Problem Set 0

Due Monday January 12 at 11:59PM

Nature laughs at the difficulties of integration.

This is ostensibly calculus review, but each problem is a piece of probability in disguise. We will illuminate these connections throughout the semester, and I will refer to Problem Set 0 often. Stay tuned!

Problem 0

Recommend some music for us to listen to while we grade this.

Problem 1

Explain why this is horrific notation:

$$\int_0^x f(x)\,\mathrm{d}x.$$

How should it be fixed?

Problem 2

Simplify this:

$$\ln\left(e^{a_1}e^{a_2}e^{a_3}\cdots e^{a_n}\right).$$

Problem 3

Assume $\lambda > 0$ is a constant and compute

$$\sum_{n=0}^{\infty} n\frac{\lambda^n}{n!}e^{-\lambda}.$$

Problem 4

Here is a very silly function:

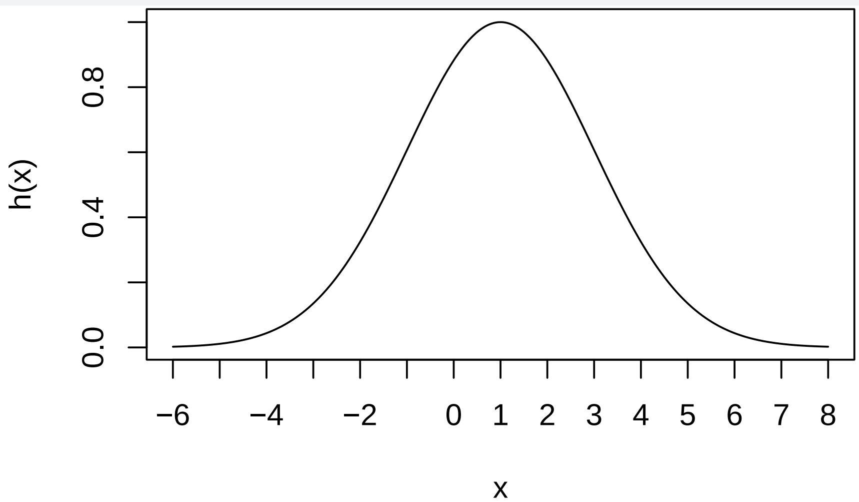
$$h(x) = \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \quad -\infty < x < \infty.$$

Treat $-\infty < \mu < \infty$ and $\sigma > 0$ as constants and compute the value(s) of x at which h has *inflection points.*

Hint

Here is an example of what h might look like in the special case where $\mu = 1$ and $\sigma = 2$:

```
m = 1
s = 2
par(mar = c(4, 4, 0.1, 4))
curve(exp(-0.5 * ((x - m) / s)^2),
      from = -6, to = 8, n = 500,
      xlab = "x", ylab = "h(x)",
      xaxt = "n")
axis(1, at = -6:8)
```



Before you start doing any math, can you use the picture to guess what the answer will be?

Problem 5

Here is another inordinately silly function:

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} \, \mathrm{d}y, \quad x > 0.$$

Prove that $\Gamma(x+1) = x\Gamma(x)$.

> **Hint**
>
> Start on the left-hand side by writing out $\Gamma(x+1)$ and evaluating the integral *by parts*.

Problem 6

Let f be any function with the following properties:

- f is twice continuously differentiable in a neighborhood of zero[1];
- $f(0) = 0$;
- $f'(0) = 0$;
- $f''(0) = 1$.

Assume t is a constant and compute

$$\lim_{x\to\infty} x f\left(\frac{t}{\sqrt{x}}\right).$$

> **Continuity?**
>
> A full credit solution must clearly explain how and why continuity is being used along the way.

Problem 7

Consider this integral:

$$\int_2^\infty \frac{1}{x(\ln x)^p} \mathrm{d}x.$$

a. Use R to create a single plot with many lines, each graphing the *integrand* for a different value of p. Consider p equal to -2, -1.5, -1, 0, 1, and 5, and make the x-axis of your plot run from 2 to 15.
b. Show that $\lim_{x\to\infty} \frac{1}{x(\ln x)^p} = 0$ for all values of $-\infty < p < \infty$.
c. For what values of p does the integral converge? When it does converge, what is its value?

[1]This means that f and its first two derivatives are all continuous functions at and around zero.

d. Consult the picture you created in part (a), and write a few sentences explaining conceptually why the integral converges for some values of p but not others.

Hint

When taking the limit or evaluating the integral, can you use the same technique for all values of p, or do you need a different technique depending on what p is?

Submission

You are free to compose your solutions for this problem set however you wish (scan or photograph written work, handwriting capture on a tablet device, LaTeX, Quarto, whatever) as long as the final product is a single PDF file. You must upload this to Gradescope and mark the pages associated with each problem.

Do not forget to include the following:

- For each problem, please acknowledge your collaborators;
- If a problem required you to code something, please include both the code and the output. "Including the code" can be as crude as a screenshot, but you might also use Quarto to get a pdf that you can merge with the rest of your submission.