Homework 1

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March 2, 2022

1 Probability Space of Tossing Coins

Problem 1

Obviously, $\varnothing \notin \{C_s\}_{\{0,1\}^n}$.
By the fact that $C_s \subset \Omega$ and $\text{Card}(\{0,1\}^n) < +\infty$, we have $\bigcup_{s\in\{0,1\}^n} C_s \subset \Omega$.
Also, $\forall \omega \in \Omega$, $\omega \in C_{(\omega_1,\dots,\omega_n)}$, which means $\Omega \subset \bigcup_{s\in\{0,1\}^n} C_s$.
$\forall s_1, s_2 \in \{0,1\}^n$, $\forall \omega_1 \in C_{s_1}$ and $\omega_2 \in C_{s_2}$, $\omega_1 \neq \omega_2$. Then $C_{s_1} \cap C_{s_2} \neq \varnothing$.
Finally, $\{C_s\}_{s\in\{0,1\}^n}$ forms a partition of Ω.

Problem 2

Firstly, I will prove a lemma and a proposition.

Defination 1. $\mathcal{S}(\mathcal{C}) \xlongequal{def} \left\{\bigcup_{i=1,\dots,m} A_i \mid m \in \{1,\cdots,|\mathcal{C}|\}, A_i \in \mathcal{C}\right\} \cup \{\varnothing\}$

Lemma 1. *If $\mathcal{C}$ forms a partition of Ω, then $\mathcal{S}(\mathcal{C})$ is a σ-algebra.*

Proof. $\varnothing, \Omega \in \mathcal{S}(\mathcal{C})$.
If $a \in \mathcal{S}(\mathcal{C})$, then for some $B_i \in \mathcal{C}$, $a = \bigcup_i B_i$. Thus $\Omega - a = \bigcup_{A_i\in\mathcal{C}} A_i - s \in \mathcal{S}(\mathcal{C})$.
If $A_i \in \mathcal{S}(\mathcal{C})$, then $\bigcup_i A_i = \bigcup_i \left(\bigcup_j B_{ij}\right) \in \mathcal{S}(\mathcal{C})$. □

Defination 2. $\mathcal{C}_n = \{C_s\}_{s\in\{0,1\}^n}$.

Proposition 1. $\mathcal{F}_n = \mathcal{S}(\mathcal{C}_n)$.

Proof. Trivially, $\mathcal{C}_n \subset \mathcal{S}(\mathcal{C}_n)$. Now we suppose that $\exists$ σ-algebra $\mathcal{F}$ s.t. $\mathcal{C}_n \subset \mathcal{F}$:
For all $A \in \mathcal{S}(\mathcal{C}_n)$. Then $A = \bigcup C_s$ for some s or $A = \varnothing$. By the defination of σ-algebra, $A \in \mathcal{F}$, which means $\mathcal{S}(\mathcal{C}_n) \in \mathcal{F}$.
Thus $\mathcal{S}(\mathcal{C}_n)$ is the minimal σ-algebra containing sets in $\mathcal{C}_n$. □

Now we have $\text{Card}(\mathcal{F}_n) = \text{Card}(\mathcal{S}(\mathcal{C}_n)) = \text{Card}(2^{\mathcal{C}_n}) = \text{Card}(2^{\{0,1\}^n}) = 2^{2^n}$. We could say that $\mathcal{F}_n \sim 2^{\{0,1\}^n}$ or $\mathcal{F}_n$ and $2^{\{0,1\}^n}$ are equinumerous. So there exists a bijection between $\mathcal{F}$ and $2^{\{0,1\}^n}$.

Problem 3

By lemma 1, $\forall n \in \mathbb{N}$, $\mathcal{F}_n = \mathcal{S}(\mathcal{C}_n)$.
And obviously $\mathcal{F}_n \neq \mathcal{F}_{n+1}$ because $\mathcal{C}_n \neq \mathcal{C}_{n+1}$ and $\text{Card}(\mathcal{C}_n) \neq \text{Card}(\mathcal{C}_{n+1})$.

$\forall A \in \mathcal{F}_n$, $A = \bigcup_i C_{s_i}$ where $s_i \in \{0,1\}^n$. Then

$$A = \bigcup_i \left(C_{[s_i,0]} \cup C_{[s_i,1]}\right),$$

where the operator $[\cdot,\cdot]$ means a concatenation. For example, $[(0,0,1),1] = (0,0,1,1)$ and $[(0,0,1),(0,1)] = (0,0,1,0,1)$. I also use this operator in Problem 6 and 7.
Ultimately $\forall n \in \mathbb{N}$, $\mathcal{F}_n = \mathcal{S}(\mathcal{C}_n) \subsetneq \mathcal{S}(\mathcal{C}_{n+1}) = \mathcal{F}_{n+1}$.

Problem 4

By defination, $\mathcal{F}_\infty = \bigcup_{n\geq 1} \mathcal{F}_n$.
For all $A, B \in \mathcal{F}_\infty$, $\exists m, n \in \mathbb{N}$, $A \in \mathcal{F}_m$ and $B \in \mathcal{F}_n$. Let $l \overset{def}{=\!=} \max(m,n)$. Then $A, B \in \mathcal{F}_l$.
Thus $A^c, B^c \in \mathcal{F}_l$ and $A + B \in \mathcal{F}_l \Longrightarrow A^c, B^c, A + B \in \mathcal{F}_l$.
The proposition that F_∞ is an algebra holds.

Lemma 2. *$\forall \omega \in \Omega$, $\forall n \in \mathbb{N}$, $\{\omega\} \notin \mathcal{F}_n$.*

Proof. $\forall A \in \mathcal{F}_n$, $\mathrm{Card}(A) = +\infty \neq \mathrm{Card}(\{\omega\}) = 1$ □

Thus $2^\Omega \neq \mathcal{F}_\infty$.

Problem 5

By lemma 2, we have $\{\omega\} \neq \mathcal{F}_\infty$. Now we just need to prove that $\{\omega\} \in \mathcal{B}(\Omega)$.
Proof by contradiction. Suppose that $\exists \omega' \notin \mathcal{B}(\Omega)$.
$\forall n \in \mathbb{N}$, $\mathcal{F}_n \in \mathcal{B}(\Omega)$.
A creative construction:

$$T \overset{def}{=\!=} \bigcup_{n=1,\ldots,\infty} \{\omega \mid \omega_1 \neq \omega'_1 \wedge \cdots \wedge \omega_n \neq \omega'_n\} \overset{def}{=\!=} \bigcup_{n=1,\ldots,\infty} T_n.$$

Here T is a countable union and every S_n is in $\mathcal{S}(\mathcal{C}_n) = \mathcal{F}_n$ ($T_n \in \mathcal{B}(\Omega)$). Hence $T \in \mathcal{B}(\Omega)$. Intuitively, $T = \Omega - \omega$.
Finnaly, $\{\omega\} = T^c \in \mathcal{B}(\Omega)$.

Problem 6

For all $A \in \mathcal{F}_\infty$, $\exists m(A) \in \mathbb{N}$ such that $A \in \mathcal{F}_{m(A)}$ and $\forall n \in \{1, 2, \cdots, m(A) - 1\}$, $A \notin \mathcal{F}_n$. For simplicity, we denote $m(A)$ as m. It is worthy of mention that m **only** depends on A.
Now we want to prove $\forall n \geq m$, $2k_n = k_{n+1}$.
If $A = C_{s_1} \cup \cdots \cup C_{s_{k_n}}$, then

$$A = \left(C_{[s_1,0]} \cup C_{[s_1,1]}\right) \cup \cdots \cup \left(C_{[s_{k_n},0]} \cup C_{[s_{k_n},1]}\right).$$

Trivially, $[s_1, 0] \neq [s_1, 1] \neq \cdots \neq [s_{n_k}, 0] \neq [s_{n_k}, 1]$. Thus we conclude $k_{n+1} = 2k_n$ and $k/2^n = Const$ for any fixed A.

Problem 7

I've talked with Liyuan Mao and Wei Jiang for days. Here I will give several attempts I make, though I could not prove the proposition successfully.

Attempt 1.

Lemma 3. *$\forall$ set A and set B, $A \cap B = A - (A - B) = A - (A^c \cup B)^c = [A^c \cup (A^c \cup B)^c]^c$.*

Proposition 2. $\mathcal{B}(\Omega) = \left\{ \bigcup_{i=1}^{\infty} \left(\bigcap_{j=1}^{\infty} A_{ij} \right) \middle| \, A_{ij} \in \mathcal{F}_\infty \right\}$.

Proof. Obviously, $\varnothing \in \mathcal{B}(\Omega)$.
$\forall A \in \mathcal{B}(\Omega)$: $A^c = \left[\bigcup_{i=1}^{\infty} \left(\bigcap_{j=1}^{\infty} A_{ij} \right) \right]^c = \bigcap_{i=1}^{\infty} \left(\bigcup_{j=1}^{\infty} A_{ij}^c \right)$. (Failure. I could not prove this.) □

Remark of the **attempt 1**: I want to construct $\mathcal{B}$ directly, while I could not prove this set is a σ-algebra. The idea is intuitively from $\forall A \in \mathcal{B}$, A could be generated from C_{s_n} and ω_n.

Attempt 2. $\forall A \in \mathcal{B}(\Omega)$:
$P_n(A) \xlongequal{def} P_n(\bigcup A_i)$ where $A_i \in F_n$ and $\bigcup A_i$ is the smallest cover of A. Then define $P(A) = \lim_{n\to\infty} P_n(A)$. (It is obvious that $P(A)$ is bounded and decreasing.)
Hoever, I could not prove the uniqueness of $P(A)$.

Finally.
I find a more general theorem called Caratheodory's extension theorem. In this theorem we could find several creative constructions:

$$\lambda(E) \xlongequal{def} \inf \left\{ \sum_{j=1}^{\infty} \Pr(A_j) : A_j \in \mathcal{F}, E \subset \bigcup_j A_j \right\}$$

and

$$\bar{\mathcal{F}} \xlongequal{def} \{E : \lambda(F) = \lambda(F \cap E) + \lambda(F \cap E^c) \; \forall F \subset \Omega\}.$$

Final remark: I found the similarity between the defination of $P_n(A)$ in Attempt 2 and $\lambda(E)$.
Reference: Fundamentals of Probability, MIT.

Problem 8

Let $G_n = \left\{ \omega \middle| \, \omega_1 = \omega_2 = \cdots = \omega_{n-1} = 0, \omega_n = 1 \text{ and } \omega \in \Omega \right\}$ and $\Omega^* = \{G_i\}_{i \in \mathbb{N}^+}$.
Then $X : \Omega' \to \mathbb{R}$, $X(\omega) = n$ such that $\omega \in G_n$. Obviously $P(X = n) = \Pr(G_n) = 1/2^n$.

2 Conditional Expectation

Problem 1

$\forall A \in \mathcal{B}$: $X^{-1}(A) = \left\{ \omega \middle| \, X(\omega) \in A \right\} \in \sigma(X)$ and $f^{-1}(A) \in \mathcal{B}$.
Now we have $\mathcal{G} \xlongequal{def} \left\{ X(\omega) \middle| \, f(X(\omega)) \in A \right\} \in \mathcal{B}$. Thus

$$[f(X)]^{-1}(A) = \left\{ \omega \middle| \, f(X(\omega)) \in A \right\} = \left\{ \omega \middle| \, X(\omega) \in \mathcal{G} \right\} \in \sigma(X)$$

which means that $f(X)$ is $\sigma(X)$-measurable.

Problem 2

Proposition 3. *$\forall \omega \in \Omega$, $\exists A \in \sigma(Y)$ such that $Y^{-1}(\omega) = A$ and $\forall B \in \sigma(Y) : A \cap B = \varnothing \vee A \subset B$.*

Proof. If not, then $\exists B \neq \varnothing$, $A \cap B \neq A$. Let $A_1 = A \cap B, A_2 = A \cup B^c$. $A_1, A_2 \in \sigma(Y)$ such that $A_1 \cup A_2 = Y^{-1}(\omega)$ and $A_1 \cap A_2 = \varnothing$. Obviously $\sigma(Y)$ is **not** the minimal σ-algebra such that Y is $\sigma(Y)$-measurable. Contradiction. $\square$

Then $\forall \omega \in \Omega$, $Y^{-1}[Y(\omega)] = Y'^{-1}[Y'(\omega)]$. (If not, let $B = Y^{-1}[Y(\omega)] \cap Y'^{-1}[Y'(\omega)]$. Then B is a set which contradicts the Proposition 3.

Problem 3

$\mathbb{E}[X \mid \mathcal{F}] = \mathbb{E}[X \mid Y]$ where $\sigma(Y) = \mathcal{F}$.
Remark: after I finnished Problem 3 and 4 I found that the existence of Y may not always hold for every $\mathcal{F}$. The existence of Y depends on whether $\mathcal{F}$ has partitions, (i.e., if we could find the partitions of $\mathcal{F}$, then we have a Y). Hoever, for more general cases I could not define it clealy (may use Lebergue Integral).

Problem 4

Suppose that $\sigma(Y_1) = \mathcal{F}_1$ and $\sigma(Y_2) = \mathcal{F}_2$.

$$\begin{aligned} \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_1] \mid \mathcal{F}_2](\omega_0) &= \sum_{a \in \mathrm{Ran}(f_Y)} a \times \Pr[f_{Y_1} = a \mid Y_2 = Y_2(\omega_0)] \\ &= f_{Y_1}(\omega_0) \times \Pr[f_{Y_1} = f_{Y_1}(\omega_0) \mid Y_2 = Y_2(\omega_0)] \\ &= f_{Y_1}(\omega_0) \end{aligned} \tag{1}$$

The second euqality of Eq.1 holds by the fact that:

$$\forall \omega \text{ such that } a = f_{Y_1}(\omega) \neq f_{Y_1}(\omega_0),\ [f_{Y_1} = a] \cap [Y_2 = Y_2(\omega_0)] = \varnothing.$$

The third euqality of Eq.1 holds by the fact that:

$$[Y_2 = Y_2(\omega_0)] \subset [f_{Y_1} = f_{Y_1}(\omega_0)].$$

Let $\mathcal{D}_0 \xlongequal{def} \left\{ b \in \mathbb{R} \mid \exists \omega \in \Omega \text{ such that } Y_2^{-1}(Y_2(\omega)) \subset Y_1^{-1}(Y_1(\omega)) \text{ and } b = f_{Y_2}(\omega) \right\}$. The interpretation of $\mathcal{D}_0$: $\forall A \in \sigma(Y_1)$, there exsists serveral sets $A_i \in \sigma(Y_2)$ such that $\bigcup A_i = A$. Here we let $A = Y_1^{-1}(\omega)$ and $\{A_i\}$ is the partition of $\sigma(Y_2)$.

$$
\begin{aligned}
\mathbb{E}[\mathbb{E}[X \mid Y_2] \mid Y_1](\omega_0) &= \frac{\sum_{b\in \mathrm{Ran}(f_{Y_2})} b \times \Pr(f_{Y_2} = b \cap Y_1 = Y_1(\omega_0))}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_{b\in \mathcal{D}_0} b \times \Pr(f_{Y_2} = b \cap Y_1 = Y_1(\omega_0))}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_{b\in \mathcal{D}_0} \mathbb{E}(X \mid Y_2 = Y_2(\omega_b)) \Pr(f_{Y_2} = b)}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_{b\in \mathcal{D}_0} \mathbb{E}(X \mid Y_2 = Y_2(\omega_b)) \Pr(Y_2 = Y_2(\omega_b))}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_{b\in \mathcal{D}_0} \sum_a \mathbb{E}(X = a \mid Y_2 \cap Y_2(\omega_b))}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_a \sum_{b\in \mathcal{D}_0} \mathbb{E}(X = a \mid Y_2 \cap Y_2(\omega_b))}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_a \Pr(X = a \cap Y_1 = Y_1(\omega_0))}{\Pr(Y_1 = Y_1(\omega))} \\
&= f_{Y_1}(\omega_0)
\end{aligned} \tag{2}
$$

Ultimately, with EQ. 1 and EQ. 2 we prove the proposition of problem 4.