

Homework 1

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1 Probability Space of Tossing Coins

Problem 1

Obviously, $\emptyset \notin \{C_s\}_{s \in \{0,1\}^n}$.

By the fact that $C_s \subset \Omega$ and $\text{Card}(\{0,1\}^n) < +\infty$, we have $\bigcup_{s \in \{0,1\}^n} C_s \subset \Omega$.

Also, $\forall \omega \in \Omega$, $\omega \in C_{(\omega_1, \dots, \omega_n)}$, which means $\Omega \subset \bigcup_{s \in \{0,1\}^n} C_s$.

$\forall s_1, s_2 \in \{0,1\}^n$, $\forall \omega_1 \in C_{s_1}$ and $\omega_2 \in C_{s_2}$, $\omega_1 \neq \omega_2$. Then $C_{s_1} \cap C_{s_2} = \emptyset$.

Finally, $\{C_s\}_{s \in \{0,1\}^n}$ forms a partition of Ω .

Problem 2

Firstly, I will prove a lemma and a proposition.

Definition 1. $\mathcal{S}(\mathcal{C}) \stackrel{\text{def}}{=} \left\{ \bigcup_{i=1, \dots, m} A_i \mid m \in \{1, \dots, |\mathcal{C}|\}, A_i \in \mathcal{C} \right\} \cup \{\emptyset\}$

Lemma 1. If \mathcal{C} forms a partition of Ω , then $\mathcal{S}(\mathcal{C})$ is a σ -algebra.

Proof. $\emptyset, \Omega \in \mathcal{S}(\mathcal{C})$.

If $a \in \mathcal{S}(\mathcal{C})$, then for some $B_i \in \mathcal{C}$, $a = \bigcup_i B_i$. Thus $\Omega - a = \bigcup_{A_i \in \mathcal{C}} A_i - a \in \mathcal{S}(\mathcal{C})$.

If $A_i \in \mathcal{S}(\mathcal{C})$, then $\bigcup_i A_i = \bigcup_i \left(\bigcup_j B_{ij} \right) \in \mathcal{S}(\mathcal{C})$. □

Definition 2. $\mathcal{C}_n = \{C_s\}_{s \in \{0,1\}^n}$.

Proposition 1. $\mathcal{F}_n = \mathcal{S}(\mathcal{C}_n)$.

Proof. Trivially, $\mathcal{C}_n \subset \mathcal{S}(\mathcal{C}_n)$. Now we suppose that \exists σ -algebra \mathcal{F} s.t. $\mathcal{C}_n \subset \mathcal{F}$:

For all $A \in \mathcal{S}(\mathcal{C}_n)$. Then $A = \bigcup C_s$ for some s or $A = \emptyset$. By the definition of σ -algebra, $A \in \mathcal{F}$, which means $\mathcal{S}(\mathcal{C}_n) \subset \mathcal{F}$.

Thus $\mathcal{S}(\mathcal{C}_n)$ is the minimal σ -algebra containing sets in \mathcal{C}_n . □

Now we have $\text{Card}(\mathcal{F}_n) = \text{Card}(\mathcal{S}(\mathcal{C}_n)) = \text{Card}(2^{\mathcal{C}_n}) = \text{Card}(2^{\{0,1\}^n}) = 2^{2^n}$. We could say that $\mathcal{F}_n \sim 2^{\{0,1\}^n}$ or \mathcal{F}_n and $2^{\{0,1\}^n}$ are [equinumerous](#). So there exists a bijection between \mathcal{F} and $2^{\{0,1\}^n}$.

Problem 3

By lemma 1, $\forall n \in \mathbb{N}$, $\mathcal{F}_n = \mathcal{S}(\mathcal{C}_n)$.

And obviously $\mathcal{F}_n \neq \mathcal{F}_{n+1}$ because $\mathcal{C}_n \neq \mathcal{C}_{n+1}$ and $\text{Card}(\mathcal{C}_n) \neq \text{Card}(\mathcal{C}_{n+1})$.

$\forall A \in \mathcal{F}_n, A = \bigcup_i C_{s_i}$ where $s_i \in \{0, 1\}^n$. Then

$$A = \bigcup_i (C_{[s_i, 0]} \cup C_{[s_i, 1]}),$$

where the operator $[\cdot, \cdot]$ means a concatenation. For example, $[(0, 0, 1), 1] = (0, 0, 1, 1)$ and $[(0, 0, 1), (0, 1)] = (0, 0, 1, 0, 1)$. I also use this operator in Problem 6 and 7.

Ultimately $\forall n \in \mathbb{N}, \mathcal{F}_n = \mathcal{S}(\mathcal{C}_n) \subsetneq \mathcal{S}(\mathcal{C}_{n+1}) = \mathcal{F}_{n+1}$.

Problem 4

By definition, $\mathcal{F}_\infty = \bigcup_{n \geq 1} \mathcal{F}_n$.

For all $A, B \in \mathcal{F}_\infty, \exists m, n \in \mathbb{N}, A \in \mathcal{F}_m$ and $B \in \mathcal{F}_n$. Let $l \stackrel{def}{=} \max(m, n)$. Then $A, B \in \mathcal{F}_l$.

Thus $A^c, B^c \in \mathcal{F}_l$ and $A + B \in \mathcal{F}_l \implies A^c, B^c, A + B \in \mathcal{F}_l$.

The proposition that \mathcal{F}_∞ is an algebra holds.

Lemma 2. $\forall \omega \in \Omega, \forall n \in \mathbb{N}, \{\omega\} \notin \mathcal{F}_n$.

Proof. $\forall A \in \mathcal{F}_n, \text{Card}(A) = +\infty \neq \text{Card}(\{\omega\}) = 1$ □

Thus $2^\Omega \neq \mathcal{F}_\infty$.

Problem 5

By lemma 2, we have $\{\omega\} \notin \mathcal{F}_\infty$. Now we just need to prove that $\{\omega\} \in \mathcal{B}(\Omega)$.

Proof by contradiction. Suppose that $\exists \omega' \notin \mathcal{B}(\Omega)$.

$\forall n \in \mathbb{N}, \mathcal{F}_n \in \mathcal{B}(\Omega)$.

A creative construction:

$$T \stackrel{def}{=} \bigcup_{n=1, \dots, \infty} \{\omega \mid \omega_1 \neq \omega'_1 \wedge \dots \wedge \omega_n \neq \omega'_n\} \stackrel{def}{=} \bigcup_{n=1, \dots, \infty} T_n.$$

Here T is a countable union and every S_n is in $\mathcal{S}(\mathcal{C}_n) = \mathcal{F}_n$ ($T_n \in \mathcal{B}(\Omega)$). Hence $T \in \mathcal{B}(\Omega)$. Intuitively, $T = \Omega - \omega$.

Finally, $\{\omega\} = T^c \in \mathcal{B}(\Omega)$.

Problem 6

For all $A \in \mathcal{F}_\infty, \exists m(A) \in \mathbb{N}$ such that $A \in \mathcal{F}_{m(A)}$ and $\forall n \in \{1, 2, \dots, m(A) - 1\}, A \notin \mathcal{F}_n$. For simplicity, we denote $m(A)$ as m . It is worthy of mention that m **only** depends on A .

Now we want to prove $\forall n \geq m, 2k_n = k_{n+1}$.

If $A = C_{s_1} \cup \dots \cup C_{s_{k_n}}$, then

$$A = (C_{[s_1, 0]} \cup C_{[s_1, 1]}) \cup \dots \cup (C_{[s_{k_n}, 0]} \cup C_{[s_{k_n}, 1]}).$$

Trivially, $[s_1, 0] \neq [s_1, 1] \neq \dots \neq [s_{k_n}, 0] \neq [s_{k_n}, 1]$. Thus we conclude $k_{n+1} = 2k_n$ and $k/2^n = \text{Const}$ for any fixed A .

Problem 7

I've talked with Liyuan Mao and Wei Jiang for days. Here I will give several attempts I make, though I could not prove the proposition successfully.

Attempt 1.

Lemma 3. \forall set A and set B , $A \cap B = A - (A - B) = A - (A^c \cup B)^c = [A^c \cup (A^c \cup B)^c]^c$.

Proposition 2. $\mathcal{B}(\Omega) = \left\{ \bigcup_{i=1}^{\infty} \left(\bigcap_{j=1}^{\infty} A_{ij} \right) \mid A_{ij} \in \mathcal{F}_{\infty} \right\}$.

Proof. Obviously, $\emptyset \in \mathcal{B}(\Omega)$.

$\forall A \in \mathcal{B}(\Omega)$: $A^c = \left[\bigcup_{i=1}^{\infty} \left(\bigcap_{j=1}^{\infty} A_{ij} \right) \right]^c = \bigcap_{i=1}^{\infty} \left(\bigcup_{j=1}^{\infty} A_{ij}^c \right)$. (**Failure.** I could not prove this.) \square

Remark of the attempt 1: I want to construct \mathcal{B} directly, while I could not prove this set is a σ -algebra. The idea is intuitively from $\forall A \in \mathcal{B}$, A could be generated from C_{s_n} and ω_n .

Attempt 2. $\forall A \in \mathcal{B}(\Omega)$:

$P_n(A) \stackrel{\text{def}}{=} P_n(\bigcup A_i)$ where $A_i \in \mathcal{F}_n$ and $\bigcup A_i$ is the smallest cover of A . Then define $P(A) = \lim_{n \rightarrow \infty} P_n(A)$. (It is obvious that $P(A)$ is bounded and decreasing.)

However, I could not prove the uniqueness of $P(A)$.

Finally.

I find a more general theorem called Caratheodory's extension theorem. In this theorem we could find several creative constructions:

$$\lambda(E) \stackrel{\text{def}}{=} \inf \left\{ \sum_{j=1}^{\infty} \Pr(A_j) : A_j \in \mathcal{F}, E \subset \bigcup_j A_j \right\}$$

and

$$\bar{\mathcal{F}} \stackrel{\text{def}}{=} \{E : \lambda(F) = \lambda(F \cap E) + \lambda(F \cap E^c) \forall F \subset \Omega\}.$$

Final remark: I found the similarity between the definition of $P_n(A)$ in Attempt 2 and $\lambda(E)$.

Reference: [Fundamentals of Probability, MIT](#).

Problem 8

Let $G_n = \{\omega \mid \omega_1 = \omega_2 = \dots = \omega_{n-1} = 0, \omega_n = 1 \text{ and } \omega \in \Omega\}$ and $\Omega = \bigcup_i G_i$.

Then $X : \Omega \mapsto \mathbb{R}$, $X(\omega) = n$ if $\omega \in G_n$. Obviously $P(X = n) = \Pr(G_n) = 1/2^n$.

2 Conditional Expectation

Problem 1

$\forall A \in \mathcal{B}$: $X^{-1}(A) = \{\omega \mid X(\omega) \in A\} \in \sigma(X)$ and $f^{-1}(A) \in \mathcal{B}$.

Now we have $\mathcal{G} \stackrel{\text{def}}{=} \{X(\omega) \mid f(X(\omega)) \in A\} \in \mathcal{B}$. Thus

$$[f(X)]^{-1}(A) = \left\{ \omega \mid f(X(\omega)) \in A \right\} = \left\{ \omega \mid X(\omega) \in \mathcal{G} \right\} \in \sigma(X)$$

which means that $f(X)$ is $\sigma(X)$ -measurable.

Problem 2

Proposition 3. $\forall \omega \in \Omega, \exists A \in \sigma(Y)$ such that $Y^{-1}(\omega) = A$ and $\forall B \in \sigma(Y) : A \cap B = \emptyset \vee A \subset B$.

Proof. If not, then $\exists B \neq \emptyset, A \cap B \neq A$. Let $A_1 = A \cap B, A_2 = A \cup B^c$. $A_1, A_2 \in \sigma(Y)$ such that $A_1 \cup A_2 = Y^{-1}(\omega)$ and $A_1 \cap A_2 = \emptyset$. Obviously $\sigma(Y)$ is **not** the minimal σ -algebra such that Y is $\sigma(Y)$ -measurable. Contradiction. \square

Then $\forall \omega \in \Omega, Y^{-1}[Y(\omega)] = Y'^{-1}[Y'(\omega)]$. (If not, let $B = Y^{-1}[Y(\omega)] \cap Y'^{-1}[Y'(\omega)]$. Then B is a set which contradicts the Proposition 3.

Problem 3

$\mathbb{E}[X \mid \mathcal{F}] = \mathbb{E}[X \mid Y]$ where $\sigma(Y) = \mathcal{F}$.

Remark: after I finished Problem 3 and 4 I found that the existence of Y may not always hold for every \mathcal{F} . The existence of Y depends on whether \mathcal{F} is generated from partitions of Ω , (i.e., if we could find the partitions generating \mathcal{F} , then we have a Y). However, for more general cases I could not define it clearly (may use Lebergue Integral).

Problem 4

Suppose that $\sigma(Y_1) = \mathcal{F}_1$ and $\sigma(Y_2) = \mathcal{F}_2$.

$$\begin{aligned} \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_1] \mid \mathcal{F}_2](\omega_0) &= \sum_{a \in \text{Ran}(f_Y)} a \times \Pr[f_{Y_1} = a \mid Y_2 = Y_2(\omega_0)] \\ &= f_{Y_1}(\omega_0) \times \Pr[f_{Y_1} = f_{Y_1}(\omega_0) \mid Y_2 = Y_2(\omega_0)] \\ &= f_{Y_1}(\omega_0) \end{aligned} \tag{1}$$

The second equality of Eq.1 holds by the fact that:

$$\forall \omega \text{ such that } a = f_{Y_1}(\omega) \neq f_{Y_1}(\omega_0), [f_{Y_1} = a] \cap [Y_2 = Y_2(\omega_0)] = \emptyset.$$

The third equality of Eq.1 holds by the fact that:

$$[Y_2 = Y_2(\omega_0)] \subset [f_{Y_1} = f_{Y_1}(\omega_0)].$$

Let $\mathcal{D}_0 \stackrel{\text{def}}{=} \{b \in \mathbb{R} \mid \exists \omega \in \Omega \text{ such that } Y_2^{-1}(Y_2(\omega)) \subset Y_1^{-1}(Y_1(\omega)) \text{ and } b = f_{Y_2}(\omega)\}$. The interpretation of \mathcal{D}_0 : $\forall A \in \sigma(Y_1)$, there exists several sets $A_i \in \sigma(Y_2)$ such that $\bigcup A_i = A$. Here we let $A = Y_1^{-1}(\omega)$ and $\{A_i\}$ is the partition of $\sigma(Y_2)$.

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[X \mid Y_2] \mid Y_1](\omega_0) &= \frac{\sum_{b \in \text{Ran}(f_{Y_2})} b \times \Pr(f_{Y_2} = b \cap Y_1 = Y_1(\omega_0))}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_{b \in \mathcal{D}_0} b \times \Pr(f_{Y_2} = b \cap Y_1 = Y_1(\omega_0))}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_{b \in \mathcal{D}_0} \mathbb{E}(X \mid Y_2 = Y_2(\omega_b)) \Pr(f_{Y_2} = b)}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_{b \in \mathcal{D}_0} \mathbb{E}(X \mid Y_2 = Y_2(\omega_b)) \Pr(Y_2 = Y_2(\omega_b))}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_{b \in \mathcal{D}_0} \sum_a \mathbb{E}(X = a \mid Y_2 \cap Y_2(\omega_b))}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_a \sum_{b \in \mathcal{D}_0} \mathbb{E}(X = a \mid Y_2 \cap Y_2(\omega_b))}{\Pr(Y_1 = Y_1(\omega))} \\
&= \frac{\sum_a \Pr(X = a \cap Y_1 = Y_1(\omega_0))}{\Pr(Y_1 = Y_1(\omega))} \\
&= f_{Y_1}(\omega_0)
\end{aligned} \tag{2}$$

Ultimately, with EQ. 1 and EQ. 2 we prove the proposition of problem 4.