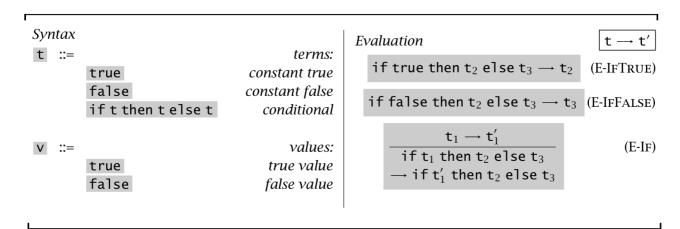
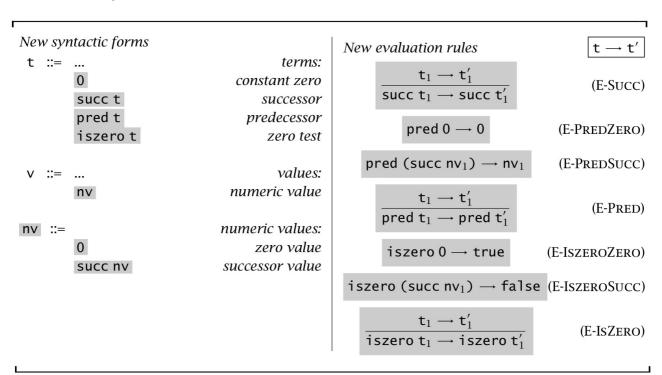
#### λ-calculus

# Rules for evaluation, typing and subtyping

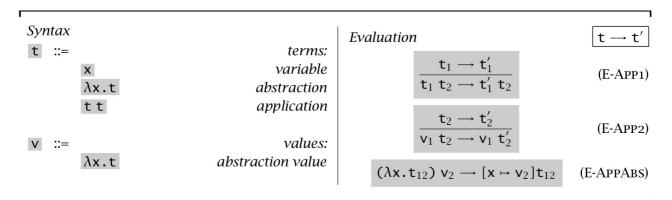
#### **Booleans**



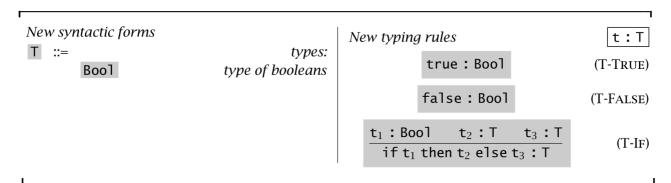
#### **Arithmetic expressions**



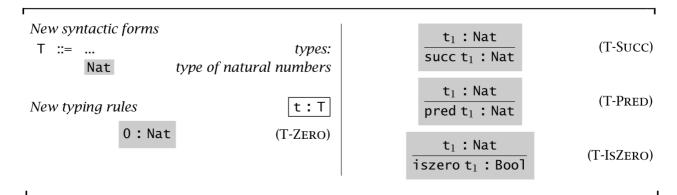
#### **Untyped lambda-calculus**



#### Typing rules for Booleans



#### Typing rules for numbers



#### Pure simply typed lambda-calculus

$$v ::= values: \lambda x : T . t$$
 abstraction value

T ::= types: 
$$T \rightarrow T$$
 type of functions

$$\Gamma$$
 ::= contexts: empty context  $\Gamma$ , x:T term variable binding

Evaluation 
$$t \rightarrow t'$$

$$\frac{\mathsf{t}_1 \longrightarrow \mathsf{t}_1'}{\mathsf{t}_1 \; \mathsf{t}_2 \longrightarrow \mathsf{t}_1' \; \mathsf{t}_2} \tag{E-APP1}$$

$$\frac{\mathsf{t}_2 \longrightarrow \mathsf{t}_2'}{\mathsf{v}_1 \; \mathsf{t}_2 \longrightarrow \mathsf{v}_1 \; \mathsf{t}_2'} \tag{E-APP2}$$

$$(\lambda x : T_{11} \cdot t_{12}) \ v_2 \longrightarrow [x \mapsto v_2] t_{12} \ (\text{E-APPABS})$$

Typing 
$$\Gamma \vdash t : T$$

$$\frac{\mathbf{x} \colon \mathsf{T} \in \Gamma}{\Gamma \vdash \mathbf{x} \colon \mathsf{T}} \tag{T-VAR}$$

$$\frac{\Gamma, \mathbf{x} \colon \mathsf{T}_1 \vdash \mathsf{t}_2 \colon \mathsf{T}_2}{\Gamma \vdash \lambda \mathbf{x} \colon \mathsf{T}_1 \cdot \mathsf{t}_2 \colon \mathsf{T}_1 \to \mathsf{T}_2} \tag{T-Abs}$$

$$\frac{\Gamma \vdash \mathsf{t}_1 : \mathsf{T}_{11} \rightarrow \mathsf{T}_{12} \qquad \Gamma \vdash \mathsf{t}_2 : \mathsf{T}_{11}}{\Gamma \vdash \mathsf{t}_1 \ \mathsf{t}_2 : \mathsf{T}_{12}} \qquad (\text{T-APP})$$

### **Unit type**

# New syntactic forms New typing rules

v ::= ... values:
unit constant unit

 $\Gamma \vdash \mathsf{unit} : \mathsf{Unit}$ 

 $\Gamma \vdash \mathsf{t} : \mathsf{T}$ 

(T-UNIT)

New derived forms

$$t_1; t_2 \stackrel{\text{def}}{=} (\lambda x: Unit. t_2) t_1$$
  
where  $x \notin FV(t_2)$ 

# **Ascription**

#### New syntactic forms

terms: ascription New typing rules

$$\Gamma \vdash \mathsf{t} : \mathsf{T}$$

New evaluation rules



(E-ASCRIBE)

$$rac{\mathsf{t}_1 o \mathsf{t}_1'}{\mathsf{t}_1 \mathsf{ as T} o \mathsf{t}_1' \mathsf{ as T}}$$

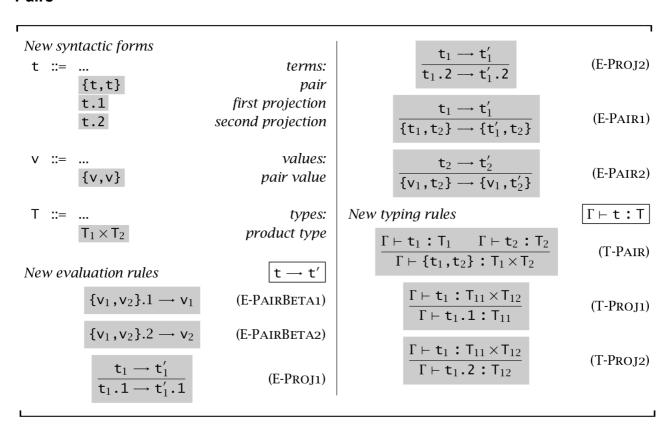
 $v_1$  as  $T \rightarrow v_1$ 

(E-ASCRIBE1)

 $\frac{\Gamma \vdash \mathsf{t}_1 : \mathsf{T}}{\Gamma \vdash \mathsf{t}_1 \text{ as } \mathsf{T} : \mathsf{T}} \tag{T-Ascribe}$ 

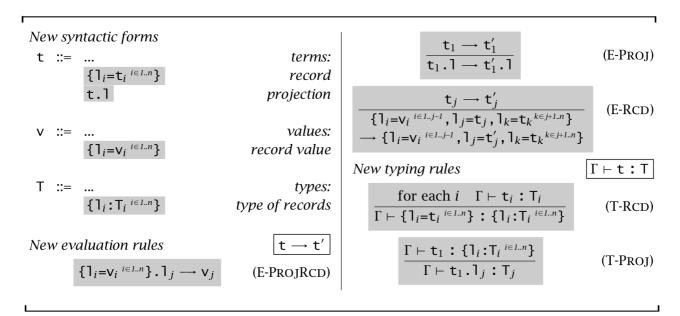
#### Let binding

#### **Pairs**



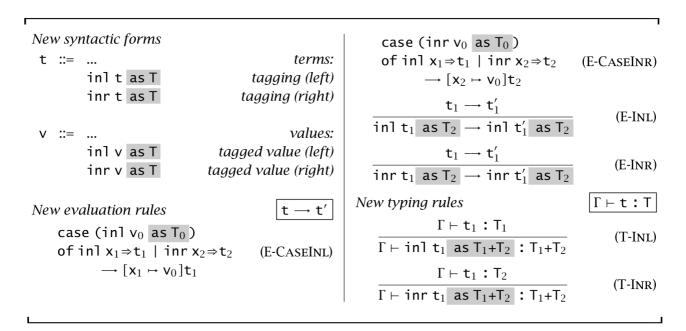
#### **Tuples**

#### Records



#### **Sums**

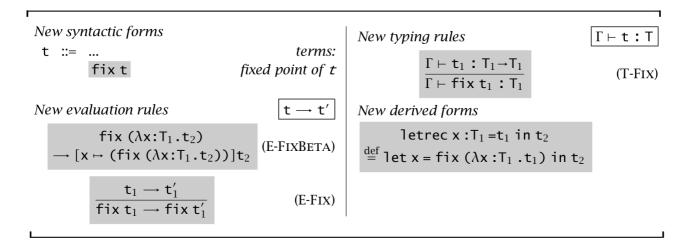
#### Sums (with unique typing)



#### **Variants**

New syntactic forms 
$$t ::= \dots \qquad terms: \\ < l = t > as \ T \qquad tagging \\ case \ t \ of \ < l_i = x_i > \Rightarrow t_i \ ^{i \in l..n} \qquad case \\ T ::= \dots \qquad types: \\ < l_i : T_i \ ^{i \in l..n} > type \ of \ variants \\ New \ evaluation \ rules \qquad t \rightarrow t' \\ case \ (< l_j = v_j > as \ T) \ of \ < l_i = x_i > \Rightarrow t_i \ ^{i \in l..n} \\ \rightarrow [x_j \mapsto v_j] t_j \qquad (E-CASEVARIANT) \\ (E-CASEVARIANT) \qquad (E-CASEVARIANT) \qquad T \mapsto t_i : T_i \ ^{i \in l..n} > t_i : T_i \ ^{i \in l..n} > t_i : T_i :$$

#### **General recursion**



#### Lists

#### New syntactic forms $\mathsf{t}_1 \to \mathsf{t}_1'$ (E-ISNIL) t ::= ... terms: $isnil[T] t_1 \rightarrow isnil[T] t'_1$ nil[T] empty list list constructor cons[T] t t head[S] (cons[T] $v_1 v_2$ ) $\rightarrow v_1$ test for empty list isnil[T] t (E-HEADCONS) head[T] t head of a list $\frac{\mathsf{t}_1 \to \mathsf{t}_1'}{\mathsf{head}[\mathsf{T}] \; \mathsf{t}_1 \to \mathsf{head}[\mathsf{T}] \; \mathsf{t}_1'}$ tail of a list tail[T] t (E-HEAD) values: v ::= ... tail[S] (cons[T] $v_1 v_2$ ) $\rightarrow v_2$ nil[T] empty list (E-TAILCONS) cons[T] v v list constructor $\frac{\mathtt{t}_1 \to \mathtt{t}_1'}{\mathtt{tail}[\mathtt{T}] \; \mathtt{t}_1 \to \mathtt{tail}[\mathtt{T}] \; \mathtt{t}_1'}$ T ::= ... (E-TAIL) types: List T type of lists New typing rules |Γ⊢ **t:** T| $t \rightarrow t'$ New evaluation rules $\Gamma \vdash \mathsf{nil} [\mathsf{T}_1] : \mathsf{List} \, \mathsf{T}_1$ (T-NIL) $\frac{\texttt{t}_1 \rightarrow \texttt{t}_1'}{\mathsf{cons}\left[\texttt{T}\right] \, \texttt{t}_1 \, \, \texttt{t}_2 \rightarrow \mathsf{cons}\left[\texttt{T}\right] \, \texttt{t}_1' \, \, \texttt{t}_2}$ (E-Cons1) $\Gamma \vdash \mathsf{t}_1 : \mathsf{T}_1 \qquad \Gamma \vdash \mathsf{t}_2 : \mathsf{List} \, \mathsf{T}_1$ (T-Cons) $\Gamma \vdash \mathsf{cons}[\mathsf{T}_1] \; \mathsf{t}_1 \; \mathsf{t}_2 : \mathsf{List} \; \mathsf{T}_1$ $t_2 \longrightarrow t_2'$ (E-Cons2) $\Gamma \vdash t_1 : List T_{11}$ $cons[T] v_1 t_2 \rightarrow cons[T] v_1 t_2'$ (T-ISNIL) $\Gamma \vdash \mathsf{isnil}[\mathsf{T}_{11}] \mathsf{t}_1 : \mathsf{Bool}$ $isnil[S] (nil[T]) \rightarrow true (E-ISNILNIL)$ $\Gamma \vdash \mathsf{t}_1 : \mathsf{List} \, \mathsf{T}_{11}$ (T-HEAD) isnil[S] (cons[T] $v_1 v_2$ ) $\rightarrow$ false $\Gamma \vdash \mathsf{head}[\mathsf{T}_{11}] \; \mathsf{t}_1 : \mathsf{T}_{11}$ (E-IsnilCons) $\Gamma \vdash \mathsf{t}_1 : \mathsf{List} \, \mathsf{T}_{11}$ (T-TAIL) $\Gamma \vdash \mathsf{tail}[\mathsf{T}_{11}] \; \mathsf{t}_1 : \mathsf{List} \; \mathsf{T}_{11}$

#### Simply typed lambda-calculus with subtyping

$$v ::= values: \lambda x:T.t$$
 abstraction value

Evaluation 
$$t \rightarrow t'$$

$$\frac{\mathsf{t}_1 \longrightarrow \mathsf{t}_1'}{\mathsf{t}_1 \; \mathsf{t}_2 \longrightarrow \mathsf{t}_1' \; \mathsf{t}_2} \tag{E-APP1}$$

$$\frac{\mathsf{t}_2 \longrightarrow \mathsf{t}_2'}{\mathsf{v}_1 \; \mathsf{t}_2 \longrightarrow \mathsf{v}_1 \; \mathsf{t}_2'} \tag{E-APP2}$$

$$(\lambda x:T_{11}.t_{12}) v_2 \rightarrow [x \mapsto v_2]t_{12}$$
 (E-APPABS)

$$S <: S$$
 (S-Refl)

$$\frac{S <: U \qquad U <: T}{S <: T}$$
 (S-Trans)

$$\frac{\mathsf{T}_1 <: \mathsf{S}_1 \qquad \mathsf{S}_2 <: \mathsf{T}_2}{\mathsf{S}_1 \rightarrow \mathsf{S}_2 <: \mathsf{T}_1 \rightarrow \mathsf{T}_2} \tag{S-ARROW}$$

*Typing* 
$$\Gamma \vdash t : T$$

$$\frac{\mathbf{x}:\mathsf{T}\in\Gamma}{\Gamma\vdash\mathbf{x}:\mathsf{T}}\tag{T-VAR}$$

$$\frac{\Gamma, x: T_1 \vdash t_2: T_2}{\Gamma \vdash \lambda x: T_1. t_2: T_1 \rightarrow T_2}$$
 (T-Abs)

$$\frac{\Gamma \vdash \mathsf{t}_1 : \mathsf{T}_{11} \rightarrow \mathsf{T}_{12} \qquad \Gamma \vdash \mathsf{t}_2 : \mathsf{T}_{11}}{\Gamma \vdash \mathsf{t}_1 \; \mathsf{t}_2 : \mathsf{T}_{12}} \qquad \text{(T-APP)}$$

$$\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}$$
 (T-SUB)

### Records and subtyping

$$\frac{\{\mathsf{k}_j\!:\!\mathsf{S}_j^{\ j\in 1..n}\}\text{ is a permutation of }\{\mathsf{l}_i\!:\!\mathsf{T}_i^{\ i\in 1..n}\}}{\{\mathsf{k}_j\!:\!\mathsf{S}_j^{\ j\in 1..n}\}\boldsymbol{<:}\{\mathsf{l}_i\!:\!\mathsf{T}_i^{\ i\in 1..n}\}}$$
 (S-RCDPERM)

# **Bottom type**

Bot

New syntactic forms 
$$T ::= ...$$
 New subtyping rules  $S <: T$ 

Bot  $S <: T > S <: T$ 

minimum type

## **Subtyping variants**

New syntactic forms

terms: tagging

 $\Gamma \vdash \mathsf{t} : \mathsf{T}$ 

New typing rules

$$\frac{\Gamma \vdash \mathsf{t}_1 : \mathsf{T}_1}{\Gamma \vdash \langle \mathsf{I}_1 = \mathsf{t}_1 \rangle : \langle \mathsf{I}_1 : \mathsf{T}_1 \rangle} \quad \text{(T-VARIANT)}$$

*New subtyping rules* 

S <: T

(S-VARIANTWIDTH)

$$\frac{\text{for each } i \quad \mathsf{S}_i <: \mathsf{T}_i}{<\mathsf{1}_i : \mathsf{S}_i \stackrel{i \in I..n}{>}} <: \quad <\mathsf{1}_i : \mathsf{T}_i \stackrel{i \in I..n}{>}$$

(S-VARIANTDEPTH)

$$\frac{<\!\mathsf{k}_j\!:\!\mathsf{S}_j^{\ j\in 1..n}\!\!>\;\mathsf{is\;a\;permutation\;of}<\!\mathsf{l}_i\!:\!\mathsf{T}_i^{\ i\in 1..n}\!\!>}{<\!\mathsf{k}_j\!:\!\mathsf{S}_j^{\ j\in 1..n}\!\!>\;<\!\!:\;<\!\!\mathsf{l}_i\!:\!\mathsf{T}_i^{\ i\in 1..n}\!\!>}$$

(S-VARIANTPERM)