Discrete Mathematics – Comprehensive Study Guide

12. Prove that G is connected iff for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

Proof ($\Rightarrow$): Suppose G is connected. Take any partition $V(G) = X \cup Y$ with $X, Y \neq \emptyset$ and $X \cap Y = \emptyset$. Since G is connected, there exists a path between any $x \in X$ and $y \in Y$. Let P be such a path from x to y.

Walking along P from x to y, there must be a first vertex in Y. The previous vertex on the path lies in X, and the edge between them has one endpoint in X and the other in Y. Thus an X–Y edge exists.

Proof ($\Leftarrow$): Suppose for every partition of $V(G)$ into nonempty sets X, Y, there exists an edge between X and Y. We show G is connected.

Assume for contradiction that G is disconnected. Then $V(G)$ can be partitioned into two nonempty sets X, Y such that no edge joins them. This contradicts the hypothesis. Hence G must be connected. ■

13. Let G be a bipartite graph with n vertices and m edges. Show that

$$m \leq \frac{n^2}{4}.$$

Proof: Let the bipartition be $V = X \cup Y$ with $|X| = a$, $|Y| = b$, and $a + b = n$. All edges go between X and Y, so the maximum possible is ab.

We maximize ab subject to $a + b = n$, $a, b \geq 0$. By AM–GM,

$$ab \leq \left(\frac{a+b}{2}\right)^2 = \frac{n^2}{4}.$$

Thus $m \leq ab \leq n^2/4$. ■

14. Show that a connected graph with n vertices has at least $n - 1$ edges.

Proof (induction):

Base case $n = 1$: A single vertex graph has $0 = n - 1$ edges.

Inductive step: Assume the statement holds for all connected graphs with fewer than n vertices. Let G be a connected graph with $n \geq 2$ vertices. Remove a vertex v of degree at least 1. Let the resulting graph be G', which has $n - 1$ vertices and $t \geq 1$ connected components $C_1, \ldots, C_t$, where $|C_i| = n_i$.

By the induction hypothesis, C_i has at least $n_i - 1$ edges. Also,

$$\sum_{i=1}^{t} n_i = n - 1.$$

Since G was connected, v had at least one neighbor in each component C_i, so at least t edges join v to G'.

Thus the total number of edges in G is at least

$$\sum_{i=1}^{t}(n_i - 1) + t = (n-1) - t + t = n - 1.$$

Hence the result. ■

1. Prove: Either G or its complement $\overline{G}$ is connected.

Proof: Suppose G is disconnected. We show $\overline{G}$ is connected.

Let u, v be any two vertices.

Case 1: u and v lie in different components of G. Then $uv \notin E(G)$, so $uv \in E(\overline{G})$.

Case 2: u and v lie in the same component of G. Pick a vertex w in a different component. Then $uw, vw \notin E(G)$, so $uw, vw \in E(\overline{G})$. Thus u and v are connected via w in $\overline{G}$.

Hence $\overline{G}$ is connected. Therefore at least one of G or $\overline{G}$ is connected. ■

2. Let G have p vertices of degree p and q vertices of degree q, with $p + q = n$. If G contains an odd-degree vertex, prove every vertex is odd-degree.

Proof: Let A be the p vertices of degree p, and B the q vertices of degree q.

The sum of degrees is

$$\sum_{v \in V} \deg(v) = p \cdot p + q \cdot q = p^2 + q^2.$$

This must be even. Hence p^2 and q^2 have the same parity. Thus p and q have the same parity.

If one vertex is odd-degree, its degree is either p or q, so that value is odd. Hence both p and q are odd. Thus every vertex in G is odd-degree. ■

3. Prove that every graph with n vertices and n edges contains a cycle.

Proof (induction on n):

Base case: The smallest simple graph with n edges and n vertices is K_3 (3 vertices, 3 edges), which clearly has a cycle.

Inductive step: Assume true for all graphs with fewer than n vertices. Let G be a graph with n vertices and n edges.

If G has a vertex of degree 0 or 1, remove it (and its incident edge, if any). Then G' has $n-1$ vertices and at least $n-1$ edges. By induction, G' contains a cycle, so G contains a cycle.

If every vertex has degree at least 2, consider a longest path. Its endpoints must have neighbors inside the path, which creates a cycle.

Thus in all cases G contains a cycle. ■

1. Even number of odd-degree vertices (Handshake Lemma) Statement. In any finite graph, the number of vertices with odd degree is even.

Proof. Let G be a graph with n vertices and m edges. By the Handshake Lemma,

$$\sum_{v \in V(G)} \deg(v) = 2m.$$

Let O be the set of odd-degree vertices and E the set of even-degree vertices. Then

$$\sum_{v \in O} \deg(v) + \sum_{v \in E} \deg(v) = 2m.$$

Since every term in the second sum is even, it follows that

$$\sum_{v \in O} \deg(v) = 2m - (\text{even}) = \text{even}.$$

But each term in $\sum_{v \in O} \deg(v)$ is odd. A sum of odd integers is even iff the number of terms is even. Hence $|O|$ is even. ■

2. Party handshake consequences Statement. At a party, the number of people who shake hands an odd number of times is even.

Proof. Model the situation as a graph: vertices = people, edges = handshakes. The degree of each vertex is the number of handshakes that person participates in. By the Handshake Lemma proved above, the number of odd-degree vertices is even. Thus the number of people who shook hands an odd number of times is even. ■

3. Degree-k preservation under isomorphism Statement. If G and H are isomorphic graphs, then for each k, the number of vertices of degree k in G equals the number in H.

Proof. Let $\varphi : V(G) \to V(H)$ be an isomorphism. Since φ is a bijection and preserves adjacency,

$$\deg_G(v) = \deg_H(\varphi(v))$$

for all $v \in V(G)$.

Thus v has degree k in G iff $\varphi(v)$ has degree k in H. Since φ is bijective, the sets

$$\{v \in V(G) : \deg(v) = k\} \quad \text{and} \quad \{u \in V(H) : \deg(u) = k\}$$

have the same cardinality. ■

5. Two-coloring a bipartite graph with k components Statement. Suppose an n-vertex bipartite graph has exactly k connected components, each of which has at least two vertices. How many proper 2-colorings does it have using two colors (red and blue)?

Proof. A connected bipartite graph has exactly two proper 2-colorings: choose any vertex and assign it a color; all others are forced, and flipping all colors gives the second coloring.

Since the components are disconnected, their colorings are independent. Thus the total number of proper 2-colorings is

$$2^k.$$

■

6. k-colorability when every component has a vertex of degree $< k$ Statement. Let G be a simple graph whose vertex degrees are all $\leq k$. Prove by induction that if every connected component of G has a vertex of degree $< k$, then G is k-colorable.

Proof (induction on $n = |V(G)|$).

Base case: $n = 1$. A single vertex has degree $0 < k$ and is trivially k-colorable.

Inductive hypothesis. Assume the statement is true for all graphs with fewer than n vertices that satisfy the given conditions.

Inductive step. Let G have n vertices. Since each component has a vertex of degree $< k$, choose such a vertex v with $\deg(v) < k$. Remove v to obtain $G' = G - v$.

In G',

- All remaining degrees are still $\leq k$, - In the component that contained v, after removal either - there is another vertex that already had degree $< k$, or - all neighbors of v had degree k in G and now have degree $k - 1 < k$.

Thus every component of G' has a vertex of degree $< k$. So by the inductive hypothesis, G' is k-colorable.

Now extend this coloring to G. Since v has fewer than k neighbors, at least one of the k colors is unused among its neighbors. Give v such a color.

Thus G is k-colorable. ■