CS2601 Linear and Convex Optimization

Homework 2

Due: 2021.10.9

1. Let $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ be an affine function from $\mathbb{R}^n$ to $\mathbb{R}^m$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Show that if $C \subset \mathbb{R}^m$ is convex, so is its inverse image $f^{-1}(C) \triangleq \{\boldsymbol{x} : f(\boldsymbol{x}) \in C\}$.

2. Suppose C_1, C_2 are two nonempty convex sets with $C_1 \cap C_2 = \emptyset$. Show $C = C_1 - C_2 = \{\boldsymbol{x}_1 - \boldsymbol{x}_2 : \boldsymbol{x}_1 \in C_1, \boldsymbol{x}_2 \in C_2\}$ is a nonempty convex set and $\boldsymbol{0} \notin C$.

3. Suppose C is a convex set.

(a). Show that its interior $\operatorname{int} C$ is convex.

(b). Show that its closure $\bar{C}$ is convex.

4. Prove the convex hull $\operatorname{conv} S$ of S is the set of all convex combinations of points in S by completing the following steps.

(a). Let

$$C = \left\{ \sum_{i=1}^{m} \theta_i \boldsymbol{x}_i : m \in \mathbb{N}; \boldsymbol{x}_i \in S, \theta_i \geq 0, i = 1, \ldots, m; \sum_{i=1}^{m} \theta_i = 1 \right\}$$

Show that C is convex.

(b). Show that $C \subset \operatorname{conv} S$ and conclude $C = \operatorname{conv} S$.

5. Let $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_K$ be distinct points in $\mathbb{R}^n$. Let V be the set of points that are closer in Euclidean distance to $\boldsymbol{x}_0$ than $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K$, called the **Vonoroi region** around $\boldsymbol{x}_0$ with respect to $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K$, i.e.

$$V = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{x}_0\|_2 \leq \|\boldsymbol{x} - \boldsymbol{x}_i\|_2, i = 1, 2, \ldots, K\}.$$

Show that V is a polyhedron by identifying $\boldsymbol{A}$ and $\boldsymbol{b}$ such that $V = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$. (You don't have to draw it in the submission, but try to visualize V for $\mathbb{R}^2$.)