Written assginment 1

Haoyu Zhen

February 27, 2022

Question 1

(a)

$\|x-y\|^2 = \int_0^T [x(t)-c]^2 \, \mathrm{d}t = \int_0^T x^2(t) \, \mathrm{d}t - 2c \int_0^T x(t) \, \mathrm{d}t + c^2 T$. To minimize the distance between x and y,

$$c = \frac{1}{T} \int_0^T x(t) \, \mathrm{d}t.$$

(b)

Let $e(t) = 1 \Big/ \sqrt{T}$ if $t \in [0, T)$ otherwise $e(t) = 0$. Then $\forall f \in V$, we have $f = f(0)e(t)$, which means that $V = \operatorname{span}(e)$. And $\int_0^T \left(1 \Big/ \sqrt{T}\right)^2 \mathrm{d}t = 1 \Longrightarrow \langle e, e \rangle = 1$.
Then

$$y(t) = \left[\int_0^T x(t)e(t) \, \mathrm{d}t\right] e(t) = \left[\frac{1}{\sqrt{T}} \int_0^T x(t) \, \mathrm{d}t\right] e(t).$$

Thus $y(t) = \frac{1}{T} \int_0^T x(t) \, \mathrm{d}t$ if $t \in [0, T)$ otherwise $e(t) = 0$.

(c)

Obviously the result in (a) coincides with that in (b). I prefer the second method because we no not need to optimize any objective function.

Question 2

(a)

Proposition 1. *$\forall f \in V$,*

$$f(t) = f(-1)\varphi_{-1}(t) + f(0)\varphi_0(t) + f(1)\varphi_1(t).$$

Proof. $\forall t \in [0, 1]$, $f(t) = [f(1) - f(0)]t + f(0) = f(0)\varphi_0(t) + f(1)\varphi_1(t)$.
And $\forall t \in [-1, 0)$, $f(t) = [f(0) - f(-1)]t + f(0) = f(-1)\varphi_{-1}(t) + f(0)\varphi_0(t)$. $\square$

Thus $f = f(-1)\varphi_{-1} + f(0)\varphi_0 + f(1)\varphi_1$.

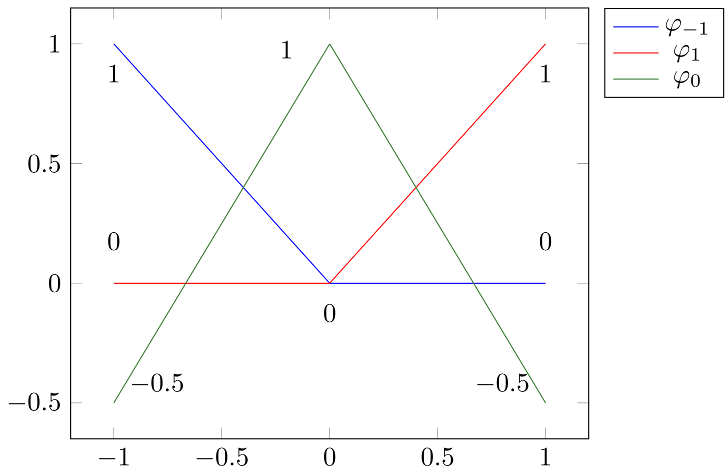
(b)

Obviously, φ_1 and φ_{-1} are orthogonal.

(c)

$\int_{-1}^{1} \varphi_{-1}(t)\varphi_0(t)\,\mathrm{d}t = \int_{-1}^{1} \varphi_1(t)\varphi_0(t)\,\mathrm{d}t = \frac{1}{6}$ and $\int_{-1}^{1} \varphi_{-1}^2(t)\,\mathrm{d}t = \int_{-1}^{1} \varphi_1^2(t)\,\mathrm{d}t = \frac{1}{3}$. Then

$$\hat{\varphi}_0 = \varphi_0 - \frac{1}{2}\varphi_{-1} - \frac{1}{2}\varphi_1 = \begin{cases} 1.5t + 1, & \text{if } t \in [-1, 0] \\ 1 - 1.5t, & \text{if } t \in (0, 1] \\ 0, & \text{otherwise} \end{cases}.$$



(d)

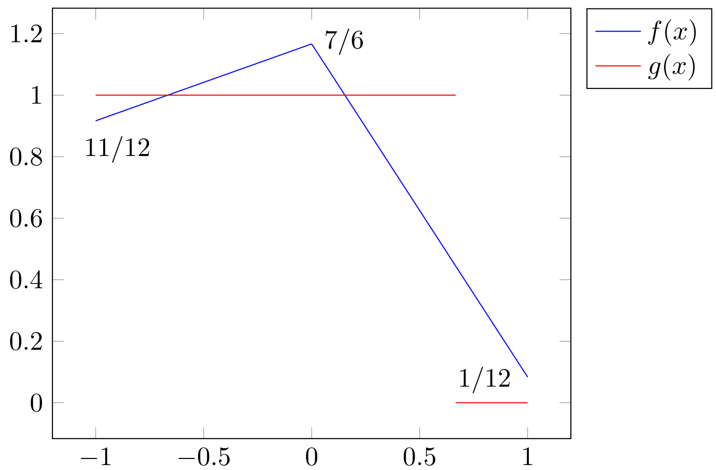
What we want to solve is:

$$f_0 = \arg\min_f \int_{-1}^{1} [g(t) - f(t)]^2\,\mathrm{d}t = \arg\min_f \left[\int_{-1}^{1} f^2(t)\,\mathrm{d}t - 2\int_{-1}^{1} g(t)f(t)\,\mathrm{d}t\right].$$

Assume that $f = a\varphi_{-1} + b\hat{\varphi}_0 + c\varphi_1$.
Then $\int_{-1}^{1} f^2(t)\,\mathrm{d}t = \frac{1}{3}a^2 + \frac{1}{2}b^2 + \frac{1}{3}c^2$ and $\int_{-1}^{1} g(t)f(t)\,\mathrm{d}t = \frac{1}{2}a + \frac{7}{12}b + \frac{2}{9}c$.
By the fact that

$$\arg\min_{[a,b,c]} \left(\frac{1}{3}a^2 + \frac{1}{2}b^2 + \frac{1}{3}c^2 - a - \frac{7}{6}b - \frac{4}{9}c\right) = \left[\frac{3}{2}, \frac{7}{6}, \frac{2}{3}\right],$$

$f = \frac{3}{2}\varphi_{-1} + \frac{7}{6}\hat{\varphi}_0 + \frac{2}{3}\varphi_1$.


$f(x)$
$g(x)$
7/6
11/12
1/12
1.2
1
0.8
0.6
0.4
0.2
0
−1
−0.5
0
0.5
1