Discrete Mathematics – Comprehensive Study Guide

Identifying Combinatorial Counting Rules (Lectures 21 & 22)

Lectures 21 and 22 introduce foundational counting techniques in combinatorics: the **Bijection Rule**, **Product Rule**, **Sum Rule**, **Generalized Product Rule**, **Division Rule**, and the **Subset Rule** (binomial coefficients).

Identifying which rule to use depends on the logical structure of the problem, particularly whether the process involves sequential independent choices, mutually exclusive alternatives, or correcting for overcounting.

1. The Bijection Rule (Claim 6.1)

Concept: If two finite, non-empty sets X and Y have a bijection between them, then they have the same number of elements:

$$|X| = |Y|.$$

How to Identify/Apply:

- Use when counting a difficult set X by mapping it bijectively to an easier set Y.
- Typical examples: counting handshakes by mapping to 2-element subsets; mapping choices to binary strings.

2. The Product Rule (Section 6.1)

Concept: If a procedure consists of n sequential independent tasks, and the i-th task has $|A_i|$ options, then the total number of outcomes is:

$$|A_1| \cdot |A_2| \cdots |A_n|.$$

How to Identify/Apply:

- Use for **sequential, independent choices**.
- Examples: number of n-bit binary strings (2^n); number of functions from A to B.

3. The Generalized Product Rule (Section 6.4)

Concept: A generalization of the Product Rule where the number of choices at step k may depend on earlier choices:

$$n_1 \cdot n_2 \cdots n_k.$$

How to Identify/Apply:

- Use for sequential choices **without replacement**.
- Example: selecting k distinct people from n for different awards:

$$n(n-1)(n-2)\cdots.$$

This counts injective functions.

4. The Sum Rule (Section 6.2)

Concept: If a task can be done in one of several mutually exclusive ways, with disjoint sets $A_1, A_2, \ldots, A_n$, then:

$$\left|\bigcup_i A_i\right| = \sum_i |A_i|.$$

How to Identify/Apply:

- Use when cases are **mutually exclusive**.
- Keyword: "*or*" between disjoint possibilities.
- Example: passwords of length 8 *or* 9 *or* 10.

5. The Division Rule (Section 6.5)

Concept: If a procedure counts each desired object exactly k times (a k-to-1 map $A \to B$), then:

$$|B| = \frac{|A|}{k}.$$

How to Identify/Apply:

- Use when an initial count **overcounts** due to symmetry or ordering.
- Examples:
  - Circular seating: $\frac{n!}{n} = (n-1)!$.
  - Handshakes counted as ordered pairs: $\frac{n(n-1)}{2}$.

6. The Subset Rule (Binomial Coefficient, Section 6.6)

Concept: When selecting k items from n distinct items with no regard to order:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

How to Identify/Apply:

- Use when order does **not** matter.
- Typical structure: "How many ways to choose k items?"
- Bijection to n-bit strings with k ones.

Strategy for Exam Questions

1. **Does order matter?**
   - If yes: use the Product Rule or Generalized Product Rule.
   - If no: use the Subset Rule $\binom{n}{k}$.
2. **Are the cases mutually exclusive?**
   - If yes: apply the Sum Rule.
   - If no: use a Product Rule (or, later, Inclusion–Exclusion).
3. **Is there overcounting due to symmetry?**
   - If yes: apply the Division Rule.
4. **Can the problem be mapped to a simpler equivalent?**
   - If yes: use the Bijection Rule.