Homework 4

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1 Doob's martingale inequality

For any given n:
Consider the stopping time $\tau = \arg\min_{0\le t\le n}\{X_t \ge \alpha\}$ or $\tau = n$ if $X_t < \alpha$ for all $0 \le t \le n$. Ostensively, $\max_{0\le t\le n} X_t \ge \alpha \iff (\exists k) X_k \ge \alpha \iff X_\tau \ge \alpha$. Thus we have:

$$\Pr\left[\max_{0\le t\le n} X_t \ge \alpha\right] = \Pr[X_\tau \ge \alpha] \le \frac{\mathbb{E}[X_\tau]}{\alpha}$$

where the inequality holds by Markov Ineq.
By defination, $\tau \le n$, which means $\mathbb{E}[X_\tau] = \mathbb{E}\not\vdash$ by Optional Stopping Theorem. Then we obtain $\Pr[\max_{0\le t\le n} X_t \ge \alpha] \le \frac{\mathbb{E}[X_0]}{\alpha}$.

2 Biased one-dimensional random walk

Subproblem 1

By defination,

$$\begin{aligned}
\mathbb{E}\big[S_{t+1}|\overline{Z_{i,n}}\big] &= \mathbb{E}\big[S_t + Z_{t+1} + 2p - 1|\overline{Z_{i,n}}\big] \\
&= S_t + 2p - 1 + (1-p) - p \\
&= S_t
\end{aligned}.$$

Subproblem 2

Simmilarly,

$$\begin{aligned}
\mathbb{E}\big[P_{t+1}|\overline{Z_{i,n}}\big] &= \mathbb{E}\left[P_t\left(\frac{p}{1-p}\right)^{Z_{t+1}}|\overline{Z_{i,n}}\right] \\
&= P_t\left[p \times \frac{1-p}{p} + (1-p) \times \frac{p}{1-p}\right] \\
&= P_t
\end{aligned}.$$

Subproblem 3

We define $p_a = \Pr[X_\tau = a]$ and $p_b = \Pr[X_\tau = b] = 1 - p_a$.
Now we want to show that $\{S_i\}$ and $\{P_i\}$ satisfy the conditions of OST theorem. Firstly we have

$$\Pr(\text{ending within the next } a+b \text{ steps}) \ge [\max(p, 1-p)]^{-a-b}$$

which means $\Pr[\tau < \infty] = 1$. Dividing the time into consecutive periods in this manner, we have $\mathbb{E}[\tau] < \infty$. Obviously, $|P_i|$ is bounded. Also we have: $\mathbb{E}[|S_{t+1} - S_t|\mathcal{F}_t] = p(2-2p)+(1-p)2p = 4p(1-p)$. By OST Thm., $\mathbb{E}[S_\tau] = \mathbb{E}[S_1]$ and $\mathbb{E}[P_\tau] = \mathbb{E}[P_1]$. These propositions entail

$$(2p-1)\mathbb{E}[\tau] + bp_b - ap_a = 0 \quad \text{and} \quad \left(\frac{p}{1-p}\right)^b p_b + \left(\frac{p}{1-p}\right)^{-a} p_a = 1.$$

Thus

$$p_a = \frac{1 - [p/(1-p)]^b}{[(1-p)/p]^a - [p/(1-p)]^b} \quad \text{and} \quad p_b = \frac{[(1-p)/p]^a - 1}{[(1-p)/p]^a - [p/(1-p)]^b}.$$

Finally,

$$\mathbb{E}[\tau] = \frac{(a+b)(1-p)^b p^a - ap^{a+b} - b(1-p)^{a+b}}{(2p-1)[(1-p)^{a+b} - p^{a+b}]}.$$

Remark: $\lim_{p\to 1/2} \mathbb{E}[\tau] = ab$.

3 Longest common subsequence

Talk with Yilin Sun and Liyuan Mao.

Subproblem 1

When $n = 2$,

$$\mathbb{E}[X] = \frac{2 \times (2+1+1+0) + 2 \times (1+2+1+1)}{16} = \frac{9}{8} \geq 2 \times \frac{9}{16}.$$

Similarly, if $n = 3$,

$$\mathbb{E}[X] = \frac{29}{16} \geq 3 \times \frac{29}{48}.$$

Since we could divide every strings into continuous subsequences whose length is 2 or 3,

$$c_1 = \frac{9}{16}.$$

For any given n:

$$\Pr[X \geq k] \leq \frac{2^{2n-k}}{2^{2n}} \binom{n}{k}^2 \approx \frac{n}{2\pi k(n-k)} \left[\frac{n^n}{k^k (n-k)^{n-k}}\right]^2 \frac{1}{2^k}.$$

Let $k = cn$. We obtain

$$\Pr[X \geq cn] \leq \frac{1}{2\pi nc(1-c)} \left[\frac{1}{\sqrt{2}^c c^c (1-c)^{1-c}}\right]^{2n}.$$

Set $c = 0.99$:

$$\Pr[X \geq 0.8n] \leq \frac{51}{\pi n}(0.76)^n \text{ when } n \to \infty.$$

Thus c_2 exists.

Subproblem 2

Denote function $f(x_1, \cdots, x_n, y_1, \cdots, y_n)$ as the LCS's length of the 2 sequences $\boldsymbol{x}$ and $\boldsymbol{y}$. Obviously f is 1-Lipschitz. By McDiarmid's Inequality:

$$\Pr(|X - \mathbb{E}[X]| \geq t) \leq 2e^{-2t^2/n}.$$