Contents

MATHEMATICS

SET, RELATION & FUNCTION

1.0 SET THEORY

The collection of well defined things is called set. Well defined means a law by which we are able to find whether a given thing is contained in the given set or not.

Illustration 1. (i) $A = \{ x ; x \text{ is a prime number}\}$ means
$A = \{ 2, 3, 5, 7, 11, 13........... \}$
(ii) $W = \{ x ; x \text{ is a whole number} \}$ means
$W = \{ 0 , 1, 2, 3, 4, 5, ... \}$

METHODS TO WRITE A SET :

(i) Roster Method : In this method a set is described by listing elements, separated by commas and enclose then by curly brackets

Ex. The set of vowels of English Alphabet may be described as $\{a, e, i, o, u\}$

(ii) Set Builder From : In this case we write down a property or rule p Which gives us all the element of the set

$A = \{x : P(x)\}$

Illustration 2. $A = \{x : x \in N \text{ and } x = 2n \text{ for } n \in N\}$
i.e. $A = \{2, 4, 6, ...\}$

Illustration 3. $B = \{x^2 : x \in z\}$
i.e. $B = \{0, 1, 4, 9, ...\}$

1.1 TYPES OF SETS

Null set or Empty set : A set having no element in it is called an Empty set or a null set or void set it is denoted by ϕ or { }

Illustrations

Illustration 4. $A = \{x \in N : 5 < x < 6\} = \phi$

A set consisting of at least one element is called a non-empty set or a non-void set.

Singleton : A set consisting of a single element is called a singleton set.

Illustration 5. Then set {0}, is a singleton set

Finite Set : A set which has only finite number of elements is called a finite set.

Illustration 6. $A = \{a, b, c\}$

Order of a finite set : The number of elements in a finite set is called the order of the set A and is denoted O(A) or n(A). It is also called cardinal number of the set.

Illustration 7. $A = \{a, b, c, d\} \Rightarrow n(A) = 4$

Infinite set : A set which has an infinite number of elements is called an infinite set.

Illustration 8. $A = \{1, 2, 3, 4, ...\}$ is an infinite set

Equal sets : Two sets A and B are said to be equal if every element of A is a member of B, and every element of B is a member of A.

If sets A and B are equal. We write A = B and A and B are not equal then $A \neq B$

Illustration 9. $A = \{1, 2, 6, 7\}$ and $B = \{6, 1, 2, 7\} \Rightarrow A = B$

Equivalent sets : Two finite sets A and B are equivalent if their number of elements are same
ie. $n(A) = n(B)$

Illustration 10. $A = \{1, 3, 5, 7\}$, $B = \{a, b, c, d\}$
$n(A) = 4$ and $n(B) = 4 \Rightarrow n(A) = n(B)$

Note : Equal set always equivalent but equivalent sets may not be equal

Subsets : Let A and B be two sets if every element of A is an element B, then A is called a subset of B if A is a subset of B. we write $A \subseteq B$

Illustration 11. $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\} \Rightarrow A \subseteq B$

The symbol "$\Rightarrow$" stands for "implies"

Proper subset : If A is a subset of B and $A \neq B$ then A is a proper subset of B. and we write $A \subset B$

Note-1 : Every set is a subset of itself i.e. $A \subseteq A$ for all A

Note-2 : Empty set ϕ is a subset of every set

Note-3 : Clearly $N \subset W \subset Z \subset Q \subset R \subset C$

Note-4 : The total number of subsets of a finite set containing n elements is 2^n

Universal set : A set consisting of all possible elements which occur in the discussion is called a Universal set and is denoted by U

Note : All sets are contained in the universal set

Illustration 12. If $A = \{1, 2, 3\}$, $B = \{2, 4, 5, 6\}$, $C = \{1, 3, 5, 7\}$ then $U = \{1, 2, 3, 4, 5, 6, 7\}$ can be taken as the Universal set.

Power set : Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A)

Illustration 13. Let $A = \{1, 2\}$ then $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Illustration 14. Let $P(\phi) = \{\phi\}$

$\because \; P(P(\phi)) = \{\phi, \{\phi\}\}$

$\because \; P(P(P(\phi)) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}$

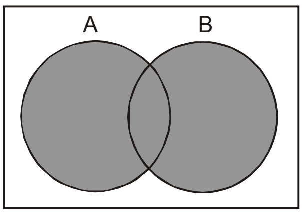
Note-1 : If $A = \phi$ then P(A) has one element

Note-2 : Power set of a given set is always non empty

1.2 OPERATIONS OF SETS

The basic operations of sets and their related results are as follows.

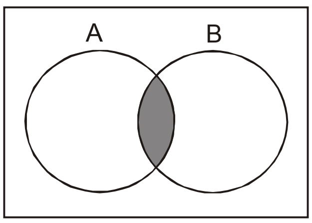
(i) Union of sets : The union of two sets is represented by $A \cup B$ or $A + B$. This set contains those elements which are in A or in B or in A and B both. So



$A \cup B$

$$A \cup B = \{ x \; ; \; x \in A \text{ or } x \in B \}$$

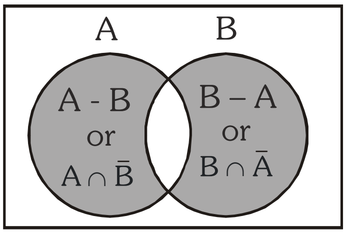
(ii) Intersection of two sets : The intersection of two sets is represented by $A \cap B$ or AB. It contains those all elements which are contained in both sets A and B both, So



$A \cap B$

$$A \cap B = \{ x \; ; \; x \in A \text{ and } x \in B \}$$

(iii) **Difference of two sets :** If A and B are two sets then A–B represents the set of those elements which are in A and not in B. In the same manner B–A represents the set of those elements which are in B and not in A. So



$$A - B = A \cap \bar{B} = \{x \,;\, x \in A \text{ and } x \notin B\}$$

$$B - A = B \cap \bar{A} = \{x \,;\, x \in B \text{ and } x \notin A\}$$

1.3 CARTESIAN PRODUCT OF TWO SETS

The cartesian product of two non-empty sets A & B is the set of all possible ordered pair of the form (a, b) where the first element comes from set A & second comes from set B.

$A \times B = \{(a, b) \mid a \in A, b \in B\}$

e.g. $A = \{1, 2, 3\}$, $B = \{p, q\}$

$A \times B = \{(1, p)(1, q), (2, p), (2, q), (3, p), (3, q)\}$

NOTE

(i) If either A or B is the null set, then $A \times B$ will also be empty set, i.e. $A \times B = \phi$

(ii) If $n(A) = p$ & $n(B) = q$, then $n(A \times B) = p \times q$, where $n(X)$ (cardinal number) denotes the number of elements in set X.

2.0 RELATION

INTRODUCTION :

Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$.

thus, R is a relation from A to B $\Leftrightarrow R \subseteq A \times B$.

Illustration 15. If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then $R = \{(1, b), (2, c), (1, a), (3, a)\}$ being a subset of $A \times B$, is a relation from A to B. Here (1, b), (2, c), (1, a) and $(3, a) \in R$, so we write 1 Rb, 2Rc, 1Ra and 3Ra. But $(2, b) \notin R$, so we write 2 $\not{R}$ b

Total Number of Realtions : Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}.

• Number of Non-empty subsets $2^{mn}-1$

2.1 Domain and Range of a relation

Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom (R) = $\{a : (a, b) \in R\}$

and, Range (R) = $\{b : (a, b) \in R\}$

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

Illustration 16. Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$ be two sets and let R be a relation from A to B defined by the phrase "$(x, y) \in R \Leftrightarrow x > y$". Under this relation R, we have 3R2, 5R2, 5R4, 7R2, 7R4 and 7R6

i.e. $R = \{(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)\}$

$\therefore$ Dom (R) = $\{3, 5, 7\}$ and Range (R) = $\{2, 4, 6\}$

2.2 Inverse Relation

Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$R^{-1} = \{(b, a) : (a, b) \in R\}$

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, $\text{Dom}(R) = \text{Range}(R^{-1})$ and $\text{Range}(R) = \text{Dom}(R^{-1})$

Illustration 17. Let A be the set of first ten natural numbers and let R be a relation on A defined by $(x, y) \in R \Leftrightarrow x + 2y = 10$, i.e. $R = \{(x, y) : x \in A, y \in A \text{ and } x + 2y = 10\}$. Express R and R^{-1} as sets of ordered pairs. Determine also (i) domain of R and R^{-1} (ii) range of R and R^{-1}

Solution. We have $(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \dfrac{10-x}{2}$, $x, y \in A$

where $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Now, $x = 1 \Rightarrow y = \dfrac{10-1}{2} = \dfrac{9}{2} \notin A$.

This shows that 1 is not related to any element in A. Similarly we can observe. that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation

Further we find that :

For $x = 2$, $y = \dfrac{10-2}{2} = 4 \in A$ $\therefore (2, 4) \in R$

For $x = 4$, $y = \dfrac{10-4}{2} = 3 \in A$ $\therefore (4, 3) \in R$

For $x = 6$, $y = \dfrac{10-6}{2} = 2 \in A$ $\therefore (6, 2) \in R$

For $x = 8$, $y = \dfrac{10-8}{2} = 1 \in A$ $\therefore (8, 1) \in R$

Thus $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

$\Rightarrow$ $R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, $\text{Dom}(R) = \{2, 4, 6, 8\} = \text{Range}(R^{-1})$

and $\text{Range}(R) = \{4, 3, 2, 1\} = \text{Dom}(R^{-1})$

2.3 TYPES OF RELATIONS

In this section we intend to define various types of relations on a given set A.

(1) Void Relation : Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

(2) Universal Relation : Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

(3) Identity Relation : Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

Illustration 18. The relation $I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on set $A = \{1, 2, 3\}$. But relations

$R_1 = \{(1, 1), (2, 2)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ are not identity relations on A, because $(3, 3) \notin R_1$ and in R_2 element 1 is related to elements 1 and 3.

(4) Reflexive Relation : A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus, R on a set A is not reflexive if there exists an element $A \in A$ such that $(a , a) \notin R$.

Illustration 19. Let A = {1, 2, 3} be a set. Then R = {(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)} is a reflexive relation on A.

But $R_1 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \notin R_1$.

Note : Every Identity relation is reflexive but every reflexive ralation is not identity.

(5) Symmetric Relation : A relation R on a set A is said to be a symmetric relation iff

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e. $a R b \Rightarrow bRa$ for all $a, b, \in A$.

Illustration 20. Let L be the set of all lines in a plane and let R be a relation defined on L by the rule $(x, y) \in R \Leftrightarrow x$ is perpendicular to y. Then R is a symmetric relation on L, because

$L_1 \perp L_2 \Rightarrow L_2 \perp L_1$

i.e. $(L_1, L_2) \in R \Rightarrow (L_2, L_1) \in R$.

Illustration 21. Let A = {1, 2, 3, 4} and Let R_1 and R_2 be realtion on A given by $R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$. Clearly, R_1 is a symmetric relation on A. However, R_2 is not so, because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$

(6) Transitive Relation : Let A be any set. A relation R on A is said to be a transitive relation iff

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e. $a R b$ and $b R c \Rightarrow a R c$ for all $a, b, c \in A$

Illustration 22. On the set N of natural numbers, the relation R defined by $x R y \Rightarrow x$ is less than y is transitive, because for any $x, y, z \in N$

$x < y$ and $y < z \Rightarrow x < z \Rightarrow x R y$ and $y R z \Rightarrow x R z$

Illustration 23. Let L be the set of all straight lines in a plane. Then the realtion 'is parallel to' on L is a transitive relation, because from any $\ell_1, \ell_2, \ell_3 \in L$.

$\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3 \Rightarrow \ell_1 \parallel \ell_3$

(7) Equivalence Relation : A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Illustration 24. Let R be a relation on the set of all lines in a plane defined by $(\ell_1, \ell_2) \in R \Leftrightarrow$ line ℓ_1 is parallel to line ℓ_2. R is an equivalence relation.

Note : It is not neccessary that every relation which is symmetric and transitive is also reflexive.

3.0 FUNCTION

A relation R from a set A to a set B is called a function if each element of A has unique image in B.

It is denoted by the symbol. $f : A \rightarrow B$ or $A \xrightarrow{f} B$ which reads 'f' is a function from A to B 'or' f maps A to B,

If an element $a \in A$ is associated with an element $b \in B$, then b is called 'the f image of a' or 'image of a under f 'or' the value of the function f at a'. Also a is called the pre-image of b or argument of b under the function f. We write it as $b = f(a)$ or $f : a \rightarrow b$ or $f : (a, b)$

Thus a function 'f' from a set A to a set B is a subset of $A \times B$ in which each 'a' belonging to A appears in one and only one ordered pair belonging to f.

Representation of Function

(a) Ordered pair – Every function from $A \to B$ satisfies the following conditions :

(i) $f \subseteq A \times B$ (ii) $\forall$ $a \in A$ there exist $b \in B$ and (iii) $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

(b) Formula based (uniformly/nonuniformly) e.g.

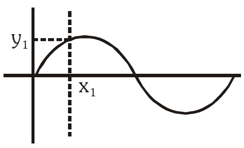
(i) $f : R \to R,\ y = f(x) = 4x$ (uniformly defined)

(ii) $f(x) = x^2$ (uniformly defined)

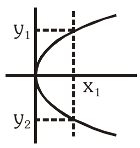
(iii) $f(x) = \begin{cases} x+1 & -1 \le x < 4 \\ -x & 4 \le x < 7 \end{cases}$ (non-uniformly defined)

(iv) $f(x) = \begin{cases} x^2 & x \ge 0 \\ -x-1 & x < 0 \end{cases}$ (non-uniformly defined)

(c) Graphical representation



Graph (1)



Graph (2)

Graph(1) represent a function but graph(2) does not represent a function.

NOTE

(i) If a vertical line cuts a given graph at more than one point then it can not be the graph of a function.

(ii) Every function is a relation but every relation is not necessarily a function.

BEGINNER'S BOX-1

1. If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is-
(A) 2^{mn} (B) $2^{mn}-1$ (C) 2mn (D) m^n

2. In the set A = {1, 2, 3, 4, 5}, a relation R is defined by R = {(x, y) | x, y ∈ A and x < y}. Then R is-
(A) Reflexive (B) Symmetric (C) Transitive (D) None of these

3. For real numbers x and y, we write $x\,R\,y \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then the relation R is-
(A) Reflexive (B) Symmetric (C) Transitive (D) none of these

4. Let X = {1, 2, 3, 4} and Y = {1, 3, 5, 7, 9}. Which of the following is relations from X to Y-
(A) $R_1 = \{(x, y) \mid y = 2 + x, x \in X, y \in Y\}$ (B) $R_2 = \{(1, 1), (2, 1), (3, 3), (4, 3), (5, 5)\}$
(C) $R_3 = \{(1, 1), (1, 3), (3, 5), (3, 7), (5, 7)\}$ (D) $R_4 = \{(1, 3), (2, 5), (2, 4), (7, 9)\}$

5. Let A = {2, 3, 4, 5} and let R = {(2, 2), (3, 3), (4, 4), (5, 5), (2, 3), (3, 2), (3, 5), (5, 3)} be a relation in A. Then R is-
(A) Reflexive and transitive (B) Reflexive and symmetric
(C) Reflexive and antisymmetric (D) none of these

6. If A = {2, 3} and B = {1, 2}, then A × B is equal to-
(A) {(2, 1), (2, 2), (3, 1), (3, 2)} (B) {(1, 2), (1, 3), (2, 2), (2, 3)}
(C) {(2, 1), (3, 2)} (D) {(1, 2), (2, 3)}

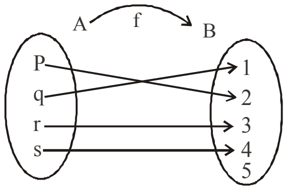
7. Let R be a relation over the set N × N and it is defined by (a, b) R (c, d) ⇒ a + d = b + c. Then R is-
(A) Reflexive only (B) Symmetric only
(C) Transitive only (D) An equivalence relation

8. If A = {1, 2, 3}, B = {1, 4, 6, 9} and R is a relation from A to B defined by 'x is greater than y'. Then range of R is-
(A) {1, 4, 6, 9} (B) {4, 6, 9} (C) {1} (D) none of these

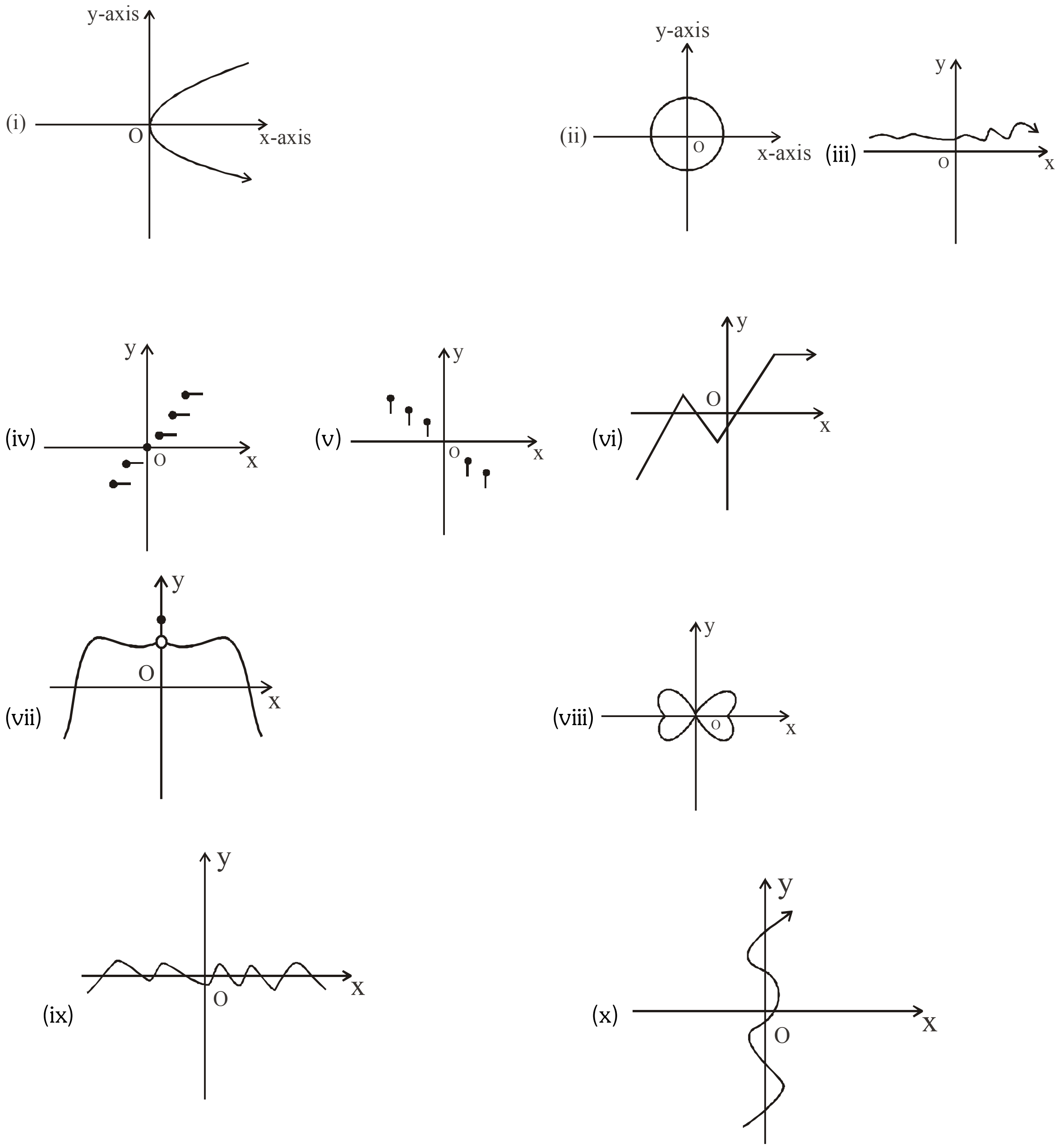
9. Let $R = \{(x, y) : x, y \in A, x + y = 5\}$ where
$A = \{1, 2, 3, 4, 5\}$ then
(A) R is not reflexive, symmetric and not transitive (B) R is an equivalence relation
(C) R is reflexive, symmetric but not transitive (D) R is not reflexive, not symmetric but transitive

10. Let R be a relation on a set A such that $R = R^{-1}$ then R is-
(A) reflexive (B) symmetric (C) transitive (D) none of these

11. In the given figure find the domain, co-domain and range.



12. Which of the following graphs are graphs of function :



13. For which of the following , y can be a function of x, $(x \in R, y \in R)$.
(i) $(x-h)^2 + (y-k)^2 = r^2$ (ii) $y^2 = 4ax$ (iii) $x^4 = y^2$ (iv) $x^6 = y^3$ (v) $3y = (\log x)^2$

14. Let A = {–2, –1, 0, 1, 2} and B = {0, 1, 2, 3, 4, 5, 6}. Consider a rule $f(x) = x^2$. Find domain & range

15. Let $f : R \to R$ be given by $f(x) = x^2 + 3$. Find (a) {x : f(x) = 28} (b) the pre-images of 39 and 2 under f.

16. If $f : R \to R$ be defined as followings : $f(x) = \begin{cases} 1, & \text{if } x \in Q \\ -1, & \text{if } x \notin Q \end{cases}$.

Find (a) f(1/2), $f(\pi), f(\sqrt{2})$ (b) Range of f (c) pre-images 1 of and –1.

17. If $a, b \in \{1, 2, 3, 4\}$, then which of the following are functions in the given set ?

(a) $f_1 = \{(x, y) : y = x + 1\}$
(b) $f_2 = \{(x, y) : x + y > 4\}$
(c) $f_3 = \{(x, y) : y < x\}$
(d) $f_4 = \{(x, y) : x + y = 5\}$

Also, in case of a function give its range.

18. Express the following functions as sets of ordered pairs and determine their ranges :

(a) $f : A \to R$, $f(x) = x^2 + 1$, where A = {–1, 0, 2, 4}

(b) $g : A \to N$, $g(x) = 2x$, where $A = \{ x : x \in N, \le 10 \}$

19. Is g = {(1, 1), (2, 3), (3, 5), (4, 7)} a function ? If this is described by the formula, $g(x) = \alpha x + \beta$, then what values should be assigned to α and β ?

20. The value of b and c for which the identity f (x + 1) – f(x) = 8x + 3 is satisfied, where $f(x) = bx^2 + cx + d$, are –

(a) b = 2 , c = 1
(b) b = 4 , c = – 1
(c) b = –1 , c = 4
(d) b = –1 , c = 1

21. Given the function f(x) = (x + 1)/(x – 1). Find f(1/x), f(2x), 2f(x), $f(x^2)$, $[f(x)]^2$.

22. Given the function $f(x) = \log\frac{1-x}{1+x}$.

Show that at $x_1, x_2 \in (-1, 1)$ the following identity holds true: $f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1x_2}\right)$

23. Given the function $f(x) = (a^x + a^{-x})/2$ (a > 0).

Show that f(x + y) + f(x – y) = 2f (x) f(y).

24. Given the function $f(x) = \begin{cases} 3^{-x} - 1, & -1 \le x < 0, \\ \tan(x/2), & 0 \le x < \pi, \\ x/(x^2 - 2), & \pi \le x \le 6, \end{cases}$

Find $f(-1),\ f\left(\frac{\pi}{2}\right), f\left(\frac{2\pi}{3}\right), f(4), f(6)$

25. The function f(x) is defined over the whole number scale by the following law:

$$f(x) = \begin{cases} 2x^3 + 1, & \text{if } \quad x \le 2, \\ 1/(x-2), & \text{if } \quad 2 < x \le 3, \\ 2x - 5, & \text{if } \quad x > 3, \end{cases}$$

Find: $f(\sqrt{2})$, $f(\sqrt{8})$, $f\left(\sqrt{\log_2 1024}\right)$.

4.0 DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION

Let $f : A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f . The set of f images of all the elements of A is known as the range of f .

Thus Domain of f = $\{a \mid a \in A, (a, f(a)) \in f\}$

Range of f = $\{f(a) \mid a \in A, f(a) \in B\}$

NOTE :

(i) It should be noted that range is a subset of co-domain.

(ii) If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range

Illustrations

Illustration 25. Find the Domain of the following function :

(i) $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$

(ii) $y = \log_{(x-4)}(x^2 - 11x + 24)$

(iii) $f(x) = \log_2\left(-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right)$

Solution (i) $\sin x \geq 0$ and $16 - x^2 \geq 0 \Rightarrow 2n\pi \leq x \leq (2n+1)\pi$ and $-4 \leq x \leq 4$

$\therefore$ Domain is $[-4, -\pi] \cup [0, \pi]$

(ii) $y = \log_{(x-4)}(x^2 - 11x + 24)$

Here 'y' would assume real value if,

$x - 4 > 0$ and $\neq 1$, $x^2 - 11x + 24 > 0 \Rightarrow x > 4$ and $\neq 5$, $(x - 3)(x - 8) > 0$

$\Rightarrow x > 4$ and $\neq 5$, $x < 3$ or $x > 8 \Rightarrow x > 8 \Rightarrow$ Domain $= (8, \infty)$

(iii) We have $f(x) = \log_2\left(-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right)$

f(x) is defined if $-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1 > 0$

or if $\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right) < -1$ or if $\left(1+\frac{1}{\sqrt[4]{x}}\right) > (1/2)^{-1}$

or if $1+\frac{1}{\sqrt[4]{x}} > 2$ or if $\frac{1}{\sqrt[4]{x}} > 1$ or if $x^{1/4} < 1$ or if $0 < x < 1$

$\therefore$ D(f) = (0, 1)

Illustration 26. Find the range of following functions :

(i) $f(x) = \dfrac{1}{8 - 3\sin x}$ (ii) $f(x) = \log_2\left(\dfrac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$

(iii) $f(x) = \log_{\sqrt{2}}\left(2 - \log_2(16\sin^2 x + 1)\right)$

Solution (i) $f(x) = \dfrac{1}{8 - 3\sin x}$

$-1 \leq \sin x \leq 1$

$\therefore$ Range of f $= \left[\frac{1}{11}, \frac{1}{5}\right]$

(ii) Let $y = \log_2\left(\dfrac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$

$\Rightarrow 2^y = \sin\left(x - \frac{\pi}{4}\right) + 3 \Rightarrow -1 \leq 2^y - 3 \leq 1$

$\Rightarrow \quad 2 \le 2^y \le 4 \quad \Rightarrow \quad y \quad \Rightarrow \quad [1, 2]$

(iii) $f(x) = \log_{\sqrt{2}}\left(2 - \log_2(16\sin^2 x + 1)\right)$

$1 \le 16\sin^2 x + 1 \le 17$

$\therefore \quad 0 \le \log_2(16\sin^2 x + 1) \le \log_2 17$

$\therefore \quad 2 - \log_2 17 \le 2 - \log_2(16\sin^2 x + 1) \le 2$

Now consider $0 < 2 - \log_2(16\sin^2 x + 1) \le 2$

$\therefore \quad -\infty < \log_{\sqrt{2}}[2 - \log_2(16\sin^2 x + 1)] \le \log_{\sqrt{2}} 2 = 2$

$\therefore$ the range is $(-\infty, 2]$

GOLDEN KEY POINTS

- To check whether relation is a function, vertical line test can be applied on its graph.
- Domain of f_1, f_2 be D_1, D_2 then domain of $f_1 + f_2$, $f_1 - f_2$, $f_1 \times f_2$ will be $D_1 \cap D_2$.
- f and g one are two functions defined for same domain, range of f is R and g is a bounded in domain. Then range of $f(x) + g(x)$ is R.
- If domain consist of discrete number of elements range can be found by direct substituting the values of x.

BEGINNER'S BOX-2

1. Find the domains of the following function:

(A) $y = \log_{10}(x+3)$ (B) $y = \sqrt{5-2x}$ (C) $y = 1/(x^2-1)$ (D) $y = \frac{1}{x^2+1}$

(E) $y = \frac{1}{x^3 - x}$ (F) $y = \frac{2x}{x^2 - 3x + 2}$

2. Find the domains of the following function:

(A) $y = 1 - \sqrt{1-x^2}$ (B) $y = 1/\sqrt{x^2 - 4x}$ (C) $y = \sqrt{x^2 - 4x + 3}$

3. Find the domains of the following function:

(A) $y = \sqrt{1 - |x|}$ (B) $y = \frac{1}{\sqrt{|x| - x}}$ (C) $y = \frac{1}{\sqrt{x - |x|}}$

4. Find the domains of the following function:

(A) $y = \sqrt{\log_{10}\left(\frac{5x - x^2}{4}\right)}$ (B) $y = \log_{10} \sin x$

(C) $y = \log_x 2$ (D) $y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

5. Find the domains of the following function:

(A) $y = \frac{3}{4 - x^2} + \log_{10}(x^3 - x)$ (B) $y = \log_{10} \sin(x-3) + \sqrt{16 - x^2}$

(C) $y = \sqrt{\sin x} + \sqrt{16 - x^2}$

6. Find the domains of the following function:

(A) $y = \sqrt{\frac{x-2}{x+2}} + \sqrt{\frac{1-x}{1+x}}$ (B) $y = \sqrt{x^2 - 3x + 2} + \frac{1}{\sqrt{3 + 2x - x^2}}$

7. Find the domains of the following function:

(A) $y=(x^2+x+1)^{-3/2}$

(B) $y=\log_{10}\left(\sqrt{x-4}+\sqrt{6-x}\right)$

(C) $y=\log_{10}\left[1-\log_{10}\left(x^2-5x+16\right)\right]$

(D) $f(x)=\sqrt{\sin\sqrt{x}}$

8. The number of integers lying in the domain of the function $f(x)=\sqrt{\log_{0.5}\left(\frac{5-2x}{x}\right)}$ is -

(A) 3 (B) 2 (C) 1 (D) 0

9. The domain of the function $f(x)=\sqrt{\frac{1-|x|}{2-|x|}}$ is -

(A) $x\in(-\infty,-1)\cup(1,\infty)$

(B) $x\in(-\infty,-2)\cup(2,\infty)$

(C) $x\in[-1,1]\cup(-\infty,-2)\cup(\infty,2)$

(D) None of these

10. Domain to function $\sqrt{\log\{(5x-x^2)/6\}}$ is -

(A) (2, 3) (B) [2, 3] (C) [1, 2] (D) [1, 3]

11. The domain of $f(x)=\sqrt{\log_{1/3}\left(\frac{3x-1}{x+2}\right)-1}$ is -

(A) $\left(\frac{1}{3},\frac{5}{8}\right]$ (B) $\left(\frac{5}{8},\infty\right)$ (C) $(-\infty,-2)\cup\left(\frac{5}{8},\infty\right)$ (D) $\left(-2,\frac{5}{8}\right)$

5.0 IMPORTANT TYPES OF FUNCTION

(a) Polynomial function

If a function 'f' is called by $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$ where n is a non negative integer and $a_0, a_1, a_2,\ldots a_n$ are real numbers and $a_0\neq 0$, then f is called a polynomial function of degree n.

NOTE

(i) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x)=ax$, $a\neq 0$

(ii) There are two polynomial functions, satisfying the relation; $f(x).\ f(1/x)=f(x)+f(1/x)$. They are (1) $f(x)=x^n+1$ & (2) $f(x)=1-x^n$, where n is a positive integer.

(iii) Domain of a polynomial function is R

(iv) Range of odd degree polynomial is R whereas range of an even degree polynomial is never R.

(b) Algebraic function

A function 'f' is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking radicals) starting with polynomials.

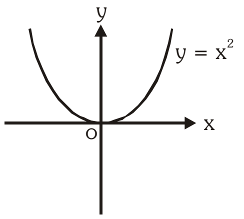
Examples – $f(x)=\sqrt{x^2+1}$; $g(x)=\frac{x^4-16x^2}{x+\sqrt{x}}+(x-2)\sqrt[3]{x+1}$

If y is an algebraic function of x, then it satisfies a polynomial equation of the form $P_0(x)y^n+P_1(x)y^{n-1}+\ldots\ldots+P_{n-1}(x)y+P_n(x)=0$, where 'n' is a positive integer and $P_0(x), P_1(x),\ldots\ldots$ are polynomial in x.

Note that all polynomial functions are Algebraic but the converse in not true. A functions that is not algebraic is called **TRANSCEDENTAL** function.

Basic algebraic function

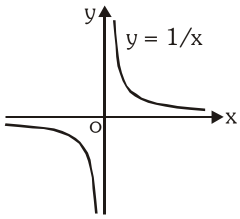
(i) $y = x^2$



Domain : R

Range : $R^+ \cup \{0\}$ or $[0,\infty)$

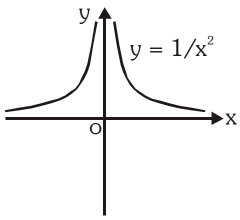
(ii) $y = \dfrac{1}{x}$



Domain : $R - \{0\}$ or R_0

Range : $R - \{0\}$

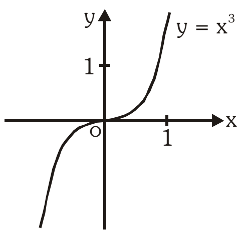
(iii) $y = \dfrac{1}{x^2}$



Domain : R_0

Range : R^+ or $(0, \infty)$

(iv) $y = x^3$



Domain : R

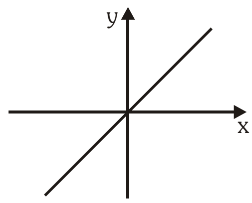
Range : R

(c) Rational function

A rational function is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$, where g(x) & h(x) are polynomials & $h(x) \neq 0$, **Domain –** $R-\{x \mid h(x)=0\}$

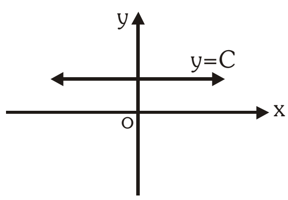
Any rational function is automatically an algebraic function.

(d) Identity function



The function $f– A \to A$ defined by $f(x) = x \ \forall\ x \in A$ is called the identity function on A and is denoted by I_A. It is easy to observe that identity function is a bijection.

(e) Constant function



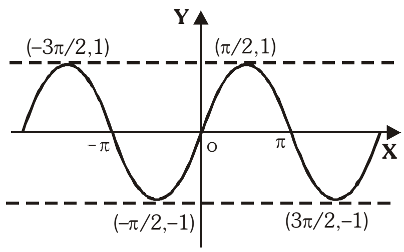
$f : A \to B$ is said to be constant function if every element of A has the same f image in B. Thus $f : A \to B$; $f(x) = c$, $\forall\ x \in A$, $c \in B$ is constant function. Note that the range of a constant function is a singleton set.

Domain – R **Range –** {C}

(f) Trigonometric functions

(i) Sine function

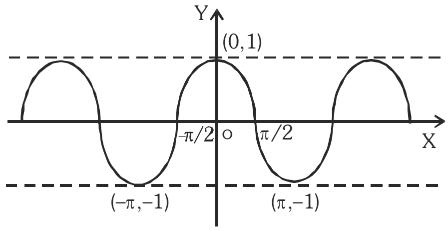
$f(x) = \sin x$



Domain – R

Range – [–1, 1], period 2π

(ii) Cosine function

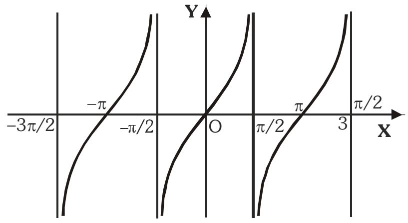


$f(x) = \cos x$

Domain : R

Range – [–1, 1], period 2π

(iii) Tangent function

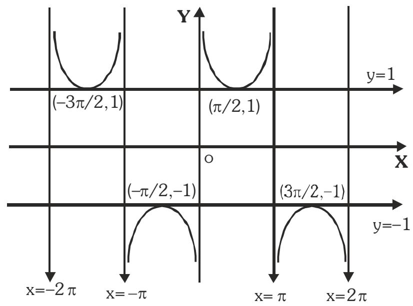


$f(x) = \tan x$

Domain – $R - \left\{x \mid x = \frac{(2n+1)\pi}{2}, n \in I\right\}$

Range – R , period π

(iv) Cosecant function

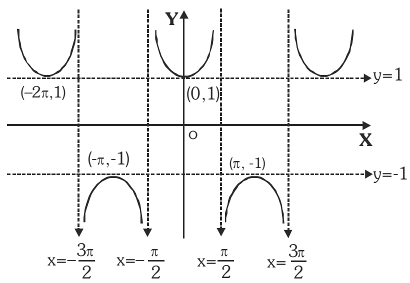


$f(x) = \text{cosec } x$

Domain – $R - \{x|x = n\pi,\ n \in I\}$

Range – $R - (-1, 1)$, period 2π

(v) Secant function

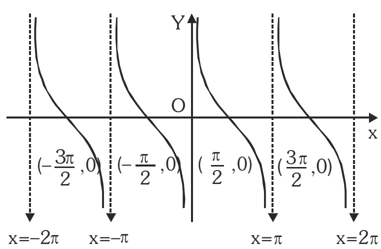


$f(x) = \sec x$

Domain – $R - \{x|x = (2n + 1)\ \pi/2 : n \in I\}$

Range – $R - (-1, 1)$, period 2π

(vi) Cotangent function



$f(x) = \cot x$

Domain – $R - \{x|x = n\pi,\ n \in I\}$

Range – R, period π

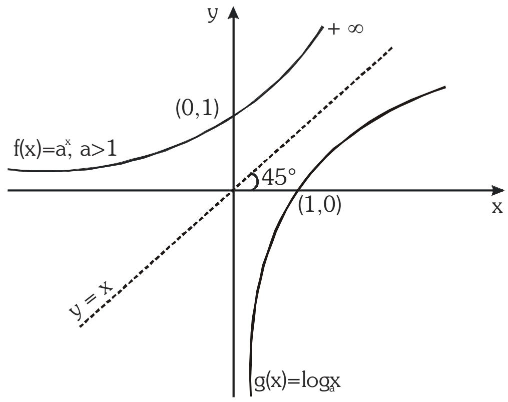
(g) Exponential and Logarithmic Function

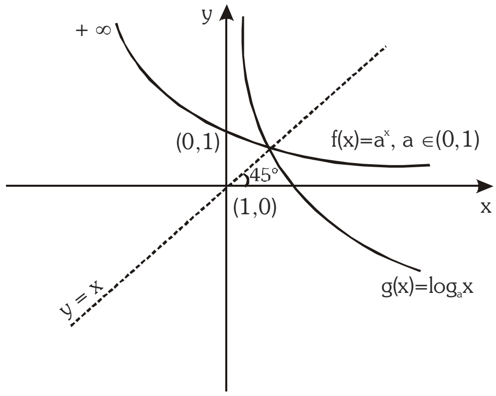
A function $f(x) = a^x (a > 0),\ a \neq 1,\ x \in R$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, i.e. $g(x) = \log_a x$.

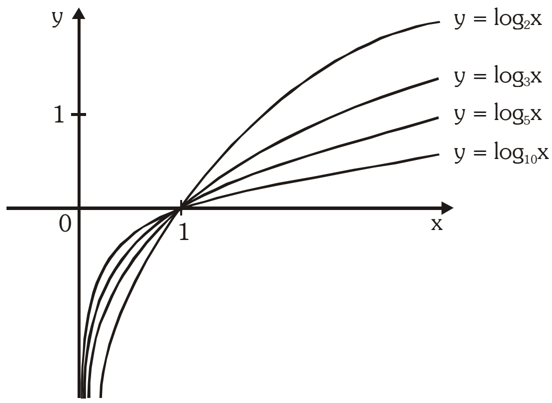
Note that f(x) & g(x) are inverse of each other & their graphs are as shown. (If functions are mirror image of each other about the line $y = x$)

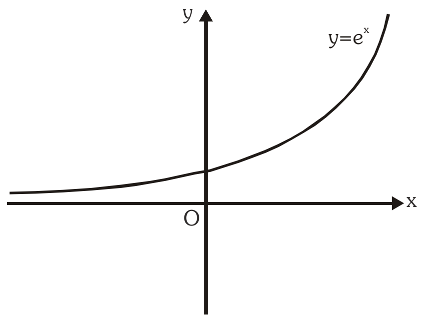
Domain of a^x is R **Range** R^+

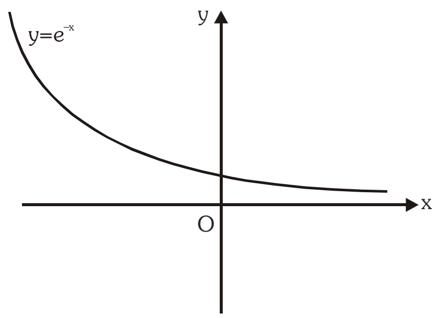
Domain of $\log_a x$ is R^+ **Range** R











Note-1 – $f(x) = a^{1/x},\ a > 0$ **Domain** – $R - \{0\}$ **Range** – $R^+ - \{1\}$

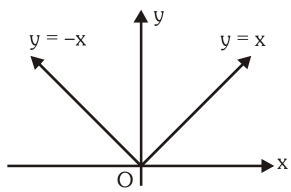
Note-2 – $f(x) = \log_x a = \dfrac{1}{\log_a x}$ **Domain** – $R^+ - \{1\}$ **Range** – $R - \{0\}$

$(a > 0)\ (a \neq 1)$

(h) Absolute value function

The absolute value (or modulus) of a real number x (written $|x|$) is a non negative real number that satisfies the conditions.

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$



Domain : R
Range : $[0, \infty)$

The properties of absolute value function are

(i) The inequality $|x| \le \alpha$ means that $-\alpha \le x \le \alpha$; if $\alpha > 0$

(ii) The inequality $|x| \ge \alpha$ means that $x \ge \alpha$ or $x \le -\alpha$ if $\alpha > 0$

(iii) $|x \pm y| \le |x| + |y|$ **(Triangle Inequality)** Equality holds when $x.y \ge 0$

(iv) $|x \pm y| \ge ||x| - |y||$ **(Triangle Inequality)** Equality holds when $x.y \ge 0$

(v) $|xy| = |x|.|y|$

(vi) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, $(y \ne 0)$

Note – $f(x) = \frac{1}{|x|}$, **Domain :** $R - \{0\}$, **Range :** R^+

Illustrations

Illustration 27. Determine the values of x satisfying the equality.

$|(x^2 + 4x + 9) + (2x - 3)| = |x^2 + 4x + 9| + |2x - 3|$.

Solution The equality $|a + b| = |a| + |b|$ is valid if and only if both summands have the same sign,

$\because$ $x^2 + 4x + 9 = (x + 2)^2 + 5 > 0$ at any values of x, the equality is satisfied at those values of x at which $2x - 3 \ge 0$, i.e. at $x \ge \frac{3}{2}$.

Illustration 28. Determine the values of x satisfying the equality $|x^4 - x^2 - 6| = |x^4 - 4| - |x^2 + 2|$.

Solution The equality $|a - b| = |a| - |b|$ holds true if and only if a and b have the same sign and $|a| \ge |b|$.

In our case the equality will hold true for the value of x at which $x^4 - 4 \ge x^2 + 2$.

Hence $x^2 - 2 \ge 1$; $|x| \ge \sqrt{3}$.

BEGINNER'S BOX-3

1. Range of the function $f(x) = 7\sin x + 8$ is

(A) [–1, 15] (B) [1, 15] (C) [–7, 7] (D) [5, 10]

2. Range of the function $f(x) = x^2 - 7$ is

(A) $[7, \infty)$ (B) $(-\infty, 7]$ (C) $[-7, \infty)$ (D) $(-\infty, -7]$

3. Range of the function $f(x) = \sin x$ where $x \in [-1, 2]$ is

(A) [sin 1, 1] (B) [–sin 1, 1] (C) [–sin 1, sin 1] (D) [–1, 1]

4. Range of the function $f(x) = \cos x$ where $x \in [-2, 2]$, is

(A) [–1, 1] (B) [cos 2, 1] (C) [– cos 2, 1] (D) [–cos 2, cos 2]

5. Sum of all the integers in the range of the function $f(x) = x^2 - 4x + 5$, where $x \in [0, 3]$, is

(A) 15 (B) 10 (C) 25 (D) None of these

6. If range of function $f(x) = e^{\sin x}$ is $[a, b]$ then $a \cdot b =$

(A) $2e$ (B) $\frac{2}{e}$ (C) $\frac{e}{2}$ (D) 1

7. The range of function $f(x)=\ln(\sin x)$ is

(A) $(-\infty, \infty)$ (B) $(-\infty, 0]$ (C) $(-\infty, \sin 1]$ (D) $[0, \infty)$

8. Range of the function $f(x)=\ln(\ln x)$ is

(A) $(-\infty, \infty)$ (B) $(-\infty, 0]$ (C) $(-\infty, 0)$ (D) $(0, \infty)$

9. Sum of natural numbers which are not contained in the range of function $f(x)=e^{\sec x}$, is

(A) 3 (B) 9 (C) 245 (D) None of these

10. Least integer in the range of function $f(x)=e^{4(\ln x)}$ is

(A) 0 (B) –4 (C) 4 (D) 1

11. Range of $F(x)=\dfrac{1}{8\sin x+5}$ is

(A) $\left(-\infty, \dfrac{1}{3}\right]$ (B) $\left(-\infty, -\dfrac{1}{3}\right]\cup\left[\dfrac{1}{13},\infty\right)$ (C) $\left(-\infty,\dfrac{1}{13}\right]$ (D) $\left[\dfrac{1}{13},\dfrac{1}{3}\right]$

12. Range of function $f(x)=\dfrac{1}{e^x-3}$ is

(A) $\left(-\infty,-\dfrac{1}{3}\right)\cup(0,\infty)$ (B) $\left(-\infty,-\dfrac{1}{3}\right)\cup\left(\dfrac{1}{3},\infty\right)$ (C) $(-\infty, \infty)$ (D) $\left(-\dfrac{1}{3},\dfrac{1}{3}\right)$

13. The range of the function $f(x) = \log_e(3x^2 - 4x + 5)$ is

(A) $\left(-\infty, \log_e\dfrac{11}{3}\right]$ (B) $\left[\log_e\dfrac{11}{3},\infty\right)$ (C) $\left[-\log_e\dfrac{11}{3}, \log_e\dfrac{11}{3}\right]$ (D)None of these

14. Range of function $f(x)=\dfrac{4}{6\sin x+5\cos x+7}$ is

(A) $\left[\dfrac{\sqrt{61}-7}{3},\infty\right)$

(B) $\left(-\infty, \dfrac{\sqrt{61}-7}{3}\right]$

(C) $\left(-\infty,-\left(\dfrac{\sqrt{61}+7}{3}\right)\right]\cup\left[\dfrac{4}{7+\sqrt{61}},\infty\right)$

(D) $\left(-\infty,\dfrac{4}{7-\sqrt{61}}\right]$

15. Number of integers in the range of function $f(x)=5\sin x+4\cos\left(x+\dfrac{\pi}{6}\right)+3$ is

(A) 8 (B) 9 (C) 10 (D) More than 10

16. The range of the function $y = 3\sin\sqrt{\frac{\pi^2}{16} - x^2}$ is

(A) $\left[0, \frac{3}{\sqrt{2}}\right]$ (B) $\left[-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right]$

(C) $\left[-\frac{3}{\sqrt{2}}, 0\right]$ (D) None of these

17. The value of the function $f(x) = \frac{x^2 - 3x + 2}{x^2 + x - 6}$ lies in the interval

(A) $(-\infty, \infty) - \left\{\frac{1}{5}, 1\right\}$ (B) $(-\infty, \infty)$ (C) $(-\infty, \infty) - \{1\}$ (D) None of these

18. Sum of least integer and greatest integer in the range of the function $f(x) = \sin^2 x - 3\sin x + 5$ is

(A) 12 (B) 10 (C) 8 (D) 6

19. Least integer in the range of the function $f(x) = 9^x + 5 \cdot 3^x + 7$ is

(A) 3 (B) 8 (C) 7 (D) 9

20. If product of least and greatest value of the function $f(x) = \frac{x^2 + x + 1}{x^2 - x + 2}$ is $\frac{a}{b}$ where *a* and *b* are coprime then $a + b =$

(A) 5 (B) 10 (C) 15 (D) 20

21. Range of the function $f(x) = \frac{x}{1 + x^2}$ is

(A) $[-1, 1]$ (B) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (C) $\left[-1, \frac{1}{2}\right]$ (D) $\left[-\frac{1}{2}, 1\right]$

22*. Let set '*S*' be the range of the function $f(x) = 9\tan^2 x + 16\cot^2 x$ then

(A) Least value of *S* is 24 (B) Number of integers in *S* is 10

(C) S is $[24, \infty)$ (D) Least value of *S* is 25

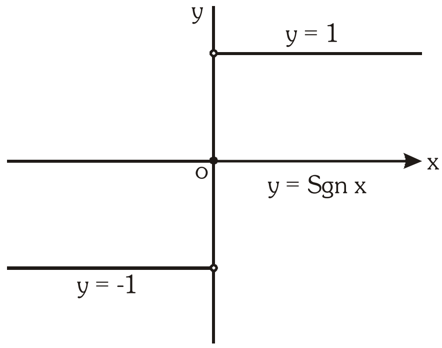
23. If $[p, q]$ is range of function $f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$ then $2p + 5q =$

24. Number of integers in the range of the function $f(x) = \cos^2 x - 5\cos x - 6$ is

(i) Signum function

Signum function y = sgn (x) is defined as follows

$$y = \begin{cases} \dfrac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{cases} = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$



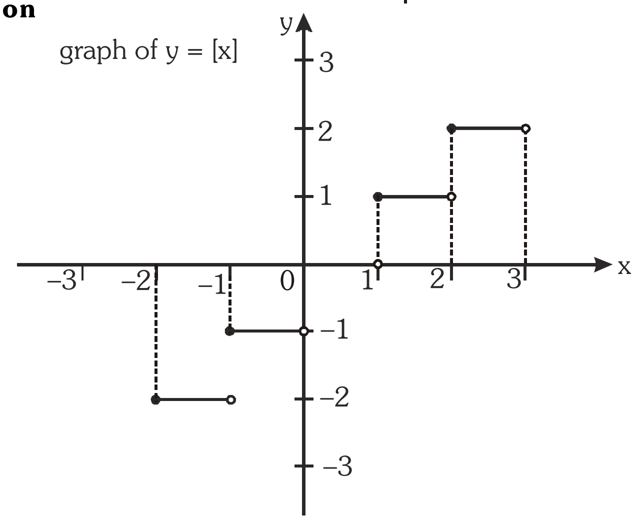
Domain – R

Range – {–1, 0, 1}

(j) Greatest integer or step up function

The function y = f(x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x. Note that for :

x	[x]
[–2,–1)	–2
[–1,0)	–1
[0,1)	0
[1,2)	1



Domain – R

Range – I

Properties of greatest integer function

(i) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x$, $0 \le x - [x] < 1$

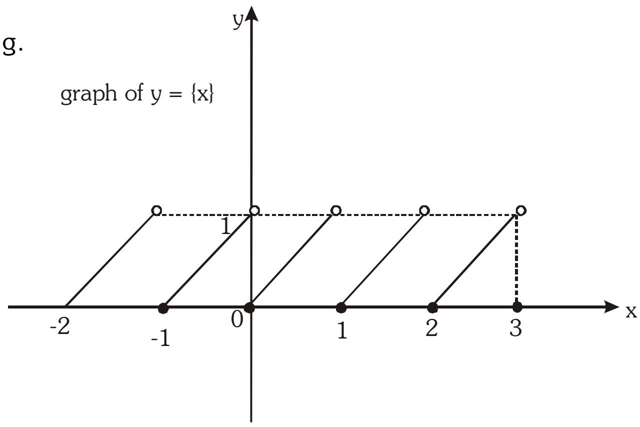
(ii) $[x + m] = [x] + m$ if m is an integer.

(iii) $[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

Note – $f(x) = \dfrac{1}{[x]}$ **Domain** – R – [0, 1) **Range** – $\{x \mid x = \dfrac{1}{n},\ n \in I_0\}$

(k) Fractional part function

It is defined as – g(x) = {x} = x – [x] e.g. the fractional part of the number 2.1 is 2.1–2 = 0.1 and the fractional part of –3.7 is 0.3 The period of this function is 1 and graph of this function is as shown.



x	{x}
[–2,–1)	x+2
[–1,0)	x+1
[0,1)	x
[1,2)	x–1

Domain – R **Range** – [0,1)

Note – $f(x) = \dfrac{1}{\{x\}}$ **Domain** – R – I **Range** – (1, ∞)

- **Properties of Fractional part function**

(i) $0 \le \{x\} < 1$ (ii) $\{[x]\} = [\{x\}] = 0$

(iii) $\{\{x\}\} = \{x\}$ (iv) $\{x+m\} = \{x\},\ m \in I$

(v) $\{x\} + \{-x\} = \begin{cases} 1, & x \notin I \\ 0, & x \in I \end{cases}$ (vi) $[x+y] = \begin{cases} [x]+[y], & \text{if } \{x\}+\{y\} < 1 \\ [x]+[y]+1, & \text{if } \{x\}+\{y\} \ge 1 \end{cases}$

Illustrations

Illustration 29. If y = 2[x] + 3 & y = 3[x – 2] + 5 then find [x + y] where [.] denotes greatest integer function.

Solution $y = 3[x-2] + 5 = 3[x] - 1$

so $3[x] - 1 = 2[x] + 3$

$[x] = 4 \Rightarrow 4 \le x < 5$

then $y = 11$

so x + y will lie in the interval [15, 16)

so $[x+y] = 15$

Illustration 30. Find the value of $\left[\frac{1}{2}\right] + \left[\frac{1}{2} + \frac{1}{1000}\right] + \ldots\ldots \left[\frac{1}{2} + \frac{2946}{1000}\right]$ where [.] denotes greatest integer function ?

Solution

$$\left[\frac{1}{2}\right] + \left[\frac{1}{2} + \frac{1}{1000}\right] + \ldots\ldots \left[\frac{1}{2} + \frac{499}{1000}\right] + \left[\frac{1}{2} + \frac{500}{1000}\right] + \ldots\ldots \left[\frac{1}{2} + \frac{1499}{1000}\right] + \left[\frac{1}{2} + \frac{1500}{1000}\right] + \ldots\ldots$$

$$+ \left[\frac{1}{2} + \frac{2499}{1000}\right] + \left[\frac{1}{2} + \frac{2500}{1000}\right] + \ldots\ldots \left[\frac{1}{2} + \frac{2946}{1000}\right]$$

$= 0 + 1 \times 1000 + 2 \times 1000 + 3 \times 447 = 3000 + 1341 = 4341$ **Ans.**

Illustration 31. Find the domain $f(x) = \dfrac{1}{\sqrt{|[|x|-5]| - 11}}$ where [.] denotes greatest integer function.

Solution $|[|x|-5]| > 11$

so $[|x|-5] > 11$ or $[|x|-5] < -11$

$[|x|] > 16$ $[|x|] < -6$

$|x| \ge 17$ or $|x| < -6$ (Not Possible)

$\Rightarrow x \le -17$ or $x \ge 17$

so $x \in (-\infty, -17] \cup [17, \infty)$

Illustration 32. Find the range of $f(x) = \dfrac{x-[x]}{1+x-[x]}$, where [.] denotes greatest integer function.

Solution $y = \dfrac{x-[x]}{1+x-[x]} = \dfrac{\{x\}}{1+\{x\}}$

$\therefore \dfrac{1}{y} = \dfrac{1}{\{x\}} + 1 \Rightarrow \dfrac{1}{\{x\}} = \dfrac{1-y}{y} \Rightarrow \{x\} = \dfrac{y}{1-y}$

$0 \le \{x\} < 1 \Rightarrow 0 \le \dfrac{y}{1-y} < 1$

Range = [0, 1/2)

Illustration 33. Solve the equation $|2x-1| = 3[x] + 2\{x\}$ where [.] denotes greatest integer and {.} denotes fractional part function.

Solution We are given that, $|2x-1| = 3[x] + 2\{x\}$

Let, $2x - 1 \le 0$ i.e. $x \le \frac{1}{2}$. The given equation yields.

$1 - 2x = 3[x] + 2\{x\}$

$\Rightarrow \quad 1 - 2[x] - 2\{x\} = 3[x] + 2\{x\} \quad \Rightarrow \quad 1 - 5[x] = 4\{x\} \Rightarrow \{x\} = \frac{1-5[x]}{4}$

$\Rightarrow \quad 0 \le \frac{1-5[x]}{4} < 1 \Rightarrow 0 \le 1 - 5[x] < 4 \Rightarrow -\frac{3}{5} < [x] \le \frac{1}{5}$

Now, $[x] = 0$ as zero is the only integer lying between $-\frac{3}{5}$ and $\frac{1}{5}$

$\Rightarrow \quad \{x\} = \frac{1}{4} \Rightarrow x = \frac{1}{4}$ which is less than $\frac{1}{2}$, Hence $\frac{1}{4}$ is one solution.

Now, let $2x - 1 > 0$ i.e $x > \frac{1}{2}$

$\Rightarrow \quad 2x - 1 = 3[x] + 2\{x\} \quad \Rightarrow \quad 2[x] + 2\{x\} - 1 = 3[x] + 2\{x\}$

$\Rightarrow \quad [x] = -1 \Rightarrow -1 \le x < 0$ which is not a solution as $x > \frac{1}{2}$

$\Rightarrow \quad x = \frac{1}{4}$ is the only solution.

BEGINNER'S BOX-4

1. Sketch the graphs of the following functions :

(i) $f(x) = x + 1$ (ii) $f(x) = x^2 - x + 5$ (iii) $f(x) = \sin x + 2$

(iv) $f(x) = \frac{2}{x-3}$ (v) $f(x) = \cos\left(\frac{\pi x}{2}\right)$ (vi) $f(x) = -\sin\left(\frac{\pi x}{3}\right)$

2. Sketch the graphs of the following functions :

(i) $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \end{cases}$ (ii) $f(x) = \begin{cases} 3-x, & x \le 1 \\ 2-x & x > 1 \end{cases}$

(iii) $f(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ x, & x \ge 0 \end{cases}$ (iv) $f(x) = \begin{cases} 1-x^2, & 0 < x \le 1 \\ 2-x^2, & 1 < x \le 2 \end{cases}$

3. Draw the garph of the function $f(x) = \frac{|\ell n x|}{\ell n x}$.

4. Find the number of solutions of the equation $y = |\sin x|$ and $x^2 + y^2 = 1$.

5. Draw the graph of the function $f(x) = |\sin x| + \sin x$ on the interval $[0, 3\pi]$.

6. Draw the graph of the function $y = x^2$ Sgn (x) where Sgn (x) $= \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$.

7. Sketch the graphs of the following functions :

(i) $f(x) = 1 + |\sin x|$ (ii) $f(x) = \dfrac{|(x-1)(x-2)|}{(x-1)(x-2)}$

8. Sketch the graphs of the given system in xy plane : $|x| + |y| = 2$

9. Sketch the graphs of the following functions :

(i) $f(x) = \text{Sgn}(\ln x)$ (ii) $f(x) = \text{Sgn}(1 - |x|)$

(iii) $f(x) = \text{Sgn}[x]$, where [.] denotes the greatest integer function.

10. Sketch the graphs of the following functions :

(i) $f(x)=\begin{cases}\sin x & -\pi \le x \le 0\\ 2, & 0<x\le 1\\ \dfrac{1}{x-1}, & 1<x\le 4\end{cases}$ (ii) $f(x)=\begin{cases}-2 & x>0\\ \dfrac{1}{2} & x=0\\ -x^3, & x<0\end{cases}$

6.0 ALGEBRAIC OPERATIONS ON FUNCTIONS

If f & g are real valued functions of x with domain set A, B respectively, f + g, f – g, (f. g) & (f/g) as follows.

(a) $(f \pm g)(x) = f(x) \pm g(x)$ domain in each case is $A \cap B$

(b) $(f.g)(x) = f(x).g(x)$ domain is $A \cap B$

(c) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ domain $A \cap B - \{x \mid g(x) = 0\}$

7.0 EQUAL OR IDENTICAL FUNCTION

Two function f & g are said to be equal if.

(a) The domain of f = the domain of g

(b) The range of f = range of g and

(c) f(x) = g (x), for every x belonging to their common domain (i.e. should have the same graph)

e.g. $f(x) = \dfrac{1}{x}$ & $g(x) = \dfrac{x}{x^2}$ are identical functions.

Illustrations

Illustration 34. The functions $f(x) = \log(x-1) - \log(x-2)$ and $g(x) = \log\left(\dfrac{x-1}{x-2}\right)$ are identical when x lies in the interval

(A) [1, 2] (B) $[2, \infty)$ (C) $(2, \infty)$ (D) $(-\infty, \infty)$

Solution Since $f(x) = \log(x-1) - \log(x-2)$.

Domain of f(x) is $x > 2$ or $x \in (2, \infty)$ .....(i)

$g(x) = \log\left(\dfrac{x-1}{x-2}\right)$ is defined if $\dfrac{x-1}{x-2} > 0 \Rightarrow x \in (-\infty, 1) \cup (2, \infty)$ .....(ii)

From (i) and (ii), $x \in (2, \infty)$. **Ans. (C)**

8.0 HOMOGENEOUS FUNCTIONS

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For examples $5x^2 + 3y^2 - xy$ is homogenous in x & y. Symbolically if, $f(tx, ty) = t^n f(x, y)$ then $f(x, y)$ is homogeneous function of degree n.

Illustrations

Illustration 35. Which of the following function is not homogeneous ?

(A) $x^3 + 8x^2y + 7y^3$ (B) $y^2 + x^2 + 5xy$ (C) $\frac{xy}{x^2+y^2}$ (D) $\frac{2x-y+1}{2y-x+1}$

Solution It is clear that (D) does not have the same degree in each term. **Ans. (D)**

9.0 BOUNDED FUNCTION

A function is said to be bounded if $|f(x)| \le M$, where M is a finite quantity.

10.0 IMPLICIT & EXPLICIT FUNCTION

A function defined by an equation not solved for the dependent variable is called an **implicit function**. e.g. the equations $x^3 + y^3 = 1$ & $x^y = y^x$, defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit function.**

Illustrations

Illustration 36. Which of the following function is implicit function ?

(A) $y = \frac{x^2+e^x+5}{\sqrt{1-\cos^{-1}x}}$ (B) $y = x^2$ (C) $xy - \sin(x + y) = 0$ (D) $y = \frac{x^2 \log x}{\sin x}$

Solution It is clear that in (C) y is not clearly expressed in x. **Ans. (C)**

11.0 ODD & EVEN FUNCTIONS

If a function is such that whenever 'x' is in it's domain '–x' is also in it's domain & it satisfies

$f(-x) = f(x)$ it is an even function

$f(-x) = -f(x)$ it is an odd function

Note

(i) A function may neither be odd nor even.

(ii) Inverse of an even function is not defined, as it is many – one function.

(iii) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

(iv) Every function which has '–x' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function .

e.g. $f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$

(v) The only function which is defined on the entire number line & even and odd at the same time is $f(x) = 0$

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$	$f(x) \cdot g(x)$	$f(x)/g(x)$	$(gof)(x)$	$(fog)(x)$
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Illustrations

Illustration 37. Which of the following functions is (are) even, odd or neither :

(i) $f(x) = x^2\sin x$ (ii) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$ (iii) $f(x) = \log\left(\frac{1-x}{1+x}\right)$

(iv) $f(x) = \sin x - \cos x$ (v) $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

(i) $f(-x) = (-x)^2 \sin(-x) = -x^2 \sin x = -f(x)$. Hence f(x) is odd.

(ii) $f(-x) = \sqrt{1+(-x)+(-x)^2} - \sqrt{1-(-x)+(-x)^2}$

$= \sqrt{1-x+x^2} - \sqrt{1+x+x^2} = -f(x)$. Hence f(x) is odd.

(iii) $f(-x) = \log\left(\frac{1-(-x)}{1+(-x)}\right) = \log\left(\frac{1+x}{1-x}\right) = -f(x)$. Hence f(x) is odd

(iv) $f(-x) = \sin(-x) - \cos(-x) = -\sin x - \cos x$.

Hence f(x) is neither even nor odd.

(v) $f(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = f(x)$. Hence f(x) is even

Illustration 38. Identify the given functions as odd, even or neither :

(i) $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$ (ii) $f(x + y) = f(x) + f(y)$ for all $x, y \in R$

Solution

(i) $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$

Clearly domain of f(x) is $R \sim \{0\}$. We have,

$$f(-x) = \frac{-x}{e^{-x} - 1} - \frac{x}{2} + 1 = \frac{-e^x.x}{1 - e^x} - \frac{x}{2} + 1 = \frac{(e^x - 1 + 1)x}{(e^x - 1)} - \frac{x}{2} + 1$$

$$= x + \frac{x}{e^x - 1} - \frac{x}{2} + 1 = \frac{x}{e^x - 1} + \frac{x}{2} + 1 = f(x)$$

Hence f(x) is an even function.

(ii) $f(x + y) = f(x) + f(y)$ for all $x, y \in R$

Replacing x, y by zero, we get $f(0) = 2f(0) \Rightarrow f(0) = 0$

Replacing y by –x, we get $f(x) + f(-x) = f(0) = 0 \Rightarrow f(x) = -f(-x)$

Hence f(x) is an odd function.

BEGINNER'S BOX-5

1. Are the following functions identical?

(a) $f(x) = \frac{x}{x}$ and $\varphi(x) \equiv 1$; (b) $f(x) = \log x^2$ and $\varphi(x) = 2\log x$;

(c) $f(x) = x$ and $\varphi(x) = \left(\sqrt{x}\right)^2$;

(d) $f(x) = 1$ and $\varphi(x) = \sin^2 x + \cos^2 x$;

(e) $f(x) = \log(x-1) + \log(x-2)$ and $\varphi(x) = \log(x-1)(x-2)$.

2. Are the following functions identical?

(a) f(x) = $\frac{x}{x^2}$ and $\varphi(x)=\frac{1}{x}$; (b) $f(x)=\frac{x^2}{x}$ and φ(x) = x; (c) f(x) = x and $\varphi(x)=\sqrt{x^2}$;

3. In what interval are the following functions identical?

(a) f (x) = x and φ (x) = $10^{\log_{10} x}$; (b) f(x) = $\sqrt{x}\sqrt{x-1}$ and φ(x) = $\sqrt{x(x-1)}$

4*. Which of the following functions are not homogeneous?

(A) $x+y\cos\frac{y}{x}$ (B) $\frac{xy}{x+y^2}$ (C) $\frac{x+y\cos x}{y\sin x+y}$ (D) $\frac{x}{y}\ell n\left(\frac{y}{x}\right)+\frac{y}{x}\ell n\left(\frac{x}{y}\right)$

5*. Which of the following homogeneouos functions are of degree zero?

(A) $\frac{x^2}{y^2}\ell n\frac{y}{x}+\frac{y^2}{x^2}\ell n\frac{x}{y}$ (B) $\frac{x(x-y)}{y(x+y)}$ (C) $\frac{x^2y}{x^3+y^3}$ (D) $x\sin\frac{y}{x}-y\cos\frac{y}{x}$

6. Determine whether even or odd:

(i) f(x) = log $\left(x+\sqrt{1+x^2}\right)$ (ii) $f(x)=x\left(\frac{a^x+1}{a^x-1}\right)$

(iii) f(x) = sin x + cos x (iv) f(x) = $x^2-|x|$

(v) f(x) = log $\left(\frac{1-x}{1+x}\right)$

(vi) f(x + y) + f(x – y) = 2f(x). f(y); where f(0) ≠ 0 and x, y ∈ R.

7. If f(x) satisfies the relation, f (x + y) = f(x) + f(y) for all x, y ∈ R and

f(1) = 5, find $\sum_{x=1}^{m} f(x)$ Also prove that f(x) is odd.

8. A function defined for all real numbers is defined for $x \ge 0$ as follows : $f(x)=\begin{cases} x|x|, & 0\le x<1 \\ 2x, & x\ge 1\end{cases}$

How is f defined for $x \le 0$ if (i) is even ? (ii) f is odd ?

9. Find out whether the given function is even, odd or neither even nor odd where, f(x) = $\begin{cases} x|x| & , \quad x\le -1 \\ [1+x]+[1-x], & -1<x<1 \\ -x|x| & , \quad x\ge 1\end{cases}$

. where | | and [ ] represents modulus and greatest integral function.

10. Find whether the given function is even or odd function, where

f(x) = $\frac{x(\sin x+\tan x)}{\left[\frac{x+\pi}{\pi}\right]-\frac{1}{2}}$ where, [ ] denotes greatest integer function.

11. Function $f(x)=x+x^3$ is

(A) Even (B) Odd (C) Neither even nor odd (D) None of these

12. $f(x)=\sin x\cdot\cos x$ is

(A) An even function (B) An odd function

(C) Neither even nor odd function (D) None of these

13. The function $f(x)=\sin x+\cos x$ is

(A) An even function (B) An odd function

(C) Neither even nor odd function (D) None of these

14. The function $f(x)=\ln\left(\dfrac{1-x}{1+x}\right)$ is

(A) An even function (B) An odd function

(C) Neither even nor odd function (D) None of these

15. The function $f(x)=\ln\left(\dfrac{1-x^2}{1+x^2}\right)$ is

(A) An even function (B) An odd function

(C) Neither even nor odd fucntion (D) None of these

16. Which of the following is not an odd function ?

(A) $g(x)-g(-x)$ (B) $\sin(g^4(x)-g^4(-x))$

(C) $\ell n\left(\dfrac{x^4+x^2+1}{x^2+x+1}\right)$ (D) $x^3g^6(x).g^6(-x)+\tan^3(\sin^5x)$

Match the entries of **column- I** with one or more entries of the elements of **column - II**.

17*.

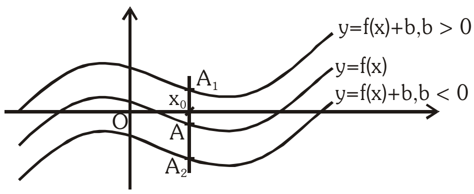
Column-I	Column-II
(A) $f(x)=\sqrt{1-x+x^4}-\sqrt{1+x+x^4}$	(P) Is even
(B) $f(x)=\ln\left(\sqrt{1+x^2}+x\right)$	(Q) Is odd
(C) $f(x)=x+\cos(x)$	(R) Is neither even nor odd
(D) $f(x)=0$	(S) Has graph symmetric about y-axis

18*.

Column-I	Column-II
(A) $f(x)=x^3+8x+9$	(P) Has graph symmetric about y-axis
(B) $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$	(Q) Has graph symmetric about origin
(C) $f(x)=\dfrac{x}{e^x-1}+\dfrac{x}{2}+1$	(R) Is neither even nor odd function
(D) $f(x)=\dfrac{g^2(x)-g^2(-x)}{g^2(x)+g^2(-x)}\forall x\in R$	(S) Is odd function

12.0 BASIC TRANSFORMATIONS ON GRAPHS

(i) Drawing the graph of $y = f(x) + b$, $b \in \mathbf{R}$, from the known graph of $y = f(x)$



It is obvious that domain of f(x) and f(x) + b are the same. Let us take any point x_0 in the domain of f(x). $y|_{x=x_0} = f(x_0)$.

The corresponding point on f(x) + b would be $f(x_0) + b$.

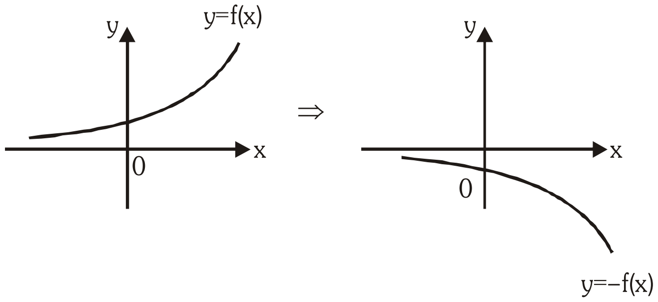
For $b > 0 \Rightarrow f(x_0) + b > f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units above the point on f(x).

For $b < 0 \Rightarrow f(x_0) + b < f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units below the point on f(x).

Accordingly the graph of f(x) + b can be obtained by translating the graph of f(x) either in the positive y-axis direction (if b > 0) or in the negative y-axis direction (if b < 0), through a distance |b| units.

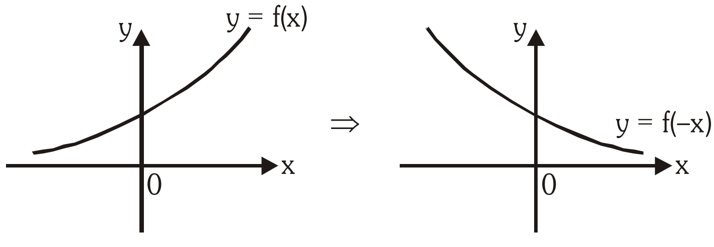
(ii) Drawing the graph of $y = -f(x)$ from the known graph of $y = f(x)$

To draw $y = -f(x)$, take the image of the curve $y = f(x)$ in the x-axis as plane mirror.



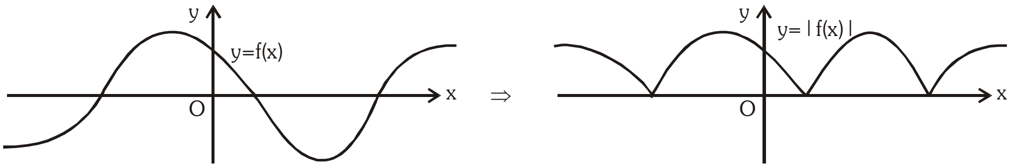
(iii) Drawing the graph of $y = f(-x)$ from the known graph of $y = f(x)$

To draw $y = f(-x)$, take the image of the curve $y = f(x)$ in the y-axis as plane mirror.



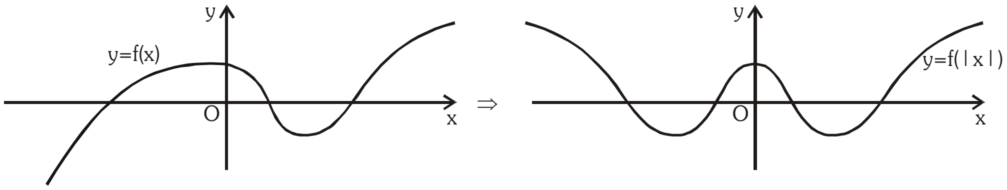
(iv) Drawing the graph of $y = |f(x)|$ from the known graph of $y = f(x)$

$|f(x)| = f(x)$ if $f(x) \geq 0$ and $|f(x)| = -f(x)$ if $f(x) < 0$. It means that the graph of f(x) and |f(x)| would coincide if $f(x) \geq 0$ and for the portions where $f(x) < 0$ graph of |f(x)| would be image of $y = f(x)$ in x-axis.



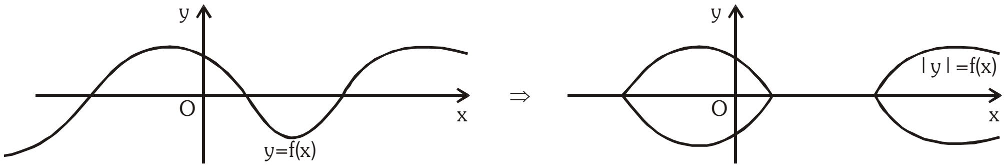
(v) Drawing the graph of y = f(|x|) from the known graph of y = f(x)

It is clear that, $f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$. Thus f(|x|) would be a even function, graph of f(|x|) and f(x) would be identical in the first and the fourth quadrants (ax $x \geq 0$) and as such the graph of f(|x|) would be symmetric about the y-axis (as (|x|) is even).

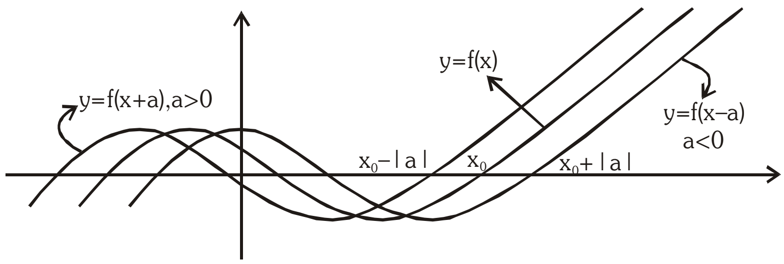


(vi) Drawing the graph of |y| = f(x) from the known graph of y = f(x)

Clearly $|y| \geq 0$. If $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$. Hence graph of $|y| = f(x)$ would exist only in the regions where f(x) is non-negative and will be reflected about the x-axis only in those regions.



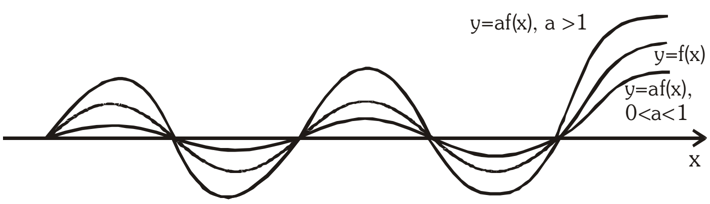
(vii) Drawing the graph of y = f(x + a), a ∈ R from the known graph of y = f(x)



(i) If $a > 0$, shift the graph of f(x) through 'a' units towards left of f(x).

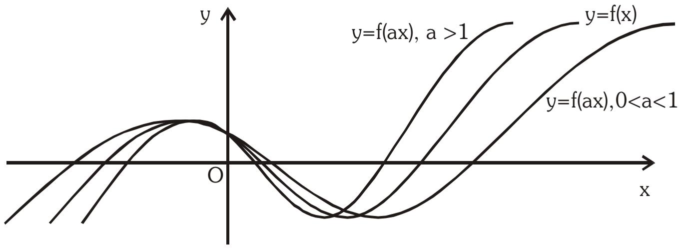
(ii) If $a < 0$, shift the graph of f(x) through 'a' units towards right of f(x).

(viii) Drawing the graph of y = af(x) from the known graph of y = f(x)



It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of 1 : a.

(ix) Drawing the graph of y = f(ax) from the known graph of y = f(x).



Let us take any point $x_0 \in$ domain of f(x). Let $ax = x_0$ or $x = \frac{x_0}{a}$.

Clearly if $0 < a < 1$, then $x > x_0$ and f (x) will stretch by $\frac{1}{a}$ units along the y-axis and if $a > 1$, $x < x_0$, then f(x) will compress by 'a' units along the y-axis.

Illustrations

Illustration 39. Find f(x) = max {1 + x, 1 – x, 2}.

Solution From the graph it is clear that

$$f(x) = \begin{cases} 1-x \;; & x < -1 \\ 2 \;; & -1 \le x \le 1 \\ 1+x \;; & x > 1 \end{cases}$$

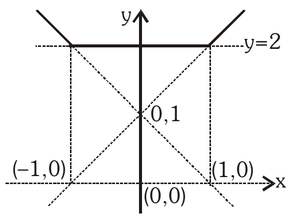
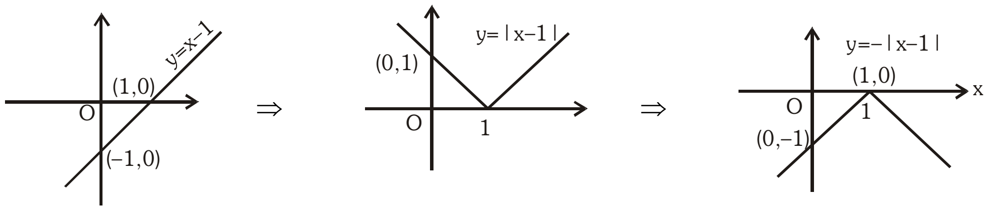


Illustration 40. Draw the graph of y = |2 –|x –1||.

Solution



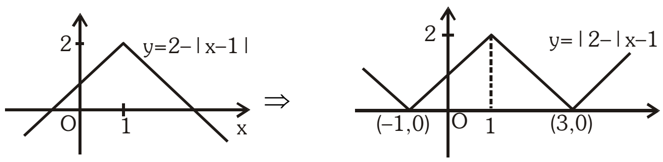
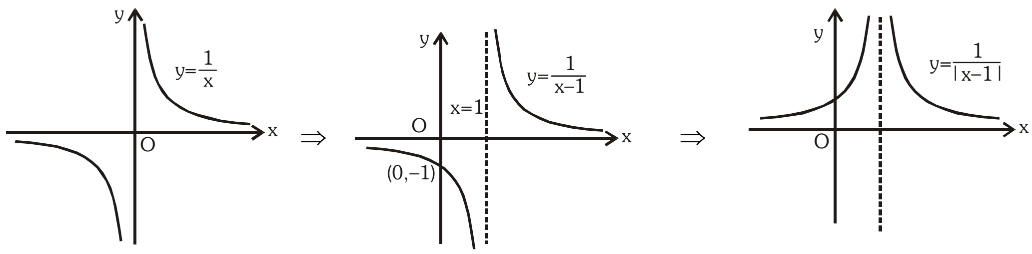


Illustration 41. Draw the graph of $y = 2 - \frac{4}{|x-1|}$

Solution



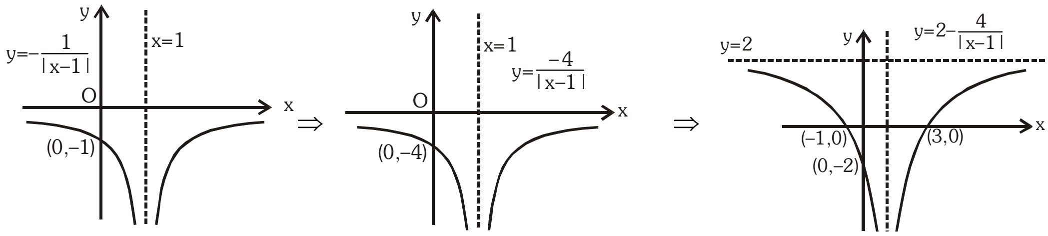


Illustration 42. Draw the graph of $y = |e^{|x|} - 2|$

Solution

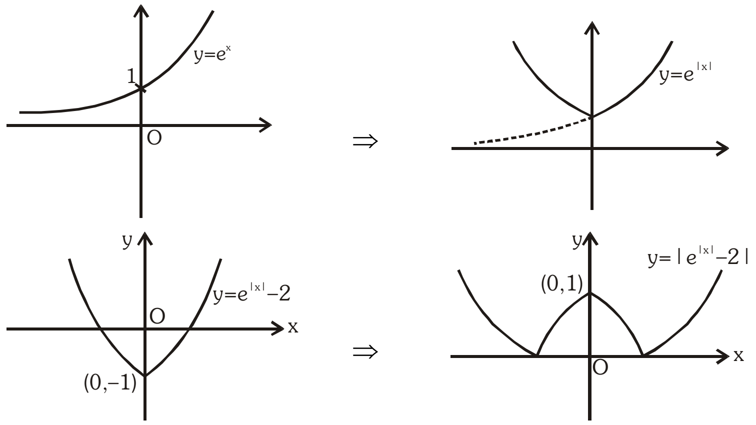
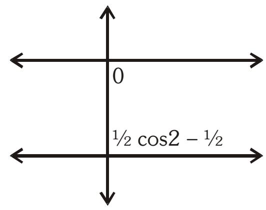


Illustration 43. Draw the graph of $f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$.

Solution $f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$

$$= \frac{1}{2}[\cos(2x+2)+\cos 2] - \frac{1}{2}[\cos(2x+2)+1]$$

$$= \frac{1}{2}\cos 2 - \frac{1}{2} < 0\,.$$



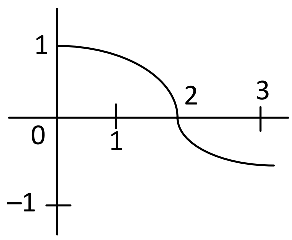
BEGINNER'S BOX–6

1. Draw the graph of the following functions using graphical transformation:

(a) $y = \left||x|-1\right|$ (b) $y = |4-x^2|$ (c) $y = x^2 - 2|x|$

(d) $y = e^{-|x|}$ (e) $y = |\log_e |x||$ (f) $y = -e^{-x}$

2. Given the curve $y = f(x)$



Draw: (i) $y = f(x + 1)$ (ii) $y = f(x/2)$

(iii) $y = |f(x)|$ (iv) $y = \frac{|f(x)| \pm f(x)}{2}$

(v) $y = \frac{|f(x)|}{f(x)}$

3. Draw the graph of $f(x) = 3|x| - x^2 - 2$

4. Draw the graph of $f(x) = |x^2 - 2|x| - 3|$

5. Draw the graph of $f(x) = \left|e^{-|x|} - \dfrac{1}{2}\right|$

6. Draw the graph of $|f(x)| = (x-1)(x-2)$

7. Sketch the graph of $f(x) = \dfrac{1}{2-|x|}$

8. Sketch the graph of $f(x) = \dfrac{|x-1|}{|x|-1}$

9. Draw the graph of the function $f(x) = \left|\left|\dfrac{1}{x}\right| - 2\right|$.

10. Let $f_1(x) = \begin{cases} x & \text{for} \quad 0 \le x \le 1 \\ 1 & \text{for} \quad x > 1 \\ 0 & \text{otherwise} \end{cases}$ and $f_2(x) = f_1(-x)$ for all x ;

$f_3(x) = -f_2(x)$ for all x ; $f_4(x) = f_3(-x)$ for all x Which of the following is necessarily ture?

(A) $f_4(x) = f_1(x)$ for all x (B) $f_1(x) = -f_3(-x)$ for all x

(C) $f_2(-x) = f_4(x)$ for all x (D) $f_1(x) + f_3(x) = 0$ for all x

11. Sketch the graphs of the following functions :

(i) $f(x) = ||x| + 1|$ (ii) $f(x) = |\ell n\, x|$ (iii) $f(x) = \tan|x|$
(iv) $f(x) = e^{|x|} - 1$

(v) $|f(x)| = \ell n\, x$ (vi) $f(x) = x + |x|$ (vii) $f(x) = -\sin|x|$

12. Find the number of solutions of the following equations

(i) $\cos x = \tan 4x$; $0 < x < \pi$ (ii) $\log_{0.5}|x| = 2|x|$

13.0 CLASSIFICATION OF FUNCTIONS

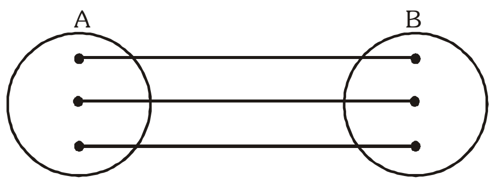
(a) One-One function (Injective mapping)

A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \ne x_2 \Leftrightarrow f(x_1) \ne f(x_2)$.

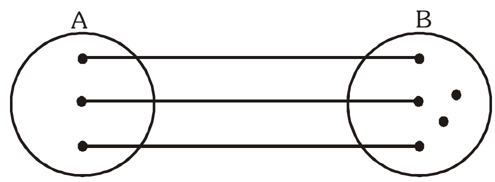
Note– (i) Any continuous function which is entirely increasing or decreasing in whole domain is one-one.

(ii) If a function is one-one, any line parallel to x-axis cuts the graph of the function at atmost one point

Diagramatically an injective mapping can be shown



or

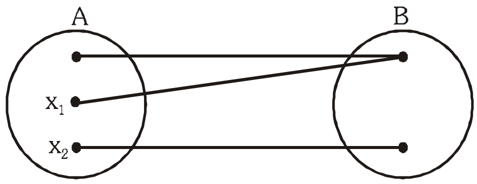


(b) Many-one function

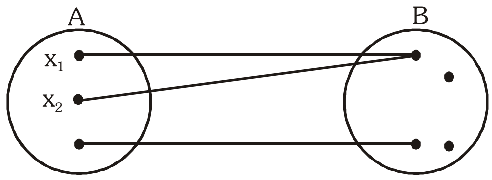
A function $f - A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f - A \to B$ is many one if $\exists\ x_1, x_2 \in A,\ f(x_1) = f(x_2)$ but $x_1 \neq x_2$

Diagramatically a many one mapping can be shown



or

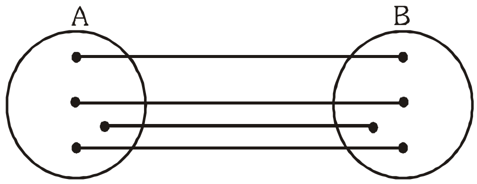


Note – If a continuous function has local maximum or local minimum, then f(x) is many-one because atleast one line parallel to x-axis will intersect the graph of function atleast twice.

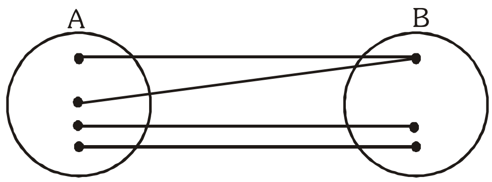
(c) Onto function (Surjective mapping)

If the function $f : A \to B$ is such that each element in B (co-domain) is the 'f' image of atleast one element in A, then we say that f is a function of A 'onto' B. Thus $f : A \to B$ is surjective if $\forall\ b \in B,\ \exists$ some $a \in A$ such that $f(a) = b$

Diagramatically surjective mapping can be shown



or

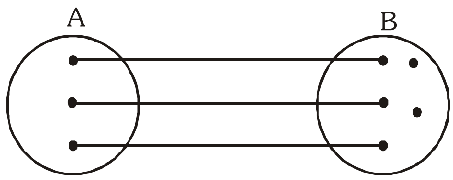


Note that – If range = co-domain, then f(x) is onto.

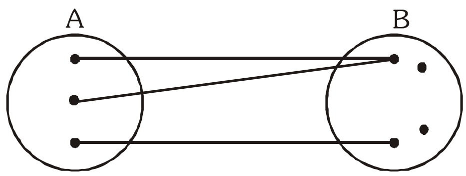
(d) Into function

If $f : A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f(x) is into.

Diagramatically into function can be shown



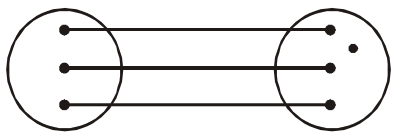
or



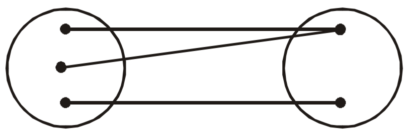
- **Thus a function can be one of these four types**

(i) one-one onto (injective & surjective)

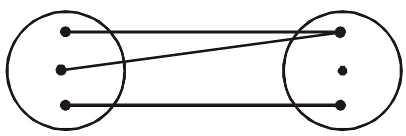
(ii) one-one into (injective but not surjective)



(iii) many-one onto (surjective but not injective)



(iv) many-one into (neither surjective nor injective)



Note – (i) If 'f' is both injective & surjective, then it is called a **Bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n! are one one and rest are many one.

(iii) $f–R \to R$ is a polynomial

(a) Of even degree, then it will neither be injective nor surjective.

(b) Of odd degree, then it will always be surjective, no general comment can be given on its injectivity.

Illustrations

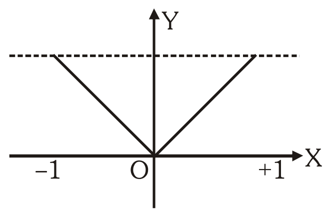
Illustration 44. Let $A = \{x : -1 \le x \le 1\} = B$ be a mapping $f : A \to B$. For each of the following functions from A to B, find whether it is surjective or bijective.

(a) $f(x) = |x|$ (b) $f(x) = x|x|$ (c) $f(x) = x^3$

(d) $f(x) = [x]$ (e) $f(x) = \sin\dfrac{\pi x}{2}$

Solution (a) $f(x) = |x|$

Graphically ;



Which shows many one, as the straight line is parallel to x-axis and cuts at two points.

Here range for $f(x) \in [0, 1]$

Which is clearly subset of co-domain i.e., $[0, 1] \subseteq [-1,1]$

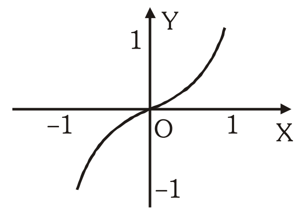
Thus, into.

Hence, function is many-one-into

$\therefore$ Neither injective nor surjective

(b) $f(x) = x|x| = \begin{Bmatrix} -x^2, & -1 < x < 0 \\ x^2, & 0 \le x < 1 \end{Bmatrix}$,

Graphically,



The graph shows f(x) is one-one, as the straight line parallel to x-axis cuts only at one point.

Here, range

$f(x) \in [-1, 1]$

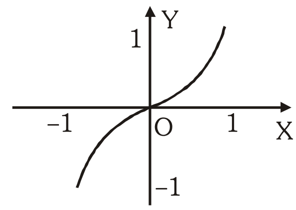
Thus, range = co-domain

Hence, onto.

Therefore, f(x) is one-one onto or (Bijective).

(c) $f(x) = x^3$,

Graphically;



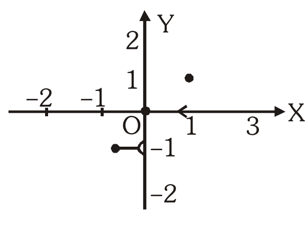
Graph shows f(x) is one-one onto

(i.e. Bijective)

[as explained in above example]

(d) $f(x) = [x]$,

Graphically;



Which shows f(x) is many-one, as the straight line parallel to x-axis meets at more than one point.

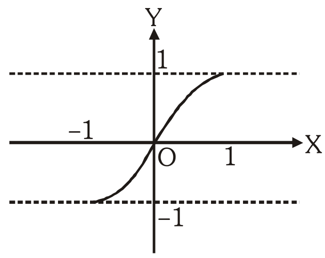
Here, range

$f(x) \in \{-1, 0, 1\}$

which shows into as range $\subseteq$ co-domain

Hence, many-one-into

(e) $f(x) = \sin\frac{\pi x}{2}$



Graphically;

Which shows f(x) is one-one and onto as range

= co-domain.

Therefore, f(x) is bijective.

Illustration 45. Let $f : R \to R$ be a function defined by $f(x) = x + \sqrt{x^2}$, then f is

(A) injective (B) surjective (C) bijective (D) None of these

Solution We have, $f(x) = x + \sqrt{x^2} = x + |x|$

Clearly, f is not one-one as $f(-1) = f(-2) = 0$ and $-1 \neq -2$

Also, f is not onto as $f(x) \geq 0\ \forall\ x \in R$

$\therefore$ range of $f = (0, \infty) \subset R$ **Ans.(D)**

Illustration 46. Let $f(x) = \frac{x^2+3x+a}{x^2+x+1}$, where $f : R \to R$. Find the value of parameter 'a' so that the given function is one-one.

Solution $f(x) = \frac{x^2+3x+a}{x^2+x+1}$

$$f'(x) = \frac{(x^2+x+1)(2x+3)-(x^2+3x+a)(2x+1)}{(x^2+x+1)^2} = \frac{-2x^2+2x(1-a)+(3-a)}{(x^2+x+1)^2}$$

Let, $g(x) = -2x^2 + 2x(1-a) + (3-a)$

$g(x)$ will be negative if $4(1-a)^2 + 8(3-a) < 0$

$\Rightarrow$ $1 + a^2 - 2a + 6 - 2a < 0 \Rightarrow (a-2)^2 + 3 < 0$

which is not possible. Therefore function is not monotonic.

Hence, no value of a is possible.

BEGINNER'S BOX-7

1*. Let $f : R^+ \to R : f(x) = x + \frac{1}{x}$ then

(A) f is injective mapping (B) f is many-one mapping

(C) f is surjective mapping (D) f is in-to mapping

2*. Let $f : R \to R : f(x) = x + 3\cos x$ then

(A) f is one-one function (B) f is many-one function

(C) f is on-to function (D) f is In-to function

3*. Let $f : R^+ \to R : f(x) = x^2 + \ln x$ then

(A) f is injective mapping (B) f is surjective mapping

(C) f is bijective mapping (D) f is singular mapping

4*. Let $f : R^+ \to R : f(x) = \frac{x}{x^2}$ then

(A) f is one-one function (B) f is many-one function

(C) f is on-to function (D) f is In-to function

Match the entries of **column- I** with one or more entries of the elements of **column - II**.

5*. **Column-I** — **Column-II**

(A) $f: R \to R : f(x) = x^3 + x^2 + 5x + 7$ — (P) One-one function

(B) $f: R \to R : f(x) = x^3 - x^2 - 3x + 2$ — (Q) Many-one function

(C) $f: R \to R : f(x) = x + \sin x$ — (R) Onto function

(D) $f: R \to R : f(x) = \sin(x^2)$ — (S) Into function

6*. **Column-I** — **Column-II**

(A) $f: R \to [-1,1] : f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ — (P) One-one mapping

(B) $f: R \to R : f(x) = x + |x|$ — (Q) Many-one mapping

(C) $f: R^+ \to R : f(x) = \sin(e^{-x})$ — (R) Onto mapping

(D) $f: R^+ \to R^+ : f(x) = x^2$ — (S) Into mapping

7. If $f: R^+ \to A : f(x) = x^2$ be any onto function then least positive integral value in set A is

8. If $f: R \to A : f(x) = \sin^3 x$ is surjective function then number of integers in the set A is

9. If $f: R \to A : f(x) = x^5 + 9x^2 + 7$ is onto function then least prime number in set A is

10. If $f: (1, \infty) \to R : f(x) = \ln(2x^2 - 5\alpha x + 3\alpha^2)$ is a bijective function then greatest value of 3α is

14.0 COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTION

Let $f - A \to B$ & $g : B \to C$ be two functions. Then the function $gof : A \to C$ defined by $(gof)(x) = g(f(x))\ \forall\ x \in A$ is called the composite of the two functions f & g.

Diagramatically $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$

Thus the image of every $x \in A$ under the function gof is the g-image of f-image of x.

Note that gof is defined only if $\forall\ x \in A$, f(x) is an element of the domain of 'g' so that we can take its g-image. Hence in gof(x) the range of 'f' must be a subset of the domain of 'g'.

- **Properties of composite functions**

  (a) In general composite of functions is not commutative i.e. $gof \neq fog$.

  (b) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh.

  (c) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

GOLDEN KEY POINTS

- f & g are two one-one function defined for all real values then both fog & gof are one-one.
- $f : R \to R$, $G : R \to R$ and range of f is A then range of fog is A.

Illustrations

Illustration 47. If f be the greatest integer function and g be the modulus function, then (gof)$\left(-\frac{5}{3}\right)$ – (fog)$\left(-\frac{5}{3}\right)$ =

(A) 1 (B) –1 (C) 2 (D) 4

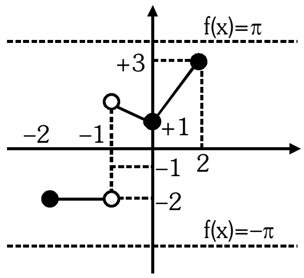
Solution Given (gof)$\left(\frac{-5}{3}\right)$ – (fog) $\left(\frac{-5}{3}\right)$ = $g\left\{f\left(\frac{-5}{3}\right)\right\}-f\left\{g\left(\frac{-5}{3}\right)\right\}$ = g(–2) – f $\left(\frac{5}{3}\right)$ = 2 – 1 = 1

Ans.(A)

Illustration 48. Find the domain and range of h(x) = g(f(x)), where

$f(x)=\begin{cases}[x], & -2\le x\le -1\\ |x|+1, & -1<x\le 2\end{cases}$ and $g(x)=\begin{cases}[x], & -\pi\le x\le 0\\ \sin x, & 0\le x\le \pi\end{cases}$, [.] denotes the greatest integer function.

Solution $h(x)=g(f(x))=\begin{cases}[f(x)], & -\pi\le f(x)<0\\ \sin(f(x)), & 0\le f(x)\le\pi\end{cases}$



From graph of f(x), we get

$h(x)=\begin{cases}[[x]], & -2\le x\le -1\\ \sin(|x|+1), & -1<x\le 2\end{cases}$

$\Rightarrow$ Domain of h(x) is [–2, 2]

and Range of h(x) is {–2, 1} $\cup$ [sin3, 1]

Illustration 49. Let $f(x)=\begin{cases}x+1, & x\le 1\\ 2x+1, & 1<x\le 2\end{cases}$ and $g(x)=\begin{cases}x^2, & -1\le x<2\\ x+2, & 2\le x\le 3\end{cases}$, find (fog)

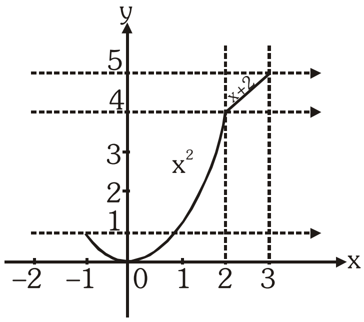
Solution $f(g(x))=\begin{cases}g(x)+1, & g(x)\le 1\\ 2g(x)+1, & 1<g(x)\le 2\end{cases}$

Here, g(x) becomes the variable that means we should draw the graph.

It is clear that g(x) ≤ 1 ; $\forall$ x $\in$ [–1, 1]

and 1 < g(x) ≤ 2 ; $\forall$ x $\in$ (1, $\sqrt{2}$]

$\Rightarrow$ $f(g(x))=\begin{cases}x^2+1, & -1\le x\le 1\\ 2x^2+1, & 1<x\le\sqrt{2}\end{cases}$



BEGINNER'S BOX-8

1. Let $f(x)=\begin{cases}1+x, & 0\le x\le 2\\ 3-x, & 2<x\le 3\end{cases}$ find (fof) (x).

2. If $f(x)=\sin^2x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right)=1$, then find (gof) (x).

3. If g_0 : R → R, $g_0(x)$ = 3 + 4x and $g_{n+1}(x)=g_0(g_n(x))$ then $g_n(x)$ =

(A) $(4^{n+1}-1)+4^{n+1}x$ (B) $4^n(x+1)$ (C) $3+4^n x$ (D) $3+4^n x^n$

4. If $f(x)=\begin{cases}x+1 & \text{if } x\le 1\\ 5-x^2 & \text{if } x>1\end{cases}$ and $g(x)=\begin{cases}x & \text{if } x\le 1\\ 2-x & \text{if } x>1\end{cases}$ then $f(g(x))$ =

(A) x + 1 if x ≤ 1 (B) 3 – x if x < 1 (C) 3 – x $\forall$ x $\in$ R (D) x + 1 if x > 1

5. If $f(x) = \frac{x}{\sqrt{1+x^2}}$, then (fofof) (x) =

(A) $\frac{3x}{\sqrt{1+x^2}}$ (B) $\frac{x}{\sqrt{1+3x^2}}$ (C) $\frac{3x}{\sqrt{1+x^2}}$ (D) none

6*. Let $f(x) = \begin{cases} x^2+1, & x<0 \\ 5-x, & x \ge 0 \end{cases}$ and $g(x) = \begin{cases} 3-x, & x<0 \\ x^2-5, & x \ge 0 \end{cases}$ then

(A) $f(g(x)) = \begin{cases} x^4+2x^2-4 & x<0 \\ x+2 & 0 \le x < \sqrt{5} \\ x^2-10 & x \ge \sqrt{5} \end{cases}$

(B) $g(f(x)) = \begin{cases} x^4+2x^2-4 & x<0 \\ x^2-10x+20 & 0 \le x \le 5 \\ x-2 & x > 5 \end{cases}$

(C) $g(f(x)) = \begin{cases} x^4-10x^2+26 & x<0 \\ x^2-10 & 0 \le x \le 5 \\ x-2 & x > 5 \end{cases}$

(D) $f(g(x)) = \begin{cases} x^4-10x^2+26 & 0 \le x < \sqrt{5} \\ x+2 & x<0 \\ 10-x^2 & x \ge \sqrt{5} \end{cases}$

7*. Let $f(x) = \begin{cases} x^2-9 & x<0 \\ 8-x & x \ge 0 \end{cases}$ and $g(x) = \begin{cases} -x-8 & x \le -1 \\ x-8 & -1<x<2 \\ 5-x^2 & x \ge 2 \end{cases}$ then

(A) $f(g(x)) = \begin{cases} (5-x^2)^2+9, & x \in (-\infty,-1] \\ x+5, & x \in (-1,2) \\ (x+8)^2+9 & x \in [2,\infty) \end{cases}$

(B) $f(g(x)) = \begin{cases} x+16 & x \in (-\infty,-8] \\ (x+8)^2-9 & x \in (-8,-1] \\ (x-8)^2-9 & x \in (-1,2) \\ x^2+3 & x \in [2,\sqrt{5}] \\ (5-x^2)^2-9 & x \in (\sqrt{5},\infty) \end{cases}$

(C) Domain of $f(g(x))$ is R (D) Range of $f(g(x))$ is R

15.0 INVERSE OF A FUNCTION

Let f – A → B be a one-one & onto function, then their exists a unique function g : B → A such that f(x) = y ⇔ g(y) = x, ∀ x ∈ A & y ∈ B. Then g is said to be inverse of f.

Thus $g = f^{-1} : B \to A = \{(f(x), x)) | (x, f(x)) \in f\}$.

- **Properties of inverse function**

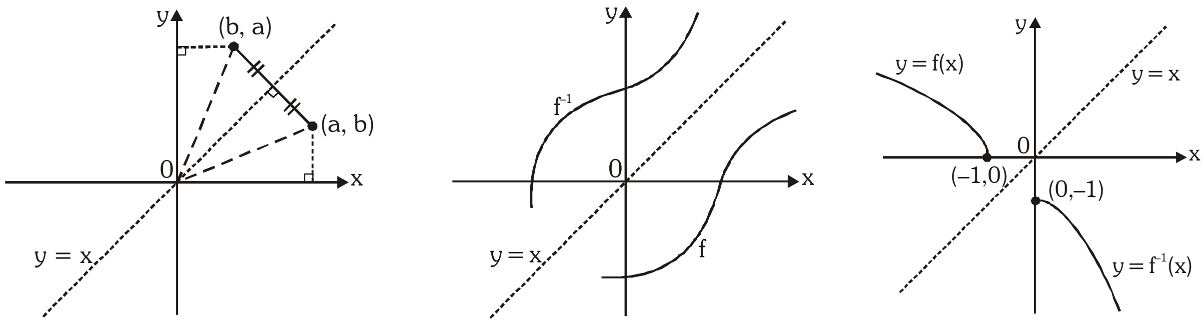
(a) The inverse of a bijection is unique.

(b) If f – A → B is a bijection & g : B → A is the inverse of f, then fog = I_B and gof = I_A, where I_A & I_B are identity functions on the sets A & B respectively. If fof = I, then f is inverse of itself.

(c) The inverse of a bijection is also a bijection.

(d) If f & g are two bijections f : A → B, g : B → C then the inverse of gof exists and $(gof)^{-1} = f^{-1} \circ g^{-1}$.

(e) Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of 'f' if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line y = x.



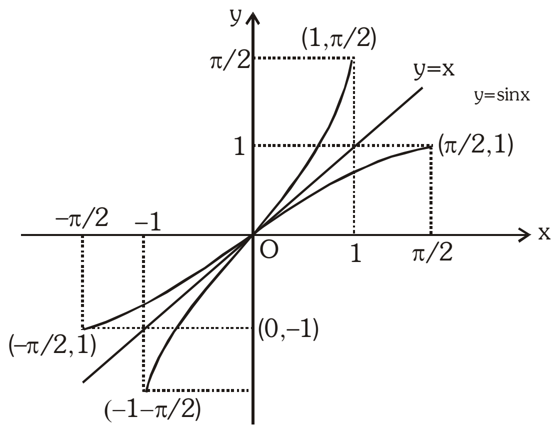
$\boxed{\text{The graph of } f^{-1} \text{ is obtained by reflecting the graph of f about the line } y = x.}$

Drawing the graph of y = f^{-1}(x) from the known graph of y = f(x)

For drawing the graph of $y = f^{-1}(x)$ we have to first of all find the interval in which the function is bijective (invertible). Then take the reflection of y = f(x) (within the invertible region) about the line y = x. The reflected part would give us the graph of $y = f^{-1}(x)$.

e.g. let us draw the graph of $y = \sin^{-1}x$. We know that y = f(x) = sinx is invertible if $f - \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

$\Rightarrow$ the inverse mapping would be $f^{-1} - [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Illustrations

Illustration 50. Let $f - R \to R$ be defined by $f(x) = (e^x - e^{-x})/2$. Is f(x) invertible ? If so, find its inverse.

Solution Let us check for invertibility of f(x) :

(a) One-One

Let $x_1, x_2 \in R$ and $x_1 < x_2$

$\Rightarrow$ $e^{x_1} < e^{x_2}$ (Because base $e > 1$) .... (i)

Also $x_1 < x_2 \Rightarrow -x_2 < -x_1$

$\Rightarrow$ $e^{-x_2} < e^{-x_1}$ (Because base $e > 1$) .... (ii)

(i) + (ii) $\Rightarrow e^{x_1} + e^{-x_2} < e^{x_2} + e^{-x_1}$

$\Rightarrow$ $\frac{1}{2}\left(e^{x_1} - e^{-x_1}\right) < \frac{1}{2}\left(e^{x_2} - e^{-x_2}\right) \Rightarrow f(x_1) < f(x_2)$ i.e. f is one-one.

(b) Onto

As x tends to larger and larger values so does f(x) and

when $x \to \infty$, $f(x) \to \infty$.

Similarly as $x \to -\infty$, $f(x) \to -\infty$ i.e. $-\infty < f(x) < \infty$ so long as $x \in (-\infty, \infty)$

Hence the range of f is same as the set R. Therefore f(x) is onto.

Since f(x) is both one-one and onto, f(x) is invertible.

(c) To find f^{-1}

Let f^{-1} be the inverse function of f, then by rule of identity $fof^{-1}(x) = x$

$$\frac{e^{f^{-1}(x)} - e^{-f^{-1}(x)}}{2} = x \Rightarrow e^{2f^{-1}(x)} - 2xe^{f^{-1}(x)} - 1 = 0$$

$$\Rightarrow \quad e^{f^{-1}(x)} = \frac{2x \pm \sqrt{4x^2+4}}{2} \Rightarrow e^{f^{-1}(x)} = x \pm \sqrt{1+x^2}$$

Since $e^{f^{-1}(x)} > 0$, hence negative sign is ruled out and

Hence $e^{f^{-1}(x)} = x + \sqrt{1+x^2}$

Taking logarithm, we have $f^{-1}(x) = \ell n(x + \sqrt{1+x^2})$.

Illustration 51. Find the inverse of the function $f(x) = \begin{cases} x; < 1 \\ x^2; 1 \le x \le 4 \\ 8\sqrt{x};\ x > 4 \end{cases}$

Solution Given $f(x) = \begin{cases} x; < 1 \\ x^2; 1 \le x \le 4 \\ 8\sqrt{x};\ x > 4 \end{cases}$

Let $f(x) = y \quad \Rightarrow \quad x = f^{-1}(y)$

$$\therefore \quad x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le \sqrt{y} \le 4 \\ \dfrac{y^2}{64}, & \dfrac{y^2}{64} > 4 \end{cases} \quad \Rightarrow \quad f^{-1}(y) = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ \dfrac{y^2}{64}, & y > 16 \end{cases}$$

Hence $f^{-1}(x) = \begin{cases} x; < 1 \\ \sqrt{x}; 1 \le x \le 16 \\ \dfrac{x^2}{64}; x > 16 \end{cases}$ **Ans.**

BEGINNER'S BOX-9

1. If $f(x) = \dfrac{1-x}{1+x}$, the domain of $f^{-1}(x)$ is

(A) R (B) R– {–1} (C) ($-\infty$, –1) (D) (–1, ∞)

2. The inverse of the function $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} + 1$ is

(A) $\log_{10}\left(\dfrac{x}{2-x}\right)$ (B) $\ln\left(\dfrac{x}{2-x}\right)$

(C) $\ln\left(\dfrac{x}{2-x}\right)^{1/2}$ (D) $\dfrac{1}{2}\log\left(\dfrac{x}{2-x}\right)$

3. The inverse of the function $f(x)=\dfrac{10^x-10^{-x}}{10^x+10^{-x}}$ is

(A) $\log_{10}(2-x)$ (B) $\dfrac{1}{2}\log_{10}\left(\dfrac{1+x}{1-x}\right)$

(C) $\dfrac{1}{2}\log_{10}(2x-1)$ (D) $\dfrac{1}{4}\log\left(\dfrac{2x}{2-x}\right)$

4. If the function $f:[1,\infty)\to[1,\infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is

(A) $\left(\dfrac{1}{2}\right)^{x(x-1)}$ (B) $\dfrac{1}{2}\{1+\sqrt{1+4\log_2 x}\}$ (C) $\dfrac{1}{2}\{1-\sqrt{1+4\log_2 x}\}$ (D) not defined

5. If $f(x) = 2x^3 + 7x - 5$ then $f^{-1}(4)$ is

(A) equal to 1 (B) equal to 2

(C) equal to 1/3 (D) non existent

6. Show that the functions $f(x) = x^2 - x + 1$, $x \geq 1/2$ and $g(x) = 1/2 + \sqrt{\left(x-\dfrac{3}{4}\right)}$ are mutually inverse and solve the equation

$x^2 - x + 1 = 1/2 + \sqrt{\left(x-\dfrac{3}{4}\right)}$.

7. For what values of the parameter m are the following functions invertible ?

(i) $f(x) = x^3 - mx^2 + 3x - 11$

(ii) $f(x) = (m + 2)x^3 - 3mx^2 + 9mx - 1$.

16.0 PERIODIC FUNCTION

A function f(x) is called periodic if there exists a least positive number T(T > 0) called the period of the function such that $f(x + T) = f(x)$, for all values of x within the domain of f(x).

e.g. The function sinx & cosx both are periodic over 2π & tan x is periodic over π.

Note – For periodic function

(i) $f(T) = f(0) = f(-T)$, where 'T' is the period.

(ii) Inverse of a periodic function does not exist.

(iii) Every constant function is periodic, but its period is not defined.

(iv) If f(x) has a period T & g(x) also has a period T then it does not mean that f(x) + g(x) must have a period T. e.g. $f(x) = |\sin x| + |\cos x|$.

(v) If f(x) has period p, then $\dfrac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(vi) If f(x) has period T then f(ax + b) has a period $T/|a|$ $(a \neq 0)$.

Illustrations

Illustration 52. Find the periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{\ell n(\sin x)} + \tan^3 x - \text{cosec}(3x-5)$ (ii) $f(x) = x - [x-b],\ b \in R$

(iii) $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$ (iv) $f(x) = \tan\dfrac{\pi}{2}[x]$

(v) $f(x) = \cos(\sin x) + \cos(\cos x)$ (vi) $f(x) = \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\text{cosec}\, x)}$

(vii) $f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+........+|\cos n\pi|}$

Solution

(i) $f(x) = e^{\ell n(\sin x)} + \tan^3 x - \text{cosec}(3x-5)$

Period of $e^{\ell n \sin x} = 2\pi$, $\tan^3 x = \pi$

$\text{cosec}\,(3x-5) = \dfrac{2\pi}{3}$

$\therefore$ Period $= 2\pi$

(ii) $f(x) = x - [x-b] = b + \{x-b\}$

$\therefore$ Period $= 1$

(iii) $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

Since period of $|\sin x + \cos x| = \pi$ and period of $|\sin x| + |\cos x|$ is $\dfrac{\pi}{2}$. Hence f(x) is periodic with π as its period

(iv) $f(x) = \tan\dfrac{\pi}{2}[x]$

$\tan\dfrac{\pi}{2}[x+T] = \tan\dfrac{\pi}{2}[x] \Rightarrow \dfrac{\pi}{2}[x+T] = n\pi + \dfrac{\pi}{2}[x]$

$\therefore$ $T = 2$

$\therefore$ Period $= 2$

(v) Let f(x) is periodic then $f(x+T) = f(x)$

$\Rightarrow$ $\cos(\sin(x+T)) + \cos(\cos(x+T)) = \cos(\sin x) + \cos(\cos x)$

If $x = 0$ then $\cos(\sin T) + \cos(\cos T)$

$=$ $\cos(0) + \cos(1) = \cos\left(\cos\dfrac{\pi}{2}\right) + \cos\left(\sin\dfrac{\pi}{2}\right)$

On comparing $T = \dfrac{\pi}{2}$

(vi) $f(x) = \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\text{cosec}\, x)} = \dfrac{(1+\sin x)(1+\cos x)\sin x}{\cos x(1+\sin x)(1+\cos x)}$

$\Rightarrow$ $f(x) = \tan x$

Hence f(x) has period π.

(vii) $f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+........+|\cos n\pi|}$

Period of $x - [x] = 1$

Period of $|\cos \pi x| = 1$

Period of $|\cos 2\pi x| = \dfrac{1}{2}$

....

Period of $|\cos n\pi x| = \dfrac{1}{n}$

So period of f(x) will be L.C.M. of all period $= 1$

Illustration 53. Find the periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{x-[x]} + \sin x$ (ii) $f(x) = \sin\frac{\pi x}{\sqrt{2}} + \cos\frac{\pi x}{\sqrt{3}}$

(iii) $f(x) = \sin\frac{\pi x}{\sqrt{3}} + \cos\frac{\pi x}{2\sqrt{3}}$

Solution (i) Period of $e^{x-[x]} = 1$

period of $\sin x = 2\pi$

$\because$ L.C.M. of rational and an irrational number does not exist.

$\therefore$ not periodic.

(ii) Period of $\sin\frac{\pi x}{\sqrt{2}} = \frac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2}$

Period of $\cos\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

$\because$ L.C.M. of two different kinds of irrational number does not exist.

$\therefore$ not periodic.

(iii) Period of $\sin\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

Period of $\cos\frac{\pi x}{2\sqrt{3}} = \frac{2\pi}{\pi/2\sqrt{3}} = 4\sqrt{3}$

$\because$ L.C.M. of two similar irrational number exist.

$\therefore$ Periodic with period $= 4\sqrt{3}$ **Ans.**

BEGINNER'S BOX-10

1. The period of $f(x) = \sin\frac{\pi x}{2} + 2\cos\frac{\pi x}{3} - \tan\frac{\pi x}{4}$, is -

(A) 6 (B) 3 (C) 4 (D) 12

2. Which of the following function has a period of 2π ?

(A) $f(x) = \sin\left(2\pi x + \frac{\pi}{3}\right) + 2\sin\left(3\pi x + \frac{\pi}{4}\right) + 3\sin 5\pi x$ (B) $f(x) = \sin\frac{\pi x}{3} + \sin\frac{\pi x}{4}$

(C) $f(x) = \sin x + \cos 2x$ (D) none

3. If $f : R \to R$ is a function satisfying the property $f(x+1) + f(x+3) = K \ \forall \ x \in R$ then the period of f (x) can be

(A) 4 (B) K (C) 1 (D) π

4. Let $f(x) = \sin\sqrt{[a]}\ x$ (where [ ] denotes the greatest integer function). If f is periodic with fundamental period π, then a belongs to -

(A) [2, 3) (B) {4, 5} (C) [4, 5] (D) [4, 5)

5. Function $f(x) = \{3x\} + \sin(\pi x)$ is [where {.} fractional part function] is

(A) Periodic with fundamental period 1 (B) Periodic with fundamental period 2

(C) Periodic with fundamental period 3 (D) Non periodic

6. $f(x) = \tan x + \sin x$ is periodic with fundamental period.

(A) π (B) 2π (C) $\frac{\pi}{2}$ (D) 3π

7. Fundamental period of $f(x)=\sin^2 x$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) 2π (D) π

8. $f(x)=\tan^2 x$ is

(A) Periodic with fundamental period $\frac{\pi}{2}$ (B) Periodic with fundamental period π

(C) Non periodic (D) None of these

9. Fundamental period of function $f(x)=\cot^2(8x)$ is

(A) $\frac{\pi}{4}$ (B) π (C) $\frac{\pi}{8}$ (D) $\frac{\pi}{16}$

10. Function $f(x)=\sin^2 x+\cos^2 x$ is

(A) Non periodic function (B) Periodic with fundamental period $\frac{\pi}{2}$

(C) Periodic with fundamental period π (D) None of these

11. Let f(x) be periodic and k be a positive real number such that

$f(x + k) + f(x) = 0$ for all $x \in R$. Prove that f(x) is a periodic with period 2k.

12. Find periods for;

(i) $\cos^4 x$ (ii) $\sin^3 x$ (iii) $\cos\sqrt{x}$ (iv) $\sqrt{\cos x}$

13. Find period of $f(x) = \tan 3x + \sin\left(\frac{x}{3}\right)$.

14. Find the period of,

$f(x) = \sin x + \tan\frac{x}{2}+\sin\frac{x}{2^2}+\tan\frac{x}{2^3}+.....+\sin\frac{x}{2^{n-1}}+\tan\frac{x}{2^n}$.

15*. Which of the following function (s) is/are periodic with period π.

(A) $f(x) = |\sin x|$ (B) $f(x) = [x + \pi]$ (C) $f(x) = \cos(\sin x)$ (D) $f(x) = \cos^2 x$

(where [.] denotes the greatest integer function)

16. Find the period of the following functions:

(i) $f(x) = \sin^4 x + \cos^4 x$ (ii) $f(x) = \cos(\sin x) + \cos(\cos x)$

(iii) $f(x) = |\sin x| + |\cos x|$ (iv) $f(x) = \sin 2x + \cos 3x$

17. Find the period of function $f(x) = \frac{1}{||\sin 2x|-|\cos 2x||}$.

18. Is the function $f(x) = 1 + \left(\frac{3}{2-\sin^2 x}\right)$ is periodic ?

If yes, mention its period.

19. Find the period of following functions:

(i) f(x) = [sin3x] + | cos6x |(where [.] is gretest integer function

(ii) $f(x) = \frac{1}{2}\left\{\frac{|\sin x|}{\cos x} + \frac{|\cos x|}{\sin x}\right\}$

(iii) $f(x) = 3\sin\frac{\pi x}{3} + 4\cos\frac{\pi x}{4}$

(iv) $f(x) = \cos 3x + \sin\sqrt{3}\pi x$

(v) $f(x) = \sin\frac{\pi x}{n!} - \cos\frac{\pi x}{(n+1)!}$

(vi) $f(x) = e^{\ln(\sin x)} + \tan^3 x - \operatorname{cosec}(3x-5)$

20. Find the period of the real valued function satisfying, f(x) + f(x + 4) = f(x + 2) + f(x + 6).

21. Check whether the function defined by :

$f(x+\lambda) = 1 + \sqrt{2f(x) - f^2(x)},\ \forall\ x \in R$ is periodic if yes, then find the period where f(x) $\in$ [0,2].

Match the entries of column- I with one or more entries of the elements of column - II.

22.

Column-I	Column-II
(A) $f(x) = \sin(\sin x)$ is	(P) Periodic with fundamental period 2π
(B) $f(x) = \cos(\sin x)$ is	(Q) Periodic with fundamental period π
(C) $f(x) = 3^{\sin x}$ is	(R) Periodic with fundamental period $\frac{\pi}{2}$
(D) $f(x) = \frac{\cos(3x)}{\sin(7x)}$ is	(S) Non periodic

23.

Column-I	Column-II
(A) $f(x) = \cos(\cos x) + \cos(\sin x)$ is	(P) Periodic with fundamental period 2π
(B) $f(x) = \|\sin x\| + \|\cos x\|$ is	(Q) Periodic with fundamental period π
(C) $f(x) = \sin^2 x + \cos^4 x + 8$ is	(R) Periodic with fundamental period $\frac{\pi}{2}$
(D) $f(x) = \sin(4x)\cos(8x)$ is	(S) Non periodic

24.

Column-I	Column-II
(A) $f(x) = \ln(\{x\})$ is (where {.} fractional part function)	(P) Is periodic with fundamental period 2π
(B) $f(x) = x - [x] + \sin^2 \pi x$ is (where [.] = G.I.F)	(Q) Is periodic with fundamental period π
(C) $f(x) = \sin^3 x - \cos^3 x$	(R) Is periodic with fundamental period 1
(D) $f(x) = \|\tan x\|$	(S) Is non periodic

17.0 GENERAL

If x, y are independent variables, then

(a) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\ell n\ x$

(b) $f(xy) = f(x)\ .\ f(y) \Rightarrow f(x) = x^n,\ n \in R$ or $f(x) = 0$

(c) $f(x+y) = f(x)\ .\ f(y) \Rightarrow f(x) = a^{kx}$ or $f(x) = 0$

(d) $f(x+y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

BEGINNER'S BOX-11

1. Let $f(x) = \dfrac{9^x}{9^x + 3}$ then find the value of the sum $f\left(\dfrac{1}{2006}\right) + f\left(\dfrac{2}{2006}\right) + f\left(\dfrac{3}{2006}\right) + \ldots\ldots\ldots\ldots + f\left(\dfrac{2005}{2006}\right)$

2. Given that f(11) = 11 and $f(x + 3) = \dfrac{f(x)-1}{f(x)+1}$ for all $x \in R$ find f(2000).

3. Assume that $f(1) = 0$ and that for all integers m and n, $f(m + n) = f(m) + f(n) + 3(4mn - 1)$ then $f(19) =$

(A) 2049 (B) 2098 (C) 1944 (D) 1998

4. If $f : R \to R$ satisfying $f(0) = 1$, $f(1) = 2$ and $f(x + 2) = 2f(x) + f(x + 1)$ then $f(6)$ is -

(A) 8(B) 32 (C) 16 (D) 64

5. Let f be a linear function for which $f(6) - f(2) = 12$. The value of $f(12) - f(2)$ is equal to -

(A) 12 (B) 18 (C) 24 (D) 30

6. A certain function $f(x)$ satisfies $f(x) + 2f(6-x) = x$ for all real numbers x. The value of $f(1)$, is -

(A) 3 (B) 2

(C) 1 (D) not possible to determine

7. If f(x) is a polynomial function satisfying the condition f(x). f(1/x) = f(x) + f(1/x) and f(2) = 9 then -

(A) 2 f(4) = 3f(6) (B) 14 f(1) = f(3) (C) 9 f(3) = f(5) (D) f(10)= f(11)

8. Which of the following is an onto function -

(A) $f : [0, \pi] \to [-1, 1]$, $f(x) = \sin x$ (B) $f : [0, \pi] \to [-1, 1]$, $f(x) = \cos x$

(C) $f : R \to R$, $f(x) = e^x$ (D) $f : Q \to Q$, $f(x) = x^3$

9. If $f : R \to R$ satisfies $f(x + y) = f(x) + f(y)$, for all $x, y \in R$ and $f(1) = 7$, then $\sum_{r=1}^{n} f(r)$ is -

(A) $\dfrac{7n}{2}$ (B) $\dfrac{7(n+1)}{2}$ (C) $7n(n + 1)$ (D) $\dfrac{7n(n+1)}{2}$

10. The number of linear functions f satisfying $f(x + f(x)) = x + f(x)$, $\forall x \in R$ is -

(A) 0 (B) 1 (C) 2 (D) 3

11. Number of polynomial functions satisfying $f(x) + f(3x) = f(2x)\ \forall\ x \in R$ is/are -

(A) 0 (B) 1 (C) 2 (D) 3

12. Let a function $f(x)$ satisfying functional rule $f(x + y) = f(x) f(y) + 3f(x - y) + 1\ \forall\ x,y \in R$, then -

A) $f(1) + f(2) = -2$ (B) $f(1) + f(2) = 0$ (C) $f(1) + f(2) = 2$ (D) $f(1) + f(2) = 4$

13. Let $f(x)$ is even and $g(x)$ is an odd function which satisfies $x^2 f(x) - 2f\left(\frac{1}{x}\right) = g(x)$,

then $f(1) + f(2) + f(3) + f(4) =$

(A) 10 (B) 0 (C) 24 (D) 4

14. If $f(x)$ is an even function defined for all x & satisfy $f(1) = 2$, $f(x + 3) = f(x) + x$ for $x \geq 0$, then $f(-4)$ is equal to -

(A) 2 (B) 3 (C) 4 (D) 5

15*. Which of the following is an odd function -

(A) $\ell n\left(\frac{x^4 + x^2 + 1}{(x^2 + x + 1)^2}\right)$

(B) sgn (sgn (x))

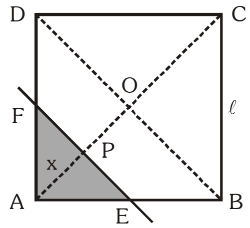
(C) $f(x)$, where $f(x) + f\left(\frac{1}{x}\right) = f(x) f\left(\frac{1}{x}\right)\ \forall\ x \in R - \{0\}$ & $f(2) = 33$

(D) $f$og(x), where $f(x)$ & g(x) both are odd function & fog(x) is defined.

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. ABCD is a square of side ℓ. A line parallel to the diagonal BD at a distance 'x' from the vertex A cuts two adjacent sides. Express the area of the segment of the square with A at a vertex, as a function of x. Find this area at $x = 1/\sqrt{2}$ and at x = 2, when $\ell = 2$.

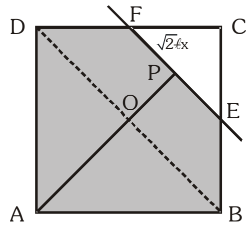
Solution There are two different situations



Case-I when x = AP ≤ OA, i.e., $x \le \dfrac{\ell}{\sqrt{2}}$

$$ar(\Delta AEF) = \frac{1}{2}x.2x = x^2 \qquad (\because PE = PF = AP = x)$$

Case-II when x = AP > OA, i.e., $x > \dfrac{\ell}{\sqrt{2}}$ but $x \le \sqrt{2}\ell$



ar(ABEFDA) = ar(ABCD) – ar(ΔCFE)

$$= \quad \ell^2 - \frac{1}{2}\left(\sqrt{2}\ell - x\right).2\left(\sqrt{2}\ell - x\right) \qquad [\because \ CP = \sqrt{2}\ell - x\,]$$

$$= \quad \ell^2 - \left(2\ell^2 + x^2 - 2\sqrt{2}\ell x\right) = 2\sqrt{2}\ell x - x^2 - \ell^2$$

$\therefore$ the required function s(x) is as follows :

$$s(x) = \begin{cases} x^2, & 0 \le x \le \dfrac{\ell}{\sqrt{2}} \\ 2\sqrt{2}\,\ell x - x^2 - \ell^2, & \dfrac{\ell}{\sqrt{2}} < x \le \sqrt{2}\ell \end{cases} \ ; \ \text{area of } s(x) = \begin{cases} \dfrac{1}{2} & \text{at } x = \dfrac{1}{\sqrt{2}} \\ 8(\sqrt{2}-1) & \text{at } x = 2 \end{cases}$$

Ans.

Illustration 2. If the function f(x) satisfies the functional rule, f(x + y) = f(x) + f(y) ∀ x,y ∈ R & f(1) = 5, then find $\sum_{n=1}^{m} f(n)$ and also prove that f(x) is odd function.

Solution Here, f(x + y) = f(x) + f(y); put x = t – 1, y = 1

f(t) = f(t – 1) + f(1) ....(1)

$\therefore$ f(t) = f(t – 1) + 5

$\Rightarrow$ f(t) = {f(t – 2) + 5} + 5

$\Rightarrow$ f(t) = f(t – 2) + 2(5)

$\Rightarrow$ f(t) = f(t – 3) + 3(5)

....

....

$\Rightarrow$ f(t) = f{t – (t –1)} + (t – 1)5

$\Rightarrow$ f(t) = f(1)+ (t – 1)5

$\Rightarrow$ f(t) = 5 + (t – 1)5

$\Rightarrow$ f(t) = 5t

$$\therefore \quad \sum_{n=1}^{m} f(n) = \sum_{n=1}^{m}(5n) = 5[1+2+3+.....+m] = \frac{5m(m+1)}{2}$$

Hence, $\sum_{n=1}^{m} f(n) = \dfrac{5m(m+1)}{2}$ .....(i)

Now putting x=0, y=0 in the given function, we have

f(0 + 0) = f(0) + f(0)

$\therefore$ f(0) = 0

Also putting (–x) for (y) in the given function.

f(x – x) = f(x) + f(– x)

$\Rightarrow$ f(0) = f(x) + f(–x)

$\Rightarrow$ 0 = f(x) + f(–x)

$\Rightarrow$ f(–x) = –f(x) ....(ii)

Thus, $\sum_{n=1}^{m} f(n) = \dfrac{5m(m+1)}{2}$ and f(x) is odd function.

Illustration 3. Range of $f(x) = 4^x + 2^x + 1$ is

Solution $f(x) = 4^x + 2^x + 1$

Let $2^x = t > 0, \forall x \in R$

$\therefore \quad f(x) = g(t) = t^2 + t + 1 \quad t > 0$

$$g(t) = \left(t+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\left(t+\frac{1}{2}\right) > \frac{1}{2} \Rightarrow \left(t+\frac{1}{2}\right)^2 > \frac{1}{4} \Rightarrow \left(t+\frac{1}{2}\right)^2 + \frac{3}{4} > 1$$

Range is $(1, \infty)$

Illustration 4. Let $f : N \to N$, where $f(x) = x + (-1)^{x-1}$, then the inverse of f is

Solution $f(x) = x + (-1)^{x-1}$

$$y = f(x) = \begin{cases} x+1 & , \text{ x is an odd natural number} \\ x-1 & , \text{ x is an even natural number} \end{cases}$$

$$x = \begin{cases} y-1 & , \text{ y is an even natural number} \\ y+1 & , \text{ y is an odd natural number} \end{cases}$$

$$f^{-1}(x) = \begin{cases} x-1 & , \text{ x is an even natural number} \\ x+1 & , \text{ x is an odd natural number} \end{cases}$$

$\therefore f^{-1}(x) = x + (-1)^{x-1}, x \in N$

Illustration 5. The domain of $f(x) = \sqrt{x^4 - x^3 + 1}$ is

Solution $f(x) = \sqrt{x^4 - x^3 + 1}$

For f(x) to be defined.

$x^4 - x^3 + 1 \geq 0$ i.e. $(1 + x^4) - x^3 \geq 0$, which is true $\forall x \in R$

$\therefore$ Domain is $(-\infty, \infty)$

Illustration 6. The value of $\left[\frac{3}{4}+\frac{1}{100}\right]+\left[\frac{3}{4}+\frac{2}{100}\right]+\left[\frac{3}{4}+\frac{3}{100}\right]+\ldots\ldots\left[\frac{3}{4}+\frac{99}{100}\right]$, where [.] represents greatest integer function:

Solution

$$\left[\frac{3}{4}+\frac{1}{100}\right]+\left[\frac{3}{4}+\frac{2}{100}\right]+\left[\frac{3}{4}+\frac{3}{100}\right]+\ldots\ldots\left[\frac{3}{4}+\frac{24}{100}\right]$$

$$+\left[\frac{3}{4}+\frac{25}{100}\right]+\left[\frac{3}{4}+\frac{26}{100}\right]+\left[\frac{3}{4}+\frac{27}{100}\right]+\ldots\ldots\left[\frac{3}{4}+\frac{99}{100}\right]$$

$= 0 + 75 = 75$

Illustration 7. Fundamental period of the function $f(x) = \dfrac{1}{\left||\sin 4x| - |\cos 4x|\right|} + \cos(\cos 6x)$ is

Solution As $|\sin 4x| - |\cos 4x|$ has period $\frac{\pi}{4}$

But on taking $\left||\sin 4x| - |\cos 4x|\right|$ as g(x) we get $g\left(x+\frac{\pi}{8}\right)=\left\|\sin\left(\frac{\pi}{2}+4x\right)\right|-\left|\cos\left(\frac{\pi}{2}+4x\right)\right\|$

$= \left\|\cos 4x\right|-\left|\sin 4x\right\| = g(x)$

$\therefore$ Fundamental period of g(x) is $\frac{\pi}{8}$.

Now h(x) = cos (cos 6x)

then $h\left(x+\frac{\pi}{6}\right)$ = cos(cos(π + 6x))

= cos (–cos 6x) = cos (cos 6x) $\therefore$ Period is $\frac{\pi}{6}$

Taking L.C.M. of $\frac{\pi}{6},\frac{\pi}{8}$ we get $\frac{\pi}{2}$

Illustration 8. If f(x) = px + sinx is bijective function then complete set of values of p is

Solution f(x) is one-one ⇒ f(x) is monotonic function ⇒ $p \in (-\infty, -1] \cup [1, \infty]$

f(x) is onto ⇒ $p \in R - \{0\}$

$\therefore$ f(x) is bijective if $p \in (-\infty, -1] \cup [1, \infty]$

Illustration 9. If x, y ∈ [0, 10], then the number of solution (x, y) of the inequality $3^{\sec^2 x-1}\sqrt{9y^2-6y+2} \le 1$ are

Solution If x, y ∈ [0, 10], then the.......

We have, $3^{\sec^2 x-1}\sqrt{9y^2-6y+2} \le 1$

$\Rightarrow \quad 3^{\sec^2 x}\sqrt{\left(y-\frac{1}{3}\right)^2+\frac{1}{9}} \le 1$

but $3^{\sec^2} \ge 3$ and $\sqrt{\left(y-\frac{1}{3}\right)^2+\frac{1}{9}} \ge \frac{1}{3}$

So, using boundness we must have $\sec^2 x = 1$ and $y-\frac{1}{3}=0$

$\Rightarrow \quad x = 0, \pi\, 2\pi, 3\pi$ and $y=\frac{1}{3}$

$\Rightarrow$ There are 4 solution.

Illustration 10. Let f (x) and g (x) be bijective functions where f: {7, 8, 9, 10} → {1, 2, 3, 4} and g:{3, 4, 5, 6} → {13, 14, 15, 16 } respectively then find the number of elements in domain and range of gof(x) ?

Solution domain of gof is {x ∈ {7, 8, 9, 10} : f(x) ∈ {3,4}}

$\therefore$ there are 2 elements in the domain of gof

Since gof is one-one, therefore, there are 2 elements in the range of gof.

EXERCISE – 1 SCQ/MCQ

SINGLE CORRECT

1. If f : R→ R, which of the following rule is **NOT** a real function-

(A) $y = 4 - x^2$ (B) $y = 3x^2$ (C) $y = \sqrt{x} - |x|$ (D) $y = 3x^2 + 5$

2. The domain of definition of $f(x) = \dfrac{\log_2(x+3)}{x^2+3x+2}$ is

(A) $R - \{-1,-2\}$ (B) $(-2,\infty)$ (C) $R - \{-1, -2, -3\}$ (D) $(-3,\infty) - \{-1,-2\}$

3. The domain of $f(x) = \log_e |\log_e x|$ is-

(A) $(0, \infty)$ (B) $(1, \infty)$ (C) $(0, 1) \cup (1, \infty)$ (D) $(-\infty, 1)$

4. The domain of the function $f(x) = \dfrac{1}{\log_{10}(1-x)} + \sqrt{x+2}$, is -

(A) $[-2, 0) \cup (0, 1)$ (B) $(-2, 0) \cup (0, 1]$ (C) $(-2, 0) \cup (0, 1]$ (D) $(-2, 0) \cup [0, 1]$

5. The domain of definition of the function y(x) given by the equation $2^x + 2^y = 2$ is

(A) $0 < x \le 1$ (B) $0 \le x \le 1$ (C) $-\infty < x \le 0$ (D) $-\infty < x < 1$

6. The domain of definition of $f(x) = \{\log_{10}(\log_{10} x) - \log_{10}(4 - \log_{10} x) - \log_{10} 3\}^{0.5}$

(A) $(10^3, 10^4)$ (B) $[10^3, 10^4]$ (C) $[10^3, 10^4)$ (D) $(10^3, 10^4]$

7. The domain of the function $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$ is-

(A) $(-\infty, -2) \cup [4, \infty)$ (B) $(-\infty, -2] \cup [4, \infty)$ (C) $(-\infty, -2) \cup (4, \infty)$ (D) none of these

8. The range of the function $f(x) = e^x - e^{-x}$, is -

(A) $[0, \infty)$ (B) $(-\infty, 0)$ (C) $(-\infty, \infty)$ (D) none

9. The range of the function $f(x) = \dfrac{1}{\sqrt{4 + 3\cos x}}$, is -

(A) $[1/\sqrt{7},\ 1]$ (B) $]1/\sqrt{7},\ 1]$ (C) $(1/\sqrt{7},\ 1]$ (D) none

10. The range of the function $f(x) = \dfrac{x^2 + x + 2}{x^2 + x + 1}$ $x \in (-\infty, \infty)$ is

(A) $[1, \infty)$ (B) $\left(1, \dfrac{11}{7}\right)$ (C) $\left(1, \dfrac{7}{3}\right]$ (D) $\left[1, \dfrac{7}{5}\right]$

11. Range of $f(x) = {}^{16-x}C_{2x-1} + {}^{20-3x}C_{4x-5}$ is

(A) $[728,\ 1474]$ (B) $\{728,\ 1474\}$ (C) $\{0,\ 728\}$ (D) $\{728,\ 1617\}$

12. The range of the function $f(x) = \dfrac{e^x - e^{|x|}}{e^x + e^{|x|}}$ is

(A) $(-\infty,\ \infty)$ (B) $[0, 1)$ (C) $(-1,\ 0]$ (D) $(-1,\ 1)$

13. If $f(x) = \ell n\left(\frac{1+x}{1-x}\right)$, then $f\left(\frac{2x}{1+x^2}\right)$ equals-

(A) $[f(x)]^2$ (B) $\{f(x)\}^3$ (C) $2f(x)$ (D) $3f(x)$

14. If $f(x) = \frac{4^x}{4^x+2}$, then $f(x) + f(1-x)$ is equal to-

(A) 0 (B) –1 (C) 1 (D) 4

15. If *f(x)* = cos (log x), then $f(x).f(y)-\frac{1}{2}\left\{f\left(\frac{x}{y}\right)+f(xy)\right\}$ is equal to

(A) 2 (B) 1 (C) 0 (D) None of these

16. Which of the following function(s) is identical to $g(x) = |x-2|$

(A) $f(x) = \sqrt{x^2-4x+4}$ (B) $f(x) = |x| - |2|$ (C) $f(x) = \frac{|x-2|^2}{|x-2|}$ (D) $f(x) = \left|\frac{x^2-x+2}{x-1}\right|$

17. Function $f(x) = \log_e(x^3 + \sqrt{1+x^6})$ is-

(A) even (B) odd (C) neither even nor odd (D) none of these

18. Which of the following is odd function ?

(A) $\sin x^2$ (B) $(a^x + 1)/(a^x - 1)$ (C) $x^2 - |x|$ (D) None of these

19. The function $f(x)=\frac{\sin^4 x+\cos^4 x}{x+\tan x}$ is-

(A) odd (B) even (C) neither even nor odd (D) odd and periodic

20. If $f : R \to R$, $f(x) = x^3 + x$, then f is-

(A) one-one onto (B) one-one into (C) many-one onto (D) many-one into

21. Let f : {x, y, z} $\to$ {a, b, c} be a one-one function and only one of the conditions

(i) f (x) $\neq$ b,. (ii) f (y) = b (iii) f (z) $\neq$ a is true then the function f is given by the set

(A) $\{(x,a),\ (y,b),\ (z,c)\}$ (B) $\{(x,a),(y,c),(z,b)\}$

(C) $\{(x,b),(y,a),(z,c)\}$ (D) $\{(x,c),(y,b),(z,a)\}$

22. Let f : **R** $\to$ **R** be a function such that $f(x) = x^3 - 6x^2 + 11x - 6$. Then

(A) f is one-one and into (B) f is many-one and into

(C) f is one-one and onto (D) f is many-one and onto

23. Let $f:R\to R$ be a function defined by $f(x)=\frac{e^{|x|}-e^{-x}}{e^x+e^{-x}}$ then

(A) 'f' is one– one and onto (B) 'f' is one –one but not onto

(C) 'f' is not one–one but onto (D) 'f' is neither one – one nor onto

24. If $f : R \to R$, $f(x) = (x + 1)^2$ and $g : R \to R$, $g(x) = x^2 + 1$ then (fog)(–3) is equal to-

(A) 121 (B) 144 (C) 112 (D) 11

25. Let $f(x) = \dfrac{\sin^{101} x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}$, where [x] denotes the integral part of x is

(A) an odd function (B) an even function

(C) neither odd nor even function (D) both odd and even function

26. Fundamental period of $f(x) = \sin\dfrac{\pi x}{(n-1)!} + \cos\dfrac{\pi x}{n!}$ is $(n \in N)$

(A) n ! (B) 2 (n !) (C) 2 (n - 1) ! (D) None of these

27. Let $f : R \to R$ be a periodic function such that $f(T+x) = 1 + [1 - 3f(x) + 3(f(x))^2 - (f(x))^3]^{1/3}$ where T is a fixed positive number, then fundamental period of f(x) can be

(A) 10T (B) 2T (C) 3 T (D) 5 T

28. The fundamental period of $\sin\frac{\pi}{4}[x] + \cos\frac{\pi x}{2} + \cot\frac{\pi}{3}[x]$, where [x] denotes the integral part of x is

(A) 8 (B) 4 (C) 3 (D) 24

29. The fundamental period of the function $f(x) = \dfrac{|\sin x| - |\cos x|}{|\sin x + \cos x|}$ is-

(A) $\frac{\pi}{2}$ (B) 2π (C) π (D) none of these

30. The number of solution of $2\cos x = |\sin x|$, $0 \le x \le 4\pi$ is

(A) 0 (B) 2 (C) 4 (D) infinite

31. The numer of solutions of the equation $\sin \pi x = |\ln|x||$ is

(A) infinite (B) 6 (B) 8 (D) 0

32. Let f(x) be defined on [–2, 2] and is given by $f(x) = \begin{cases} -1 & -2 \le x \le 0 \\ x-1 & 0 < x \le 2 \end{cases}$ and $g(x) = f(|x|) + |f(x)|$. Then g(x) is equal to

(A) $\begin{cases} -x & -2 \le x < 0 \\ 0 & 0 \le x < 1 \\ x-1 & 1 \le x \le 2 \end{cases}$ (B) $\begin{cases} -x & -2 \le x < 0 \\ 0 & 0 \le x < 1 \\ 2(x-1) & 1 \le x \le 2 \end{cases}$ (C) $\begin{cases} -x & -2 \le x < 0 \\ x-1 & 0 \le x \le 2 \end{cases}$ (D) None of these

33. The function $f(x) = \max\{(1-x), (1+x), 2\}$, $x \in (-\infty, \infty)$ is equivalent to

(A) $f(x) = \begin{cases} 1-x & x \le -1 \\ 2 & -1 < x < 1 \\ 1+x & x \ge 1 \end{cases}$ (B) $f(x) = \begin{cases} 1+x & x \le -1 \\ 2 & -1 < x < 1 \\ 1-x & x \ge 1 \end{cases}$

(C) $f(x) = \begin{cases} 1-x & x \le -1 \\ 1 & -1 < x < 1 \\ 1+x & x \ge 1 \end{cases}$ (D) None of these

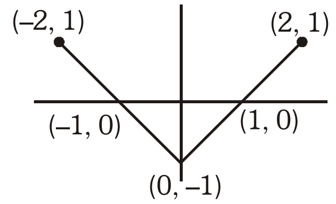
34. If $2f(x) - 3f\left(\frac{1}{x}\right) = x^2$, x is not equal to zero, then f(2) is equal to-

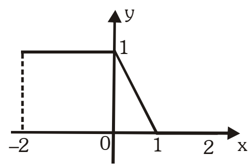
(A) $-\frac{7}{4}$ (B) $\frac{5}{2}$ (C) –1 (D) none of these

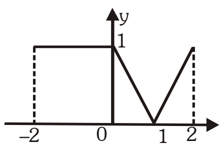
35. If $2\, f(x^2) + 3\, f(1/x^2) = x^2 - 1\ \forall\ x \in R_0$ then $f(x^2)$ is -

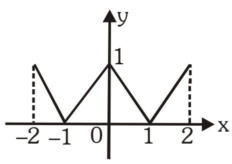
(A) $\frac{1-x^4}{5x^2}$ (B) $\frac{1-x^2}{5x}$ (C) $\frac{5x^2}{1-x^4}$ (D) $\frac{3-2x^4-x^2}{5x^2}$

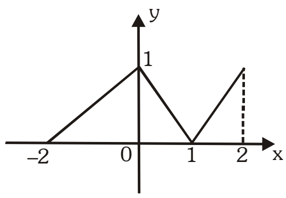
36. The graph of the function y = f(x) is as shown in the figure. Then which of the following could represent the graph of the function $y = |f(x)|$?



(A) 

(B) 

(C) 

(D) 

37. If $f(x) = \frac{4^x}{4^x+2}$, then $f\left(\frac{1}{1997}\right)+f\left(\frac{2}{1997}\right)+....+f\left(\frac{1996}{1997}\right)$ is equal to

(A) 1997 (B) 998 (C) 0 (D) None of these

38. The value of the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfies the relation $f(x+y) = f(x).f(y)$ for all natural numbers x, y and further $f(1) = 2$, is

(A) 3 (B) 4 (C) 2 (D) None of these

39. Let f : R $\to$ R, g : R $\to$ R be two functions given by f (x) = 2x – 3, g(x) = x^3 + 5. Then $(fog)^{-1}(x)$ is equal to

(A) $\left(\frac{x-7}{2}\right)^{1/3}$ (B) $\left(\frac{x+7}{2}\right)^{1/3}$ (C) $\left(x-\frac{7}{2}\right)^{1/3}$ (D) $\left(\frac{x-2}{7}\right)^{1/3}$

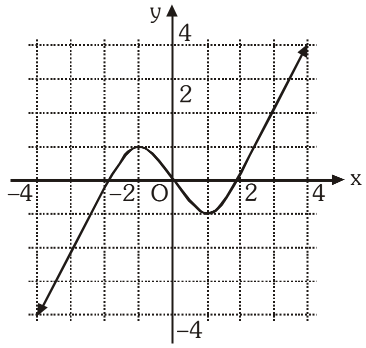
40. If $f(n+1) = \frac{2f(n)+1}{2}, n = 1, 2, ...$ and $f(1) = 2$, then $f(101)$ equals

(A) 52 (B) 49 (C) 48 (D) 51

41. If f(x) is defined on (0, 1) then the domain of definition of $f(e^x) + f(\ln |x|)$ is -

(A) (–e, –1) (B) $(-e, -1) \cup (1, e)$ (C) $(-\infty, -1) \cup (1, \infty)$ (D) (–e, e)

42. The graph of $\phi(x)$ is given then the number of positive solution of $\left||\phi(x)|-1\right| = 1$ are-

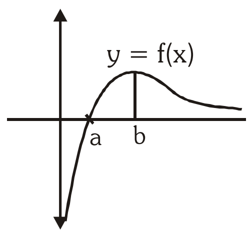


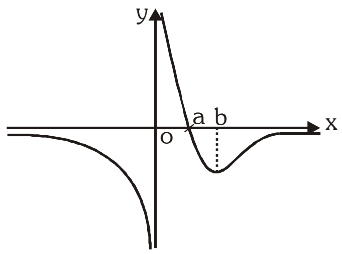
(A) 5 (B) 2 (C) 3 (D) 1

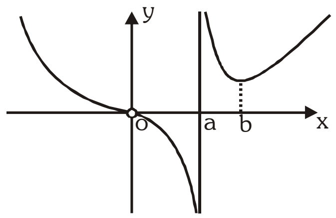
43. If g(x) is a polynomial satisfying $g(x)\,g(y) = g(x) + g(y) + g(xy) - 2$ for all real x and y and $g(2) = 5$ then g(3) is equal to -

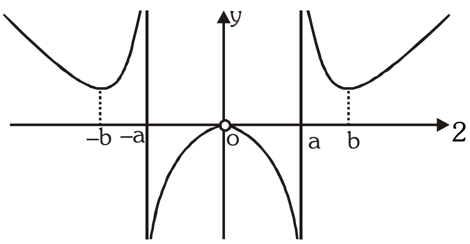
(A) 10 (B) 24 (C) 21 (D) none of these

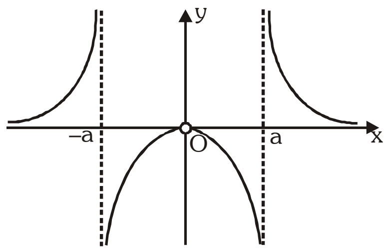
44. The graph of function f(x) is as shown, adjacently. Then the graph of $\dfrac{1}{f(|x|)}$ is -



(A)


(B)


(C)


(D)


MORE THAN ONE OPTION CORRECT

45. Which of the following function is periodic (where, [x] denotes the greatest integer function)

(A) $\text{sgn}\left(e^{-x}\right)$

(B) $\sin x + |\sin x|$

(C) $\min\{\sin x, |x|\}$

(D) $\left[x+\frac{1}{2}\right]+\left[x-\frac{1}{2}\right]+2[-x]$

46. The equation $\sin x - \frac{\pi}{2} + 1 = 0$ has one root in interval.

(A) $\left(0,\frac{\pi}{2}\right)$ (B) $\left(\frac{\pi}{2},\pi\right)$ (C) $\left(\pi,\frac{3\pi}{2}\right)$ (D) None of these

47. Which of the following functions is not injective ?

(A) $f(x)=|x+1|, x\in[-1,0]$ (B) $f(x)=x+1/x, x\in(0,\infty)$

(C) $f(x)=x^2+4x-5$ (D) $f(x)=e^{-x}, x\in[0,\infty)$

48. If $f : R \to R$ is a real valued function given by, $f(x) = \left|x^2-4|x|+12\right|$, then $f(x) = a$ has,

(A) two distinct real roots if $a > 12$ (B) four distinct real roots if $8 < a < 12$

(C) can have at most 4 real roots (D) have sum of real roots to be zero, if $a > 8$.

49. Which of the following functions are homogeneous ?

(A) $x\sin y + y\sin x$ (B) $x\,e^{y/x} + y\,e^{x/y}$ (C) $x^2 - xy$ (D) $\sin^{-1}xy$

50. Which of the following pairs of functions are identical ?

(A) $f(x) = \log_x e$, $g(x) = \frac{1}{\log_e x}$

(B) $\operatorname{sgn}(x^2+1)$; $g(x) = \sin^2 x + \cos^2 x$

(C) $f(x) = \sec^2 x - \tan^2 x$; $g(x) = \operatorname{cosec}^2 x - \cot^2 x$

(D) $f(x) = \frac{1}{|x|}$; $g(x) = \sqrt{x^{-2}}$

51. A function whose graph is **NOT** symmetrical about the origin is given by -

(A) $f(x) = e^x + e^{-x}$ (B) $f(x) = \sin(\sin(\cos(\sin x)))$

(C) $f(x+y) = f(x) + f(y)$ (D) $\sin x + \sin|x|$

52. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ then let us define a function $f(x) = \det(A^T A^{-1})$ then which of the following can be the value of $\underbrace{f(f(f(f\ldots\ldots\ldots f(x))))}_{n\text{ times}}$ $(n \ge 2)$

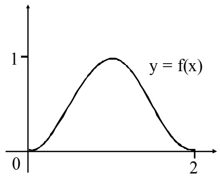
(A) $f^n(x)$ (B) 1 (C) $f^{n-1}(x)$ (D) $nf(x)$

EXERCISE – 2 MISCELLANEOUS

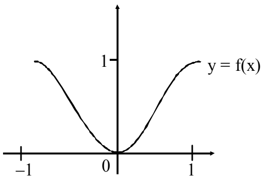
Comprehension Based Questions

Comprehension-1

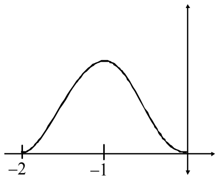
The accompanying figure shows the graph of a function f(x) with domain [0, 2] and range [0, 1]



(A)



(B)



(C)

1. Figure B represents the graph of the function

(A) – f(x) (B) –f(x – 1) + 1 (C) – f(x + 1) – 1 (D) – f(x + 1) + 1

2. [1, 3] and [0, 1] are the domain and range (respectively) of the function

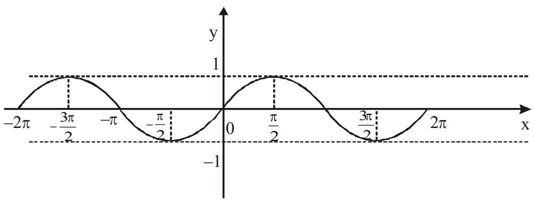
(A) –f(x) (B) f(x – 1) (C) –f(x + 1) + 1 (D) – f(x + 1)

3. Figure C represents the graph of function

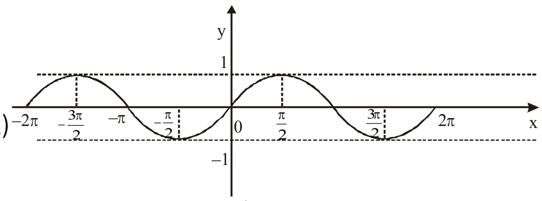
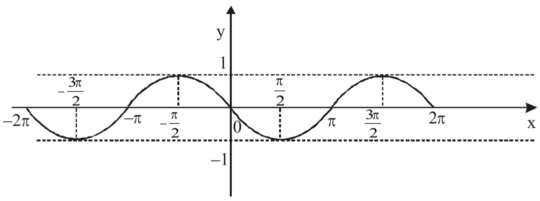
(A) 2f(x) (B) f(x – 2) (C) f(x + 2) (D) f(x – 2) + 1

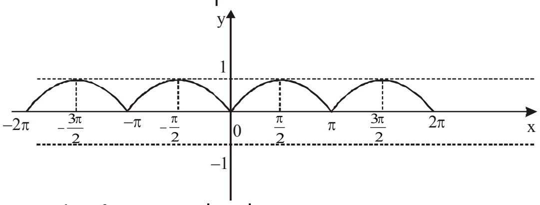
Comprehension-2

The graph of y = sin x is

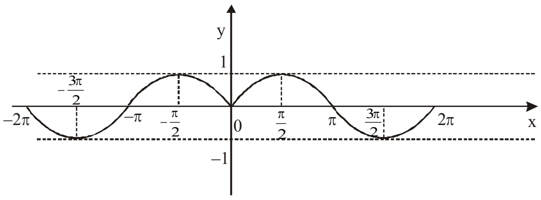
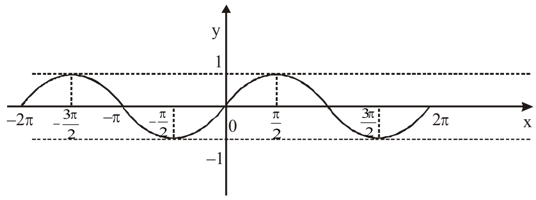


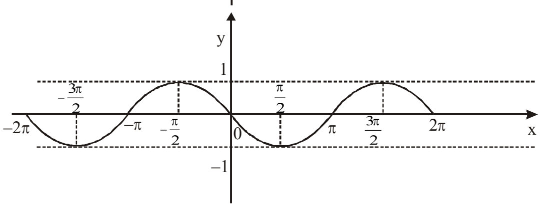
4. The graph of y = | sin x | is

(A)  (B) 

(C)  (D) None of these

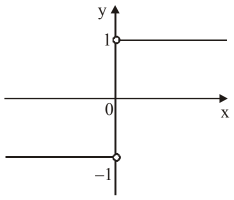
5. The graph of y = sin | x | is

(A)  (B) 

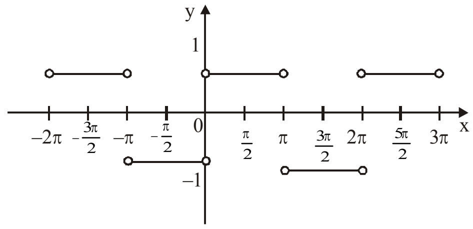
(C)  (D) None of these

6. The graph of $y = \dfrac{|\sin x|}{\sin x}$ is

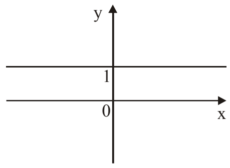
(A)



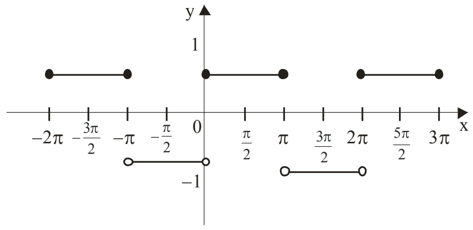
(B)



(C)



(D)



Comprehension-3

Consider the function $y = 2|x + 2| - |3 - x| - 3x + 3$.

7. Number of real solutions of the equation $y = 0$ is

(A) 1 (B) 2 (C) 3 (D) infinite

8. Number of real solutions of the equation $y = 4$ is

(A) 1 (B) 2 (C) 3 (D) infinite

9. Number of positive values of x satisfying the equation $y = -1$ is

(A) 0 (B) 1 (C) 2 (D) infinite

10. Domain of given function is

(A) $x \in R$ (B) $x \in [0, \infty)$ (C) $x \in (0, \infty)$ (D) $x \in [-2, \infty)$

11. Range of given function is

(A) $y \in [-2, \infty)$ (B) $y \in (0, \infty)$ (C) $y \in [0, \infty)$ (D) $y \in R$

Comprehension–4

If $f(x) = \begin{cases} x+1, & \text{if } x \le 1 \\ 5-x^2, & \text{if } x > 1 \end{cases}$ & $g(x) = \begin{cases} x, & \text{if } x \le 1 \\ 2-x, & \text{if } x > 1 \end{cases}$

On the basis of above information, answer the following questions

12. The range of $f(x)$ is -

(A) $(-\infty, 4)$ (B) $(-\infty, 5)$ (C) R (D) $(-\infty, 4]$

13. If $x \in (1, 2)$, then $g(f(x))$ is equal to -

(A) $x^2 + 3$ (B) $x^2 - 3$ (C) $5 - x^2$ (D) $1 - x$

14. Number of negative integral solutions of $g(f(x)) + 2 = 0$ are -

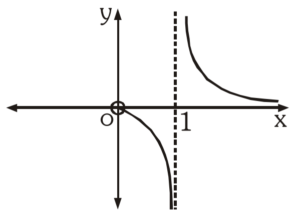
(A) 0 (B) 3 (C) 1 (D) 2

Match the column

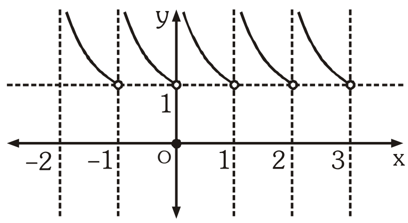
Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

15. Column - I **Column - II**

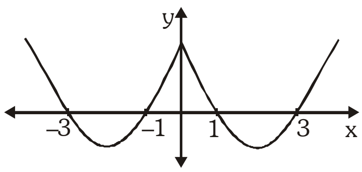
(A) If $f(x) = x^2 - 4x + 3$, then graph of $f(|x|)$ is

(p) 

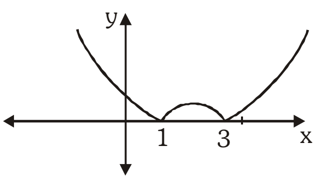
(B) If $g(x) = \dfrac{1}{\ln x}$, then it's graph is

(q) 

(C) If $f(x) = x^2 - 4x + 3$, then graph of $|f(x)|$ is

(r) 

(D) If $k(x) = \dfrac{1}{\{x\}}$, then its graph is

(s) 

16*. Column - I **Column - II**

(A) $f : R \to R$
$f(x) = (x - 1)(x - 2)........(x - 11)$ — (p) one one

(B) $f : R - \{-4/3\} \to R$ — (q) onto
$$f(x) = \frac{2x+1}{3x+4}$$

(C) $f : R \to R$ — (r) many one
$f(x) = e^{\sin x} + e^{-\sin x}$

(D) $f : R \to R$ — (s) into
$f(x) = \log(x^2 + 2x + 3)$

INTEGER/SUBJECTIVE TYPE QUESTIONS

17. Let $f(x) = (x + 1)(x + 2)(x + 3)(x + 4) + 5$ where $x \in [-6, 6]$. If the range of the function is $[a, b]$ where $a, b \in N$ then find the value of $(a + b)$.

18. The set of real values of 'x' satisfying the equality $\left[\dfrac{3}{x}\right] + \left[\dfrac{4}{x}\right] = 5$ (where [ ] denotes the greatest integer function) belongs to the interval $\left(a, \dfrac{b}{c}\right]$ where $a, b, c \in N$ and $\dfrac{b}{c}$ is in its lowest form. Find the value of $a + b + c + abc$.

19. Suppose p(x) is a polynomial with integer coefficients. The remainder when p(x) is divided by x – 1 is 1 and the remainder when p(x) is divided by x – 4 is 10. If r(x) is the remainder when p(x) is divided by (x –1)(x – 4), find the value of r(2006).

20. Find the domains of definitions of the following functions :
(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively)
(a) $f(x) = \log_7\log_5\log_3\log_2(2x^3 + 5x^2 - 14x)$

(b) $f(x) = \dfrac{1}{\sqrt{4x^2-1}} + \ell n x(x^2 - 1)$ (c) $f(x) = \sqrt{\log_{\frac{1}{2}} \dfrac{x}{x^2-1}}$

(d) $f(x) = \log_x \sin x$

(e) If $f(x) = \sqrt{x^2 - 5x + 4}$ & $g(x) = x + 3$, then find the domain of $\dfrac{f}{g}(x)$

21. Solve the following problems from (a) to (e) on functional equation.

(a) The function f(x) defined on the real numbers has the property that f(f(x)) . (1 + f(x)) = –f(x) for all x in the domain of f. If the number 3 is in the domain and range of f, compute the value of f(3).

(b) Suppose f is a real functional satisfying f(x + f(x)) = 4f(x) and f(1) = 4. Find the value of f(21).

(c) Let'f' be a function defined from $R^+ \to R^+$. If $[f(xy)]^2 = x(f(y))^2$ for all positive numbers x and y and f(2) = 6, find the value of f(50).

(d) Let f(x) be a function with two properties
(i) For any two real number x and y, f(x + y) = x + f(y) and
(ii) f(0) = 2.
Find the value of f(100)

(e) Let f be a function such that f(3) = 1 and f(3x) = x + f(3x – 3) for all x. Then find the value of f(300).

22. $f(x) = \begin{bmatrix} 1-x & \text{if} & x \le 0 \\ x^2 & \text{if} & x > 0 \end{bmatrix}$ and $g(x) = \begin{bmatrix} -x & \text{if} & x < 1 \\ 1-x & \text{if} & x \ge 1 \end{bmatrix}$ find (fog)(x) and (gof) (x)

23. Compute the inverse of the functions:

(a) $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$ (b) $f(x) = 2^{\frac{x}{x-1}}$ (c) $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$

24. Let f : R → R – {3} be a function with the property that there exist T > 0 such that $f(x + T) = \dfrac{f(x)-5}{f(x)-3}$ for every $x \in R$. Prove that f(x) is periodic.

25. Let $f(x) = x^{135} + x^{125} - x^{115} + x^5 + 1$. If f(x) is divided by $x^3 - x$ then the remainder is some function of x say g(x). Find the value of g(10).

26. Let {x} & [x] denote the fractional and integral part of a real number x respectively. Solve 4{x} = x + [x]

27. Let $f(x) = \dfrac{9^x}{9^x+3}$ then find the value of the sum $f\left(\dfrac{1}{2008}\right) + f\left(\dfrac{2}{2008}\right) + f\left(\dfrac{3}{2008}\right) + \dots\dots f\left(\dfrac{2007}{2008}\right)$

28. Let f be a one-one function with domain {x, y, z} and range {1, 2, 3}. It is given that exactly one of the following statements is true and the remaining two are false.
f(x) = 1; $f(y) \ne 1$; $f(z) \ne 2$. Determine $f^{-1}(1)$

29. A function f : R → R is such that $f\left(\dfrac{1-x}{1+x}\right) = x$ for all $x \ne -1$. Prove the following.

(a) f(f(x)) = x (b) f(1/x) = –f(x), $x \ne 0$ (c) f(–x – 2) = –f(x) – 2.

NCERT CORNER

1. Find x and y, if $(x+3, 5) = (6, 2x+y)$.

2. If $\left(\frac{x}{3}+1, y-\frac{2}{3}\right) = \left(\frac{5}{3}, \frac{1}{3}\right)$, find the values of x and y.

3. If the set A has 3 elements and the set $B = \{3, 4, 5\}$, then find the number of elements in $(A \times B)$.

4. If $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$, find A and B.

5. Find the domain of the function $f(x)$ defined by : $f(x) = \sqrt{4-x} + \frac{1}{\sqrt{x^2-1}}$

6. Find the domain of the function $f(x)$ defined by : $f(x) = \frac{1}{\sqrt{x+|x|}}$

7. Let $f : R \to R$ be defined by : $f(x) = x + |x|$ Determine whether or not f is onto.

8. Find the domain and range of the real functions : $f(x) = \sqrt{9-x^2}$

9. A function f is defined by $f(x) = 2x - 5$. Write down the values of :

(i) $f(0)$ (ii) $f(7)$ (iii) $f(-3)$

10. Find the range of the function : $f(x) = 2 - 3x, x \in R, x > 0$.

11. Find the range of the function : $f(x) = x^2 + 2$, x is a real number.

Short Answer (4 Mark)

12. If $A = \{1, 2\}, B = \{3, 4\}$ and $C = \{5\}$, verify that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

13. If $G = \{7, 8\}$ and $H = \{5, 4, 2\}$ find $G \times H$ and $H \times G$.

14. If $A = \{-1, 1\}$, find $A \times A \times A$.

15. Let $f : R \to R$ be given by $f(x) = x^2 + 3$. Find

(i) $\{x : f(x) = 28\}$

(ii) The pre-images of 39 and 2 under f.

16. Let $f : R \to R$ be a function given by $f(x) = x^2 + 1$. Find :

(i) $f^{-1}\{26\}$

(ii) $f^{-1}\{10, 37\}$

17. Let $f : R \to R$ be a function given by $f(x) = x^2 + 1$. Find :

$f^{-1}\{-5\}$

18. Find the domain and range of each of the following functions given by :

(i) $f(x)=\dfrac{1}{\sqrt{x-[x]}}$ (ii) $f(x)=1-|x-3|$

19. Draw the graphs of the function :
$$f(x)=\begin{cases} x\ ,\ 0\le 0x\le 1 \\ \dfrac{3-x}{2}\ ,\ 1<x<3 \\ -x+3\ ,\ 3\le x\le 5 \end{cases}$$

20. Draw the graphs of the function :
$$f(x)=\begin{cases} x^2,\ x\le 0 \\ x,\ 0<x<1 \\ \dfrac{1}{x},\ x\ge 1 \end{cases}$$

21. Find the domain of the function: $f(x)=\dfrac{x^2+2x+1}{x^2-8x+12}$.

22. Find the domain and the range of the real function *f* defined by $f(x)=\sqrt{(x-1)}$.

23. Let $f=\left\{\left(x,\dfrac{x^2}{1+x^2}\right),: x\in R\right\}$ be a function from *R* into *R*. Determine the range of *f*.

24. Let $f, g: R\to R$ be defined respectively by $f(x)=x+1, g(x)=2x-3$. Find $f+g, f-g$ and $\dfrac{f}{g}$.

25. Let $f=\{(1,1),(2,3),(0,-1),(-1,-3)\}$ be a function from *Z* to *Z* defined by $f(x)=ax+b,$ for some integers *a*, *b*. Determine *a*, *b*.

26. Let $A=\{1,2,3,4\}, B=\{1,5,9,11,15,16\}$ and

$f=\{(1,5),(2,9),(3,1),(4,5),(2,11)\}$. Are the following true?

(i) *f* is a relation from *A* to *B*

(ii) *f* is a function from *A* to *B*

Justify your answer in each case.

ANSWER KEY

BEGINNER'S BOX-1

1	(A)	2	(C)	3	(A)	4	(A)	5	(B)	6	(A)
7	(D)	8	(C)	9	(A)	10	(B)				

11. Domain of f = {p, q, r, s}. Co-domain of f = {1, 2, 3, 4, 5}. Range of f = {1, 2, 3, 4}

12. (iii), (iv), (vi), (vii), (ix)

13. (iv, v)

14. Domain = {–2, –1, 0, 1, 2}, Range = {0, 1, 4}

15. (a) {–5, 5} (b) Pre-image of 39 are ± 6. Does not have any pre-image of 2 under f.

16. (a) f(1/2) = 1, $f(\pi) = -1$, $f(\sqrt{2}) = -1$ (b) {1, –1}

(c) Pre-image of 1 are rational numbers, Pre-image of –1 are irrational numbers

17. f_4 is the function in the given set. Range of f = {1, 2, 3, 4}

18. (a) f = $\{(x, f(x)) : x \in A\} \Rightarrow f = \{(-1, 2), (0, 1), (2, 5), (4, 17)\}$; Range = {2, 1, 5, 17}

(b) Range of g = $\{(x, g(x)) : x \in A\}$; Range of g = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20}

19. g is a function. $\alpha = 2$, $\beta = -1$.

20. (b)

21. $f\left(\frac{1}{x}\right) = \frac{1+x}{1-x}$, $f(2x) = \frac{2x+1}{2x-1}$, $2f(x) = 2\left(\frac{1+x}{x-1}\right)$, $f(x^2) = \frac{x^2+1}{x^2-1}$ $(f(x))^2 = \left(\frac{x+1}{x-1}\right)^2$

24. f(–1) = 2, $f\left(\frac{\pi}{2}\right) = 1$, $f\left(\frac{2\pi}{3}\right) = \sqrt{3}$, f(4) = 2/7, f(6) = 3/17

25. $f\left(\sqrt{2}\right) = 4\sqrt{2}+1$, $f\left(\sqrt{8}\right) = \frac{1}{2\sqrt{2}-2}$, $f\left(\sqrt{\log_2 1024}\right) = 2\sqrt{10}-5$

BEGINNER'S BOX-2

1. (A) $x > -3$ (B) $x \le 5/2$ (C) R – {1, - 1}

(D) R (E) R – {0, 1, - 1} (F) R – {1, 2}

2. (A) [–1, 1] (B) $(-\infty, 0) \cup (4, \infty)$ (C) $(-\infty, 1] \cup [3, \infty)$

3. (A) [–1, 1] (B) x < 0 (C) nullset

4. (A) [1, 4] (B) $(2n\pi, (2n+1)\pi)$ (C) $(0, 1) \cup (1, \infty)$

(D) $(1, 0) \cup (0, -2]$ (E) {1}

5. (A) $(-1, 0) \cup (1,2) \cup (2, \infty)$ (B) $(3 - \pi, 3 - 2\pi) \cup (3, 4]$

(C) $[-4, -\pi] \cup [0, \pi]$

6. (A) nullset (B) $(-1, 1] \cup [2, 3)$

7. (A) R (B) [4, 6] (C) (2, 3) (D) $4k^2\pi^2 \le x \le (2k+1)^2\pi^2$

8. (C) **9.** (C) **10.** (B) **11.** (A)

BEGINNER'S BOX-3

1.	(B)	2.	(C)	3.	(B)	4.	(B)	5.	(A)	6.	(D)
7.	(B)	8.	(A)	9.	(A)	10.	(D)	11.	(B)	12.	(A)
13.	(B)	14.	(C)	15.	(B)	16.	(A)	17.	(A)	18.	(A)
19.	(B)	20.	(B)	21.	(B)	22.	(AC)	23.	(12)	24.	(11)

BEGINNER'S BOX-4

4. **(2)**

BEGINNER'S BOX-5

1. (a) No (b) No (c) No (d) Yes (e) No

2. (a) Yes (b) No (c) No

3. (a) $(0, \infty)$ (b) $[1, \infty)$

4. (B,C)

5. (A,B,C)

6. (i) odd (ii) even (iii) neither even nor odd

(iv) even (v) odd (vi) even

7. $\frac{5}{2}m(m+1)$

8. (i) for even $f(x) = \begin{cases} -x|x| & -1 < x \le 0 \\ -2x & x \le -1 \end{cases}$ (ii) for odd $f(x) = \begin{cases} x|x| & -1 \le x \le 0 \\ 2x & x \le -1 \end{cases}$

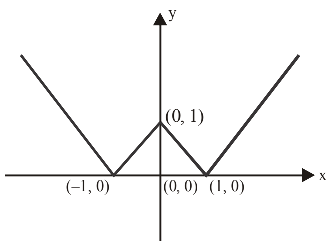
9. Even **10.** Odd **11.** (B) **12.** (B) **13.** (C) **14.** (B)

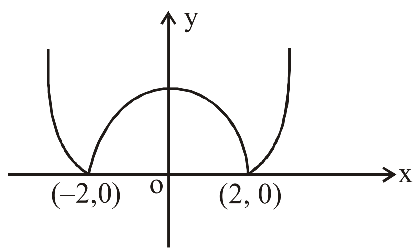
15. (A) **16.** (C)

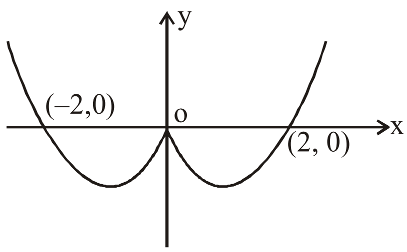
17. [(A)-Q ; (B)-Q ; (C)-R ; (D)-P,Q,S] **18.** [(A)-R ; (B)-Q, S ; (C)-P ; (D)-Q, S]

BEGINNER'S BOX-6

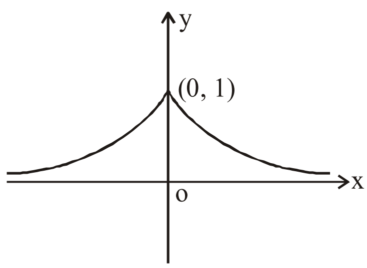
1. (a) $y = ||x|-1|$ (b) $y = |4-x^2|$ (c) $y = x^2 - 2|x|$

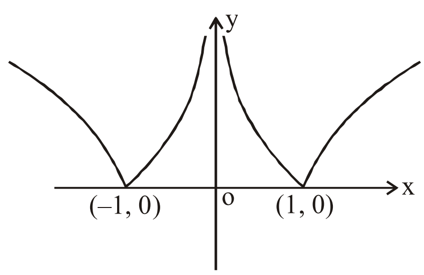


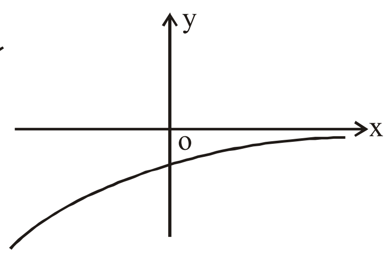




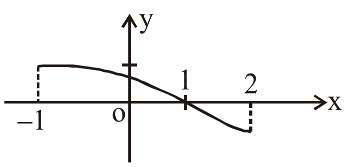
(d) $y = e^{-|x|}$ (e) $y = |\log_e|x||$ (f) $y = -e^{-x}$

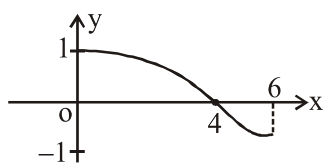


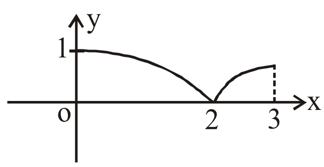




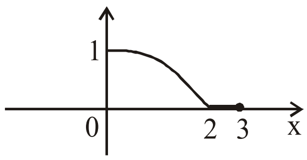
2. **(i) y = f(x + 1)** **(ii) y = f(x/2)** **(iii) $y = |f(x)|$**

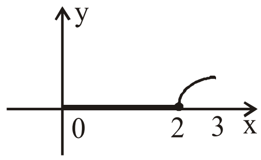


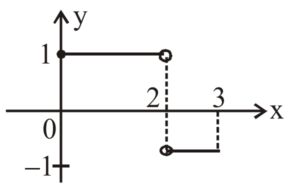




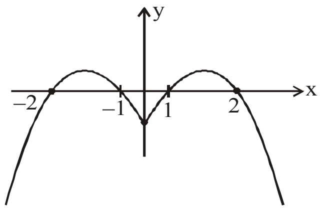
(iv) $y=\dfrac{|f(x)|+f(x)}{2}$, $y=\dfrac{|f(x)|-f(x)}{2}$ **(v)** $y=\dfrac{|f(x)|}{f(x)}$



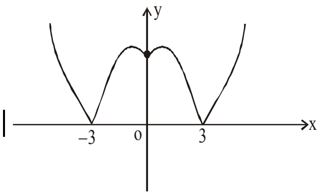




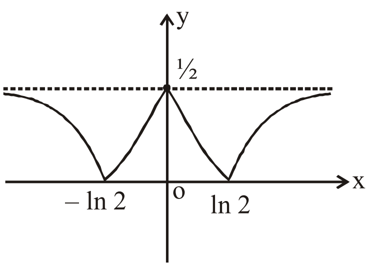
3. f(x) = 3| x | – x^2 – 2



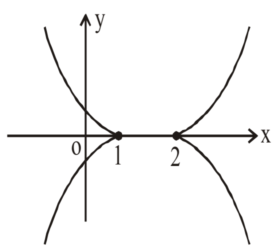
4. f(x) = | x^2– 2 | x | – 3 |



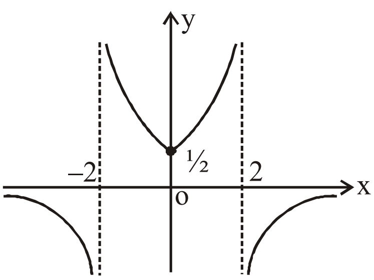
5. $f(x)=\left|e^{-|x|}-\dfrac{1}{2}\right|$



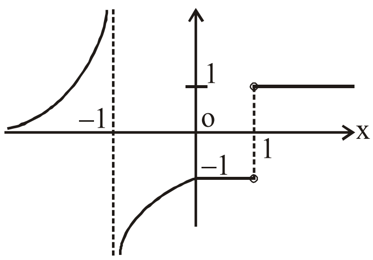
6. | f (x) | = (x – 1) (x – 2)



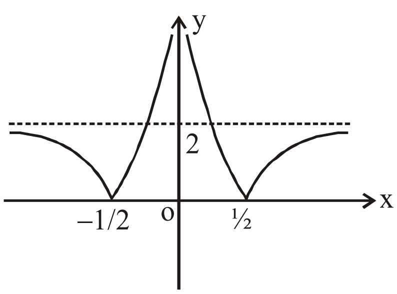
7. f(x) = $\dfrac{1}{2-|x|}$



8. f(x) = $\dfrac{|x-1|}{|x|-1}$



9. f(x) = $\left|\left|\dfrac{1}{x}\right|-2\right|$.



10. (B) **12.** (i) 5, (ii) 2

BEGINNER'S BOX-7

1. **(BD)** **2.** **(BC)** **3.** **(ABC)** **4.** **(AD)**

5. **[(A)-P, R ; (B)-Q, R ; (C)-P, R ; (D)-Q, S]**

6. **[(A)-P, S ; (B)-Q,S ; (C)-P, S ; (D)-P, R]**

7. **(1)** **8.** **(3)** **9.** **(2)** **10.** **(2)**

BEGINNER'S BOX-8

1. $\text{fof}(x)=\begin{cases}4-x;\ 2<x\le 3\\ 2+x;\ 0\le x\le 1\\ 2-x;\ 1\le x\le 2\end{cases}$ **2.** (1)

3. (A) **4.** (A) **5.** **(B)** **6.** **(BD)** **7.** **(BCD)**

BEGINNER'S BOX-9

1. (B) **2.** (C) **3** (B) **4.** (B) **5.** (A) **6.** $x = 1$

7. (i) [–3, 3] (ii) $(-\infty, -3] \cup [0, \infty)$

BEGINNER'S BOX-10

1. **(D)** **2.** **(C)** **3.** **(A)** **4.** **(D)** **5.** **(B)** **6.** **(B)**

7. **(D)** **8.** **(B)** **9.** **(C)** **10.** **(D)**

12. (i) π (ii) 2π (iii) Not periodic (iv) 2π **13.** 6π **14.** $2^n\pi$

15. (ACD) **16.** (i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{2}$ (iii) $\frac{\pi}{2}$ (iv) 2π

17. $\frac{\pi}{4}$ **18.** π **19.** (i) $\frac{2\pi}{3}$ (ii) 2π (iii) 24 (iv) does not exist (v) $2\underline{|n+1}$

(vi) 2π **20.** 8 **21.** 2λ **22.** [(A)-P ; (B)-Q ; (C)-P ; (D)-Q]

23. [(A)-R ; (B)-R ; (C)-R ; (D)-R] **24.** [(A)-R ; (B)-R ; (C)-P ; (D)-Q]

BEGINNER'S BOX-11

1. 1002.5 **2.** –6/5 **3.** (D)

4. **(D)** **5.** (D) **6.** (A) **7.** **(B)**

8. **(B)** **9.** **(D)** **10.** **(C)** **11.** **(B)**

12. **(A)** **13.** **(B)** **14.** **(B)** **15.** **(A,B,D)**

EXERCISE-1

SINGLE CORRECT & MORE THAN ONE OPTION CORRECT

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	D	C	A	D	C	A	C	A	C
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	D	C	C	C	C	A	B	B	A	A
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	C	D	D	A	B	B	B	D	C	C
Que.	31	32	33	34	35	36	37	38	39	40
Ans.	B	B	A	A	D	C	B	A	A	A
Que.	41	42	43	44	45	46	47	48	49	50
Ans.	A	B	A	C	ABCD	AB	BC	ABCD	BC	ABD
Que.	51	52								
Ans.	ABD	ABC								

EXERCISE-2

- **Comprehension Based Questions**

Comprehension – 1	**1.** D	**2.** B	**3.** C		
Comprehension – 2	**4.** C	**5.** A	**6.** B		
Comprehension – 3	**7.** A	**8.** D	**9.** B	**10.** A	**11.** D
Comprehension – 4	**12.** A	**13.** B	**14.** C		

- **Match the Column**

15. (A) $\to$ (r); (B) $\to$ (p); (C) $\to$ (s); (D) $\to$ (q)

16. (A) $\to$ (r, q); (B) $\to$ (p, s); (C) $\to$ (r, s); (D) $\to$ (r, s)

- **INTEGER/SUBJECTIVE TYPE QUESTIONS**

17. (5049) **18.** (20) **19.** (6016)

20. (a) $\left(-4, -\frac{1}{2}\right) \cup (2, \infty)$

(b) $(-1 < x < -1/2)$ U $(x > 1)$ (c) $\left[\frac{1-\sqrt{5}}{2}, 0\right) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$

(d) $2K\pi < x < (2K + 1)\pi$ but $x \neq 1$ where K is non-negative integer

(e) $(-\infty, -3) \cup (-3, 1] \cup [4, \infty)$

21. (a) $-\frac{3}{4}$ (b) 64 (c) 30 (d) 102 (e) 5050

22. $(gof)(x) = \begin{cases} x & \text{if} \quad x \le 0 \\ -x^2 & \text{if} \quad 0 < x < 1; \\ 1-x^2 & \text{if} \quad x \ge 1 \end{cases}$ $(fog)(x) = \begin{cases} x^2 & \text{if} \quad x < 0 \\ 1+x & \text{if} \quad 0 \le x < 1 \\ x & \text{if} \quad x \ge 1 \end{cases}$

23. (A) $\frac{e^x - e^{-x}}{2}$; (B) $\frac{\log_2 x}{\log_2 x - 1}$ (C) $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

25. (21) **26.** $0, \frac{5}{3}$ **27.** 1003.5 **28.** y

Node-1\Target-2022-23\1.JEE(M+A)\Module\Enthusiast\English\Maths\EM-1\17.Set, Relation & Function\Ex.

NCERT CORNER

1. $x = 3$ and $y = -1$

2. $x = 2$ **and** $y = 1$.

3. **9**

4. $A = \{a, b\}$ **and** $B = \{x, y\}$.

5. Domain $(f) = (-\infty, -1) \cup (1, 4]$.

6. Domain $(f) = (0, \infty)$.

7. ***f* is not onto.**

8. **Domain of** f = [-3, 3], **Range** = [0, 3]

9. **(i)** $f(0) = 2\times 0 - 5 = -5$ **(ii)** $f(7) = 2\times 7 - 5 = 9$ **(iii)** $f(-3) = 2\times(-3) - 5 = -11$.

10. $R_f = (-\infty, 2)$.

11. $R_f = [2, \infty)$.

13. $G\times H = \{(7,5),(7,4),(7,2),(8,5),(8,4),(8,2)\}$, $H\times G = \{(5,7),(5,8),(4,7),(4,8),(2,7),(2,8)\}$.

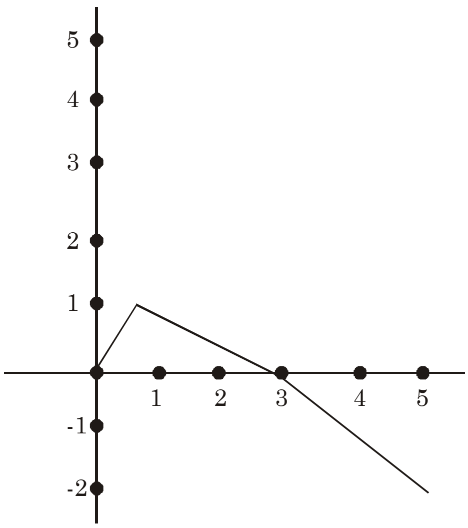
14. $A\times A\times A = \{(-1,-1,-1),(-1,-1,1),(-1,1,-1),(-1,1,1),\ (1,-1,-1),(1,-1,1),(1,1,-1),(1,1,1)\}$

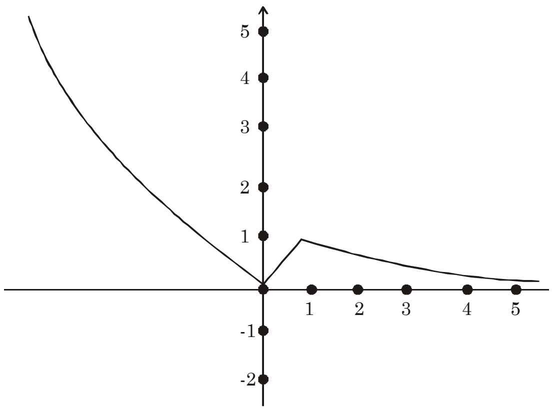
15. (i) $\{-5, 5\}$. (ii) Pre-images of 39 are –6 and 6, 2 does not have any pre-image under *f*.

16. **(i)** Does not exist. **(ii)** Does not exist.

17. ϕ

18. (i) Domain $(f) = R - Z$, Range $(f) = (1, \infty)$. (ii) Domain $(f) = R$, Range $(f) = (-\infty, 1]$.

19. 

20. 

21. $D_f = R - \{2, 6\}$

22. $D_f = [1, \infty)$, $R_f = [0, \infty)$

23. $R_f = [0, 1)$. $[\because\ y \neq 1]$

24. $(f+g) = 3x - 2$, $(f-g) = 4 - x$, $\left(\dfrac{f}{g}\right)(x) = \dfrac{x+1}{2x-3}$

25. **a = 2, b = –1**

26. (i) True (ii) False

* * * *

INVERSE TRIGONOMETRIC FUNCTIONS

1.0 INTRODUCTION

The inverse trigonometric functions, denoted by $\sin^{-1}x$ or (arc sinx), $\cos^{-1}x$ etc., denote the angles whose sine, cosine etc, is equal to x. The angles are usually the numerically smallest angles, except in the case of $\cot^{-1}x$, and if positive & negative angles have same numerical value, the positive angle has been chosen.

It is worthwhile noting that the functions sinx, cosx etc are in general not invertible. Their inverse is defined by choosing an appropriate domain & co-domain so that they become invertible. For this reason the chosen value is usually the simplest and easy to remember.

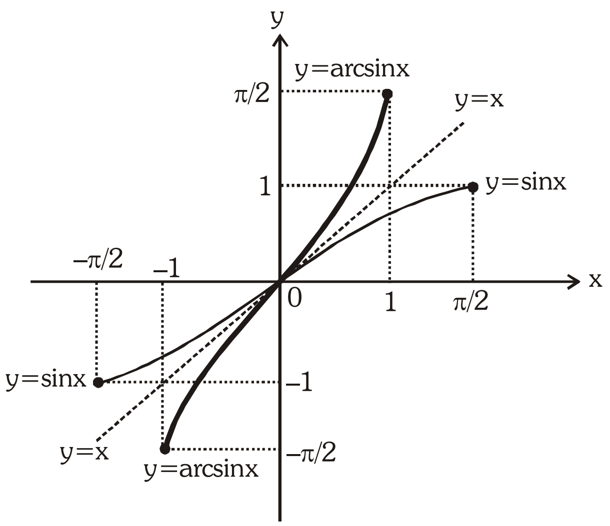
2.0 DOMAIN & RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

S.No	f (x)	Domain	Range
(1)	$\sin^{-1}x$	$\lvert x\rvert \le 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(2)	$\cos^{-1}x$	$\lvert x\rvert \le 1$	$[0, \pi]$
(3)	$\tan^{-1}x$	$x \in R$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(4)	$\sec^{-1}x$	$\lvert x\rvert \ge 1$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$ or $\left[0, \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2}, \pi\right]$
(5)	$\text{cosec}^{-1}x$	$\lvert x\rvert \ge 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$
(6)	$\cot^{-1}x$	$x \in R$	$(0, \pi)$

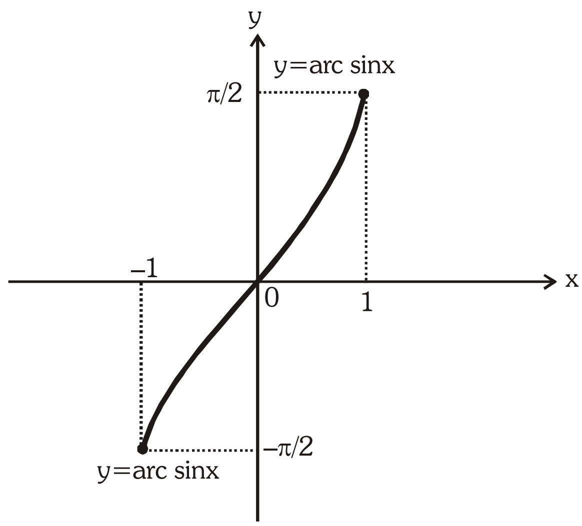
3.0 GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS

(a) $f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ $\qquad$ $f^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$

$f(x) = \sin x$ $\qquad$ $f^{-1}(x) = \sin^{-1}(x)$



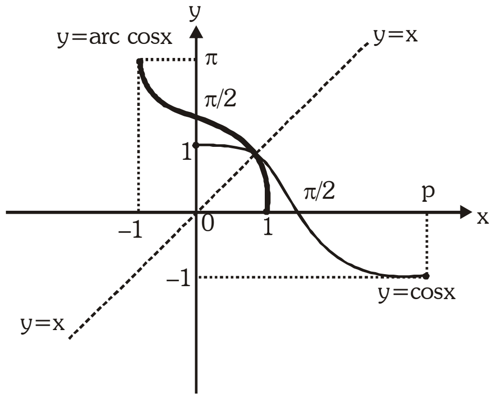
(Taking image of sin x about y = x to get $\sin^{-1}x$)



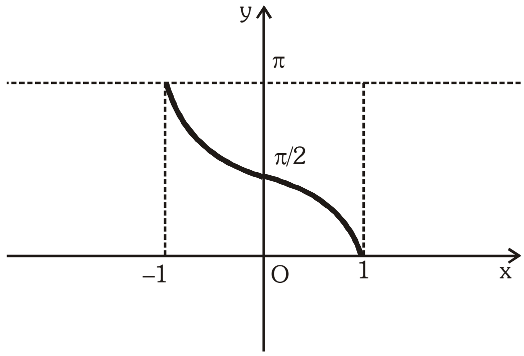
($y = \sin^{-1}x$)

(b) $f : [0, \pi] \rightarrow [-1, 1]$ $\qquad$ $f^{-1} : [-1, 1] \rightarrow [0, \pi]$

$f(x) = \cos x$ $\qquad$ $f^{-1}(x) = \cos^{-1} x$



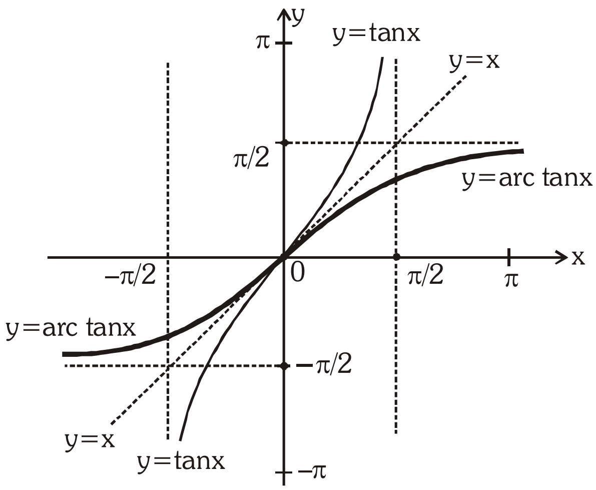
(Taking image of cos x about y = x)



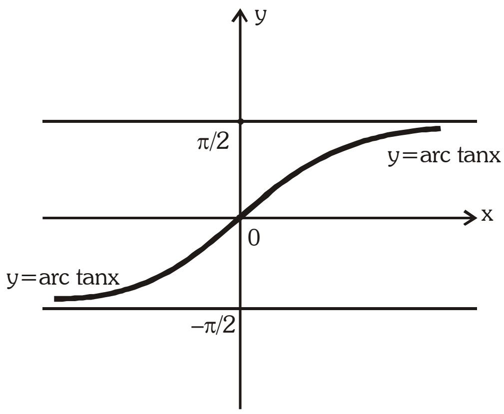
($y = \cos^{-1}x$)

(c) $f : (-\pi/2, \pi/2) \rightarrow R$ $\qquad$ $f^{-1} : R \rightarrow (-\pi/2, \pi/2)$

$f(x) = \tan x$ $\qquad$ $f^{-1}(x) = \tan^{-1} x$



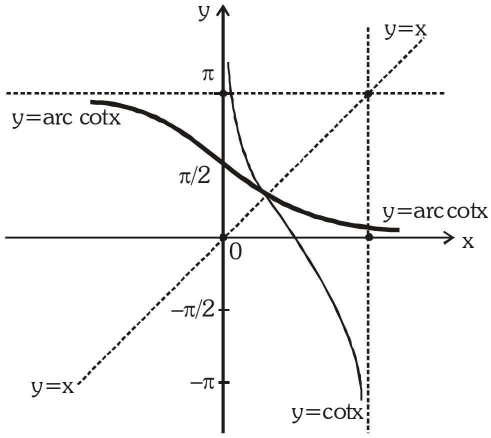
(Taking image of tan x about y = x)



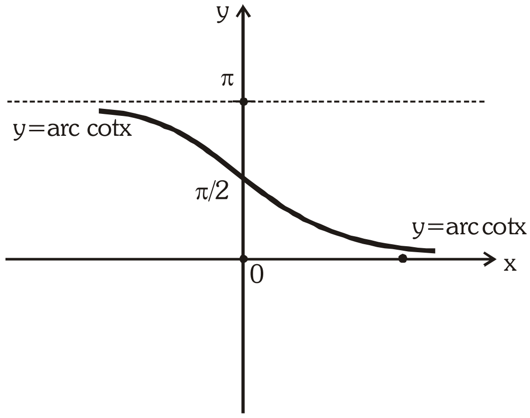
($y = \tan^{-1}x$)

(d) $f : (0, \pi) \rightarrow R$ $\qquad$ $f^{-1} : R \rightarrow (0, \pi)$

$f(x) = \cot x$ $\qquad$ $f^{-1}(x) = \cot^{-1} x$



(Taking image of cot x about y = x)



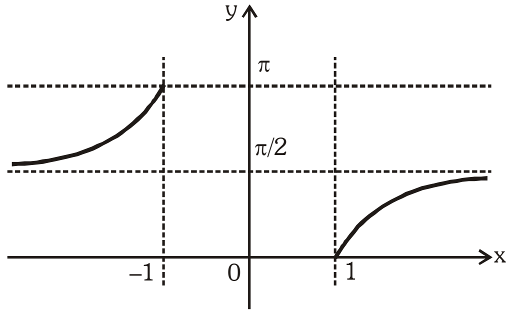
($y = \cot^{-1}x$)

(e) $f : [0, \pi/2) \cup (\pi/2, \pi] \rightarrow (-\infty, -1] \cup [1, \infty)$

$f(x) = \sec x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \rightarrow [0, \pi/2) \cup (\pi/2, \pi]$

$f^{-1}(x) = \sec^{-1} x$

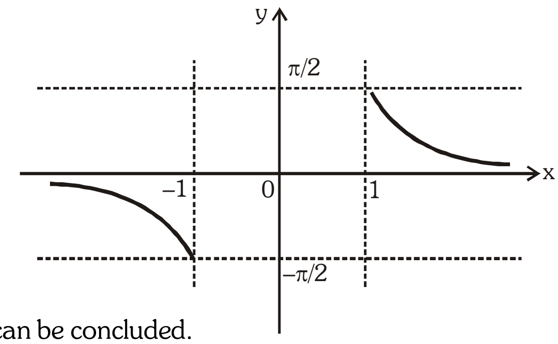


(f) $f : [-\pi/2, 0) \cup (0, \pi/2] \rightarrow (-\infty, -1] \cup [1, \infty)$

$f(x) = \text{cosec } x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \rightarrow [-\pi/2, 0) \cup (0, \pi/2]$

$f^{-1}(x) = \text{cosec}^{-1} x$



From the above discussions following IMPORTANT points can be concluded.

(i) All the inverse trigonometric functions represent an angle.

(ii) If $x \geq 0$, then all six inverse trigonometric functions viz $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$, $\cot^{-1}x$ represent an acute angle.

(iii) If $x < 0$, then $\sin^{-1}x$, $\tan^{-1}x$ & $\text{cosec}^{-1}x$ represent an angle from $-\pi/2$ to 0 (IV[th] quadrant)

(iv) If $x < 0$, then $\cos^{-1} x$, $\cot^{-1}x$ & $\sec^{-1}x$ represent an obtuse angle. (II[nd] quadrant)

(v) III[rd] quadrant is never used in inverse trigonometric function.

Illustrations

Illustration 1. The value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{5\pi}{12}$ (C) $\frac{3\pi}{4}$ (D) $\frac{13\pi}{12}$

Solution $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$ **Ans.(C)**

Illustration 2. If $\sum_{i=1}^{2n} \cos^{-1} x_i = 0$ then find the value of $\sum_{i=1}^{2n} x_i$

Solution We know, $0 \leq \cos^{-1} x \leq \pi$

Hence, each value $\cos^{-1}x_1, \cos^{-1}x_2, \cos^{-1}x_3, \ldots\ldots, \cos^{-1}x_{2n}$ are non-negative their sum is zero only when each value is zero.

i.e., $\cos^{-1}x_i = 0$ for all i

$\Rightarrow$ $x_i = 1$ for all i

$\therefore$ $\sum_{i=1}^{2n} x_i = x_1 + x_2 + x_3 + \ldots\ldots + x_{2n} = \underbrace{\{1 + 1 + 1 \ldots\ldots\ldots + 1\}}_{2n\text{ times}} = 2n$ {using (i)}

$\Rightarrow$ $\sum_{i=1}^{2n} x_i = 2n$ **Ans.**

BEGINNER'S BOX-1

1. $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}$ is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

2. The value of $\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$ is

(A) 1 (B) –1 (C) 0 (D) None of these

3. $\tan\left[\cos^{-1}\frac{1}{2}+\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$ is equal to

4. The value of $\sin\left[\tan^{-1}(-\sqrt{3})+\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ is

(A) 1 (B) –1 (C) 0 (D) None of these

5. If α, β are roots of the equation $6x^2 + 11x + 3 = 0$, then

(A) both $\cos^{-1}\alpha$ and $\cos^{-1}\beta$ are real (B) both $\operatorname{cosec}^{-1}\alpha$ and $\operatorname{cosec}^{-1}\beta$ are real

(C) both $\cot^{-1}\alpha$ and $\cot^{-1}\beta$ are real (D) none of these

6. If $\sin^{-1}x + \sin^{-1}y = \pi$ and $x = ky$, then find the value of $39^{2k} + 5^k$.

7. $\tan^{-1}\left(1-x^2-\frac{1}{x^2}\right)+\sin^{-1}\left(x^2+\frac{1}{x^2}-1\right)$ (where $x \neq 0$) is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{3\pi}{4}$ (D) π

8. Let $f(x)=\cot^{-1}x+\operatorname{cosec}^{-1}x$. Then f(x) is real for

(A) $x\in[-1,1]$ (B) $x\in(-\infty,-1]\cup[1,\infty)$ (C) $x\in(-\infty,\infty)$ (D) None of these

9. The domain of the function $\sin^{-1}\left(\log_2\left(\frac{x}{3}\right)\right)$ is-

(A) $\left[\frac{1}{2},3\right]$ (B) $\left[\frac{1}{2},3\right]$ (C) $\left[\frac{3}{2},6\right]$ (D) $\left[\frac{1}{2},2\right]$

10. $\sec^{-1}(\sin x)$ is real , if

(A) $x\in(-\infty,\infty)$ (B) $x\in[-1,1]$ (C) $x=(2n+1)\frac{\pi}{2},n\in I$ (D) $x=n\pi,n\in Z$

11. The domain of the function $f(x)=\cos^{-1}(x+[x])$, where $[\cdot]$ denotes the greatest integer function, is

(A) [–1, 1] (B) [0, 1) (C) (–1, 0) (D) None of these

12. Domain of the function $f(x) = \log_e \cos^{-1}\{\sqrt{x}\}$ is, where $\{.\}$ represents fractional part function -

(A) $x \in R$ (B) $x \in [0, \infty)$ (C) $x \in (0, \infty)$ (D) $x \in R - \{x \mid x \in I\}$

13. The range of the function $f(x) = \sin^{-1}(\log_2(-x^2 + 2x + 3))$ is -

(A) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (B) $\left[-\frac{\pi}{2}, 0\right]$ (C) $\left[0, \frac{\pi}{2}\right]$ (D) $[-1, 1]$

14. Range of $f(x) = \cot^{-1}(\log_e(1 - x^2))$ is -

(A) $(0,\pi)$ (B) $\left(0, \frac{\pi}{2}\right)$ (C) $\left[\frac{\pi}{2}, \pi\right)$ (D) $\left(0, \frac{\pi}{2}\right]$

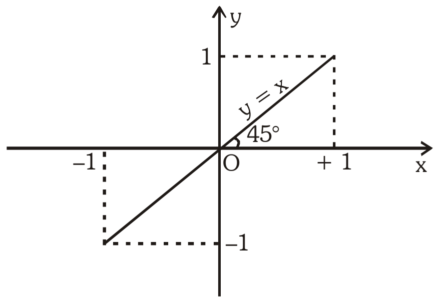
15. Find the domain and range of the following functions .
(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $f(x) = \cot^{-1}(2x - x^2)$ (ii) $f(x) = \sec^{-1}(\log_3 \tan x + \log_{\tan x} 3)$

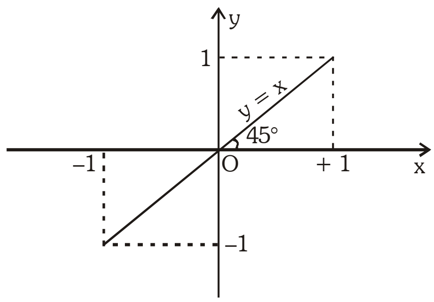
(iii) $f(x) = \cos^{-1}\left(\frac{\sqrt{2x^2+1}}{x^2+1}\right)$ (iv) $f(x) = \tan^{-1}\left(\log_{\frac{4}{5}}\left(5x^2 - 8x + 4\right)\right)$

4.0 PROPERTIES OF INVERSE CIRCULAR FUNCTIONS

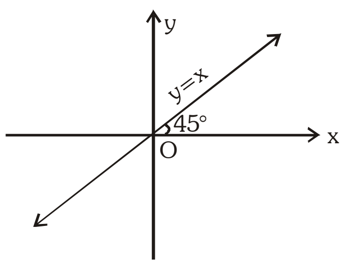
P-1 (i) $y = \sin(\sin^{-1}x) = x$
$x \in [-1,1], y \in [-1,1]$



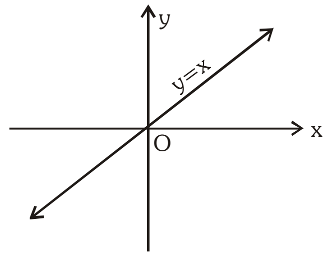
(ii) $y = \cos(\cos^{-1} x) = x$
$x \in [-1,1], y \in [-1,1]$



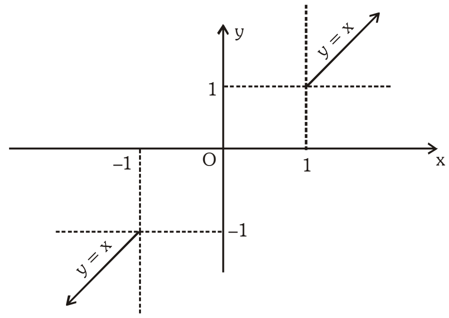
(iii) $y = \tan(\tan^{-1} x) = x$
$x \in R, y \in R$



(iv) $y = \cot(\cot^{-1} x) = x$,
$x \in R; y \in R$

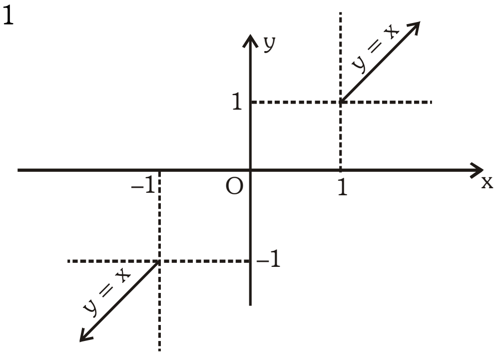


(v) $y = \operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$,



$|x| \geq 1, |y| \geq 1$

(vi) $y = \sec(\sec^{-1} x) = x$ $\quad |x| \geq 1 ; |y| \geq 1$



Note – All the above functions are aperiodic.

Illustrations

Illustration 3. Evaluate the following :

(i) $\sin(\cos^{-1}3/5)$ (ii) $\cos(\tan^{-1} 3/4)$ (iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

Solution (i) Let $\cos^{-1} 3/5 = \theta$. Then,

$\cos\theta = 3/5 \Rightarrow \sin\theta = 4/5$

$\therefore \sin(\cos^{-1} 3/5) = \sin \theta = 4/5$

(ii) Let $\tan^{-1} 3/4 = \theta$. Then,

$\tan\theta = 3/4$

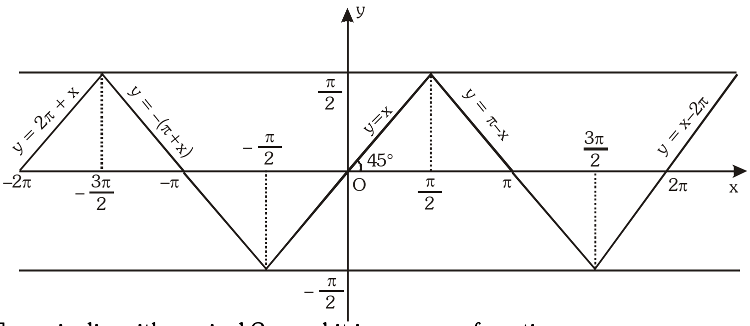
$\Rightarrow \cos\theta = \frac{4}{5}$ $\quad \left\{\because \text{ as } \cos^2\theta = \frac{1}{1+\tan^2\theta}\right\}$

$\therefore \cos(\tan^{-1} 3/4) = \cos\theta = 4/5$

(iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{2} - \left(-\frac{\pi}{6}\right)\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ **Ans.**

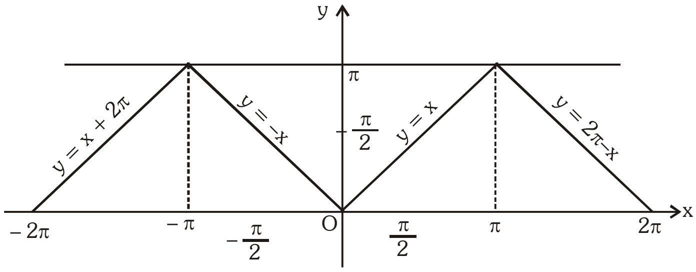
P-2 (i) $y = \sin^{-1}(\sin x),\ x \in R,\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ periodic with period 2π and it is an odd function.

$$\sin^{-1}(\sin x) = \begin{cases} -\pi - x, & -\pi \le x \le -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi \end{cases}$$



(ii) $y = \cos^{-1}(\cos x),\ x \in R,\ y \in [0,\pi]$, periodic with period 2π and it is an even function.

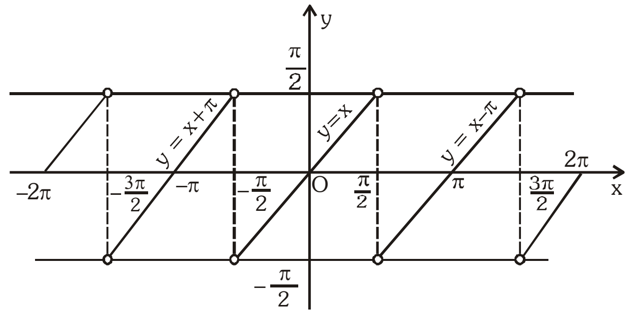
$$\cos^{-1}(\cos x) = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 < x \le \pi \end{cases}$$



(iii) $y = \tan^{-1}(\tan x)$

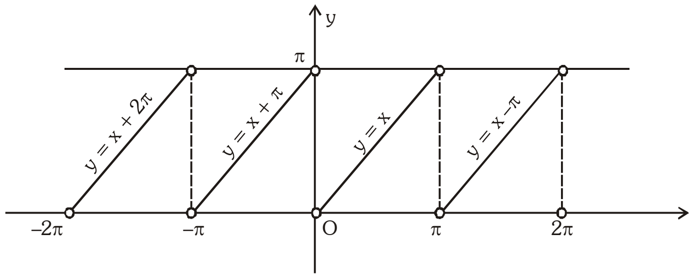
$x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\};\quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, periodic with period π and it is an odd function.

$$\tan^{-1}(\tan x) = \begin{cases} x + \pi, & -\frac{3\pi}{2} < x < -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x - \pi, & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

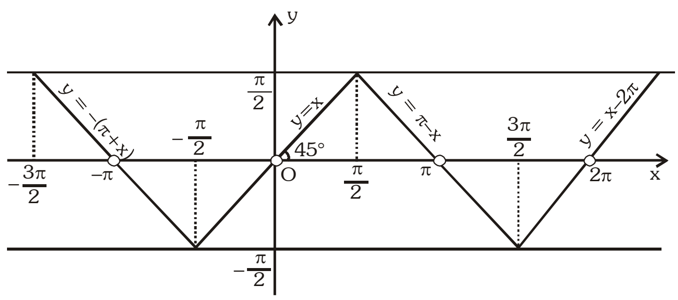


(iv) $y = \cot^{-1}(\cot x),\ x \in R - \{n\pi, n \in I\},\ y \in (0, \pi)$, periodic with period π and neither even nor odd function.

$$\cot^{-1}(\cot x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \\ x - \pi, & \pi < x < 2\pi \end{cases}$$



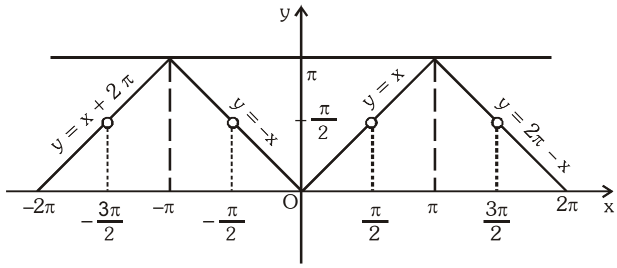
(v) $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x),\ x \in R - \{n\pi, n \in I\}\ y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, is periodic with period 2π and it is an odd function.



(vi) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π

and it is an even function.

$x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



Illustrations

Illustration 4. The value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6))$ is -

(A) $5\pi/6$ (B) $\pi/2$ (C) $3\pi/2$ (D) none of these

Solution $\sin^{-1}\left(-\sqrt{3}/2\right) = -\sin^{-1}\left(\sqrt{3}/2\right) = -\pi/3$

and $\cos^{-1}(\cos(7\pi/6) = \cos^{-1}\cos(2\pi - 5\pi/6) = \cos^{-1}\cos(5\pi/6) = 5\pi/6$

Hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos 7\pi/6) = -\frac{\pi}{3} + \frac{5\pi}{6} = \frac{\pi}{2}$ **Ans.(B)**

Illustration 5. Evaluate the following :

(i) $\sin^{-1}(\sin(\pi/4))$ (ii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Solution (i) $\sin^{-1}(\sin(\pi/4)) = \frac{\pi}{4}$

(ii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$, because $\frac{7\pi}{6}$ does not lie between 0 and π.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{5\pi}{6}\right)\right)$ $\left\{\because \frac{7\pi}{6} = 2\pi - \frac{5\pi}{6}\right\}$

$= \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$ **Ans.**

Illustration 6. Evaluate the following :

(i) $\sin^{-1}(\sin 10)$ (ii) $\tan^{-1}(\tan(-6))$ (iii) $\cot^{-1}(\cot 4)$

Solution (i) We know that $\sin^{-1}(\sin\theta) = \theta$, if $-\pi/2 \le \theta \le \pi/2$

Here, $\theta = 10$ radians which does not lie between $-\pi/2$ and $\pi/2$

But, $3\pi - \theta$ i.e., $3\pi - 10$ lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Also, $\sin(3\pi - 10) = \sin 10$

$\therefore$ $\sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = (3\pi - 10)$

(ii) We know that,

$\tan^{-1}(\tan\theta) = \theta$, if $-\pi/2 < \theta < \pi/2$. Here, $\theta = -6$, radians which does not lie between $-\pi/2$ and $\pi/2$. We find that $2\pi - 6$ lies between $-\pi/2$ and $\pi/2$ such that;

$\tan(2\pi - 6) = -\tan 6 = \tan(-6)$

$\therefore$ $\tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi - 6)) = (2\pi - 6)$

(iii) $\cot^{-1}(\cot 4) = \cot^{-1}(\cot(\pi + (4 - \pi))) = \cot^{-1}(\cot(4 - \pi)) = (4 - \pi)$ **Ans.**

Illustration 7. Prove that $\sec^2(\tan^{-1}2) + \operatorname{cosec}^2(\cot^{-1}3) = 15$

Solution We have,

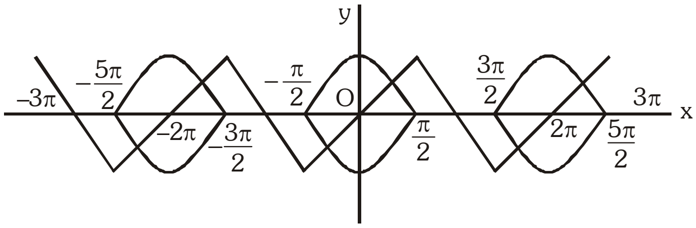
$\sec^2(\tan^{-1}2) + \operatorname{cosec}^2(\cot^{-1}3)$

$$= \{\sec(\tan^{-1}2)\}^2 + \{\operatorname{cosec}(\cot^{-1}3)\}^2 = \left\{\sec\left(\tan^{-1}\frac{2}{1}\right)\right\}^2 + \left\{\operatorname{cosec}\left(\cot^{-1}\frac{3}{1}\right)\right\}^2$$

$$= \{\sec(\sec^{-1}\sqrt{5})\}^2 + \{\operatorname{cosec}(\operatorname{cosec}^{-1}\sqrt{10})\}^2 = (\sqrt{5})^2 + (\sqrt{10})^2 = 15$$

Illustration 8. Find the number of solutions of (x, y) which satisfy $|y| = \cos x$ and $y = \sin^{-1}(\sin x)$, where $|x| \le 3\pi$.

Solution Graphs of $y = \sin^{-1}(\sin x)$ and $|y| = \cos x$ meet exactly six times in $[-3\pi, 3\pi]$.



BEGINNER'S BOX-2

1. Evaluate the following :

(a) $\tan\left(\cos^{-1}\left(\frac{8}{17}\right)\right)$ (b) $\sin\left(\frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right)\right)$ (c) $\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$

(d) $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right)$ (e) $\cos\left(\sin^{-1}\frac{1}{2}\right)$ (f) $\sin\left(\cos^{-1}\frac{3}{5}\right)$

2. Evaluate the following

(a) $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ (b) $\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)$ (c) $\sin^{-1}(\sin 2)$ (d) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

(e) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$ (f) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ (g) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

3. $\cos\left(\cos^{-1}\cos\left(\frac{8\pi}{7}\right) + \tan^{-1}\tan\left(\frac{8\pi}{7}\right)\right)$ has the value equal to

(A) 1 (B) –1 (C) $\cos\frac{\pi}{7}$ (D) 0

4. The value of $\sec\left[\sin^{-1}\left(-\sin\frac{50\pi}{9}\right) + \cos^{-1}\cos\left(-\frac{31\pi}{9}\right)\right]$ is equal to

(A) $\sec\frac{10\pi}{9}$ (B) $\sec\frac{\pi}{9}$ (C) 1 (D) –1

5. The value of $\sin^2\left(\cos^{-1}\frac{1}{2}\right)+\cos^2\left(\sin^{-1}\frac{1}{3}\right)$ is

(A) $\frac{17}{36}$ (B) $\frac{59}{36}$ (C) $\frac{36}{59}$ (D) None of these

6*. The value of $\cos\left[\frac{1}{2}\cos^{-1}\left(\cos\left(-\frac{14\pi}{5}\right)\right)\right]$ is :

(A) $\cos\left(-\frac{7\pi}{5}\right)$ (B) $\sin\left(\frac{\pi}{10}\right)$ (C) $\cos\left(\frac{2\pi}{5}\right)$ (D) $-\cos\left(\frac{3\pi}{5}\right)$

7. The value of $\cot^{-1}\left[\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right]$ is ($x\in\left(0,\frac{\pi}{2}\right)$)

(A) $\pi - x$ (B) $2\pi - x$ (C) $\frac{x}{2}$ (D) $\pi-\frac{x}{2}$

8. Find the value of:

$$\sin^{-1}\left(\sin\frac{33\pi}{7}\right)+\cos^{-1}\left(\cos\frac{46\pi}{7}\right)+\tan^{-1}\left(-\tan\frac{13\pi}{8}\right)+\cot^{-1}\left(\cot\left(\frac{-19\pi}{8}\right)\right)$$

9. Find the value of $\sin^{-1}(\sin 5)+\cos^{-1}(\cos 10)+\tan^{-1}[\tan(-6)]+\cot^{-1}[\cot(-10)]$

10. Let $y = \sin^{-1}(\sin 8) - \tan^{-1}(\tan 10) + \cos^{-1}(\cos 12) - \sec^{-1}(\sec 9) + \cot^{-1}(\cot 6) - \mathrm{cosec}^{-1}(\mathrm{cosec}\, 7)$. If y simplifies to $a\pi + b$ then find $(a - b)$.

11. If $2 \le a < 3$, then the value of $\cos^{-1}\cos[a]+\mathrm{cosec}^{-1}\,\mathrm{cosec}[a]+\cot^{-1}\cot[a]$, (where [.] denotes greatest integer less than equal to x) is equal to

(A) $2-\pi$ (B) $2+\pi$ (C) π (D) 6

12. If x takes negative permissible value, then $\sin^{-1}x =$

(A) $\cos^{-1}\sqrt{1-x^2}$ (B) $-\cos^{-1}\sqrt{1-x^2}$ (C) $\cos^{-1}\sqrt{x^2-1}$ (D) $\pi-\cos^{-1}\sqrt{1-x^2}$

13. $\tan(\cos^{-1}x)$ is equal to

(A) $\pm\sqrt{\frac{1-x^2}{x}}, x\neq 0$ (B) $\frac{\sqrt{1+x^2}}{x}, x\neq 0$ (C) $\frac{x}{\sqrt{1+x^2}}$ (D) $\frac{\sqrt{1-x^2}}{x}, x\neq 0$

14. Prove that $\sin\cot^{-1}\cos\tan^{-1}x = \sqrt{\frac{x^2+1}{x^2+2}}$

15. Prove that : $\sin\cot^{-1}\tan\cos^{-1}x = \sin\mathrm{cosec}^{-1}\cot\tan^{-1}x = x$ where $x\in(0,1]$

Node-1\Target-2022-23\1 JEE(M+A)\Module\Enthusiast\English\Maths\EM-1118.ITF\Th.

P-3 (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ $\quad -1 \le x \le 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ $\quad x \in R$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$ $\quad |x| \ge 1$

P-4 (i) $\sin^{-1}(-x) = -\sin^{-1} x$, $\quad -1 \le x \le 1$

(ii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$, $\quad x \le -1$ or $x \ge 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$, $\quad x \in R$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, $\quad x \in R$

(v) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $\quad -1 \le x \le 1$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, $\quad x \le -1$ or $x \ge 1$

P-5 (i) $\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$; $\quad x \le -1, x \ge 1$

(ii) $\sec^{-1} x = \cos^{-1}\frac{1}{x}$; $\quad x \le -1, x \ge 1$

(iii) $$\cot^{-1} x = \begin{cases} \tan^{-1}\frac{1}{x} & ; \quad x > 0 \\ \pi + \tan^{-1}\frac{1}{x} & ; \quad x < 0 \end{cases}$$

Illustrations

Illustration 9. Prove that $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \begin{cases} \pi/2 & , \text{ if } x > 0 \\ -\pi/2 & , \text{ if } x < 0 \end{cases}$

Solution We have , $$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & , \text{ for } x > 0 \\ -\pi + \cot^{-1} x & , \text{ for } x < 0 \end{cases}$$

$$\therefore \quad \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \tan^{-1} x + \cot^{-1} x = \pi/2 & , \text{ if } x > 0 \\ \tan^{-1} x + \cot^{-1} x - \pi = \pi/2 - \pi = -\pi/2 & , \text{ if } x < 0 \end{cases}$$

BEGINNER'S BOX-3

1. Prove that $\tan(\tan^{-1} x + \tan^{-1} y + \tan^{-1} z) = \cot(\cot^{-1} x + \cot^{-1} y + \cot^{-1} z)$

2. Find the value of $\cos(2\cos^{-1} x + \sin^{-1} x)$ at $x = \frac{1}{5}$

3. Find the value of $\sin(\tan^{-1} a + \tan^{-1}\frac{1}{a})$; $a \neq 0$

4*. If range of $f(x) = \tan^{-1} x + \cot^{-1} x + \sin^{-1} x$ is [a, b] then ($x \in R$)

(A) $a = 0$ (B) $b = \frac{\pi}{2}$ (C) $a = \frac{\pi}{4}$ (D) $b = \pi$

5. Number of integral ordered pairs (a,b) for which $\sin^{-1}(1+b+b^2+\ldots\infty)+\cos^{-1}\left(a-\frac{a^2}{3}+\frac{a^3}{9}-\ldots\infty\right)=\frac{\pi}{2}$ is -

(A) 0 (B) 4 (C) 9 (D) Infinitely many

6. Prove the following

(a) $\cos^{-1}\left(\frac{5}{13}\right)=\tan^{-1}\left(\frac{12}{5}\right)$

(b) $\sin^{-1}\left(-\frac{4}{5}\right)=\tan^{-1}\left(-\frac{4}{3}\right)=\cos^{-1}\left(-\frac{3}{5}\right)-\pi$

7. Evaluate : $\sin[\cos^{-1}(3/5)+\tan^{-1}(-2)]$

8. Evaluate : $\tan[\sin^{-1}(-3/5)+\cot^{-1}3]$

9. Evaluate : $\cos[\sin^{-1}(-24/25)+\tan^{-1}(5/12)]$

10*. If α is only real root of the equation $x^3+(\cos1)x^2+(\sin1)x+1=0$, then $\left(\tan^{-1}\alpha+\tan^{-1}\frac{1}{\alpha}\right)$ **cannot** be equal to-

(A) 0 (B) $\pi/2$ (C) $-\pi/2$ (D) π

P-6 (i) (a)
$$\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\frac{x+y}{1-xy} & \text{where } x>0,\ y>0\ \&\ xy<1\\ \pi+\tan^{-1}\frac{x+y}{1-xy} & \text{where } x>0,\ y>0\ \&\ xy>1\\ \frac{\pi}{2}, & \text{where } x>0,\ y>0\ \&\ xy=1\end{cases}$$

(b) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}$ where $x>0,\ y>0$

(c) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if $x>0,\ y>0,\ z>0\ \&\ xy+yz+zx<1$

(ii) (a)
$$\sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}[x\sqrt{1-y^2}+y\sqrt{1-x^2}] & \text{where } x>0,\ y>0\ \&\ (x^2+y^2)\le 1\\ \pi-\sin^{-1}[x\sqrt{1-y^2}+y\sqrt{1-x^2}] & \text{where } x>0,\ y>0\ \&\ x^2+y^2>1\end{cases}$$

(b) $\sin^{-1}x-\sin^{-1}y=\sin^{-1}[x\sqrt{1-y^2}-y\sqrt{1-x^2}]$ where $x>0,\ y>0$

(iii) (a) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}[xy-\sqrt{1-x^2}\sqrt{1-y^2}]$ where $x>0,\ y>0$

(b)
$$\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right) & ;\ x<y,\ \ x,y>0\\ -\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right) & ;\ x>y,\ \ x,y>0\end{cases}$$

Note – In the above results x & y are taken positive. In case if these are given as negative, we first apply P-4 and then use above results.

Illustrations

Illustration 10. Prove that

(i) $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$ (ii) $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Solution (i) L.H.S. $=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}$

$$=\tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right\} \qquad \left\{\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right); \text{ if } xy<1\right\}$$

$$=\tan^{-1}\left(\frac{20}{90}\right)=\tan^{-1}\left(\frac{2}{9}\right)=\text{R.H.S.}$$

(ii) $\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$

$$=\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) \qquad =\tan^{-1}\left(\frac{6}{17}\right)+\tan^{-1}\left(\frac{11}{23}\right)$$

$$=\tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)=\tan^{-1}\left(\frac{325}{325}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$ **Ans.**

Illustration 11. Prove that $\sin^{-1}\frac{12}{13}+\cot^{-1}\frac{4}{3}+\tan^{-1}\frac{63}{16}=\pi$

Solution We have,

$$\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}$$

$$=\tan^{-1}\frac{12}{5}+\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{63}{16} \qquad \left[\because \sin^{-1}\frac{12}{13}=\tan^{-1}\frac{12}{5}\text{ and }\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}\right]$$

$$=\pi+\tan^{-1}\left\{\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right\}+\tan^{-1}\frac{63}{16} \qquad \left[\because \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right),\text{if } xy>1\right]$$

$$=\pi+\tan^{-1}\left(\frac{63}{-16}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi-\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{63}{16} \qquad \left[\because \tan^{-1}(-x)=-\tan^{-1}x\right]$$

$$=\pi$$

Illustration 12. Prove that : $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Solution We have, $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{3}{5}$ $\left[\because \cos^{-1}\frac{12}{13}=\sin^{-1}\frac{5}{13}\right]$

$$=\sin^{-1}\left\{\frac{5}{13}\times\sqrt{1-\left(\frac{3}{5}\right)^2}+\frac{3}{5}\times\sqrt{1-\left(\frac{5}{13}\right)^2}\right\}=\sin^{-1}\left\{\frac{5}{13}\times\frac{4}{5}+\frac{3}{5}\times\frac{12}{13}\right\}=\sin^{-1}\frac{56}{65}$$

Illustration 13. If $x = \text{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$ and $y = \sec(\cot^{-1}(\sin(\tan^{-1}(\text{cosec}(\cos^{-1}a)))))$, where $a \in [0, 1)$. Find the relationship between x and y in terms of 'a'

Solution Here,

$x = \text{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$ $\left\{ \text{Let } \sin\theta = a \Rightarrow \sec\theta = \dfrac{1}{\sqrt{1-a^2}} \right.$

$= \text{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec\theta))))$

$\Rightarrow x = \text{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\dfrac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$ $\left\{ \text{Let } \cot^{-1}\left(\dfrac{1}{\sqrt{1-a^2}}\right) = \phi \Rightarrow \cot\phi = \dfrac{1}{\sqrt{1-a^2}} \right.$

$= \text{cosec}(\tan^{-1}(\cos\phi))$ therefore $\cos\phi = \dfrac{1}{\sqrt{2-a^2}}$

$\Rightarrow x = \text{cosec}\left(\tan^{-1}\left(\dfrac{1}{\sqrt{2-a^2}}\right)\right)$ $\left\{ \text{Let, } \tan^{-1}\left(\dfrac{1}{\sqrt{2-a^2}}\right) = \psi \Rightarrow \tan\psi = \dfrac{1}{\sqrt{2-a^2}} \right.$

$= \text{cosec}\,\psi$ therefore $\text{cosec}\,\psi = \sqrt{3-a^2}$

$\Rightarrow x = \sqrt{3-a^2}$ ...... (i)

and $y = \sec(\cot^{-1}(\sin(\tan^{-1}(\text{cosec}(\cos^{-1}a)))))$ $\left\{ \text{Let } \cos^{-1}a = \alpha \Rightarrow \cos\alpha = a \Rightarrow \text{cosec}\alpha = \dfrac{1}{\sqrt{1-a^2}} \right.$

$= \sec(\cot^{-1}(\sin(\tan^{-1}(\text{cosec}\alpha))))$

$\Rightarrow y = \sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\dfrac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$ $\left\{ \text{Let, } \tan^{-1}\dfrac{1}{\sqrt{1-a^2}} = \beta \Rightarrow \tan\beta = \dfrac{1}{\sqrt{1-a^2}} \right.$

$= \sec(\cot^{-1}(\sin(\beta)))$ $\Rightarrow \sin\beta = \dfrac{1}{\sqrt{2-a^2}}$

$\Rightarrow y = \sec\left(\cot^{-1}\left(\dfrac{1}{\sqrt{2-a^2}}\right)\right)$ $\left\{ \text{Let } \cot^{-1}\dfrac{1}{\sqrt{2-a^2}} = \gamma \Rightarrow \cot\gamma = \dfrac{1}{\sqrt{2-a^2}} \Rightarrow \sec\gamma = \sqrt{3-a^2} \right.$

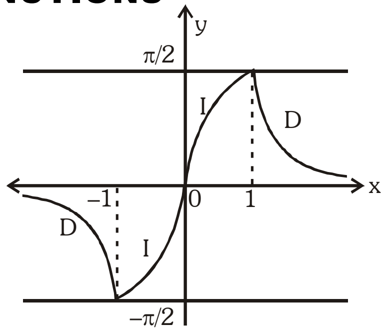
$= \sec\gamma$

$\Rightarrow y = \sqrt{3-a^2}$ ...... (ii)

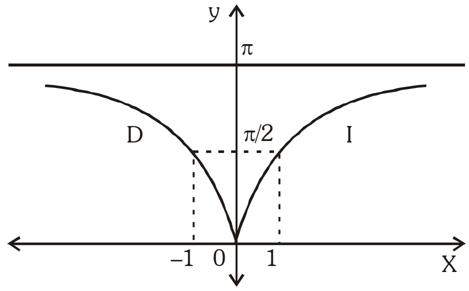
from (i) and (ii), $x = y = \sqrt{3-a^2}$. **Ans.**

5.0 SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS

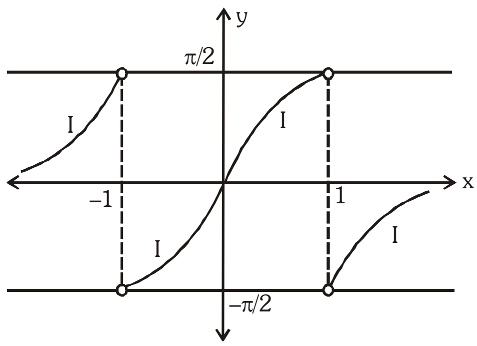
(a) $y = f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$



(b) $y = f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

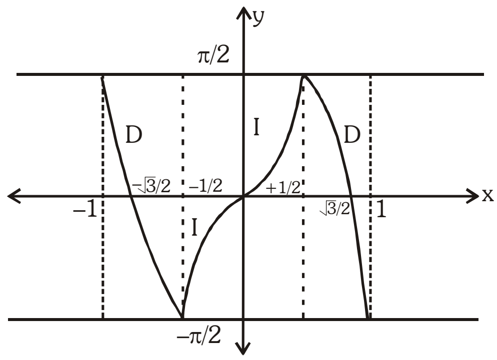


(c) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{cases}$



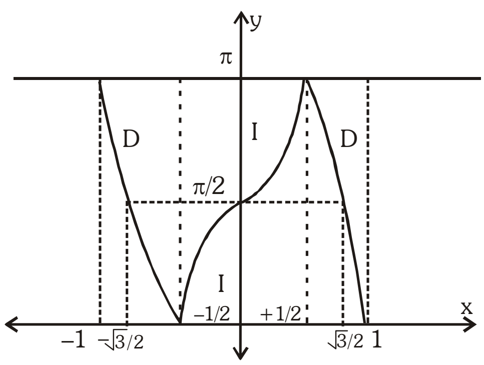
(d) $y = f(x) = \sin^{-1}(3x - 4x^3)$

$= \begin{cases} -(\pi + 3\sin^{-1}x) & \text{if } -1 \le x < -\dfrac{1}{2} \\ 3\sin^{-1}x & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if } \dfrac{1}{2} < x \le 1 \end{cases}$

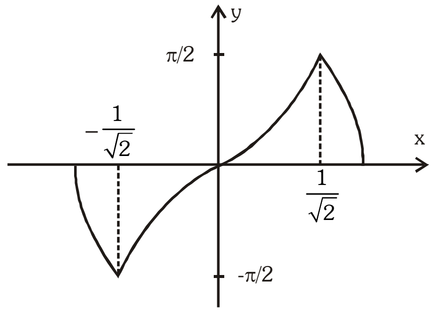


(e) $y = f(x) = \cos^{-1}(4x^3 - 3x)$

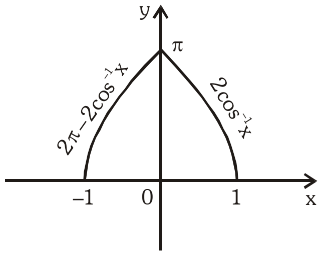
$= \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x < -\dfrac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1}x & \text{if } \dfrac{1}{2} < x \le 1 \end{cases}$



(f) $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\begin{cases}-(\pi+2\sin^{-1}x) & -1\le x<-\frac{1}{\sqrt{2}}\\ 2\sin^{-1}x & -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}\\ \pi-2\sin^{-1}x & \frac{1}{\sqrt{2}}<x\le 1\end{cases}$



(g) $\cos^{-1}\left(2x^2-1\right)=\begin{cases}2\cos^{-1}x & 0\le x\le 1\\ 2\pi-2\cos^{-1}x & -1\le x<0\end{cases}$



Illustrations

Illustration 14. Evalulate : (i) $\tan\left\{2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right\}$ (ii) $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\}$

Solution (i) $\tan\left\{2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right\}=\tan\left\{\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right)-\tan^{-1}1\right\}$ $\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right),\text{ if } |x|<1\right]$

$$=\tan\left\{\tan^{-1}\frac{5}{12}-\tan^{-1}1\right\}=\tan\left\{\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)\right\}=\tan\left\{\tan^{-1}\left(\frac{-7}{17}\right)\right\}=\frac{-7}{17}$$

(ii) Let $\cos^{-1}\frac{\sqrt{5}}{3}=\theta$. Then, $\cos\theta=\frac{\sqrt{5}}{3}$, $0<\theta<\pi/2$

Now, $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

$$=\tan\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\frac{\sqrt{1-\frac{\sqrt{5}}{3}}}{\sqrt{1+\frac{\sqrt{5}}{3}}}$$

$$=\sqrt{\frac{3-\sqrt{5}}{3+\sqrt{5}}}=\sqrt{\frac{(3-\sqrt{5})^2}{(3+\sqrt{5})(3-\sqrt{5})}}=\sqrt{\frac{(3-\sqrt{5})^2}{9-5}}=\frac{3-\sqrt{5}}{2}$$

Illustration 15. Prove that : $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Solution We have, $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$

$$=\tan^{-1}\left\{\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right\}+\tan^{-1}\frac{1}{7}\qquad\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right),\text{if } -1<x<1\right]$$

$$=\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left\{\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right\}=\tan^{-1}\frac{31}{17}$$

Illustration 16. Prove that $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),x\in[0,1]$

Solution We have, $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left\{\frac{1-\left(\sqrt{x}\right)^2}{1+\left(\sqrt{x}\right)^2}\right\}=\frac{1}{2}\times 2\tan^{-1}\sqrt{x}=\tan^{-1}\sqrt{x}.$

Alter : Putting $\sqrt{x}=\tan\theta$, we have $\Rightarrow\theta\in\left[0,\frac{\pi}{4}\right]$

$$\text{RHS}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\frac{1}{2}\cos^{-1}(\cos 2\theta)=\theta\qquad\because\left(2\theta\in\left[0,\frac{\pi}{2}\right]\right)$$

$$=\tan^{-1}\sqrt{x}=\text{LHS}$$

Illustration 17. Prove that : (i) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}$

(ii) $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Solution (i) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=2\left\{2\tan^{-1}\frac{1}{5}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$

$$=2\left\{\tan^{-1}\frac{2\times 1/5}{1-(1/5)^2}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}\qquad\left[\begin{array}{c}\because 2\tan^{-1}x\\=\tan^{-1}\frac{2x}{1-x^2},\text{if } |x|<1\end{array}\right]$$

$$=2\tan^{-1}\frac{5}{12}-\left\{\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}\right\}=\tan^{-1}\left\{\frac{2\times 5/12}{1-(5/12)^2}\right\}-\tan^{-1}.\left\{\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right\}$$

$$=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{29}{6931}=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{1}{239}=\tan^{-1}\left\{\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right\}=\tan^{-1}1=\frac{\pi}{4}$$

(ii) $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=2\left\{\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right\}+\sec^{-1}\frac{5\sqrt{2}}{7}$

$$=2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right\}+\tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2-1} \qquad \left[\because \sec^{-1}x=\tan^{-1}\sqrt{x^2-1}\right]$$

$$=2\tan^{-1}\frac{13}{39}+\tan^{-1}\frac{1}{7}=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left\{\frac{2\times 1/3}{1-(1/3)^2}\right\}+\tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2},\ \text{if} |x|<1\right]$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right\}=\tan^{-1}1=\frac{\pi}{4}$$

BEGINNER'S BOX-4

1. Prove that:

(i) $\tan^{-1}2+\tan^{-1}3=\frac{3\pi}{4}$

(ii) $\cot^{-1}9+\operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}=\frac{\pi}{4}$

(iii) $arc\ \cos\sqrt{\frac{2}{3}}-arc\ \cos\frac{\sqrt{6}+1}{2\sqrt{3}}=\frac{\pi}{6}$

(iv) $4\tan^{-1}(1/5)-\tan^{-1}(1/70)+\tan^{-1}(1/99)=\pi/4$.

(v) $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=2[\tan^{-1}1+\tan^{-1}1/2+\tan^{-1}1/3]$

(vi) $2\cos^{-1}\frac{3}{\sqrt{13}}+\cot^{-1}\frac{16}{63}+\frac{1}{2}\cos^{-1}\frac{7}{25}=\pi$

2. Prove the following results

(a) $2\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\tan^{-1}\left(\frac{4}{7}\right)$

(b) $2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)=\frac{\pi}{4}$

3. Prove the following

(a) $\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{8}{17}\right)=\cos^{-1}\left(\frac{36}{85}\right)$

(b) $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{8}{19}\right)=\frac{\pi}{4}$

(c) $\tan^{-1}\left(\frac{2}{11}\right)+\tan^{-1}\left(\frac{7}{24}\right)=\tan^{-1}\left(\frac{1}{2}\right)$

4. If $\tan^{-1}3+\tan^{-1}x=\tan^{-1}8$, then x =

(A) 5 (B) 1/5 (C) 5/14 (D) 14/5

5. $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\frac{x-y}{x+y}$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$ or $-\frac{3\pi}{4}$

6. Prove the following **:** $\tan^{-1} a + \tan^{-1} b = \cos^{-1}\left[\frac{1-ab}{\sqrt{\{(1+a^2)(1+b^2)\}}}\right]$ (where a,b>0)

7. Prove the following **:** $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]=\frac{2a}{1-a^2}$

8. Prove that: $2\tan^{-1}\left[\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\frac{1}{2}\theta\right]=\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)$.

9. Prove the following : $2\tan^{-1}(\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x)=\tan^{-1}x$

10. If $\sin^{-1}x+\tan^{-1}x=\pi/2$, prove that $2x^2+1=\sqrt{5}$

11. Prove that:

$$\tan^{-1}\sqrt{\left\{\frac{a}{bc}(a+b+c)\right\}}+\tan^{-1}\sqrt{\left\{\frac{b}{ca}(a+b+c)\right\}}+\tan^{-1}\sqrt{\left\{\frac{c}{ab}(a+b+c)\right\}}=\pi$$

12. Prove that: $\tan^{-1}\left(\frac{yz}{xr}\right)+\tan^{-1}\left(\frac{zx}{yr}\right)+\tan^{-1}\left(\frac{xy}{zr}\right)=\frac{\pi}{2}$, *where* $r^2=x^2+y^2+z^2$.

13. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, show that $x^2+y^2+z^2+2xyz=1$.

6.0 EQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Illustrations

Illustration 18. The equation $2\cos^{-1}x+\sin^{-1}x=\frac{11\pi}{6}$ has

(A) no solution (B) only one solution

(C) two solutions (D) three solutions

Solution Given equation is $2\cos^{-1}x+\sin^{-1}x=\frac{11\pi}{6}$

$\Rightarrow \cos^{-1}x+(\cos^{-1}x+\sin^{-1}x)=\frac{11\pi}{6} \Rightarrow \cos^{-1}x+\frac{\pi}{2}=\frac{11\pi}{6} \Rightarrow \cos^{-1}x=4\pi/3$

which is not possible as $\cos^{-1}x\in[0,\pi]$ **Ans.(A)**

Illustration 19. If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=5\pi^2/8$, then x is equal to-

(A) –1 (B) 0

(C) 1 (D) none of these

Solution The given equation can be written as $(\tan^{-1}x+\cot^{-1}x)^2-2\tan^{-1}x\cot^{-1}x=5\pi^2/8$

Since $\tan^{-1}x+\cot^{-1}x=\pi/2$ we have

$(\pi/2)^2 - 2\tan^{-1} x\,(\pi/2 - \tan^{-1} x) = 5\pi^2/8$

$\Rightarrow \quad 2(\tan^{-1} x)^2 - 2\,(\pi/2)\tan^{-1} x - 3\pi^2/8 = 0 \quad \Rightarrow \tan^{-1} x = -\pi/4 \Rightarrow x = -1$ **Ans. (A)**

Illustration 20. Solve the equation : $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

Solution $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

taking tangent on both sides

$$\Rightarrow \quad \tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right)\right) = 1$$

$$\Rightarrow \quad \frac{\tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)\right) + \tan\left(\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)}{1 - \tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)\right)\tan\left(\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)} = 1$$

$$\Rightarrow \quad \frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x-1}{x-2}\cdot\frac{x+1}{x+2}} = 1$$

$$\Rightarrow \quad \frac{(x-1)(x+2) + (x-2)(x+1)}{x^2 - 4 - (x^2 - 1)} = 1$$

$$\Rightarrow \quad 2x^2 - 4 = -3 \qquad \Rightarrow \qquad x = \pm\frac{1}{\sqrt{2}}$$

Now verify $x = \frac{1}{\sqrt{2}}$

$$= \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}} - 1}{\frac{1}{\sqrt{2}} - 2}\right) + \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}} + 1}{\frac{1}{\sqrt{2}} + 2}\right) = \tan^{-1}\left(\frac{\sqrt{2}-1}{2\sqrt{2}-1}\right) + \tan^{-1}\left(\frac{\sqrt{2}+1}{2\sqrt{2}+1}\right)$$

$$= \tan^{-1}\left(\frac{(2\sqrt{2}+1)(\sqrt{2}-1) + (2\sqrt{2}-1)(\sqrt{2}+1)}{(2\sqrt{2}-1)(2\sqrt{2}+1) - (\sqrt{2}-1)(\sqrt{2}+1)}\right) = \tan^{-1}\left(\frac{6}{6}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$x = -\frac{1}{\sqrt{2}}$

$$= \tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}} - 1}{-\frac{1}{\sqrt{2}} - 2}\right) + \tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}} + 1}{-\frac{1}{\sqrt{2}} + 2}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}+1}{2\sqrt{2}+1}\right) + \tan^{-1}\left(\frac{\sqrt{2}-1}{2\sqrt{2}-1}\right) \text{ \{same as above\}}$$

$$= \tan^{-1}(1) = \frac{\pi}{4}$$

$\therefore \quad x = \pm\frac{1}{\sqrt{2}}$ are solutions **Ans.**

Illustration 21. Solve the equation : $2\tan^{-1}(2x + 1) = \cos^{-1}x$.

Solution Here, $2\tan^{-1}(2x + 1) = \cos^{-1}x$

or $\quad \cos(2\tan^{-1}(2x+1)) = x \qquad \left\{\text{We know } \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right\}$

$\therefore \quad \dfrac{1-(2x+1)^2}{1+(2x+1)^2} = x$

$\Rightarrow \quad (1-2x-1)(1+2x+1) = x(4x^2+4x+2)$

$\Rightarrow \quad -2x\,.\,2(x+1) = 2x(2x^2+2x+1) \Rightarrow 2x(2x^2+2x+1+2x+2) = 0$

$\Rightarrow \quad 2x(2x^2+4x+3) = 0 \quad \Rightarrow \quad x = 0$ or $2x^2+4x+3 = 0$ {No solution}

Verify $x = 0$

$2\tan^{-1}(1) = \cos^{-1}(1) \quad \Rightarrow \dfrac{\pi}{2} = \dfrac{\pi}{2}$

$\therefore$ $x = 0$ is only the solution **Ans.**

7.0 INEQUALITIES INVOLVING INVERSE TRIGONOMETRIC FUNCTION

Illustrations

Illustration 22. Find the complete solution set of $\sin^{-1}(\sin 5) > x^2 - 4x$.

Solution $\sin^{-1}(\sin 5) > x^2 - 4x \quad \Rightarrow \quad \sin^{-1}[\sin(5-2\pi)] > x^2 - 4x$

$\Rightarrow \quad x^2 - 4x < 5 - 2\pi \quad \Rightarrow \quad x^2 - 4x + (2\pi - 5) < 0$

$\Rightarrow \quad 2-\sqrt{9-2\pi} < x < 2+\sqrt{9-2\pi} \quad \Rightarrow \quad x \in (2-\sqrt{9-2\pi},\ 2+\sqrt{9-2\pi})$ **Ans.**

Illustration 23. Find the complete solution set of $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$, where [.] denotes the greatest integer function.

Solution $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$

$\Rightarrow \quad ([\cot^{-1}x]-3)^2 \le 0 \ \Rightarrow \ [\cot^{-1}x] = 3 \quad \Rightarrow \quad 3 \le \cot^{-1}x < 4 \ \Rightarrow \ x \in (-\infty, \cot 3]$

Illustration 24. If $\cot^{-1}\dfrac{n}{\pi} > \dfrac{\pi}{6}, n \in N$, then the maximum value of n is -

(A) 1 (B) 5 (C) 9 (D) None of these

Solution $\cot^{-1}\dfrac{n}{\pi} > \dfrac{\pi}{6}$

$\Rightarrow \quad \cot\left(\cot^{-1}\left(\dfrac{n}{\pi}\right)\right) < \cot\left(\dfrac{\pi}{6}\right) \Rightarrow \dfrac{n}{\pi} < \sqrt{3}$

$\Rightarrow \quad n < \pi\sqrt{3} \quad \Rightarrow \quad n < 5.5$ (approx)

$\Rightarrow \quad n = 5 \quad \because (n \in N)$ **Ans. (B)**

8.0 SUMMATION OF SERIES

Illustrations

Illustration 25. Prove that :

$$\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\ldots+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)=\tan^{-1}\left(\frac{x}{y}\right)$$

Solution L.H.S.

$$\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\ldots+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}.\frac{1}{c_1}}\right)+\tan^{-1}\left(\frac{\frac{1}{c_1}-\frac{1}{c_2}}{1+\frac{1}{c_1}.\frac{1}{c_2}}\right)+\tan^{-1}\left(\frac{\frac{1}{c_2}-\frac{1}{c_3}}{1+\frac{1}{c_2}.\frac{1}{c_3}}\right)+\ldots+\tan^{-1}\left(\frac{\frac{1}{c_{n-1}}-\frac{1}{c_n}}{1+\frac{1}{c_{n-1}}.\frac{1}{c_n}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$= \left(\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{1}{c_1}\right) + \left(\tan^{-1}\frac{1}{c_1} - \tan^{-1}\frac{1}{c_2}\right) + \left(\tan^{-1}\frac{1}{c_2}\tan^{-1}\frac{1}{c_3}\right) + \ldots\ldots$$

$$+\left(\tan^{-1}\frac{1}{c_{n-1}} - \tan^{-1}\frac{1}{c_n}\right) + \tan^{-1}\left(\frac{1}{c_n}\right) = \tan^{-1}\left(\frac{x}{y}\right) = \text{R.H.S.}$$

BEGINNER'S BOX-5

1. Solve the following equation for x

(a) $\sin\left[\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}x\right] = 1$ **(b)** $\cos^{-1}x + \sin^{-1}\frac{x}{2} = \frac{\pi}{6}$

(c) $\cot^{-1}x - \cot^{-1}(x+2) = \frac{\pi}{12}$, where x > 0.

2. Solve the following equations / system of equations :

(i) $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$

(ii) $\tan^{-1}\frac{1}{1+2x} + \tan^{-1}\frac{1}{1+4x} = \tan^{-1}\frac{2}{x^2}$

(iii) $\tan^{-1}(x-1) + \tan^{-1}(x) + \tan^{-1}(x+1) = \tan^{-1}(3x)$

(iv) $\sin^{-1}\frac{1}{\sqrt{5}} + \cos^{-1}x = \frac{\pi}{4}$

(v) $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$ & $\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$

3. Solve the following equations / system of equations:

(a) $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}(8/31)$.

(b) $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$

(c) $\tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) = \tan^{-1}(-7)$.

4. If $(\sin^{-1}x)^2 + (\cos^{-1}x)^2 = \frac{5\pi^2}{8}$, then x is equal to

(A) 1 (B) –1 (C) $\frac{1}{\sqrt{2}}$ (D) $-\frac{1}{\sqrt{2}}$

5. $\tan^{-1}x > \cot^{-1}x$ holds for

(A) x > 1 (B) x < 1 (C) x = 1 (D) all values of x

6. If $\tan^{-1}\frac{x}{\pi} < \frac{\pi}{3}, x \in \mathbb{N}$, then the maximum value of x is

(A) 2 (B) 5 (C) 7 (D) None of these

7. Evaluate : $\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2}{1+(2r+1)(2r-1)}\right)$

8. If $a > b > c > 0$ then find the value of : $\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right)$.

9. Evaluate: $\sum_{1}^{\infty} \tan^{-1} \frac{1}{(n^2+n+1)}$.

10. Find the sum of the series:

(a) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + \ldots\ldots + \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}} + \ldots\ldots \infty$

(b) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} + \tan^{-1}\frac{1}{32} + \ldots\ldots \infty$

11. If $a_1, a_2, a_3, \ldots\ldots$ form an A. P. with a common difference d, $a_i > 0$, $d > 0$.

Prove that $\tan^{-1}\frac{d}{1+a_1a_2} + \tan^{-1}\frac{d}{1+a_2a_3} + \tan^{-1}\frac{d}{1+a_3a_4} + \ldots$to n terms $= \tan^{-1}\frac{nd}{1+a_1a_{n+1}}$.

Hence find the sum upto infinity.

12. Evaluate: $\tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13}$ to n terms.

13. Evaluate: $\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\frac{\sqrt{2}-1}{\sqrt{6}} + \ldots\ldots + \sin^{-1}\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}} + \ldots\ldots \infty$

14. Evaluate: $\cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21 + \cot^{-1}31 + \ldots\ldots$ to n terms.

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. If $\tan^{-1} y = 4\tan^{-1} x, \left(|x| < \tan\frac{\pi}{8}\right)$, find y as an algebraic function of x and hence prove that $\tan\frac{\pi}{8}$ is a root of the equation $x^4 - 6x^2 + 1 = 0$.

Solution We have $\tan^{-1} y = 4\tan^{-1} x$

$\Rightarrow \tan^{-1} y = 2\tan^{-1}\frac{2x}{1-x^2}$ (as $|x| < 1$)

$= \tan^{-1}\frac{\frac{4x}{1-x^2}}{1-\frac{4x^2}{(1-x^2)^2}} = \tan^{-1}\frac{4x(1-x^2)}{x^4-6x^2+1}$ $\left(\text{as}\left|\frac{2x}{1-x^2}\right| < 1\right)$

$\Rightarrow y = \frac{4x(1-x^2)}{x^4-6x^2+1}$

If $x = \tan\frac{\pi}{8} \Rightarrow \tan^{-1} y = 4\tan^{-1} x = \frac{\pi}{2} \Rightarrow$ y is not defined $\Rightarrow x^4 - 6x^2 + 1 = 0$ **Ans.**

Illustration 2. If $A = 2\tan^{-1}\left(2\sqrt{2}-1\right)$ and $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$, then show A > B.

Solution We have, $A = 2\tan^{-1}\left(2\sqrt{2}-1\right) = 2\tan^{-1}(1.828)$

$\Rightarrow \quad A > 2\tan^{-1}\left(\sqrt{3}\right) \quad \Rightarrow \quad A > \dfrac{2\pi}{3}$ ..... (i)

also we have, $\sin^{-1}\left(\dfrac{1}{3}\right) < \sin^{-1}\left(\dfrac{1}{2}\right) \Rightarrow \sin^{-1}\left(\dfrac{1}{3}\right) < \dfrac{\pi}{6}$

$\Rightarrow \quad 3\sin^{-1}\left(\dfrac{1}{3}\right) < \dfrac{\pi}{2}$

also, $3\sin^{-1}\left(\dfrac{1}{3}\right) = \sin^{-1}\left(3.\dfrac{1}{3} - 4\left(\dfrac{1}{3}\right)^3\right) = \sin^{-1}\left(\dfrac{23}{27}\right) = \sin^{-1}(0.852)$

$\Rightarrow \quad 3\sin^{-1}(1/3) < \sin^{-1}\left(\sqrt{3}/2\right) \quad \Rightarrow \quad 3\sin^{-1}(1/3) < \pi/3$

also, $\sin^{-1}(3/5) = \sin^{-1}(0.6) < \sin^{-1}\left(\sqrt{3}/2\right) \quad \Rightarrow \quad \sin^{-1}(3/5) < \pi/3$

Hence, $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5) < \dfrac{2\pi}{3}$ .... (ii)

From (i) and (ii), we have A > B.

Illustration 3. Solve for x : If $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$, where [.] denotes the greatest integer function.

Solution We have, $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$

$\Rightarrow \quad 1 \le \sin^{-1}.\cos^{-1}.\sin^{-1}.\tan^{-1}x \le \dfrac{\pi}{2} \quad \Rightarrow \quad \sin 1 \le \cos^{-1}.\sin^{-1}.\tan^{-1}x \le 1$

$\Rightarrow \quad \cos\sin 1 \ge \sin^{-1}.\tan^{-1}x \ge \cos 1 \quad \Rightarrow \quad \sin\cos\sin 1 \ge \tan^{-1}x \ge \sin\cos 1$

$\Rightarrow \quad \tan\sin\cos\sin 1 \ge x \ge \tan\sin\cos 1$

Hence, $x \in [\tan\sin\cos 1, \tan\sin\cos\sin 1]$ **Ans.**

Illustration 4. If $\theta = \tan^{-1}(2\tan^2\theta) - \dfrac{1}{2}\sin^{-1}\left(\dfrac{3\sin 2\theta}{5+4\cos 2\theta}\right)$ then find the sum of all possible values of $\tan\theta$.

Solution $\theta = \tan^{-1}(2\tan^2\theta) - \dfrac{1}{2}\sin^{-1}\left(\dfrac{3\sin 2\theta}{5+4\cos 2\theta}\right) \quad \Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta)$

$-\dfrac{1}{2}\sin^{-1}\left(\dfrac{6\tan\theta}{9+\tan^2\theta}\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \dfrac{1}{2}\sin^{-1}\left[\dfrac{2\left(\dfrac{1}{3}\tan\theta\right)}{1+\left(\dfrac{1}{3}\tan\theta\right)^2}\right] \Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta)$

$-\dfrac{2}{2}\tan^{-1}\left(\dfrac{1}{3}\tan\theta\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}\left(\frac{1}{3}\tan\theta\right)$ .... (i)

taking tangent on both sides

$$\Rightarrow \quad \tan\theta = \frac{6\tan^2\theta - \tan\theta}{3 + 2\tan^3\theta}$$

$\Rightarrow \quad 2\tan^4\theta - 6\tan^2\theta + 4\tan\theta = 0$

$\Rightarrow \quad 2\tan\theta(\tan^3\theta - 3\tan\theta + 2) = 0$

$\Rightarrow \quad 2\tan\theta(\tan\theta - 1)^2(\tan\theta + 2) = 0$

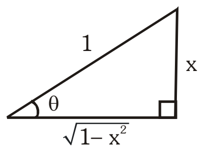
$\Rightarrow \quad \tan\theta = 0, 1, -2$ which satisfy equation (i)

$\therefore \quad \text{sum} = 0 + 1 - 2 = -1$ **Ans.**

Illustration 5. Transform $\sin^{-1}x$ in other inverse trigonometric functions, where $x \in (-1, 1) - \{0\}$

Solution **Case -I :** $0 < x < 1$

Let $\sin^{-1}x = \theta, \quad \theta \in \left(0, \frac{\pi}{2}\right)$



Now, $\cos\theta = \sqrt{1 - \sin^2\theta} \quad \Rightarrow \quad \theta = \cos^{-1}\sqrt{1 - x^2}$

$$\Rightarrow \quad \sin^{-1}x = \cos^{-1}\sqrt{1 - x^2} = \sec^{-1}\left(\frac{1}{\sqrt{1 - x^2}}\right)$$

$$\tan\theta = \frac{x}{\sqrt{1 - x^2}}$$

$$\Rightarrow \quad \theta = \tan^{-1}\frac{x}{\sqrt{1 - x^2}} \qquad \Rightarrow \quad \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1 - x^2}}$$

$$\Rightarrow \quad \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1 - x^2}} = \cot^{-1}\left(\frac{\sqrt{1 - x^2}}{x}\right)$$

Hence, $\sin^{-1}x = \cos^{-1}\sqrt{1 - x^2}$

$$= \quad \sec^{-1}\frac{1}{\sqrt{1 - x^2}} = \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1 - x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right), \; 0 < x < 1$$

Case-II : $-1 < x < 0$

Let $\sin^{-1}x = \theta \quad \theta \in \left(-\frac{\pi}{2}, 0\right)$, Then $x = \sin\theta$

$\Rightarrow \quad \cos\theta = \sqrt{1 - x^2} \qquad \Rightarrow \quad \cos(-\theta) = \sqrt{1 - x^2}$

$$\Rightarrow \quad \theta = -\cos^{-1}\sqrt{1 - x^2} \qquad \Rightarrow \quad \sin^{-1}x = -\cos^{-1}\sqrt{1 - x^2} = -\sec^{-1}\left(\frac{1}{\sqrt{1 - x^2}}\right)$$

$$\text{Again, } \tan\theta = \frac{x}{\sqrt{1 - x^2}} \quad \Rightarrow \quad \theta = \tan^{-1}\frac{x}{\sqrt{1 - x^2}} \quad \Rightarrow \quad \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1 - x^2}}$$

$$\Rightarrow \quad \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = -\pi + \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\left[\because \ \tan^{-1}x = -\pi + \cot^{-1}\left(\frac{1}{x}\right), x<0\right]$$

Hence, $\sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}$

$$= -\sec^{-1}\frac{1}{\sqrt{1-x^2}} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = -\pi + \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right), -1<x<0$$

Illustration 6. Find the value of $\cos^{-1}\left(-\sin\frac{5\pi}{6}\right) + \sin^{-1}\left(\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right)$?

Solution Given expression $= \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\frac{1}{2} = \frac{2\pi}{3} + \frac{\pi}{6} = \frac{5\pi}{6}$

Illustration 7. Find the value of $\cos^{-1}\left(\cos\left(2\tan^{-1}\left(\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}\right)\right)\right)$?

Solution

$$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}} = \frac{\frac{\sqrt{5}-1}{4}}{\frac{\sqrt{10+2\sqrt{5}}}{4}} = \frac{\sin\frac{\pi}{10}}{\cos\frac{\pi}{10}} = \tan\frac{\pi}{10}$$

$\therefore$ Given expression $= \cos^{-1}\left(\cos 2\,.\,\frac{\pi}{10}\right) = \cos^{-1}\left(\cos\,.\frac{\pi}{5}\right) = \frac{\pi}{5}$

Illustration 8. Solution of $2\tan^{-1}\frac{x}{2} + \sin^{-1}\left(\frac{4x}{4+x^2}\right) = \pi$ is

Solution

$$\sin^{-1}\left(\frac{2\frac{x}{2}}{1+\left(\frac{x}{2}\right)^2}\right) = \begin{cases} -\pi - 2\tan^{-1}\frac{x}{2} & : \quad x<-2 \\ 2\tan^{-1}\frac{x}{2} & : \quad -2 \le x \le 2 \\ \pi - 2\tan^{-1}\frac{x}{2} & : \quad x>2 \end{cases}$$

$x < -2$ will not satisfy

If $-2 \le x \le 2$, then $4\tan^{-1}\frac{x}{2} = \pi$ i.e. $\frac{x}{2} = 1$ i.e. $x = 2$

If $x > 2$, then $\pi = \pi$

$\therefore \quad x \in [2, \infty)$

Illustration 9. Number of solutions of the equation $\sin^{-1}|x| = |\cos^{-1}x|$, is

Solution $\sin^{-1}|x| = |\cos^{-1}x| = \cos^{-1}x$

$\therefore$ 1 solution

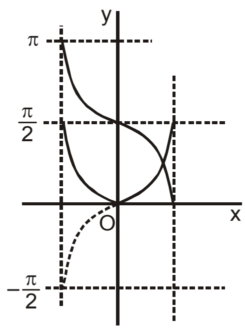


Illustration 10. The number of real solution of the equation $\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x), -\pi \le x \le \pi$ is

Solution $\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x), -\pi \le x \le \pi$

$\Rightarrow \sqrt{2}\,|\cos x| = \sqrt{2}\sin^{-1}(\sin x)$

$\Rightarrow |\cos x| = \sin^{-1}(\sin x)$

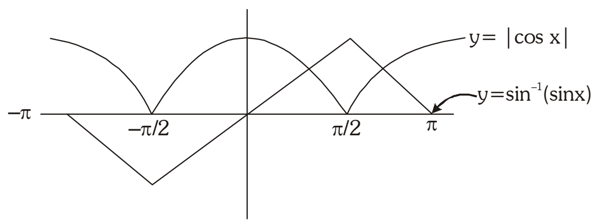


Illustration 11. The range of the function $f(x) = \cot^{-1}\left(\frac{\sqrt{3}x^2}{1+x^2}\right)$ is (a, b], then the value of $\frac{b}{a}$ is

Solution $0 \le \frac{\sqrt{3}x^2}{1+x^2} < \sqrt{3} \quad \Rightarrow \quad \frac{\pi}{6} < \cot^{-1}\left(\frac{\sqrt{3}x^2}{1+x^2}\right) \le \frac{\pi}{2}$

Illustration 12. If the sum of n terms of the series

$$S_n = \operatorname{cosec}^{-1}\sqrt{10} + \operatorname{cosec}^{-1}\sqrt{50} + \operatorname{cosec}^{-1}\sqrt{170} + \dots\dots\dots + \operatorname{cosec}^{-1}\sqrt{\left(n^2+1\right)\left(n^2+2n+2\right)}.$$

The value of $\left[\lim_{x\to\infty} s_n\right]$ is ...... ([.] is G.I.F)

Solution $\operatorname{cosec}^{-1}\sqrt{(n^2+1)(n^2+2n+2)} = \tan^{-1}(n+1) - \tan^{-1}n$

So $S_n = \tan^{-1}(n+1) - \frac{\pi}{4}$

$\lim_{n\to\infty} S_n = \frac{\pi}{4}$

Illustration 13. No. of solution of the equstion $2(\sin^{-1}x)^2 - (\sin^{-1}x) - 6 = 0$ is ...

Solution
$\sin^{-1}x = y$
$2y^2 - y - 6 = 0$
$y = -1.5, 2$
$\sin^{-1}x = -1.5, 2$
Hence, $\sin^{-1}x = -1.5$ (rejecting 2)
Only one solution

Illustration 14. The range of the function $y = \dfrac{\pi}{\sin^{-1}x}$ is

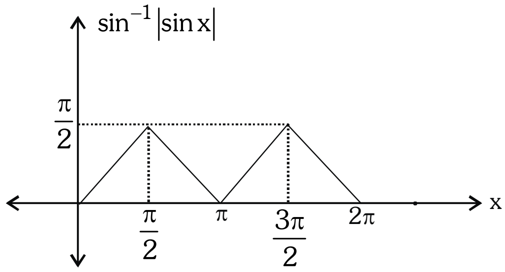
Solution $\dfrac{-\pi}{2} \le \sin^{-1}x \le \dfrac{\pi}{2}$

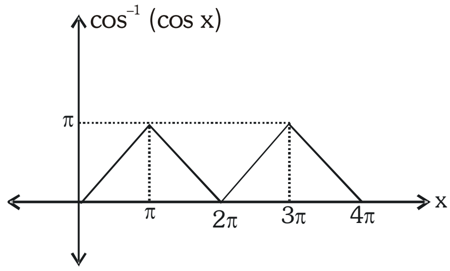
$$\Rightarrow \quad \frac{1}{\sin^{-1}x} \in \left(-\infty, \frac{-2}{\pi}\right] \cup \left[\frac{2}{\pi}, \infty\right)$$

$$\Rightarrow \quad \frac{\pi}{\sin^{-1}x} \in (-\infty, -2] \cup [2, \infty)$$

Illustration 15. The number of integral roots of the equaiton. $\sin^{-1}|\sin x| = \cos^{-1}(\cos x)$ in $x \in [0, 4\pi]$ are

Solution





Clearly, $\sin^{-1}|\sin x| = \cos^{-1}(\cos x)$

for $x \in \left[0, \dfrac{\pi}{2}\right] \cup \left[\dfrac{3\pi}{2}, 2\pi\right] \cup \left(2\pi, \dfrac{5\pi}{2}\right) \cup \left[\dfrac{7\pi}{2}, 4\pi\right]$

so, integral roots are $x = \{0, 1, 5, 6, 7, 11, 12\}$

EXERCISE – 1 **SCQ/MCQ**

SINGLE CORRECT

1. $\tan^{-1}\left[\cos\left(2\tan^{-1}\frac{3}{4}\right)+\sin\left(2\cot^{-1}\frac{1}{2}\right)\right]$ is

(A) not real (B) equal to $\pi/4$ (C) greater then $\pi/4$ (D) less than $\pi/4$

2. If $x=\sin\left(2\tan^{-1}2\right)$ and $y=\sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$, then

(A) $x = y^2$ (B) $y^2=1-x$ (C) $x^2=y/2$ (D) $x < y$

3. The three angles given by $\alpha=2\tan^{-1}\left(\sqrt{2}-1\right)$, $\beta=3\sin^{-1}\frac{1}{\sqrt{2}}+\sin^{-1}\left(-\frac{1}{2}\right)$ and $\gamma=\cos^{-1}\frac{1}{3}$.

(A) $\alpha>\beta$ (B) $\beta>\gamma>\alpha$ (C) $\alpha>\gamma$ (D) None of these

4. Let $f(x)=a+b\cos^{-1}x\,(b>0)$. If domain and range of f(x) are the same set then (b – a) is equal to

(A) $1-\frac{1}{\pi}$ (B) $\frac{2}{\pi}+1$ (C) $1-\frac{2}{\pi}$ (D) 2

5. Range of f(x) = $\sin^{-1}x+\tan^{-1}x+\sec^{-1}x$, is

(A) $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$ (B) $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$ (C) $\left\{\frac{\pi}{4},\frac{3\pi}{4}\right\}$ (D) None of these

6. If $\left[\sin^{-1}x\right]+\left[\cos^{-1}x\right]=0$, where 'x' is a non-negative real number and [.] denotes the greatest integer function, then complete set of values of x is

(A) $(\cos 1,1)$ (B) $(-1,\cos 1)$ (C) $(\sin 1,1)$ (D) $(\cos 1,\sin 1)$

7. The set of values of a for which $x^2+ax+\sin^{-1}\left(x^2-4x+5\right)+\cos^{-1}\left(x^2-4x+5\right)=0$ has at least one solution is

(A) $(-\infty,-\sqrt{2\pi}]\cup[\sqrt{2\pi},\infty)$ (B) $\left(-\infty,-\sqrt{2\pi}\right)\cup\left(\sqrt{2\pi},\infty\right)$

(C) R (D) None of these

8. The complete solution set of $\sin^{-1}(\sin 5)>x^2-4x$, is

(A) $|x-2|<\sqrt{9-2\pi}$ (B) $|x-2|>\sqrt{9-2\pi}$ (C) $|x|<\sqrt{9-2\pi}$ (D) $|x|>\sqrt{9-2\pi}$

9. Which one of the following quantities is negative ?

(A) $\cos\left(\tan^{-1}(\tan 4)\right)$ (B) $\sin\left(\cot^{-1}(\cot 4)\right)$

(C) $\tan\left(\cos^{-1}(\cos 5)\right)$ (D) $\cot\left(\sin^{-1}(\sin 4)\right)$

10. If $\left(\tan^{-1} x\right)^2 + \left(\cot^{-1} x\right)^2 = \frac{5\pi^2}{8}$, then x equals

(A) –1 (B) 1 (C) 0 (D) None of these

11. The maximum value of $\left(\sec^{-1} x\right)^2 + \left(\operatorname{cosec}^{-1} x\right)^2$ is equal to

(A) $\frac{\pi^2}{2}$ (B) $\frac{\pi^2}{4}$ (C) π^2 (D) $\frac{5\pi^2}{4}$

12. $\tan\left(\text{arc}\ \tan\left(\frac{-2}{3}\right) + arc\ \tan(5)\right)$ equals

(A) $-\sqrt{3}$ (B) –1 (C) 1 (D) $\sqrt{3}$

13. The value of $\cos^{-1}\sqrt{\frac{2}{3}} - \cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}$ is equal to

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{6}$

14. If $3\sin^{-1} x = -\pi - \sin^{-1}\left(3x - 4x^3\right)$, then

(A) $x \in \left[-1, -\frac{1}{2}\right]$ (B) $x \in \left[\frac{1}{2}, 1\right]$ (C) $|x| \le \frac{1}{2}$ (D) None of these

15. The value of x satisfying the equation $\sin(\tan^{-1} x) = \cos(\cot^{-1}(x+1))$ is

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) $\sqrt{2}-1$ (D) no finite value

16. $\sum_{r=1}^{n} \tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right)$ is equal to

(A) $\tan^{-1}\left(2^n\right)$ (B) $\tan^{-1}\left(2^n\right) - \frac{\pi}{4}$ (C) $\tan^{-1}\left(2^{n+1}\right)$ (D) $\tan^{-1}\left(2^{n+1}\right) - \frac{\pi}{4}$

17. $\sum_{r=1}^{n} \sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)$ is equal to

(A) $\tan^{-1}(\sqrt{n})-\frac{\pi}{4}$ (B) $\tan^{-1}(\sqrt{n+1})-\frac{\pi}{4}$ (C) $\tan^{-1}(\sqrt{n})$ (D) $\tan^{-1}(\sqrt{n+1})$

18. Point P(x, y) satisfying the equation $\sin^{-1}x+\cos^{-1}y+\cos^{-1}(2xy)=\frac{\pi}{2}$ lies on

(A) the bisector of the first and third quadrant

(B) bisector of the second and fourth quadrant

(C) the rectangle formed by the lines $x=\pm1$ and $y=\pm1$

(D) lies on a unit circle with centre at the origin.

19. $(\sin^{-1}x)^2+(\sin^{-1}y)^2+2(\sin^{-1}x)(\sin^{-1}y)=\pi^2$, then x^2+y^2 is equal to -

(A) 1 (B) 3/2 (C) 2 (D) 1/2

20. If $\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{2}{3}=\sin^{-1}x$, then the value of x is-

(A) 0 (B) $\frac{(\sqrt{5}-4\sqrt{2})}{9}$ (C) $\frac{(\sqrt{5}+4\sqrt{2})}{9}$ (D) $\frac{\pi}{2}$

21. The solution of the inequality $(\tan^{-1}x)^2-3\tan^{-1}x+2\geq 0$ is -

(A) $(-\infty,\tan1]\cup[\tan2,\infty)$ (B) $(-\infty,\ \tan1]$ (C) $(-\infty,-\tan1]\cup[\tan2,\infty)$ (D) $[\tan2,\ \infty)$

22. If $a_1,a_2,a_3........a_n$ is in A.P. with common difference d, then

$\tan\left[\tan^{-1}\frac{d}{1+a_1a_2}+\tan^{-1}\frac{d}{1+a_2a_3}+....+\tan^{-1}\frac{d}{1+a_{n-1}a_n}\right]$ is equal to-

(A) $\frac{(n-1)d}{a_1+a_n}$ (B) $\frac{(n-1)d}{1+a_1a_n}$ (C) $\frac{nd}{1+a_1a_n}$ (D) $\frac{a_n-a_1}{a_n+a_1}$

23. $\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\frac{2r+1}{r^4+2r^3+r^2+1}$ is equal to -

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $-\frac{\pi}{8}$

24. $\sum_{r=0}^{\infty}\tan^{-1}\left(\frac{r((r+1)!)}{(r+1)+((r+1)!)^2}\right)$ is equal to -

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\cot^{-1}3$ (D) $\tan^{-1}2$

25. The value of $\tan^{-1}\left(\frac{1}{2}\tan2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3A)$ for $0<A<(\pi/4)$ is -

(A) $4\tan^{-1}(1)$ (B) $2\tan^{-1}(2)$ (C) 0 (D) none

26. The value of $\left[\tan\left\{\frac{\pi}{4}+\frac{1}{2}\sin^{-1}\left(\frac{a}{b}\right)\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\sin^{-1}\left(\frac{a}{b}\right)\right\}\right]^{-1}$, where $(0 < a < b)$, is -

(A) $\frac{b}{2a}$ (B) $\frac{a}{2b}$ (C) $\frac{\sqrt{b^2-a^2}}{2b}$ (D) $\frac{\sqrt{b^2-a^2}}{2a}$

27. Value of k for which the point $(\alpha, \sin^{-1}\alpha)(\alpha > 0)$ lies inside the triangle formed by $x + y = k$ with co-ordinate axes is -

(A) $\left(1+\frac{\pi}{2},\infty\right)$ (B) $\left(-\left(1+\frac{\pi}{2}\right), \left(1+\frac{\pi}{2}\right)\right)$ (C) $\left(-\infty, 1+\frac{\pi}{2}\right)$ (D) $(-1-\sin 1, 1+\sin 1)$

28. Solution set of the inequality $\sin^{-1}\left(\sin\frac{2x^2+3}{x^2+1}\right) \le \pi - \frac{5}{2}$ is -

(A) $(-\infty, 1) \cup (1, \infty)$ (B) $[-1, 1]$ (C) $(-1, 1)$ (D) $(-\infty, -1] \cup [1, \infty)$

MORE THAN ONE OPTION CORRECT

29. $\sin^{-1}x > \cos^{-1}x$ holds for

(A) all values of x (B) $x \in \left(0, \frac{1}{\sqrt{2}}\right)$ (C) $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$ (D) $x = 0.75$

30. If the numerical value of tan $\left\{\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right\}$ is $\frac{a}{b}$, then (where a and b are coprime number)

(A) $a + b = 23$ (B) $a - b = 11$ (C) $3b = a + 1$ (D) $2a = 3b$

31. $\cos^{-1}x = \tan^{-1}x$ then -

(A) $x^2 = \left(\frac{\sqrt{5}-1}{2}\right)$ (B) $x^2 = \left(\frac{\sqrt{5}+1}{2}\right)$

(C) $\sin(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$ (D) $\tan(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$

32. Identify the pair(s) of functions which are identical -

(A) $y = \tan(\cos^{-1}x)$; $y = \frac{\sqrt{1-x^2}}{x}$ (B) $y = \tan(\cot^{-1}x)$; $y = \frac{1}{x}$

(C) $y = \sin(\text{arc}\tan x)$; $y = \frac{x}{\sqrt{1+x^2}}$ (D) $y = \cos(\text{arc}\tan x)$; $y = \sin(\text{arc}\cot x)$

33. The sum of the infinite terms of the series -

$\cot^{-1}\left(1^2+\frac{3}{4}\right)+\cot^{-1}\left(2^2+\frac{3}{4}\right)+\cot^{-1}\left(3^2+\frac{3}{4}\right)+\ldots\ldots\ldots.$ is equal to -

(A) $\tan^{-1}(1)$ (B) $\tan^{-1}(2)$ (C) $\tan^{-1}(3)$ (D) $\frac{3\pi}{4}-\tan^{-1}3$

EXERCISE – 2 MISCELLANEOUS

Comprehension Based Questions

Comprehension – 1

Consider the two equations in x ; (i) $\sin\left(\frac{\cos^{-1} x}{y}\right) = 1$ (ii) $\cos\left(\frac{\sin^{-1} x}{y}\right) = 0$

The sets $X_1, X_2 \subseteq [-1, 1]$; $Y_1, Y_2 \subseteq I - \{0\}$ are such that

X_1 : the solution set of equation (i)

X_2 : the solution set of equation (ii)

Y_1 : the set of all integral values of y for which equation (i) possess a solution

Y_2 : the set of all intergral values of y for which equation (ii) possess a solution

Let : C_1 be the correspondence : $X_1 \to Y_1$ such that x C_1y for $x \in X_1$, $y \in Y_1$ & (x, y) satisfy (i).

C_2 be the correspondence : $X_2 \to Y_2$ such that x C_2y for $x \in X_2$, $y \in Y_2$ & (x, y) satisfy (ii).

On the basis of above information, answer the following questions

1. The number of ordered pair (x, y) satisfying correspondence C_1 is
(A) 1 (B) 2 (C) 3 (D) 4

2. The number of ordered pair (x, y) satisfying correspondence C_2 is
(A) 1 (B) 2 (C) 3 (D) 4

3*. $C_1 : X_1 \to Y_1$ is a function which is -
(A) one-one (B) many-one (C) onto (D) into

Comprehension – 2

Let $h_1(x) = \sin^{-1}(3x - 4x^3)$; $h_2(x) = \cos^{-1}(4x^3 - 3x)$ & $f(x) = h_1(x) + h_2(x)$

when $x \in [-1, \frac{-1}{2}]$; let $f(x) = a\cos^{-1}x + b\pi$; $a, b \in Q$

$h_1(x) = p\sin^{-1}x + q\pi$; $p, q \in Q$

$h_2(x) = r\cos^{-1}x + s\pi$; $r, s \in Q$

Let C_1 be the circle with centre (p, q) & radius 1 & C_2 be the circle with centre (r, s) & radius 1.

On the basis of above information, answer the following questions

4. $p + r + 2q - s =$
(A) 0 (B) 1 (C) 2 (D) 4

5. If $b.\log_{|s|}|p + q| = k.a$, then value of k is -
(A) $\frac{9}{2}$ (B) 6 (C) $\frac{-3}{2}$ (D) none of these

6. Radical axis of circle C_1 & C_2 is -
(A) $12x - 2y - 3 = 0$ (B) $12x + 2y - 3 = 0$ (C) $-12x + 2y - 3 = 0$ (D) none of these

Match the Column

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

7.

Column-I	**Column-II**
(A) $\sin^{-1}\left(\sin\frac{33\pi}{7}\right)$	(p) $-2\pi/7$
(B) $\cos^{-1}\left(\cos\frac{46\pi}{7}\right)$	(q) $2\pi/7$
(C) $\tan^{-1}\left(\tan\left(\frac{-33\pi}{7}\right)\right)$	(r) $3\pi/7$
(D) $\cot^{-1}\left(\cot\left(\frac{-46\pi}{7}\right)\right)$	(s) $4\pi/7$

8.

Column-I	**Column-II**
(A) $\sin(\tan^{-1}x)$	(p) x
(B) $\cos(\tan^{-1}x)$	(q) $\frac{x}{\sqrt{x^2+1}}$
(C) $\sin(\cot^{-1}(\tan(\cos^{-1}x)))$; $x \in (0,1]$	(r) $\frac{1}{\sqrt{x^2+1}}$
(D) $\sin(\text{cosec}^{-1}(\cot(\tan^{-1}x)))$; $x \in (0,1]$	(s) $\sqrt{1-x^2}$

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

9*. $x \geq 0$, $y \geq 0$, $z \geq 0$ and $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = k$, the possible value(s) of k, if

Column-I	**Column-II**
(A) $xy + yz + zx = 1$, then	(p) $k = \frac{\pi}{2}$
(B) $x + y + z = xyz$, then	(q) $k = \pi$
(C) $x^2 + y^2 + z^2 = 1$ and $x + y + z = \sqrt{3}$, then	(r) $k = 0$
(D) $x = y = z$ and $xyz \geq 3\sqrt{3}$, then k can be equal to	(s) $k = \frac{7\pi}{6}$

INTEGER/SUBJECTIVE TYPE QUESTIONS

10. Find the domain of definition the following functions.

(Read the symbols [*] and { * } as greatest integers and fractional part functions respectively)

(a) $f(x) = \cos^{-1}\dfrac{2}{2+\sin x}$

(b) $f(x) = \dfrac{1}{x} + 2^{\arcsin x} + \dfrac{1}{\sqrt{x-2}}$

(c) $e^{\cos^{-1}x} + \cot^{-1}\left[\dfrac{x}{2}-1\right] + \dfrac{1}{2}\ \ln\{x\}$

(d) $f(x) = \sin^{-1}\left(\dfrac{x-3}{2}\right) - \log_{10}(4-x)$

(e) $f(x) = \dfrac{\sqrt{1-\sin x}}{\log_5(1-4x^2)} + \cos^{-1}(1-\{x\})$

(f) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\dfrac{3-2x}{5}\right) + \log_6(2|x|-3) + \sin^{-1}(\log_2 x)$

11. Draw the graph of the following functions :

(a) $f(x) = \sin^{-1}(x+2)$

(b) $g(x) = [\cos^{-1}x]$, where [ ] denotes greatest integer function.

(c) $h(x) = -|\tan^{-1}(3x)|$

12. Express $f(x) = \arccos x + \arccos\left(\dfrac{x}{2} + \dfrac{1}{2}\sqrt{3-3x^2}\right)$ in simplest form and hence find the values of

(a) $f\left(\dfrac{2}{3}\right)$

(b) $f\left(\dfrac{1}{3}\right)$

13. Prove that : $3\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{1}{20} = \dfrac{\pi}{4} - \tan^{-1}\dfrac{1}{1985}$

14. If α and β are the root of the equation $x^2 + 5x - 49 = 0$, then find the value of $\cot(\cot^{-1}\alpha + \cot^{-1}\beta)$.

15. Let $\cos^{-1}x + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$. If x satisfies the cubic $ax^3 + bx^2 + cx - 1 = 0$, then find the value of $a + b + c$.

16. Solve the following equations

(a) $\sin^{-1}x = \cos^{-1}x + \sin^{-1}(3x-2)$

(b) $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$

(c) $2\tan^{-1}x = \cos^{-1}\dfrac{1-a^2}{1+a^2} - \cos^{-1}\dfrac{1-b^2}{1+b^2}\quad a > 0,\ b > 0$

(d) $\cos^{-1}\dfrac{x^2-1}{x^2+1} + \tan^{-1}\dfrac{2x}{x^2-1} = \dfrac{2\pi}{3}$

17. Find all values of k for which there is a triangle whose angles have measure $\tan^{-1}\left(\dfrac{1}{2}\right)$, $\tan^{-1}\left(\dfrac{1}{2}+k\right)$ and $\tan^{-1}\left(\dfrac{1}{2}+2k\right)$.

18. The value of $\tan\left(\sum_{k=1}^{10}\cot^{-1}(1+k+k^2)\right) = \dfrac{a}{b}$ where a and b are coprime, find the value of (a + b).

19. If the total area between the curves $f(x) = \cos^{-1}(\sin x)$ and $g(x) = \sin^{-1}(\cos x)$ on the interval $[-7\pi, 7\pi]$ is A, find the value of 49A. (Take $\pi = 22/7$)

20. Solve the following system of inequations

$4(\tan^{-1}x)^2 - 8\tan^{-1}x + 3 < 0$ and $4\cot^{-1}x - (\cot^{-1}x)^2 - 3 > 0$

21. Find the set of values of 'a' for which the equation $2\cos^{-1}x = a + a^2(\cos^{-1}x)^{-1}$ posses a solution.

NCERT CORNER

1. Find the principal value of : $\cos^{-1}\left(-\frac{1}{2}\right)$

2. Find the principal value of : $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

3. Find the principal value of : $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

4. Evaluate: $\cos\left(\sec^{-1}\frac{5}{3}\right)$

5. Evaluate: $\sin\left(2\cos^{-1}\left(-\frac{3}{5}\right)\right)$

6. Evaluate : $\tan(\sec^{-1}(-1))$.

7. Find the value of: $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

8. Prove that : $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

9. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x.

10. If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$, then find the value of x.

11. Find the value of $\sin^{-1}(\cos(\sin^{-1}x))+\cos^{-1}(\sin(\cos^{-1}x))$.

12. Prove that $\cos^{-1}x=2\sin^{-1}\sqrt{\frac{1-x}{2}}=2\cos^{-1}\sqrt{\frac{1+x}{2}}$.

13. Find the value of: $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

14. Prove that

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}=\frac{\pi}{4}-\frac{x}{2}$$

if $\pi<x<\frac{3\pi}{2}$.

15. If $y=\cot^{-1}(\sqrt{\cos x})-\tan^{-1}(\sqrt{\cos x})$, prove that $\sin y=\tan^2\frac{x}{2}$.

ANSWER KEY

BEGINNER'S BOX-1

1. (D) **2.** (B) **3.** $\frac{1}{\sqrt{3}}$ **4.** (A) **5.** (C) **6.** (1526)

7. (B) **8.** (B) **9.** (C) **10.** (C) **11.** (B) **12.** (B)

13. (A) **14.** (C)

15. **(i)** D : x ε R R : $[\pi/4, \pi)$

(ii) D: $x \in \left(n\pi, n\pi+\frac{\pi}{2}\right)-\left\{x \middle| x=n\pi+\frac{\pi}{4}\right\}$ $n \in I$; R : $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]-\left[\frac{\pi}{2}\right]$

(iii) D : $x \in R$ R : $\left[0, \frac{\pi}{2}\right)$ **(iv)** D : $x \in R$ R : $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$

BEGINNER'S BOX-2

1. **(a)** $\frac{15}{8}$ **(b)** $\frac{1}{\sqrt{10}}$ **(c)** $\frac{4}{5}$ **(d)** 1 **(e)** $\frac{\sqrt{3}}{2}$ **(f)** $\frac{4}{5}$

2. **(a)** $\frac{\pi}{6}$ **(b)** $\frac{\pi}{6}$ **(c)** $\pi - 2$ **(d)** $\frac{\pi}{6}$ **(e)** $-\frac{\pi}{3}$ **(f)** $\frac{-\pi}{4}$

(g) $\frac{2\pi}{3}$

3. (B) **4.** (D) **5.** (B) **6.** (B,C,D) **7.** (D) **8.** $\frac{13\pi}{7}$

9. $8\pi - 21$ **10.** (53) **11.** (B) **12.** (B) **13.** (D)

BEGINNER'S BOX-3

2. $-\frac{2\sqrt{6}}{5}$ **3.** $\begin{cases} 1, & \text{if } a>0 \\ -1, & \text{if } a<0 \end{cases}$ **4.** (A,D) **5.** **(A)** **7.** $\frac{-2}{5\sqrt{5}}$

8. –1/3 **9.** 204/325 **10.** (ABD)

BEGINNER'S BOX-4

4. B **5.** D

BEGINNER'S BOX-5

1. **(a)** $\frac{1}{5}$ **(b)** 1 **(c)** $\sqrt{3}$

2. (i) $x = \frac{1}{2}\sqrt{\frac{3}{7}}$ (ii) x =3 (iii) $x = 0, \frac{1}{2}, -\frac{1}{2}$ (iv) $x=\frac{3}{\sqrt{10}}$ (v) $x=\frac{1}{2}, y=1$

3. (a) 1/4 (b) $\pm 1, -1 \pm \sqrt{2}, 1 \pm \sqrt{2}$ (c) no solution

4. (D) **5.** (A) **6.** (B) **7.** $\pi/4$ **8.** π **9.** $\pi/4$

10. **(a)** $\frac{\pi}{4}$ **(b)** $\frac{\pi}{4}$ **11.** $\cot^{-1}(a_1)$ **12.** arc tan (x + n) - arc tan x

13. $\frac{\pi}{2}$ **14.** arc cot$\left[\frac{2n+5}{n}\right]$

EXERCISE-1

SINGLE CORRECT & MORE THAN ONE CORRECT

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	B	B	B	C	D	D	A	D	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	D	C	D	A	D	B	C	D	C	C
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	B	B	A	B	A	C	A	B	CD	ABC
Que.	31	32	33							
Ans.	AC	ABCD	BD							

EXERCISE-2

- **Comprehension Based Questions**

Comprehension # 1 : **1.** B **2.** D **3.** A,C

Comprehension # 2 : **4.** A **5.** C **6.** A

- **Match the Column**

7. (A) $\rightarrow$ (q), (B) $\rightarrow$ (s), (C) $\rightarrow$ (q), (D) $\rightarrow$ (r)

8. (A) $\rightarrow$ (q), (B) $\rightarrow$ (r), (C) $\rightarrow$ (p), (D) $\rightarrow$ (p)

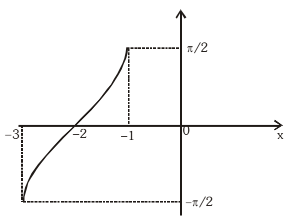
9. (A) $\rightarrow$ (p), (B) $\rightarrow$ (q,r), (C) $\rightarrow$ (p), (D) $\rightarrow$ (q,s)

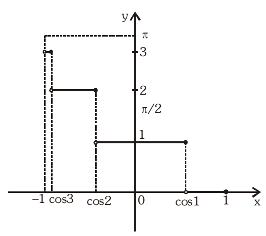
- **INTEGER/SUBJECTIVE TYPE QUESTINS**

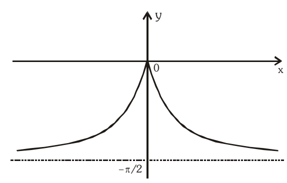
10. **(a)** $[2n\pi, (2n+1)\pi]$; $n \in I$ **(b)** ϕ (not defined for any real x)

(c) $(-1, 1) - \{0\}$ **(d)** $1 \le x < 4$

(e) $x \in (-1/2, 1/2)$, $x \neq 0$ **(f)** $\left(\frac{3}{2}, 2\right]$

12. **(a)** 

(b) 

(c) 

13. **(a)** $\frac{\pi}{3}$ **(b)** $2\cos^{-1}\left(\frac{1}{3}\right) - \frac{\pi}{3}$

14. (10) **15.** (26)

16. **(a)** $x = 1, \frac{1}{2}$ **(b)** $x = 0, \frac{1}{2}$ **(c)** $x = \frac{a-b}{1+ab}$ **(d)** $x = 2-\sqrt{3}$ OR $\sqrt{3}$

17. $k = \frac{11}{4}$ **18.** 0011 **19.** 3388

20. $x = \left(\tan\frac{1}{2}, \cot 1\right)$ **21.** $a \in [-2\pi, \pi] - \{0\}$

NCERT CORNER

1. $\frac{2\pi}{3}$ **2.** $\frac{\pi}{3}$ **3.** $\frac{\pi}{6}$ **4.** $\frac{3}{5}$ **5.** $-\frac{24}{25}$

6. (0) **7.** $\frac{3\pi}{4}$ **9.** $\frac{1}{5}$ **10.** $\pm\frac{1}{\sqrt{2}}$ **11.** $\frac{\pi}{2}$ **13.** $\frac{\pi}{4}$

* * * * * *

IMPORTANT NOTES

LIMIT

1.0 INTRODUCTION

The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry. We use limits to describe the way a function f varies. Some functions vary continuously; small changes in x produce only small changes in f(x). Other functions can have values that jump or vary erratically. We also use limits to define tangent lines to graphs of functions. This geometric application leads at once to the important concept of derivative of a function.

2.0 DEFINITION

Let f(x) be defined on an open interval about 'a' except possibly at 'a' itself. If f(x) gets arbitrarily close to L (a finite number) for all x sufficiently close to 'a' we say that f(x) approaches the limit L as x approaches 'a' and we write $\lim\limits_{x\to a} f(x) = L$ and say "the limit of f(x), as x approaches a, equals L".

This implies if we can make the value of f(x) arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

3.0 LEFT HAND LIMIT AND RIGHT HAND LIMIT OF A FUNCTION

The value to which f(x) approaches, as x tends to 'a' from the left hand side ($x \to a^-$) is called left hand limit of f(x) at x = a. Symbolically, LHL = $\lim\limits_{x\to a^-} f(x) = \lim\limits_{h\to 0} f(a-h)$.

The value to which f(x) approaches, as x tends to 'a' from the right hand side ($x \to a^+$) is called right hand limit of f(x) at x = a. Symbolically, RHL = $\lim\limits_{x\to a^+} f(x) = \lim\limits_{h\to 0} f(a+h)$.

Limit of a function f(x) is said to exist as, $x \to a$ when $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$ = Finite quantity.

Example :

Graph of y = f(x)

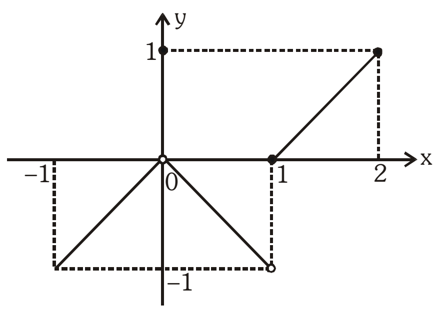


Fig. 1

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{h\to 0} f(-1+h) = f(-1^+) = -1$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h) = f(0^-) = 0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h) = f(0^+) = 0$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} f(1-h) = f(1^-) = -1$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{h\to 0} f(1+h) = f(1^+) = 0$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{h\to 0} f(2-h) = f(2^-) = 1$

$\lim\limits_{x\to 0} f(x) = 0$ and $\lim\limits_{x\to 1} f(x)$ does not exist.

Important note

In $\lim\limits_{x\to a} f(x)$, $x \to a$ necessarily implies $x \neq a$. That is while evaluating limit at x = a, we are not concerned with the value of the function at x = a. In fact the function may or may not be defined at x = a.

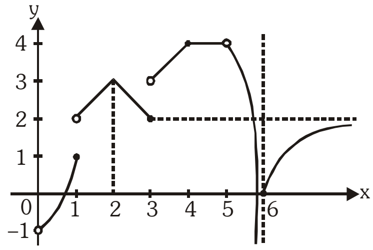
Also it is necessary to note that if f(x) is defined only on one side of **'x = a'**, one sided limits are good enough to establish the existence of limits, & if f(x) is defined on either side of **'a'** both sided limits are to be considered.

As in $\lim\limits_{x\to 1} \cos^{-1} x = 0$, though f(x) is not defined for x > 1, even in it's immediate vicinity.

Illustrations

Illustration 1. Consider the adjacent graph of $y = f(x)$

Find the following :



(a) $\lim\limits_{x\to 0^-} f(x)$ (b) $\lim\limits_{x\to 0^+} f(x)$ (c) $\lim\limits_{x\to 1^-} f(x)$

(d) $\lim\limits_{x\to 1^+} f(x)$ (e) $\lim\limits_{x\to 2^-} f(x)$ (f) $\lim\limits_{x\to 2^+} f(x)$

(g) $\lim\limits_{x\to 3^-} f(x)$ (h) $\lim\limits_{x\to 3^+} f(x)$ (i) $\lim\limits_{x\to 4^-} f(x)$

(j) $\lim\limits_{x\to 4^+} f(x)$ (k) $\lim\limits_{x\to \infty} f(x) = 2$ (l) $\lim\limits_{x\to 6^-} f(x) = -\infty$

Solution (a) As $x \to 0^-$: limit does not exist (the function is not defined to the left of $x = 0$)

(b) As $x \to 0^+ : f(x) \to -1 \Rightarrow \lim\limits_{x\to 0^+} f(x) = -1$. (c) As $x \to 1^- : f(x) \to 1 \Rightarrow \lim\limits_{x\to 1^-} f(x) = 1$.

(d) As $x \to 1^+ : f(x) \to 2 \Rightarrow \lim\limits_{x\to 1^+} f(x) = 2$. (e) As $x \to 2^- : f(x) \to 3 \Rightarrow \lim\limits_{x\to 2^-} f(x) = 3$.

(f) As $x \to 2^+ : f(x) \to 3 \Rightarrow \lim\limits_{x\to 2^-} f(x) = 3$. (g) As $x \to 3^- : f(x) \to 2 \Rightarrow \lim\limits_{x\to 3^-} f(x) = 2$.

(h) As $x \to 3^+ : f(x) \to 3 \Rightarrow \lim\limits_{x\to 3^+} f(x) = 3$. (i) As $x \to 4^- : f(x) \to 4 \Rightarrow \lim\limits_{x\to 4^-} f(x) = 4$.

(j) As $x \to 4^+ : f(x) \to 4 \Rightarrow \lim\limits_{x\to 4^+} f(x) = 4$. (k) As $x \to \infty : f(x) \to 2 \Rightarrow \lim\limits_{x\to \infty} f(x) = 2$.

(l) As $x \to 6^-$, $f(x) \to -\infty \Rightarrow \lim\limits_{x\to 6^-} f(x) = -\infty$ limit does not exist because it is not finite.

4.0 FUNDAMENTAL THEOREMS ON LIMITS

Let $\operatorname*{Lim}\limits_{x\to a} f(x) = l$ & $\operatorname*{Lim}\limits_{x\to a} g(x) = m$. If l & m exists finitely then :

(a) Sum rule : $\operatorname*{Lim}\limits_{x\to a} \{f(x) + g(x)\} = l + m$

(b) Difference rule : $\operatorname*{Lim}\limits_{x\to a} \{f(x) - g(x)\} = l - m$

(c) Product rule : $\operatorname*{Lim}\limits_{x\to a} f(x).g(x) = l.m$

(d) Quotient rule : $\operatorname*{Lim}\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, provided $m \neq 0$

(e) Constant multiple rule : $\operatorname*{Lim}\limits_{x\to a} kf(x) = k\operatorname*{Lim}\limits_{x\to a} f(x)$; where k is constant.

(f) Power rule : If m and n are integers then $\operatorname*{Lim}\limits_{x\to a} [f(x)]^{m/n} = l^{m/n}$ provided $l^{m/n}$ is a real number.

(g) $\operatorname*{Lim}\limits_{x\to a} f[g(x)] = f\left(\operatorname*{Lim}\limits_{x\to a} g(x)\right) = f(m)$; provided f(x) is continuous at x = m.

For example : $\operatorname*{Lim}\limits_{x\to a} \ell n(g(x)) = \ell n[\operatorname*{Lim}\limits_{x\to a} g(x)] = \ell n\,(m)$; provided ℓnx is continuous at x = m, $m = \lim\limits_{x\to a} g(x)$.

5.0 INDETERMINATE FORMS

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty - \infty,\ 0 \times \infty,\ 1^{\infty},\ 0^0,\ \infty^0.$$

Initially we will deal with first five forms only and the other two forms will come up after we have gone through differentiation.

6.0 GENERAL METHODS TO BE USED TO EVALUATE LIMITS

6.1 Factorization

Important factors

(i) $x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + ... + a^{n-1}),\ n \in N$

(ii) $x^n + a^n = (x + a)(x^{n-1} - ax^{n-2} + ... + a^{n-1})$, n is an odd natural number.

Illustrations

Illustration 2. Evaluate : $\lim\limits_{x\to 2}\left[\dfrac{1}{x-2}-\dfrac{2(2x-3)}{x^3-3x^2+2x}\right]$

Solution We have

$$\lim_{x\to 2}\left[\frac{1}{x-2}-\frac{2(2x-3)}{x^3-3x^2+2x}\right]=\lim_{x\to 2}\left[\frac{1}{x-2}-\frac{2(2x-3)}{x(x-1)(x-2)}\right]=\lim_{x\to 2}\left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$=\lim_{x\to 2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right]=\lim_{x\to 2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right]=\lim_{x\to 2}\left[\frac{x-3}{x(x-1)}\right]=-\frac{1}{2}$$ **Ans.**

Illustration 3. $\lim\limits_{x\to 2}\dfrac{2^x+2^{3-x}-6}{2^{-x/2}-2^{1-x}}$ is equal to

(A) 8 (B) 4 (C) 2 (D) none of these

Solution

$$\lim_{x\to 2}\frac{2^x+2^{3-x}-6}{2^{-x/2}-2^{1-x}}=\lim_{x\to 2}\frac{\dfrac{\left(2^{2x}+2^3-6.2^x\right)}{2^x}}{\dfrac{1}{2^{x/2}}-\dfrac{2}{2^x}}=\lim_{x\to 2}\frac{2^{2x}-6.2^x+8}{2^{x/2}-2}=\lim_{x\to 2}\frac{\left(2^x-4\right)\left(2^x-2\right)}{\left(2^{x/2}-2\right)}$$

$$=\lim_{x\to 2}\left(2^{\frac{x}{2}}+2\right)\left(2^x-2\right)=(2+2).(4-2)=8$$ **Ans. (A)**

Illustration 4. Evaluate : $\lim\limits_{x\to 1}\dfrac{x^{P+1}-(P+1)x+P}{(x-1)^2}$

Solution

$$\lim_{x\to 1}\frac{x^{P+1}-(P+1)x+P}{(x-1)^2}\quad\left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to 1}\frac{x^{P+1}-Px-x+P}{(x-1)^2}=\lim_{x\to 1}\frac{x(x^P-1)-P(x-1)}{(x-1)^2}$$

Dividing numerator and denominator by $(x-1)$, we get

$$=\lim_{x\to 1}\frac{\dfrac{x(x^P-1)}{x-1}-P}{(x-1)}=\lim_{x\to 1}\frac{(x+x^2+x^3+....+x^P)-P}{(x-1)}$$

$$=\lim_{x\to 1}\frac{(x+x^2+x^3+....+x^P)-(1+1+1+.........\text{upto P times})}{(x-1)}$$

$$=\lim_{x\to 1}\left\{\frac{(x-1)}{(x-1)}+\frac{(x^2-1)}{(x-1)}+\frac{(x^3-1)}{(x-1)}+......+\frac{(x^P-1)}{(x-1)}\right\}$$

$$=1+2(1)^{2-1}+3(1)^{3-1}+.......+P(1)^{P-1}=1+2+3+.......+P=\frac{P(P+1)}{2}$$ **Ans.**

6.2 Rationalization or Double Rationalization

If any surd or two surds are given which are involved in indeterminate form must be rationalized.

Illustrations

Illustration 5. Evaluate : $\lim\limits_{x\to 1} \dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$

Solution $\lim\limits_{x\to 1} \dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}} = \lim\limits_{x\to 1} \dfrac{(4-\sqrt{15x+1})(2+\sqrt{3x+1})(4+\sqrt{15x+1})}{(2-\sqrt{3x+1})(4+\sqrt{15x+1})(2+\sqrt{3x+1})}$

$= \lim\limits_{x\to 1} \dfrac{(15-15x)}{(3-3x)} \times \dfrac{2+\sqrt{3x+1}}{4+\sqrt{15x+1}} = \dfrac{5}{2}$ **Ans.**

Illustration 6. Evaluate : $\lim\limits_{x\to 1} \left(\dfrac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\right)$

Solution This is of the form $\dfrac{3-3}{2-2} = \dfrac{0}{0}$ if we put $x = 1$

To eliminate the $\dfrac{0}{0}$ factor, multiply by the conjugate of numerator and the conjugate of the denominator

$\therefore$ Limit $= \lim\limits_{x\to 1} (\sqrt{x^2+8} - \sqrt{10-x^2}) \dfrac{(\sqrt{x^2+8}+\sqrt{10-x^2})}{(\sqrt{x^2+8}+\sqrt{10-x^2})} \times$

$\dfrac{(\sqrt{x^2+3}+\sqrt{5-x^2})}{(\sqrt{x^2+3}+\sqrt{5-x^2})(\sqrt{x^2+3}-\sqrt{5-x^2})}$

$= \lim\limits_{x\to 1} \dfrac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}} \times \dfrac{(x^2+8)-(10-x^2)}{(x^2+3)-(5-x^2)} = \lim\limits_{x\to 1} \left(\dfrac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\right) \times 1$

$= \dfrac{2+2}{3+3} = \dfrac{2}{3}$ **Ans.**

6.3 Infinite Limit

Limit when $x \to \infty$

(i) Divide by greatest power of x in numerator and denominator.

(ii) Put $x = 1/y$ and apply $y \to 0$

Illustrations

Illustration 7. Evaluate : $\operatorname{Lim}\limits_{x\to\infty} \dfrac{x^2+x+1}{3x^2+2x-5}$

Solution $\operatorname{Lim}\limits_{x\to\infty} \dfrac{x^2+x+1}{3x^2+2x-5}$, $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

Put $x = \dfrac{1}{y}$

Limit $= \operatorname{Lim}\limits_{y\to 0} \dfrac{1+y+y^2}{3+2y-5y^2} = \dfrac{1}{3}$ **Ans.**

Illustration 8. If $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2$, then

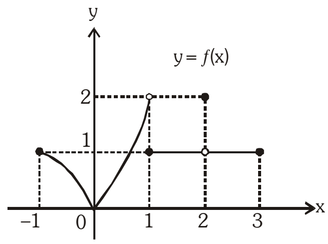
(A) a = 1, b = 1 (B) a = 1, b = 2 (C) a = 1, b = –2 (D) none of these

Solution $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2 \Rightarrow \lim\limits_{x\to\infty}\dfrac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}=2$

$\Rightarrow \lim\limits_{x\to\infty}\dfrac{x(1-a)-b-\dfrac{a}{x}+\dfrac{(1-b)}{x^2}}{1+\dfrac{1}{x^2}}=2 \Rightarrow 1-a=0,-b=2 \Rightarrow a=1,b=-2$ **Ans. (C)**

BEGINNER'S BOX-1

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false ?



(a) $\lim\limits_{x\to-1^+} f(x)=1$ (b) $\lim\limits_{x\to2} f(x)$ does not exist

(c) $\lim\limits_{x\to2} f(x)=2$ (d) $\lim\limits_{x\to1^-} f(x)=2$

(e) $\lim\limits_{x\to1} f(x)$ does not exist (f) $\lim\limits_{x\to0^+} f(x)=\lim\limits_{x\to0^-} f(x)$

(g) $\lim\limits_{x\to c} f(x)$ exists at every $c \in (-1, 1)$

(h) $\lim\limits_{x\to c} f(x)$ exists at every $c \in (1, 3)$ (i) $\lim\limits_{x\to1^-} f(x)=0$

2. Evaluate : $\lim\limits_{x\to1}\dfrac{x-1}{2x^2-7x+5}$

3. Evaluate : $\lim\limits_{x\to0}\dfrac{\sqrt{p+x}-\sqrt{p-x}}{\sqrt{q+x}-\sqrt{q-x}}$

4. Evaluate : $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$, $a>0$

5. If $G(x) = -\sqrt{25-x^2}$, then find the $\lim\limits_{x\to1}\left(\dfrac{G(x)-G(1)}{x-1}\right)$

6. Evaluate: $\lim\limits_{x\to1/2}\dfrac{8x^3-1}{16x^4-1}$

7. Evalulate: $\lim\limits_{x\to1}\left(\dfrac{2}{1-x^2}+\dfrac{1}{x-1}\right)$

8. Evaluate: $\lim\limits_{x\to1}\left(\dfrac{1}{x^2+x-2}-\dfrac{x}{x^3-1}\right)$.

9. Evaluate: $\lim\limits_{x\to\sqrt2}\dfrac{x^4-4}{x^2+3x\sqrt2-8}$

10. Evaluate: **(i)** $\lim\limits_{x\to2}\dfrac{x^3-3x^2+4}{x^4-8x^2+16}$ **(ii)** $\lim\limits_{x\to2}\dfrac{x^3-6x^2+11x-6}{x^2-6x+8}$.

11. Evaluate: $\lim\limits_{x\to 3}\dfrac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27}$.

12. Evalulate: **(i)** $\lim\limits_{x\to\sqrt{3}}\dfrac{x^2-3}{x^2+3\sqrt{3}x-12}$ **(ii)** $\lim\limits_{x\to 2}\left(\dfrac{x}{x-2}-\dfrac{4}{x^2-2x}\right)$

(iii) $\lim\limits_{x\to 2}\left(\dfrac{1}{x-2}-\dfrac{4}{x^3-2x^2}\right)$

13. Evaluate: $\lim\limits_{x\to 4}\dfrac{x^2-16}{\sqrt{x^2+9}-5}$

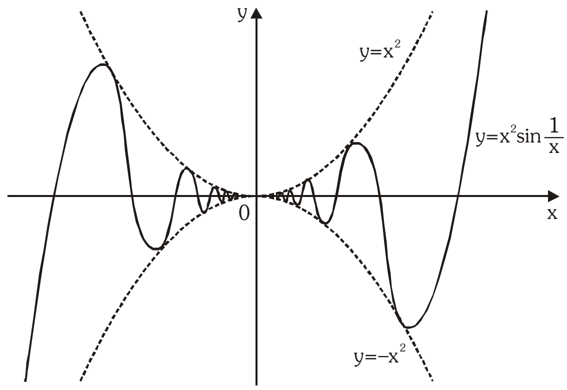
14. Evaluate: $\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$

15. Evaluate: **(i)** $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x+x^2}-\sqrt{x+1}}{2x^2}$ **(ii)** $\lim\limits_{x\to 1}\dfrac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}$

6.4 Squeeze Play Theorem (Sandwich Theorem)

Statement – If $f(x)\le g(x)\le h(x)$; $\forall x$ in the neighbourhood at x = a and

$$\lim_{x\to a} f(x)=\ell=\lim_{x\to a} h(x) \text{ then } \lim_{x\to a} g(x)=\ell,$$

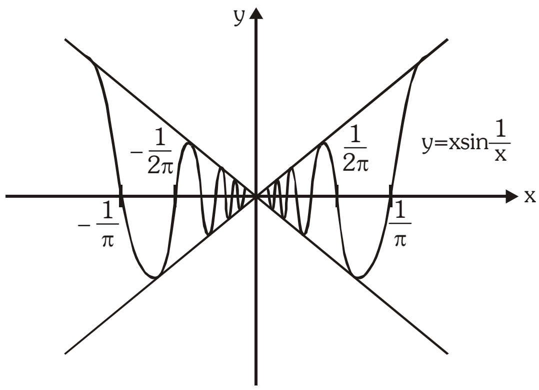


Ex.1 $\lim\limits_{x\to 0} x^2\sin\dfrac{1}{x}=0$,

$\because$ $\sin\left(\dfrac{1}{x}\right)$ lies between –1 & 1

$\Rightarrow$ $-x^2\le x^2\sin\dfrac{1}{x}\le x^2$

$\Rightarrow$ $\lim\limits_{x\to 0} x^2\sin\dfrac{1}{x}=0$ as $\lim\limits_{x\to 0}(-x^2)=\lim\limits_{x\to 0} x^2=0$



Ex.2 $\lim\limits_{x\to 0} x\sin\dfrac{1}{x}=0$

$\because$ $\sin\left(\dfrac{1}{x}\right)$ lies between –1 & 1

$\Rightarrow$ $-x\le x\sin\dfrac{1}{x}\le x$

$\Rightarrow$ $\lim\limits_{x\to 0} x\sin\dfrac{1}{x}=0$ as $\lim\limits_{x\to 0}(-x)=\lim\limits_{x\to 0} x=0$

Illustrations

Illustration 9. Evaluate : $\lim\limits_{n\to\infty}\dfrac{[x]+[2x]+[3x]+.....[nx]}{n^2}$ Where [.] denotes the greatest integer function.

Solution We know that $x-1<[x]\le x$

$$\Rightarrow \quad x + 2x + ...nx - n < \sum_{r=1}^{n}[rx] \le x + 2x + ...+ nx$$

$$\Rightarrow \quad \frac{xn}{2}(n+1) - n < \sum_{r=1}^{n}[rx] \le \frac{x.n(n+1)}{2} \quad \Rightarrow \quad \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} < \frac{1}{n^2}\sum_{r=1}^{n}[rx] \le \frac{x}{2}\left(1+\frac{1}{n}\right)$$

Now, $\lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right) = \frac{x}{2}$ and $\lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} = \frac{x}{2}$

Thus, $\lim_{n\to\infty}\frac{[x]+[2x]+......+[nx]}{n^2} = \frac{x}{2}$ **Ans.**

GOLDEN KEY POINTS

- We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number It does not obey the laws of elementary algebra,

  (a) $\infty + \infty \to \infty$ (b) $\infty \times \infty \to \infty$ (c) $\infty^{\infty} \to \infty$ (d) $0^{\infty} \to 0$

- $\operatorname{Lim}_{x\to a}\frac{x^n - a^n}{x - a} = na^{n-1}$

BEGINNER'S BOX- 2

1. Evaluate: $\lim_{x\to 4}\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$

2. Evaluate: **(i)** $\lim_{x\to 9}\frac{x^{3/2}-27}{x-9}$ **(ii)** $\lim_{x\to a}\frac{x^m - a^m}{x^n - a^n}$ **(iii)** $\lim_{x\to a}\frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$

3. Evaluate:

(i) $\lim_{x\to 1}\frac{(x+x^2+x^3+......+x^n)-n}{x-1}$ **(ii)** If $\lim_{x\to a}\frac{x^9-a^9}{x-a} = \lim_{x\to 5}(4+x)$, find all possible values of a.

4. Evaluate: **(i)** $\lim_{x\to 0}\frac{(1+x)^6-1}{(1+x)^2-1}$ **(ii)** $\lim_{x\to a}\frac{x^{2/3}-a^{2/3}}{x^{3/4}-a^{3/4}}$ **(iii)** $\lim_{x\to 27}\frac{(x^{1/3}+3)(x^{1/3}-3)}{x-27}$

5. Evaluate: $\lim_{x\to\infty}\frac{\sqrt{x^2+a^2}-\sqrt{x^2+b^2}}{\sqrt{x^2+c^2}-\sqrt{x^2+d^2}}$

6. Evaluate: $\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$

7. If $\lim_{x\to\infty}\left\{\frac{x^2+1}{x+1}-(ax+b)\right\} = 2$, find a, b.

8. Evaluate: **(i)** $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)$ **(ii)** $\lim_{x\to-\infty}\left(\sqrt{25x^2-3x}+5x\right)$

9. Evaluate: **(i)** $\lim_{x\to\infty}\left[\sqrt{x+1}-\sqrt{x}\right]\sqrt{x+2}$ **(ii)** $\lim_{x\to-\infty}\left(\sqrt{4x^2+7x}+2x\right)$

10. Evaluate:

(i) $\lim_{n\to\infty}\frac{1^2+2^2+......+n^2}{n^3}$

(ii) $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+....+\frac{n-1}{n^2}\right)$

(iii) $\lim_{n\to\infty}\frac{1^3+2^3+.....+n^3}{n^4}$

11. If [x] denotes the greatest integer less than or equal to x, then $\lim_{n\to\infty}\frac{[1^2x]+[2^2x]+[3^2x]+...+[n^2x]}{n^3}$ equals

(A) x/2 (B) x/3 (C) x/6 (D) 0

12. Evaluate : $\lim_{n\to\infty}\frac{\underline{|n+2}+\underline{|n+1}}{\underline{|n+2}-\underline{|n+1}}$

13. Evaluate : $\lim_{n\to\infty}(n-\sqrt{n^2+n})$

7.0 LIMIT OF TRIGONOMETRIC FUNCTIONS

$\lim_{x\to0}\frac{\sin x}{x}=1=\lim_{x\to0}\frac{\tan x}{x}=\lim_{x\to0}\frac{\tan^{-1}x}{x}=\lim_{x\to0}\frac{\sin^{-1}x}{x}$ [where x is measured in radians]

Note – If $\lim_{x\to a}f(x)=0$, then $\lim_{x\to a}\frac{\sin f(x)}{f(x)}=1$, e.g. $\lim_{x\to1}\frac{\sin(\ell nx)}{(\ell nx)}=1$

Illustrations

Illustration 10. Evaluate : $\lim_{x\to0}\frac{x^3\cot x}{1-\cos x}$

Solution $\lim_{x\to0}\frac{x^3\cos x}{\sin x(1-\cos x)}=\lim_{x\to0}\frac{x^3\cos x(1+\cos x)}{\sin x.\sin^2 x}=\lim_{x\to0}\frac{x^3}{\sin^3 x}.\cos x(1+\cos x)=2$ **Ans.**

Illustration 11. Evaluate : $\lim_{x\to0}\frac{(2+x)\sin(2+x)-2\sin2}{x}$

Solution $\lim_{x\to0}\frac{2(\sin(2+x)-\sin2)+x\sin(2+x)}{x}=\lim_{x\to0}\left(\frac{2.2.\cos\left(2+\frac{x}{2}\right)\sin\frac{x}{2}}{x}+\sin(2+x)\right)$

$=\lim_{x\to0}\frac{2\cos\left(2+\frac{x}{2}\right)\sin\frac{x}{2}}{\frac{x}{2}}+\lim_{x\to0}\sin(2+x)=2\cos2+\sin2$ **Ans.**

Illustration 12. Evaluate : $\lim_{n\to\infty} \frac{\sin\frac{a}{n}}{\tan\frac{b}{n+1}}$

Solution As $n \to \infty, \frac{1}{n} \to 0$ and $\frac{a}{n}$ also tends to zero

$\sin\frac{a}{n}$ should be written as $\frac{\sin\frac{a}{n}}{\frac{a}{n}}$ so that it looks like $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$

The given limit $= \lim_{n\to\infty}\left(\frac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\frac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right)\cdot\frac{a(n+1)}{n.b}$

$$= \lim_{n\to\infty}\left(\frac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\frac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right)\cdot\frac{a}{b}\left(1+\frac{1}{n}\right) = 1\times 1\times\frac{a}{b}\times 1 = \frac{a}{b}$$ **Ans.**

Illustration 13. $\lim_{x\to\infty} x\cos\left(\frac{\pi}{4x}\right)\sin\left(\frac{\pi}{4x}\right)$ is equal to -

(A) $\pi/2$ (B) $\pi/4$ (C) 1 (D) none of these

Solution $\lim_{x\to\infty}\frac{x}{2}\left(2\sin\frac{\pi}{4x}\cos\frac{\pi}{4x}\right) = \lim_{x\to\infty}\frac{x}{2}\sin\left(\frac{\pi}{2x}\right)$

$$= \lim_{x\to\infty}\frac{\sin\left(\frac{\pi}{2x}\right)}{\frac{\pi}{2x}}\cdot\frac{\pi}{4} = \frac{\pi}{4} \quad \lim_{y\to 0}\frac{\sin y}{y} = \frac{\pi}{4}, \quad \text{where } y = \frac{\pi}{2x}$$ **Ans. (B)**

Illustration 14. Evaluate : $\lim_{x\to\frac{\pi}{2}}(\sec x - \tan x)$

Solution $\lim_{x\to\frac{\pi}{2}}(\sec x - \tan x); (\infty - \infty \text{ form})$

$\lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin x}{\cos x}\right); \left(\text{now in } \frac{0}{0} \text{ form}\right)$

Put $x = \left(\frac{\pi}{2}+h\right)$

$$\therefore \quad \text{Limit} = \lim_{h\to 0}\left[\frac{1-\sin\left(\frac{\pi}{2}+h\right)}{\cos\left(\frac{\pi}{2}+h\right)}\right] = \lim_{h\to 0}\left[\frac{1-\cos h}{-\sin h}\right]$$

$$= \lim_{h\to 0}\left[\frac{2\sin^2\frac{h}{2}}{-2\sin\frac{h}{2}\cos\frac{h}{2}}\right] = \lim_{h\to 0}\left[\frac{\sin\frac{h}{2}}{-\cos\frac{h}{2}}\right] = 0$$ **Ans.**

Node-1\Target-2022-23\1.JEE(M+A)\Module\Enthusiast\English\Maths\EM-2\19.LIMIT\Th. & Ex.

BEGINNER'S BOX- 3

1. Evaluate: **(i)** $\lim\limits_{x\to 0}\frac{3\sin x-4\sin^3 x}{x}$ **(ii)** $\lim\limits_{x\to 0}\frac{\tan 8x}{\sin 2x}$

2. Evaluate: **(i)** $\lim\limits_{x\to 0}\frac{7x\cos x-3\sin x}{4x+\tan x}$ **(ii)** $\lim\limits_{x\to 0}\frac{\cos ax-\cos bx}{\cos cx-\cos dx}$

3. Evaluate: **(i)** $\lim\limits_{x\to 0}\frac{\sin x^2(1-\cos x^2)}{x^6}$ **(ii)** $\lim\limits_{x\to 0}\frac{\tan 3x-2x}{3x-\sin^2 x}$

4. Evaluate: **(i)** $\lim\limits_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{x^2}$ **(ii)** $\lim\limits_{y\to 0}\frac{(x+y)\sec(x+y)-x\sec x}{y}$

5. Evaluate: $\lim\limits_{x\to 0}\frac{\tan x+4\tan 2x-3\tan 3x}{x^2\tan x}$

6. Evaluate: $\lim\limits_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}$

7. Evaluate: **(i)** $\lim\limits_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}$ **(ii)** $\lim\limits_{x\to\frac{\pi}{6}}\frac{2-\sqrt{3}\cos x-\sin x}{(6x-\pi)^2}$

8. Evaluate: $\lim\limits_{x\to\pi}\frac{\sin 3x-3\sin x}{(\pi-x)^3}$

9. $\lim\limits_{x\to\infty}2^{x-1}\tan\left(\frac{a}{2^x}\right)$

10. Evaluate: **(i)** $\lim\limits_{n\to\infty}n\sin\left(\frac{\pi}{4n}\right)\cos\left(\frac{\pi}{4n}\right)$ **(ii)** $\lim\limits_{n\to\infty}2^{n-1}\sin\left(\frac{a}{2^n}\right)$

11. Evaluate: **(i)** $\lim\limits_{x\to\frac{\pi}{2}}\frac{\sqrt{2}-\sqrt{1+\sin x}}{\sqrt{2}\cos^2 x}$ **(ii)** $\lim\limits_{x\to\frac{\pi}{4}}\frac{\sec^2 x-2}{\tan x-1}$

12. If $\lim\limits_{x\to 0}\frac{\sin 3x}{x}=\alpha$ and $\lim\limits_{x\to 0}\frac{\tan kx}{x}=\beta$ and $|\alpha+\beta|=4$ then $k=$

(A) 0, 7 (B) 1, – 7 (C) 4, 3 (D) none of these

13. **(i)** Evaluate: $\lim\limits_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$ **(ii)** Evaluate : $\lim\limits_{x\to\frac{\pi}{4}}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}$

14. $\lim\limits_{x\to 0}\frac{1-\cos^3 x}{x\sin 2x}=$

(A) $-\frac{3}{4}$ (B) $-\frac{1}{2}$ (C) $\frac{4}{3}$ (D) $\frac{3}{4}$

8.0 LIMIT OF EXPONENTIAL FUNCTIONS

(a) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ell na(a>0)$ In particular $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$.

In general if $\lim\limits_{x\to a}f(x)=0$, then $\lim\limits_{x\to a}\dfrac{a^{f(x)}-1}{f(x)}=\ell na,\ a>0$

(b) $\lim\limits_{x\to 0}\dfrac{\ell n(1+x)}{x}=1$

Illustrations

Illustration 16. Evaluate : $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^x}{\tan x-x}$

Solution $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^x}{\tan x-x}=\lim\limits_{x\to 0}\dfrac{e^x\times e^{(\tan x-x)}-e^x}{\tan x-x}$

$=\lim\limits_{x\to 0}\dfrac{e^x(e^{\tan x-x}-1)}{\tan x-x}=\lim\limits_{x\to 0 y\to 0}\dfrac{e^x(e^y-1)}{y}$ where $y=\tan x-x$ and $\lim\limits_{y\to 0}\dfrac{e^y-1}{y}=1$

$=e^0\times 1$ [as $x\to 0$, $\tan x-x\to 0$]

$=1\times 1=1$ **Ans.**

BEGINNER'S BOX-4

1. Evaluate : **(a)** $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\tan\beta x}$ **(b)** $\lim\limits_{x\to y}\dfrac{\sin^2 x-\sin^2 y}{x^2-y^2}$

(c) $\lim\limits_{h\to 0}\dfrac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

2. Evaluate : $\lim\limits_{x\to a}\dfrac{e^x-e^a}{x-a}$

3. Evaluate : $\lim\limits_{x\to 0}\dfrac{2^x-1}{(1+x)^{1/2}-1}$

4. Evaluate: $\lim\limits_{x\to 1}\dfrac{1-\sqrt{x}}{(\cos^{-1}x)^2}$

5. **(i)** $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}$ **(ii)** $\lim\limits_{x\to\infty}x\left(\tan^{-1}\dfrac{x+1}{x+4}-\dfrac{\pi}{4}\right)$

6. Evaluate: **(i)** $\lim\limits_{x\to 0}\dfrac{10^x-2^x-5^x+1}{x\tan x}$ **(ii)** $\lim\limits_{x\to 0}\dfrac{3^x+3^{-x}-2}{x^2}$

7. Evaluate: **(i)** $\lim\limits_{x\to 0}\dfrac{2^{3x}-3^x}{\sin 3x}$ **(ii)** Evaluate: $\lim\limits_{x\to 2}\dfrac{3^x+3^{3-x}-12}{3^{3-x}-3^{x/2}}$

8. Evaluate: **(i)** $\lim\limits_{x\to e}\dfrac{\ln x-1}{x-e}$ **(ii)** $\lim\limits_{x\to 0}\dfrac{\log(5+x)-\log(5-x)}{x}$

9. Evaluate: **(i)** $\lim\limits_{x\to\pi/2}\dfrac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}$ **(ii)** $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{3x^2-4}-\dfrac{x^2}{3x+2}\right)$

10. $\lim\limits_{x\to 1}\left(\dfrac{1+x}{2+x}\right)^{\left(1-\sqrt{x}\right)/(1-x)}$

11. **(i)** $\lim\limits_{x\to 0}\dfrac{\cos x+4\tan x}{2-x-2x^4}$ **(ii)** $\lim\limits_{x\to 1}\dfrac{\sin(1-x)}{\sqrt{x}-1}$

12. **(i)** $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$ **(ii)** $\lim\limits_{x\to\frac{\pi}{4}}\tan 2x\tan\left(\dfrac{\pi}{4}-x\right)$

13. $\lim\limits_{x\to 0}\dfrac{a^x+b^x+c^x-3}{x}(a,b,c>0)$

14. $\lim\limits_{x\to 2}\dfrac{\sin(e^{x-2}-1)}{\ell n(x-1)}$

15. $\lim\limits_{x\to\infty}x(e^{1/x}-1)$

9.0 LIMIT OF THE FORM OF 1^∞

(i) $\lim\limits_{x\to 0}(1+x)^{1/x}=e=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$ (Note : The base and exponent depends on the same variable)

In general, if $\lim\limits_{x\to a}f(x)=0$, then $\lim\limits_{x\to a}(1+f(x))^{1/f(x)}=e$

(ii) If $\lim\limits_{x\to a}f(x)=1$ and $\lim\limits_{x\to a}\phi(x)=\infty$, then ; $\lim\limits_{x\to a}\left[f(x)\right]^{\phi(x)}=e^k$ where $k=\lim\limits_{x\to a}\phi(x)\,[f(x)-1]$

(iii) If $\lim\limits_{x\to a}f(x)=A>0$ & $\lim\limits_{x\to a}\phi(x)=B$ (a finite quantity) then ; $\lim\limits_{x\to a}\left[f(x)\right]^{\phi(x)}=e^{B\ln A}=A^B$

Illustrations

Illustration 17. Evaluate $\lim\limits_{x\to 1}(\log_3 3x)^{\log_x 3}$

Solution $\lim\limits_{x\to 1}(\log_3 3x)^{\log_x 3}=\lim\limits_{x\to 1}(\log_3 3+\log_3 x)^{\log_x 3}$

$=\lim\limits_{x\to 1}(1+\log_3 x)^{1/\log_3 x}=e$ $\quad\because\ \log_b a=\dfrac{1}{\log_a b}$ **Ans.**

Illustration 18. Evaluate : $\lim\limits_{x\to 0}\dfrac{x\ell n(1+2\tan x)}{1-\cos x}$

Solution $\lim\limits_{x\to 0}\dfrac{x\ell n(1+2\tan x)}{1-\cos x} = \lim\limits_{x\to 0}\dfrac{x\ell n(1+2\tan x)}{\dfrac{1-\cos x}{x^2}.x^2}.\dfrac{2\tan x}{2\tan x} = 4$ **Ans.**

Illustration 19. Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{2x^2-1}{2x^2+3}\right)^{4x^2+2}$

Solution Since it is in the form of 1^{∞}

$$\lim_{x\to\infty}\left(\frac{2x^2-1}{2x^2+3}\right)^{4x^2+2} = e^{\lim\limits_{x\to\infty}\left(\frac{2x^2-1-2x^2-3}{2x^2+3}\right)(4x^2+2)} = e^{-8}$$ **Ans.**

Illustration 20. $\lim\limits_{x\to a}\left(\dfrac{\sin x}{\sin a}\right)^{\frac{1}{x-a}}, a\neq n\pi, n$ is an integer, equals -

(A) $e^{\cot a}$ (B) $e^{\tan a}$ (C) $e^{\sin a}$ (D) $e^{\cos a}$

Solution $\lim\limits_{x\to a}\left(\dfrac{\sin x}{\sin a}\right)^{\frac{1}{x-a}} = \lim\limits_{x\to a}\left(1+\dfrac{\sin x-\sin a}{\sin a}\right)^{\frac{1}{x-a}} = \lim\limits_{x\to a}\left[\left\{1+\left(\dfrac{\sin x-\sin a}{\sin a}\right)\right\}^{\frac{\sin a}{\sin x-\sin a}}\right]^{\frac{\sin x-\sin a}{(x-a)\sin a}}$

$= \quad \lim\limits_{e^{x\to a}}\dfrac{2}{x-a}\cos\left(\dfrac{x+a}{2}\right)\sin\left(\dfrac{x-a}{2}\right).\dfrac{1}{\sin a} = e^{\frac{\cos a}{\sin a}} = e^{\cot a}$ **Ans. (A)**

Illustration 21. $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{1/x} =$

(A) abc (B) $\sqrt{abc}$ (C) $(abc)^{1/3}$ (D) none of these

Solution $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{1/x} = \lim\limits_{x\to 0}\left(1+\dfrac{a^x+b^x+c^x-3}{3}\right)^{1/x}$

$= \quad \lim\limits_{x\to 0}\left[\left(1+\dfrac{(a^x-1)}{3}+\dfrac{(b^x-1)}{3}+\dfrac{(c^x-1)}{3}\right)^{\frac{3}{(a^x-1)+(b^x-1)+(c^x-1)}}\right]^{\frac{a^x-1+b^x-1+c^x-1}{3x}}$

$= \quad e^{1/3}\lim\limits_{x\to 0}\left[\dfrac{a^x-1}{x}+\dfrac{b^x-1}{x}+\dfrac{c^x-1}{x}\right] = e^{1/3\ (\log a+\log b+\log c)} = e^{\log (abc)1/3} = (abc)^{1/3}$

Ans. (C)

Node-1\Target-2022-23\1.JEE(M+A)\Module\Enthusiast\English\Maths\EM-2\19.LIMIT\Th. & Ex.

Illustration 22. Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{7x^2+1}{5x^2-1}\right)^{\frac{x^5}{1-x^3}}$

Solution Here $f(x)=\dfrac{7x^2+1}{5x^2-1}$, $\phi(x)=\dfrac{x^5}{1-x^3}=\dfrac{x^2.x^3}{1-x^3}=\dfrac{x^2}{\dfrac{1}{x^3}-1}$

$\therefore \quad \lim\limits_{x\to\infty} f(x)=\dfrac{7}{5}$ & $\lim\limits_{x\to\infty}\phi(x)\to-\infty$

$\Rightarrow \quad \lim\limits_{x\to\infty}(f(x))^{\phi(x)}=\left(\dfrac{7}{5}\right)^{-\infty}=0$ **Ans.**

10.0 LIMIT USING SERIES EXPANSION

Expansion of function like binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx should be remembered by heart which are given below :

(a) $a^x=1+\dfrac{x\,\ell na}{1!}+\dfrac{x^2\ell n^2 a}{2!}+\dfrac{x^3\ell n^3 a}{3!}+.........a>0$

(b) $e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+..........$

(c) $\ell n(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+.......\text{for}-1<x\le 1$

(d) $\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+..........$

(e) $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+..........$

(f) $\tan x=x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+.......$

(g) $\tan^{-1}x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+.......$

(h) $\sin^{-1}x=x+\dfrac{1^2}{3!}x^3+\dfrac{1^2.3^2}{5!}x^5+\dfrac{1^2.3^2.5^2}{7!}x^7+.......$

(i) $\sec^{-1}x=1+\dfrac{x^2}{2!}+\dfrac{5x^4}{4!}+\dfrac{61x^6}{6!}+.......$

(j) $(1+x)^n=1+nx+\dfrac{n(n-1)}{2!}x^2+..........\ n\in Q$

Illustrations

Illustration 15. $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$

Solution $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x} \Rightarrow \lim\limits_{x\to 0}\dfrac{1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+......-\left(1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+.....\right)-2x}{x-\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}.....\right)}$

$\Rightarrow \quad \lim\limits_{x\to 0}\dfrac{2.\dfrac{x^3}{6}+2.\dfrac{x^5}{5!}+......}{\dfrac{x^3}{6}+\dfrac{x^5}{5!}.....} \Rightarrow \lim\limits_{x\to 0}\dfrac{x^3\left(\dfrac{1}{3}+\dfrac{1}{60}x^2+.....\right)}{x^3\left(\dfrac{1}{6}+\dfrac{1}{120}x^2+.....\right)}=\dfrac{1/3}{1/6}=2$ **Ans.**

GOLDEN KEY POINTS

- To evaluate limit try to remove indeterminacy.
- In the problems of limit, the limiting value should be put only once.

BEGINNER'S BOX-5

1. $\lim\limits_{x\to\infty}\left(\dfrac{2x^2+3}{2x^2+5}\right)^{8x^2+3}$

2. $\lim\limits_{x\to 0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{1/\sin x}$

3. $\lim\limits_{x\to\infty}\dfrac{a^x}{a^x+1}\ (a>0)$

4. $\lim\limits_{x\to\infty}\dfrac{a^x-a^{-x}}{a^x+a^{-x}}(a>0)$

5. $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^x$

6. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{\frac{x+1}{x}}$

7. $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x-2}\right)^{2x-1}$

8. $\lim\limits_{x\to\infty}\left(\dfrac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$

9. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x^2-1}\right)^{x^2}$

10. $\lim\limits_{x\to\infty}\left(\dfrac{x^2-2x+1}{x^2-4x+2}\right)^x$

11. $\lim\limits_{x\to 0}(1+\sin x)^{\cot x}$

12. $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin x}}$

13. $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$

14. $\lim\limits_{x\to 0}\left\{(1+x)^{\frac{2}{x}}\right\}$ (where $\{.\}$ denotes the fractional part of x) is equal to-

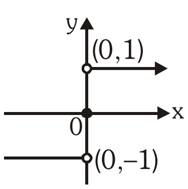
(A) e^2-7 (B) e^2-8 (C) e^2-6 (D) none of these

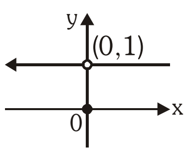
15. Evaluate : $\lim\limits_{x\to 0}\dfrac{\cos(\sin x)-\cos x}{x^4}$

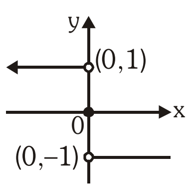
16. Evaluate : $\lim\limits_{x\to 0}\dfrac{x-\sin x}{\sin(x^3)}$

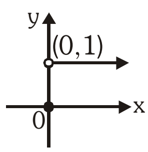
17. Evaluate : $\lim\limits_{x\to 0}\dfrac{x-\tan^{-1}x}{x^3}$

18. Which one of the following best represents the graph of the function $f(x)=\lim\limits_{n\to\infty}\dfrac{2}{\pi}\tan^{-1}(nx)$?

(A)


(B)


(C)


(D)


SOME WORKED OUT ILLUSTRATIONS

Illustration 1. Find $\lim\limits_{n\to\infty} \dfrac{\sqrt{1-x_0^2}}{x_1x_2x_3x_4.....x_n}$, where $x_{r+1} = \sqrt{\dfrac{1+x_r}{2}}$, $0 \le r \le (n-1)$, $r \in I$, $n \in N$

Solution Let $x_0 = \cos\theta$ then $x_1 = \sqrt{\dfrac{1+x_0}{2}} = \cos\dfrac{\theta}{2}$

$x_2 = \sqrt{\dfrac{1+x_1}{2}} = \cos\dfrac{\theta}{2^2}$,.........$x_n = \cos\dfrac{\theta}{2^n}$

$\therefore$ Limits $= \dfrac{\sin\theta}{\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}\cos\dfrac{\theta}{8}.....\cos\dfrac{\theta}{2^n}}$, $n \to \infty$

$= \lim\limits_{n\to\infty}\dfrac{\sin\theta}{\sin\theta}2^n\sin\dfrac{\theta}{2^n} = \lim\limits_{n\to\infty}\theta.\dfrac{\sin\dfrac{\theta}{2^n}}{\dfrac{\theta}{2^n}} = \theta = \cos^{-1}x_0$ **Ans.**

Illustration 2. Evaluate : $\lim\limits_{x\to 0}\dfrac{\cos^2\{1-\cos^2(1-\cos^2(............(1-\cos^2(x))))\}}{\sin\left[\pi\left(\dfrac{\sqrt{x+4}-2}{x}\right)\right]}$

Solution Let $A = \lim\limits_{x\to 0}\dfrac{\cos^2\{1-\cos^2(1-\cos^2(............(1-\cos^2(x))))\}}{\sin\left[\pi\left(\dfrac{\sqrt{x+4}-2}{x}\right)\right]}$

$= \dfrac{\cos^2\{\sin^2(\sin^2(......(1-\cos^2(x))))\}}{\sin\left(\pi\left(\dfrac{\sqrt{x+4}-2}{x}.\dfrac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right)\right)} = \dfrac{\cos^2 0}{1}\lim\limits_{x\to 0}\dfrac{1}{\sin\left(\pi\dfrac{(x+4-4)}{x(\sqrt{x+4}+2)}\right)} = \dfrac{1}{\sin\dfrac{\pi}{4}} = \sqrt{2}$ **Ans.**

Illustration 3. Evaluate the following limits, if exist $\lim\limits_{n\to\infty} n^{-n^2}\left\{(n+1)\left(n+\dfrac{1}{2}\right)\left(n+\dfrac{1}{2^2}\right)...........\left(n+\dfrac{1}{2^{n-1}}\right)\right\}^n$

Solution $\lim\limits_{n\to\infty} n^{-n^2}\left\{(n+1)\left(n+\dfrac{1}{2}\right)......\left(n+\dfrac{1}{2^{n-1}}\right)\right\}^n = \lim\limits_{n\to\infty}\left(\dfrac{(n+1)\left(n+\dfrac{1}{2}\right)......\left(n+\dfrac{1}{2^{n-1}}\right)}{n^n}\right)^n$

$= \lim\limits_{n\to\infty}\left(\dfrac{n+1}{n}\right)^n.\left(\dfrac{n+\dfrac{1}{2}}{n}\right)^n....\left(\dfrac{n+\dfrac{1}{2^{n-1}}}{n}\right)^n = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n.\left(1+\dfrac{1}{2n}\right)^{\frac{2n}{2}}......\left(1+\dfrac{1}{2^{n-1}n}\right)^{\frac{2^{n-1}n}{2^{n-1}}}$

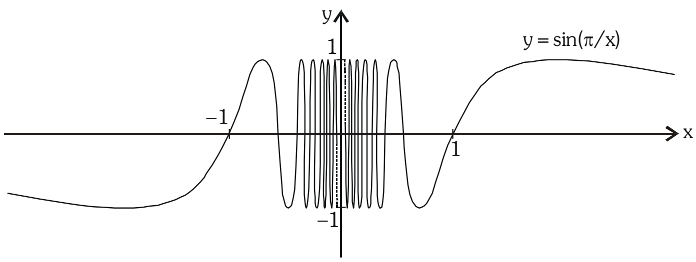
$= e.e^{\frac{1}{2}}.e^{\frac{1}{4}}...... = e^{\left(1+\frac{1}{2}+\frac{1}{4}..........\right)} = e^2$ **Ans.**

Illustration 4: Evaluate $\lim\limits_{x \to 0} \sin\dfrac{\pi}{x}$.

Solution : Again the function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x, we get $f(A) = \sin\pi = 0$, $\quad f\left(\dfrac{1}{2}\right) = \sin 2\pi = 0$,

$$f(0.1) = \sin 10\pi = 0, \qquad f(0.01) = \sin 100\pi = 0.$$

On the basis of this information we might be tempted to guess that $\lim\limits_{x \to 0} \sin\dfrac{\pi}{x} = 0$ but this time our guess is wrong. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n, it is also true that $f(x) = 1$ for infinitely many values of x that approach 0. [In fact, $\sin(\pi/x) = 1$ when $\dfrac{\pi}{x} = \dfrac{\pi}{2} + 2n\pi$ and solving for x, we get $x = 2/(4n + 1)$]. The graph of f is given in following figure



The dashed line indicate that the values of $\sin(\pi/x)$ oscillate between 1 and –1 infinitely often as x approaches 0. Since the values of f(x) do not approach a fixed number as x approaches 0,

$\Rightarrow \quad \lim\limits_{x \to 0} \sin\dfrac{\pi}{x}$ does not exist.

EXERCISE – 1 SCQ/MCQ

SINGLE CORRECT

1. $\lim\limits_{n\to\infty}\left(\frac{1}{1.3}+\frac{1}{3.5}+.......+\text{ to } n \text{ terms}\right)$ is equal to -

(A) 1/4 (B) 1/2 (C) 1 (D) 2

2. $\lim\limits_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]$ is equal to -

(A) 0 (B) 1 / 2 (C) log 2 (D) e^4

3. $\lim\limits_{x\to1}\left[\left(\frac{4}{x^2-x^{-1}}-\frac{1-3x+x^2}{1-x^3}\right)^{-1}+3\left(\frac{x^4-1}{x^3-x^{-1}}\right)\right]$ is equal to -

(A) 2 (B) 3 (C) 4 (D) none of these

4. $\lim\limits_{x\to0}\frac{\tan x-\sin x}{x^3}$ is equal to -

(A) 1 / 2 (B) – 1 (C) 2 (D) – 2

5. $\lim\limits_{x\to0}\left(\frac{3}{1+\sqrt{4+x}}\right)^{\operatorname{cosec} x}$ has the value equal to -

(A) $e^{-1/12}$ (B) $e^{-1/6}$ (C) $e^{-1/4}$ (D) $e^{-1/3}$

6. $\lim\limits_{x\to0}\frac{\left[\frac{x}{\sin x}\right]-\left[\frac{\sin x}{x}\right]}{\left[\frac{\tan x}{x}\right]}$ is equal to ([.] represents the greatest integer function) -

(A) 1 (B) -1 (C) 0 (D) does not exists

7. $\lim\limits_{x\to1}(1-x)\tan\left(\frac{\pi x}{2}\right)$ is equal to -

(A) $2/\pi$ (B) $\pi/2$ (C) π (D) none of these

8. $\lim\limits_{n\to\infty}\cos\left(\pi\sqrt{n^2+n}\right)$ when n is an integer -

(A) is equal to 1 (B) is equal to –1 (C) is equal to zero (D) does not exist

9. $\lim\limits_{x\to0}\frac{\sqrt[3]{1+3x}-1-x}{(1+x)^{101}-1-101x}$ has the value equal to -

(A) $-\frac{3}{5050}$ (B) $-\frac{1}{5050}$ (C) $\frac{1}{5051}$ (D) $\frac{1}{4950}$

10. $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin x}{\cos^{-1}\left[\frac{1}{4}(3\sin x-\sin 3x)\right]}$ where [ ] denotes greatest integer function, is -

(A) $\dfrac{2}{\pi}$ (B) 1 (C) $\dfrac{4}{\pi}$ (D) does not exist

11. $\lim\limits_{n\to\infty} \dfrac{1\cdot 1!+2\cdot 2!+3\cdot 3!+.......+n\cdot n!}{(n+1)!} =$

(A) 0 (B) does not exist (C) 1 (D) None of these

12. $\lim\limits_{n\to\infty} \prod\limits_{r=3}^{n}\left(\dfrac{r^3-1}{r^3+1}\right)$

(A) $\dfrac{1}{3}$ (B) $\dfrac{6}{7}$ (C) $-\dfrac{2}{3}$ (D) None of these

MORE THAN ONE OPTION CORRECT

13. The value of $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+\sin x}-\sqrt[3]{1-\sin x}}{x}$ is -

(A) $\dfrac{2}{3}$ (B) $-\dfrac{2}{3}$ (C) $\dfrac{3}{2}$ (D) $-\dfrac{3}{2}$

14. $\lim\limits_{x\to c}$ f(x) does not exist when -

(A) $f(x) = [x] - [2x-1]$, $c = 3$

(B) $f(x) = [x] - x$, $c = 1$

(C) $f(x) = \{x\}^2 - \{-x\}^2$, $c = 0$

(D) $f(x) = \dfrac{\tan(\operatorname{sgn} x)}{\operatorname{sgn} x}$, $c = 0$

where [x] denotes step up function & {x} fractional part function.

15. Identify the true statement(s) -

(A) $\lim\limits_{n\to\infty}\left[\sum\limits_{r=1}^{n}\dfrac{1}{2^r}\right] = 1$, where [.] denotes the greatest integer function.

(B) If $f(x) = (x-1)\{x\}$, then limit of f(x) does not exist at all integers.

(C) $\lim\limits_{x\to 0^+}\left[\dfrac{\tan x}{x}\right] = 1$, where [.] denotes the greatest integer function.

(D) $\left[\lim\limits_{x\to 0^+}\dfrac{\tan x}{x}\right] = 1$, where [.] denotes the greatest integer function.

16. For a > 0, let $\ell = \lim\limits_{x\to\frac{\pi}{2}}\dfrac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}$ and $m = \lim\limits_{x\to-\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2-ax}\right)$ then -

(A) 'ℓ' is always greater than 'm' for all values of a >0

(B) 'ℓ' is always greater than 'm' only when a ≥1

(C) 'ℓ' is always greater than 'm' for all values of 'a' satisfying $a > e^{-a}$

(D) $e^{\ell} + m = 0$

17. Which of the following limits vanishes ?

(A) $\lim\limits_{x\to\infty} x^{\frac{1}{4}} \sin\dfrac{1}{\sqrt{x}}$ (B) $\lim\limits_{x\to\pi/2}(1-\sin x).\tan x$ (C) $\lim\limits_{x\to\infty}\dfrac{2x^2+3}{x^2+x-5}.\text{sgn}(x)$ (D) $\lim\limits_{x\to 3^+}\dfrac{[x]^2-9}{x^2-9}$

where [ ] denotes greatest integer function

18. Consider the function $f(x)=\left(\dfrac{ax+1}{bx+2}\right)^x$ where a, b > 0 then $\lim\limits_{x\to\infty} f(x)$ -

(A) exists for all values of a and b

(B) is zero for a < b

(C) is non existent for a > b

(D) is $e^{-\left(\frac{1}{a}\right)}$ or $e^{-\left(\frac{1}{b}\right)}$ if a = b

19. Which of the following limits is/are unity ?

(A) $\lim\limits_{t\to 0}\dfrac{\sin(\tan t)}{\sin t}$ (B) $\lim\limits_{x\to\pi/2}\dfrac{\sin(\cos x)}{\cos x}$ (C) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$ (D) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2}}{x}$

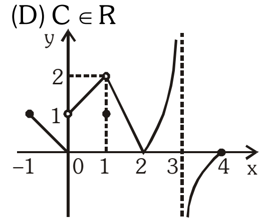
20. Let $f(x) = \dfrac{x.2^x - x}{1-\cos x}$ & $g(x) = 2^x \sin\left(\dfrac{\ell n 2}{2^x}\right)$ then -

(A) $\lim\limits_{x\to 0} f(x) = \ell n 2$ (B) $\lim\limits_{x\to\infty} g(x) = \ell n 4$ (C) $\lim\limits_{x\to 0} f(x) = \ell n 4$ (D) $\lim\limits_{x\to\infty} g(x) = \ell n 2$

21. If $\lim\limits_{x\to 3}\dfrac{x^3+Cx^2+5x+12}{x^2-7x+12} = \ell$ (finite real number) then -

(A) $\ell = 4$ (B) C = –6 (C) C = 4 (D) $C \in R$

22. The graph of the function $y = f(x)$ is shown in the adjacent figure, then correct statement is -



(A) $\lim\limits_{x\to 0^+} f(x) = 1$

(B) $\lim\limits_{x\to 1} f(x) = 2$

(C) $\lim\limits_{x\to 3} f(x)$ does not exist.

(D) $\lim\limits_{x\to 4} f(x) = 0$

23. For $\lim\limits_{x\to 0}\dfrac{\cot^{-1}\left(\dfrac{1}{x}\right)}{x}$

(A) R.H.L exists

(B) L.H.L does not exists

(C) limit does not exists as R.H.L is 1 and L.H.L is –1

(D) limit does not exists as R.H.L and L.H.L both are non-existent.

24. If $\ell = \lim\limits_{x\to\infty}\left(\dfrac{x+1}{x-1}\right)^x$, then $\{\ell\}$ and $[\ell]$ (where {.} & [.] denotes the fractional part function & greatest integer function respectively), is/are -

(A) 7 (B) 7 – e^2 (C) – 7 (D) e^2 – 7

25. Which of the following limits is/are unity ?

(A) $\lim\limits_{x\to\pi}\dfrac{\sin(\pi-x)}{(\pi-x)}$ (B) $\lim\limits_{x\to0}\dfrac{\tan(\sin x)}{\tan x}$ (C) $\lim\limits_{x\to\infty}\dfrac{x^2+x}{x^2-x}$ (D) $\lim\limits_{x\to1}\dfrac{\tan(\pi x)}{\pi(x-1)}$

26. If $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & [x]\neq0\\ 0, & [x]=0\end{cases}$ where [x] denotes the greatest integer less than or equal to x, then

(A) $\lim\limits_{x\to0^-}f(x)=\sin1$ (B) $\lim\limits_{x\to0^+}f(x)=0$

(C) limit does not exist at x = 0 (D) limit exist at x = 0

27. Which of the following limits vanishes ?

(A) $\lim\limits_{x\to0}\left(\dfrac{1}{x}-\dfrac{1}{\tan x}\right)$ (B) $\lim\limits_{x\to\infty}\left(\dfrac{3x^2+1}{2x^2-1}\right)^{\frac{x^3}{1-x}}$ (C) $\lim\limits_{x\to\frac{\pi}{4}^+}\left[\tan\left(x+\dfrac{\pi}{8}\right)\right]^{\tan2x}$ (D) $\lim\limits_{x\to1}\dfrac{x^4-2x^2+1}{x^3-1}$

EXERCISE – 2 MISCELLANEOUS

Comprehension Based Questions

Comprehension – 1

Consider two functions $f(x)=\lim\limits_{n\to\infty}\left(\cos\dfrac{x}{\sqrt{n}}\right)^n$ and $g(x) = -x^{4b}$, where $b=\lim\limits_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)$.

On the basis of above information, answer the following questions

1. $f(x)$ is -

(A) e^{-x^2} (B) $e^{\frac{-x^2}{2}}$ (C) e^{x^2} (D) $e^{\frac{x^2}{2}}$

2. g(x) is -

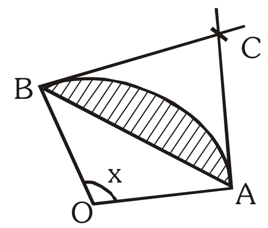
(A) $-x^2$ (B) x^2 (C) x^4 (D) $-x^4$

3. Number of solutions of $f(x) + g(x) = 0$ is -

(A) 2 (B) 4 (C) 0 (D) 1

Comprehension – 2

A circular arc of radius 1 subtends an angle of x radians as shown in figure. The centre of the circle is O and the point C is the intersection of two tangent lines at A and B. Let T(x) be the area of triangle ABC and let s(x) be the area of shaded region



On the basis of above information, answer the following questions

4. $\lim\limits_{x\to 0}\dfrac{T(x)}{x^3}$

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{8}$

5. $\lim\limits_{x\to 0}\dfrac{s(x)}{x}$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) none of these

6. $\lim\limits_{x\to 0}\dfrac{T(x)}{s(x)}$

(A) $\frac{1}{4}$ (B) $\frac{3}{4}$ (C) $\frac{3}{2}$ (D) 0

Match the Column

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

7. **Column-I** | **Column-II**

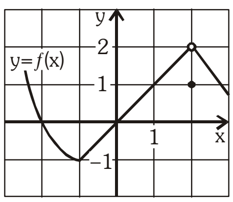
(A) If $L = \lim\limits_{x \to \pi/2^+} \dfrac{\cos(\tan^{-1}(\tan x))}{x - \pi/2}$ then $\cos(2\pi L)$ is — (p) 1

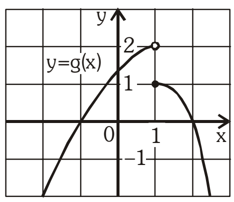
(B) Number of solutions of the equation $\operatorname{cosec}\theta = k$ in $[0, \pi]$ where $k = \lim\limits_{n \to \infty} \prod\limits_{r=2}^{n} \dfrac{r^3 - 1}{r^3 + 1}$ — (q) –1

(C) If c satisfies the equation $\lim\limits_{x \to \infty} \left(\dfrac{x+c}{x-c}\right)^x = 4$ then $-\dfrac{e^c}{2}$ is — (r) 0

(D) If $\lim\limits_{x \to -\infty} \dfrac{(3x^4 + 2x^2)\sin(1/x) + |x|^3 + 5}{|x|^3 + |x|^2 + |x| + 1} = k$, then $\dfrac{k}{2}$ is — (s) 2

8. The graphs of f and g are given. Use them to evaluate





each limit.

Column-I | **Column-II**

(A) $\lim\limits_{x \to 1} f(g(x))$ — (p) 1

(B) $\lim\limits_{x \to 2} \sqrt{3f(x) - 2}$ — (q) does not exist

(C) $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} + f(x)g(x)$ — (r) 0

(D) $\lim\limits_{x \to 1^+} \dfrac{3f(x) - g(x)}{f(x) + g(x)}$ — (s) 2

9. **Column-I** | **Column-II**

(A) $\operatorname*{Lim}\limits_{n \to \infty} \dfrac{(n+1)^2}{2n^2}$ — (p) $\dfrac{4}{3}$

(B) $\operatorname*{Lim}\limits_{n \to \infty} \dfrac{1 + \frac{1}{2} + \frac{1}{4} + \ldots.. \frac{1}{2^n}}{1 + \frac{1}{3} + \frac{1}{9} + \ldots.. + \frac{1}{3^n}}$ — (q) $\dfrac{1}{3}$

(C) $\operatorname*{Lim}\limits_{n \to \infty} \dfrac{1}{n^2}(1 + 2 + 3 + \ldots..n)$ — (r) 2

(D) $\operatorname*{Lim}\limits_{x \to 0} \dfrac{\sqrt[3]{1 + x^2} - 1}{x^2}$ — (s) $\dfrac{1}{2}$

10. Column-I **Column-II**

(A) $\displaystyle\lim_{x\to\infty}\frac{\sqrt[3]{x^4+3}-\sqrt[5]{x^3+4}}{\sqrt[3]{x^7+1}}=\alpha$ then $|\alpha|=$ (p) 101

(B) $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x}=\beta$ then $\beta^2=$ (q) 5

(C) $\displaystyle\lim_{x\to 2}\frac{x^2+5}{x^2-3}=\alpha^\beta$ where $\alpha,\beta\in I$ then $|\alpha+\beta|=$ (r) 201

(D) $\displaystyle\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+(x+3)^{10}+....+(x+100)^{10}}{x^{10}+10^{10}}=\frac{\alpha}{\beta}$ (s) 0

where α, β is coprime then the value of $|\alpha+\beta^2+\alpha\beta|$

11. Column-I **Column-II**

(A) $\displaystyle\lim_{x\to\infty}\frac{x^2-1}{2x^2+1}=\frac{a}{b}$ then $a^2+b^2=$ (p) 1

(B) $\displaystyle\lim_{x\to 1}\frac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1}=\frac{a}{b}$ and $b>0$, then $a+\sqrt{b}=$ (q) 5

(C) $\displaystyle\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}=\frac{a}{b}$ where a and b are minimum then $a+b=$ (r) 3

(s) 0

12. Column-I **Column-II**

(A) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+.....\frac{1}{(n-1)n}\right)=$ (P) 0

(B) $\displaystyle\lim_{x\to 1}\frac{x^2-2x+1}{x^3-x}=$ (Q) 1/2

(C) $\displaystyle\lim_{x\to 1}\frac{(x-1)\sqrt{2-x}}{x^2-1}=$ (R) 5

(D) $\displaystyle\lim_{x\to 0}\frac{\sqrt[3]{(1+x^2)}-\sqrt[4]{1-2x}}{x+x^2}=$ (S) 1

INTEGER/SUBJECTIVE TYPE QUESTIONS

13. The value of $\lim\limits_{n\to\infty}\frac{(2n+1)^4-(n-1)^4}{(2n+1)^4+(n-1)^4}$ is $\frac{p}{q}$ where p and q are co-prime then $|\,p-q\,| =$

14. The value of $\lim\limits_{x\to-2}\frac{x^3+3x^2+2x}{x^2-x-6}=\frac{\alpha}{\gamma}$ and $\lim\limits_{x\to a}\frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}=\frac{1}{\beta a\sqrt{a-b}}$ then $|\alpha|+|\beta|-(\alpha\beta\gamma)=$

15. If $L=\lim\limits_{n\to\infty}\frac{(n+1)^3-(n-1)^3}{(n+1)^2+(n-1)^2}$ and $M=\lim\limits_{n\to\infty}\frac{\sqrt[3]{n^3+2n-1}}{n+2}$, $N=\lim\limits_{n\to\infty}\frac{\left(\sqrt{n^2+1}+n\right)^2}{\sqrt[3]{n^6+1}}$ *then* $L^2+M^2+N^2=$

16. $\lim\limits_{x\to1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}=\frac{p}{q}$ where p and q are in lowest terms and $p,\ q\in I$ then $p^2+q^2-pq=$

17. Using Sandwich theorem, evaluate $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots\ldots+\frac{1}{\sqrt{n^2+2n}}\right)$

18. Evaluate $\lim\limits_{x\to1}\left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$

19. ABC is an isosceles triangle inscribed in a circle of radius r. If AB = AC and h is the altitude from A to BC, then evaluate $\lim\limits_{h\to0}\frac{\Delta}{P^3}$, where Δ is area of the triangle and P it's perimeter.

20. If $f(x)=\begin{cases}\sin x, & x\neq n\pi,\ n=0,\pm1,\pm2,\ldots\ldots\\ 2, & \text{otherwise}\end{cases}$ & $g(x)=\begin{cases}x^2+1, & x\neq 0,2\\ 4, & x=0\\ 5, & x=2\end{cases}$, then find the $\lim\limits_{x\to0} g(f(x))$

ANSWER KEY

BEGINNER'S BOX-1

1. **(a)** T **(b)** F **(c)** F **(d)** T **(e)** T **(f)** T **(g)** T **(h)** T **(i)** F

2. $-\frac{1}{3}$ **3.** $\frac{\sqrt{q}}{\sqrt{p}}$ **4.** $\frac{2}{3\sqrt{3}}$ **5.** $\frac{1}{\sqrt{24}}$ **6.** $\frac{3}{4}$ **7.** $\frac{1}{2}$

8. $-\frac{1}{9}$ **9.** $\frac{8}{5}$ **10.** **(i)** $\frac{3}{16}$; **(ii)** $\frac{1}{2}$

11. $\frac{2}{9}$ **12.** **(i).** $\frac{2}{5}$; **(ii).** 2; **(iii).** 1 **13.** 10 **14.** $-\frac{1}{10}$ **15. (i)** $\frac{1}{4}$; **(ii)** $\frac{1}{2}$

BEGINNER'S BOX - 2

1. $-\frac{1}{3}$ **2.** **(i)** $\frac{9}{2}$; **(ii)** $\frac{m}{n}a^{m-n}$; **(iii)** $\frac{5}{3}(a+2)^{2/3}$ **3.** **(i)** $\frac{n(n+1)}{2}$; **(ii)** 1, –1

4. **(i)** 3; **(ii)** $\frac{8}{9}a^{-1/12}$; **(iii)** $\frac{2}{9}$ **5.** $\frac{a^2-b^2}{c^2-d^2}$ **6.** 1

7. **(i)** a = 1 and b = – 3 **8.** **(i)** $\frac{1}{2}$; **(ii)** $\frac{3}{10}$ **9.** **(i)** $\frac{1}{2}$; **(ii)** $-\frac{7}{4}$

10. **(i)** $\frac{1}{3}$; **(ii)** $\frac{1}{2}$; **(iii)** $\frac{1}{4}$ **11.** b **12.** 1 **13.** $-\frac{1}{2}$

BEGINNER'S BOX-3

1. **(i)** 3; **(ii)** 4 **2.** **(i)** $\frac{4}{5}$; **(ii)** $\frac{a^2-b^2}{c^2-d^2}$ **3.** **(i)** $\frac{1}{2}$; **(ii)** $\frac{1}{3}$

4. **(i)** $\frac{1}{4\sqrt{2}}$; **(ii)** $x\tan x\sec x+\sec x$ **5.** –16 **6.** $\frac{3}{2}$ **7.** (i) $\frac{1}{2}$; (ii) $\frac{1}{36}$

8. – 4 **9.** $\frac{a}{2}$ **10.** (i) $\frac{\pi}{4}$; (ii) $\frac{a}{2}$ **11.** (i) $\frac{1}{8}$; (ii) 2 **12.** (b)

13. (i) $\frac{1}{4\sqrt{2}}$; (ii) –4 **14.** (d)

BEGINNER'S BOX-4

1. **(a)** $\frac{\alpha}{\beta}$ **(b)** $\frac{\sin 2y}{2y}$ **(c)** 2asina + a²cosa

2. e^a **3.** $2\ell n2$ **4.** $\frac{1}{4}$ **5.** (i) 1; (ii) $-\frac{3}{2}$ **6.** (i) (ln 5) (ln 2); (ii) $(\ln 3)^2$

7. (i) ln 2 – $\frac{1}{3}$ ln 3 (ii) $-\frac{4}{3}$ **8.** (i) $\frac{1}{e}$; (ii) $\frac{2}{5}$ **9.** (i) log a**;** (ii) 2/9

10. $\sqrt{\frac{2}{3}}$ **11.** (i) 1/2; (ii) –2 **12.** (i) 0; (ii) 1/2 **13.** $ln(abc)$

14. 1 **15.** 1

BEGINNER'S BOX-5

1. e^{-8} **2** 1 **3.** $\begin{cases} 0 & a \in (0,1) \\ 1/2 & a = 1 \\ 1 & a \in (1,\infty) \end{cases}$ **4.** $\begin{cases} -1 & a \in (0,1) \\ 0 & a = 1 \\ 1 & a \in (1,\infty) \end{cases}$ **5.** $\frac{1}{e}$

6. 1 **7.** e^6 **8.** $e^{-2/3}$ **9.** e^2 **10.** e^2 **11.** e **12.** 1

13. $\frac{1}{e}$ **14.** (A) **15.** $\frac{1}{6}$ **16.** $\frac{1}{6}$ **17.** $\frac{1}{3}$ **18.** (A)

EXERCISE-1
(SINGLE CORRECT & MORE THAN ONE OPTION CORRECT)

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	B	B	A	A	A	A	C	B	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	C	B	A	BC	CD	CD	ABD	BCD	ABC	CD
Que.	21	22	23	24	25	26	27			
Ans.	AB	ABCD	AB	AD	ABCD	ABC	ABCD			

EXERCISE-2 (MISCELLANEOUS)

- **Comprehension Based Questions**

  Comprehension – 1 **1.** B **2.** A **3.** A

  Comprehension – 2 **4.** D **5.** A **6.** C

- **Match the Column**

  7. (A) $\rightarrow$ (p); (B) $\rightarrow$ (r); (C) $\rightarrow$ (q); (D) $\rightarrow$ (q)

  8. (A) $\rightarrow$ (q); (B) $\rightarrow$ (s); (C) $\rightarrow$ (r); (D) $\rightarrow$ (p)

  9. (A) $\rightarrow$ (s); (B) $\rightarrow$ (p); (C) $\rightarrow$ (s); (D) $\rightarrow$ (q)

  10. (A) $\rightarrow$ (s); (B) $\rightarrow$ (s); (C) $\rightarrow$ (q); (D) $\rightarrow$ (r)

  11. (A) $\rightarrow$ (q); (B) $\rightarrow$ (p); (C) $\rightarrow$ (q);

  12. (A) $\rightarrow$ (s); (B) $\rightarrow$ (p); (C) $\rightarrow$ (q); (D) $\rightarrow$ (q)

- **INTEGER/SUBJECTIVE TYPE QUESTIONS**

  13. (2) **14.** (46) **15.** (26) **16.** (7) **17.** 2 ; **18.** e^{-1}

  19. $\frac{1}{128r}$ **20.** 1

* * * * *

IMPORTANT NOTES