Information Processing Letters 115 (2015) 104–111

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

The hamiltonicity of generalized honeycomb torus networks

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ARTICLE INFO

Article history:
Received 18 November 2013
Received in revised form 21 July 2014
Accepted 21 July 2014
Available online 11 August 2014
Communicated by M. Chrobak

Keywords:
Interconnection network
Generalized honeycomb torus
Hamiltonian cycle
Parallel computing

ABSTRACT

Yang et al. (2004) [8] proved that the generalized honeycomb torus $GHT(m, n, d)$ is hamiltonian, but their proofs are not sufficient when the width m is odd. In this paper, we propose a series of procedures for constructing hamiltonian cycles in generalized honeycomb tori, which apply to every instance of $GHT(m, n, d)$ with odd width m.

1. Introduction

The advent of very large scale integrated circuit technology has enabled the construction of very complex and large parallel computing systems. By most accounts, a supercomputer achieves its gains by increasing the number of processing elements, rather than by using faster processors. The most difficult technical problem in constructing a supercomputer will be the design of the interconnection network through which the processors communicate and exchange data with each other. Therefore, selecting an appropriate and adequate topological structure of interconnection networks will become a critical issue in the field of parallel computing [6].

There exist a lot of mutually conflicting requirements in designing the topology of an interconnection network, such that it is almost impossible to design a network which is optimal from all aspects. One has to design a suitable network according to the requirements and its properties. Many efficient algorithms were originally designed based on rings for solving a variety of algebraic problems, graph problems and some parallel applications, such as those in image and signal processing. Thus, it is important to have an effective cycle in a network, preferably the hamiltonian cycle.

Stojmenovic [7] introduced three classes of honeycomb torus networks: hexagonal honeycomb torus, rectangular honeycomb torus and parallelogramic honeycomb torus. Megson et al. [4,5] proved that a hexagonal honeycomb torus is hamiltonian and fault-tolerant hamiltonian with two adjacent faulty vertices. Cho and Hsu [1] proposed the generalized honeycomb torus, which includes the above mentioned honeycomb tori as special instances. Yang et al. [8] proved that all generalized honeycomb tori are hamiltonian. However, we found that their proofs are not sufficient when the width m is odd.

In this paper, we propose a series of procedures for constructing the hamiltonian cycles in generalized honeycomb tori, which apply to every instance of $GHT(m, n, d)$ with odd width m. The rest of this paper is organized as follows. Section 2 gives definitions and notations. Section 3 presents the main result of the paper. Section 4 makes the concluding remarks.

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http://dx.doi.org/10.1016/j.ipl.2014.07.011

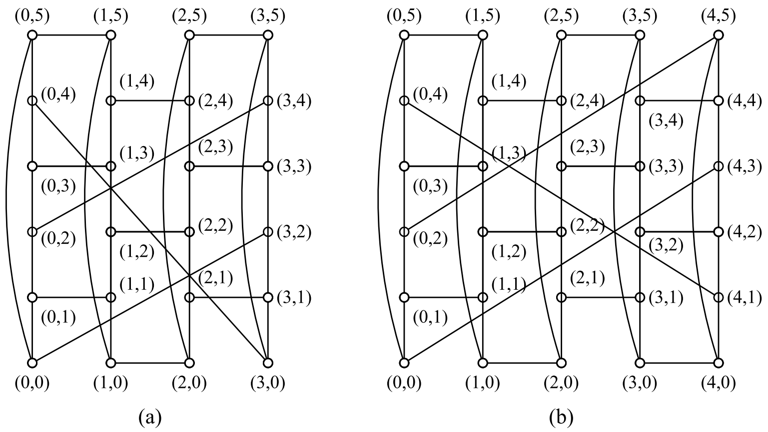


Fig. 1. Two examples of generalized honeycomb tori. (a) $GHT(4,6,2)$. (b) $GHT(5,6,3)$.

2. Definitions and notations

The topological structure of an interconnection network can be modeled by a graph $G=(V,E)$, where vertices and edges correspond to processors and communication links between processors, respectively. This fact has been universally accepted and used by computer scientists and engineers. Moreover, practically it has been demonstrated that graph theory is a fundamental and powerful mathematical tool for designing and analyzing topological structure of interconnection networks.

A hamiltonian cycle of a graph is a cycle that traverses every vertex of the graph exactly once. A graph is hamiltonian if it contains a hamiltonian cycle. We follow [2] for graph-theoretical terminology and notations not defined here.

Definition 2.1. (See [1].) Let n be a positive even integer, $m \geq 2$ be a positive integer, and d be a nonnegative integer which is less than n and of the same parity with m. An (m,n,d) generalized honeycomb torus, denoted by $GHT(m,n,d)$, is a graph with the vertex set

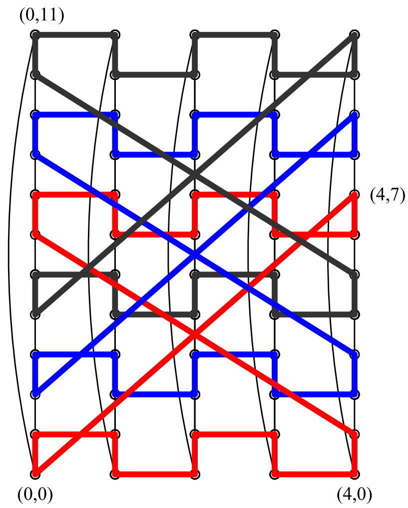
$$V\big(GHT(m,n,d)\big)=\big\{(i,j):0 \leq i \leq m-1, 0 \leq j \leq n-1\big\}.$$

m, n and d are named the width, height and slope of $GHT(m,n,d)$. For a vertex (i,j) of $GHT(m,n,d)$, i and j are called its first and the second component, respectively. Here and in what follows, all arithmetic operations carried out on the first and second components are modulo m and n, respectively. Two vertices (x_1,y_1) and (x_2,y_2) with $x_1 \leq x_2$ are adjacent if and only if one of the follow conditions is satisfied:

(1) $(x_2,y_2)=(x_1,y_1+1)$ or $(x_2,y_2)=(x_1,y_1-1)$;
(2) $0 \leq x_1 \leq m-2$, x_1+y_1 is odd, and $(x_2,y_2)=(x_1+1,y_1)$;
(3) $x_1=0$, y_1 is even, and $(x_2,y_2)=(m-1,y_1+d)$.

Fig. 1 gives two examples of generalized honeycomb tori. We can easily see that generalized honeycomb tori are 3-regular bipartite graphs.

Yang et al. [8] proved that every generalized honeycomb torus is hamiltonian. However, when m is odd, their scheme for constructing hamiltonian cycles is only valid when $gcd(\frac{2n}{gcd(n,d+1)},\frac{2(d+1)}{gcd(n,d+1)-1})=1$. For example in $GHT(5,12,7)$, they get three vertex-disjoint cycles instead of a hamiltonian cycle (see Fig. 2).



GHT(5,12,7)

Fig. 2. Three vertex-disjoint cycles in $GHT(5,12,7)$.

Definition 2.2. Given two positive integers a and b where $a \geq b$, $GC(a,b)$ is a graph defined by the vertex set $\{0,1,\ldots,a-1\}$ and the edge set $\{(i,i+b):0 \leq i \leq a-1\}$, where the arithmetic is modulo a [8].

Given two positive integers a and b where $a \geq b$, let $gcd(a,b)$ denote the greatest common divisor of a and b. The following lemmas will be useful in this paper.

Lemma 2.1. *If $gcd(a,b)=1$, then $GC(a,b)$ is a cycle [8].*

Lemma 2.2. *If $gcd(a,b)=c \geq 2$, then $GC(a,b)$ is composed of c vertex-disjoint cycles $\langle 0,b,2b,3b,\ldots,(a/c-1)b,0\rangle$, $\langle 1,b+1,2b+1,3b+1,\ldots,(a/c-1)b+1,1\rangle$, $\langle 2,b+2,2b+2,3b+2,\ldots,(a/c-1)b+2,2\rangle,\ldots,$ and $\langle c-1,b+c-1,2b+c-1,3b+c-1,\ldots,(a/c-1)b+c-1,c-1\rangle$.*

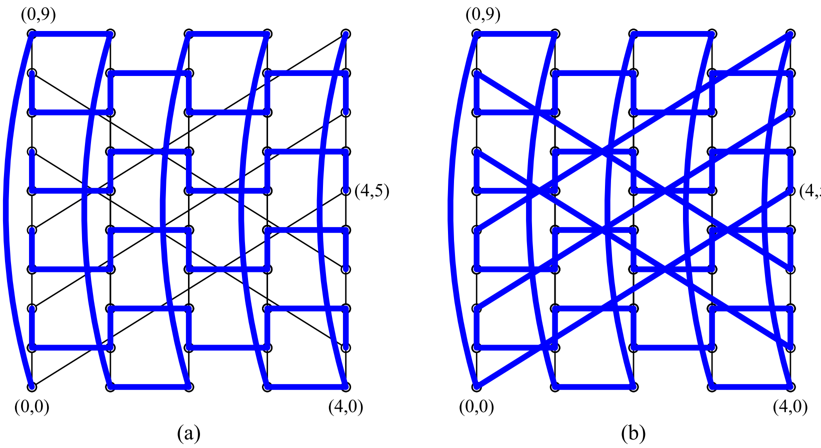


Fig. 3. Illustration of Procedure ODD_HC_1 on $GHT(5, 10, 5)$: (a) Step 2, (b) Step 3.

Proof. Note that $gcd(a/c, b) = 1$, then by Lemma 2.1, the sequence $\langle i, b+i, 2b+i, 3b+i, \dots, (a/c-1)b+i, i\rangle$ forms a closed cycle, where $0 \le i \le c-1$. These c cycles are clearly vertex-disjoint, thus the lemma is proved. □

Next we will give some useful notations of paths and cycles in $GHT(m, n, d)$.

Given two vertices (i, j) and (i, k) of $GHT(m, n, d)$, the path $(i, j) \to (i, j+1) \to (i, j+2) \to \dots \to (i, k)$ is denoted as $(i, j) \uparrow (i, k)$, and the path $(i, j) \to (i, j-1) \to (i, j-2) \to \dots \to (i, k)$ is denoted as $(i, j) \downarrow (i, k)$.

Given a vertex (i, j) of $GHT(m, n, d)$, where $i + j$ is odd. Then the hexagonal cycle $(i, j) \uparrow (i, j+2) \to (i+1, j+2) \downarrow (i+1, j) \to (i, j)$ is denoted as $C_{hex}(i, j)$.

Let k and h be two positive even integers satisfying $0 \le k \le n-2$ and $2 \le h \le n$. For odd m, $GHT(m, n, d)$ contains a path $P(k, h)$ starting from vertex $(0, k)$ and terminating at vertex $(m-1, k+h-1)$, and a path $Q(k, h)$ starting from vertex $(0, k)$ and terminating at vertex $(m-1, k-h+1)$:

$$\begin{aligned} P(k,h) &= (0,k) \uparrow (0,k+h-1) \to (1,k+h-1) \downarrow (1,k) \\ &\quad \to (2,k) \uparrow (2,k+h-1) \to (3,k+h-1) \\ &\quad \downarrow (3,k) \to (4,k) \uparrow (4,k+h-1) \to \dots \\ &\quad \to (m-1,k+h-1), \\ Q(k,h) &= (0,k) \downarrow (0,k-h+1) \to (1,k-h+1) \uparrow (1,k) \\ &\quad \to (2,k) \downarrow (2,k-h+1) \to (3,k-h+1) \\ &\quad \uparrow (3,k) \to (4,k) \downarrow (4,k-h+1) \to \dots \\ &\quad \to (m-1,k-h+1). \end{aligned}$$

Definition 2.3. A path decomposition of graph G is a set of vertex-disjoint paths $P_1, P_2, \dots, P_k$ in G satisfying $\bigcup_{i=1}^{k} V(P_i) = V(G)$, where $V(P_i)$ denotes the set of vertices on P_i.

Figs. 3(a), 4(a) and 5(a) give three examples of path decomposition of generalized honeycomb tori.

3. Construction of hamiltonian cycles

In this section, we present a series of procedures which can effectively construct hamiltonian cycles in $GHT(m, n, d)$ with odd m. The discussion will proceed by distinguishing the following three cases.

Case 1. $gcd(n, d+1) = 2$.

Procedure ODD_HC_1
INPUT: $GHT(m, n, d)$ where m is odd and $gcd(n, d+1) = 2$.
OUTPUT: The edge set E of a hamiltonian cycle in $GHT(m, n, d)$.
BEGIN
Step 1. Set E as an empty set.
Step 2. Add the edges of paths $\{Q(2i, 2) : 0 \le i \le n/2 - 1\}$ into E.
Step 3. Add into E the edges $\{((0, 2i), (m-1, 2i+d)) : 0 \le i \le n/2 - 1\}$.
END

Fig. 3 illustrates the steps of Procedure ODD_HC_1 on $GHT(5, 10, 5)$.

Lemma 3.1. *If m is odd, then the paths $\{Q(2i, 2) : 0 \le i \le n/2 - 1\}$ constitute a path decomposition of $GHT(m, n, d)$.*

Theorem 3.2. *Procedure ODD_HC_1 produces a hamiltonian cycle in $GHT(m, n, d)$, where m is odd and $gcd(n, d+1) = 2$.*

Proof. By Lemma 3.1, the paths $\{Q(2i, 2) : 0 \le i \le n/2 - 1\}$ constitute a path decomposition of $GHT(m, n, d)$. It is easy to verify that every edge $((0, 2i), (m-1, 2i+d))$ connects $Q(2i, 2)$ to $Q(2i+d+1, 2)$ of $GHT(m, n, d)$.

We construct a graph $GC(\frac{n}{2}, \frac{d+1}{2})$ in the way given in Definition 2.2. By Lemma 2.1, $GC(\frac{n}{2}, \frac{d+1}{2})$ is a cycle with the sequence of neighboring vertices $\langle 0, \frac{d+1}{2}, 2 \times \frac{d+1}{2}, \dots, (\frac{n}{2}-1) \times \frac{d+1}{2}, 0\rangle$.

For each i with $0 \le i \le \frac{n}{2} - 1$, if each path $Q(2i, 2)$ of $GHT(m, n, d)$ is mapped to the vertex i of $GC(\frac{n}{2}, \frac{d+1}{2})$, then

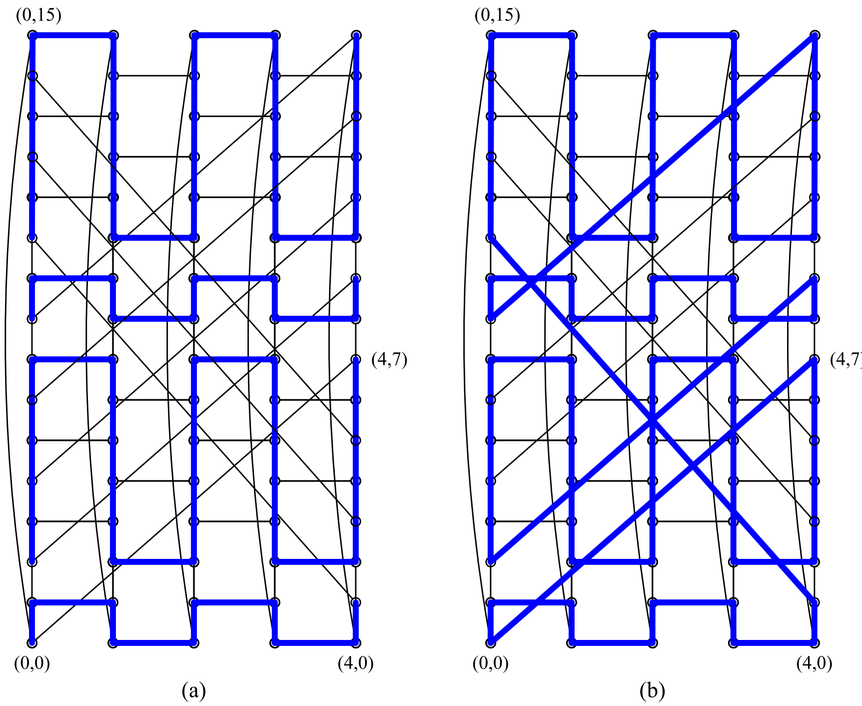


Fig. 4. Illustration of Procedure ODD_HC_2_1 on $GHT(5, 16, 7)$: (a) Step 2, (b) Step 3.

the edges $((0, 2i), (m-1, 2i+d))$ of $GHT(m, n, d)$ are exactly corresponding to the edges $(i, i+\frac{d+1}{2})$ of $GC(\frac{n}{2}, \frac{d+1}{2})$. Therefore, Procedure ODD_HC_1 produces a hamiltonian cycle in $GHT(m, n, d)$ with odd m. $\square$

Case 2. $gcd(n, d+1) = hgt \geq 6$.

For convenience, let $p = \frac{n}{gcd(n,d+1)}$ and $q = \frac{d+1}{gcd(n,d+1)}$.

Case 2.1. $gcd(2p, 2q-1) = 1$.

Procedure ODD_HC_2_1
INPUT: $GHT(m, n, d)$ where m is odd, $gcd(n, d+1) = hgt \geq 6$ and $gcd(2p, 2q-1) = 1$.
OUTPUT: The edge set E of a hamiltonian cycle in $GHT(m, n, d)$.
BEGIN

Step 1. Set E as an empty set.

Step 2. Add the edges of paths $\{P(i \times hgt, 2) : 0 \leq i \leq p-1\} \cup \{P(i \times hgt + 2, hgt - 2) : 0 \leq i \leq p-1\}$ into E.

Step 3. Add the edges $\{((0, i \times hgt), (m-1, i \times hgt + d)) : 0 \leq i \leq p-1\} \cup \{((0, i \times hgt + 2), (m-1, i \times hgt + 2 + d)) : 0 \leq i \leq p-1\}$ into E.

END

Fig. 4 illustrates the steps of Procedure ODD_HC_2_1 on $GHT(5, 16, 7)$.

Lemma 3.3. *If m is odd, then the paths $\{P(i \times hgt, 2) : 0 \leq i \leq p-1\} \cup \{P(i \times hgt + 2, hgt - 2) : 0 \leq i \leq p-1\}$ constitute a path decomposition of $GHT(m, n, d)$.*

Theorem 3.4. *Procedure ODD_HC_2_1 produces a hamiltonian cycle in $GHT(m, n, d)$, where m is odd, $gcd(n, d+1) = hgt \geq 6$ and $gcd(2p, 2q-1) = 1$.*

Proof. By Lemma 3.3, the paths $\{P(i \times hgt, 2) : 0 \leq i \leq p-1\} \cup \{P(i \times hgt + 2, hgt - 2) : 0 \leq i \leq p-1\}$ constitute a path decomposition of $GHT(m, n, d)$. It is easy to verify that every edge $((0, i \times hgt), (m-1, i \times hgt + d))$ connects $P(i \times hgt, 2)$ to $P((i+q-1) \times hgt + 2, hgt - 2)$ of $GHT(m, n, d)$.

We construct a graph $GC(2p, 2q-1)$ in the way given in Definition 2.2. By Lemma 2.1, $GC(2p, 2q-1)$ is a cycle with the sequence of neighboring vertices $\langle 0, 2q-1, 2 \times (2q-1), 3 \times (2q-1), \ldots, (2p-1)(2q-1), 0 \rangle$.

For each i with $0 \leq i \leq p-1$, if each path $P(i \times hgt, 2)$ ($P(i \times hgt + 2, hgt - 2)$, respectively) of $GHT(m, n, d)$ is mapped to the vertex $2i$ ($2i+1$, respectively) of $GC(2p, 2q-1)$, then the edges $((0, i \times hgt), (m-1, i \times hgt + d))$ ($((0, i \times hgt + 2), (m-1, i \times hgt + 2 + d))$, respectively) of $GHT(m, n, d)$ are exactly corresponding to the edges $(2i, 2i + 2q - 1)$ ($(2i+1, 2i+2q)$, respectively) of $GC(2p, 2q-1)$. Therefore, Procedure ODD_HC_2_1 produces a hamiltonian cycle in $GHT(m, n, d)$ with odd m. $\square$

Case 2.2. $gcd(2p, 2q-1) = noc \geq 3$.

Procedure ODD_HC_2_2
INPUT: $GHT(m, n, d)$ where m is odd, $gcd(n, d+1) = hgt \geq 6$ and $gcd(2p, 2q-1) = noc \geq 3$.
OUTPUT: The edge set E of a hamiltonian cycle in $GHT(m, n, d)$.
BEGIN

Steps 1–3 are the same as those of Procedure ODD_HC_2_1.

Step 4. For $k = 0$ to $\frac{noc-3}{2}$, do the following edge-exchange operations:

Substep 4.1. Look at the 6-cycle $C_{hex}(0, k \times hgt + 1)$. Exactly 3 pairwise nonadjacent edges on the cycle

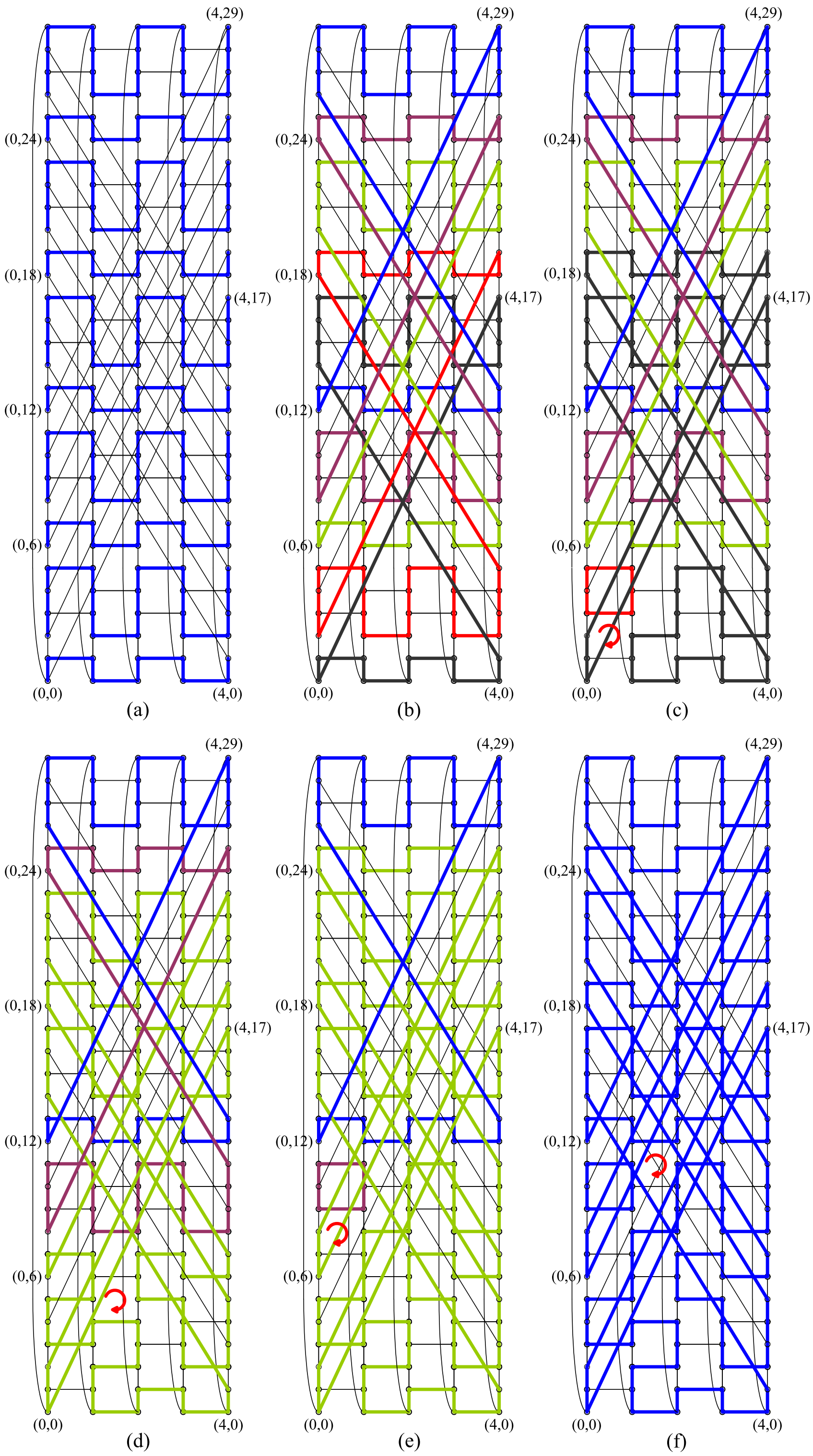


Fig. 5. Illustration of Procedure ODD_HC_2_2 on $GHT(5, 30, 17)$: (a) Step 2, (b) Step 3, (c) Substep 4.1 for $k = 0$, (d) Substep 4.2 for $k = 0$, (e) Substep 4.1 for $k = 1$, (f) Substep 4.2 for $k = 1$.

belong to E. Remove these three edges from E, and add the remaining three edges on the cycle to E.
Substep 4.2. Look at the 6-cycle $C_{hex}(1, (k+1) \times hgt - 2)$. Exactly 3 pairwise nonadjacent edges on the cycle belong to E. Remove these three edges from E, and add the remaining three edges on the cycle to E.

END

Fig. 5 illustrates the steps of Procedure ODD_HC_2_2 on $GHT(5, 30, 17)$.

Theorem 3.5. *The ODD_HC_2_2 procedure produces a hamiltonian cycle in $GHT(m, n, d)$ where m is odd, $gcd(n, d+1) = hgt \geq 6$ and $gcd(2p, 2q-1) = noc \geq 3$.*

Proof. By Lemma 3.3, the paths $\{P(i \times hgt, 2) : 0 \leq i \leq p-1\} \cup \{P(i \times hgt + 2, hgt - 2) : 0 \leq i \leq p-1\}$ constitute a path decomposition of $GHT(m, n, d)$. It is easy to verify that every edge $((0, i \times hgt), (m-1, i \times hgt + d))$ connects $P(i \times hgt, 2)$ to $P((i+q-1) \times hgt + 2, hgt - 2)$ of $GHT(m, n, d)$, and every edge $((0, i \times hgt + 2), (m-1, i \times hgt + 2 + d))$ connects $P(i \times hgt + 2, hgt - 2)$ to $P((i+q) \times hgt, 2)$ of $GHT(m, n, d)$.

We construct a graph $GC(2p, 2q-1)$ in the way given in Definition 2.2. By Lemma 2.2, $GC(2p, 2q-1)$ is composed of noc vertex-disjoint cycles $\langle 0, 2q-1, 2 \times (2q-1), 3 \times (2q-1), \ldots, (2p/noc - 1) \times (2q-1), 0\rangle$, $\langle 1, (2q-1)+1, 2 \times (2q-1)+1, 3 \times (2q-1)+1, \ldots, (2p/noc-1) \times (2q-1)+1, 1\rangle$, $\langle 2, (2q-1)+2, 2 \times (2q-1)+2, 3 \times (2q-1)+2, \ldots, (2p/noc-1) \times (2q-1)+2, 2\rangle, \ldots$, and $\langle noc-1, (2q-1)+noc-1, 2 \times (2q-1)+noc-1, 3 \times (2q-1)+noc-1, \ldots, (2p/noc-1) \times (2q-1)+noc-1, noc-1\rangle$.

For each i with $0 \leq i \leq p-1$, if each path $P(i \times hgt, 2)$ ($P(i \times hgt + 2, hgt - 2)$, respectively) of $GHT(m, n, d)$ is mapped to the vertex $2i$ ($2i+1$, respectively) of $GHT(m, n, d)$, then the edges $((0, i \times hgt), (m-1, i \times hgt + d))$ ($((0, i \times hgt + 2), (m-1, i \times hgt + 2 + d))$, respectively) of $GHT(m, n, d)$ are exactly corresponding to the edges $(2i, 2i+2q-1)$ ($(2i+1, 2i+2q)$, respectively) of $GC(2p, 2q-1)$. In this way, we get a family of noc vertex-disjoint cycles in $GHT(m, n, d)$, which are

$$C_0 = P(0,2) \to P\big((q-1) \times hgt + 2, hgt - 2\big) \to P\big((2q-1) \times hgt, 2\big) \to P\big((3q-2) \times hgt + 2, hgt - 2\big) \to \ldots \to P\left(\left(2q * \frac{p}{noc} - \frac{p}{noc} - q\right) \times hgt + 2, hgt - 2\right) \to P(0,2),$$

$$C_1 = P(2, hgt-2) \to P(q \times hgt, 2) \to P\big((2q-1) \times hgt + 2, hgt - 2\big) \to P\big((3q-1) \times hgt, 2\big) \to \ldots \to P\left(\left(2q * \frac{p}{noc} - \frac{p}{noc} - q + 1\right) \times hgt, 2\right) \to P(2, hgt-2),$$

$$C_2 = P(hgt, 2) \to P(q \times hgt + 2, hgt - 2) \to P(2q \times hgt, 2) \to P\big((3q-1) \times hgt + 2, hgt - 2\big) \to \ldots \to P\left(\left(2q * \frac{p}{noc} - \frac{p}{noc} - q + 1\right) \times hgt + 2, hgt - 2\right) \to P(hgt, 2), \quad \text{and}$$

$\vdots$

$$C_{noc-1} = P\left(\frac{noc-1}{2} \times hgt, 2\right) \to P\left(\frac{2q+noc-3}{2} \times hgt + 2, hgt - 2\right) \to P\left(\frac{4q+noc-3}{2} \times hgt, 2\right) \to P\left(\frac{6q+noc-5}{2} \times hgt + 2, hgt - 2\right) \to \ldots \to P\left(\frac{4q * \frac{p}{noc} - 2\frac{p}{noc} - 2q + noc - 1}{2} \times hgt + 2, hgt - 2\right) \to P\left(\frac{noc-1}{2} \times hgt, 2\right).$$

In Substep 4.1 most edges of C_{2k+1} are merged into a longer cycle C_{2k}, and C_{2k+1} is truncated into a shorter cycle (see Fig. 6 for illustration). In Substep 4.2, the two cycles C_{2k}, C_{2k+1} are incorporated into a longer cycle C_{2k+2} in the edge-exchange way shown in Fig. 7. Repeating Step 4 until $k = \frac{noc-3}{2}$, we get a hamiltonian cycle in $GHT(m, n, d)$. □

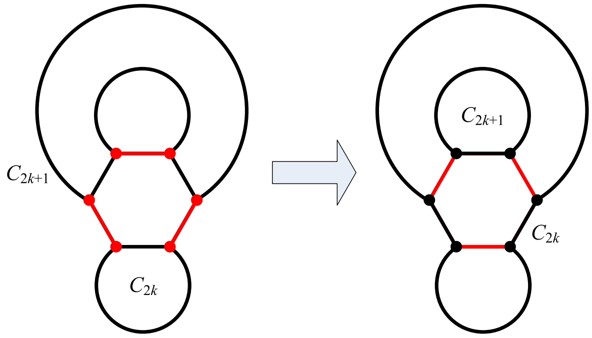


Fig. 6. Edge reassignment on two cycles through edge-exchange operation on a shared 6-cycle.

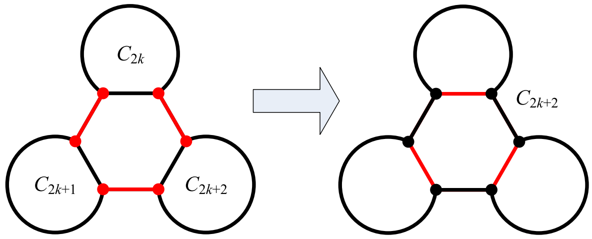


Fig. 7. Merge three disjoint cycles into a single one through edge-exchange operation on a shared 6-cycle.

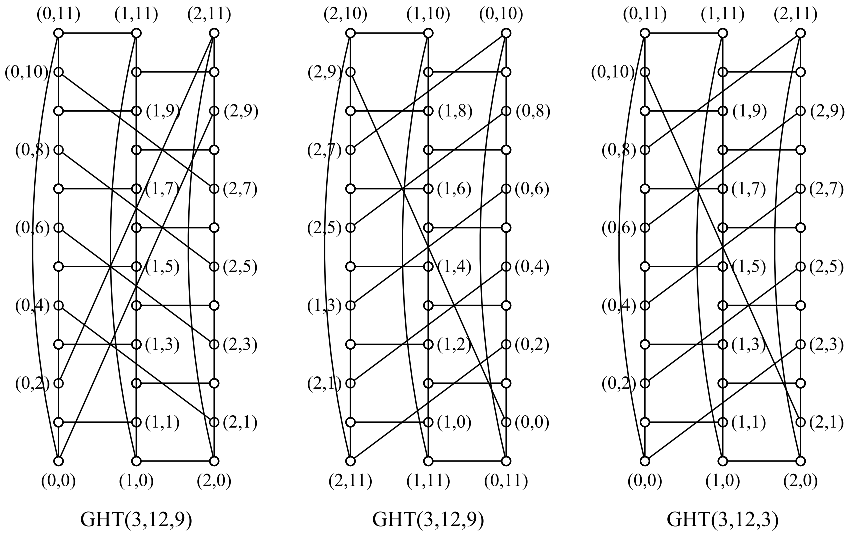


Fig. 8. Isomorphism from $GHT(3, 12, 9)$ to $GHT(3, 12, 3)$.

Case 3. $gcd(n, d+1) = 4$.

Lemma 3.6. *Let a and b be two positive integers with $a \geq b+1$. If $gcd(a, b+1) = 4$, then $gcd(a, a-b+1) \neq 4$.*

Proof. Since $gcd(a, b+1) = 4$, we can assume that $a = 4p$ and $b+1 = 4q$, where p and q are two positive integers. Clearly, $a - b + 1 = 4p - (4q-1) + 1 = 4(p-q) + 2$ is not a multiple of 4, therefore $gcd(a, a-b+1) \neq 4$. $\square$

Theorem 3.7. *$GHT(m, n, d)$ is isomorphic to $GHT(m, n, n-d)$.*

Proof. Let h be a mapping from the vertex set of $GHT(m, n, d)$ into the vertex set of $GHT(m, n, n-d)$ by setting $h(x, y) = (x', y') = (m-1-x, y-1)$. It is easy to check that h is a one-to-one mapping from the vertex set of $GHT(m, n, d)$ to the vertex set of $GHT(m, n, n-d)$.

To prove that h is an isomorphism, we need to check that h preserves the adjacency, that is, for each edge $((x_1, y_1), (x_2, y_2)) \in E(GHT(m, n, d))$, $((x'_1, y'_1), (x'_2, y'_2)) \in E(GHT(m, n, n-d))$ is true. According to Definition 2.1, we discuss the following three scenarios.

Scenario 1. $(x_2, y_2) = (x_1, y_1+1)$ or $(x_2, y_2) = (x_1, y_1-1)$.

It is easy to verify that $(x'_2, y'_2) = (x'_1, y'_1+1)$ if $(x_2, y_2) = (x_1, y_1+1)$, and $(x'_2, y'_2) = (x'_1, y'_1-1)$ if $(x_2, y_2) = (x_1, y_1-1)$. According to Definition 2.1, vertices (x'_1, y'_1) and (x'_2, y'_2) are adjacent in $GHT(m, n, n-d)$.

Scenario 2. $0 \leq x_1 \leq m-2$, $x_1 + y_1$ is odd, and $(x_2, y_2) = (x_1+1, y_1)$.

$x'_2 + y'_2 = (m-1-x_2) + (y_2-1) = (m-1-(x_1+1)) + (y_1-1) = m + (x_1+y_1) - 2x_1 - 3$. Note that m and $x_1 + y_1$ are both odd, thus $x'_2 + y'_2$ is odd. What's more, $(x'_1, y'_1) = (m-1-x_1, y_1-1) = (m-1-x_2+1, y_2-1) = (x'_2+1, y'_2)$. According to Definition 2.1, vertices (x'_1, y'_1) and (x'_2, y'_2) are adjacent in $GHT(m, n, n-d)$.

Scenario 3. $x_1 = 0$, y_1 is even, and $(x_2, y_2) = (m-1, y_1+d)$.

$x'_2 = m-1-x_2 = m-1-(m-1) = 0$. Note that d is odd and y_1 is even, thus $y'_2 = y_2 - 1 = y_1 + d - 1$ is even. $(x'_1, y'_1) = (m-1-x_1, y_1-1) = (m-1, y_2-1-d) = (m-1, y'_2-d) = (m-1, y'_2+(n-d))$. According to Definition 2.1, vertices (x'_1, y'_1) and (x'_2, y'_2) are adjacent in $GHT(m, n, n-d)$. $\square$

Fig. 8 illustrates the isomorphism from $GHT(3, 12, 9)$ to $GHT(3, 12, 3)$.

Theorem 3.8. *If m is odd and $gcd(n, d+1) = 4$, then $GHT(m, n, d)$ is hamiltonian.*

Proof. By Lemma 3.6, we know $gcd(n, n-d+1) \neq 4$. According to Theorems 3.2, 3.4 and 3.5, $GHT(m, n, n-d)$ is hamiltonian. By Theorem 3.7, $GHT(m, n, d)$ is isomorphic to $GHT(m, n, n-d)$, thus $GHT(m, n, d)$ is hamiltonian. $\square$

Combining Theorems 3.2, 3.4, 3.5 and 3.8, we get the main theorem of this paper.

Theorem 3.9. *If m is odd, then $GHT(m, n, d)$ is hamiltonian.*

4. Concluding remarks

In this paper, we give a corrected proof on the hamiltonicity of $GHT(m, n, d)$ when m is odd. In [3], Dong et al. proved that a generalized honeycomb torus is fault-tolerant hamiltonian even with two conditional faulty vertices. We believe that the method proposed in this paper will be useful for checking the fault-tolerant hamiltonicity of a generalized honeycomb torus with any two faulty vertices belonging to different bipartite sets.

Acknowledgements

The authors would like to express their gratitude to the editor and the two anonymous referees for their valuable suggestions that have greatly improved the quality of the paper.

This work was supported by National Natural Science Foundation of China (Nos. 61300018, 61103109), Research Fund for the Doctoral Program of Higher Education of China (No. 20120185120017), China Postdoctoral Science Foundation (Nos. 2013M531951, 2014T70860), Fundamental Research Funds for the Central Universities (No. ZYGX2012J071), and Special Project of Sichuan Youth Science and Technology Innovation Research Team (No. 2013TD0006).

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