Exercise Sheet 20

Discrete Mathematics, 2020.12.15

1. ([R], Page 557, Exercise 8) In a survey of 270 college students, it is found that 64 like brussels sprouts, 94 like broccoli, 58 like cauliflower, 26 like both brussels sprouts and broccoli, 28 like both brussels sprouts and cauliflower, 22 like both broccoli and cauliflower, and 14 like all three vegetables. How many of the 270 students do not like any of these vegetables?
2. ([R], Page 558, Exercise 22) Prove the principle of inclusion–exclusion using mathematical induction.
3. ([R], Page 565, Exercise 13) How many derangements are there of a set with seven elements?
4. ([R], Page 565, Exercise 16) A group of n students is assigned seats for each of two classes in the same classroom. How many ways can these seats be assigned if no student is assigned the same seat for both classes?
5. ([R], Page 565, Exercise 17) How many ways can the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 be arranged so that no even digit is in its original position?

1. ([R], Page 557, Exercise 8) In a survey of 270 college students, it is found that 64 like brussels sprouts, 94 like broccoli, 58 like cauliflower, 26 like both brussels sprouts and broccoli, 28 like both brussels sprouts and cauliflower, 22 like both broccoli and cauliflower, and 14 like all three vegetables. How many of the 270 students do not like any of these vegetables?

Let U be the set of all students, $A_1 = \{a \mid a \text{ likes BS}\}$, $A_2 = \{a \mid a \text{ likes Br}\}$ and $A_3 = \{a \mid a \text{ likes Ca}\}$

$$|U| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1| - |A_1 \cap A_2 \cap A_3|$$

$$= 270 - 64 - 94 - 58 + 26 + 28 + 22 - 14$$

$$= 116$$

2. ([R], Page 558, Exercise 22) Prove the principle of inclusion–exclusion using mathematical induction.

① if $n = 1, 2$, obviously $|A_1| = |A_1|$, $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

② Suppose that when $n = m$ we have $\left|\bigcup_{i=1}^{m} A_i\right| = \sum_{l=1}^{m} \sum_{1 \le k_1 < \cdots < k_l \le m} (-1)^{l+1} \cdot \left|\bigcap_{i=1}^{l} A_{k_l}\right|$

③ When $n = m+1$

$$\left|\bigcup_{i=1}^{m+1} A_i\right| = \left|\left(\bigcup_{i=1}^{m} A_i\right) \cup A_{m+1}\right|$$

$$= \left|\bigcup_{i=1}^{m} A_i\right| + |A_{m+1}| - \left|\left(\bigcup_{i=1}^{m} A_i\right) \cap A_{m+1}\right|$$

$$= \left|\bigcup_{i=1}^{m} A_i\right| + |A_{m+1}| - \left|\bigcup_{i=1}^{m} (A_i \cap A_{m+1})\right|$$

$$= \sum_{l=1}^{m} \sum_{1 \le k_1 < \cdots < k_l \le m} (-1)^{l+1} \cdot \left|\bigcap_{i=1}^{l} A_{k_l}\right| + |A_{m+1}| - \sum_{l=1}^{m} \sum_{1 \le k_1 < \cdots < k_l \le m} (-1)^{l+1} \cdot \left|\bigcap_{i=1}^{l} (A_{k_i} \cap A_{m+1})\right|$$

$$= \sum_{l=1}^{m} \sum_{1 \le k_1 < \cdots < k_l \le m} (-1)^{l+1} \cdot \left|\bigcap_{i=1}^{l} A_{k_l}\right| + |A_{m+1}| - \sum_{l=1}^{m} \sum_{1 \le k_1 < \cdots < k_l \le m} (-1)^{l+1} \cdot \left|\left(\bigcap_{i=1}^{l} A_{k_l}\right) \cap A_{m+1}\right|$$

$$= \underbrace{\sum_{l=1}^{m} \sum_{1 \le k_1 < \cdots < k_l < m+1} (-1)^{l+1} \cdot \left|\bigcap_{i=1}^{l} A_{k_l}\right|}_{A_{m+1}\text{不在其中的所有情况}} + \underbrace{|A_{m+1}| + \sum_{l=1}^{m} \sum_{1 \le k_1 < \cdots < k_l < k_{l+1} = m+1} (-1)^{l+2} \cdot \left|\bigcap_{i=1}^{l+1} A_{k_i}\right|}_{A_{m+1}\text{在其中的所有情况}}$$

$$= \sum_{l=1}^{m+1} \sum_{1 \le k_1 < \cdots < k_l \le m+1} (-1)^{l+1} \cdot \left|\bigcap_{i=1}^{l} A_{k_l}\right|$$

3. ([R], Page 565, Exercise 13) How many derangements are there of a set with seven elements?

$$7! - 7 \times 6! + C_7^2 \cdot 5! - C_7^3 \cdot 4! + C_7^4 \cdot 3! - C_7^5 \cdot 2! + C_7^6 \cdot 1! - C_7^7$$

$$= 1854$$

4. ([R], Page 565, Exercise 16) A group of n students is assigned seats for each of two classes in the same classroom. How many ways can these seats be assigned if no student is assigned the same seat for both classes?

First step: assign students seats for the first class

we have $n!$ cases. (Permutation)

Second step: assign students seats for the first class

we have $n! \cdot \sum_{i=0}^{n} (-1)^i \frac{1}{i!}$ cases. (Derangement)

By multiplication: answer is $(n!)^2 \cdot \sum_{i=0}^{n} (-1)^i \frac{1}{i!}$

2170680

5. ([R], Page 565, Exercise 17) How many ways can the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 be arranged so that no even digit is in its original position?

Let U be the set of all the ways with no constraint.

A_i = the set of ways in which i is in its original position.

Obviously $|A_0| = 0$, $|U| = 9 \cdot 9!$

$|A_{2i}| = 8 \times 8!$, $|A_{2i} \cap A_{2j}| = 7 \cdot 7!$, $|A_{2i} \cap A_{2j} \cap A_{2k}| = 6 \cdot 6!$ $|A_{2i} \cap A_{2j} \cap A_{2k} \cap A_{2l}| = 5 \cdot 5!$

Thus, $9 \cdot 9! - 4 \cdot 8 \times 8! + C_4^2 \cdot 7 \cdot 7! - 4 \cdot 6 \cdot 6! + 5 \cdot 5!$

$$= 2170680$$