Chapter 1 - Real and Complex Number Systems

Ordered Sets

Quick Definitions

- A order on set S is a relation denoted by $<$ with following properties
  - if $x, y \in S$, only one of following statements are true: $x < y, x = y, y < x$
  - If $x, y, z \in S$ and $x < y, y < z$ then $x < z$
- Suppose S is an ordered set and $E \subset S$ then if $\exists \beta \in S, \forall x \in E, x \leq \beta$. Then E is bounded above by β
- Least upperbound property: α is upperbound of E and if $\gamma < \beta$ γ is not upperbound of E. Call $\alpha = \sup E$.

Proofs

Theorem 1. *Suppose S is an ordered set with LUB property, $B \subset S$ and B is not an empty set and B is bounded below. Let L be the set of all lower bounds of B, then $\alpha = \sup L = \inf B \in S$.*

Proof. Let L be the set of all lowerbounds of B. Since B is bounded below, L is not empty. Then since S has LUB property, sup L $\in S$.
We now prove $\alpha \in L$. For all beta ¡ alpha, notice beta is not an upperbound of L by definition of upperbounds. Then, beta is not in B because every element in B is an upperbound of L. Thus, we have $\beta >= \alpha, \forall \beta \in S$, then by definition of L, $\alpha in L$. We now prove $\forall \beta < \alpha$, β is not greatest lower bound because $\alpha > \beta$ is α is a lower bound of B.
Thus, $\alpha = \inf B$. $\square$