Homework 2

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1 Optimal Coupling

Assume $\Omega = \{1, 2, \cdots, n\}$. Our goal is to fill the $n \times n$ probabilty table, which means we need to determine every value of $P_{i,j}$ such that

$$\sum_i P(i,j) = \nu(j) \text{ and } \sum_j P_{i,j} = \mu(i) \tag{1}$$

Constructe $\boldsymbol{P}$ as follows:

1. Let $\Pr_{(X,Y)\sim\omega}(X = Y = i) = \min[\mu(i), \nu(i)]$.
2. Then
$$\text{If } i \neq j,\ P_{i,j} = \frac{(\mu(i) - P_{i,i})(\nu(j) - P_{j,j})}{1 - \sum_k P_{k,k}}.$$

Reference: Markov Chains and Coupling, Duke University. And the proof is not mentioned in this lecture notes.
Now we want to prove that the construction satisfies the EQ. 1. If $\nu(i) = P_{i,i}$:

$$\begin{aligned}
&P_{i,i} + \sum_{j\neq i} \frac{(\mu(i) - P_{i,i})(\nu(j) - P_{j,j})}{1 - \sum_k P_{k,k}} \\
=&P_{i,i} + (\mu(i) - P_{i,i})\frac{\sum_{j\neq i}(\nu(j) - P_{j,j})}{1 - \sum_k P_{k,k}} \\
=&P_{i,i} + (\mu(i) - P_{i,i})\frac{\sum_{j}(\nu(j) - P_{j,j})}{1 - \sum_k P_{k,k}} \\
=&P_{i,i} + (\mu(i) - P_{i,i})\frac{1 - \sum_j P_{j,j}}{1 - \sum_k P_{k,k}} \\
=&\mu(i)
\end{aligned}.$$

If $\mu(i) = P_{i,i}$, obviously $\sum_j P_{i,j} = P_{i,i} = \mu(i)$. Similarly, $\sum_i P_{i,j} = \nu(j)$. And the EQ.1 holds.
Thus

$$\Pr_{(X,Y)\sim\omega}[X \neq Y] = 1 - \sum_i \min(\mu(i), \nu(i)) = \sum_i [\mu(i) - \min(\mu(i), \nu(i))] = \max_{A\subset\Omega} |\mu(A) - \nu(A)| = D_{TV}(\mu, \nu).$$

2 Stochastic Dominance

Problem 1

"$\Longrightarrow$": $\Pr[X = n] \geq \Pr[Y = n]$, which means that $p^n \geq q^n$. Thus $p \geq q$.

"$\Longleftarrow$", if $p \geq q$:
For simplicity, $\Pr[X \geq k] - \Pr[Y \geq k] \triangleq A_k$. $A_0 = 0$ and $A_n = p^n - q^n$. Then

$$A_{k+1} - A_k = q^k(1-q)^{n-k} - p^k(1-p)^{n-k}.$$

Let $k_0 = \dfrac{n\log(1-q/1-p)}{\log[p(1-q)/q(1-p)]}$.

If $k \leq k_0$, then $A_{k+1} - A_k \geq 0$. And if $k_0 \leq k \leq n$, then $A_{k+1} - A_k \leq 0$.

Thus $0 = A_0 \nearrow A_{k_0} \searrow A_n = p^n - q^n > 0$ for some k_0, which means that $A_k > 0$.

Finally

$$\Pr[X \geq k] - \Pr[Y \geq k] \geq 0.$$

Problem 2

Assume $\Omega = \{1, 2, \cdots, n\}$.

"$\Longleftarrow$": $P_{i,j}$ denote $\omega(i,j)$. $P_{i,j} = 0$ if $j > i$.

$$\Pr[X \geq k] = \sum_{i=k}^{n}\sum_{j=0}^{i} P_{i,j} \leq \sum_{i=k}^{n}\sum_{j=k}^{i} P_{i,j} = \sum_{j=k}^{i}\sum_{i=k}^{n} P_{i,j} = \sum_{j=k}^{n}\sum_{i=0}^{j} P_{i,j} = \Pr[Y \geq k].$$

"$\Longrightarrow$":
Design the coupling by the following steps:

1. $\forall i, j,\ P_{i,j} = 0$.
2. Let (i,j) traverse: $(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\cdots, (n,n)$ and update $P_{i,j}$:

$$P_{i,j} = \min\left\{\left[\mu(i) - \sum_{k=1}^{i} P_{i,k}\right], \left[\nu(j) - \sum_{k=i}^{j} P_{k,j}\right]\right\}.$$

For any i:
If $\exists j$, $P_{i,j} = \mu(i) - \sum_{k=1}^{i} P_{i,k}$, then $\sum_j P_{i,j} = \mu(i)$.
If not, then $\forall j$, $P_{i,j} = \nu(j) - \sum_{k=i}^{j} P_{k,j}$, which means $\sum_{k=i+1}^{n}\mu(k) = \sum_{k=i+1}^{n}\nu(k)$. Thus

$$\begin{aligned}
\sum_{j=1}^{n} P_{i,j} &= \sum_{j=1}^{i} P_{i,j} = \sum_{j=1}^{i}\left[\nu(j) - \sum_{k=i}^{j} P_{k,j}\right] \\
&= \sum_{k=1}^{i}\nu(j) - \sum_{k=1}^{i-1}\mu(k) = \sum_{k=1}^{i}\mu(j) - \sum_{k=1}^{i-1}\mu(k) \\
&= \mu(i)
\end{aligned}.$$

Problem 3

I refer to the lecture notes last year.

	Not Connected	Connected	G
Not Connected	P_{11}	P_{12}	Pr[G is not connected]
Connected	P_{21}	P_{22}	Pr[G is connected]
H	Pr[H is not connected]	Pr[H is connected]	

We generate $G \sim \mathcal{G}(n,p)$ and $H \sim \mathcal{G}(n,q)$ simultaneously. Let $r \sim \mathrm{U}(0,1)$. For every edge (u,v):

$$\begin{cases} (u,v)\text{exisits in } G \text{ and } H & \text{if } r \in [0,q] \\ (u,v)\text{exisits only in } G & \text{if } r \in (q,p] \\ (u,v)\text{does not exisit} & \text{if } r \in (p,1] \end{cases}.$$

Then for all H, H is the subgraph of G, which means that $P_{12} = 0$. By 2 we have

$$\Pr_{G\sim\mathcal{G}(n,p)}[\text{G is connected}] \geq \Pr_{H\sim\mathcal{G}(n,q)}[\text{H is connected}].$$

3 Total Variation Distance is Non-Increasing

By coupling lemma, for every $X^t \sim \mu^t$ and $Y^t \sim \pi$, we have a coupling such that $\Delta(t) = \Pr(X^t \neq Y^t)$. Then we construct X^{t+1} and Y^{t+1} by following rules:

$$\begin{cases} X^{t+1} = X^t \text{ and } Y^{t+1} = Y^t & \text{, if } X^t = Y^t \\ X^t \to X^{t+1} \text{ and } Y^t \to Y^{t+1} \textbf{ independently} & \text{, otherwise} \end{cases}.$$

Thus

$$\Delta(t+1) \leq \Pr\big(X^{t+1} \neq y^{t+1}\big) \leq \Pr\big(X^t \neq Y^t\big) = \Delta(t).$$

Reference: Markov Chains and Coupling, Duke University.