Complex Ginzburg-Landau equation, its solutions and their physical relevance

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PAP721 Nonlinear Dynamics
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Outline

Complex Ginzburg-Landau equation

of a family of reaction-diffusion equations

complex Ginzburg-Landau equation is a complex PDE with a nonlinear term:

$$\frac{\partial A}{\partial t} = (1 + i\alpha)\frac{\partial^2 A}{\partial x^2} + A - (1 + i\beta)|A|^2 A,$$

A – complex wave amplitude; α, β – linear and nonlinear dispersion

if $\alpha, \beta = 0 \rightarrow$ real GL eqn, describes superconductivity and superfluidity
$\alpha, \beta \rightarrow$ inf – nonlinear Shrodinger equation (solution - soliton)

was initially derived in superconductivity studies, and Ginzburg received a Nobel Prize for it in 2003

General solution

nonlinear plane wave

this eqn belongs to a family of "reaction-diffusion" eqns with second-order spatial derivative

$$u_t = \sigma \Delta u + N(u)$$

with $N(u)$ – nonlinear term; generally, solution should be a wave in a form

$$u = u_1 e^{ikx + i\omega t} + u_2 e^{-ikx + i\omega t} \equiv c^+ + c^-$$

– running waves, symmetric under $x \to -x$ and phase invariant $\cdot e^{i\phi}$

they are "fixed points" of the system

homoclinic and heteroclinic connections between these fixed points correspond to localized coherent structures

Localizes solutions

soliton-like

theoretical stability analysis is quite complicated 😰 but some features[1]:

- 1D – wave solutions are linearly unstable for $\alpha\beta < -1$: Benjamin-Feir instability;
- 1D – "hole" solutions analytically derived by Bekki and Nozaki are sources and are dynamically stable;
- 2D – spirals and holes (vortices) possess dynamical stability;
- 3D – wires (vortex filaments) possess dynamical stability.

all of the stated solutions are coherent structures, so they interact when they collide

[1] W. van Saarloos, Physica D 56, pp303-367, 1992.

Stability properties

some of them

with change of α and β system undergoes a supercritical Hopf bifurcation and quickly becomes chaotic

$\rightarrow$ system is said to have oscillatory instability
$\rightarrow$ so waves growth rate $>$ 0 because of interaction in time
$\rightarrow$ the only possibility for solutions in the system to be stable is dynamical stability

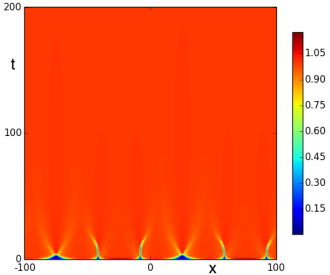
We can try to observe dynamically stable solutions in 1D, 2D and 3D simulations and draw connections with physical systems

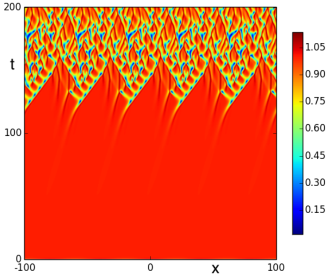
methods: `python+MATLAB` simulation, pseudospectral ETD

1D CGLE

wave solutions and instability onset

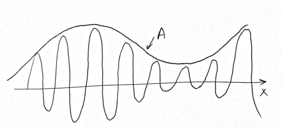
input: gaussian noise / two solitons given as $cosh(c \cdot x)^{-2}$ + noise / periodic wave packet + noise; result is the same:





left: stable plane wave at $\alpha = 1, \beta = 2$, system saturates to some nonzero constant amplitude

right: BF instability onset at $\alpha = 1, \beta = -1.9$ after stable propagation

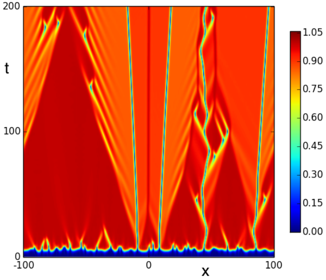


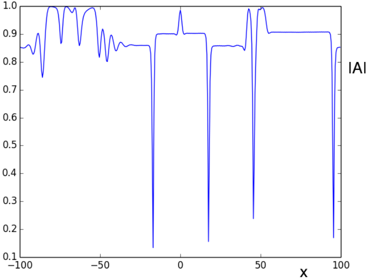
remark: we're observing amplitude modulation

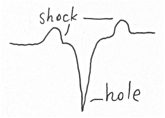
1D CGLE

holes: dynamically stable solutions

finally, we're able to observe dynamically stable Bekki-Nozaki holes:





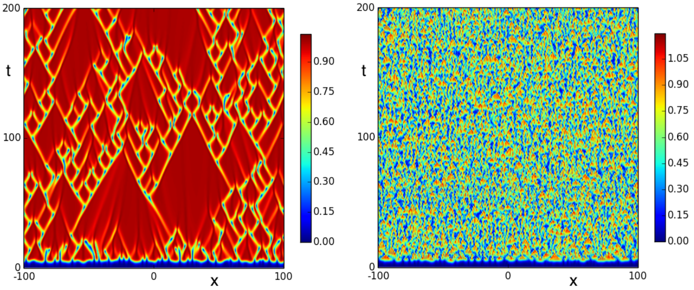


Bekki-Nozaki holes (drops in A + "shocks") stabily propagating at $\alpha = 0, \beta = 1.3$; initial conditions: two solitons, given as $cosh(c \cdot x)^{-2}$ + gaussian noise. Holes are formed randomly in $\alpha \in [-0.2, 0.2], \beta \in [1.2, 1.5]$

1D CGLE

intermittency and chaos

and, finally, intermittency and chaos:



LEFT: intermittency regime for $\alpha = 0.5, \beta = -1.5$; RIGHT: chaos at $\alpha = 1, \beta = -4$

initial conditions: two solitons, given as $cosh(c \cdot x)^{-2}$ + gaussian noise

A couple of words about Turing patterns

Alan Turing has not only become the creator of cryptography, but inspired the whole branch of biophysical and chemical research on reaction-diffusion systems. In 1952, he published a seminal work "The chemical basis of morphogenesis" with some ideas on pattern formation in nature.

[1] picture from P. Ball DOI: 10.1098/rstb.2014.0218

2D CGLE

vortices, spirals and their relevance

- in flames[2], where the variables of the system are masses or concentrations of the fuel and oxidizer – tried to simulate with CGLE (success);
- in superconductors, known as Abrikosov vortices[3]– they act as single magnetic flux quanta – tried to simulate with CGLE (success).

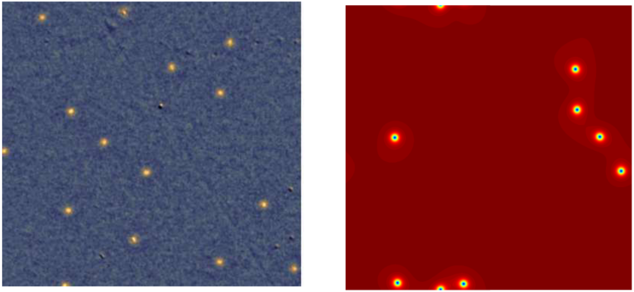
Abrikosov shared Nobel Prize in physics in 2003 with Ginzburg for this theory

[2] S.K. Scott, J. Wang, K. Showalter, Journ. Chem. Soc. Farad. 93(9), pp1733-1739, 1997.

[3] A.A. Abrikosov, Journ. Phys. Chem. Solids 2(3), pp199-208, 1957.

2D CGLE

vortices and their relevance

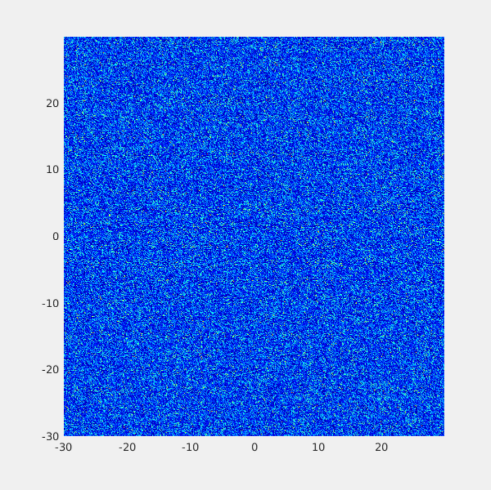


LEFT: Abrikosov vortices, picture taken from[4]; RIGHT: CGLE simulation for $\alpha = 2, \beta = 4$, $|A|_x, |A|_y$ shown.

[4] I. S. Veshchunov et al., Nature Comm., 2016.

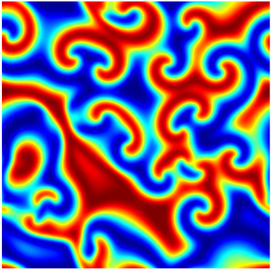
2D CGLE

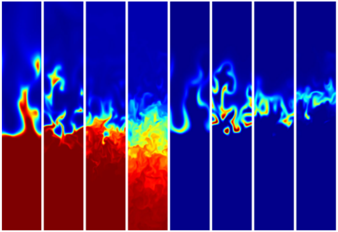
vortex formation



2D CGLE

vortices and their relevance





LEFT: CGLE wave with phase turbulence simulation, $\alpha = 0, \beta = 0$, ReA_x, ReA_y shown; RIGHT: slices from precise chemical and thermodynamical reaction-diffusion simulation of the 2D bunsen turbulent flames[5].

[5] https://ccse.lbl.gov/Research/Combustion/TurbFlameInteractions/index.html

3D CGLE

vortex filaments – "3D solitons"

in 3D main observed solutions are vortex strings – "3D solitons". Experimentally observed in:

- superfluidity – vortex filaments[6];
- chemical systems – strands with higher density[7]– tried to simulate with CGLE (failed);
- and even in quantum field[8]and string theory – tried to simulate with CGLE (success).

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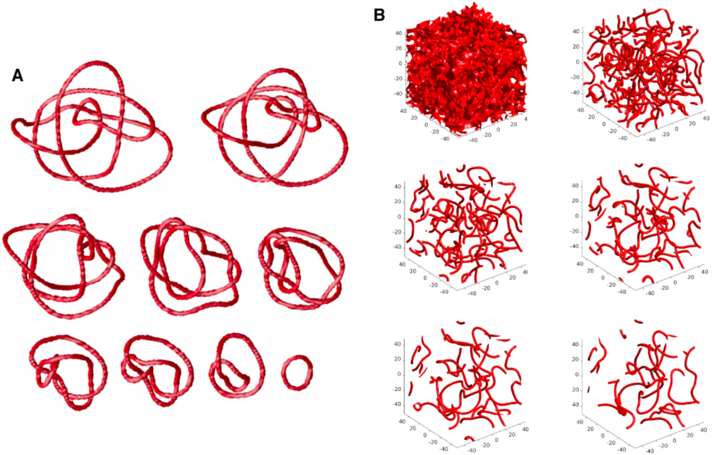
[6] H. Salman, Phys. Rev. Lett. 111, 2013.

[7] Z. Tan, S. Chen, X. Peng, L. Zhang, C. Gao, Science 360(6388), pp518-521, 2018.

[8] F. Maucher, P. Sutcliffe, Phys Rev Lett. 116(17), pp178101, 2016.

3D CGLE

vortex filaments and their physical relevance

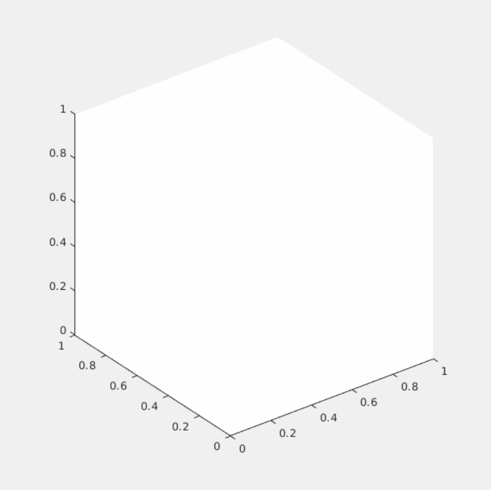


(A) knots detangling from[9]. (B) tubes pattern formation with topology preservation in 3D CGLE with $\alpha = -1.5, \beta = -2.5$.

[8] F. Maucher, P. Sutcliffe, Phys Rev Lett. 116(17), pp178101, 2016.

3D CGLE

vortex filaments and their physical relevance



Summary

- we managed to numerically analyze complex Ginzburg-Landau equation in 1D, 2D and 3D;
- we observed dynamically stable solutions – holes, vortices and filaments;
- we tried to simulate some physically relevant processes: successfully for Abrikosov vortices, flames and filaments in QFT; as a bonus, we observed latter to unknot topologically smoothly

Summary

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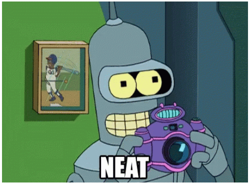
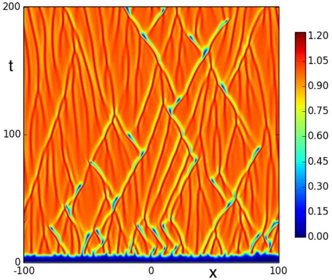


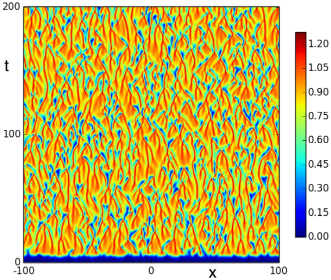
THANK YOU ☺

1D CGLE

instability leading to turbulence

with α, β increase we enter turbulence regime:

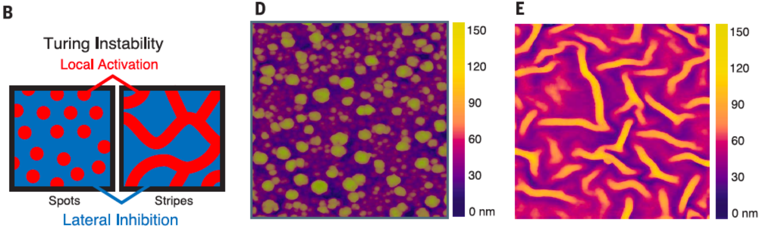




LEFT: amplitude “phase instability” with $A = 0$ – defects at $\alpha = 2, \beta = -1$;
RIGHT: “phase turbulence” – no defects at $\alpha = 2, \beta = -1.4$

3D CGLE

vortex filaments and their physical relevance



Turing patterns formation principle (B) and nanoscale pictures of the formed patterns (D, E)[10].

hard to simulate with CGLE, structures are too coherent; here they are separated

[8] Z. Tan, S. Chen, X. Peng, L. Zhang, C. Gao, Science 360(6388), pp518-521, 2018.