Practice set - set theory, induction and graph theory

Disclaimer: There questions have been taken from multiple sources and all the details have not been verified due to the time constraint. So, there might be typographical and other errors. Also, the difficulty level of these questions have nothing to do with the difficulty level of the exam.

1 Set theory

1. Prove or disprove:
$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

2. If $A \subseteq B$, prove:
$$A \cup (B \setminus A) = B.$$

3. For sets A, B, C, prove:
$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C).$$

4. Prove:
$$A = B \iff A \subseteq B \text{ and } B \subseteq A.$$

5. Show:
$$A \subseteq B \iff A \cap B^c = \varnothing.$$

6. Determine whether
$$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$$
holds for all sets A, B, C. Justify your answer.

7. Let A be any set. Show that A can be written as a union of pairwise disjoint singleton subsets of A.

8. Prove or disprove:
$$A = B \iff A \cup B = A \cap B.$$

9. Describe the set
$$\mathrm{pow}(A) \setminus \mathrm{pow}(B)$$
in terms of how elements of A differ from those of B.

10. Let $A \subsetneq B$. Prove:
$$\mathrm{pow}(A) \subsetneq \mathrm{pow}(B).$$

11. Determine all sets A, B for which:
$$\text{pow}(A \cup B) = \text{pow}(A) \cup \text{pow}(B).$$

12. Let A, B, C be sets. Prove or disprove:
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

13. Prove that:
$$(A \setminus B) \cap C = (A \cap C) \setminus B.$$

14. Show that:
$$(A \cup B)^c = A^c \cap B^c.$$

15. Show that:
$$(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B).$$

16. Let $A \Delta B = (A \setminus B) \cup (B \setminus A)$ be the symmetric difference. Prove:
$$(A \cap B) \Delta (A \cap C) \subseteq A \cap (B \Delta C).$$

17. Show that if
$$A \cap B = A \cap C \quad \text{and} \quad A \cup B = A \cup C,$$
then $B = C$.

18. Let A, B, C be sets. Determine whether:
$$A \setminus (B \Delta C) = (A \setminus B) \Delta (A \setminus C)$$
(always, sometimes, or never) holds. Here $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$.

19. Prove that symmetric difference is associative:
$$(A \Delta B) \Delta C = A \Delta (B \Delta C),$$
where Δ is defined as above.

20. Show that if $\text{pow}(A) = \text{pow}(B)$, then $A = B$.

21. For sets A, B, C, prove:
$$(A \cap B^c) \cup (A^c \cap C) = \varnothing \quad \Longrightarrow \quad A \subseteq B \ \text{ or } \ C \subseteq A.$$

22. Let A, B, C be sets. Prove or refute:
$$(A \cap B) \cup (A^c \cap C) = (A \cup C) \cap (A^c \cup B).$$

23. Show that:
$$(A \cup B) \cap (B \cup C) \cap (C \cup A)$$
is equal to the union of all pairwise intersections among A, B, C.

24. Determine the necessary and sufficient conditions on sets A, B so that:

$$(A \setminus C) \subseteq (B \setminus C) \quad \text{for all sets } C.$$

25. Let A, B be sets. Characterize all sets C such that:

$$A \subseteq B \cup C.$$

26. Let $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Characterize all pairs B, C such that:

$$A \Delta B = A \Delta C.$$

27. For sets A, B, C, determine necessary and sufficient conditions for:

$$A \cup B = A \cup C \quad \text{and} \quad A \cap B = A \cap C$$

to imply $B = C$.

28. Let $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Prove or refute:

$$(A \Delta B) \cap (A \Delta C) = A \Delta (B \cap C).$$

29. Describe all triples (A, B, C) such that:

$$(A \setminus B) \cup (C \setminus A) = (A \setminus C) \cup (B \setminus A).$$

30. Find all sets A, B, C for which:

$$(A \setminus B) \cap (B \setminus C) = (A \setminus C) \cap (C \setminus B).$$

31. Let $f : A \to B$ and $g : A \to B$. If $f(x) = g(x)$ for all $x \in A$, prove that $f(S) = g(S)$ for all $S \subseteq A$.

32. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$. Determine whether f is injective or surjective.

33. Let $f : A \to B$ and $S \subseteq T \subseteq A$. Show that $f(S) \subseteq f(T)$.

34. For $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n + 5$, determine whether it is injective, surjective, and bijective.

35. Let $f : A \to B$ be surjective. Show that every $b \in B$ is $f(a)$ for some $a \in A$.

36. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$. Determine whether it is injective and surjective.

37. Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is injective, prove that f is injective.

38. Give an example where $g \circ f$ is surjective but neither f nor g is surjective.

39. Prove that for any $S, T \subseteq A$, $f(S \cup T) = f(S) \cup f(T)$.

40. Let $f : A \to B$ be injective. Show that $f(S) = f(T)$ implies $S = T$.

41. Give an example where $f(S \cap T) \neq f(S) \cap f(T)$ even though f is injective.

42. For $f(x) = x^2$, describe the image of f.

43. Let $f : A \to B$ be bijective. Prove that

$$f(S \triangle T) = f(S) \triangle f(T)$$

for all $S, T \subseteq A$.

44. Determine whether injectivity implies $f(S \setminus T) = f(S) \setminus f(T)$.

45. Let $f : A \to B$ be surjective. Show that if $U \subseteq B$ is nonempty, then some $a \in A$ satisfies $f(a) \in U$.

46. Show that $f : A \to B$ is injective iff for all subsets $S, T \subseteq A$,

$$f(S \cap T) = f(S) \cap f(T).$$

47. Suppose for distinct $x, y \in A$, the sets $\{f(x)\}$ and $\{f(y)\}$ are disjoint. Prove f is injective.

48. For any family $\{S_i\}_{i \in I}$, prove

$$f\left(\bigcap_{i \in I} S_i\right) \subseteq \bigcap_{i \in I} f(S_i),$$

and give an example where the inclusion is strict.

49. Construct injective f and g such that $g \circ f$ is not injective.

50. Suppose $S \cap T = \emptyset$ implies $f(S) \cap f(T) = \emptyset$. Prove f is injective.

51. For $f(x) = x^3 - x$, determine injectivity and surjectivity.

52. For $f(n) = 2n$, characterize all $S \subseteq \mathbb{Z}$ for which $f(S) = f(T)$ implies $S = T$.

53. Suppose $f(S \triangle T) = f(S) \triangle f(T)$ for all $S, T \subseteq A$. Prove f is injective.

54. If f is not injective, prove that there exist $S \neq T$ with $f(S) = f(T)$.

55. For $F : \mathcal{P}(A) \to \mathcal{P}(B)$ defined by $F(S) = f(S)$, prove that F is injective iff f is injective.

56. Let f satisfy $f(S \cup T) = f(S) \cup f(T)$ and $f(S \cap T) = f(S) \cap f(T)$. Prove f is injective.

57. Suppose $f(A \setminus S) = B \setminus f(S)$ for all $S \subseteq A$. Prove f is bijective.

58. Determine when $F(S) = f(S)$ preserves arbitrary unions.

59. Prove that $f(S) = f(T)$ implies $S = T$ is stronger than injectivity.

60. If $A = S \cup T$ with $S \cap T = \emptyset$, and $f(S), f(T)$ are disjoint and partition $f(A)$, prove that f is injective and characterize surjectivity.

61. Among any 25 distinct integers, show that two have a difference ≤ 4.

62. In any group of 10 people, each person selects a favorite color from a list of 6 colors. Show that at least two people picked the same color.

63. Place 9 points in a unit square. Prove that two points lie in a sub-square of side length $\frac{1}{\sqrt{5}}$.

64. Given any 20 points inside a 2×2 square, prove that some unit square inside it contains at least 6 points.

65. A class has 26 students, and each chooses one of the 20 sports offered in school. Prove that two students choose the same sport.

66. Place 12 points on a circle. Prove that two adjacent arcs between consecutive points have lengths whose difference is at most $\frac{1}{6}$ of the circle's circumference.

67. Choose 51 numbers from the set $\{1, 2, \ldots, 100\}$. Prove that two chosen numbers must be consecutive integers.

68. In a basket of 13 fruits, each labeled apple or banana, show that one type must occur at least 7 times.

69. Fifty candies are distributed among 16 children. Prove that one child receives at least 4 candies.

70. Given 10 points inside a triangle of area 1, prove that two points lie in a region of area $\leq \frac{1}{5}$ (via partitioning).

71. In any group of 20 people, prove that two people have the same number of acquaintances within the group.

72. You cut a stick of length 1 into 9 pieces (not necessarily equal). Prove that one piece must have length at least $\frac{1}{9}$.

73. Suppose 100 students each choose exactly 12 clubs from a list of 40 clubs. Prove that two students have at least 4 clubs in common.

74. In a group of 10 people seated in 8 chairs arranged on a circle (some chairs empty), show that two seated people occupy adjacent chairs.

75. Place 14 points anywhere in a rectangle of size 3×2. Prove that two points lie in the same region when it is divided into 12 rectangles of size 1×0.5.

76. Among any 15 books placed on 5 shelves, show that one shelf contains at least 3 books.

77. You place 15 socks into 10 drawers. Prove that one drawer contains at least 2 socks.

78. Place 30 real numbers on the number line. Show that two of them lie within some interval of length 3, assuming the span (max–min) is ≤ 85.

79. Choose any 18 integers. Prove that two of them lie within distance 10 on the number line.

80. Let $a_1, \ldots, a_{60}$ be positive integers with total sum ≤ 500. Prove that there exist two disjoint non-empty subsets with equal sum.

81. Let R be the relation on $\mathbb{Z}$ defined by $a\,R\,b$ iff $a - b$ is even. Describe the partition of $\mathbb{Z}$ induced by R.

82. On the set $X = \{1, 2, 3, 4, 5, 6\}$, define R by

$$a\,R\,b \iff a \equiv b \pmod 3.$$

List all equivalence classes. Is this relation a partition of X?

83. Let R be the relation on $\mathbb{Q}$ defined by $a\,R\,b$ iff $a - b \in \mathbb{Z}$. Show that R is an equivalence relation and describe its equivalence classes.

84. Define a relation R on $\mathbb{R}$ by
$$x\,R\,y \iff |x - y| \le 1.$$

Is R an equivalence relation? If not, which property fails?

85. Consider the set $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Suppose a relation R partitions X into exactly four equivalence classes. Give one possible partition and write down a relation R that induces it.

86. Let R be the relation on $\mathbb{Z}$ defined by $a\,R\,b$ iff $a - b$ is divisible by 4. Describe the set of equivalence classes.

87. Define R on $\mathbb{R}$ by
$$x\,R\,y \iff \lfloor x \rfloor = \lfloor y \rfloor.$$

Describe the induced partition explicitly.

88. Suppose R is an equivalence relation on a nonempty set X. Prove that any two equivalence classes are either identical or disjoint.

89. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Partition X into equivalence classes of equal size except possibly one. Write an explicit equivalence relation that yields your partition.

90. On $\mathbb{Z}$, define $a\,R\,b$ iff a and b leave the same remainder when divided by 5. How many equivalence classes are there? Describe them.

91. Let R be a relation on $\mathbb{R}$ defined by

$$x\,R\,y \iff x - y \in \{0, 1, -1\}.$$

Is this an equivalence relation? Justify.

92. Let R be the relation on $\mathbb{Z}$ defined by $a\,R\,b$ iff a and b have the same number of prime factors (counted with multiplicity). Is R an equivalence relation? If yes, describe two distinct equivalence classes.

93. Partition the set $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ into three equivalence classes, each of size three. Write a relation R whose equivalence classes are exactly these.

94. Let R be the relation on $\mathbb{Q}$ defined by

$$a\,R\,b \iff a - b = \frac{m}{10} \text{ for some } m \in \mathbb{Z}.$$

Is this relation an equivalence relation? If yes, describe the partition.

95. For a fixed $k \in \mathbb{N}$, define $a\,R\,b$ on $\mathbb{Z}$ iff a and b differ by a multiple of k. Show that the partition consists of exactly k equivalence classes. Give the classes when $k = 7$.

96. Define R on $\mathbb{R}$ by $x\,R\,y$ iff $\sin x = \sin y$. Describe the equivalence classes of R.

97. Let R be the relation on $\mathbb{N}$ defined by
$$a\,R\,b \iff a \text{ and } b \text{ have the same parity.}$$
Describe the partition.

98. Let $X = \{1, \dots, 12\}$. Partition X into equivalence classes such that each class has size either 2 or 3. Write an explicit equivalence relation realizing your partition.

99. On $\mathbb{Z}$, define $a\,R\,b$ iff a and b have the same remainder when divided by 8. Describe the set of equivalence classes.

100. Let R be defined on the plane $\mathbb{R}^2$ by
$$(x, y)\,R\,(u, v) \iff x^2 + y^2 = u^2 + v^2.$$
Is this an equivalence relation? If yes, describe its equivalence classes.

101. Define R on $\mathbb{R}$ by
$$x\,R\,y \iff x - y \in \mathbb{Q}.$$
Show that this is an equivalence relation and describe the induced partition.

102. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$. Suppose R is an equivalence relation on X with exactly one class of size 3 and the others of size 2. Give an example of such a partition and write the corresponding R.

103. Let R be the relation on $\mathbb{Z}$ defined by $a\,R\,b$ iff a and b have the same absolute value. Is this an equivalence relation? If yes, describe its classes.

104. Consider the set of all strings over $\{0, 1\}$. Define $u\,R\,v$ iff u and v have the same length. Show that this is an equivalence relation and describe its partition.

105. On $\mathbb{R}$, define $x\,R\,y$ iff $x - y$ is an integer multiple of 0.5. Describe the equivalence classes.

106. Let $X = \mathbb{Z} \setminus \{0\}$. Define $a\,R\,b$ iff $\mathrm{sgn}(a) = \mathrm{sgn}(b)$. Is R an equivalence relation? Describe the partition.

107. Define R on $\mathbb{N}$ by
$$a\,R\,b \iff \lfloor a/3 \rfloor = \lfloor b/3 \rfloor.$$
Describe the equivalence classes.

108. Let R be the relation on $\mathbb{Z}$ defined by
$$a\,R\,b \iff 6 \mid (a - b).$$
Describe the partition of $\mathbb{Z}$ induced by R. How many equivalence classes are there?

109. Consider $X = \{1, 2, 3, 4, 5, 6\}$. Suppose a relation R on X has the following equivalence classes:

$$\{1, 4\}, \quad \{2, 3, 5\}, \quad \{6\}.$$

Give an explicit condition (in terms of a, b) defining such a relation R.

110. Let a relation R on $\mathbb{R}$ be defined by

$$x \, R \, y \iff \begin{cases} x = y, & x, y \notin \mathbb{Z}, \\ \lfloor x \rfloor = \lfloor y \rfloor, & x, y \in \mathbb{Z}. \end{cases}$$

Is R an equivalence relation? If yes, describe its partition; if not, identify which property fails.

111. Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$. Compute $|A \cup B|$ and $|A \cap B|$.

112. If a finite set X has 12 subsets, determine $|X|$.

113. Let A, B be finite sets with $|A| = 7$, $|B| = 5$, and $|A \cap B| = 2$. Find $|A \cup B|$.

114. If a finite set X satisfies $|\mathcal{P}(X)| = 1024$, find $|X|$.

115. Let A, B, C be finite sets with

$$|A| = 10, \quad |B| = 8, \quad |C| = 6, \quad |A \cap B| = 4, \quad |B \cap C| = 3, \quad |A \cap C| = 2, \quad |A \cap B \cap C| = 1.$$

Compute $|A \cup B \cup C|$.

116. A finite set S is partitioned into k blocks of equal size. If $|S| = 48$ and each block has size 6, find k.

117. Let A, B be finite sets. Suppose $|A \triangle B| = 10$ and $|A \cup B| = 17$. Determine $|A \cap B|$. (Recall: $A \triangle B = (A \setminus B) \cup (B \setminus A)$, and it is disjoint.)

118. Let X be a finite set such that

$$|\{Y \subseteq X : |Y| \text{ is even}\}| = 64.$$

Determine $|X|$.

119. Let A, B, C be finite sets with

$$|A \cup B| = 30, \qquad |A \cup C| = 32, \qquad |B \cup C| = 28, \qquad |A \cap B \cap C| = 5.$$

Find $|A| + |B| + |C|$.

120. A finite set X is written as the disjoint union of three sets A, B, C. Suppose

$$|\mathcal{P}(A)| + |\mathcal{P}(B)| + |\mathcal{P}(C)| = 40.$$

Determine all possible triples $(|A|, |B|, |C|)$.

121. Let A, B be finite sets. Show that

$$|A \cap B| = \frac{|A| + |B| - |A \triangle B|}{2}.$$

Then compute $|A \cap B|$ when $|A| = 19$, $|B| = 23$, and $|A \triangle B| = 14$.

122. A finite set X has the property that the number of subsets of odd size equals the number of subsets of even size. Prove that $|X| \geq 1$, and determine all possible values of $|X|$.

123. Let A, B, C be finite sets such that

$$|A \cup B| = |B \cup C| = |C \cup A| = 20.$$

Find all possible values of $|A \cap B \cap C|$.

124. Let X be a finite set, and suppose every element of X belongs to exactly 5 subsets in a family $\mathcal{F} \subseteq \mathcal{P}(X)$, and $|\mathcal{F}| = 12$. If the average size of the sets in $\mathcal{F}$ is 8, determine $|X|$.

125. Let X be a finite set and let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a collection such that every pair of distinct sets in $\mathcal{C}$ has intersection of size exactly 2. If each set in $\mathcal{C}$ has size 6 and $|X| = 20$, determine the maximum possible value of $|\mathcal{C}|$.

2 Mathematical Induction

1. Prove by induction that for all $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Use strong induction to prove that every integer $n \geq 2$ can be factored into primes.

3. Prove that for all $n \geq 1$,

$$n! \geq 2^{n-1}.$$

4. Define a sequence by $b_1 = 1$, $b_2 = 4$ and

$$b_n = 4b_{n-1} - 4b_{n-2}.$$

Prove by induction that $b_n = n^2$.

5. Use induction to prove that for all $n \geq 1$,

$$7^n - 1 \text{ is divisible by } 6.$$

6. Let $f(1) = 1$ and

$$f(n) = f(\lfloor n/2 \rfloor) + n.$$

Using strong induction, prove that $f(n) \leq 2n$.

7. Prove by induction that for all $n \geq 1$,

$$\binom{2n}{n} \geq 2^n.$$

8. Use strong induction to show that every integer $n \geq 1$ has a base-3 representation.

9. Prove by induction that for all $n \geq 1$,

$$\prod_{k=1}^{n}(2k-1) \leq (2n-1)^n.$$

10. For the sequence defined by $a_1 = 1$ and $a_{n+1} = 3a_n + 1$, prove that

$$a_n = \frac{3^n - 1}{2}.$$

11. Let $a_1 = 3$ and $a_n = \frac{a_{n-1}}{2} + 3$. Prove that $a_n < 6$ for all n.

12. Prove by induction that for all $n \geq 1$,

$$\sum_{k=1}^{n} k(k+2) = \frac{n(n+1)(2n+7)}{6}.$$

13. Use strong induction to prove that every integer $n \geq 2$ can be expressed as a product

$$n = a_1 a_2 \cdots a_k,$$

where $a_i \geq 2$ and $a_1 \leq a_2 \leq \cdots \leq a_k$.

14. Prove that for all $n \geq 4$,

$$2^n > n^2.$$

15. Prove by induction that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

16. Let the Fibonacci sequence be defined by

$$F_1 = 1, \ F_2 = 1, \ F_n = F_{n-1} + F_{n-2} \quad (n \geq 3).$$

Use induction to prove that for all $n \geq 1$,

$$F_{n+2} \geq \phi^n, \quad \text{where } \phi = \frac{1 + \sqrt{5}}{2}.$$

17. Show by induction that

$$3^n \geq 1 + 2n$$

for all $n \geq 1$.

18. Use strong induction to prove that any postage amount ≥ 12 can be formed using only 4-cent and 5-cent stamps.

19. Prove by induction that
$$\sum_{k=1}^{n} 2^k = 2^{n+1} - 2.$$

20. Prove that for integers $n \geq 1$,
$$n^3 - n$$
is divisible by 6.

21. Let $b_1 = 2$, $b_2 = 5$, and $b_n = b_{n-1} + b_{n-2}$. Prove that every b_n is odd.

22. Prove by induction that for all integers $n \geq 1$,
$$n! > n^2$$
for all $n \geq 4$.

23. Use strong induction to show that any integer $n \geq 1$ can be expressed as a sum of distinct powers of 2.

24. Prove that for all $n \geq 1$,
$$\left(1 + \frac{1}{n}\right)^n < 3.$$

25. Prove by induction that
$$\sum_{k=1}^{n} (2k-1)^3 = n^2(2n-1)^2.$$

26. Use induction to show that for all $n \geq 1$,
$$\prod_{k=1}^{n} (k+1) \leq (n+1)^n.$$

27. Prove by induction that for all $n \geq 1$,
$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

28. Let $c_1 = 1$ and $c_{n+1} = c_n + 2n$. Prove that $c_n = n^2$.

29. Use strong induction to prove that every integer $n \geq 2$ has a divisor d such that
$$d \leq \sqrt{n}.$$

30. Prove by induction that for $n \geq 1$,
$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1).$$

31. Show by induction that for all integers $n \geq 1$,
$$2^n \geq n + 1.$$

32. Prove that for all odd integers $n \geq 1$,
$$n^3 \equiv n \pmod{24}.$$

33. Use strong induction to prove that every connected graph with n vertices has at least $n - 1$ edges.

34. Let the Fibonacci sequence be defined by
$$F_1 = 1, \ F_2 = 1, \ F_n = F_{n-1} + F_{n-2} \quad (n \geq 3).$$

Prove that the sum of the first n Fibonacci numbers is
$$F_{n+2} - 1.$$

35. Prove that for all $n \geq 2$,
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n} - 1).$$

36. For $n \geq 1$, prove that
$$n! \leq n^n.$$

37. Prove by induction that for all $n \geq 1$,
$$4^n + 6n - 1 \text{ is divisible by } 9.$$

38. Let a sequence be defined by $x_1 = 2$ and
$$x_{n+1} = x_n^2 - x_n + 1.$$

Show using induction that x_n is odd for all n.

39. Use strong induction to show that every integer $n \geq 8$ can be written as
$$n = 3a + 5b$$

for some nonnegative integers a, b.

40. Prove by induction that for all $n \geq 1$,
$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}.$$

3 Basic Graph Theory

1. Determine if a 6-cycle can be colored with 2 colors. (A 6-cycle is 6 vertices connected in a closed loop.)

2. Determine whether a 4-cycle is a tree. (4-cycle: 4 vertices connected in a closed loop.)

3. Show that any simple graph with at least 2 vertices has at least 2 vertices of the same degree. (Use the concept of degree sequence: list of all vertex degrees in non-increasing order.)

4. Determine if a tree with 7 vertices can have 5 leaves. (A leaf is a vertex of degree 1.)

5. Show that in any simple graph with all vertices degree 2, the graph is a union of cycles. (All cycles: vertices connected in closed loops.)

6. Determine whether a 7-cycle plus one extra edge is bipartite. (7-cycle: 7 vertices connected in a closed loop.)

7. Determine if a graph with vertices $\{1, 2, 3, 4, 5\}$ and edges $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$ has a perfect matching. (Perfect matching: set of edges covering all vertices exactly once.)

8. Show that a simple graph with maximum degree Δ can be colored with at most $\Delta + 1$ colors.

9. Determine whether a 3-cycle is bipartite. (3-cycle: triangle, 3 vertices in a closed loop.)

10. Determine the chromatic number of a 5-cycle. (5-cycle: 5 vertices in a closed loop; chromatic number: minimum colors needed to color vertices so adjacent vertices have different colors.)

11. Determine whether a 7-vertex cycle plus one extra edge is bipartite. (7-cycle: 7 vertices in a closed loop.)

12. Determine if a 4-regular graph with 5 vertices exists. (Regular graph: all vertices same degree.)

13. Determine if a 4-vertex tree can have 3 leaves. (Leaf: vertex of degree 1.)

14. Determine all non-isomorphic trees with 6 vertices.

15. Determine if a 6-vertex, 10-edge graph can have no cycles. (Cycle: sequence of vertices connected in a closed loop.)

16. Show that a graph with degree sequence $(2, 2, 2, 2, 2, 2)$ is regular. (Regular graph: all vertices same degree; degree sequence: list of all vertex degrees.)

17. Determine if a 5-cycle can be colored with 2 colors. (5-cycle: 5 vertices in a closed loop.)

18. Show that any tree with at least two vertices has at least two leaves. (Leaf: vertex of degree 1.)

19. Determine whether a 4-cycle with diagonals is bipartite. (4-cycle with diagonals: 4 vertices forming a cycle plus both diagonals.)

20. Find the degree sequence of K_4. (Degree sequence: list of degrees of all vertices.)

21. Determine if a 5-vertex graph with degree sequence $(3, 3, 2, 2, 2)$ is possible. (Degree sequence: list of all vertex degrees.)

22. Show that a graph with all vertices of degree 2 is a regular graph and a disjoint union of cycles. (Regular: all vertices same degree.)

23. Determine the degree sequence of $K_{3,3}$. (Degree sequence: list of degrees in non-increasing order; $K_{3,3}$: complete bipartite graph with sets of sizes 3 and 3.)

24. Determine if a 6-cycle is bipartite. (6-cycle: 6 vertices in a closed loop.)

25. Determine if the complete bipartite graph $K_{2,3}$ has a perfect matching. (Complete bipartite: two disjoint sets with all edges between sets; perfect matching: edges cover all vertices.)

26. Determine if a 6-cycle can be colored with 2 colors. (6-cycle: 6 vertices in a closed loop.)

27. Determine the number of non-isomorphic trees with 5 vertices.

28. Determine if K_5 has a perfect matching. (Perfect matching: set of edges covering all vertices exactly once.)

29. Determine if a 4-vertex tree is a 4-cycle. (4-cycle: 4 vertices connected in a closed loop.)

30. Determine if a 3-cycle is bipartite. (3-cycle: triangle, 3 vertices in a closed loop.)

31. Determine whether a 4-regular graph with 4 vertices exists. (Regular graph: all vertices have the same degree.)

32. Determine whether a 5-vertex graph has a perfect matching.

33. Determine if a 6-cycle is bipartite. (6-cycle: 6 vertices in a closed loop.)

34. Determine the chromatic number of a 6-cycle. (6-cycle: 6 vertices in a closed loop.)

35. Determine if K_5 is a tree. (Complete graph: every pair of distinct vertices connected.)

36. Determine if a 7-cycle plus one extra edge is bipartite. (7-cycle: 7 vertices connected in a closed loop.)

37. Determine if a 4-vertex tree can have 3 leaves. (Leaf: vertex of degree 1.)

38. Determine if a 6-vertex graph with 10 edges can have no cycles. (Cycle: vertices connected in a closed loop.)

39. Show that a simple graph with degree sequence $(3, 3, 3, 3, 3, 3)$ is regular. (Regular: all vertices same degree; degree sequence: list of degrees.)

40. Determine the chromatic number of a 5-cycle. (5-cycle: 5 vertices in a closed loop.)

41. Determine the degree sequence of K_4. (Degree sequence: list of vertex degrees.)

42. Show that in a complete bipartite graph $K_{4,5}$, every vertex in the 4-set has degree 5 and every vertex in the 5-set has degree 4. (Complete bipartite: two disjoint sets, edges only between sets.)

43. Determine whether $K_{3,3}$ has a perfect matching. (Complete bipartite graph: all edges between two sets.)

44. Show that any complete bipartite graph $K_{m,n}$ satisfies $\chi(K_{m,n}) = 2$. (Chromatic number: minimum vertex colors so adjacent vertices differ.)

45. Determine if a 5-cycle can be colored with 2 colors. (5-cycle: 5 vertices in a closed loop.)

46. Show that in any simple graph with all vertices degree 2, the graph is a union of cycles. (Cycle: vertices connected in a closed loop.)

47. Show that a simple graph with maximum degree Δ can be colored with at most $\Delta + 1$ colors.

48. Show that in a simple graph with 10 vertices and 15 edges, there exists at least one vertex of degree at least 3.

49. Show that a tree with n vertices has exactly $n - 1$ edges.

50. Show that a connected graph with n vertices and $n - 1$ edges is a tree.

51. Show that in a bipartite graph, all cycles have even length.

52. Show that any simple graph with at least 2 vertices has at least 2 vertices of the same degree. (Degree sequence: list of all vertex degrees.)

53. Show that any tree with at least two vertices has at least two leaves. (Leaf: vertex of degree 1.)

54. Determine all non-isomorphic simple graphs with 4 vertices and 3 edges.

55. Show that a graph is bipartite if and only if it has no cycles of odd length.

56. Show that in any graph, the sum of degrees of all vertices equals twice the number of edges.

57. Show that a graph with all vertices of degree 2 is a regular graph and a disjoint union of cycles.

58. Show that a graph with degree sequence $(2, 2, 2, 2, 2, 2)$ is regular. (Regular: all vertices same degree.)

59. Determine the number of non-isomorphic trees with 5 vertices.

60. Show that a simple graph with degree sequence $(3, 3, 2, 2, 2, 1)$ is possible. (Degree sequence: list of all vertex degrees.)

61. Show that a simple graph with 6 vertices and 9 edges must contain a cycle.

62. Show that a simple graph with 6 vertices and 10 edges has a vertex of degree at least 4.