CS2601 Linear and Convex Optimization
Homework 5

Due: 2021.11.4

1. Let $\bar{B} = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq 1\}$ be the closed unit 2-norm ball. Recall the projection of a point $\boldsymbol{x}_0$ onto $\bar{B}$ is the solution to the following problem,

$$\min_{\boldsymbol{x} \in \bar{B}} f(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}_0\|^2$$

Use the first-order optimality condition to show that the projection of $\boldsymbol{x}_0 \notin \bar{B}$ is $\hat{\boldsymbol{x}}_0 = \frac{\boldsymbol{x}_0}{\|\boldsymbol{x}_0\|}$.

For the rest of of this assignment, we need CVXPY (https://www.cvxpy.org/).

2. Consider the optimization problem

$$\begin{aligned} \min_{x_1, x_2} \quad & f(x_1, x_2) \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Sketch the feasible set. For the objective functions in (a)-(c), first find the set of optimal solutions and the optimal value graphically, and then solve them using CVXPY. For the objective functions in (d) and (e), solve them using CVXPY. Show the outputs of CVXPY. Check the first example at https://www.cvxpy.org/tutorial/intro/index.html. You can use `cvxpy.maximum(x,y)` to get the maximum of two scalar variables `x` and `y`, or `cvxpy.max(x)` to get the maximum of the components of a vector variables `x`.

(a). $f(x_1, x_2) = x_1 + x_2$

(b). $f(x_1, x_2) = -x_1 - x_2$

(c). $f(x_1, x_2) = x_1$

(d). $f(x_1, x_2) = \max\{x_1, x_2\}$

(e). $f(x_1, x_2) = x_1^2 + 9x_2^2$

3. Consider the following optimization problem with $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{b} \in \mathbb{R}^m$.

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_1 \\ \text{s.t.} \quad & \|\boldsymbol{x}\|_\infty \le 1 \end{aligned} \tag{1}$$

(a). Reformulate problem (1) as an LP.

(b). Solve the original problem in (1) using CVXPY for $m = 3$, $n = 2$,

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \\ 1 & 2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 5 \\ 10 \\ -5 \end{pmatrix}.$$

Show the output of CVXPY. To see how to use vectors and matrices in CVXPY, check the examples at `https://www.cvxpy.org/tutorial/intro/index.html#vectors-and-matrices`. The p-norm of a vector `x` is given by `cvxpy.norm(x, p)`. Use `"inf"` to get the ∞-norm. Alternatively, you can use `cvxpy.norm2(x), cvxpy.norm1(x), cvxpy.norm_inf(x)` to get the 2-, 1-, and ∞-norms, respectively.

(c). Solve the LP problem you find in part (a) using CVXPY. Show the output of CVXPY. You can check the example here `https://www.cvxpy.org/examples/basic/linear_program.html`.

4. Let $\boldsymbol{w} \in \mathbb{R}^2$ and

$$\boldsymbol{X} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}.$$

Solve problem (a) by hand and problems (b), (c) with CVXPY and show the outputs.

(a). Linear least squares regression

$$\min_{\boldsymbol{w}} \|\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}\|_2^2$$

(b). Lasso

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \|\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}\|_2^2 \\ \text{s.t.} \quad & \|\boldsymbol{w}\|_1 \le t \end{aligned}$$

for $t = 1$ and $t = 10$. In each case, is the solution the same as that of (a)? Does the solution have zero components? (Note that there are numerical errors, which you should ignore when answering the questions.)

(c). Ridge regreesion

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \|\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}\|_2^2 \\ \text{s.t.} \quad & \|\boldsymbol{w}\|_2^2 \le t \end{aligned}$$

for $t = 1$ and $t = 100$. In each case, is the solution the same as that of (a)? Does the solution have zero components? (Again ignore the numerical errors.)