) w.r.t. g(sec x) at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'(\sqrt{2}) = 4$, is

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 1 (D) None of these

4. Find $\frac{dy}{dx}$ if $3x^2 - 2y^2 = 1$ for (1, 1) and (1, –1)

5. Find $\frac{dy}{dx}$ if $(\cos x)^y = (\sin y)^x$

6. Find $\frac{dy}{dx}$ if y and x are related by the equation $x^2 \sin y = y^2 \sin x$.

7. Find $\frac{dy}{dx}$, if $x + y = \sin(x - y)$

8. If $x^2 + xe^y + y = 0$, find y', also find the value of y' at point (0,0).

9. The expression of $\frac{dy}{dx}$ of the function $y = a^{x^{a^{x^{\cdots\infty}}}}$ is

(A) $\frac{y^2}{x(1-y\ell n\,x)}$ (B) $\frac{y^2\ell n\,y}{x(1-y\ell n\,x)}$ (C) $\frac{y^2\ell n\,y}{x(1-y\ell n\,x\ \ell n\,y)}$ (D) $\frac{y^2\ell n\,y}{x(1+y\ell n\,x\,\ell n\,y)}$

10. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + ...\text{to } \infty}}}$, then the value of $\frac{dy}{dx} =$

(A) $\sqrt{\frac{\sin x}{y+1}}$ (B) $\frac{\sin x}{y+1}$ (C) $\frac{\cos x}{2y+1}$ (D) $\frac{\cos x}{2y-1}$

11. Let $f(x) = x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{2x + ...\infty}}}$

Compute the value of f(100). f'(100).

12. If g is the inverse of f and $f'(x) = \frac{1}{1+x^3}$, then g'(x) is equal to

(A) $1 + [g(x)]^3$ (B) $\frac{1}{1+[g(x)]^3}$ (C) $[g(x)]^3$ (D) None of these

13. If g is inverse of f and $f(x) = 2x + \sin x$; then g'(x) equals:

(A) $-\frac{3}{x^2} + \frac{1}{\sqrt{1-x^2}}$ (B) $2 + \sin^{-1}x$ (C) $2 + \cos g(x)$ (D) $\frac{1}{2+\cos(g(x))}$

14. If $y = \tan^{-1}\frac{x-\sqrt{1-x^2}}{x+\sqrt{1-x^2}}$, then $\frac{dy}{dx}$ is equal to

(A) $\frac{-1}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{-x}{\sqrt{1-x^2}}$ (D) None of these

15. The differential coefficient of $\tan^{-1}\frac{2x\sqrt{1-x^2}}{1-2x^2}$ w.r.t. $\sec^{-1}\frac{1}{2x^2-1}$ at $x = \frac{1}{2}$ is equal to

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) -1 (D) None of these

16. The derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t. $\sqrt{1-x^2}$ at $x=\frac{1}{2}$ is :

(A) 4 (B) 1/4 (C) 1 (D) none

17. If $y=\sin^{-1}\frac{2x}{1+x^2}$ $then \left.\frac{dy}{dx}\right]_{x=-2}$ is :

(A) $\frac{2}{5}$ (B) $\frac{2}{\sqrt{5}}$ (C) $-\frac{2}{5}$ (D) none

18. If $y = \cos^{-1}(4x^3 - 3x)$

Then find (A) $f'\left(-\frac{\sqrt{3}}{2}\right)$, (B) $f'(0)$, (C) $f'\left(\frac{\sqrt{3}}{2}\right)$

19. Find the derivative of $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ when $-\infty < x < 0,$ using the substitution $x = \tan\theta$.

20. If $y=\sin^{-1}\frac{x^2-1}{x^2+1}+\sec^{-1}\frac{x^2+1}{x^2-1}, |x|>1$ then $\frac{dy}{dx}$ is equal to :

(A) $\frac{x}{x^4-1}$ (B) $\frac{x^2}{x^4-1}$ (C) 0 (D) 1

11.0 HIGHER ORDER DERIVATIVES

Let a function $y = f(x)$ be defined on an interval (a, b). If $f(x)$ is differentiable function, then its derivative $f'(x)$ [or (dy/dx) or y'] is called the first derivative of y w.r.t. x. If $f'(x)$ is again differentiable function on (a, b), then its derivative $f''(x)$ [or d^2y/dx^2 or y''] is called second derivative of y w.r.t. x. Similarly, the 3rd order derivative of y w.r.to x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ and denoted by $f'''(x)$ or y''' and so on.

Note – If $x = f(\theta)$ and $y = g(\theta)$ where 'θ' is a parameter then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ & $\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{d\theta}$

In general $\frac{d^ny}{dx^n} = \frac{d}{d\theta}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)\Big/\frac{dx}{d\theta}$

Illustrations

Illustration 16. If $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all $x \in R$. Then find f(x) independent of f'(1), f''(2) and f'''(3).

Solution Here, $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$

put $f'(1) = a,\ f''(2) = b,\ f'''(3) = c$ ....(i)

$\therefore$ $f(x) = x^3 + ax^2 + bx + c$

$\Rightarrow$ $f'(x) = 3x^2 + 2ax + b$ or $f'(1) = 3 + 2a + b$ ....(ii)

$\Rightarrow$ $f''(x) = 6x + 2a$ or $f''(2) = 12 + 2a$ ....(iii)

$\Rightarrow$ $f'''(x) = 6$ or $f'''(3) = 6$ ....(iv)

from (i) and (iv), c = 6

from (i), (ii) and (iii) we have, a = –5, b = 2

$\therefore$ $f(x) = x^3 - 5x^2 + 2x + 6$ **Ans.**

Illustration 17. If x = a (t + sin t) and y = a(1 – cos t), find $\dfrac{d^2y}{dx^2}$.

Solution Here x = a (t + sin t) and y = a (1–cos t)
Differentiating both sides w.r.t. t, we get :

$$\frac{dx}{dt} = a(1 + \cos t) \quad \text{and} \quad \frac{dy}{dt} = a(\sin t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin\frac{t}{2}.\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\left(\frac{t}{2}\right)$$

Again differentiating both sides, we get,

$$\frac{d^2y}{dx^2} = \sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{2}\cdot\frac{dt}{dx} = \frac{1}{2}\sec^2(t/2)\cdot\frac{1}{a(1+\cos t)} = \frac{1}{2a}\cdot\frac{\sec^2\left(\frac{t}{2}\right)}{2\left(\cos^2\frac{t}{2}\right)}$$

Hence, $\dfrac{d^2y}{dx^2} = \dfrac{1}{4a}\cdot\sec^4\left(\dfrac{t}{2}\right)$ **Ans.**

Illustration 18. y = f(x) and x = g(y) are inverse functions of each other then express g'(y) and g''(y) in terms of derivative of f(x).

Solution $\dfrac{dy}{dx} = f'(x)$ and $\dfrac{dx}{dy} = g'(y)$

$\Rightarrow$ $g'(y) = \dfrac{1}{f'(x)}$ ...(i)

Again differentiating w.r.t. to y

$$g''(y) = \frac{d}{dy}\left(\frac{1}{f'(x)}\right) = \frac{d}{dx}\left(\frac{1}{f'(x)}\right).\frac{dx}{dy} = -\frac{f''(x)}{(f'(x))^2}.\left(\frac{1}{f'(x)}\right)$$

$\Rightarrow$ $g''(y) = -\dfrac{f''(x)}{(f'(x))^3}$ ....(ii)

Which can also be remembered as $\boldsymbol{\dfrac{d^2x}{dy^2} = -\dfrac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}}$ **Ans.**

BEGINNER'S BOX-3

1. If $y = xe^{x^2}$ then find y".

2. Find y" at $x = \pi/4$, if $y = x \tan x$.

3. If $x^2y + y^3 = 2$ then the value of $\frac{d^2y}{dx^2}$ at the point (1, 1) is :

(A) $-\frac{3}{4}$ (B) $-\frac{3}{8}$ (C) $-\frac{5}{12}$ (D) none

4. If $y = x + e^x$ then $\frac{d^2x}{dy^2}$ is :

(A) e^x (B) $-\frac{e^x}{(1+e^x)^3}$ (C) $-\frac{e^x}{(1+e^x)^2}$ (D) $\frac{-1}{(1+e^x)^3}$

5. If $x = f(t)$, $y = \phi(t)$, then $\frac{d^2y}{dx^2}$ is equal to

(A) $\frac{f_1\phi_2 - \phi_1 f_2}{f_1^2}$ (B) $\frac{f_1\phi_2 - \phi_1 f_2}{f_1^3}$ (C) $\frac{\phi_1 f_2 - f_1\phi_2}{f_1^3}$ (D) None of these

6. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$, where $0 < \theta < \frac{\pi}{2}$, then $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$ is equal to

(A) $\frac{4\sqrt{2}}{a\pi}$ (B) $\frac{8\sqrt{2}}{a\pi}$ (C) $\frac{4}{a\pi\sqrt{2}}$ (D) None of these

7. If $y = \tan^{-1}\left(\frac{\ell n \frac{e}{x^2}}{\ell n e x^2}\right) + \tan^{-1}\frac{3+2\ell nx}{1-6\ell nx}$ then $\frac{d^2y}{dx^2} =$

(A) 2 (B) 1 (C) 0 (D) –1

8. Prove that the function $y = e^x \sin x$ satisfies the relationship $y'' - 2y' + 2y = 0$.

9. If $y = at^2 + 2bt + c$ and $t = ax^2 + 2bx + c$, then $\frac{d^3y}{dx^3}$ equals

(A) $24a^2(at + b)$ (B) $24a(ax + b)^2$ (C) $24a(at + b)^2$ (D) $24a^2(ax + b)$

10. If $y = e^{\tan x}$, then $\cos^2x \frac{d^2y}{dx^2} =$

(A) $(1 - \sin 2x)\frac{dy}{dx}$ (B) $-(1 + \sin 2x)\frac{dy}{dx}$ (C) $(1 + \sin 2x)\frac{dy}{dx}$ (D) None of these

11. If $y = a\cos(\ell n\, x) + b\sin(\ell n\, x)$, then $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} =$

(A) 0 (B) y (C) – y (D) None of these

12. If $y = (\sin^{-1} x)^2$, then $(1 - x^2)\dfrac{d^2y}{dx^2}$ is equal to

(A) $x\dfrac{dy}{dx} + 2$ (B) $x\dfrac{dy}{dx} - 2$ (C) $-x\dfrac{dy}{dx} + 2$ (D) None of these

13. If $y = e^{ax}\sin bx$, then $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx}$ is equal to

(A) $-(a^2 + b^2)\,y$ (B) $(a^2 + b^2)\,y$ (C) – y (D) None of these

14. If $y = x\ln[(ax)^{-1} + a^{-1}]$, prove that $x(x+1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = y - 1$.

15. If $y = (A + Bx)\,e^{mx} + (m-1)^{-2}\,e^x$ then $\dfrac{d^2y}{dx^2} - 2m\dfrac{dy}{dx} + m^2y$ is equal to:

(A) e^x (B) e^{mx} (C) e^{-mx} (D) $e^{(1-m)x}$

12.0 DIFFERENTIATION OF DETERMINANTS

If $f(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h. l, m, n, u, v, w are differentiable functions of x then

$$f'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Illustrations

Illustration 19. If $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, find $f'(x)$.

Solution Here, $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

On differentiating, we get,

$$\Rightarrow f'(x) = \begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ \frac{d}{dx}(1) & \frac{d}{dx}(2x) & \frac{d}{dx}(3x^2) \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ \frac{d}{dx}(0) & \frac{d}{dx}(2) & \frac{d}{dx}(6x) \end{vmatrix}$$

or $\quad f'(x) = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$

As we know if any two rows or columns are equal, then value of determinant is zero.

$= \quad 0 + 0 + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix} \quad \therefore \quad f'(x) = 6\,(2x^2 - x^2)$

Therefore, $\quad f'(x) = 6x^2$ **Ans.**

13.0 L'HÔPITAL'S RULE

(A) This rule is applicable for the indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function f(x) and g(x) are differentiable in certain neighbourhood of the point 'a', except, may be, at the point 'a' itself and $g'(x) \neq 0$, and if

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$,

then $\quad \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

provided the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (L' Hôpital's rule). The point 'a' may be either finite or improper ($+\infty$ or $-\infty$).

(B) Indeterminate forms of the type $0.\infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

(C) Indeterminate forms of the type $1^\infty, \infty^0$ or 0^0 are reduced to forms of the type $0 \times \infty$ by taking logarithms or by the transformation $[f(x)]^{\phi(x)} = e^{\phi(x).\ell nf(x)}$.

Illustrations

Illustration 20. Evaluate $\lim_{x\to 0} |x|^{\sin x}$

Solution $\quad \lim_{x\to 0} |x|^{\sin x} = \lim_{x\to 0} e^{\sin x \log_e |x|} = e^{\lim_{x\to 0} \frac{\log_e |x|}{\operatorname{cosec} x}}$

$= e^{\lim_{x\to 0} \frac{1/x}{-\operatorname{cosec} x \cot x}}$ (applying L'Hôpital's rule)

$= e^{\lim_{x\to 0} -\frac{\sin^2 x}{x \cos x}} = e^{\lim_{x\to 0} -\left(\frac{\sin x}{x}\right)^2 \cdot \left(\frac{x}{\cos x}\right)} = e^{-(1)^2.(0)} = e^0 = 1$

Ans.

Illustration 21. Solve $\lim_{x\to 0^+} \log_{\sin x} \sin 2x$.

Solution $\quad$ Here $\lim_{x\to 0^+} \log_{\sin x} \sin 2x$

$= \lim_{x\to 0^+} \frac{\log \sin 2x}{\log \sin x} \qquad \left(\frac{-\infty}{-\infty} \text{form}\right)$

$= \lim_{x\to 0^+} \frac{\frac{1}{\sin 2x} \cdot 2\cos 2x}{\frac{1}{\sin x} \cdot \cos x} \qquad$ {applying L'Hôpital's rule}

$= \lim_{x\to 0^+} \frac{\left(\frac{(2x)}{\sin(2x)}\right) \cos 2x}{\left(\frac{x}{\sin x}\right) \cos x} = \lim_{x\to 0^+} \frac{\cos 2x}{\cos x} = 1$ **Ans.**

Illustration 22. Evaluate $\lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}$.

Solution Here, $A = \lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}$ $(\infty^0 \text{ form})$

$$\therefore \quad \log A = \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{e^n}{\pi}\right) = \lim_{n\to\infty}\frac{n\log e - \log\pi}{n} \quad \left(\frac{\infty}{\infty}\text{form}\right)$$

$$= \lim_{n\to\infty}\frac{\log e - 0}{1} \quad \{\text{applying L'Hôpital's rule}\}$$

$$\log A = 1 \Rightarrow A = e^1 \text{ or } \lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n} = e$$ **Ans.**

Interesting fact

In 1694 John Bernoulli agreed to accept a retainer of 300 pounds per year from his former student L'Hôpital to solve problems for him and keep him up to date on calculus. One of the problems was the so-called 0/0 problem, which Bernoulli solved as agreed. When L'Hôpital published his notes on calculus in book form in 1696, the 0/0 rule appeared as a theorem. L'Hôpital acknowledged his debt to Bernoulli and, to avoid claiming authorship of the book's entire contents, had the book published anonymously. Bernoulli nevertheless accused L'Hôpital of plagiarism, an accusation inadvertently supported after L'Hôpital's death in 1704 by the publisher's promotion of the book as L'Hôpital's. By 1721, Bernoulli, a man so jealous he once threw his son Daniel out of the house for accepting a mathematics prize from the French Academy of Sciences, claimed to have been the author of the entire work. As puzzling and fickle as ever, history accepted Bernoulli's claim (until recently), but still named the rule after L'Hôpital.

BEGINNER'S BOX-4

1. If $y = \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix}$, then $\frac{dy}{dx}$ is equal to

(A) 1 (B) –1 (C) 0 (D) None of these

2. If $f(x) = \begin{vmatrix} e^x & x^2 \\ \ell nx & \sin x \end{vmatrix}$, then find $f'(1)$.

3. If $f(x) = \begin{vmatrix} 2x & x^2 & x^3 \\ x^2+2x & 1 & 3x+1 \\ 2x & 1-3x^2 & 5x \end{vmatrix}$ then find $f'(1)$.

4. If f(x), g(x), h(x) are polynomials in x of degree 2 and $F(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$, then $F'(x)$ is equal to

(A) 1 (B) 0 (C) –1 (D) None of these

5. Let $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$ then $f'\left(\frac{\pi}{2}\right) =$

(A) 0 (B) –12 (C) 4 (D) 12

6. If $f(x)=\begin{vmatrix}\cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2)\\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2)\\ \sin 2x & 0 & \sin 2x^2\end{vmatrix}$ then find f'(x).

7. Let $f(x)=\begin{vmatrix}\cos x & x & 1\\ 2\sin x & x^2 & 2x\\ \tan x & x & 1\end{vmatrix}$. Then $\lim\limits_{x\to 0}\dfrac{f'(x)}{x}=$

(A) 2 (B) –2 (C) –1 (D) 1

8. Using L'Hôpital's rule verify that :

(A) $\lim\limits_{x\to 0}\dfrac{\sin x-\tan x}{x^3}=-\dfrac{1}{2}$ (B) $\lim\limits_{x\to 0}\dfrac{\ell n(1+x)}{x}=1$

9. Using L'Hôpital's rule find

(A) $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x^3}$ (B) $\lim\limits_{x\to 0}\dfrac{e^x-x-1}{x^2}$

10. $\lim\limits_{x\to\pi/6}\left[\dfrac{3\sin x-\sqrt{3}\cos x}{6x-\pi}\right]=$

(A) $\sqrt{3}$ (B) $1/\sqrt{3}$ (C) $-\sqrt{3}$ (D) $-1/\sqrt{3}$

11. If f (1) = 1, f′(1) = 2, then $\lim\limits_{x\to 1}\dfrac{\sqrt{f(x)}-1}{\sqrt{x}-1}$ is

(A) 2 (B) 4 (C) 1 (D) 1/2

12. If f (a) = 2, f ′(a) = 1, g(a) = –1 ; g′(a) = 2, then $\lim\limits_{x\to a}\dfrac{g(x)f(a)-g(a)f(x)}{x-a}=$

(A) 3 (B) 5 (C) 0 (D) –3

13. Let f(2) = 4 and f′(2) = 4. Then $\lim\limits_{x\to 2}\dfrac{xf(2)-2f(x)}{x-2}$ is given by

(A) 2 (B) –2 (C) –4 (D) 3

14. If f(4) = g(4) = 2; f'(4) = 9, g'(4) = 6 then $\lim\limits_{x\to 4}\dfrac{\sqrt{f(x)}-\sqrt{g(x)}}{\sqrt{x}-2}$ is equal to :

(A) $3\sqrt{2}$ (B) $\dfrac{3}{\sqrt{2}}$ (C) 0 (D) none

15. $\lim\limits_{n\to\infty}\left[\log_{n-1}(n).\log_n(n+1).\log_{n+1}(n+2).......\log_{n^k-1}\left(n^k\right)\right]$ is equal to

(A) ∞ (B) n (C) k (D) None of these

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, then prove that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$ -

Solution Put $x = \sin\alpha \Rightarrow \alpha = \sin^{-1}(x)$

$y = \sin\beta \Rightarrow \beta = \sin^{-1}(y)$

$\Rightarrow \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$

$$\Rightarrow 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)=2a\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \cot\left(\frac{\alpha-\beta}{2}\right)=a$$

$$\Rightarrow \alpha-\beta=2\cot^{-1}(a)$$

$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}(A)$

differentiating w.r.t. to x.

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$ hence proved **Ans.**

Illustration 2. Find second order derivative of $y = \sin x$ with respect to $z = e^x$.

Solution

$$\frac{dy}{dz}=\frac{dy/dx}{dz/dx}=\frac{\cos x}{e^x}$$

$$\Rightarrow \frac{d^2y}{dz^2}=\frac{d}{dx}\left(\frac{\cos x}{e^x}\right).\frac{dx}{dz} = \frac{-e^x\sin x-\cos x e^x}{\left(e^x\right)^2}.\frac{1}{e^x}$$

$$\Rightarrow \frac{d^2y}{dz^2}=-\frac{(\sin x+\cos x)}{e^{2x}}$$ **Ans.**

Illustration 3. If $y = (\tan^{-1}x)^2$ then prove that $(1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}=2$

Solution $y = (\tan^{-1}x)^2$

Differentiating w.r.t. x

$$\frac{dy}{dx}=\frac{2\tan^{-1}x}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)\frac{dy}{dx}=2\tan^{-1}(x)$$

Again differentiating w.r.t. x

$$\Rightarrow \left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=\frac{2}{\left(1+x^2\right)} \Rightarrow \left(1+x^2\right)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}=2$$ **Ans.**

Illustration 4. Obtain differential coefficient of $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ with respect to $\cos^{-1}\sqrt{\dfrac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

Solution Assume $u = \tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$, $v = \cos^{-1}\sqrt{\dfrac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

The function needs simplification before differentiation Let $x = \tan\theta$

$$\therefore \quad u = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$v = \cos^{-1}\sqrt{\frac{1+\sec\theta}{2\sec\theta}} = \cos^{-1}\sqrt{\frac{1+\cos\theta}{2}} = \cos^{-1}\left(\cos\frac{\theta}{2}\right) = \frac{\theta}{2} \Rightarrow \quad u = v$$

$$\therefore \quad \frac{du}{dv} = 1.$$ **Ans.**

Node-1\Target-2022-23\1. JEE(M+A)\Module\Enthusiast\English\Maths\EM-2\22.Method of Differentiation\Th.

EXERCISE – 1 **SCQ/MCQ**

SINGLE CORRECT

1. Let f(x) = x +3 ln(x – 2) & g(x) = x + 5 ln(x - 1), then the set of x satisfying the inequality f'(x) < g'(x) is -

(A) $\left(2, \frac{7}{2}\right)$ (B) $(1, 2)\cup\left(\frac{7}{2}, \infty\right)$ (C) $(2, \infty)$ (D) $\left(\frac{7}{2}, \infty\right)$

2. Differential coefficient of $\left(x^{\frac{\ell+m}{m-n}}\right)^{\frac{1}{n-\ell}} \cdot \left(x^{\frac{m+n}{n-\ell}}\right)^{\frac{1}{\ell-m}} \cdot \left(x^{\frac{n+\ell}{\ell-m}}\right)^{\frac{1}{m-n}}$ w.r.t. x is -

(A) 1 (B) 0 (C) –1 (D) x^{lmn}

3. If f(x) = $\underset{n\to\infty}{\text{Lt}}\left(\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}......\cos\frac{x}{2^n}\right)$, then f ′(x) at x = $\frac{\pi}{2}$ is

(A) $\frac{4}{\pi^2}$ (B) $-\frac{4}{\pi^2}$ (C) 0 (D) None of these

4. If $f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $0\le x<\frac{\pi}{2}$, then $f'\left(\frac{\pi}{6}\right)$ is

(A) $-\frac{1}{4}$ (B) $-\frac{1}{2}$ (C) $\frac{1}{4}$ (D) $\frac{1}{2}$

5. If f (x) = $\sin^{-1}(\sin x) + \cos^{-1}(\sin x)$ and φ (x) = f (f(f(x))), then φ′(x) is equal to

(A) 1 (B) sin x (C) 0 (D) None of these

6. If f(x) = $\prod_{n=1}^{100}(x-n)^{n(101-n)}$; then $\frac{f(101)}{f'(101)}$ =

(A) 5050 (B) $\frac{1}{5050}$ (C) 10010 (D) $\frac{1}{10010}$

7. If x = secθ – cos θ, y = $\sec^{10}\theta - \cos^{10}\theta$ and $(x^2+4)\left(\frac{dy}{dx}\right)^2 = k(y^2+4)$, then k is equal to

(A) $\frac{1}{100}$ (B) 1 (C) 10 (D) 100

8. If u = f(x^3), v = g(x^2), f ′(x) = cos x and g ′(x) = sin x, then $\frac{du}{dv}$ =

(A) $\frac{1}{2}x\cos x^3\cdot\text{cosec}x^2$ (B) $\frac{3}{2}x\cos x^3\,\text{cosec}\,x^2$ (C) $\frac{1}{2}x\sec x^3\sin x^2$ (D) $\frac{3}{2}x\sec x^3\text{cosec}\,x^2$

9. If f(x) = x + tan x and f is inverse of g then g′ (x) equal to

(A) $\frac{1}{1+[g(x)-x]^2}$ (B) $\frac{1}{2-[g(x)-x]^2}$ (C) $\frac{1}{2+[g(x)-x]^2}$ (D) None of these

10. Derivative of $\tan^{-1}\dfrac{2x}{1-x^2}$ w.r.t $\sin^{-1}\dfrac{2x}{1+x^2}$ is (x ∈ (–1, 1))

(A) $\dfrac{1}{1+x^2}$ (B) $\dfrac{1}{1-x^2}$ (C) 0 (D) 1

11. Let $f(x) = \begin{vmatrix} \sin 3x & 1 & 2\left(\cos\left(\frac{3x}{2}\right)+\sin\left(\frac{3x}{2}\right)\right)^2 \\ \cos 3x & -1 & 2\left(\cos^2\left(\frac{3x}{2}\right)-\sin^2\left(\frac{3x}{2}\right)\right) \\ \tan 3x & 4 & 1+2\tan 3x \end{vmatrix}$ then the value of $f'(x)$ at x = (2n + 1)π, n ∈ I

(the set of integers) is equal to

(A) $(-1)^n$ (B) 3 (C) $(-1)^{n+1}$ (D) 9

12. If $f(x) = (|x|)^{|\sin x|}$, then $f'\left(-\dfrac{\pi}{4}\right)$ is -

(A) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\left(\dfrac{\sqrt{2}}{2}\log\dfrac{4}{\pi}-\dfrac{2\sqrt{2}}{\pi}\right)$ (B) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\left(\dfrac{\sqrt{2}}{2}\log\dfrac{4}{\pi}+\dfrac{2\sqrt{2}}{\pi}\right)$

(C) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\left(\dfrac{\sqrt{2}}{2}\log\dfrac{\pi}{4}-\dfrac{2\sqrt{2}}{\pi}\right)$ (D) $\left(\dfrac{\pi}{4}\right)^{1/\sqrt{2}}\left(\dfrac{\sqrt{2}}{2}\log\dfrac{\pi}{4}+\dfrac{2\sqrt{2}}{\pi}\right)$

13. Let g is the inverse function of f & $f'(x) = \dfrac{x^{10}}{(1+x^2)}$. If g (B) = a then g'(B) is equal to -

(A) $\dfrac{5}{2^{10}}$ (B) $\dfrac{1+a^2}{a^{10}}$ (C) $\dfrac{a^{10}}{1+a^2}$ (D) $\dfrac{1+a^{10}}{a^2}$

14. If $f(x) = x + \dfrac{x^2}{1!}+\dfrac{x^3}{2!}+\dots\dots\dfrac{x^n}{(n-1)!}$, then $f(0) + f'(0) + f''(0) + \dots\dots + f''''^{\dots\dots\text{ n times}}(0)$ is equal to -

(A) $\dfrac{n(n+1)}{2}$ (B) $\dfrac{(n^2+1)}{2}$ (C) $\left(\dfrac{n(n+1)}{2}\right)^2$ (D) $\dfrac{n(n+1)(2n+1)}{6}$

15. If f is differentiable in (0, 6) & f'(4) =5 then $\lim\limits_{x\to 2}\dfrac{f(4)-f(x^2)}{2-x} =$

(A) 5 (B) 5/4 (C) 10 (D) 20

MORE THAN ONE OPTION CORRECT

16. If $y = \tan x \tan 2x \tan 3x$ then $\frac{dy}{dx}$ is equal to-

(A) $3\sec^2 3x \tan x \tan 2x + \sec^2 x \tan 2x \tan 3x + 2\sec^2 2x \tan 3x \tan x$

(B) $2y(\text{cosec } 2x + 2\text{ cosec } 4x + 3\text{ cosec } 6x)$

(C) $3\sec^2 3x - 2\sec^2 2x - \sec^2 x$

(D) $\sec^2 x + 2\sec^2 2x + 3\sec^2 3x$

17. If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$ then $\frac{dy}{dx}$ equals -

(A) $\frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2\sqrt{x}}$ (B) $\frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2x}$ (C) $\frac{1}{2\sqrt{x}}\sqrt{y^2-4}$ (D) $\frac{1}{2\sqrt{x}}\sqrt{y^2+4}$

18. Let $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\infty}}}$ then $\frac{dy}{dx}$ -

(A) $\frac{1}{2y-1}$ (B) $\frac{x}{x-2y}$ (C) $\frac{1}{\sqrt{1+4x}}$ (D) $\frac{y}{2x+y}$

19. If $2^x + 2^y = 2^{x+y}$ then $\frac{dy}{dx}$ has the value equal to -

(A) $-\frac{2^y}{2^x}$ (B) $\frac{1}{1-2^x}$ (C) $1-2^y$ (D) $\frac{2^x(1-2^y)}{2^y(2^x-1)}$

20. The functions $u = e^x \sin x$; $v = e^x \cos x$ satisfy the equation -

(A) $v\frac{du}{dx} - u\frac{dv}{dx} = u^2 + v^2$ (B) $\frac{d^2u}{dx^2} = 2v$ (C) $\frac{d^2v}{dx^2} = -2u$ (D) none of these

21. If $y^2 + b^2 = 2xy$, then $\frac{dy}{dx}$ equals -

(A) $\frac{1}{xy-b^2}$ (B) $\frac{y}{y-x}$ (C) $\frac{xy-b^2}{(y-x)^2}$ (D) $\frac{xy-b^2}{y}$

22. If $\sqrt{y+x} + \sqrt{y-x} = c$, then $\frac{dy}{dx}$ is equal to -

(A) $\frac{2x}{c^2}$ (B) $\frac{x}{y+\sqrt{y^2-x^2}}$ (C) $\frac{y-\sqrt{y^2-x^2}}{x}$ (D) $\frac{c^2}{2y}$

23. If $y + \ell n(1 + x) = 0$, which of the following is true ?

(A) $e^y = xy' + 1$ (B) $y' = -\frac{1}{(x+1)}$ (C) $y' + e^y = 0$ (D) $y' = e^y$

24. If g is inverse of f and $f(x) = x^2 + 3x - 3$ $(x > 0)$ then $g'(1)$ equals-

(A) $\frac{1}{2g(1)+3}$ (B) -1 (C) $\frac{1}{5}$ (D) $\frac{-f'(1)}{(f(1))^2}$

25. If $f : R \to R$ is a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3), x \in \mathbf{R}$, then $f(1) - f(0)$ is equal to

(A) –2 (B) f (2) (C) f (3) (D) –f (2)

26. If $f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\beta) & \sin(\alpha-\beta) \end{vmatrix}$ then $f(\alpha), f(\beta), f(\gamma)$ are in

(A) A.P. (B) G.P. (C) H.P. (D) None of these

27. If $y = \dfrac{\sqrt{(1+t^2)} - \sqrt{(1-t^2)}}{\sqrt{(1+t^2)} + \sqrt{(1-t^2)}}$ and $x = \sqrt{(1-t^4)}$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{-1}{t^2\left\{1+\sqrt{(1-t^4)}\right\}}$ (B) $\dfrac{\left\{\sqrt{(1-t^4)}-1\right\}}{t^6}$ (C) $\dfrac{1}{t^2\{1+\sqrt{(1-t^4)}\}}$ (D) $\dfrac{1-\sqrt{(1-t^4)}}{t^6}$

28. Differential coefficient of $\sin^{-1} x$ w.r.t. $\sin^{-1}(3x - 4x^3)$ is

(A) $\dfrac{1}{3}$ if $-\dfrac{\pi}{8} < x < \dfrac{\pi}{8}$ (B)3 if $\dfrac{-\pi}{8} < x < \dfrac{\pi}{8}$ (C) $\dfrac{1}{3}$ if $-\dfrac{\pi}{9} < x < \dfrac{\pi}{9}$ (D) 3 if $\dfrac{-\pi}{9} < x < \dfrac{\pi}{9}$

29. Let $f(x) = x^n$, n being a non-negative integer, the value of *n* for which the equality $f'(a + b) = f'(a) + f'(b)$ is valid for all a, b> 0 is

(A) 0 (B) 1 (C) 2 (D) None of these

EXERCISE – 2 **MISCELLANEOUS**

Comprehension Based Questions

Comprehension – 1

Let $\frac{f(x+y)-f(x)}{2} = \frac{f(y)-1}{2} + xy$, $(x,y \in R)$. f(x) is differentiable and $f'(0) = 1$. Let g(x) be a derivable function at x = 0 and follows the functional rule $g\left(\frac{x+y}{k}\right) = \frac{g(x)+g(y)}{k}$ $(k \in R, k \neq 0, 2)$

Let $g'(0) = \lambda \neq 0$

On the basis of above information, answer the following questions

1. Domain of $\ell n(f(x))$ is-

(A) R^+ (B) $R - \{0\}$ (C) R (D) R^-

2. Range of $y = \log_{3/4}(f(x))$

(A) $(-\infty, 1]$ (B) $\left[\frac{3}{4}, \infty\right)$ (C) $(-\infty, \infty)$ (D) R

3*. If the graphs of y = f(x) and y = g(x) intersect in coincident points the λ can take values-

(A) 3 (B) 1 (C) –1 (D) 4

Comprehension – 2

Left hand derivative and right hand derivative of a function f(x) at a point x = a are defined as

$$f'(a^-) = \lim_{h\to 0^+}\frac{f(a)-f(a-h)}{h} = \lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h} \text{ and}$$

$$f'(a^+) = \lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^-}\frac{f(a)-f(a-h)}{h} = \lim_{x\to a^+}\frac{f(a)-f(x)}{a-x} \text{ respectively}$$

Let f be a twice differentiable function. We also know that derivative of an even function is odd function and derivative of an odd function is even function.

On the basis of above information, answer the following questions

4. If f is odd, which of the following is Left hand derivative of f at x = –a

(A) $\lim_{h\to 0^-}\frac{f(a-h)-f(a)}{-h}$ (B) $\lim_{h\to 0^-}\frac{f(h-a)-f(a)}{h}$

(C) $\lim_{h\to 0^+}\frac{f(a)+f(a-h)}{-h}$ (D) $\lim_{h\to 0^-}\frac{f(-a)-f(-a-h)}{-h}$

5. If f is even, which of the following is Right hand derivative of f' at x = a

(A) $\lim_{h\to 0^-}\frac{f'(a)+f'(-a+h)}{h}$ (B) $\lim_{h\to 0^+}\frac{f'(a)+f'(-a-h)}{h}$

(C) $\lim_{h\to 0^-}\frac{-f'(-a)+f'(-a-h)}{-h}$ (D) $\lim_{h\to 0^+}\frac{f'(a)+f'(-a+h)}{-h}$

6. The statement $\lim_{h\to 0}\frac{f(-x)-f(-x-h)}{h} = \lim_{h\to 0}\frac{f(x)-f(x-h)}{-h}$ implies that for all $x \in R$

(A) f is odd (B) f is even

(C) f is neither odd nor even (D) nothing can be concluded

Comprehension – 3

If y = f(x) be a differentiable function of x such that whose second, third,......, nth derivatives exist.

i.e, nth derivative of y is denoted by $y_n, \frac{d^n y}{dx^n}, D^n y, y^n, f^n(x)$

$\Rightarrow \quad \frac{d^n y}{dx^n} = \lim_{h\to 0} \frac{f^{n-1}(x+h) - f^{n-1}(x)}{h}$

On the basis of above information, answer the following questions :

7. If $y = e^{3x+7}$, then the value of $y_n(0)$ is

(A) 1 (B) 3^n (C) $3^n . e^7$ (D) $3^n . e^7 . 7!$

8. If $y = \frac{\ln x}{x}$, then the value of $y''(e)$ is

(A) 1 (B) $-\frac{1}{e}$ (C) $-\frac{1}{e^2}$ (D) $-\frac{1}{e^3}$

9. If x = sin t, y = sin kt, then the value of $(1 - x^2)y_2 - xy_1$ is

(A) k^2y (B) $-k^2y$ (C) ky^2 (D) $-ky^2$

Comprehension – 4

If $D^* f(x) = \lim_{h\to 0} \frac{f^2(x+h) - f^2(x)}{h}$

Where $f^2(x) = \{f(x)\}^2$

10. If u = f(x), v = g(x), then the value of D*(u . v) is

(A) (D*u)v + (D*v)u (B) $u^2D^*v + v^2D^*u$ (C) D*u + D*v (D) uvD*(u + v)

11. If u = f(x), v = g(x) then the value of $D^*\left\{\frac{u}{v}\right\}$ is

(A) $\frac{u^2D^*v - v^2D^*u}{v^4}$ (B) $\frac{uD^*v - vD^*u}{v^2}$ (C) $\frac{v^2D^*u - u^2D^*v}{v^4}$ (D) $\frac{vD^*u - uD^*v}{v^2}$

12. D* (tan x) is equal to

(A) $\sec^2 x$ (B) $2\sec^2 x$ (C) $\tan x \sec^2 x$ (D) $2\tan x \sec^2 x$

Match the Column

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

13.

	Column-I		Column-II
(A)	If $f(x) = x^3 + x + 1$, then $f'(x^2 + 1)$ at x = 0 is	(p)	1
(B)	If $f(x) = \log_{x^2}(\ell n x)$, then $f'(e^e)$ is equal to	(q)	0
(C)	For the function $y = \ell n \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$ if $\frac{dy}{dx}$ = secx + p, then p is equal to	(r)	28
(D)	If $f(x) = \lvert x^3 - x^2 + x - 1\rvert \sin x$, then $4f'(28f(f(\pi)))$ is equal to	(s)	4

14*. **Column-I** **Column-II**

(A) $y = \cos^{-1}(4x^3 - 3x)$, then $\frac{dy}{dx}$ is

(B) $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, then $\frac{dy}{dx}$ is

(p) $\frac{3}{1+x^2}, x \in \left(\frac{1}{\sqrt{3}}, \infty\right)$

(q) $\frac{3}{\sqrt{1-x^2}}, x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

(r) $\frac{3}{1+x^2}, x \in \left(-\infty, -\frac{1}{\sqrt{3}}\right)$

(s) $-\frac{3}{\sqrt{1-x^2}}, x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$

15*. **Column-I** **Column-II**

(A) If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then $\frac{dy}{dx}$ is

(B) If $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is

(C) If $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then $\frac{dy}{dx}$ is

(p) $\frac{2}{1+x^2}, |x| < 1$

(q) $\frac{2}{1+x^2}, |x| \neq 1$

(r) $-\frac{2}{1+x^2}, x < 0$

(s) $-\frac{2}{1+x^2}, |x| > 1$

(t) Non-existent, $|x| = 1$

INTEGER/SUBJECTIVE TYPE QUESTIONS

16. The function $f: R \to R$ satisfies $f(x^2).f''(x) = f'(x).f'(x^2)$ for all real x. Given that $f(1) = 1$ and $f'''(1) = 8$, compute the value of $f'(1) + f''(1)$.

17. If $f'(x) = \sqrt{2x^2 - 1}$ and $y = f(x^2)$ then find $\frac{dy}{dx}$ at $x = 1$.

18. If $f(x) = \begin{vmatrix} (x-a)^4 & (x-a)^3 & 1 \\ (x-b)^4 & (x-b)^3 & 1 \\ (x-c)^4 & (x-c)^3 & 1 \end{vmatrix}$ then $f'(x) = \lambda.\begin{vmatrix} (x-a)^4 & (x-a)^2 & 1 \\ (x-b)^4 & (x-b)^2 & 1 \\ (x-c)^4 & (x-c)^2 & 1 \end{vmatrix}$ Find the value of λ .

19. If $y = \tan^{-1}\frac{u}{\sqrt{1-u^2}}$ & $x = \sec^{-1}\frac{1}{2u^2 - 1}$, $u \in \left(0, \frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$, find $\frac{dy}{dx} =$

20. Let $f(x) = x^2 - 4x - 3$, $x > 2$ and let g be the inverse of f. Find the value of $g'(2)$.

21. If $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$, then find $\frac{dy}{dx}$ for $x \in (-1, 1)$.

NCERT CORNER

Very Short Answer

1. Differentiate each of the following w.r.t. x :

(a) $\tan\sqrt{x}$ (b) $(ax + b)^m$ (c) $\sin 3x \cos 5x$

2. Differentiate each of the following w.r.t. x :

(a) $\log(\log x)$, $x > 1$ (b) $\sin(\tan^{-1} x)$ (c) $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$

3. (a) If $x.\sqrt{1+y} + y\sqrt{1+x} = 0,$ prove that $\dfrac{dy}{dx} = \dfrac{-1}{(1+x)^2}$

(b) If $y = \dfrac{5^x}{x^{5'}}$ find $\dfrac{dy}{dx}$

(c) If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + ...to\ \infty}}}$, prove that $\dfrac{dy}{dx} = \dfrac{\cos x}{(2y-1)}$.

4. (a) Find $\dfrac{dy}{dx}$, when x = a (t + sin t) and y = a (1- cos t)

(b) If y = (tan x + sec x), prove that $\dfrac{d^2y}{dx^2} = \dfrac{\cos x}{(1-\sin x)^2}$

5. For the function $f(x) = \dfrac{x^{100}}{100} + \dfrac{x^{99}}{99} + ... + \dfrac{x^2}{2} + x + 1$.

Prove that $f'(1) = 100f'(0)$.

Short Answer

6. Find the derivative of the function from first principles : y = $\sin(x+1)$

7. Find the derivative of the function (it is to be understood that a, p and q are fixed non-zero constants) : $(ax^2 + \sin x)(p + q\cos x)$

8. Find the derivative of the function : $\dfrac{4x + 5\sin x}{3x + 7\cos x}$

9. Find the derivatives of the function from first principles : y = $\cot(3x+1)$

10. If $y = \cos^2 x^2$, find $\frac{dy}{dx}$,

11. If $y = \sqrt{e^{\sqrt{x}}}$, find $\frac{dy}{dx}$.

12. Differentiate $\sqrt{\cot^{-1}\sqrt{x}}$ w.r.t. x.

13. Differentiate w.r.t. x :

$\tan^{-1}(\sec x + \tan x)$

14. Differentiate $\cot^{-1}\left(\sqrt{1+x^2}+x\right)$ w.r.t. x.

15. If sin y = x sin (a + y), prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

16. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

17. If $y = \left(\sqrt{x}\right)^{\left(\sqrt{x}\right)^{\left(\sqrt{x}\right)\ldots\ldots\infty}}$, prove that $x\left(\frac{dy}{dx}\right) = \frac{y^2}{(2 - y\log x)}$.

18. If x = 3 sin t - sin 3t, y = 3 cos t - cos 3t, find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$.

19. If y = $(\tan^{-1} x)^2$, prove that $(1 + x^2)^2 y_2 + 2x(1 + x^2) y_1 = 2$.

20. If $y = \sqrt{x^2+1} - \log\left\{\frac{1}{x} + \sqrt{1+\frac{1}{2^2}}\right\}$. Find $\frac{dy}{dx}$.

Long Answer

21. Find the derivative of the function from first principles : $\frac{x+1}{x-1}$

22. Find the derivative of $\cos x^2$ from first principles.

23. Differentiate the function $\frac{(x+2)(1-3x)}{2x+1}, x \neq -\frac{1}{2}$ w.r.t. x.

24. If $y=\sqrt{\frac{1-x}{1+x}}$, find the value of $(1-x^2)\frac{dy}{dx}+y, -1<x<1$.

25. Show that $\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]=\sqrt{a^2-x^2}$

26. If $y=\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$, Find $\frac{dy}{dx}$

27. If $y=\sqrt{\frac{(x-3)(x^2+4)}{(3x^2+4x+5)}}$, Find $\frac{dy}{dx}$

28. If $y=(x)^{\cos x}+(\cos x)^{\sin x}$, find $\frac{dy}{dx}$.

29. If x = a sin 2t (1 + cos 2t) and y = b cos 2t (1 - cos 2t), find that $\left(\frac{dy}{dx}\right)_{at\ t=\frac{\pi}{4}}$.

30. If $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$, show that $(1-x^2)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-y=0$

ANSWER KEY

BEGINNER'S BOX-1

1. $3x^2 + 12x + 11$ **2.** (B) **3.** (A) **4.** (B) **5.** (B)

6. (A) **7.** $5e^{5x}\tan(x^2+2) + 2xe^{5x}\sec^2(x^2+2)$ **8.** $x^2e^{x^2}\sin 2x\left(3+2x^2+2x\cot 2x\right)$

9. (B) **10.** (A) **11.** (C) **12.** (B) **13.** $x^x(\ell nx+1)$

14. $(\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x}-\sin x\ln\sin x\right)$ **15.** $y(1 + 2x + 3x^2 + 4x^3)$

16. $\frac{\left(57x^2-302x+361\right)}{20(x-2)^{\frac{3}{4}}(x-3)^{\frac{7}{5}}}.(x+1)^2$ **17.** $x^{x^x}.x^x\left(\ln^2 x+\ln x+\frac{1}{x}\right)$

18. -1 **19.** (B) **20.** (B)

BEGINNER'S BOX-2

1. $2(x^{\ell nx})(\ell nx)$

2. $-\frac{(\log x)^{\tan x}\left(\sec^2 x\ln(\ln x)+\frac{\tan x}{x\ln x}\right)\sqrt{1-x^2}}{m\cos(m\cos^{-1}x)}$ **3.** (A)

4. 3/2, –3/2 **5.** $\frac{(\sin y)^x\left((\ln\sin y)+\tan x\right)}{(\cos x)^y\left(\ln\cos x - x\cot y\right)}$

6. $\frac{2x\sin y-y^2\cos x}{2y\sin x-x^2\cos y}$ **7.** $\frac{\cos(x-y)-1}{\cos(x-y)+1}$

8. $y' = -\left(\frac{2x+e^y}{xe^y+1}\right)$, -1

9. (C) **10.** (D) **11.** 100 **12.** (A) **13.** (D) **14.** (B)
15. (C) **16.** (A) **17.** (C) **18.** **(a)** – 6, **(b)** 3, **(c)** –6

19. $-\frac{2}{1+x^2}$ **20.** (C)

BEGINNER'S BOX-3

1. $y'' = 4y + 2xy'$ **2.** $\pi + 4$ **3.** (B) **4.** (B)

5. (B) **6.** (B) **7.** (C) **9.** (D) **10.** (C) **11.** (C)

12. (A) **13.** (A) **15.** (A)

BEGINNER'S BOX-4

1. (A) **2.** $e(\sin 1 + \cos 1) - 1$ **3.** 9 **4.** (B)
5. (C) **6.** $2(1 + 2x)$, $\cos 2(x + x^2)$, **7.** (B)

9. **(a)** $\frac{1}{3}$, **(b)** $\frac{1}{2}$ **10.** (B) **11.** (A) **12.** (B) **13.** (C)

14. (A) **15.** (C)

EXERCISE-1 (SINGLE CORRECT & MORE THAN ONE OPTION CORRECT)

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	D	B	B	D	C	B	D	B	C	D
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	D	A	B	A	D	ABC	AC	ACD	ABCD	ABC
Que.	21	22	23	24	25	26	27	28	29	
Ans.	BC	ABC	ABC	AC	AB	ABC	AB	AC	AC	

EXERCISE-2 (MISCELLANEOUS)

- **Comprehension Based Questions**

Comprehension – 1   **1.** C   **2.** A   **3.** A,C

Comprehension – 2   **4.** A   **5.** B   **6.** B

Comprehension – 3   **7.** C   **8.** D   **9.** B

Comprehension – 4   **10.** B   **11.** C   **12.** D

- **Match the Column**

13. (A) → (s); (B) → (q); (C) → (q); (D) → (s)

14. (A) → (q,s); (B) → (p, r)

15. (A) → (p,s,t); (B) → (r); (C) → (p,q,t)

- **INTEGER/SUBJECTIVE TYPE QUESTIONS**

16. (6)   **17.** (2)   **18.** (3)

19. (–1/2)   **20.** (1/6)   **21.** $\frac{1-2x}{2\sqrt{1-x^2}}$

NCERT CORNER

1. (a) $=\frac{\sec^2\sqrt{x}}{2\sqrt{x}}$ (b) ma (ax + b)$^{m-1}$ (c) (4 cos 8x - cos 2x

2.(a) $\frac{1}{(x\log x)}$ (b) $\frac{1}{\left(1+x^2\right)^{3/2}}$ (c) -1

3. (b) $\Rightarrow \frac{dy}{dx}=\frac{5^x}{x^5}\left(\log 5-\frac{5}{x}\right)$ **4.** (a) $\tan\frac{t}{2}$

6. $\cos(x+1)$

7. $q(\cos^2 x-\sin^2 x)+p\cos x+2axq\cos x+2axp\ -aqx^2\sin x$

8. $\frac{15(x\cos x-\sin x)+28(\cos x+x\sin x)+35}{(3x+7\cos x)^2}$ **9.** $-3\operatorname{cosec}^2(3x+1)$

10. - 4 x cos x^2 sin x^2

11. $\frac{e^{\frac{1}{2}\sqrt{x}}}{4\sqrt{x}}$ **12.** $\frac{-1}{4\left(\sqrt{\cot^{-1}\sqrt{x}}\right)(1+x)\sqrt{x}}$ **13.** $\frac{1}{2}$

14. $\dfrac{-1}{2\left(1+x^{2}\right)}$

18. $\dfrac{-16}{27}$

20. $\dfrac{\sqrt{x^{2}+1}}{x}$ **21.** $\dfrac{-2}{(x-1)^{2}}$ **22.** $-2x\sin x^{2}$.

23. $\dfrac{-3(2x^{2}+2x+3)}{(2x+1)^{2}}$. **24.** 0

26. $\dfrac{x}{\sqrt{1-x^{4}}}$

27. $\dfrac{1}{2}.\sqrt{\dfrac{(x-3)\left(x^{2}+4\right)}{\left(3x^{2}+4x+5\right)}}\left\{\dfrac{1}{(x-3)}+\dfrac{2x}{\left(x^{2}+4\right)}-\dfrac{(6x+4)}{\left(3x^{2}+4x+5\right)}\right\}$

28. $\dfrac{dy}{dx}=(x)^{\cos x}.\left\{\dfrac{\cos x}{x}-(\log x)\sin x\right\}+(\cos x)^{\sin x}.\{-\sin x\tan x+\cos x.\log(\cos x)\}$

29. $\dfrac{b}{a}$

* * * * * *

PERMUTATION & COMBINATION

1.0 FUNDAMENTAL PRINCIPLE OF COUNTING

(counting without actual counting)

If an event A can occur in 'm' different ways and another event B can occur in 'n' different ways, then the total number of different ways of-

(A) simultaneous occurrence of both events in a definite order is $m \times n$. This can be extended to any number of events (known as multiplication principle).

(B) happening exactly one of the events is $m + n$ (known as addition principle).

Example – There are 15 IITs in India and let each IIT has 10 branches, then the IITJEE topper can select the IIT and branch in $15 \times 10 = 150$ number of ways.

Example – There are 15 IITs & 20 NITs in India, then a student who cleared both IITJEE & AIEEE exams can select an institute in $(15 + 20) = 35$ number of ways.

Illustrations

Illustration 1. A college offers 6 courses in the morning and 4 in the evening. The possible number of choices with the student if he wants to study one course in the morning and one in the evening is-

(A) 24 (B) 2 (C) 12 (D) 10

Solution The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways $6 \times 4 = 24$. **Ans.(A)**

Illustration 2. A college offers 6 courses in the morning and 4 in the evening. The number of ways a student can select exactly one course, either in the morning or in the evening-

(A) 6 (B) 4 (C) 10 (D) 24

Solution The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways $6 + 4 = 10$. **Ans. (C)**

2.0 PERMUTATION & COMBINATION

2.1 Factorial

A Useful Notation : $n! = n.(n-1).(n-2)..............3.\ 2.\ 1;\ \ n! = n.\ (n-1)!$ where $n \in N$

2.2 Permutation

Each of the arrangements in a definite order which can be made by taking some or all of the things at a time is called a PERMUTATION. In permutation, order of appearance of things is taken into account; when the order is changed, a different permutation is obtained.

Generally, it involves the problems of arrangements (standing in a line, seated in a row), problems on digit, problems on letters from a word etc.

${}^nP_r$ denotes the number of permutations of n **different** things, taken r at a time $(n \in N, r \in W, r \le n)$

$${}^nP_r = n\,(n-1)\,(n-2)\,............\,(n-r+1) = \frac{n!}{(n-r)!}$$

2.3 Combination

Each of the groups or selections which can be made by taking some or all of the things without considering the order of the things in each group is called a COMBINATION.

Generally, involves the problem of selections, choosing, distributed groups formation, committee formation, geometrical problems etc.

${}^nC_r$ denotes the number of combinations of n different things taken r at a time $(n \in N, r \in W, r \le n)$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

GOLDEN KEY POINTS

- 0! = 1! = 1
- Factorials of negative integers are not defined.
- n! is also denoted by $\underline{|n}$
- $(2n)! = 2^n.n!\,[1.\,3.\,5.\,7........(2n-1)]$
- Prime factorisation of n! : Let p be a prime number and n be a positive integer, then exponent of p in n! is denoted by $E_p(n!)$ and is given by

  $$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ... + \left[\frac{n}{p^k}\right]$$

  where $p^k \le n < p^{k+1}$ and [x] denotes the integral part of x.
  If we isolate the power of each prime contained in any number n, then n can be written as

  $$n = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4} \;....$$ where α_i are whole numbers.
- ${}^nP_n = n!$, ${}^nP_0 = 1$, ${}^nP_1 = n$
- Number of arrangements of n **distinct** things taken all at a time = n!
- ${}^nP_r$ is also denoted by A_r^n or P(n,r).
- ${}^nC_r$ is also denoted by $\binom{n}{r}$ or C (n, r).
- ${}^nP_r = {}^nC_r.\,r!$

Illustrations

Illustration 3. Find the exponent of 6 in 50!

Solution

$$E_2(50!) = \left[\frac{50}{2}\right]+\left[\frac{50}{4}\right]+\left[\frac{50}{8}\right]+\left[\frac{50}{16}\right]+\left[\frac{50}{32}\right]+\left[\frac{50}{64}\right]$$ (where [ ] denotes integral part)

$E_2(50!) = 25 + 12 + 6 + 3 + 1 + 0 = 47$

$$E_3(50!) = \left[\frac{50}{3}\right]+\left[\frac{50}{9}\right]+\left[\frac{50}{27}\right]+\left[\frac{50}{81}\right]$$

$E_3(50!) = 16 + 5 + 1 + 0 = 22$

$\Rightarrow$ 50! can be written as $50! = 2^{47}.\,3^{22}.........$

Therefore exponent of 6 in 50! = 22 **Ans.**

Illustration 4. If a denotes the number of permutations of (x + 2) things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of (x – 11) things taken all at a time such that a = 182 bc, then the value of x is

(A) 15 (B) 12 (C) 10 (D) 18

Solution

$${}^{x+2}P_{x+2} = a \Rightarrow a = (x+2)!$$

$${}^{x}P_{11} = b \Rightarrow b = \frac{x!}{(x-11)!}$$

and $${}^{x-11}P_{x-11} = c \Rightarrow c = (x-11)!$$

$\therefore \quad a = 182bc$

$$(x+2)! = 182\frac{x!}{(x-11)!}(x-11)! \Rightarrow (x+2)(x+1) = 182 = 14\times 13$$

$\therefore \quad x+1 = 13 \Rightarrow x = 12$ **Ans. (B)**

Illustration 5. A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be drawn so that there are atleast two balls of each colour ?

Solution The selections of 6 balls, consisting of atleast two balls of each colour from 5 red and 6 white balls, can be made in the following ways

Red balls (5)	White balls (6)	Number of ways
2	4	${}^5C_2 \times {}^6C_4 = 150$
3	3	${}^5C_3 \times {}^6C_3 = 200$
4	2	${}^5C_4 \times {}^6C_2 = 75$

Therefore total number of ways = 425 **Ans.**

Illustration 6. How many 4 letter words can be formed from the letters of the word 'ANSWER' ? How many of these words start with a vowel ?

Solution Number of ways of arranging 4 different letters from 6 different letters are ${}^6C_4 4! = \frac{6!}{2!} = 360$.

There are two vowels (A & E) in the word 'ANSWER'.

Total number of 4 letter words starting with A : A _ _ _ = ${}^5C_3 3! = \frac{5!}{2!} = 60$

Total number of 4 letter words starting with E : E _ _ _ = ${}^5C_3 3! = \frac{5!}{2!} = 60$

$\therefore$ Total number of 4 letter words starting with a vowel = 60 + 60 = 120. **Ans.**

Illustration 7. If all the letters of the word 'RAPID' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RAPID'.

Solution First of all, arrange all letters of given word alphabetically : 'ADIPR'

Total number of words starting with A _ _ _ _ = 4! = 24
Total number of words starting with D _ _ _ _ = 4! = 24
Total number of words starting with I _ _ _ _ = 4! = 24
Total number of words starting with P _ _ _ _ = 4! = 24
Total number of words starting with RAD _ _ = 2! = 2
Total number of words starting with RAI _ _ = 2! = 2
Total number of words starting with RAPD _ = 1
Total number of words starting with RAPI _ = 1

$\therefore$ Rank of the word RAPID = 24 + 24 + 24 + 24 + 2 + 2 + 1 + 1 = 102 **Ans.**

3.0 PROPERTIES OF ${}^nP_r$ and ${}^nC_r$

(A) The number of permutation of n different objects taken r at a time, when p particular objects are always to be included is $r!.{}^{n-p}C_{r-p}$ $(p \le r \le n)$

(B) The number of permutations of n different objects taken r at a time, when repetition is allowed any number of times is n^r.

(C) Following properties of ${}^nC_r$ should be remembered :

(i) ${}^nC_r = {}^nC_{n-r}$; ${}^nC_0 = {}^nC_n = 1$

(ii) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iv) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots\ldots + {}^nC_n = 2^n$

(v) ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$

(vi) ${}^nC_r$ is maximum when $r = \frac{n}{2}$ if n is even & $r = \frac{n-1}{2}$ or $r = \frac{n+1}{2}$ if n is odd.

(D) The number of combinations of n different things taking r at a time,

(i) when p particular things are always to be included = ${}^{n-p}C_{r-p}$

(ii) when p particular things are always to be excluded = ${}^{n-p}C_r$

(iii) when p particular things are always to be included and q particular things are to be excluded = ${}^{n-p-q}C_{r-p}$

Illustrations

Illustration 8. There are 6 pockets in the coat of a person. In how many ways can he put 4 pens in these pockets?

(A) 360 (B) 1296 (C) 4096 (D) none of these

Solution First pen can be put in 6 ways.

Similarly each of second, third and fourth pen can be put in 6 ways.

Hence total number of ways = $6 \times 6 \times 6 \times 6 = 1296$ **Ans.(B)**

Illustration 9. A delegation of four students is to be selected from a total of 12 students. In how many ways can the delegation be selected, if-

(A) all the students are equally willing ?

(B) two particular students have to be included in the delegation ?

(C) two particular students do not wish to be together in the delegation ?

(D) two particular students wish to be included together only ?

(E) two particular students refuse to be together and two other particular students wish to be together only in the delegation ?

Solution (A) Formation of delegation means selection of 4 out of 12.

Hence the number of ways = ${}^{12}C_4 = 495$.

(B) If two particular students are already selected. Here we need to select only 2 out of the remaining 10. Hence the number of ways = ${}^{10}C_2 = 45$.

(C) The number of ways in which both are selected = 45. Hence the number of ways in which the two are not included together = 495 – 45 = 450

(D) There are two possible cases

(i) Either both are selected. In this case, the number of ways in which the selection can be made = 45.

(ii) Or both are not selected. In this case all the four students are selected from the remaining ten students. This can be done in ${}^{10}C_4 = 210$ ways.

Hence the total number of ways of selection = 45 + 210 = 255

(E) We assume that students A and B wish to be selected together and students C and D do not wish to be together. Now there are following 6 cases.

(i) (A, B, C) selected, (D) not selected

(ii) (A, B, D) selected, (C) not selected

(iii) (A, B) selected, (C, D) not selected

(iv) (C) selected, (A, B, D) not selected

(v) (D) selected, (A, B, C) not selected

(vi) A, B, C, D not selected

For (i) the number of ways of selection = ${}^8C_1 = 8$

For (ii) the number of ways of selection = ${}^8C_1 = 8$

For (iii) the number of ways of selection = ${}^8C_2 = 28$

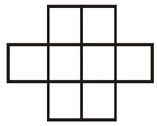
For (iv) the number of ways of selection = ${}^8C_3 = 56$

For (v) the number of ways of selection = ${}^8C_3 = 56$

For (vi) the number of ways of selection = ${}^8C_4 = 70$

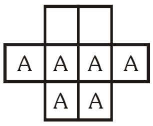
Hence total number of ways = 8 + 8 + 28 + 56 + 56 + 70 = 226. **Ans.**

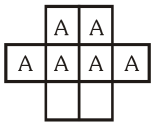
Illustration 10. In the given figure of squares, 6 A's should be written in such a manner that every row contains at least one 'A'. In how many number of ways is it possible ?



(A) 24 (B) 25 (C) 26 (D) 27

Solution There are 8 squares and 6 'A' in given figure. First we can put 6 'A' in these 8 squares by ${}^{8}C_{6}$ number of ways.

(I) 

(II) 

According to question, atleast one 'A' should be included in each row. So after subtracting these two cases, number of ways are $= ({}^{8}C_{6} - 2) = 28 - 2 = 26$. **Ans. (C)**

Illustration 11. There are three coplanar parallel lines. If any p points are taken on each of the lines, the maximum number of triangles with vertices at these points is :

(A) $3p^2(p-1)+1$ (B) $3p^2(p-1)$ (C) $p^2(4p-3)$ (D) none of these

Solution The number of triangles with vertices on different lines $= {}^{p}C_1 \times {}^{p}C_1 \times {}^{p}C_1 = p^3$

The number of triangles with two vertices on one line and the third vertex on any one of the other two lines $= {}^{3}C_1\{{}^{p}C_2 \times {}^{2p}C_1\} = 6p.\dfrac{p(p-1)}{2}$

So, the required number of triangles $= p^3 + 3p^2(p-1) = p^2(4p-3)$ **Ans. (C)**

Illustration 12. There are 10 points in a row. In how many ways can 4 points be selected such that no two of them are consecutive ?

Solution Total number of remaining non-selected points = 6

. . . . . .

Total number of gaps made by these 6 points = 6 + 1 = 7

If we select 4 gaps out of these 7 gaps and put 4 points in selected gaps then the new points will represent 4 points such that no two of them are consecutive.

x . . x . x . . x .

Total number of ways of selecting 4 gaps out of 7 gaps $= {}^{7}C_4$ **Ans.**

In general, total number of ways of selection of r points out of n points in a row such that no two of them are consecutive : ${}^{n-r+1}C_r$

BEGINNER'S BOX-1

1. Four visitors A, B, C & D arrive at a town which has 5 hotels. In how many ways can they disperse themselves among 5 hotels, if 4 hotels are used to accommodate them.

2. If the letters of the word "VARUN" are written in all possible ways and then are arranged as in a dictionary, then the rank of the word VARUN is :

(A) 98 (B) 99 (C) 100 (D) 101

3. How many natural numbers are their from 1 to 1000 which have none of their digits repeated.

4. 3 different railway passes are allotted to 5 students. The number of ways this can be done is :

(A) 60 (B) 20 (C) 15 (D) 10

5. There are 6 roads between A & B and 4 roads between B & C.

(i) In how many ways can one drive from A to C by way of B ?

(ii) In how many ways can one drive from A to C and back to A, passing through B on both trips ?

(iii) In how many ways can one drive the circular trip described in (ii) without using the same road more than once.

6. (i) How many car number plates can be made if each plate contains 2 different letters of English alphabet, followed by 3 different digits.

(ii) Solve the problem, if the first digit cannot be 0.

7. (i) Find the number of four letter word that can be formed from the letters of the word HISTORY. (each letter to be used atmost once)

(ii) How many of them contain only consonants?

(iii) How many of them begin & end in a consonant?

(iv) How many of them begin with a vowel?

(v) How many contain the letters Y?

(vi) How many begin with T & end in a vowel?

(vii) How many begin with T & also contain S?

(viii) How many contain both vowels?

8. If repetitions are not permitted

(i) How many 3 digit numbers can be formed from the six digits 2, 3, 5, 6, 7 & 9 ?

(ii) How many of these are less than 400 ?

(iii) How many are even ?

(iv) How many are odd ?

(v) How many are multiples of 5 ?

9. Every telephone number consists of 7 digits. How many telephone numbers are there which do not include any other digits but 2 , 3 , 5 & 7 ?

10. (A) In how many ways can four passengers be accommodated in three railway carriages, if each carriage can accommodate any number of passengers.

(B) In how many ways four persons can be accommodated in 3 different chairs if each person can occupy only one chair.

11. How many odd numbers of five distinct digits can be formed with the digits 0,1,2,3,4 ?

12. Number of natural numbers between 100 and 1000 such that at least one of their digits is 7, is

(A) 225 (B) 243 (C) 252 (D) none

13. How many four digit numbers are there which are divisible by 2 ?

14. In a certain strange language, words are written with letters from the following six-letter alphabet : A, G, K, N, R, U. Each word consists of six letters and none of the letters repeat. Each combination of these six letters is a word in this language. The word "KANGUR" remains in the dictionary at,

(A) 248^{th} (B) 247^{th} (C) 246^{th} (D) 253^{rd}

15. Find the number of 7 lettered palindromes which can be formed using the letters from the English alphabets.

16. Number of 4 digit numbers of the form N = abcd which satisfy following three conditions :

(i) $4000 \le N < 6000$ (ii) N is multiple of 5 (iii) $3 \le b < c \le 6$

is equal to

(A) 12 (B) 18 (C) 24 (D) 48

17. How many 10 digit numbers can be made with odd digits so that no two consecutive digits are same.

BEGINNER'S BOX–2

1. $(n-r+1)\,{}^{n}P_{r-1} =$

(A) ${}^{n-1}P_r$ (B) ${}^{n+1}P_r$ (C) ${}^{n}P_r$ (D) ${}^{n}P_{r-1}$

2. If ${}^{K+5}P_{K+1} = \dfrac{11(K-1)}{2}\ {}^{K+3}P_K$ then the values of K are

(A) 2 and 6 (B) 2 and 11 (C) 7 and 11 (D) 6 and 7

3. The value of $2^n\{1.3.5.....(2n-3)(2n-1)\}$ is

(A) $\frac{(2n)!}{n!}$ (B) $\frac{(2n)!}{2^n}$ (C) $\frac{n!}{(2n)!}$ (D) None of these

4. How many words can be formed form the letters of the word COURTESY, whose first letter is C and the last letter is Y

(A) 6 ! (B) 8 ! (C) 2 (6) ! (D) 2 (7) !

5. How many words can be made from the letters of the word INSURANCE, if all vowels come together

(A) 18270 (B) 17280 (C) 12780 (D) None of these

6. In how many ways can 5 boys and 5 girls stand in a row so that no two girls may be together

(A) $(5!)^2$ (B) $5!\times 4!$ (C) $5!\times 6!$ (D) $6\times 5!$

7. The number of ways in which 5 boys and 3 girls can be seated in a row so that each girl in between two boys

(A) 2880 (B) 1880 (C) 3800 (D) 2800

8. How many numbers between 5000 and 10,000 can be formed using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 each digit appearing not more than once in each number

(A) $5\times {}^8P_3$ (B) $5\times {}^8C_3$ (C) $5!\times {}^8P_3$ (D) $5!\times {}^8C_3$

9. How many words can be formed by taking 3 consonants and 2 vowels out of 5 consonants and 4 vowels

(A) ${}^5C_3\times {}^4C_2$ (B) $\frac{{}^5C_3\times {}^4C_2}{5}$ (C) ${}^5C_3\times {}^4C_3$ (D) $({}^5C_3\times {}^4C_2)(5)!$

10. How many of the 900 three digit numbers have at least one even digit?

(A) 775 (B) 875 (C) 450 (D) 750

11. All possible three digits even numbers which can be formed with the condition that if 5 is one of the digit, then 7 is the next digit is :

(A) 5 (B) 325 (C) 345 (D) 365

12. Number of 5 digit numbers which are divisible by 5 and each number containing the digit 5, digits being all different is equal to k(4!), the value of k is

(A) 84 (B) 168 (C) 188 (D) 208

13. The number of six digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6 & 7 so that digits do not repeat and the terminal digits are even is :

(A) 144 (B) 72 (C) 288 (D) 720

14. A 5 digit number divisible by 3 is to be formed using the numerals 0, 1, 2, 3, 4 & 5 without repetition. The total number of ways this can be done is :

(A) 3125 (B) 600 (C) 240 (D) 216

15. Number of 9 digits numbers divisible by nine using the digits from 0 to 9 if each digit is used atmost once is K · 8! , then K has the value equal to ______.

16. A new flag is to be designed with six vertical strips using some or all of the colours yellow, green, blue and red. Then, the number of ways this can be done such that no two adjacent strips have the same colour is -

(A) 12 × 81 (B) 16 × 192 (C) 20 × 125 (D) 24 × 216

17. The kindergarten teacher has 25 kids in her class . She takes 5 of them at a time, to zoological garden as often as she can, without taking the same 5 kids more than once. Then the number of visits, the teacher makes to the garden exceeds that of a kid by :

(A) ${}^{25}C_5 - {}^{24}C_5$ (B) ${}^{24}C_5$ (C) ${}^{24}C_4$ (D) none

18. Number of permutations of 1, 2, 3, 4, 5, 6, 7, 8 and 9 taken all at a time, such that the digit
1 appearing somewhere to the left of 2
3 appearing to the left of 4 and
5 somewhere to the left of 6, is
(*e.g.* 815723946 would be one such permutation)

(A) $9 \cdot 7!$ (B) $8!$ (C) $5! \cdot 4!$ (D) $8! \cdot 4!$

19. The number of triangles that can be formed by 5 points in a line and 3 points on a parallel line is

(A) ${}^8C_3$ (B) ${}^8C_3 - {}^5C_3$ (C) ${}^8C_3 - {}^5C_3 - 1$ (D) None of these

20. The maximum number of points of intersection of 20 straight lines will be

(A) 190 (B) 220 (C) 200 (D) None of these

21. There are 720 permutations of the digits 1, 2, 3, 4, 5, 6. Suppose these permutations are arranged from smallest to largest numerical values, beginning from 1 2 3 4 5 6 and ending with 6 5 4 3 2 1.

(A) What number falls on the 124^{th} position?

(B) What is the position of the number 321546?

22. All the five digits number in which each successive digit exceeds its predecessor are arranged in the increasing order of their magnitude. The 97^{th} number in the list does not contain the digit

(A) 4 (B) 5 (C) 7 (D) 8

23. The exponent of 3 in 100 ! is

(A) 33 (B) 44 (C) 48 (D) 52

24. Find the exponent of 10 in ${}^{75}C_{25}$.

25. Let P_n denotes the number of ways in which three people can be selected out of 'n' people sitting in a row, if no two of them are consecutive. If, $P_{n+1} - P_n = 15$ then the value of 'n' is :

(A) 7 (B) 8 (C) 9 (D) 10

4.0 FORMATION OF GROUPS

(A) (i) The number of ways in which (m + n) different things can be divided into two groups such that one of them contains m things and other has n things, is $\frac{(m+n)!}{m!\ n!}$ $(m \neq n)$.

(ii) If m = n, it means the groups are equal & in this case the number of divisions is $\frac{(2n)!}{n!\ n!\ 2!}$. As in any one way it is possible to interchange the two groups without obtaining a new distribution.

(iii) If 2n things are to be divided equally between two persons then the number of ways : $\frac{(2n)!}{n!\ n!\ (2!)} \times 2!$.

(B) (i) Number of ways in which (m + n + p) different things can be divided into three groups containing m, n & p things respectively is : $\frac{(m+n+p)!}{m!\ n!\ p!}$, $m \neq n \neq p$.

(ii) If m = n = p then the number of groups = $\frac{(3n)!}{n!\ n!\ n!\ 3!}$.

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

(C) In general, the number of ways of dividing n distinct objects into ℓ groups containing p objects each and m groups containing q objects each is equal to $\frac{n!(\ell+m)!}{(p!)^{\ell}(q!)^{m}\ \ell!m!}$

Here $\ell p + mq = n$

Illustrations

Illustration 13. Find the number of ways of dividing 52 cards among 4 players equally such that each gets exactly one Ace.

Solution Total number of ways of dividing 48 cards (Excluding 4Aces) in 4 groups $= \dfrac{48!}{(12!)^4 4!}$

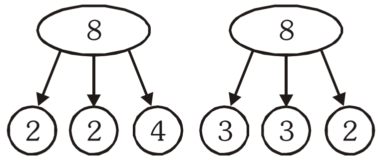
Now, distribute exactly one Ace to each group of 12 cards. Total number of ways $= \dfrac{48!}{(12!)^4 4!} \times 4!$

Now, distribute these groups of cards among four players

$= \dfrac{48!}{(12!)^4 4!} \times 4!4! = \dfrac{48!}{(12!)^4} \times 4!$ **Ans.**

Illustration 14. In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books ?

Solution If each receives at least two books, then the division trees would be as shown below :



(i) (ii)

The number of ways of division for tree in figure (i) is $\left[\dfrac{8!}{(2!)^2 4!2!}\right]$.

The number of ways of division for tree in figure (ii) is $\left[\dfrac{8!}{(3!)^2 2!2!}\right]$.

The total number of ways of distribution of these groups among 3 students

is $\left[\dfrac{8!}{(2!)^2 4!2!} + \dfrac{8!}{(3!)^2 2!2!}\right] \times 3!$. **Ans.**

BEGINNER'S BOX–3

1. An old man while dialing a 7 digit telephone number remembers that the first four digits consists of one 1's, one 2's and two 3's. He also remembers that the fifth digit is either a 4 or 5 while has no memorising of the sixth digit, he remembers that the seventh digit is 9 minus the sixth digit. Maximum number of distinct trials he has to try to make sure that he dials the correct telephone number, is

(A) 360 (B) 240 (C) 216 (D) none

2. A committee of 5 is to be chosen from a group of 9 people. Number of ways in which it can be formed if two particular persons either serve together or not at all and two other particular persons refuse to serve with each other, is

(A) 41 (B) 36 (C) 47 (D) 76

3. A women has 11 close friends. Find the number of ways in which she can invite 5 of them to dinner, if two particular of them are not on speaking terms & will not attend together.

4. In a certain algebraical exercise book there are 4 examples on arithmetical progressions, 5 examples on permutation-combination and 6 examples on binomial theorem . Number of ways a teacher can select for his pupils atleast one but not more than 2 examples from each of these sets, is ______.

5. If m denotes the number of 5 digit numbers if each successive digits are in their descending order of magnitude and n is the corresponding figure, when the digits are in their ascending order of magnitude then $(m - n)$ has the value

(A) ${}^{10}C_4$ (B) ${}^{9}C_5$ (C) ${}^{10}C_3$ (D) ${}^{9}C_3$

6. There are m points on a straight line AB & n points on the line AC none of them being the point A. Triangles are formed with these points as vertices, when

(i) A is excluded (ii) A is included. The ratio of number of triangles in the two cases is:

(A) $\frac{m+n-2}{m+n}$ (B) $\frac{m+n-2}{m+n-1}$ (C) $\frac{m+n-2}{m+n+2}$ (D) $\frac{m\,(n-1)}{(m+1)\,(n+1)}$

7. A rack has 5 different pairs of shoes. The number of ways in which 4 shoes can be chosen from it , so that there will be no complete pair is :

(A) 1920 (B) 200 (C) 110 (D) 80

8. A library has *a* copies of one book, *b* copies of each of two books, *c* copies of each of three books and single copies of *d* books. The total number of ways in which these books can be distributed is

(A) $\frac{(a+b+c+d)!}{a!b!c!}$ (B) $\frac{(a+2b+3c+d)!}{a!(b!)^2\,(c!)^3}$ (C) $\frac{(a+2b+3c+d)!}{a!b!c!}$ (D) None of these

9. The number of ways in which *mn* students can be distributed equally among *n* sections is

(A) $(mn)^n$ (B) $\frac{(mn)!}{(m!)^n}$ (C) $\frac{mn}{m!}$ (D) $\frac{mn}{m!n!}$

10. In how many ways can 10 balls be divided between two boys, one of them receiving two and the other eight balls

(A) 45 (B) 75 (C) 90 (D) None of these

11. The number of ways in which six different prizes can be distributed among three children each receiving at least one prize is

(A) 270 (B) 540 (C) 1080 (D) 2160

12. Choose the correct number of ways in which 15 different books can be divided into five heaps of equal number of books

(A) $\frac{15!}{5!(3!)^5}$ (B) $\frac{15!}{(3!)^5}$ (C) ${}^{15}C_5$ (D) ${}^{15}P_5$

13. Number of ways in which 8 people can be arranged in a line if A and B must be next each other and C must be somewhere behind D, is equal to

(A) 10080 (B) 5040 (C) 5050 (D) 10100

14. Seven different coins are to be divided amongst three persons . If no two of the persons receive the same number of coins but each receives atleast one coin & none is left over, then the number of ways in which the division may be made is

(A) 420 (B) 630 (C) 710 (D) none

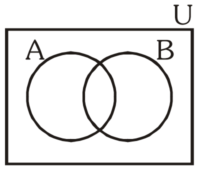
15. The number of ways in which 8 distinguishable apples can be distributed among 3 boys such that every boy should get atleast 1 apple & atmost 4 apples is $K \cdot {}^7P_3$ where K has the value equal to

(A) 14 (B) 66 (C) 44 (D) 22

16. Number of ways in which 9 different toys be distributed among 4 children belonging to different age groups in such a way that distribution among the 3 elder children is even and the youngest one is to receive one toy more, is :

(A) $\frac{(5\,!)^2}{8}$ (B) $\frac{9\,!}{2}$ (C) $\frac{9\,!}{3\,!\,(2\,!)^3}$ (D) none

5.0 PRINCIPLE OF INCLUSION AND EXCLUSION

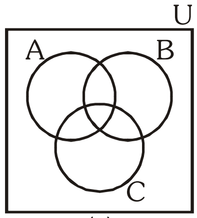


(i)

In the Venn's diagram (i), we get

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A' \cap B') = n(U) - n(A \cup B)$



(ii)

In the Venn's diagram (ii), we get

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

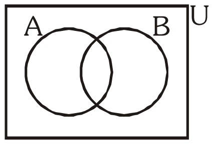
$n(A' \cap B' \cap C') = n(U) - n(A \cup B \cup C)$

In general, we have $n(A_1 \cup A_2 \cup .... \cup A_n)$

$$= \sum n(A_i) - \sum_{i \neq j} n(A_i \cap A_j) + \sum_{i \neq j \neq k} n(A_i \cap A_j \cap A_k) + ... + (-1)^n \sum n(A_1 \cap A_2 \cap ... \cap A_n)$$

Illustrations

Illustration 15. Find the number of permutations of letters a,b,c,d,e,f,g taken all at a time if neither 'beg' nor 'cad' pattern appear.



Solution The total number of permutations without any restrictions; $n(U) = 7!$

b e g a c d f

Let A be the set of all possible permutations in which 'beg' pattern always appears : $n(A) = 5!$

c a d b e f g

Let B be the set of all possible permutations in which 'cad' pattern always appears : $n(B) = 5!$

c a d b e g f

$n(A \cap B)$: Number of all possible permutations when both 'beg' and 'cad' patterns appear.

$n(A \cap B) = 3!$.

Therefore, the total number of permutations in which 'beg' and 'cad' patterns do not appear

$n(A' \cap B') = n(U) - n(A \cap B) = n(U) - n(A) - n(B) + n(A \cap B)$

$= \quad 7! - 5! - 5! + 3!$. **Ans.**

6.0 PERMUTATIONS OF ALIKE OBJECTS

6.1 Taken all at a time

The number of permutations of n things taken all at a time : when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining $n - (p + q + r)$ are all different is : $\dfrac{n!}{p!\ q!\ r!}$.

Illustrations

Illustration 16. In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative position of vowels & consonants.

Solution The consonants in their positions can be arranged in $\dfrac{4!}{2!} = 12$ ways.

The vowels in their positions can be arranged in $\dfrac{3!}{2!} = 3$ ways

$\therefore$ Total number of arrangements $= 12 \times 3 = 36$ **Ans.**

Illustration 17. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?

(A) 17 (B) 18 (C) 19 (D) 20

Solution There are 4 odd digits (1, 1, 3, 3) and 4 odd places (first, third, fifth and seventh). At these places the odd digits can be arranged in $\frac{4!}{2!2!} = 6$ ways

Then at the remaining 3 places, the remaining three digits (2, 2, 4) can be arranged in $\frac{3!}{2!} = 3$ ways

$\therefore$ The required number of numbers $= 6 \times 3 = 18$. **Ans. (B)**

Illustration 18. (A) How many permutations can be made by using all the letters of the word HINDUSTAN ?

(B) How many of these permutations begin and end with a vowel ?

(C) In how many of these permutations, all the vowels come together ?

(D) In how many of these permutations, none of the vowels come together ?

(E) In how many of these permutations, do the vowels and the consonants occupy the same relative positions as in HINDUSTAN ?

Solution (A) The total number of permutations = Arrangements of nine letters taken all at a time $= \frac{9!}{2!} = 181440$.

(B) We have 3 vowels and 6 consonants, in which 2 consonants are alike. The first place can be filled in 3 ways and the last in 2 ways. The rest of the places can be filled in $\frac{7!}{2!}$ ways.

Hence the total number of permutations $= 3 \times 2 \times \frac{7!}{2!} = 15120$.

(C) Assume the vowels (I, U, A) as a single letter. The letters (IUA), H, D, S, T, N, N can be arranged in $\frac{7!}{2!}$ ways. Also IUA can be arranged among themselves in $3! = 6$ ways.

Hence the total number of permutations $= \frac{7!}{2!} \times 6 = 15120$.

(D) Let us divide the task into two parts. In the first, we arrange the 6 consonants as shown below in $\frac{6!}{2!}$ ways.

$\times\, C \times C \times C \times C \times C \times C \times$ (Here C stands for a consonant and $\times$ stands for a gap between two consonants)

Now 3 vowels can be placed in 7 places (gaps between the consonants) in ${}^7C_3.3! = 210$ ways.

Hence the total number of permutations $= \frac{6!}{2!} \times 210 = 75600$.

(E) In this case, the vowels can be arranged among themselves in $3! = 6$ ways.

Also, the consonants can be arranged among themselves in $\frac{6!}{2!}$ ways.

Hence the total number of permutations $= \frac{6!}{2!} \times 6 = 2160$. **Ans.**

Illustration 19. If all the letters of the word 'PROPER' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'PROPER' .

Solution First of all, arrange all letters of given word alphabetically : EOPPRR

Total number of words starting with-

E _ _ _ _ _ $= \frac{5!}{2!2!} = 30$ O _ _ _ _ _ $= \frac{5!}{2!2!} = 30$

PE _ _ _ _ $= \frac{4!}{2!} = 12$ PO _ _ _ _ $= \frac{4!}{2!} = 12$

PP _ _ _ _ $= \frac{4!}{2!} = 12$ PRE _ _ _ $= 3! = 6$

PROE _ _ $= 2! = 2$

PROPER $= 1 = 1$

Rank of the word PROPER = 105 **Ans.**

6.2 Taken some at a time

Illustrations

Illustration 20. Find the total number of 4 letter words formed using four letters from the word "PARALLELOPIPED".

Solution Given letters are PPP, LLL, AA, EE, R, O, I, D.

Cases	No. of ways of selection	No. of ways of arrangements	Total
All distinct	$^8C_4$	$^8C_4 \times 4!$	1680
2 alike, 2 distinct	$^4C_1 \times {}^7C_2$	$^4C_1 \times {}^7C_2 \times \frac{4!}{2!}$	1008
2 alike, 2 other alike	$^4C_2$	$^4C_2 \times \frac{4!}{2!2!}$	36
3 alike, 1 distinct	$^2C_1 \times {}^7C_1$	$^2C_1 \times {}^7C_1 \times \frac{4!}{3!}$	56
		Total	2780

Ans.

Illustration 21. Find the number of all 6 digit numbers such that all the digits of each number are selected from the set {1,2,3,4,5} and any digit that appears in the number appears at least twice.

Solution

Cases	No. of ways of selection	No. of ways of arrangements	Total
All alike	$^5C_1$	$^5C_1 \times 1$	5
4 alike + 2 other alike	$^5C_2 \times 2!$	$^5C_2 \times 2 \times \frac{6!}{2!4!}$	300
3 alike + 3 other alike	$^5C_2$	$^5C_2 \times \frac{6!}{3!3!}$	200
2 alike + 2 other alike +2 other alike	$^5C_3$	$^5C_3 \times \frac{6!}{2!2!2!}$	900
		Total	1405

Ans.

BEGINNER'S BOX-4

1. Number of different natural numbers which are smaller than two hundred million & using only the digits 1 or 2 is :

(A) $(3).2^8-2$ (B) $(3).2^8-1$ (C) $2(2^9-1)$ (D) none

2. The number of permutations of the letters xxyyyyzzz will be

(A) $\frac{9!}{2!4!}$ (B) $\frac{9!}{2!4!3!}$ (C) $\frac{9!}{4!3!}$ (D) $9!$

3. The number of different arrangements which can be made from the letters of the word SERIES taken all together is

(A) $\frac{6!}{2!2!}$ (B) $\frac{6!}{4!}$ (C) $6!$ (D) None of these

4. How many words can be formed with the letters of the word MATHEMATICS by rearranging them

(A) $\frac{11!}{2!2!}$ (B) $\frac{11!}{2!}$ (C) $\frac{11!}{2!2!2!}$ (D) $11!$

5. How many different nine-digit numbers can be formed from the digits of the number 223355888 by rearrangement of the digits so that the odd digits occupy even places

(A) 16 (B) 36 (C) 60 (D) 180

6 A shelf contains 20 different books of which 4 are in single volume and the others form sets of 8, 5 and 3 volumes respectively. Number of ways in which the books may be arranged on the shelf, if the volumes of each set are together and in their due order is

(A) $\frac{20!}{8!\,5!\,3!}$ (B) 7! (C) 8! (D) 7 . 8!

7. In how many ways can the letters of the word 'ALLEN' be arranged ? Also find its rank if all these words are arranged as they are in dictionary.

8. The number of ways of arranging the letter AAAAA BBB CCC D EE F in a row when no two C's are together is

(A) $\frac{15!}{5!3!3!2!}-3!$ (B) $\frac{15!}{5!3!3!2!}-\frac{13!}{5!3!2!}$ (C) $\frac{12!}{5!3!2!}\times\frac{^{13}P_3}{3!}$ (D) $\frac{12!}{5!3!\,2!}\times{}^{13}P_3$

9. Find the number of permutations of the word "AUROBIND" in which vowels appear in an alphabetical order.

10 In how many ways the letters of the word "COMBINATORICS" can be arranged if

(i) All the vowels are always grouped together to form a contiguous block.

(ii) All vowels and all consonants are alphabetically ordered.

11. There are six periods in each working day of a school. Number of ways in which 5 subjects can be arranged if each subject is allotted at least one period and no period remains vacant is

(A) 210 (B) 1800 (C) 360 (D) 3600

12. The number of ways in which 6 rings can be worn on the four fingers of one hand is

(A) 4^6 (B) $^6C_4$ (C) 6^4 (D) None of these

BEGINNER'S BOX–5

1. There are 10 red balls of different shades & 9 green balls of identical shades . Then the number of arranging them in a row so that no two green balls are together is :

(A) $(10!) . {}^{11}P_9$ (B) $(10!) . {}^{11}C_9$ (C) $10!$ (D) $10!\ 9!$

2. The number of combination of 16 things, 8 of which are alike and the rest different, taken 8 at a time is ______.

3. Number of different words that can be formed using all the letters of the word "DEEPMALA" if two vowels are together and the other two are also together but separated from the first two is

(A) 960 (B) 1200 (C) 2160 (D) 1440

4. Find the number of 4 digit numbers starting with 1 and having exactly two identical digits.

5. Number of ways in which 7 green bottles and 8 blue bottles can be arranged in a row if exactly 1 pair of green bottles is side by side, is (Assume all bottles to be alike except for the colour).

(A) 84 (B) 360 (C) 504 (D) none

6. There are n identical red balls & m identical green balls . The number of different linear arrangements consisting of "n red balls but not necessarily all the green balls" is ${}^{x}C_y$ then

(A) $x = m + n,\ y = m$ (B) $x = m + n + 1,\ y = m$

(C) $x = m + n + 1, y = m + 1$ (D) $x = m + n,\ y = n$

7. The total number of ways of selecting five letters from the letters of the word 'INDEPENDENT' is

(A) 72 (B) 3320 (C) 120 (D) None of these

8. Messages are conveyed by arranging 4 white, 1 blue and 3 red flags on a pole. Flags of the same colour are alike. If a message is transmitted by the order in which the colours are arranged then the total number of messages that can be transmitted if exactly 6 flags are used is

(A) 45 (B) 65 (C) 125 (D) 185

9. An ice cream parlour has ice creams in eight different varieties . Number of ways of choosing 3 ice creams taking atleast two ice creams of the same variety, is :

(Assume that ice creams of the same variety are identical & available in unlimited supply)

(A) 56 (B) 64 (C) 100 (D) none

10. Consider the word W = MISSISSIPPI

(A) If N denotes the number of different selections of 5 letters from the word W = MISSISSIPPI then N belongs to the set

(A) {15, 16, 17, 18, 19} (B) {20, 21, 22, 23, 24}

(C) {25, 26, 27, 28, 29} (D) {30, 31, 32, 33, 34}

(B) Number of ways in which the letters of the word W can be arranged if atleast one vowel is separated from rest of the vowels

(A) $\frac{8!\cdot 161}{4!\cdot 4!\cdot 2!}$ (B) $\frac{8!\cdot 161}{4\cdot 4!\cdot 2!}$ (C) $\frac{8!\cdot 161}{4!\cdot 2!}$ (D) $\frac{8!}{4!\cdot 2!}\cdot\frac{165}{4!}$

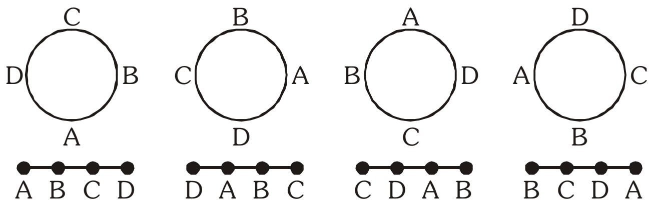
(C) If the number of arrangements of the letters of the word W if all the S's and P's are separated is (K) $\left(\frac{10!}{4!\cdot 4!}\right)$, then K equals -

(A) $\frac{6}{5}$ (B) 1 (C) $\frac{4}{3}$ (D) $\frac{3}{2}$

11. The number of different ways in which five 'dashes' and eight 'dots' can be arranged, using only seven of these 13 'dashes' & 'dots' is :

(A) 1287 (B) 119 (C) 120 (D) 1235520

7.0 CIRCULAR PERMUTATION



(a) (b) (c) (d)

Let us consider that persons A,B,C,D are sitting around a round table. If all of them (A,B,C,D) are shifted by one place in anticlockwise order, then we will get Fig.(B) from Fig.(A). Now, if we shift A,B,C,D in anticlockwise order, we will get Fig.(C). Again, if we shift them, we will get Fig.(D) and in the next time, Fig.(A).

Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements, thus obtained will be the same, because anticlockwise order of A,B,C,D does not change.

But if A,B,C,D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 things, then for each circular arrangement number of linear arrangements is 4.

Similarly, if n different things are arranged along a circle, for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things is n $\times$ (number of circular arrangements of n different things). Hence, the number of circular arrangements of n different things is -

1/n $\times$ (number of linear arrangements of n different things) $= \frac{n!}{n} = (n-1)!$

Therefore note that –

(i) The number of circular permutations of n different things taken all at a time is : $(n-1)!$.

If clockwise & anti-clockwise circular permutations are considered to be same, then it is : $\frac{(n-1)!}{2}$.

(ii) The number of circular permutations of n different things taking r at a time distinguishing clockwise & anticlockwise arrangements is : $\frac{{}^{n}P_{r}}{r}$

Illustrations

Illustration 22. In how many ways can 5 boys and 5 girls be seated at a round table so that no two girls are together?

(A) $5! \times 5!$ (B) $5! \times 4!$ (C) $\frac{1}{2}(5!)^2$ (D) $\frac{1}{2}(5! \times 4!)$

Solution Leaving one seat vacant between two boys, 5 boys may be seated in 4! ways. Then at remaining 5 seats, 5 girls sit in 5! ways. Hence the required number of ways $= 4! \times 5!$ **Ans. (B)**

Illustration 23. The number of ways in which 7 girls can stand in a circle so that they do not have same neighbours in any two arrangements ?

(A) 720 (B) 380 (C) 360 (D) none of these

Solution Seven girls can stand in a circle by $\frac{(7-1)!}{2!}$ number of ways, because there is no difference in anticlockwise and clockwise order of their standing in a circle.

$\therefore \quad \frac{(7-1)!}{2!} = 360$ **Ans. (C)**

Illustration 24. The number of ways in which 20 different pearls of two colours can be set alternately on a necklace, there being 10 pearls of each colour, is

(A) $9! \times 10!$ (B) $5(9!)^2$ (C) $(9!)^2$ (D) none of these

Solution Ten pearls of one colour can be arranged in $\frac{1}{2}.(10-1)!$ ways. The number of arrangements of 10 pearls of the other colour in 10 places between the pearls of the first colour = 10!

$\therefore$ The required number of ways $=\frac{1}{2}\times 9!\times 10! = 5\,(9!)^2$ **Ans. (B)**

Illustration 25. A person invites a group of 10 friends at dinner. They sit

(i) 5 on one round table and 5 on other round table,

(ii) 4 on one round table and 6 on other round table.

Find the number of ways in each case in which he can arrange the guests.

Solution : (i) The number of ways in which 10 persons can be divided into two groups of five person is $\frac{10!}{5!\times 5!\times 2!}$

The total number of permutations of 5 guests at a round table is 4!. Hence, the total number of arrangements is $\frac{10!}{5!\times 5!\times 2!}\times 4!\times 4! = \frac{10!4!4!}{5!5!2!}=\frac{10!}{50}$

(ii) The number of ways of selection of 6 guests is $^{10}C_6$.

The number of ways of permutations of 6 guests on round table is 5!. The number of permutations of 4 guests on round table is 3!

Therefore, total number of arrangements is : $^{10}C_6 5!\times 3! = \frac{(10)!}{6!4!}5!3! = \frac{(10)!}{24}$ **Ans.**

8.0 TOTAL NUMBER OF COMBINATIONS

(A) Given n different objects , the number of ways of selecting atleast one of them is, $^nC_1 + {^nC_2} + {^nC_3} + \dots\dots + {^nC_n} = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(B) (i) Total number of ways in which it is possible to make a selection by taking some or all out of p + q + r +......things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by : (p + 1) (q + 1) (r + 1).........–1.

(ii) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is given by :

$(p + 1)(q + 1)(r + 1)\,2^n - 1$

Illustrations

Illustration 26. A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q = \phi$ is :-

(A) $2^{2n} - {^{2n}C_n}$ (B) 2^n (C) $2^n - 1$ (D) 3^n

Solution Let $A = \{a_1, a_2, a_3, \dots\dots a_n\}$. For $a_i \in A$, we have the following choices :

(i) $a_i \in P$ and $a_i \in Q$ (ii) $a_i \in P$ and $a_i \notin Q$

(iii) $a_i \notin P$ and $a_i \in Q$ (iv) $a_i \notin P$ and $a_i \notin Q$

Out of these only (ii), (iii) and (iv) imply $a_i \notin P \cap Q$. Therefore, the number of ways in which none of $a_1, a_2, \dots a_n$ belong to $P \cap Q$ is 3^n. **Ans.(D)**

Illustration 27. A student is allowed to select at most n books from a collection of (2n + 1) books. If the total number of ways in which he can select books is 63, find the value of n.

Solution Given student selects at most n books from a collection of (2n + 1) books. It means that he selects one book or two books or three books or .... or n books. Hence, by the given condition-

$^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n = 63$ ...(i)

But we know that

$^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_{2n+1} = 2^{2n+1}$ ...(ii)

Since $^{2n+1}C_0 = {}^{2n+1}C_{2n+1} = 1$, equation (ii) can also be written as

$2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n) +$

$({}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + {}^{2n+1}C_{n+3} + ... + {}^{2n+1}C_{2n-1} + {}^{2n+1}C_{2n}) = 2^{2n+1}$

$\Rightarrow 2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n)$

$+ ({}^{2n+1}C_n + {}^{2n+1}C_{n-1} + ... + {}^{2n+1}C_2 + {}^{2n+1}C_1) = 2^{2n+1}$

$(\because {}^{2n+1}C_r = {}^{2n+1}C_{2n+1-r})$

$\Rightarrow 2 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n) = 2^{2n+1}$ [from (i)]

$\Rightarrow 2 + 2.63 = 2^{2n+1}$ $\Rightarrow 1 + 63 = 2^{2n}$

$\Rightarrow 64 = 2^{2n} \Rightarrow 2^6 = 2^{2n}$ $\therefore 2n = 6$

Hence, n = 3. **Ans.**

Illustration 28. There are 3 different books of mathematics, 4 different books of science and 5 different books of english. How many different collections can be made such that each collection consists of-

(i) one book of each subject ?

(ii) at least one book of each subject ?

(iii) at least one book of english ?

Solution (i) $^3C_1 \times {}^4C_1 \times {}^5C_1 = 60$

(ii) $(2^3-1)(2^4-1)(2^5-1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5-1)(2^3)(2^4) = 31 \times 128 = 3968$ **Ans.**

Illustration 29. Find the number of groups that can be made from 5 red balls, 3 green balls and 4 black balls, if at least one ball of all colours is always to be included. Given that all balls are identical except colours.

Solution After selecting one ball of each colour, we have to find total number of combinations that can be made from 4 red. 2 green and 3 black balls. These will be (4 + 1) (2 + 1) (3 + 1) = 60 **Ans.**

9.0 DIVISORS

Let $N = p^a . q^b . r^c$ .... where p, q, r........ are distinct primes & a, b, c....... are natural numbers then :

(A) The total numbers of divisors of N including 1 & N is = (a + 1) (b + 1) (c + 1).......

(B) The sum of these divisors is = $(p^0 + p^1 + p^2 + ...+ p^a)(q^0 + q^1 + q^2 + ...+ q^b)(r^0 + r^1 + r^2 + ...+ r^c)$...

(C) Number of ways in which N can be resolved as a product of two factor is =

$\frac{1}{2}(a+1)(b+1)(c+1)......$ if N is not a perfect square

$\frac{1}{2}[(a+1)(b+1)(c+1)......+1]$ if N is a perfect square

(D) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

Illustrations

Illustration 30. Find the number of proper divisors of the number 38808. Also find the sum of these divisors.

Solution

(i) The number $38808 = 2^3 . 3^2 . 7^2 . 11$

Hence the total number of divisors (excluding 1 and itself i.e.38808)

$= (3 + 1)(2 + 1)(2 + 1)(1 + 1) - 2 = 70$

(ii) The sum of these divisors

$=(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)(7^0 + 7^1 + 7^2)(11^0 + 11^1) - 1 - 38808$

$= (15)(13)(57)(12) - 1 - 38808 = 133380 - 1 - 38808 = 94571.$ **Ans.**

Illustration 31. In how many ways the number 18900 can be split in two factors which are relative prime (or coprime) ?

Solution Here $N = 18900 = 2^2 . 3^3 . 5^2 . 7^1$

Number of different prime factors in $18900 = n = 4$

Hence number of ways in which 18900 can be resolved into two factors which are relative prime (or coprime) $= 2^{4-1} = 2^3 = 8.$ **Ans.**

Illustration 32. Find the total number of proper factors of the number 35700. Also find

(i) sum of all these factors,

(ii) sum of the odd proper divisors,

(iii) the number of proper divisors divisible by 10 and the sum of these divisors.

Solution $35700 = 5^2 \times 2^2 \times 3^1 \times 7^1 \times 17^1$

The total number of factors is equal to the total number of selections from (5,5), (2,2), (3), (7) and (17), which is given by $3 \times 3 \times 2 \times 2 \times 2 = 72$.

These include 1 and 35700. Therefore, the number of proper divisors (excluding 1 and 35700) is $72 - 2 = 70$

(i) Sum of all these factors (proper) is :

$(5^0 + 5^1 + 5^2)(2^0 + 2^1 + 2^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 1 - 35700$

$= 31 \times 7 \times 4 \times 8 \times 18 - 1 - 35700 = 89291$

(ii) The sum of odd proper divisors is :

$(5^0 + 5^1 + 5^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 1$

$= 31 \times 4 \times 8 \times 18 - 1 = 17856 - 1 = 17855$

(iii) The number of proper divisors divisible by 10 is equal to number of selections from (5,5), (2,2), (3), (7), (17) consisting of at least one 5 and at least one 2 and 35700 is to be excluded and is given by $2 \times 2 \times 2 \times 2 \times 2 - 1 = 31$.

Sum of these divisors is :

$(5^1 + 5^2)(2^1 + 2^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 35700$

$= 30 \times 6 \times 4 \times 8 \times 18 - 35700 = 67980$ **Ans.**

GOLDEN KEY POINTS

- Every natural number except 1 has atleast 2 divisors. If it has exactly two divisors then it is called a prime. System of prime numbers begin with 2. All primes except 2 are odd.
- A number having more than 2 divisors is called composite. 2 is the only even number which is not composite.
- Two natural numbers are said to be relatively prime or coprime if their HCF is one. For two natural numbers to be relatively prime, it is not necessary that one or both should be prime. It is possible that they both are composite but still coprime, eg. 4 and 25.
- 1 is neither prime nor composite however it is co-prime with every other natural number.
- Two prime numbers are said to be twin prime numbers if their non-negative difference is 2 (e.g.5 & 7, 19 & 17 etc).
- All divisors except 1 and the number itself are called proper divisors.

BEGINNER'S BOX-6

1. n gentlemen can be made to sit on a round table in

(A) $\frac{1}{2}(n+1)!$ ways (B) $(n-1)!$ ways (C) $\frac{1}{2}(n-1)!$ ways (D) $(n+1)!$ ways

2. In how many ways can 12 gentlemen sit around a round table so that three specified gentlemen are always together

(A) 9 ! (B) 10 ! (C) 3 ! 10! (D) 3 ! 9 !

3. In how many ways can 15 members of a council sit along a circular table, when the Secretary is to sit on one side of the Chairman and the Deputy secretary on the other side

(A) $2\times 12!$ (B) 24 (C) 2×15 ! (D) None of these

4. 12 persons are to be arranged to a round table. If two particular persons among them are not to be side by side, the total number of arrangements is

(A) 9(10 !) (B) 2 (10 !) (C) 45 (8 !) (D) 10 !

5. There are 20 persons among whom two are brothers . The number of ways in which we can arrange them round a circle so that there is exactly one person between the two brothers , is

(A) 18 ! (B) 2 (18!) (C) 2 (19 !) (D) None of these

6. A gentleman invites a party of $m + n$ $(m \neq n)$ friends to a dinner & places m at one table T_1 and n at another table T_2 , the table being round . If not all people shall have the same neighbour in any two arrangement, then the number of ways in which he can arrange the guests, is

(A) $\frac{(m+n)!}{4\ mn}$ (B) $\frac{1}{2}\frac{(m+n)!}{mn}$ (C) $2\frac{(m+n)!}{mn}$ (D) none

7. Six persons A, B, C, D, E and F are to be seated at a circular table. The number of ways this can be done if A must have either B or C on his right and B must have either C or D on his right is :

(A) 36 (B) 12 (C) 24 (D) 18

8. The number of proper divisors of $2^p.6^q.15^r$ is

(A) $(p+q+1)(q+r+1)\ (r+1)$ (B) $(p+q+1)(q+r+1)(r+1)-2$

(C) $(p+q)(q+r)r-2$ (D) None of these

9. The number of odd proper divisors of $3^p.6^m.21^n$ is

(A) $(p+1)(m+1)(n+1)-2$ (B) $(p+m+n+1)(n+1)-1$

(C) $(p+1)(m+1)(n+1)-1$ (D) None of these

10. The number of even proper divisors of 1008 is

(A) 23 (B) 24 (C) 22 (D) None of these

11. The number of divisors of 1800 which are also divisible by 10, is

(A) 18 (B) 34 (C) 27 (D) None of these

12. The number of all possible selections of one or more questions from 10 given questions, each equestion having an alternative is :

(A) 3^{10} (B) $2^{10}-1$ (C) $3^{10}-1$ (D) 2^{10}

13. A team of 8 students goes on an excursion, in two cars, of which one can seat 5 and the other only 4. If internal arrangement inside the car does not matter then the number of ways in which they can travel, is

(A) 91 (B) 182 (C) 126 (D) 3920

14. In a unique hockey series between India & Pakistan, they decide to play on till a team wins 5 matches. The number of ways in which the series can be won by India, if no match ends in a draw is :

(A) 126 (B) 252 (C) 225 (D) none

15. A four digit number is called a doublet if any of its digit is the same as only one neighbour. For example, 1221 is a doublet but 1222 is not. Number of such doublets are

(A) 2259 (B) 2268 (C) 2277 (D) 2349

10.0 TOTAL DISTRIBUTION

(A) Distribution of distinct objects – Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them is given by : p^n

(B) Distribution of alike objects – Number of ways to distribute n alike things among p persons so that each may get none, one or more thing(s) is given by ${}^{n+p-1}C_{p-1}$.

Illustrations

Illustration 33. In how many ways can 5 different mangoes, 4 different oranges & 3 different apples be distributed among 3 children such that each gets alteast one mango ?

Solution 5 different mangoes can be distributed by following ways among 3 children such that each gets atleast 1 :

3 1 1

2 2 1

Total number of ways : $\left(\dfrac{5!}{3!1!1!2!}+\dfrac{5!}{2!2!2!}\right)\times 3!$

Now, the number of ways of distributing remaining fruits (i.e. 4 oranges + 3 apples) among 3 children = 3^7 (as each fruit has 3 options).

$\therefore$ Total number of ways $=\left(\dfrac{5!}{3!2!}+\dfrac{5!}{(2!)^3}\right)\times 3!\times 3^7$ **Ans.**

Illustration 34. In how many ways can 12 identical apples be distributed among four children if each gets atleast 1 apple and not more than 4 apples.

Solution Let x,y,z & w be the number of apples given to the children.

$\Rightarrow \quad x + y + z + w = 12$

Giving one-one apple to each

Now, $x + y + z + w = 8$ ....(i)

Here, $0 \le x \le 3,\ 0 \le y \le 3,\ 0 \le z \le 3,\ 0 \le w \le 3$

$x = 3 - t_1,\ y = 3 - t_2,\ z = 3 - t_3,\ w = 3 - t_4.$

Putting value of x, y, z, w in equation (i)

Put $12 - 8 = t_1 + t_2 + t_3 + t_4$

$\Rightarrow \quad t_1 + t_2 + t_3 + t_4 = 4$

(Here max. value that t_1, t_2, t_3 & t_4 can attain is 3, so we have to remove those cases when any of t_i getting value 4)

$= \quad {}^7C_3 -$ (all cases when atleast one is 4)

$= \quad {}^7C_3 - 4 = 35 - 4 = 31$ **Ans.**

Illustration 35. Find the number of non negative integral solutions of the inequation $x + y + z \le 20$.

Solution Let w be any number ($0 \le w \le 20$), then we can write the equation as :

$x + y + z + w = 20$ (here $x, y, z, w \ge 0$)

Total ways = ${}^{23}C_3$ **Ans.**

Illustration 36. Find the number of integral solutions of $x + y + z + w < 25$, where $x > -2, y > 1, z \ge 2, w \ge 0$.

Solution Given $x + y + z + w < 25$

$x + y + z + w + v = 25$ ....(i)

Let $x = -1 + t_1,\ y = 2 + t_2,\ z = 2 + t_3,\ w = t_4,\ v = 1 + t_5$ where $(t_1, t_2, t_3, t_4 \ge 0)$

Putting value of x, y, z, w, v in equation (i)

$\Rightarrow \quad t_1 + t_2 + t_3 + t_4 + t_5 = 21.$

Number of solutions = ${}^{25}C_4$ **Ans.**

Illustration 37. Find the number of positive integral solutions of the inequation $x + y + z \geq 150$, where $0 < x \leq 60$, $0 < y \leq 60$, $0 < z \leq 60$.

Solution Let $x = 60 - t_1$, $y = 60 - t_2$, $z = 60 - t_3$ (where $0 \leq t_1 \leq 59$, $0 \leq t_2 \leq 59$, $0 \leq t_3 \leq 59$)

Given $x + y + z \geq 150$

or $x + y + z - w = 150$ (where $0 \leq w \leq 30$) ....(i)

Putting values of x, y, z in equation (i)

$60 - t_1 + 60 - t_2 + 60 - t_3 - w = 150$

$30 = t_1 + t_2 + t_3 + w$

Total solutions $= {}^{33}C_3$ **Ans.**

Illustration 38. Find the number of positive integral solutions of $xy = 12$

Solution $xy = 12$

$xy = 2^2 \times 3^1$

(i) 3 has 2 ways either 3 can go to x or y

(ii) 2^2 can be distributed between x & y as distributing 2 identical things between 2 persons (where each person can get 0, 1 or 2 things). Let two person be ℓ_1 & ℓ_2

$\Rightarrow$ $\ell_1 + \ell_2 = 2$

$\Rightarrow$ ${}^{2+1}C_1 = {}^3C_1 = 3$

So total ways $= 2 \times 3 = 6$.

Alternatively –

$xy = 12 = 2^2 \times 3^1$

$x = 2^{a_1}3^{a_2}$ $\quad 0 \leq a_1 \leq 2$

$0 \leq a_2 \leq 1$

$y = 2^{b_1}3^{b_2}$ $\quad 0 \leq b_1 \leq 2$

$0 \leq b_2 \leq 1$

$2^{a_1+b_1}3^{a_2+b_2} = 2^2 3^1$

$\Rightarrow$ $a_1 + b_1 = 2 \rightarrow {}^3C_1$ ways

$a_2 + b_2 = 1 \rightarrow {}^2C_1$ ways

Number of solutions $= {}^3C_1 \times {}^2C_1 = 3 \times 2 = 6$ **Ans.**

Illustration 39. Find the number of solutions of the equation $xyz = 360$ when **(i)** $x,y,z \in N$ **(ii)** $x,y,z \in I$

Solution **(i)** $xyz = 360 = 2^3 \times 3^2 \times 5$ $(x,y,z \in N)$

$x = 2^{a_1}3^{a_2}5^{a_3}$ (where $0 \leq a_1 \leq 3$, $0 \leq a_2 \leq 2$, $0 \leq a_3 \leq 1$)

$y = 2^{b_1}3^{b_2}5^{b_3}$ (where $0 \leq b_1 \leq 3$, $0 \leq b_2 \leq 2$, $0 \leq b_3 \leq 1$)

$z = 2^{c_1}3^{c_2}5^{c_3}$ (where $0 \leq c_1 \leq 3$, $0 \leq c_2 \leq 2$, $0 \leq c_3 \leq 1$)

$\Rightarrow$ $2^{a_1}3^{a_2}5^{a_3}.2^{b_1}3^{b_2}5^{b_3}.2^{c_1}3^{c_2}5^{c_3} = 2^3 \times 3^2 \times 5^1$

$\Rightarrow$ $2^{a_1+b_1+c_1}.3^{a_2+b_2+c_2}.5^{a_3+b_3+c_3} = 2^3 \times 3^3 \times 5^1$

$\Rightarrow$ $a_1 + b_1 + c_1 = 3 \rightarrow {}^5C_2 = 10$

$a_2 + b_2 + c_2 = 2 \rightarrow {}^4C_2 = 6$

$a_3 + b_3 + c_3 = 1 \rightarrow {}^3C_2 = 3$

Total solutions $= 10 \times 6 \times 3 = 180$.

(ii) If $x,y,z \in I$ then, (A) all positive (B) 1 positive and 2 negative.

Total number of ways $= 180 + {}^3C_2 \times 180 = 720$ **Ans.**

BEGINNER'S BOX-7

1. In how many ways can 12 identical apples be distributed among 4 boys. (A) If each boy receives any number of apples. (B) If each boy receives atleast 2 apples.

2. Find the number of non-negative integral solutions of the equation $x + y + z = 10$.

3. Find the number of integral solutions of $x + y + z = 20$, if $x \geq -4$, $y \geq 1$, $z \geq 2$

4. The number of way in which 10 identical apples can be distributed among 6 children so that each child receives atleast one apple is -

(A) 126 (B) 252 (C) 378 (D) none of these

5. The number of ways in which thirty five apples can be distributed among 3 boys so that each can have any number of apples, is

(A) 1332 (B) 666 (C) 333 (D) None of these

6. Number of ways in which 25 identical pens can be distributed among Keshav, Madhav, Mukund and Radhika such that at least 1, 2, 3 and 4 pens are given to Keshav, Madhav, Mukund and Radhika respectively, is -

(A) ${}^{18}C_4$ (B) ${}^{28}C_3$ (C) ${}^{24}C_3$ (D) ${}^{18}C_3$

7. A lift with 7 people stops at 10 floors. People varying from zero to seven go out at each floor. The number of ways in which the lift can get emptied, assuming each way only differs by the number of people leaving at each floor, is

(A) ${}^{16}C_6$ (B) ${}^{17}C_7$ (C) ${}^{16}C_7$ (D) none

8. Number of ways in which four different toys and five indistinguishable marbles can be distributed between Amar, Akbar and Anthony, if each child receives atleast one toy and one marble, is

(A) 42 (B) 100 (C) 150 (D) 216

9. Number of positive integral solutions satisfying the equation $(x_1 + x_2 + x_3)(y_1 + y_2) = 77$, is

(A) 150 (B) 270 (C) 420 (D) 1024

10. If x_1, x_2, x_3 are the whole numbers and gives remainders 0,1,2 respectively, when divided by 3 then total number of different solutions of the equation $x_1 + x_2 + x_3 = 33$ are k, then $\frac{k}{11}$ is equal to

11. A commitee of 10 members is to be formed with members chosen from the faculties of Arts, Economics, Education, Engineering, Medicine and Science. Number of possible ways in which the faculties representation be distributed on this committee, is .

12. The sum of all numbers greater than 1000 formed by using the digits 1, 3, 5, 7 such that no digit is being repeated in any number is -

(A) 72215 (B) 83911 (C) 106656 (D) 114712

13. Three digit numbers in which the middle one is a perfect square are formed using the digits 1 to 9. Their sum is

(A) 134055 (B) 270540 (C) 17055 (D) none of these

14. Distinct 3 digit numbers are formed using only the digits 1, 2, 3 and 4 with each digit used at most once in each number thus formed. The sum of all possible numbers so formed is

(A) 6660 (B) 3330 (C) 2220 (D) none

11.0 DEARRANGEMENT

There are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is

$$n!\left[1-\frac{1}{1!}+\frac{1}{2!}+.....+\frac{(-1)^n}{n!}\right]$$

Proof – n letters are denoted by 1,2,3,........,n. Let A_i denote the set of distribution of letters in envelopes (one letter in each envelope) so that the i^{th} letter is placed in the corresponding envelope. Then,

$n(A_i) = 1 \times (n-1)!$ [since the remaining n–1 letters can be placed in n –1 envelops in (n–1)! ways]

Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes. Then,

$n(A_i \cap A_j) = 1 \times 1 \times (n-2)!$

Also $n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$

Hence, the required number is

$n(A_1' \cup A_2' \cup ..... \cup A_n') = n! - n(A_1 \cup A_2 \cup .... \cup A_n)$

$= n! - \left[\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + ... + (-1)^n \sum n(A_i \cap A_2 ..... \cap A_n)\right]$

$= n! - [{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + ... + (-1)^{n-1} \times {}^nC_n 1]$

$= n! - \left[\frac{n!}{1!(n-1)!}(n-1)! - \frac{n!}{2!(n-2)!}(n-2)! + ... + (-1)^{n-1}\right] = n!\left[1-\frac{1}{1!}+\frac{1}{2!}+........+\frac{(-1)^n}{n!}\right]$

Illustrations

Illustration 40. A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that–

(i) all the letters are in the wrong envelopes.

(ii) at least two of them are in the wrong envelopes.

Solution (i) The number of ways is which all letters be placed in wrong envelopes

$= 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right) = 720\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}\right)$

$= 360 - 120 + 30 - 6 + 1 = 265.$

(ii) The number of ways in which at least two of them in the wrong envelopes

$= {}^6C_4 . 2!\left(1-\frac{1}{1!}+\frac{1}{2!}\right) + {}^6C_3 . 3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right) + {}^6C_2 . 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)$

$+ {}^6C_1 . 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right) + {}^6C_0 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$

$= 15 + 40 + 135 + 264 + 265 = 719.$ **Ans.**

BEGINNER'S BOX-8

1. There are 10 seats in a double decker bus, 6 in the lower deck and 4 on the upper deck. Ten passengers board the bus, of them 3 refuse to go to the upper deck and 2 insist on going up. The number of ways in which the passengers can be accommodated is _____. (Assume all seats to be duly numbered)

2. In how many different ways a grandfather along with two of his grandsons and four grand daughters can be seated in a line for a photograph so that he is always in the middle and the two grandsons are never adjacent to each other.

3. Six married couple are sitting in a room. Find the number of ways in which 4 people can be selected so that

(A) they do not form a couple (B) they form exactly one couple

(C) they form at least one couple (D) they form atmost one couple

4. One hundred management students who read at least one of the three business magazines are surveyed to study the readership pattern. It is found that 80 read Business India, 50 read Business world, and 30 read Business Today. Five students read all the three magazines. How many read exactly two magazines?

(A) 50 (B) 10 (C) 95 (D) 65

5. Find the number of 10 digit numbers using the digits 0, 1, 2, ... 9 without repetition. How many of these are divisible by 4.

6. Let A be a set containing 10 distinct elements. Then the total number of distinct functions from A to A, is

(A) $10!$ (B) 10^{10} (C) 2^{10} (D) $2^{10}-1$

7. Let P_n denotes the number of ways of selecting 3 people out of 'n' sitting in a row, if no two of them are consecutive and Q_n is the corresponding figure when they are in a circle. If $P_n - Q_n = 6$, then 'n' is equal to :

(A) 8 (B) 9 (C) 10 (D) 12

8. Find the number of ways in which the number 94864 can be resolved as a product of two factors.

9. Find the number of order pair of (x, y) of solution of xy = 1440.

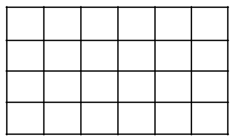
10. Product of all the even divisors of N = 1000, is

(A) $32 \cdot 10^2$ (B) $64 \cdot 2^{14}$ (C) $64 \cdot 10^{18}$ (D) $128 \cdot 10^6$

11. The 9 horizontal and 9 vertical lines on an 8×8 chessboard form 'r' rectangles and 's' squares. The ratio $\frac{s}{r}$ in its lowest terms is -

(A) $\frac{1}{6}$ (B) $\frac{17}{108}$ (C) $\frac{4}{27}$ (D) none

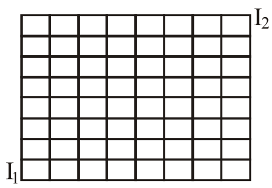
12. Number of rectangles in the grid shown which are not squares is



(A) 160 (B) 162

(C) 170 (D) 185

13. On the normal chess board as shown, I_1 & I_2 are two insects which starts moving towards each other. Each insect moving with the same constant speed. Insect I_1 can move only to the right or upward along the lines while the insect I_2 can move only to the left or downward along the lines of the chess board. Find the total number of ways the two insects can meet at same point during their trip.



14. A person writes letters to his 5 friends and addresses the corresponding envelopes. Number of ways in which the letters can be placed in the envelope, so that atleast two of them are in the wrong envelopes, is,

(A) 1 (B) 2 (C) 118 (D) 119

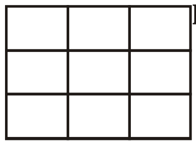
15. A disarranged number from the set 1 - 9 is an arrangement of all these numbers so that all numbers take up its usual position. (e.g. 1 is in any place other than the first position, 2 is in any place other than the second position........ all the way to 9). Number of ways in which at least six numbers take up their unusual positions, is

(A) 84 (B) 168 (C) 205 (D) none

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. In how many ways can a person go from point A to point B if he can travel only to the right or upward along the lines (Grid Problem) ?

B(3,3)



A(0,0)

Solution To reach the point B from point A, a person has to travel along 3 horizontal and 3 vertical strips.

Therefore, we have to arrange 3H and 3V in a row. Total number of ways = $\frac{6!}{3!3!} = 20$ ways

Ans.

Illustration 2. Find sum of all numbers formed using the digits 2,4,6,8 taken all at a time and no digit being repeated.

Solution All possible numbers = 4! = 24

If 2 occupies the unit's place then total numbers = 6

Hence, 2 comes at unit's place 6 times.

Sum of all the digits occuring at unit's place

$= \quad 6 \times (2 + 4 + 6 + 8)$

Same summation will occur for ten's, hundred's & thousand's place. Hence required sum

$= \quad 6 \times (2 + 4 + 6 + 8) \times (1 + 10 + 100 + 1000) = 133320$ **Ans.**

Illustration 3. Find the sum of all the numbers greater than 1000 using the digits 0,1,2,2.

Solution

(i) When 1 is at thousand's place, total numbers formed will be = $\frac{3!}{2!} = 3$

(ii) When 2 is at thousand's place, total numbers formed will be = 3! = 6

(iii) When 1 is at hundred's, ten's or unit's place then total numbers formed will be-
Thousand's place is fixed i.e. only the digit 2 will come here, remaining two places can be filled in 2! ways.
So total numbers = 2!

(iv) When 2 is at hundred's, ten's or unit's place then total numbers formed will be-
Thousand's place has 2 options and other two places can be filled in 2 ways.
So total numbers = 2 × 2 = 4

Sum = $10^3 (1 \times 3 + 2 \times 6) + 10^2 (1 \times 2 + 2 \times 4) + 10^1(1 \times 2 + 2 \times 4) + (1 \times 2 + 2 \times 4)$

$= 15 \times 10^3 + 10^3 + 10^2 + 10$

$= 16110$ **Ans.**

Illustration 4. Find the number of positive integral solutions of $x + y + z = 20$, if $x \neq y \neq z$.

Solution

$x \geq 1$

$y = x + t_1 \qquad t_1 \geq 1$

$z = y + t_2 \qquad t_2 \geq 1$

$x + x + t_1 + x + t_1 + t_2 = 20$

$3x + 2t_1 + t_2 = 20$

(i) $x = 1 \qquad 2t_1 + t_2 = 17$
$t_1 = 1, 2 \ldots\ldots\ldots 8 \Rightarrow 8$ ways

(ii) $x = 2 \qquad 2t_1 + t_2 = 14$
$t_1 = 1, 2 \ldots\ldots\ldots 6 \Rightarrow 6$ ways

(iii) $x = 3 \qquad 2t_1 + t_2 = 11$
$t_1 = 1, 2 \ldots\ldots\ldots 5 \Rightarrow 5$ ways

(vi) $x = 4 \qquad 2t_1 + t_2 = 8$
$t_1 = 1, 2, 3 \Rightarrow 3$ ways

(v) $x = 5 \qquad 2t_1 + t_2 = 5$
$t_1 = 1, 2 \Rightarrow 2$ ways

Total = 8 + 6 + 5 + 3 + 2 = 24

But each solution can be arranged by 3! ways.

So total solutions = 24 × 3! = 144. **Ans.**

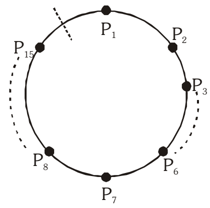
Illustration 5. A regular polygon of 15 sides is constructed. In how many ways can a triangle be formed using the vertices of the polygon such that no side of triangle is same as that of polygon ?

Solution Select one point out of 15 point, therefore total number of ways = ${}^{15}C_1$

Suppose we select point P_1. Now we have to choose 2 more point which are not consecutive. since we can not select P_2 & P_{15}.

Total points left are 12.

Now we have to select 2 points out of 12 points which are not consecutive



Total ways = ${}^{12-2+1}C_2 = {}^{11}C_2$

Every select triangle will be repeated 3 times.

So total number of ways $= \dfrac{{}^{15}C_1 \times {}^{11}C_2}{3} = 275$

Alternative –

First of all let us cut the polygon between points P_1 & P_{15}. Now there are 15 points on a straight line and we have to select 3 points out of these, such that the selected points are not consecutive.

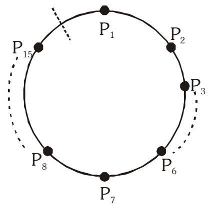
x O y O z O w

Here bubbles represents the selected points,

x represents the number of points before first selected point,

y represents the number of points between Ist & IInd selected point,

z represents the number of points between IInd & IIIrd selected point and w represents the number of points after IIIrd selected point.



$x + y + z + w = 15 - 3 = 12$

here $x \geq 0$, $y \geq 1$, $z \geq 1$, $w \geq 0$

Put $y = 1 + y'$ & $z = 1 + z'$ ($y' \geq 0$, $z' \geq 0$)

$\Rightarrow x + y' + z' + w = 10$

Total number of ways = ${}^{13}C_3$

These selections include the cases when both the points P_1 & P_{15} are selected. We have to remove those cases. Here a represents number of points between P_1 & 3^{rd} selected point & b represents number of points between 3^{rd} selected point and P_{15}

$\Rightarrow a + b = 15 - 3 = 12 \quad (a \geq 1, b \geq 1)$

put $a = 1 + t_1$ & $b = 1 + t_2$

$t_1 + t_2 = 10$

Total number of ways = ${}^{11}C_1 = 11$

Therefore required number of ways = ${}^{13}C_3 - {}^{11}C_1 = 286 - 11 = 275$ **Ans.**

Illustration 6. Find the number of ways in which three numbers can be selected from the set $\{5^1, 5^2, 5^3, ...5^{11}\}$ so that they form a G.P.

Solution Any three selected numbers which are in G.P. have their powers in A.P.

Set of powers is = $\{1,2,.........6,7,.....11\}$

By selecting any two numbers from $\{1,3,5,7,9,11\}$, the middle number is automatically fixed.

Total number of ways = ${}^{6}C_2$

Now select any two numbers from $\{2,4,6,8,10\}$ and again middle number is automatically fixed.

Total number of ways = ${}^{5}C_2$

$\therefore$ Total number of ways are = ${}^{6}C_2 + {}^{5}C_2 = 15 + 10 = 25$ **Ans.**

EXERCISE – 1 **SCQ/MCQ**

SINGLE CORRECT

1. Numbers greater than 1000 but not greater than 4000 which can be formed with the digits 0, 1, 2, 3, 4 (repetition of digits is allowed), are

(A) 350 (B) 375 (C) 450 (D) 576

2. A five digit number divisible by 3 has to formed using the numerals 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways in which this can be done is

(A) 216 (B) 240 (C) 600 (D) 3125

3. Total number of four digit odd numbers that can be formed using 0, 1, 2, 3, 5, 7 are

(A) 192 (B) 375 (C) 400 (D) 720

4. The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by

(A) $6! \times 5!$ (B) 30 (C) $5! \times 4!$ (D) $7! \times 5!$

5. A student is to answer 10 out of 13 questions in an examination such that he must choose at least 4 from the first five question. The number of choices available to him is

(A) 140 (B) 196 (C) 280 (D) 346

6. How many ways are there to arrange the letters in the word 'GARDEN' with the vowels in alphabetical order?

(A) 120 (B) 240 (C) 360 (D) 480

7. The number of ways of distributing 8 identical balls in 3 distinct boxes so that none of the boxes is empty is

(A) 5 (B) 21 (C) 3^8 (D) ${}^8C_3$

8. The number of different seven digit numbers that can be written using only three digits 1, 2 & 3 under the condition that the digit 2 occurs exactly twice in each number is -

(A) 672 (B) 640 (C) 512 (D) none of these

9. Number of ways in which 9 different prizes can be given to 5 students, if one particular student receives 4 prizes and the rest of the students can get any numbers of prizes is -

(A) ${}^9C_4 . 2^{10}$ (B) ${}^9C_5 . 5^4$ (C) $4 . 4^5$ (D) none of these

10. Ten different letters of alphabet are given. Words with four letters are formed from these letters, then the number of words which have at least one letter repeated is -

(A) 10^4 (B) ${}^{10}P_4$ (C) ${}^{10}C_4$ (D) 4960

11. Given 11 points, of which 5 lie on one circle, other than these 5, no 4 lie on one circle. Then the maximum number of circles that can be drawn so that each contains atleast three of the given points is:

(A) 216 (B) 156 (C) 172 (D) none

12. Number of numbers divisible by 25 that can be formed using only the digits 1, 2, 3, 4, 5 & 0 taken five at a time is

(A) 2 (B) 32 (C) 42 (D) 52

13. Define a 'good word' as a sequence of letters that consists only of the letters A, B and C and in which A never immidiately followed by B, B is never immediately followed by C, and C is never immediately followed by A. If the number of n-letter good words are 384, then the value of n is

(A) 6 (B)10 (C) 8 (D) 12

14. There are 100 different books in a shelf. Number of ways in which 3 books can be selected so that no two of which are neighbours is

(A) ${}^{100}C_3 - 98$ (B) ${}^{97}C_3$ (C) ${}^{96}C_3$ (D) ${}^{98}C_3$

15. There are (p + q) different books on different topics in Mathematics. ($p \neq q$) If L = The number of ways in which these books are distributed between two students x and y such that x get p books and y gets q books.

M = The number of ways in which these books are distributed between two students X and Y such that one of them gets p books and another gets q books.

N = The number of ways in which these books are divided into two groups of p books and q books then,

(A) L = M = N (B) L = 2M = 2N (C) 2L = M = 2N (D) L = M = 2N

16. The 120 permutations of MAHES are arranged in dictionary order, as if each were an ordinary five-letter word. The last letter of the 86^{th} word in the list is
(A) A (B) H (C) S (D) E

17. 5 Indian & 5 American couples meet at a party & shake hands. If no wife shakes hands with her own husband & no indian wife shakes hands with a male, then the number of hand shakes that takes place in the party is
(A) 95 (B) 110 (C) 135 (D) 150

18. There are counters available in x different colours. The counters are all alike except for the colour. The total number of arrangements consisting of y counters, assuming sufficient number of counters of each colour, if no arrangement consists of all counters of the same colour is
(A) $x^y - x$ (B) $x^y - y$ (C) $y^x - x$ (D) $y^x - y$

19. An English school and a Vernacular school are both under one superintendent. Suppose that the superintendentship, the four different teachership of English and Vernacular school each, are vacant, if there be altogether 11 candidates for the appointments, 3 of whom apply exclusively for the superintendentship and 2 exclusively for the appointment in the English school, the number of ways in which the different appointments can be disposed is
(A) 4320 (B) 268 (C) 1080 (D) 25920

20. In an election three districts are to be canvassed by 2, 3 & 5 men respectively . If 10 men volunteer, the number of ways they can be alloted to the different districts is :
(A) $\frac{10!}{2!\,3!\,5!}$ (B) $\frac{10!}{2!\,5!}$ (C) $\frac{10!}{(2!)^2\ 5!}$ (D) $\frac{10!}{(2!)^2\ 3!\,5!}$

21. The number of positive integers not greater than 100, which are not divisible by 2, 3 or 5 is
(A) 26 (B) 18 (C) 31 (D) none

22. Six people are going to sit in a row on a bench. A and B are adjacent, C does not want to sit adjacent to D. E and F can sit any where. Number of ways in which these six people can be seated, is
(A) 200 (B) 144 (C) 120 (D) 56

23. Three vertices of a convex *n* sided polygon are selected. If the number of triangles that can be constructed such that none of the sides of the triangle is also the side of the polygon is 30, then the polygon is a
(A) Heptagon (B) Octagon (C) Nonagon (D) Decagon

24. There are 12 guests at a dinner party . Supposing that the master and mistress of the house have fixed seats opposite one another, and that there are two specified guests who must always, be placed next to one another ; the number of ways in which the company can be placed, is:
(A) 20 . 10 ! (B) 22 . 10 ! (C) 44 . 10 ! (D) none

25. In a conference 10 speakers are present. If S_1 wants to speak before S_2 & S_2 wants to speak after S_3, then the number of ways all the 10 speakers can given their speeches with the above restriction if the remaining seven speakers have no objection to speak at any number is
(A) $^{10}C_3$ (B) $^{10}P_8$ (C) $^{10}P_3$ (D) $\frac{10!}{3}$

26. Let m denote the number of ways in which 4 different books are distributed among 10 persons, each receiving none or one only and let n denote the number of ways of distribution if the books are all alike. Then
(A) m = 4n (B) n = 4m (C) m = 24n (D) none

27. How many five digit numbers can be formed from 1, 2, 3, 4, 5 (without repetition), when the digit at the unit place must be greater than that in the tenth place?
(A) 54 (B) 60 (C) 5! / 3 (D) $2 \times 4!$

28. You are given an unlimited supply of each of the digits 1, 2, 3 or 4. Using only these four digits, you construct *n* digit numbers will be called L E G I T I M A T E if it contains the digit 1 either an even number times or not at all. Number of *n* digit legitimate numbers are
(A) $2^n + 1$ (B) $2^{n+1} + 2$ (C) $2^{n+2} + 4$ (D) $2^{n-1}(2^n + 1)$

29. The number 916238457 is an example of nine digit number which contains each of the digit 1 to 9 exactly once. It also has the property that the digits 1 to 5 occur in their natural order, while the digits 1 to 6 do not. Number of such numbers are
(A) 2268 (B) 2520 (C) 2975 (D) 1560

30. Number of functions defined from f :{1, 2, 3, 4, 5, 6} $\rightarrow$ {7, 8, 9, 10}such that the sum f (1) + f (2) + f (3) + f (4) + f (5) + f (6) is odd, is

(A) 2^{10} (B) 2^{11} (C) 2^{12} (D) $2^{12}-1$

31. The maximum number of different permutations of 4 letters of the word "EARTHQUAKE" is -

(A) 2910 (B) 2550 (C) 2190 (D) 2091

32. The number of ways in which we can arrange n ladies & n gentlemen at a round table so that 2 ladies or 2 gentlemen may not sit next to one another is -

(A) (n – 1)! (n – 2)! (B) (n)! (n – !)! (C) (n + 1)! (n)! (D) none of these

33. If as many more words as possible be formed out of the letters of the word "DOGMATIC" then the number of words in which the relative order of vowels and consonants remain unchanged is .

(A) 719 (B) 720 (C) 360 (D) 120

MORE THAN ONE OPTION CORRECT

34. $N = 2^2 . 3^3 . 5^4 . 7$, then -

(A) Number of proper divisors of N(excluding 1 & N) is 118

(B) Number of proper divisors of N(excluding 1 & N) is 120

(C) Number of positive integral solutions of xy = N is 60

(D) Number of positive integral solutions of xy = N is 120

35. The number of five digit numbers that can be formed using all the digits 0, 1, 3, 6, 8 which are -

(A) divisible by 4 is 30

(B) greater than 30,000 and divisible by 11 is 12

(C) smaller than 60,000 when digit 8 always appears at ten's place is 6

(D) between 30,000 and 60,000 and divisible by 6 is 18.

36. All the 7 digit numbers containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once and not divisible by 5 are arranged in the increasing order. Then -

(A) 1800^{th} number in the list is 3124567 (B) 1897^{th} number in the list is 4213567

(C) 1994^{th} number in the list is 4312567 (D) 2001^{th} number in the list is 4315726

37. Number of dissimilar terms in the expansion of $(x_1 + x_2 + ... + x_n)^3$ is -

(A) $\frac{n^2(n+1)^2}{4}$ (B) $\frac{n(n+1)(n+2)}{6}$ (C) ${}^{n+1}C_2 + {}^{n+1}C_3$ (D) $\frac{n^3+3n^2}{4}$

38. A persons wants to invite one or more of his friend for a dinner party. In how many ways can he do so if he has eight friends : -

(A) 2^8 (B) $2^8 - 1$

(C) 8^2 (D) ${}^8C_1 + {}^8C_2 + ...+ {}^8C_8$

39. If P(n, n) denotes the number of permutations of n different things taken all at a time then P(n, n) is also identical to :-

(A) n.P(n – 1, n – 1) (B) P(n, n – 1) (C) r! . P(n, n – r) (D) (n – r) . P(n, r)

where $0 \le r \le n$

40. Which of the following statement(s) is/are true :-

(A) ${}^{100}C_{50}$ is not divisible by 10

(B) n(n – 1)(n – 2) ....(n – r + 1) is always divisible by r! ($n \in N$ and $0 \le r \le n$)

(C) Morse telegraph has 5 arms and each arm moves on 6 different positions including the position of rest. Number of different signals that can be transmitted is $5^6 - 1$.

(D) There are 5 different books each having 5 copies. Number of different selections is $6^5 - 1$.

EXERCISE – 2 MISCELLANEOUS

Comprehension Based Questions

Comprehension – 1

S = {0, 2, 4, 6, 8}. A natural number is said to be divisible by 2 if the digit at the unit place is an even number. The number is divisible by 5, if the number at the unit place is 0 or 5. If four numbers are selected from S and a four digit number ABCD is formed.

On the basis of above information, answer the following questions

1. The number of such numbers which are even (all digits are different) is

(A) 60 (B) 96 (C) 120 (D) 204

2. The number of such numbers which are even (all digits are not different) is

(A) 404 (B) 500 (C) 380 (D) none of these

3. The number of such numbers which are divisible by two and five (all digits are not different) is

(A) 125 (B) 76 (C) 65 (D) 100

Comprehension – 2

Let p be a prime number and n be a positive integer, then exponent of p is n! is denoted by E_p (n!) and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ... + \left[\frac{n}{p^k}\right]$$

where $p^k < n < p^{k+1}$

and [x] denotes the integral part of x.

If we isolate the power of each prime contained in any number N, then N can be written as

$$N = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4} ....$$

where α_i are whole numbers.

On the basis of above information, answer the following questions

4. The exponent of 7 in ${}^{100}C_{50}$ is -

(A) 0 (B) 1 (C) 2 (D) 3

5. The number of zeros at the end of 108! is -

(A) 10 (B) 13 (C) 25 (D) 26

6. The exponent of 12 in 100! is -

(A) 32 (B) 48 (C) 97 (D) none of these

Match the Column

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

7. 5 balls are to be placed in 3 boxes. Each box can hold all the 5 balls. Number of ways in which the balls can be placed so that no box remains empty, if :

	Column-I		Column-II
(A)	balls are identical but boxes are different	(p)	2
(B)	balls are different but boxes are identical	(q)	25
(C)	balls as well as boxes are identical	(r)	50
(D)	balls as well as boxes are identical but boxes are kept in a row	(s)	6

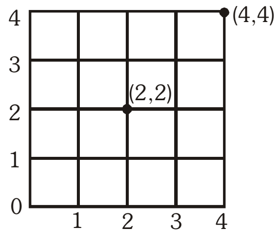
8. Consider all the different words that can be formed using the letters of the word HAVANA, taken 4 at a time.

	Column-I		Column-II
(A)	Number of such words in which all the 4 letters are different	(p)	36
(B)	Number of such words in which there are 2 alike letters & 2 different letters.	(q)	42
(C)	Number of such words in which A's never appear together	(r)	37
(D)	If all such 4 letters words are written, by the rule of dictionary then the rank of the word HANA	(s)	24

9. **Column-I** **Column-II**

(A) ${}^{24}C_2 + {}^{23}C_2 + {}^{22}C_2 + {}^{21}C_2 + {}^{20}C_2 + {}^{20}C_3$ is equal to (p) 102

(B) In the adjoining figure number of progressive (q) 2300



ways to reach from (0,0) to (4, 4) passing through point (2, 2) are (particle can move on horizontal or vertical line)

(C) The number of 4 digit numbers that can be made with the digits 1, 2, 3, 4, 3, 2 (r) 82

(D) If $\left\{\frac{500!}{14^k}\right\} = 0,$ then the maximum natural value of k is equal to (s) 36

(where {.} is fractional part function)

Integer/Subjective Type Questions

10. Number of ways can a team of 6 horses be selected out of a stud of 16, so that there shall always be 3 out of A B C A′ B′ C′ , but never AA′, BB′ or CC′ together is 15×2^m find m.

11. An examination paper consists of 12 questions divided into parts A & B. Part-A contains 7 questions & part-B contains 5 questions. A candidate is required to attempt 8 questions selecting atleast 3 from each part. Maximum ways can the candidate select the questions is m (m + 1) find m/10.

12. There are 2 women participating in a chess tournament. Every participant played 2 games with the other participants. The number of games that the men played between themselves exceeded by 66 as compared to the number of games that the men played with the women. The number of participants is 2n + 1 and the total numbers of games played in the tournament is m(m + 1) find n, m/3.

13. Number of different ways can 15 Candy bars be distributed between Ram, Shyam, Ghanshyam and Balram, if Ram can not have more than 5 candy bars and Shyam must have at least two. Assume all Candy bars to be alike is m find sum of digits of m.

14. There are 5 white, 4 yellow, 3 green, 2 blue & 1 red ball. The balls are all identical except for colour. These are to be arranged in a line in 5 places. Number of distinct arrangements is m2111 find m. (where m is a digit).

15. Number of arrangements each consisting 2 vowels & 2 consonants can be made out of the letters of the word **DEVASTATION'** is (m – 1) (m + 2) find $\frac{m}{10}$.

16. Number of integral solutions are there for the equation; $x + y + z + w = 29$ when $x > 0, y > 1, z > 2$ & $w \geq 0$ is $1300 \times k$ find k.

17. A shop sells 6 different flavours of ice-cream. Number of ways can a customer choose 4 ice-cream cones if

(i) they are all of different flavours is

(ii) they are non necessarily of different flavours is

(iii) they contain only 3 different flavours is

(iv) they contain only 2 or 3 differnet flavours is

18. There are 2n guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another, and that there are two specified guests who must not be placed next to one another. The number of ways in which the company can be placed is $(\lambda n - \lambda)! \,.\, (An^2 + Bn + c)$. Find $\lambda + A + B + C$.

19. Number of ways can the letters of the word **MULTIPLE** be arranged (λ and μ are numerical digits)

(i) without changing the order of the vowels is $\lambda\mu$ 59 find $\lambda + \mu$

(ii) keeping the position of each vowel fixed is $\lambda\mu$ 59 find $\lambda + \mu$

(iii) without changing the relative order/position of vowels & consonants is $\lambda\mu$ 59 find $\lambda + \mu$

20. The number of ways in which the number 30 can be partitioned into three unequal parts, each part being a natural number is λ. This number would be μ if equal parts are also included find $\frac{\mu - \lambda}{2}$.

NCERT CORNER

Very Short Answer

1. Find x, if $\frac{1}{8!}+\frac{3}{7!}=\frac{x}{9!}$.

2. How many 4 - letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated ?

3. How many 4-letter words, with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed ?

4. Find n, if ${}^{n-1}P_3 : {}^{n}P_4 = 1 : 9$.

5. How many numbers between 20,000 and 30,000 can be formed using the digits 2, 3, 5, 6, 7, if each digit may be repeated any number of times in any number.

6. In how many ways can 6 beads of different colours form a necklace ?

7. From a committee of 8 persons, in how many ways can be choose a chairman and a vice chairman assuming one person can not hold more than one position ?

8. If ${}^{n}C_{r-1} = 36, {}^{n}C_{r} = 84$ and ${}^{n}C_{r+1} = 126$, then find ${}^{r}C_2$.

9. Determine n, if ${}^{2n}C_3 : {}^{n}C_3 = 11 : 1$.

10. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.

11. How many words with or without meaning each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER ?

Short Answer

12. If $\frac{n!}{2!(n-2)!}$ and $\frac{n!}{4!(n-4)!}$ are in the ratio 2 : 1, find the value of n.

13. How many 3 - digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated ?

14. Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them ?

15. How many 3 digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated ?

16. How many words can be formed from the letters of the word 'DAUGHTER' so that

(i) The vowels always come together ? (ii) The vowels never come together ?

17. If all the letters of the word 'AGAIN' be arranged as in a dictionary, what is the fiftieth word ?

18. There are 6 gentlemen and 3 Iadies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together ?

19. A bag contains 4 black and 5 red balls, 6 balls are drawn, determine the number of ways in which a 3 black and 3 red balls can be drawn.

20. A committee of 6 is to be formed from 6 boys and 4 girls. In how many ways can this be done if the committee contains

(i) 2 girls (ii) Atleast 2 girls ?

21. In how many ways can a cricket eleven be chosen out of a batch of 15 players if

(i) There is no restriction on the selection.

(ii) A particular player is always chosen.

(iii) A particular player is never chosen.

22. In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus signs are together ?

Long Answer

23. How many numbers are there between 100 and 1000 such that :

(i) Every digit is either 2 or 5 ?

(ii) The digit in the unit's place is 5 ?

(iii) Atleast one of the digits is 5 ?

(iv) Exactly one of the digits is 5 ?

24. Find the number of 4 - digit numbers that can be formed using the digits 1, 2, 3, 4, 5, if no digit is repeated. How many of these will be even ?

25. The principal wants to arrange 5 students on the platform such that the boy 'SALIM' occupies the second position and such that the girl 'SITA' is always adjacent to the girl 'RITA'. How many such arrangements are possible ?

26. In how many ways can be letters of the word PERMUTATIONS be arranged if the

(i) Words start with P and end with S.

(ii) Vowels are all together.

(iii) There are always 4 letters between P and S.

27. There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the

(i) number of straight lines obtained from the pairs of these points

(ii) number of triangles that can be formed with the vertices as these points.

28. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of

(i) Exactly 3 girls ?

(ii) At least 3 girls ?

(iii) At most 3 girls ?

ANSWER KEY

BEGINNER'S BOX - 1

1. (120) **2.** (C) **3.** (738) **4.** (A) **5.** (i) 24 (ii) 576 (iii) 360

6. (i) 468000 (ii) 421200 **7.** (i) 840 (ii) 120 (iii) 400 (iv) 240 (v) 480 (vi) 40 (vii) 60 (viii) 240

8. (i) 120 (ii) 40 (iii) 40 (iv) 80 (v) 20 **9.** 4^7 **10.** (A) 3^4 (B) 24

11. 36 **12.** (C) **13.** 4500 **14.** (A) **15.** 26^4 **16.** (C)

17. 5.4^9

BEGINNER'S BOX - 2

1. (C) **2.** (D) **3.** (A) **4.** (A) **5.** (D) **6.** (C)
7. (A) **8.** (A) **9.** (D) **10.** (A) **11.** (D) **12.** (B)
13. (D) **14.** (D) **15.** (17) **16.** (A) **17.** (B) **18.** (A)
19. (C) **20.** (A) **21.** (A) 213564 (B) 267^{th} **22.** (B) **23.** (C)
24. (0) **25.** (B)

BEGINNER'S BOX - 3

1. (B) **2.** (A) **3.** (378) **4.** (3150) **5.** (B) **6.** (A)
7. (D) **8.** (B) **9.** (B) **10.** (C) **11.** (B) **12.** (A)
13. (B) **14.** (B) **15.** (D) **16.** (C)

BEGINNER'S BOX - 4

1. (A) **2.** (B) **3.** (A) **4.** (C) **5.** (C)

6. (C) **7.** 60, 6^{th} **8.** (C) **9.** $({}^{8}C_4.4!)$

10. (i) $\dfrac{(9!)(5!)}{(2!)^3}$; (ii) $\dfrac{(13!)}{(8!)(5!)}$ **11.** (B) **12.** (A)

BEGINNER'S BOX - 5

1. (B) **2.** 256 **3.** (D) **4.** 432 **5.** (C) **6.** (B)

7. (A) **8.** (D) **9.** (B) **10.** (A) - (C), (B) - (B), (C) - (B)

11. (C)

BEGINNER'S BOX - 6

1. (B) **2.** (D) **3.** (A) **4.** (A) **5.** (B) **6.** (A)

7. (D) **8.** (B) **9.** (B) **10.** (A) **11.** (A) **12.** (C)
13. (C) **14.** (A) **15.** (B)

BEGINNER'S BOX - 7

1. (A) ${}^{15}C_3$ (B) ${}^{7}C_3$ **2.** ${}^{12}C_2$ **3.** ${}^{23}C_2$ **4.** (A) **5.** (B)

6. (D) **7.** (C) **8.** (D) **9.** (C) **10.** (6) **11.** 3003

12. (C) **13.** (A) **14.** (A)

BEGINNER'S BOX - 8

1. (172800) **2.** (528) **3.** (A) 240, (B) 240, (C) 255, (D) 480

4. (A) **5.** (20.8!) **6.** (B) **7.** (C) **8.** 23

9. 36 **10.** (C) **11.** (B) **12.** (A) **13.** 12870

14. (D) **15.** (C)

EXERCISE-1 (SINGLE CORRECT & MORE THAN ONE OPTION CORRECT)

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	A	A	A	B	C	B	A	A	D
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	C	C	D	C	D	C	A	D	A
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	A	B	C	A	D	C	B	D	B	B
Que.	31	32	33	34	35	36	37	38	39	40
Ans.	C	B	A	AD	ABD	BD	BC	BD	ABC	ABD

EXERCISE-2 (MISCELLANEOUS)

- **Comprehension Based Questions**

  Comprehension – 1 **1.** B **2.** A **3.** B

  Comprehension – 2 **4.** A **5.** C **6.** B

- **Match the Column** **7.** (A) $\rightarrow$ (s), (B) $\rightarrow$ (q), (C) $\rightarrow$ (p), (D) $\rightarrow$ (s)

  8. (A) $\rightarrow$ (s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (r)

  9. (A)$\rightarrow$(q), (B)$\rightarrow$(s), (C)$\rightarrow$(p), (D)$\rightarrow$(r)

- **Integer/Subjective Type Questions**

10. [6] **11.** [2] **12.** [6, 4] **13.** [8]

14. [1] **15.** [4] **16.** [2] **17.** (i) ${}^{6}C_4$ (ii) 126 (iii) 60 (iv) 105

18. [4] **19.** (i) [6] (ii) [0] (iii) [3] **20.** [7]

NCERT CORNER

1.	225	**2**.	5040	**3**.	5040	**4**.	9	**5**.	625	**6**.	60
7.	56	**8**.	36/84	**9**.	6	**10**.	2000	**11**.	3600	**12**.	5
13.	108	**14**.	172800	**15**.	60	**16**.	(i) 4320		(ii) 36000		
17.	NAAIG	**18**.	14400	**19**.	40	**20**.	(i) 90	(ii) 185			
21.	(i) 1365		(ii) 1001		(iii) 364	**22**.	56				
23.	(i) 8		(ii) 90		(iii) 252		(iv) 225	**24**.	48	**25**.	8
26.	(i) 1814400		(ii) 2419200		(iii) 25401600.	**27**.	(i) 40	(ii) 116			
28.	(i) 504		(ii) 588		(iii) 1632						

IMPORTANT NOTES

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