Algorithmic Foundations 2

Propositional Logic (cont.)

Simon Fowler & Gethin Norman

School of Computing Science
University of Glasgow

Recap – Propositional logic

Propositions are the basic building blocks of logic

- declarative sentences that are either `true` or `false`
- denoted using lower case letters: p,q,…

Compound propositions (logical formula) generated using logical connectives to combine propositions

- denoted using upper case letters: P,Q,R,…
- negation (not): $\neg p$
- conjunction (and): $p \land q$
- disjunction (or): $p \lor q$
- exclususive or: $p \oplus q$
- conditional (implication): $p \rightarrow q$
- biconditional: $p \leftrightarrow q$

Outline

Propositions

Connectives

Tautologies and contradictions

Logical equivalence

- introduction
- laws of logical equivalence
- examples

Logical equivalence

Two syntactically (i.e. textually) different compound propositions may be semantically identical (i.e. have the same meaning)

- it such cases they are called logically equivalent

The statement $P \equiv Q$ expresses that P is logically equivalent to Q

- given any assignment to the propositions appearing in P and Q the truth values of P and Q are the same

Logical equivalence can be proved by:

- by laws of logical equivalence (this lecture)
- by a truth table (i.e. show truth table columns are the same)
  - number of rows of a truth table is 2^n where n is number of propositions
- by some other line of reasoning

Logical equivalence

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To show two compound propositions are not logically equivalent

- need to give one assignment to the propositions that makes one of the formulae **true** and the other formula **false**
  - notice do not need to give the full truth table just one row

Example: $(p \lor q) \land r$ and $p \lor (q \land r)$ are not logically equivalent

- if p is **true** while both q and r are **false**, then
- $(p \lor q) \land r$ evaluates to $(\mathbf{true} \lor \mathbf{false}) \land \mathbf{false}$
- $p \lor (q \land r)$ evaluates to $\mathbf{true} \lor (\mathbf{false} \land \mathbf{false})$

Logical equivalence

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- if p is **true** while both q and r are **false**, then
- $(p \lor q) \land r$ evaluates to **true**$\land$**false**
- $p \lor (q \land r)$ evaluates to **true**$\lor$**false**

Logical equivalence

Two syntactically (i.e. textually) different compound propositions may be semantically identical (i.e. have the same meaning)

- it such cases they are called logically equivalent

To show two compound propositions are **not** logically equivalent

- need to give one assignment to the propositions that makes one of the formulae **true** and the other formula **false**
  - notice do not need to give the full truth table just one row

Example: **$(p \vee q) \wedge r$** and **$p \vee (q \wedge r)$** are not logically equivalent

- if p is **true** while both q and r are **false**, then
- $(p \vee q) \wedge r$ evaluates to **false**
- $p \vee (q \wedge r)$ evaluates to **true**

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Laws of logical equivalence

These are similar to the arithmetic identities such as:

- $x \cdot 0 = 0$
- $x \cdot 1 = x$
- $x + y = y + x$
- $x \cdot (y + z) = x \cdot y + x \cdot z$
- $x + (y + z) = (x + y) + z$

The identities hold for all possible values of x, y and z

Equality replaced by logical equivalence and applied to compound propositions (formulae)

- provides a pattern or template that can be used to match all or part of a much more complicated proposition and to find an equivalence for it
  - examples to come
- can replace pattern on either side of the law with that on the other

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Laws – Identity, domination, idempotent & double negation

Identity laws:

- $P \wedge \mathbf{true} \equiv P$
- $P \vee \mathbf{false} \equiv P$

Domination laws:

- $P \vee \mathbf{true} \equiv \mathbf{true}$
- $P \wedge \mathbf{false} \equiv \mathbf{false}$

Idempotent laws:

- $P \wedge P \equiv P$
- $P \vee P \equiv P$

Double negation law:

- $\neg(\neg P) \equiv P$

In either case: result only depends on P

Value of P is irrelevant: true dominates ∨ and false dominates ∧

Multiple mentions of P does not change the meaning

```
¬¬true  = ¬false = true
¬¬false = ¬true  = false
```

Notice P is a compound proposition so example applications are

$((p \vee q) \wedge r) \wedge \mathbf{true} \equiv ((p \vee q) \wedge r)$ identity law

$(\neg q \rightarrow p) \wedge \mathbf{false} \equiv \mathbf{false}$ domination law

Laws – Commutative & associative

Commutative laws:

- $P \land Q \equiv Q \land P$
- $P \lor Q \equiv Q \lor P$

Ordering using connective does not matter

Associative laws:

- $(P \land Q) \land R \equiv P \land (Q \land R)$
- $(P \lor Q) \lor R \equiv P \lor (Q \lor R)$

Grouping using **same** connective does not matter

Laws – Distributive

Note: no way for me to go to beach and neither eat ice cream nor play video games!

Distributive laws:

- $P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R)$

I play video games, or I go to the beach and eat ice cream

≡

I play video games or go to the beach, and I play video games or eat ice cream

- $P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$

I will walk the dog and either play Truck Simulator or Stardew Valley

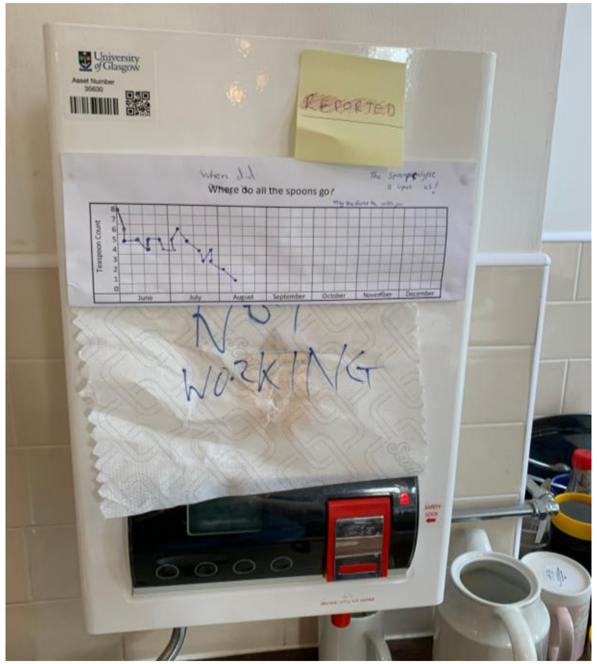
≡

I will walk the dog and play Truck Simulator, or I will walk the dog and play Stardew Valley

Laws – De Morgan

Let P be the proposition *"The coffee machine is working"*

Let Q be the proposition *"The water boiler is working"*



Disaster!

De Morgan laws:

$$\neg(P \land Q) \equiv \neg P \lor \neg Q$$

If it is not the case that both the coffee machine and the boiler work, then either the coffee machine is broken or the boiler is broken

$$\neg(P \lor Q) \equiv \neg P \land \neg Q$$

If it is not the case that **either** machine works, then both the coffee machine and the boiler must be broken

Laws – De Morgan, Contradiction, tautology & implication

Contradiction and tautology laws:

- $P \land \neg P \equiv$ **false**
- $P \lor \neg P \equiv$ **true**

Proposition cannot be true and false at the same time!

Proposition must either be true or false!

Implication law:

- $P \rightarrow Q \equiv \neg P \lor Q$

p	q	p→q	¬p∨q
0	**0**	**1**	**1**
0	**1**	**1**	**1**
1	0	0	0
1	**1**	**1**	**1**

Laws – Exclusive OR and Biconditional

Exclusive or and biconditional laws:

- $P \oplus Q \equiv (P \vee Q) \wedge \neg(P \wedge Q)$
- $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$

Focus on **understanding** the laws and using them, rather than rote learning. Practice makes perfect.

The exam will include a formula sheet, so you don't need to learn the rules off by heart!

Slight detour…

Recall the double negation and de Morgan laws

- $\neg(\neg P) \equiv P$
- $\neg(P \land Q) \equiv \neg P \lor \neg Q$
- $\neg(P \lor Q) \equiv \neg P \land \neg Q$

therefore we have

- $P \land Q \equiv \neg(\neg(P \land Q))$ double negation law
  $\equiv \neg(\neg P \lor \neg Q)$ De Morgan law

so can define conjunction using disjunction and negation

also we have

- $P \lor Q \equiv \neg(\neg(P \lor Q))$ double negation law
  $\equiv \neg(\neg P \land \neg Q)$ De Morgan law

so can define disjunction using conjunction and negation

Slight detour…

Recall also the implication law

- $P \rightarrow Q \equiv \neg P \vee Q$

it follows that

- $P \vee Q \equiv \neg(\neg P) \vee Q$ double negation
  $\equiv \neg P \rightarrow Q$ implication law

- $P \wedge Q \equiv \neg(\neg(P \wedge Q))$ double negation
  $\equiv \neg(\neg P \vee \neg Q)$ De Morgan law
  $\equiv \neg(P \rightarrow \neg Q)$ implication law

so we can define disjunction and conjunction using implication and negation

Slight detour…

Recall also the implication law

- $P \rightarrow Q \equiv \neg P \lor Q$

it follows that

- $P \lor Q \equiv \neg(\neg P) \lor Q \equiv \neg P \rightarrow Q$
- $P \land Q \equiv \neg(\neg(P \land Q)) \equiv \neg(\neg P \lor \neg Q) \equiv \neg(P \rightarrow \neg Q)$

so we can define disjunction and conjunction using implication and negation

We can also define negation with implication (and **`false`**)

- $\neg P$ is logically equivalent to $P \rightarrow$**`false`**
  - for $P \rightarrow$**`false`** to be **`true`**, then P must be **`false`**

This is very important for proof assistants (e.g., Lean / Agda)

It follows all operators can be defined using implication (and **`false`**)

When reducing one logic formula to another…

General guidelines

- start with the more complex side and "reduce" to the simpler formula
  - if both sides complex attempt to reduce both to the same simple form

- **Eliminate**
  - almost all rules concern only conjunction, disjunction and negation
  - first remove other connectives using the corresponding rules
    that reduce them to conjunction, disjunction and negation
- **Simplify**
  - simplify using the Identity, Domination, Idempotent, Contradiction and Tautology laws
- **Shuffle**
  - to be able to simplify using the above laws, move "related" propositions close together using the Commutative, Associative, Distributive and de Morgan Laws

Example 1 – $\neg(P \lor (\neg P \land Q)) \equiv \neg P \land \neg Q$

$\neg(P \lor (\neg P \land Q)) \equiv \neg P \land \neg Q$

Recall negation has the highest precedence so this is equivalent to showing $\neg(P \lor ((\neg P) \land Q)) \equiv (\neg P) \land (\neg Q)$

Example 1 – $\neg(P \vee (\neg P \wedge Q)) \equiv \neg P \wedge \neg Q$

$\neg(P \vee (\neg P \wedge Q)) \equiv \neg P \wedge \neg Q$

- we start with either the left or right hand side and apply laws of logical equivalence to derive the other side

Example 1 – $\neg(P \lor (\neg P \land Q)) \equiv \neg P \land \neg Q$

$\neg(P \lor (\neg P \land Q)) \equiv \neg P \land \neg Q$

- we start with either the left or right hand side and apply laws of logical equivalence to derive the other side

$\neg(P \lor (\neg P \land Q)) \equiv \neg((P \lor \neg P) \land (P \lor Q))$ distributive law

Distributive law: $P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R)$

- note pattern matching from law set $P=P$, $Q=\neg P$ and $R=Q$

Example 1 – $\neg(P\vee(\neg P\wedge Q)) \equiv \neg P\wedge\neg Q$

$\neg(P\vee(\neg P\wedge Q)) \equiv \neg P\wedge\neg Q$

- we start with either the left or right hand side and apply laws of logical equivalence to derive the other side

$\neg(P\vee(\neg P\wedge Q)) \equiv \neg((P\vee\neg P)\wedge(P\vee Q))$ distributive law

$\equiv \neg(\mathbf{true}\wedge(P\vee Q))$ tautology law

Tautology law: $P\vee\neg P \equiv \mathbf{true}$

- notice have removed the extra paretheses: (**true**) replaced by **true**

Example 1 – $\neg(P \lor (\neg P \land Q)) \equiv \neg P \land \neg Q$

$\neg(P \lor (\neg P \land Q)) \equiv \neg P \land \neg Q$

- we start with either the left or right hand side and apply laws of logical equivalence to derive the other side

$\neg(P \lor (\neg P \land Q))$	$\equiv$	$\neg((P \lor \neg P) \land (P \lor Q))$	distributive law
	$\equiv$	$\neg(\mathbf{true} \land (P \lor Q))$	tautology law
	$\equiv$	$\neg((P \lor Q) \land \mathbf{true})$	commutative law

Commutative law: $P \land Q \equiv Q \land P$

- note pattern matching again $P=\mathbf{true}$ and $Q=P \lor Q$

Example 1 – $\neg(P\lor(\neg P\land Q)) \equiv \neg P\land\neg Q$

$\neg(P\lor(\neg P\land Q)) \equiv \neg P\land\neg Q$

- we start with either the left or right hand side and apply laws of logical equivalence to derive the other side

$\neg(P\lor(\neg P\land Q))$	$\equiv$	$\neg((P\lor\neg P)\land(P\lor Q))$	distributive law
	$\equiv$	$\neg(\mathtt{true}\land(P\lor Q))$	tautology law
	$\equiv$	$\neg((P\lor Q)\land\mathtt{true})$	commutative law
	$\equiv$	$\neg(P\lor Q)$	identity law

Identity law: $P \land \mathtt{true} \equiv P$

- again removed extra parentheses: $\neg((P\lor Q))$ replaced by $\neg(P\lor Q)$

Example 1 – $\neg(P \lor (\neg P \land Q)) \equiv \neg P \land \neg Q$

$\neg(P \lor (\neg P \land Q)) \equiv \neg P \land \neg Q$

- we start with either the left or right hand side and apply laws of logical equivalence to derive the other side

$\neg(P \lor (\neg P \land Q))$	$\equiv$	$\neg((P \lor \neg P) \land (P \lor Q))$	distributive law
	$\equiv$	$\neg(\mathrm{true} \land (P \lor Q))$	tautology law
	$\equiv$	$\neg((P \lor Q) \land \mathrm{true})$	commutative law
	$\equiv$	$\neg(P \lor Q)$	identity law
	$\equiv$	$\neg P \land \neg Q$	De Morgan law

De Morgan law: $\neg(P \lor Q) \equiv \neg P \land \neg Q$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv$ `true`

We start with either the left or right hand side and apply laws of logical equivalence to derive the other side

- when showing something is equivalent to **`true`** or **`false`** easier to start with that side
- otherwise hard to know where to begin
  - i.e. which law to apply first

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \texttt{true}$

$(P \wedge Q) \rightarrow (P \vee Q)$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \mathtt{true}$

$(P \wedge Q) \rightarrow (P \vee Q) \quad \equiv \quad \neg(P \wedge Q) \vee (P \vee Q)$ implication law

Implication law: $P \rightarrow Q \equiv \neg P \vee Q$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \mathbf{true}$

$(P \wedge Q) \rightarrow (P \vee Q) \equiv \neg(P \wedge Q) \vee (P \vee Q)$ implication law

$\equiv (\neg P \vee \neg Q) \vee (P \vee Q)$ De Morgan law

De Morgan law: $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \mathbf{true}$

$(P \wedge Q) \rightarrow (P \vee Q)$	$\equiv$	$\neg(P \wedge Q) \vee (P \vee Q)$	implication law
	$\equiv$	$(\neg P \vee \neg Q) \vee (P \vee Q)$	De Morgan law
	$\equiv$	$\neg P \vee (\neg Q \vee (P \vee Q))$	associative law

Associative law: $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

- pattern matching $P = \neg P$, $Q = \neg Q$ and $R = (P \vee Q)$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \mathbf{true}$

$(P \wedge Q) \rightarrow (P \vee Q)$	$\equiv$	$\neg(P \wedge Q) \vee (P \vee Q)$	implication law
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	$\equiv$	$\neg P \vee (\neg Q \vee (P \vee Q))$	associative law
	$\equiv$	$\neg P \vee (\neg Q \vee (Q \vee P))$	commutative law

Commutative law: $P \vee Q \equiv Q \vee P$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \mathtt{true}$

$(P \wedge Q) \rightarrow (P \vee Q)$	$\equiv$	$\neg(P \wedge Q) \vee (P \vee Q)$	implication law
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	$\equiv$	$\neg P \vee (\neg Q \vee (P \vee Q))$	associative law
	$\equiv$	$\neg P \vee (\neg Q \vee (Q \vee P))$	commutative law
	$\equiv$	$\neg P \vee ((\neg Q \vee Q) \vee P)$	associative law

Associative law: $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \texttt{true}$

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	$\equiv$	$\neg P \vee (\neg Q \vee (Q \vee P))$	commutative law
	$\equiv$	$\neg P \vee ((\neg Q \vee Q) \vee P)$	associative law
	$\equiv$	$\neg P \vee (\texttt{true} \vee P)$	tautology & commutative

Tautology and commutative laws: $\neg P \vee P \equiv P \vee \neg P \equiv \texttt{true}$

- can apply two laws at once if not confusing

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \mathbf{true}$

$(P \wedge Q) \rightarrow (P \vee Q)$	$\equiv$	$\neg(P \wedge Q) \vee (P \vee Q)$	implication law
	$\equiv$	$(\neg P \vee \neg Q) \vee (P \vee Q)$	De Morgan law
	$\equiv$	$\neg P \vee (\neg Q \vee (P \vee Q))$	associative law
	$\equiv$	$\neg P \vee (\neg Q \vee (Q \vee P))$	commutative law
	$\equiv$	$\neg P \vee ((\neg Q \vee Q) \vee P)$	associative law
	$\equiv$	$\neg P \vee (\mathbf{true} \vee P)$	tautology & commutative
	$\equiv$	$\neg P \vee \mathbf{true}$	domination & commutative

Domination and commutative laws: $\mathbf{true} \vee P \equiv P \vee \mathbf{true} \equiv \mathbf{true}$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \text{true}$

$(P \wedge Q) \rightarrow (P \vee Q)$	$\equiv$	$\neg(P \wedge Q) \vee (P \vee Q)$	implication law
	$\equiv$	$(\neg P \vee \neg Q) \vee (P \vee Q)$	De Morgan law
	$\equiv$	$\neg P \vee (\neg Q \vee (P \vee Q))$	associative law
	$\equiv$	$\neg P \vee (\neg Q \vee (Q \vee P))$	commutative law
	$\equiv$	$\neg P \vee ((\neg Q \vee Q) \vee P)$	associative law
	$\equiv$	$\neg P \vee (\text{true} \vee P)$	tautology & commutative
	$\equiv$	$\neg P \vee \text{true}$	domination & commutative
	$\equiv$	true	domination law

Domination law: $P \vee \text{true} \equiv \text{true}$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \texttt{true}$

Derivation is not necessarily unique and can often show equivalence a number of different ways

- one way might require less steps
- but this does not mean it is any more correct

Example 2 – $(P \land Q) \rightarrow (P \lor Q) \equiv \texttt{true}$

$(P \land Q) \rightarrow (P \lor Q) \quad \equiv \quad \neg(P \land Q) \lor (P \lor Q)$ implication law

Implication law: $P \rightarrow Q \equiv \neg P \lor Q$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \mathbf{true}$

$(P \wedge Q) \rightarrow (P \vee Q) \equiv \neg(P \wedge Q) \vee (P \vee Q)$ implication law

$\equiv (\neg P \vee \neg Q) \vee (P \vee Q)$ De Morgan law

De Morgan law: $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \mathbf{true}$

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	$\equiv$	$(\neg P \vee \neg Q) \vee (P \vee Q)$	De Morgan law
	$\equiv$	$((\neg P \vee \neg Q) \vee P) \vee Q$	associative law

Associative law: $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

- now applied the other way around
- pattern matching $P = (\neg P \vee \neg Q)$, $Q = P$ and $R = Q$

Example 2 – $(P \wedge Q) \rightarrow (P \vee Q) \equiv \mathtt{true}$

$(P \wedge Q) \rightarrow (P \vee Q)$	$\equiv$	$\neg(P \wedge Q) \vee (P \vee Q)$	implication law
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Commutative law: $P \vee Q \equiv Q \vee P$

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Associative law: $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

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Tautology and commutative laws: $\neg P \vee P \equiv P \vee \neg P \equiv \mathbf{true}$

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	$\equiv$	$(\neg Q \vee \mathbf{true}) \vee Q$	tautology & commutative
	$\equiv$	$\mathbf{true} \vee Q$	domination & commutative

Domination and commutative laws: $\mathbf{true} \vee P \equiv P \vee \mathbf{true} \equiv \mathbf{true}$

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	$\equiv$	$(\neg Q \vee \mathbf{true}) \vee Q$	tautology & commutative
	$\equiv$	$\mathbf{true} \vee Q$	domination & commutative
	$\equiv$	$\mathbf{true}$	domination & commutative

Domination and commutative laws: $\mathbf{true} \vee P \equiv P \vee \mathbf{true} \equiv \mathbf{true}$

Example 3

If we have `true=1` and `false=0`, then we can define negation and conjuction by:

- **`not`**`(P) = 1-P`
- **`and`**`(P,Q) = P·Q`

But what about disjunction, i.e. what is `or(P,Q)`?

- be careful, what is P+Q?

Can be derived using logical equivalences

- by expressing disjunction in terms of a logically equivalent formula including only conjunction and negation
- then applying above definitions for **`not`**`(P)` and **`and`**`(P,Q)`

Example 3

We showed earlier that $P \lor Q \equiv \neg(\neg P \land \neg Q)$

Using this equivalence we have

```
or(P,Q)   = not(and(not(P),not(Q)))
```

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We showed earlier that $P \lor Q \equiv \neg(\neg P \land \neg Q)$

Using this equivalence we have

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or(P,Q)   = not(and(not(P),not(Q))
          = not(and(1-P,1-Q))
```

Using `not(P) = 1-P`

Example 3

We showed earlier that $P \lor Q \equiv \neg(\neg P \land \neg Q)$

Using this equivalence we have

```
or(P,Q)   = not(and(not(P),not(Q))
          = not(and(1-P,1-Q))
          = not((1-P)·(1-Q))
```

Using `and(P,Q) = P·Q`

Example 3

We showed earlier that $P \vee Q \equiv \neg(\neg P \wedge \neg Q)$

Using this equivalence we have

```
or(P,Q)   = not(and(not(P),not(Q)))
          = not(and(1-P,1-Q))
          = not((1-P)·(1-Q))
          = 1 - (1-P)·(1-Q)
```

Using `not(P) = 1–P`

Example 3

We showed earlier that $P \lor Q \equiv \neg(\neg P \land \neg Q)$

Using this equivalence we have

$$
\begin{aligned}
\mathtt{or(P,Q)} &= \mathtt{not(and(not(P),not(Q))} \\
&= \mathtt{not(and(1-P,1-Q))} \\
&= \mathtt{not((1-P)\cdot(1-Q))} \\
&= 1 - (1-P)\cdot(1-Q) \\
&= 1 - (1 - P - Q + P\cdot Q)
\end{aligned}
$$

Example 3

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Using this equivalence we have

```
or(P,Q)   = not(and(not(P),not(Q)))
          = not(and(1-P,1-Q))
          = not((1-P)·(1-Q))
          = 1 - (1-P)·(1-Q)
          = 1 - (1 - P - Q + P·Q)
          = P + Q - P·Q
```

Summary

Atomic propositions: p, q, r

Boolean connectives: $\neg \; \wedge \; \vee \; \oplus \; \rightarrow \; \leftrightarrow$

Compound propositions

- e.g. $P = (p \wedge \neg q) \vee r$

Logical equivalences and laws of equivalence

- (there is a pdf on moodle summarising all the laws)

Proving equivalences: use truth tables or laws of logical equivalence

Please look at the LaTeX exercise

LaTeX is a typesetting system that makes it easy to write out mathematical formulae

You will need to submit the answers to your assessed exercises using LaTeX

The first week's tutorial is a LaTeX exercise (designed by 4^{th} year students in the CS Education course).

- Designed to be entertaining!