Homework 5

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Problem 1

By defination, $n \sim \mathrm{Pois}(s\lambda)$. Thus

$$\Pr(\text{achieve his goal}) = \Pr(n = 1) = s\lambda \exp(-s\lambda).$$

Then

$$s_{\text{opt}} = \arg\max_s \Big[s\lambda \exp(-s\lambda) \Big] = \frac{1}{\lambda}.$$

Problem 2

(a)

Apparently

$$\begin{aligned} & \Pr[X = \lambda + k] \geq \Pr[X = \lambda - k - 1] \\ \iff & \lambda^{\lambda+k} \big/ (\lambda + k)! \geq \lambda^{\lambda-k-1} \big/ (\lambda - k - 1)! \\ \iff & \lambda^{2k+1} \geq \prod_{i=-k}^{k} (\lambda + i) \end{aligned}$$

And the last line holds by the fact that $(\lambda - i)(\lambda + i) \leq \lambda^2$. Then we have

$$2\Pr[X \geq \lambda] \geq \Pr[X \geq \lambda] + \Pr[X < \lambda] = 1$$

which entails that $\Pr[X \geq \lambda] \geq 1/2$

(b)

Let $\mathbb{E}_m[X]$ denotes the expectation of X with respect to m. Intuitively and inspired by Cor.4,

$$\begin{aligned}
\mathbb{E}_m[f(Y_1,\cdots,Y_n)] &= \sum_{k=0}^{\infty} \mathbb{E}\left[f(Y_1,\cdots,Y_n)|\sum_i^n Y_i = k\right] \Pr\left[\sum_i^n Y_i = k\right] \\
&= \sum_{k=0}^{\infty} \mathbb{E}_k[f(X_1,\cdots,X_n)] \Pr\left[\sum_i^n Y_i = k\right] \\
&\geq \sum_{k=m}^{\infty} \mathbb{E}_k[f(X_1,\cdots,X_n)] \Pr\left[\sum_i^n Y_i = k\right] \\
&\geq \mathbb{E}_m[f(X_1,\cdots,X_n)] \sum_{k=m}^{\infty} \Pr\left[\sum_i^n Y_i = k\right] \\
&\geq \frac{1}{2}\mathbb{E}_m[f(X_1,\cdots,X_n)]
\end{aligned}$$

where the second inequality holds by $E_m[f(X_1,\cdots,X_n)]$ is monotonically increasing in m and the third inequlity holds by (a).

(c)

Firstly we define some useful symbols. Let X_i be the number of students whose birthday is on day i ($i \in [365]$). Then $f(X_1,\cdots,X_m) \triangleq \mathbb{1}(\exists i \in [m], X_i \geq 4)$ where $\mathbb{1}(\cdot)$ is an indicator. Hence we could deduce that $\Pr[f(X_1,\cdots,X_m)=1] = \mathbb{E}[f(X_1,\cdots,X_m)]$. And obviously, $\Pr[f(X_1,\cdots,X_m)=1]$ is monotonically increasing in n (the advent of a new student will increase the probability absolutely).

$$\Pr\left[\exists i \in [m], X_i \geq 4\right] \leq 2\Pr\left[\exists i \in [m], Y_i \geq 4\right] = 2\Big(1 - \Pr[Y_1 < 4]^{365}\Big).$$

Trivilly,

$$\Pr[Y_1 < 4] = \left(1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6}\right)\exp(-\lambda).$$

When $\lambda = 50/365$

$$\Pr\left[\exists i \in [m], X_i \geq 4\right] \leq 9.58 \times 10^{-3} \leq 1\%.$$