A Novel Parallel Triangle Counting Algorithm with Reduced Communication

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Abstract—Counting and finding triangles in graphs is often used in real-world analytics for characterizing the cohesiveness and identifying communities in graphs. In this paper, we present novel sequential and parallel triangle counting algorithms based on identifying horizontal-edges in a breadth-first search (BFS) traversal of the graph. The BFS allows our algorithm to drastically reduce the number of edges examined for set intersections. Our new approach is the first communication-optimal parallel algorithm that asymptotically reduces the communication on massive graphs such as from real social networks and synthetic graphs from the Graph500 Benchmark. In our estimate from massive-scale Graph500 graphs, our new algorithms reduces the communication by 21.8x on a scale 36 and by 180x on a scale 42. Because communication is known to be the dominant cost of parallel triangle counting, our new parallel algorithm, to our knowledge, is now the fastest method for counting triangles in large graphs.

Index Terms—Graph Algorithms, Parallel Algorithms, High Performance Data Analytics

I. INTRODUCTION

Let G = (V, E) be an undirected graph with n = |V| vertices and m = |E| edges. A *triangle* in the graph is a set of three vertices $\{v_a, v_b, v_c\} \in V$ such that $\{\langle v_a, v_b \rangle, \langle v_a, v_c \rangle, \langle v_b, v_c \rangle\} \in E$. Triangle counting is defined as finding the number of unique triangles in G and is one of the fundamental problems in graph analytics. In this paper we consider the problem of both counting and finding triangles in G. Triangle counting is used in graph analytics such as clustering coefficients [1], k-truss [2], and triangle centrality [3], and its importance is recognized by its use in high performance computing benchmarks such as Graph500 [4] as well as in future architecture systems design (e.g. IARPA AGILE [5]).

Triangle counting has been studied extensively in the literature for both sequential and parallel algorithms. Latapy [6] provides an extensive literature of sequential triangle counting and finding algorithms. The fastest sequential triangle counting algorithm for sparse graphs due to Itai and Rodeh [7] runs in

 $\Theta\left(m^{\frac{3}{2}}\right)$ time and $\Theta(n^2)$ space, which is cost-prohibitive for practical use on large graphs.

Triangle counting algorithms are often based on techniques such as list intersection, matrix multiplication and subgraph matching [8]. The three main list intersection based triangle counting algorithms are summarized as: the node iterator, the edge iterator and the forward algorithm (detailed in Section III).

Cohen [9] designed a novel map-reduce parallelization of triangle counting that generates *open wedges* between triples of vertices in the graph, and determines if a closing edge exists that completes a triangle. Most parallel approaches for triangle counting [10], [11] partition the sparse graph data structure across the compute nodes, and follow this strategy of generating open wedges that are sent to other compute nodes to find whether or not a closing edge exists. As such, the running time of parallel triangle counting is often dominated by the communication time for these open wedges.

In fact, most parallel triangle counting algorithms are based on generating open wedges (also known as 2-paths) and checking whether or not a closing edge exists. This approach results in an overabundance of communication because each graph edge is sent repeatedly, the repetition factor being the degree of one of its endpoints. One way to view our new approach is that rather than send all open wedges in the graph, we reverse this and send potential closing edges once, resulting in asymptotically less communication. In this paper, we present novel sequential and parallel triangle counting algorithms. Our contributions include:

- A new sequential triangle counting and finding algorithm that runs in $O(m \cdot d_{max})$ time and O(n+m) space, where d_{max} is the maximal degree of a vertex $v \in V$. The algorithm uses breadth-first search to identify horizontal-edges, and uses the intersection of the endpoints of each horizontal-edge to determine triangles.
- A novel communication-optimal parallel algorithm for triangle counting and finding that asymptotically reduces the communication from all prior parallel approaches.
 The new approach has a total communication volume of O(m) and is the first parallel algorithm to achieve this.

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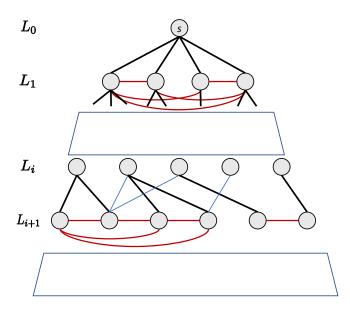


Fig. 1. Example BFS tree of Graph G. The tree-edges are black, strut-edges are blue, and horizontal-edges are red.

This is an asymptotic improvement from prior wedgequery based approaches that require a communication volume $O(n \cdot d_{max}^2) = O(n^3)$.

The remainder of the paper is organized as follows. Section II gives additional notation. Section III discusses related work. Section IV presents our new approach for triangle counting. In Section V, we present our new parallel triangle counting. Lastly, in Section VI, we conclude the paper.

II. NOTATION

Let G=(V,E) be an undirected graph with n=|V| vertices and m=|E| edges. We will use $N(v)=\{u\in V, \langle v,u\rangle\in E\}$ to denote the *neighborhood* of vertex $v\in V$. The degree of vertex $v\in V$ is d(v)=|N(v)|, and d_{\max} is the maximal degree in graph G. Given a source vertex s, the level L(v) of a vertex v is the distance from s to v. After breadth-first search (BFS) of an undirected graph, edges can be classified as 1) tree-edges: in the BFS-tree, 2) strut-edges: non-tree edges whose endpoints are on two adjacent levels, and 3) tree-edges: non-tree edges whose endpoints are on the same level. See Fig. 1 for an example.

III. RELATED WORK

A. Sequential Algorithms

The naïve approach for triangle counting uses brute-force: find all the triplets $\{v_a, v_b, v_c\}$, that is, permutations of three arbitrary vertices in the graph, and check whether each edge in the triplet exists. The time complexity is $\Omega(n^3)$. Latapy [6] and Schank and Wagner [12] provide surveys of faster sequential algorithms. Triangle counting algorithms generally fall into three approaches: list intersection, matrix multiplication and subgraph matching.

The three main **intersection-based** triangle counting algorithms are: 1) the node-iterator algorithm iterates over all vertices and tests for each pair of neighbors whether they are connected by an edge, 2) the edge-iterator algorithm iterates over all edges and searches for common neighbors of the two endpoints of each edge, and 3) the forward algorithm is a refinement of the edge-iterator algorithm that computes the intersection of a subset of neighborhoods by using an orientation of the graph. The time complexity of node-iterator and edge-iterator are both $O(m \cdot d_{max})$ and the forward algorithm is $O(m^{\frac{3}{2}})$, which is significantly better performance [6].

When performing the intersection of two lists, the commonly used techniques are merge-path, binary search and hashing-based algorithms. Merge-path algorithms (e.g., [13], [14]) use two pointers to scan through neighbor lists of two endpoints from beginning to end in order to find the list intersection. During the scan, the pointer that points to a smaller value will be incremented. A triangle is enumerated if both pointers are incremented (i.e., they both point to the same vertex). Binary-search algorithms (e.g., [15], [16]) organize the longer list as a binary tree and use the shorter list as search keys. For each search key, it descends through the binary-search tree in order to find the equal entry, which is a triangle. Hashing-based algorithms (e.g., [8], [14]) construct a hash table for one list and use the other list as search keys to find the common elements in the hash table. We use a hash table here to find the intersection of two adjacency lists, so it is not necessary to sort all the adjacency lists to find all the triangles. The running time is proportional to the size of the two adjacency lists.

Triangle counting using **matrix multiplication** [17] relies on a linear algebra formulation for triangle counting. For the graph's adjacency matrix A, the approach performs $B=A\times A$, which counts the number of wedges, then the elementwise multiplication $A\odot B$ determines whether the wedge is closed. The method finds the triangle count after scaling. This approach can be optimized [18] using matrix decomposition by decomposing A into lower and upper triangular matrices L and U, and then computing $(L\times U)\odot L$, or $(L\times L)\odot L$ to determine the number of triangle.

A **matching-based** approach for triangle counting searches for all occurrences of a query graph, which is a triangle, in the input graph. Wang and Owens [19] use breadth-first search to update the subgraph matching approach by pruning more invalid vertices based on neighborhood encoding information, and using optimizations like k-step look-ahead to reduce unwanted intermediate results.

B. Parallel Algorithms

Map-reduce is a standard platform for large scale distributed computation. Cohen [9] first demonstrated the capability of map-reduce to solve triangle counting in an approach that generates *open wedges* between triples of vertices in the graph and determines if a closing edge exists that completes a triangle. Suri *et al.* [20] implemented triangle counting using map-reduce that ranks vertices by degree and distributes

them across hosts. Pearce [10] developed an algorithm that is based on creating an augmented degree-ordered directed graph, where the original undirected edges are directed from low-degree to high degree, and implemented this approach in the distributed asynchronous graph processing framework HavoqGT. DistTC [16] is a distributed triangle counting implementation for multiple machines that uses mirror proxy on each partition to eliminate almost all the inner-host communication. TriCore [15] partitions the graph held in a compressedsparse row (CSR) data structure for multiple GPUs and uses stream buffers to load edge lists from CPU memory to GPU memory on-the-fly and then uses binary search to find the intersection. TriC [11] exploits the vertex-based distributed triangle counting and sends vertices rather the edges (vertex pairs), and then the remote processor could translate the sequence of vertex IDs to correct combination of vertices as edges to reduce communication. An enhancement is then presented to Tric [21] that added a user-defined buffer to improve the flexibility of controlling the memory usage for large data sets and used a probabilistic data structure to optimize the edge lookups by trading off the accuracy. Strausz et al. [22] use CLaMPI, a software caching layer that caches data retrieved through MPI remote memory access operations, to reduce the overall communication cost. Zeng et al. [23] proposed a triangle counting algorithm that adaptively selects vertex-parallel and edge-parallel paradigm.

IV. NEW APPROACH

We observe the following about a triangle $\{v_a,v_b,v_c\}$ in an undirected graph G=(V,E). (We assume G is connected. If not, it is trivial to extend this approach to each component.) Given an arbitrary vertex $s\in V$, the shortest path *distance* from s to v_a , $d(s,v_a)$, and the distances to the other two triangle vertices, $d(s,v_b)$ and $d(s,v_c)$ differ by at most 1. (A simple proof by contradiction is left to the reader.) This motivates our new sequential approach given in Alg. 1.

Algorithm 1 Triangle Counting

```
Input: Graph G = (V, E)
Output: Triangle Count T
 1: Select an arbitrary vertex s \in V
 2: Compute the breadth-first search of G from s and label
    each edge e \in E as a tree-, strut-, or horizontal- edge
 3: for all v \in V do
         L(v) \leftarrow d(s, v)
 5: end for
 6: for all horizontal-edges \langle u, w \rangle do
         I \leftarrow N(u) \cap N(w)
 7:
 8:
         for all v \in I do
                                                 \triangleright triangle \{u, v, w\}
             if L(v) \neq L(u) then c_1 \leftarrow c_1 + 1
 9:
             else c_2 \leftarrow c_2 + 1
10:
             end if
11:
         end for
12:
13: end for
14: T \leftarrow c_1 + c_2/3
```

Lemma 1. Each triangle $\{u, v, w\}$ must contain at least one horizontal-edge.

Proof. (By contradiction.) A triangle is a path of length 3 that has the same beginning and ending vertex. If no horizontal-edges are in the triangle, then each edge in the path (i.e., a tree- or strut-edge) increases or decreases the level by one. Since the path must end on the same level as the starting vertex, there must be the same number of edges in the path decreasing the level as there are edges in the path increasing the level. This requires an even path length. However, this is a contradiction since a triangle has an odd path length of 3. □

Lemma 2. Each triangle $\{u, v, w\}$ must contain either one or three horizontal-edges.

Proof. It follows from the proof of Lemma 1 that the path must include an even number of tree- and strut-edges; hence, 0 or 2 tree- or strut-edges in each triangle. In the first case with 0 tree- or strut-edges, all three triangle edges must be horizontal-edges. In the second case with 2 tree- or strut-edges, then the triangle contains exactly one horizontal-edge. □

Theorem 1 (Novel Counting Method). The triangle count of a graph G is the sum of the number of triangle apex vertices on a different level from the horizontal-ends endpoints and one-third of the number of that are on the same level.

Proof. Lemma 2 proves that for a triangle either 1) the two endpoint vertices of the horizontal-edge are on the same level and the apex vertex is on a different level, or 2) all three triangle vertices are in the same level. For the triangle $\{v_a, v_b, v_c\}$, assume (w.l.o.g.) that $\langle v_a, v_b \rangle$ is a horizontaledge (so $L(v_a) \equiv L(v_b)$) and that v_c is the apex vertex. If $L(v_c) \neq L(v_a)$, then we have the first case; otherwise $L(v_c) \equiv L(v_a) \equiv L(v_c)$ and we have the second case. Each unique triangle is defined by a horizontal-edge and an apex vertex from the list intersection of the horizontal-edge's endpoint vertices. A triangle is triply-counted when the second case occurs: once for each of the triangle's three horizontaledges. Hence, the total number of triangles is the sum of the number of the apex vertices in the first case and one-third the number of apex vertices in the second case.

A. Time Complexity

Computing breadth-first search, the level of each vertex, and marking horizontal-edges, takes O(n+m) time. There are at most O(m) horizontal-edges, and the time complexity for finding the list intersection for each is $O(m \cdot d_{max})$. Thus, the time complexity is $O(m \cdot d_{max})$.

V. PARALLEL ALGORITHM

In this section, we present our communication-optimal parallel algorithm for counting triangles in massive graph on a p-processor distributed-memory parallel computer. Similar with prior approaches, the input graph G is stored in compressed sparse row (CSR) format. The vertices are partitioned non-uniformly to the p processors such that each processor stores approximately 2m/p edge endpoints.

Our parallel algorithm (see Alg. 2) is based on the horizontal-edge approach of our sequential algorithm. This is the first triangle counting algorithm to our knowledge that uses the breadth-first search followed by a transpose of the sparse edge arrays to significantly reduce the communication for set intersections.

We illustrate the parallel triangle counting algorithm using a real dataset of a very unusual social community of bottlenose dolphins living in a fjord, a geographically-isolated environment at the southern most extreme of the species' range. A research team systematically surveyed the social interactions of this animal community for seven years, from November 1994 to November 2001 in Doubtful Sound, Fiordland, New Zealand [24]. In Figure 2 we show 62 individual dophins from this survey and their interactions in an undirected social network graph.

The algorithm first runs parallel breadth-first search (BFS) (line 2) on the graph G and assigns a level L(v) to each $v \in V$. If the diameter of the graph is D, then $\lceil \log D \rceil$ bits are needed per vertex to store its level. During the BFS, for each edge in the graph we assign a bit as to whether the edge is a horizontal-edge or not. Figure 3 gives an example of the breadth-first search and identifying horizontal-edges in the dolphin social network graph. The horizontal-edges from this example are given in Figure 4.

Earlier we defined $N(v) = \{u \in V \mid \langle v, u \rangle \in E\}$ to denote the neighborhood of vertex $v \in V$. Let the modified neighborhood of v be $\hat{N}(v) = N(v) - \{w \in V \mid \langle v, w \rangle \leftarrow$ horizontal-edge, $v < w\}$. That is, we remove all vertices adjacent to v along a horizontal-edge where v's label is less than w's. The modified neighborhoods break symmetry and will be used to prevent triple-counting of triangles comprised of three horizontal-edges in the graph. In line 4 of Alg. 2 the modified neighborhoods are created. In Figure 5 we list the modified neighborhoods of the dolphin social network graph and their assignments to four processors.

The approach performs a sparse matrix transpose of the modified neighborhoods (lines 22 to 28). This transpose allows intersections of the respective sublists to occur locally without further communication. The transpose must partition the n vertices to the p processors. Simply assigning blocks of n/p vertices to each processor; that is, $i \cdot n/p$ to (i+1)n/p-1 to processor p_i , for $0 \le i \le p-1$; potentially results in a load imbalance. Instead, our approach uses techniques first developed for parallel sorting (e.g., [25]–[27]) to ensure that each processor receives at most twice the average $(2 \cdot (2-k)m/p)$ number of elements. By oversampling, we

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Algorithm 2 Parallel Triangle Counting
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```
Input: Graph G = (V, E)
Output: Triangle Count T
 1: Select an arbitrary vertex s \in V
 2: Compute the breadth-first search in parallel of G from s
     and set X(\langle v, w \rangle) if \langle v, w \rangle is a horizontal-edge
 3: for all v \in V in parallel do
         \hat{N}(v) \leftarrow N(v) - \{ w \in V \mid X(\langle v, w \rangle), v < w \}
 5: end for
 6: for all i \in 0 \dots p-1 in parallel do
         z_i = \Sigma |\hat{N}_i(*)|
 7:
 8: end for
 9: for all i \in 0 \dots p-1 in parallel do
         On p_i perform a multi-way merge of \hat{N}_i(*) and
10:
         select elements at positions |j \cdot z_i/(p+1)|
11:
         for 1 \le j \le p as samples.
12:
13: end for
14: for all i \in 0 \dots p-1 in parallel do
         p_i sends its p samples to p_0
15:
16: end for
17: p_0 creates a multi-way merge of the p sorted lists, and
18: selects elements at positions j \cdot p for 1 \le j \le (p-1)
    as the p-1 splitters.
20: p_0 broadcasts the p-1 splitters
21: Splitter[0] \leftarrow 0; Splitter[p] \leftarrow n
22: for all i \in 0 \dots p-1 in parallel do
         for j = 0 to p - 1 do
23:
             p_i sends p_{i \oplus j} all \hat{N}_i(*) sublists with values
24:
              > Splitter[i \oplus j] and \leq Splitter[(i \oplus j) + 1]
25:
              and receives its sublists from p_{i \oplus i}
26:
27:
         end for
28: end for
29: for all i \in 0 \dots p-1 in parallel do
30:
31: end for
32: for all i \in 0 \dots p-1 in parallel do
         for each horizontal-edge \langle v, w \rangle with v < w on p_i do
33:
              t_i = t_i + |\hat{N}_i(v) \cap \hat{N}_i(w)|
34:
35:
         end for
         for j = 1 to p - 1 do
36:
             Processors p_i and p_{i \oplus j} swap
37:
              horizontal-edges \langle v, w \rangle where v < w.
38:
              for each edge \langle v, w \rangle p_i receives from p_{i \oplus i} do
39:
                  t_i = t_i + |\hat{N}_i(v) \cap \hat{N}_i(w)|
40:
              end for
41:
         end for
42:
43: end for
44: T \leftarrow \text{Reduce}(t_i, +)
```

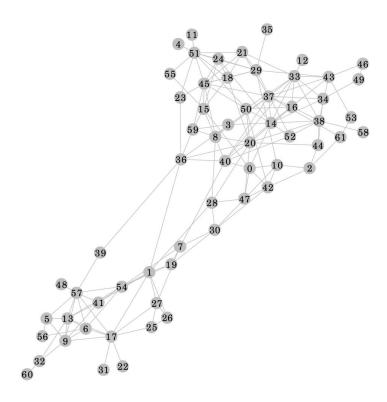


Fig. 2. Interactions of 62 bottlenose dolphins represented by an undirected graph.

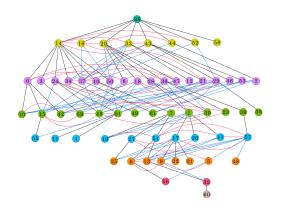


Fig. 3. Breadth-first search of the dolphin social network graph. The vertices are organized by level with tree-edges in black, strut-edges in blue, and horizontal-edges in red.

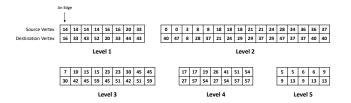


Fig. 4. The horizontal-edges in the breadth-first search of the dolphin social network graph, organized by level.

can reduce the maximum number of elements received by any processor to $(2-k)m/p + (2-k)m/\beta - p$ where β is the number of samples selected per processor [26] and $p \leq \beta \leq (2-k)m/p$. In lines 6-8 we compute z_i , the total number of elements in processor p_i 's modified neighborhoods. Each processor performs a multi-way merge of its sorted modified neighborhoods and selects p samples at positions $|j \cdot z_i/(p+1)|$ for $1 \le j \le p$ (lines 9-13). Duplicates in the samples can be handled with a slight modification to the algorithm [26]. See Figure 6 for the selection of 16 samples from the modified neighborhoods. These p sorted sublists of samples are sent to p_0 (lines 14-16). Processor p_0 merges them, and selects elements $j \cdot p$ for $1 \leq j \leq (p-1)$ as the p-1 splitters (lines 17-19). Figure 7 shows the selection of 3 pivots on the first processor from the dolphin graph's 16 samples. Finally, p_0 broadcasts these p-1 splitters to the other processors (line 20). Figure 8 shows the partitioning of the dolphin graph's neighborhoods based on the splitters.

Next, the km horizontal-edges $\langle v,w\rangle$ where v< w are shared between each pair of processors (lines 32 to 43). By considering the orientation of the edge where v< w, the algorithm ensures that the horizontal-edge is used only once to count its incident triangles. Given a horizontal-edge $\langle v,w\rangle$ a processor p_i will compute an intersection of the local sublists of v and w in the range of (Splitter[i]... Splitter[i+1]] and increment its local triangle count by the size of the intersection. We do not need to retain the received horizontal-edges.

Finally, a reduction over all the triangle counters across the

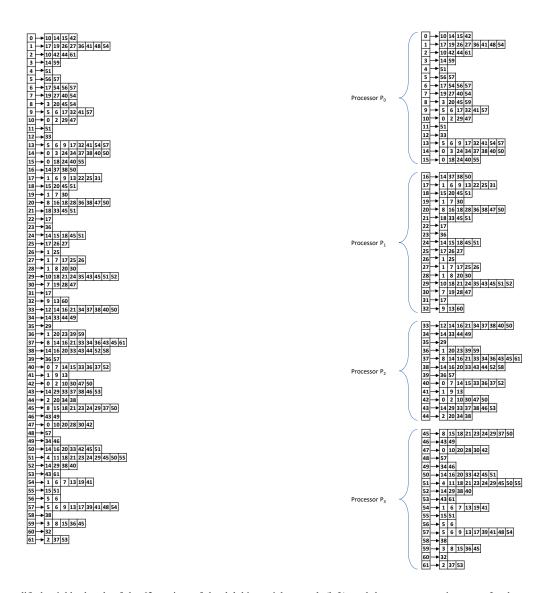


Fig. 5. The modified neighborhoods of the 62 vertices of the dolphin social network (left), and the processor assignments for these modified neighborhoods (right).

system is performed and the total number of triangles is $T = \sum_{i=0}^{p-1} t_i$.

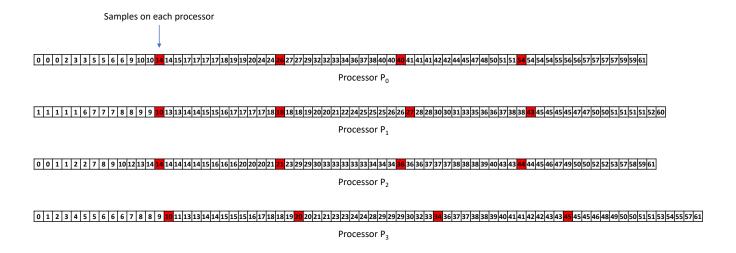


Fig. 6. Selection of the 16 samples from the modified neighbors of the dolphin graph.

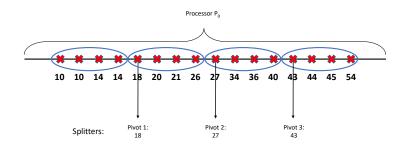


Fig. 7. Finding the 3 pivots for p = 4 processors from the samples.

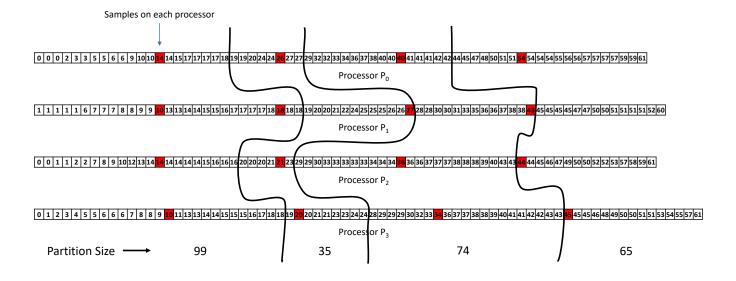


Fig. 8. The partitioning of the dolphin graph's modified neighborhoods on each processor using the three pivots.

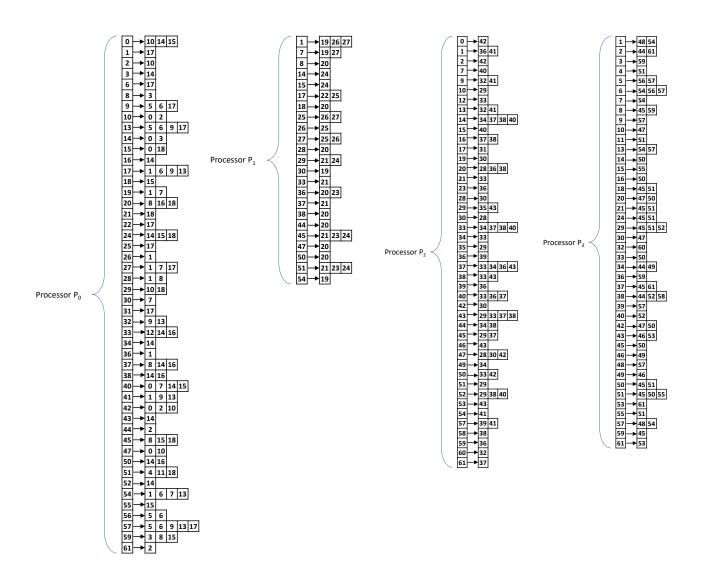


Fig. 9. The transposed modified neighborhood sets of the dolphin graph on p = 4 processors.

A. Cost Analysis

1) Space: In addition to the input graph data structure, an additional bit is needed per edge (for marking a horizontal-edge) and $O(\lceil \log D \rceil)$ bits per vertex to store its level. This is a total of at most $m + n \lceil \log D \rceil$ bits across the p processors, which is negligible in practice. Preserving the graph requires additional O(n+m) space for the graph, otherwise the space can be reclaimed during the transpose of the modified neighborhoods. Note that the horizontal-edges that are swapped between each of the processors are used in local computation and do not need to be stored.

2) Communication: In our analysis for communication cost for BFS and transpose, we measure the total communication volume independent of the number of processors. Thus, this is a conservative over-estimate of communication since a fraction (e.g., 1/p) of accesses will be on the same compute node versus message traffic between nodes.

The cost of breadth-first search is m edge traversals with $\lceil \log D \rceil + 3 \lceil \log n \rceil$ bits communicated per edge traversal for the level information, pair of vertex ids, and vertex degree, yielding $m \cdot (\lceil \log D \rceil + 3 \lceil \log n \rceil)$ bits for the BFS.

Determining the p-1 splitters requires p^2 samples to be sent and for p_0 to broadcast the p-1 splitters. The communication cost is $(p^2 + (p-1)p)\lceil \log n \rceil = (2p^2 - p)\lceil \log n \rceil$ bits.

Due to the symmetry-breaking of the horizontal-edges, km vertices, one for each horizontal-edge, are removed from the neighborhoods N(*) to form the modified neighborhoods $\hat{N}(*)$. Hence, the total size of the modified neighborhoods is (2-k)m vertices. Thus, the transpose of the modified neighborhoods requires the communication of $(2-k)m\lceil\log n\rceil$ bits.

Swapping the km horizontal-edges requires $kmp\lceil \log n \rceil$ bits, where p is the number of processors.

The final reduction to find the total number of triangles requires $(p-1)\lceil \log n \rceil$ bits.

Hence, the total communication volume is $m \cdot (\lceil \log D \rceil + 3\lceil \log n \rceil) + (2p^2 - p)\lceil \log n \rceil + (2-k)m\lceil \log n \rceil + kmp\lceil \log n \rceil + (p-1)\lceil \log n \rceil = m \cdot (\lceil \log D \rceil + (kp-k+5)\lceil \log n \rceil) + (2p^2-1)\lceil \log n \rceil$ bits. Hence, on a given parallel machine, the communication is O(m) words, which is communication-optimal.

B. Analysis on Real and Synthetic Graphs

In this section we analyze the performance of the parallel triangle counting algorithm on both real and synthetic graphs. The summary is given in Table I. For the real graphs, we find the actual value of k, the percentage of graph edges that are horizontal-edges, for an arbitrary breadth-first search, and set the number p of processors to a reasonable number given the size of the graph. For the synthetic graphs, we use large Graph500 RMAT graphs [28] with parameters a=0.57, b=0.19, c=0.19, and d=0.05, for scale 36 and 42 with $n=2^{\rm scale}$ and m=16n, similar with the IARPA AGILE benchmark graphs, and set p according to estimates of potential system sizes with sufficient memory to hold these large instances. For these graphs,

For comparison, most prior parallel algorithms for triangle counting operate on the graph as follows. A parallel loop over the vertices $v \in V$ produces all 2-paths (wedges) where $\langle v, v_1 \rangle, \langle v, v_2 \rangle \in E$ and (w.l.o.g.) $v_1 < v_2$. The processor that produces this wedge will send a open wedge query message containing the vertex ids of v_1 and v_2 to the processor that owns vertex v_1 . If the consumer processor that receives this query message finds an edge $\langle v_1, v_2 \rangle \in E$, then a local triangle counter is incremented. After producers and consumers complete all work, a global reduction over the p triangle counts computes the total number of triangles in G.

C. Graph500 RMAT Graphs

Pearce [10] shows that for large Graph500 graphs, the total running time closely tracks the wedge checking time. Their implementation for scale 36 takes 3960s on 1.5M CPUs of IBM BG/Q to count triangles. The result finds 2.7×10^{13} triangles and checks 1.05×10^{15} wedges. Since each wedge requires 72 bits, the total data volume of checks is $8.32\text{PB}^{\,1}$.

We estimate the number of triangles and wedges for the scale 42 graph by extrapolating from counts up to scale 34 in [10] and set triangles to $2^{42-36} \cdot \# Triangles_{36}$ and wedges to $2^{(42-36)*2} \cdot \# Wedges_{36}$, resulting in the scale 42 estimate of 8.64×10^{14} triangles and 1.08×10^{18} wedges. With 84 bits/wedge, the total volume of wedge checks is 9.84EB.

Beamer *et al.* [29] find a typical BFS on a scale 27 Graph500 RMAT graph has 7 levels, so 4 bits is a reasonable estimate for $\log D$ in our analyses of scale 36 and 42 graphs. In our experimental analyses of these RMAT graphs for scales from 10 to 24, we found the fraction k of horizontal-edges to be approximately 0.65.

In our new approach for scale 36, where the communication cost is $m \cdot (\lceil \log D \rceil + (kp-k+5)\lceil \log n \rceil) + (2p^2-1)\lceil \log n \rceil$ bits. With $\lceil \log D \rceil = 4$, and assuming p=128 processors, we have a total communication volume of 394TB, for a communication reduction of $21.8 \times$.

For scale 42, and assuming p = 256 processors, we estimate the communication of our new triangle counting algorithm as 56.1PB, for a communication reduction of $180 \times$.

VI. CONCLUSIONS

In this paper, we present novel sequential and parallel algorithms for counting and finding triangles in graphs. The parallel algorithm is the first communication-optimal triangle counting algorithm and is an asymptotic improvement upon all prior approaches and significantly reduces the communication volume on massive graphs of practical interest. Our approach uses the breadth-first search to drastically reduce the number of edges examined and a sparse transpose in the parallel algorithm that minimizes the communication required for set intersections. The parallel algorithm achieves an order of magnitude or more of speedup for large graphs as communication is the main bottleneck for triangle counting on distributed memory systems.

 $^{^{1}\}mathrm{Throughout}$ this paper, a petabyte (PB) is 2^{50} bytes and an exabyte (EB) is 2^{60} bytes.

Graph	n	m	# Triangles	# Wedges	k	p	Previous	This paper	Speedup
ca-GrQc	5242	14484	48260	165798	0.522	4	514KB	156KB	3.37
ca-HepTh	9877	25973	28339	277389	0.423	4	926KB	288KB	3.29
as-caida20071105	26475	53381	36365	776895	0.225	4	2.78MB	574KB	4.96
facebook_combined	4039	88234	1612010	17051688	0.914	4	48.8MB	1.01MB	48.4
ca-CondMat	23133	93439	173361	1567373	0.511	4	5.61MB	1.13MB	4.98
ca-HepPh	12008	118489	3358499	5081984	0.621	4	17.0MB	1.40MB	12.1
email-Enron	36692	183831	727044	5933045	0.478	4	22.6MB	2.32MB	9.75
ca-AstroPh	18772	198050	1351441	8451765	0.667	4	30.2MB	2.55MB	11.9
loc-brightkite_edges	58228	214078	494728	6956250	0.441	4	26.5MB	2.66MB	9.98
soc-Epinions1	75879	405740	1624481	21377935	0.498	4	86.7MB	5.49MB	15.8
amazon0601	403394	2443408	3986507	96348699	0.529	8	436MB	49.0MB	8.90
com-Youtube	1134890	2987624	3056386	209811585	0.347	8	1.03GB	56.6MB	18.6
RMAT-36	68719476736	1.09951E+12	2.7E+13	1.05E+15	0.65	128	8.39PB	394TB	21.8
RMAT-42	4.39805E+12	7.03687E+13	8.64E+14	1.08E+18	0.65	256	9.84EB	56.1PB	180.

TABLE I

COMMUNICATION COSTS FOR REAL AND SYNTHETIC GRAPH. THE SYNTHETIC GRAPHS ARE GRAPH500 RMAT GRAPHS OF SCALE 36 AND 42. THE COLUMN **PREVIOUS** REPRESENTS THE COMMUNICATION VOLUME OF THE BEST PRIOR PARALLEL ALGORITHMS THAT USE WEDGE-CHECKING BASED ALGORITHMS AND **THIS PAPER** REPRESENTS THE COMMUNICATION COST OF OUR NEW APPROACH. **SPEEDUP** REPRESENTS THE COMMUNICATION REDUCTION BETWEEN THESE TWO, AND THUS, THE EXPECTED SPEEDUP OF THE PARALLEL ALGORITHM.

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