Homework 3

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Problem 1 (FTMC for countably infinite chains)

(1). We just need to prove [F]+[I] $\Longrightarrow$ [PR].
By [I] and [F], $\forall i \in [n]$, $\exists t_1, t_2, \alpha, \beta$ such that: $\mathbf{P}_i[X_{t_1} = i] = \alpha$ and $\forall j \neq i$, $\mathbf{P}_j[X_{t_2} = i] > \beta$.
Then we have $\exists t$, $\mathbf{P}_i[T_i \leq t] > \alpha$ and $\mathbf{P}_j[T_i \leq t] > \beta$. Thus

$$\mathbf{P}_i[T_i < nt] = 1 - (1-\alpha)\beta(1-\beta)^n \implies \mathbf{P}_i[T_i < \infty] = 1$$

and

$$\mathbb{E}[T_i] < \alpha(1 + \cdots + t) + \sum_{n=0}^{\infty} \beta(1-\alpha)(1-\beta)^n \left[\frac{t(t+1)}{2} + (n-1)t^2\right] < \infty.$$

Nota Bene: I use $\sum_{n=0}^{\infty} n\alpha^n < \infty$ where $|\alpha| < 1$. Finally,

$$[\mathrm{F}]+[\mathrm{A}]+[\mathrm{I}] \implies [\mathrm{PR}]+[\mathrm{A}]+[\mathrm{I}] \implies [\mathrm{S}]+[\mathrm{U}]+[\mathrm{C}].$$

(2). Let Q be the transition function of the markov chain on Ω^2. Q has several properties:

- Q has a stationary $\pi_{(i,j)} = \pi_i \pi_j$ by the fact that

$$\pi_i \pi_j = \sum_k P(k,i)\pi_k \times \sum_l P(l,j)\pi_l = \sum_k \sum_l P(k,i)P(l,j)\pi_k \pi_l.$$

- Q is irreducible: for all $i, j, k, l \in \mathbb{N}$, there exists $t \in \mathbb{N}$ such that

$$Q^t[(i,j),(k,l)] = P^t(i,k)P^t(j,l) > 0.$$

Then we have $\pi_{(i,j)} = 1/\mathbb{E}_{(i,j)}[T_{(i,j)}] > 0$, which means that $P_{(i,j)}[T_{(i,j)} < \infty] = 1$. This could entail $P_{(i,j)}[T_{(k,k)} < \infty] = 1$.

(3). We have proved [PR]+[I] $\Longrightarrow$ [S]+[U]. Now we construct a coupling and update X^t and Y^t by following rules: ($Y^0 = \pi$ and $X^0 = \mu$)

$$\begin{cases} X^{t+1} = X^t,\ Y^{t+1} = Y^t & \text{if } X^t = Y^t \\ X^t \to X^{t+1},\ Y^t \to Y^{t+1} \text{ Independently} & \text{if } X^t \neq Y^t \end{cases}.$$

By what we've proved in (2), $\forall i, j$, $\exists T$, $(X^t = i, Y^t = j)$ and $X^{t+T} = Y^{t+T} = k$. Thus we have $D_{TV}(\mu, \pi) \leq \lim_{t \to \infty} \Pr(X^t \neq Y^t) = 0$. Then [C] holds for countably infinite chain.

Problem 2 (A Randomized Algorithm for 3-SAT)

(1). Fistly we have $\mathbf{P}_{2n^2} \triangleq \Pr\left(\exists t \in [0, 2n^2], \text{ s.t.} Y_t = n\right) \geq 1/2$. Then

$$\mathbf{P}[\text{the new ALGO outputs the correct answer}] = 1 - (1 - \mathbf{P}_{2n^2})^{50} \leq 1 - 0.5^{50}.$$

(2). Suppose the clause is $p \vee q \vee r$. Let $\sigma(a)$ denote the ground truth of a. If $\sum_{p,q,r} 1[\sigma(i) = \text{True}] = i$, then $\Pr[X_{t+1} = X_t + 1] = i/3$. Thus

$$\Pr[X_{t+1} = X_t + 1] \geq \frac{1}{3} \text{ and } \Pr[X_{t+1} = X_t - 1] \leq \frac{2}{3}.$$

(3). Consider the 1-D randowalk $\{Y_t\}$:

$$Y_{t+1} = \begin{cases} Y_t + 1 & \text{w.p. } 1/3 \\ Y_t - 1 & \text{w.p. } 2/3 \end{cases}$$

where $Y_t \neq 0$ or n. If $Y_t = 0$, $Y_{t+1} = Y_t + 1$ w.p. 1 and if $Y_t = n$,then $Y_{t+1} = Y_t - 1$ w.p. 1. Then

$$\mathbb{E}[T_{i\to i+1}] = \frac{1}{3} + \frac{2}{3}(1 + \mathbb{E}[T_{i-1\to i}] + \mathbb{E}[T_{i\to i+1}]),$$

which entails

$$\mathbb{E}[T_{i\to i+1}] = 2\mathbb{E}[T_{i-1\to i}] + 3.$$

By $\mathbb{E}[t_{0\to 1}] = 1$:

$$\mathbb{E}[T_{i\to i+1}] = 2^{i+2} - 3.$$

Thus

$$\mathbb{E}[T_{i\to n}] = \sum_{k=i}^{n-1} \mathbb{E}[T_{k\to k+1}] = 2^{n+2} - 2^i - 3(n - i) < 2^{n+2}.$$

Finally,

$$1 - \Pr\left(\exists t \in [0, 400 \cdot 2^n] : Y_t = n\right) = \Pr[T_{Y_0\to n} > 400 \cdot 2^n] \leq \frac{\mathbb{E}[T_{Y_0\to n}]}{400 \cdot 2^n} = 0.01.$$

(4). $Y_0 = n - i$.

$$\begin{aligned}\Pr\left(\exists t \in [0, 3n] : Y_t = n\right) \geq& \Pr\left(Y_{3i} = n\right) \\ =& \binom{3i}{i}\left(\frac{1}{3}\right)^{2i}\left(\frac{2}{3}\right)^{i} \\ \geq& \sqrt{\frac{3}{4\pi i}}\left(\frac{27}{4}\right)^{i}\left(\frac{1}{3}\right)^{2i}\left(\frac{2}{3}\right)^{i}. \\ =& \sqrt{\frac{3}{4\pi i}}\left(\frac{1}{2}\right)^{i}\end{aligned}$$

The second inequality holds by stiling formula.

(5). By some tricks of inequality:

$$\begin{aligned}
&\Pr[\text{Output a satisfying assignment}] \\
=&\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\sqrt{\frac{3}{4\pi i}}\left(\frac{1}{2}\right)^i \\
\geq&\sqrt{\frac{3}{4\pi n}}\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(\frac{1}{2}\right)^i \\
=&\sqrt{\frac{3}{4\pi n}}\frac{1}{2^n}\left(\frac{3}{2}\right)^n \\
=&\sqrt{\frac{3}{4\pi n}}\left(\frac{3}{4}\right)^n
\end{aligned}.$$

(6). In order to:

$$\left[1-\sqrt{\frac{3}{4\pi n}}\left(\frac{3}{4}\right)^n\right]^k \leq 0.01.$$

k should satisfy:

$$k \geq -2 \bigg/ \log_{10}\left[1-\sqrt{\frac{3}{4\pi n}}\left(\frac{3}{4}\right)^n\right].$$

By the fact that $-\ln(1-x) > x$,

$$k = \sqrt{\frac{16\pi}{3}}\ln(10)\times\sqrt{n}\left(\frac{4}{3}\right)^n = \mathcal{O}\left[n^{0.5}\left(\frac{4}{3}\right)^n\right].$$

The algoithm should be:
Repeat for k times: "repeat the flipping process for $3n$ times, starting with some σ_0 which is uniform at random from all $2n$ assignments of the variables".
The complexity is

$$\mathcal{O}(nk) = \mathcal{O}\left[n^{1.5}\left(\frac{4}{3}\right)^n\right].$$

Reference of problem 2: Randomized Algorithms, National Tsing Hua University.