Assignment 2, Part 2 Math derivations

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Given:

$$w_t = \frac{1}{t+1} \tag{1}$$

Definition of sample mean:

$$m_t = \frac{1}{t}\sum_{i=1}^{t} x_i \tag{2}$$

$$tm_t = \sum_{i=1}^{t} x_i \tag{3}$$

Deriving unbiased recursive sample mean:

$$\begin{aligned}
m_{t+1} &= \frac{1}{t+1}\sum_{i=1}^{t+1} x_i && (4)\\
&= \frac{1}{t+1}\left(x_{t+1} + \sum_{i=1}^{t} x_i\right) && (5)\\
&= \frac{1}{t+1}(x_{t+1} + tm_t) && \text{See eq 3} \quad (6)\\
&= \frac{1}{t+1}x_{t+1} + \frac{t}{t+1}m_t && (7)\\
&= \frac{1}{t+1}x_{t+1} + \frac{t+1-1}{t+1}m_t && (8)\\
&= \frac{1}{t+1}x_{t+1} + \left(\frac{t+1}{t+1} - \frac{1}{t+1}\right)m_t && (9)\\
&= \frac{1}{t+1}x_{t+1} + \left(1 - \frac{1}{t+1}\right)m_t && (10)\\
&= w_t x_{t+1} + (1-w_t)m_t && \text{See eq 1} \quad (11)
\end{aligned}$$

Additional equations used to derive formula for unbiased recursive sample variance:

$$\begin{aligned}
m_{t+1} &= \frac{1}{t+1}x_{t+1} + \left(1 - \frac{1}{t+1}\right)m_t && \text{See eq 10} \quad (12)\\
m_{t+1} &= m_t + \left(\frac{1}{t+1}\right)(x_{n+1} - m_t) && (13)\\
m_{t+1} - m_t &= \frac{x_{n+1} - m_t}{t+1} && (14)
\end{aligned}$$

Definition of Sample variance:

$$s_t^2 = \frac{1}{t-1}\sum_{i=1}^{t}(x_i - m_t)^2 \tag{15}$$

Deriving unbiased recursive sample variance:

$$\begin{aligned}
s_{t+1}^2 &= \frac{1}{(t+1)-1}\sum_{i=1}^{t+1}(x_i - m_{t+1})^2 && (16)\\
&= \frac{1}{t}\sum_{i=1}^{t+1}(x_i - m_{t+1})^2 && (17)\\
ts_{t+1}^2 &= \sum_{i=1}^{t+1}(x_i - m_{t+1})^2 && (18)\\
&= \sum_{i=1}^{t+1}\left[(x_i - m_t) + (m_t - m_{t+1})\right]^2 && (19)\\
&= \sum_{i=1}^{t+1}(x_i - m_t)^2 + 2\sum_{i=1}^{t+1}(x_i - m_t)(m_t - m_{t+1}) + \sum_{i=1}^{t+1}(m_t - m_{t+1})^2 && (20)
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{t+1}(x_i - m_t)^2 &= \sum_{i=1}^{t}(x_i - m_t)^2 + (x_{t+1} - m_t)^2 && \text{Part of eq 20} \quad (21)\\
&= (t-1)s_t^2 + (x_{t+1} - m_t)^2 \text{See of eq 15} && (22)
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{t+1}(x_i - m_t)(m_t - m_{t+1}) &= (m_t - m_{t+1})\left[\sum_{i=1}^{t+1}(x_i - m_t)\right] && \text{Part of eq 20} \quad (23)\\
&= (-m_{t+1} + m_t)\left[\sum_{i=1}^{t}(x_i - m_t) + (x_{t+1} - m_t)\right] && (24)\\
&= (-1)(m_{t+1} - m_t)\left[\sum_{i=1}^{t} x_i - tm_t + x_{t+1} - m_t\right] && (25)\\
&= -\left(\frac{x_{n+1} - m_t}{t+1}\right)\left[\sum_{i=1}^{t} x_i - t\left(\frac{1}{t}\sum_{i=1}^{t} x_i\right) + x_{t+1} - m_t\right] && \text{See eqs 3,14} \quad (26)\\
&= -\left(\frac{x_{n+1} - m_t}{t+1}\right)\left[\sum_{i=1}^{t} x_i - \sum_{i=1}^{t} x_i + x_{t+1} - m_t\right] && (27)\\
&= -\left(\frac{x_{n+1} - m_t}{t+1}\right)(x_{t+1} - m_t) && (28)\\
&= -\frac{(x_{n+1} - m_t)^2}{t+1} && (29)
\end{aligned}$$

$$\sum_{i=1}^{t+1}(m_t - m_{t+1})^2 = (t+1)(m_t - m_{t+1})^2 \qquad \text{Part of eq 20} \tag{30}$$

$$= (t+1)(m_{t+1} - m_t)^2 \tag{31}$$

$$= (t+1)\left(\frac{x_{n+1} - m_t}{t+1}\right)^2 \qquad \text{See eq 14} \tag{32}$$

$$= \frac{1}{t+1}(x_{n+1} - m_t)^2 \tag{33}$$

$$ts_{t+1}^2 = \sum_{i=1}^{t+1}(x_i - m_t)^2 + 2\sum_{i=1}^{t+1}(x_i - m_t)(m_t - m_{t+1}) + \sum_{i=1}^{t+1}(m_t - m_{t+1})^2 \qquad \text{Eq 20} \tag{34}$$

$$= \left[(t-1)s_t^2 + (x_{t+1} - m_t)^2\right] + 2\left[-\frac{(x_{n+1} - m_t)^2}{t+1}\right] + \left[\frac{1}{t+1}(x_{n+1} - m_t)^2\right] \qquad \text{Eqs 22,29,33} \tag{35}$$

$$= (t-1)s_t^2 + (x_{t+1} - m_t)^2 - 2\frac{1}{t+1}(x_{n+1} - m_t)^2 + \frac{1}{t+1}(x_{n+1} - m_t)^2 \tag{36}$$

$$= (t-1)s_t^2 + \left(1 - 2\frac{1}{t+1} + \frac{1}{t+1}\right)(x_{t+1} - m_t)^2 \tag{37}$$

$$= (t-1)s_t^2 + \left(1 - \frac{1}{t+1}\right)(x_{t+1} - m_t)^2 \tag{38}$$

$$= (t-1)s_t^2 + \left(\frac{t+1}{t+1} - \frac{1}{t+1}\right)(x_{t+1} - m_t)^2 \tag{39}$$

$$= (t-1)s_t^2 + \left(\frac{t+1-1}{t+1}\right)(x_{t+1} - m_t)^2 \tag{40}$$

$$= (t-1)s_t^2 + \left(\frac{t}{t+1}\right)(x_{t+1} - m_t)^2 \tag{41}$$

$$s_{t+1}^2 = \frac{1}{t}(t-1)s_t^2 + \frac{1}{t}\left(\frac{t}{t+1}\right)(x_{t+1} - m_t)^2 \tag{42}$$

$$= \frac{t-1}{t}s_t^2 + \left(\frac{1}{t+1}\right)(x_{t+1} - m_t)^2 \tag{43}$$