CS2601 Linear and Convex Optimization

Homework 10

Due: 2021.12.9

1. Consider the following problem,

$$\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^2} \quad & f(\boldsymbol{x}) = x_1^2 + x_2^2 \\
\text{s.t.} \quad & g_1(\boldsymbol{x}) = (x_1-1)^2 + (x_2-1)^2 - 1 \le 0 \\
& g_2(\boldsymbol{x}) = (x_1-1)^2 + x_2^2 - 1 \le 0
\end{aligned}$$

Write down the KKT conditions and find the optimal point $\boldsymbol{x}^*$ and the corresponding Lagrange multipliers.

2. Consider the following problem

$$\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^2} \quad & f(\boldsymbol{x}) = x_1^2 + x_2^2 \\
\text{s.t.} \quad & (x_1-1)^2 + (x_2-1)^2 \le 1 \\
& (x_1-1)^2 + (x_2+1)^2 \le 1
\end{aligned}$$

(a). Sketch the feasible set and the level set of the objective function. Find the optimal point $\boldsymbol{x}^*$ and the optimal value f^*.

(b). Write down the KKT conditions. Do there exist Lagrange multipliers that satisfy the KKT conditions? Is $\boldsymbol{x}^*$ regular?

3. Consider the following problem

$$\begin{aligned}
\min \quad & (x_1 - \frac{9}{4})^2 + (x_2-2)^2 \\
\text{s.t.} \quad & -x_1^2 + x_2 \ge 0 \\
& x_1 + x_2 \le 6 \\
& x_1, x_2 \ge 0
\end{aligned}$$

For each of the following points, determine whether it is an optimal solution to the above problem and show your arguments,

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} \frac{9}{4} \\ 2 \end{bmatrix}, \boldsymbol{x}^{(2)} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \boldsymbol{x}^{(3)} = \begin{bmatrix} \frac{3}{2} \\ \frac{9}{4} \end{bmatrix}$$

Hint: Try if you can find Lagrange multipliers satisfying the KKT conditions. Note you can easily check which constraints are active.

4. Consider the following optimization problem,

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \frac{1}{2}\|\boldsymbol{x}-\boldsymbol{z}\|^2 \\ \text{s.t.} \quad & \boldsymbol{y}^T\boldsymbol{x} = 0 \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{z} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \{1, -1\}^n$ are constants.

(a). Show that there exists a $\lambda \in \mathbb{R}$ s.t. the solution $\boldsymbol{x}^*$ takes the form

$$x_i^* = (z_i - \lambda y_i)^+, \quad i = 1, 2, \dots, n$$

where λ satisfies

$$\sum_{i=1}^{n} y_i(z_i - \lambda y_i)^+$$

You can assume the KKT conditions hold even at a critical optimal point.

Remark. The KKT conditions hold even at critical optimal points because the constraint functions here are all affine. In our discussion of the Lagrange condition for equality constrained convex problems, we assumed the matrix $\boldsymbol{A}$ has full row rank for simplicity. When $\boldsymbol{A}$ has dependent rows, we can eliminate the redundant ones and set the corresponding Lagrange multipliers to zero.

(b). Let $I_+ = \{i : y_i = 1\}$ and $I_- = \{i : y_i = -1\}$. Relabel elements in $\{z_i : i \in I_+\}$ as $u_1 \leq u_2 \leq \cdots \leq u_p$, and those of $\{-z_i : i \in I_-\}$ as $w_1 \leq w_2 \leq \cdots \leq w_m$. We also use the convention $u_0 = w_0 = -\infty$ and $u_{p+1} = w_{m+1} = +\infty$. Then λ satisfies

$$\sum_{i=1}^{p}(u_i - \lambda)^+ = \sum_{j=1}^{m}(\lambda - w_j)^+$$

If $u_k \leq \lambda \leq u_{k+1}$ and $w_\ell \leq \lambda \leq w_{\ell+1}$, then

$$\lambda = \frac{\sum_{i=k+1}^{p} u_i + \sum_{j=1}^{\ell} w_j}{p - k + \ell}$$

If λ indeed satisfies $u_k \leq \lambda \leq u_{k+1}$ and $w_\ell \leq \lambda \leq w_{\ell+1}$, then it is the solution. Once you have λ, you can find $\boldsymbol{x}^*$. Write a python function that takes an arbitrary $\boldsymbol{z}$ and $\boldsymbol{y}$ as input and outputs the optimal solution $\boldsymbol{x}^*$. Show the result for $\boldsymbol{z} = (1, 2, 1)^T$ and $\boldsymbol{y} = (1, 1, -1)^T$.