Question 1

$A[n] = \{a_0, a_1, \cdots, a_m\}$ denotes $A[0] = a_0, A[1] = a_1, \cdots, A[m] = a_m$.

$$x[n] = \{1, 0, -1, 0\} \text{ and } h[n] = \{1, 2, 4, 8\}.$$

(a).

$$X[k] = \sum_n x[n]e^{-j\pi kn/2} = \{0, 2, 0, 2\}.$$

(b).

$$H[k] = \sum_n h[n]e^{-j\pi kn/2} = \{15, -3+6j, -5, -3-6j\}.$$

(c). By defination,

$$x[n] \text{ ④ } h[n] = \{-3, -6, 3, 6\}.$$

(d). $Y[n] = X[n] \cdot H[n] = \{0, -6+12j, 0, -6-12j\}$. Thus

$$y[n] = \frac{1}{N}\{\text{DFT}(Y^*[n])\}^* = \frac{1}{4}\{-12, -24, 12, 24\} = \{-3, -6, 3, 6\}.$$

Question 2

(a). Case I: $\forall r \in \mathbb{Z}, m-k \neq r(N+1)$.

$$\begin{aligned}
\text{LHS} &= \sum_{n=N/2}^{N/2} \exp\left[-\frac{2j\pi n(m-k)}{N+1}\right] \\
&= \left\{\exp\left[\frac{j\pi(m-k)N}{N+1}\right] - \exp\left[\frac{j\pi(m-k)(N+2)}{N+1}\right]\right\} \Big/ \left\{1 - \exp\left[-\frac{2j\pi(m-k)}{N+1}\right]\right\}. \\
&= 0
\end{aligned}$$

Case II: $\exists r \in \mathbb{Z}, m-k = r(N+1)$.

$$\text{LHS} = \sum_{n=N/2}^{N/2} 1 = N+1.$$

Thus LHS = RHS.

(b). I use Dirac Braket $\langle\cdot|\cdot\rangle$ to represent inner product.
Then

$$\vec{F} = \frac{1}{N+1}\sum_{k=-N/2}^{N/2} \langle f|\hat{e}_k\rangle\, \hat{e}_k.$$

By the orthogonality property that $\langle\hat{e}_k|\hat{e}_m\rangle = 0$ if $m \neq k$ otherwise $\langle\hat{e}_k|\hat{e}_m\rangle = N+1$:

$$f_m = \langle F|\hat{e}_m\rangle = \frac{1}{N+1}\sum_{k=-N/2}^{N/2} \langle f|\hat{e}_k\rangle \langle\hat{e}_k|\hat{e}_m\rangle = \langle f|\hat{e}_m\rangle\,,$$

which means

$$f_m = \sum_{k=-N/2}^{N/2} F_k W_{N+1}^{mk}.$$

(c).

$$\lim_{N\to\infty} x_{\pm N/2} = \lim_{N\to\infty} \pm\frac{NA}{2(N+1)} = \pm\frac{A}{2}.$$

Question 3

By defination of DFT:

$$F[k] = \mathrm{DFT}(\bar{f}_n)[k] = \begin{cases} 1 & k=0 \\ (-2-\sqrt{2}/2) + (2+3\sqrt{2}/2)j & k=1 \\ j & k=2 \\ (-2+\sqrt{2}/2) + (-2+3\sqrt{2}/2)j & k=3 \\ -1 & k=4 \\ (-2+\sqrt{2}/2) + (2-3\sqrt{2}/2)j & k=5 \\ -j & k=6 \\ (-2-\sqrt{2}/2) - (2+3\sqrt{2}/2)j & k=7 \end{cases},$$

which is not odd and imaginary. This is because $f(x)$ is not well-defined when implementing periodic extension. I.e., $f(-0.5) \neq f(-0.5+1) = f(0.5)$. To ameliorate it, the input sequence should be:

$$\bar{f}_n = \{0,\,-1,\,-1,\,-1,\,0,\,1,\,1,\,1\}.$$

Then

$$F = \begin{cases} 0 & k=0,2,4,6 \\ (2+2\sqrt{2})j & k=1 \\ (-2+2\sqrt{2})j & k=3 \\ (2-2\sqrt{2})j & k=5 \\ (-2-2\sqrt{2})j & k=7 \end{cases},$$

which is odd and imaginary.

Question 4

(a). Trivially, $n = 50 + 10 - 1 = 59$.

(b). Let $y_1 = x[n] \textcircled{50}\, h[n]$, $y_2 = x[n] * h[n]$ for the first 5 points and $y = x[n] * h[n]$.
Case I: Obviously, $y[n] = y_2[n] = 5$ when $0 \leq n \leq 4$.
Case II: Also, $y[n] = y_1[n] = 10$ when $9 \leq n \leq 49$.
Case III: If $50 \leq n \leq 54$, then $y[n] = y_1[n] - y_2[n - 50] = 5$.
For other n, $y[n]$ could not be determined.