Exercise Sheet 12

Discrete Mathematics, 2020.11.3

1. We call f a choice function of S if $f : S \to \bigcup S$ and $f(X) \in X$ for any $X \in S$. Thus, the axiom of choice actually says: if $\emptyset \notin S$ then S has at least a choice function. Now, please determine whether f is a choice function of S in the following examples. (If yes, you only need to say yes. If no, briefly explain why.)

   a) $S = \{\{0\}, \{1\}, \{2\}\}$, $f = \{(\{0\}, 0), (\{1\}, 1), (\{2\}, 2)\}$

   b) $S = \{\{0\}, \{1\}, \{2\}\}$, $f = \{(0, 0), (1, 1), (2, 2)\}$

   c) $S = \emptyset$, $f = \emptyset$

   d) $S = \mathcal{P}(\mathbb{N})$, $f = \{(\{m \in \mathbb{N} \mid m \geq n\}, n) \mid n \in \mathbb{N}\}$

2. Suppose R is an equivalence relation on A. Prove that there exists an injection from $\{[a]_R \mid a \in A\}$ into A, based on the axiom of choice.

1. We call f a choice function of S if $f : S \to \bigcup S$ and $f(X) \in X$ for any $X \in S$. Thus, the axiom of choice actually says: if $\emptyset \notin S$ then S has at least a choice function. Now, please determine whether f is a choice function of S in the following examples. (If yes, you only need to say yes. If no, briefly explain why.)

   a) $S = \{\{0\}, \{1\}, \{2\}\}$, $f = \{(\{0\}, 0), (\{1\}, 1), (\{2\}, 2)\}$
   b) $S = \{\{0\}, \{1\}, \{2\}\}$, $f = \{(0, 0), (1, 1), (2, 2)\}$
   c) $S = \emptyset$, $f = \emptyset$
   d) $S = \mathcal{P}(\mathbb{N})$, $f = \{(\{m \in \mathbb{N} \mid m \geq n\}, n) \mid n \in \mathbb{N}\}$

(a) Yes

(b) No. $0 \notin S$. f is not a function from S to $\bigcup S$.

(c) Yes

(d) No. $\{1\} \in \mathcal{P}(\mathbb{N})$, but $\{1\} \notin \{\{m \in \mathbb{N} \mid m \geq n\} \mid n \in \mathbb{N}\}$

So f is not a function from S to $\bigcup S$.

(Actually, $\emptyset \in S$)

2. Suppose R is an equivalence relation on A. Prove that there exists an injection from $\{[a]_R \mid a \in A\}$ into A, based on the axiom of choice.

$S = \{[a]_R \mid a \in A\}$, $\bigcup S = A$. $\emptyset \notin S$

Then exist a function $f : S \to A$, s.t $f(X) \in X$, for any $X \in S$

Now we want to prove that f is a injection:

$\forall X_1, X_2 \in S$, $\exists a \in A$

$f(X_1) = f(X_2) = a \Rightarrow$

$a \in X_1 \land a \in X_2 \Rightarrow$

$X_1 \cap X_2 \neq \emptyset \Rightarrow$

$X_1 = X_2$

Q.E.D.