Homework 1

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1 Probability Space of Tossing Coins

Problem 1

Obviously, $\varnothing \notin \{C_s\}_{\{0,1\}^n}$.
By the fact that $C_s \subset \Omega$ and $\mathrm{Card}(\{0,1\}^n) < +\infty$, we have $\bigcup_{s\in\{0,1\}^n} C_s \subset \Omega$.
Also, $\forall \omega \in \Omega$, $\omega \in C_{(\omega_1,\ldots,\omega_n)}$, which means $\Omega \subset \bigcup_{s\in\{0,1\}^n} C_s$.
$\forall s_1, s_2 \in \{0,1\}^n$, $\forall \omega_1 \in C_{s_1}$ and $\omega_2 \in C_{s_2}$, $\omega_1 \neq \omega_2$. Then $C_{s_1} \cap C_{s_2} \neq \varnothing$.
Finally, $\{C_s\}_{s\in\{0,1\}^n}$ forms a partition of Ω.

Problem 2

Firstly, I will prove a lemma and a proposition.

Lemma 1. *If $\mathcal{C}$ forms a partition of Ω, then $2^{\mathcal{C}}$ is a σ-algebra.*

Proof. $\varnothing \in 2^{\mathcal{C}}$.
If $A \in 2^{\mathcal{C}}$, then $\Omega - A \subset \mathcal{C} \Longrightarrow \Omega - A \in 2^{\mathcal{C}}$.
If $A_i \in 2^{\mathcal{C}}$, then $\bigcup_i A_i \subset \mathcal{C}$, which means that $\bigcup_i A_i \in 2^{\mathcal{C}}$. □

Proposition 1. *$\mathcal{F}_n = 2^{\mathcal{C}_n}$ where $\mathcal{C}_n = \{C_s\}_{s\in\{0,1\}^n}$.*

Proof. Trivially, $\mathcal{C}_n \subset 2^{\mathcal{C}_n}$. Now we suppose that $\exists$ σ-algebra $\mathcal{F}$ s.t. $\mathcal{C}_n \subset \mathcal{F}$:
For all $A \in 2^{\mathcal{C}_n}$, $A \subset \mathcal{C}$. Then $A = \bigcup C_s$ for some s or $A = \varnothing$. By the defination of σ-algebra, $A \in \mathcal{F}$, which means $2^{\mathcal{C}_n} \in \mathcal{F}$.
Thus $2^{\mathcal{C}_n}$ is the minimal σ-algebra containing sets in $\mathcal{C}$. □

Now we have $\mathrm{Card}(\mathcal{F}_n) = \mathrm{Card}(2^{\mathcal{C}_n}) = \mathrm{Card}(2^{\{0,1\}^n}) = 2^{2^n}$. We could say that $\mathcal{F}_n \sim 2^{\{0,1\}^n}$ or $\mathcal{F}_n$ and $2^{\{0,1\}^n}$ are equinumerous. So there exists a bijection between $\mathcal{F}$ and $2^{\{0,1\}^n}$.

Problem 3

By lemma 1, $\forall n \in \mathbb{N}$, $\mathcal{F}_n = 2^{\mathcal{C}_n}$.
And obviously $\mathcal{F}_n \neq \mathcal{F}_{n+1}$ because $\mathcal{C}_n \neq \mathcal{C}_{n+1}$ and $\mathrm{Card}(\mathcal{C}_n) \neq \mathrm{Card}(\mathcal{C}_{n+1})$.
$\forall A \in \mathcal{F}_n$, $A \subset \mathcal{C}_n$. Assume that $A = \{\omega \in \Omega \mid \omega_1 = s_1, \cdots, \omega_n = s_n\}$. Then

$$\{\omega \in \Omega \mid \omega_1 = s_1, \cdots, \omega_n = s_n, \omega_{n+1} = 0\} \cup \{\omega \in \Omega \mid \omega_1 = s_1, \cdots, \omega_n = s_n, \omega_{n+1} = 1\} = A,$$

where the 2 elements of LHS are in $\mathcal{C}_{n+1}$. Thus $A \in \mathcal{C}_{n+1}$.
Ultimately $\forall n \in \mathbb{N}$, $\mathcal{F}_n = 2^{\mathcal{C}_n} \subsetneq 2^{\mathcal{C}_{n+1}} = \mathcal{F}_{n+1}$.

Problem 4

By defination, $\mathcal{F}_\infty = \bigcup_{n\geq 1} \mathcal{F}_n$.
For all $A, B \in \mathcal{F}_\infty$, $\exists m, n \in \mathbb{N}$, $A \in \mathcal{F}_m$ and $B \in \mathcal{F}_n$. Let $l \xlongequal{def} \max(m, n)$. Then $A, B \in \mathcal{F}_l$.
Thus $A^c, B^c \in \mathcal{F}_l$ and $A + B \in \mathcal{F}_l \Longrightarrow A^c, B^c, A + B \in \mathcal{F}_l$.
The proposition that F_∞ is an algebra holds.

Lemma 2. *$\forall \omega \in \Omega$, $\forall n \in \mathbb{N}$, $\omega \notin \mathcal{F}_n$.*

Proof. $\forall A \in \mathcal{F}_n$, $\mathrm{Card}(A) = +\infty \neq \mathrm{Card}(\{\omega\}) = 1$ □

Thus $2^\Omega \neq \mathcal{F}_\infty$.

Problem 5

By lemma 2, we have $\{\omega\} \neq \mathcal{F}_\infty$. If not, there exists $n \in \mathbb{N}$ such that $\omega \notin \mathcal{F}_n$, which lead to a contradiction. Now we just need to prove that $\{\omega\} \in \mathcal{B}(\Omega)$.