Optimisation methods

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1 P4 (max points 25)

Consider the following optimisation problem:

$$\begin{array}{rl} \text{minimise} & f(\mathbf{x}) = c + \mathbf{g}^\top \mathbf{x} + \frac{1}{2}\mathbf{x}^\top \mathbf{H}\mathbf{x} \\ \text{over} & \mathbf{x} \in \mathbb{R}^n \end{array} \tag{1}$$

where c is a real value, $\mathbf{g}$ is a n-dimensional real-valued vector and $\mathbf{H}$ is an $n \times n$ real-symmetric matrix.

P4-i (max points 5) Assume that

$$\mathbf{H} = \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}$$

where $\mathbf{D}$ is a diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_{n-2}$ and $\mathbf{B} = \begin{pmatrix} a_{n-1} & b \\ b & a_n \end{pmatrix}$.

What are the necessary and sufficient conditions for the optimisation problem (1) to be a convex optimisation problem?

P4-ii (max points 5) Write down the update equation for the gradient descent algorithm with a constant step size η for the optimisation problem (1).

P4-iii (max points 5) Assume that $\mathbf{g} = \mathbf{0}$. Show that f is a β-smooth function and express the value of β as a function of the eigenvalues of the Hessian matrix $\mathbf{H}$.

P4-iv (max points 5) Assume that $\mathbf{g} = \mathbf{0}$ and

$$\mathbf{H} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Given initial value $\mathbf{x}_0$, let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T, \ldots$ be updated according to gradient descent algorithm with a constant step size, and let $\mathbf{x}^*$ denote a minimiser of f.

What is the best upper bound that can be guaranteed for $f(\mathbf{x}_T) - f(\mathbf{x}^*)$?

P4-v (max points 5) Consider the same question as in P4-iv but under assumptions that $\mathbf{g} = \mathbf{0}$ and

$$\mathbf{H} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$