Homework 12

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Problem 1

(a) By the fact that

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} e^{x_1} & 2e^{x_2} & 2e^{x_3} \end{bmatrix}^T$$
$$\nabla^2 f(\boldsymbol{x}) = \operatorname{diag}(e^{x_1}, 4e^{x_2}, 4e^{x_3})$$

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{d} = (d_1, d_2, d_3)$. The KKT system is:

$$\begin{bmatrix} e^{x_1} & 0 & 0 & 1 \\ 0 & 4e^{x_2} & 0 & 1 \\ 0 & 0 & 4e^{x_3} & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \lambda \end{bmatrix} = - \begin{bmatrix} e^{x_1} \\ 2e^{x_2} \\ 2e^{x_3} \\ 0 \end{bmatrix}$$

Thus we have $\lambda = -\dfrac{8}{4e^{-x_1} + e^{-x_2} + e^{-x_3}}$.

Finally,

$$d_1 = \frac{4e^{-x_1} - e^{-x_2} - e^{-x_3}}{4e^{-x_1} + e^{-x_2} + e^{-x_3}},$$

$$d_2 = \frac{-2e^{-x_1} + 3/2e^{-x_2} - 1/2e^{-x_3}}{4e^{-x_1} + e^{-x_2} + e^{-x_3}}$$

and

$$d_3 = \frac{-2e^{-x_1} - 1/2e^{-x_2} + 3/2e^{-x_3}}{4e^{-x_1} + e^{-x_2} + e^{-x_3}}.$$

(b) The output of my code:

```
iteration 0:  [0.  1.  0.]
iteration 1:  [ 0.55783402 0.55270748 -0.1105415 ]
iteration 2:  [0.74171111 0.22388047 0.03440841]
iteration 3:  [0.83735858 0.09139269 0.07124873]
iteration 4:  [0.8464719 0.07685719 0.07667091]
iteration 5:  [0.84657358 0.07671322 0.0767132 ]
iteration 6:  [0.84657359 0.0767132 0.0767132 ]
optimal value:  4.663287963194248
```

Problem 2

(a)

$$
\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \boldsymbol{c}^T\boldsymbol{x} - \frac{1}{t}\sum_{i=1}^{n}\log x_i \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}
\end{aligned}
$$

(b) Let $f(\boldsymbol{x}) = \boldsymbol{c}^T\boldsymbol{x} - \frac{1}{t}\sum_{i=1}^{n}\log x_i$. Then we have:

$$\nabla f(\boldsymbol{x}) = \boldsymbol{c} - \frac{1}{t}\frac{1}{\boldsymbol{x}}$$

where $\frac{1}{\boldsymbol{x}} = \left[\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right]^T$.

Also,

$$\nabla^2 f(\boldsymbol{x}) = \operatorname{diag}\left(\frac{1}{t\boldsymbol{x}_1^2}, \frac{1}{t\boldsymbol{x}_2^2}, \ldots, \frac{1}{t\boldsymbol{x}_n^2}\right).$$

(c) (d)

Let $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$. Then we attain the standard form:

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & -x_1 - 3x_2 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 = 6 \\
& x_1 - 2x_2 - x_4 = -8 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

The output of "p2.py":

```
iteration 0:  [2.  1.  3.  8.]
iteration 1:  [1.63307563 3.90402064 0.46290373 1.82503435]
iteration 2:  [1.34600004 4.59597677 0.05802319 0.1540465 ]
iteration 3:  [1.3343604 4.65966009 0.00597951 0.01504022]
iteration 4:  [1.33343360e+00 4.66596660e+00 5.99794358e-04 1.50040181e-03]
iteration 5:  [1.33334334e+00 4.66659667e+00 5.99948876e-05 1.49996563e-04]
iteration 6:  [1.33333433e+00 4.66665967e+00 5.99937452e-06 1.49985317e-05]
iteration 7:  [1.33333343e+00 4.66666597e+00 5.99939244e-07 1.49984906e-06]
iteration 8:  [1.33333334e+00 4.66666660e+00 5.93972887e-08 1.48493250e-07]
iteration 9:  [1.33333333e+00 4.66666666e+00 3.99866757e-09 9.99667352e-09]
optimal value:  -15.333333320004442
```

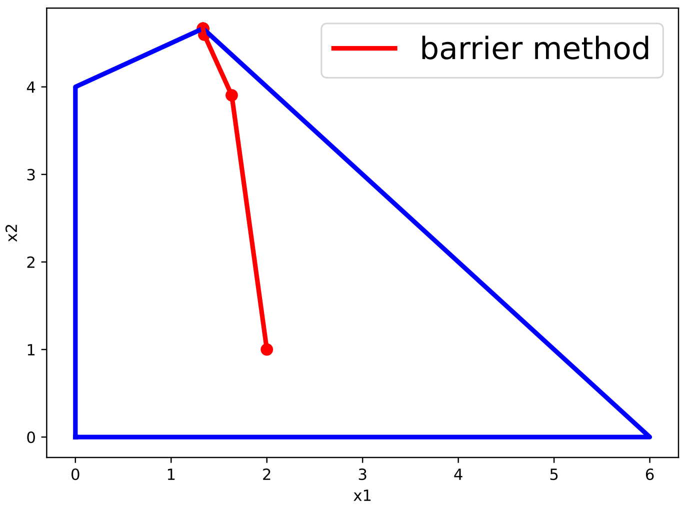


Figure 1: The projection of the iterates onto the x_1, x_2 coordinates

Problem 3

(a) Let $\boldsymbol{\mu} = \left[\mu_1, \mu_2, \mu_3, \mu_4\right]^T$. Then we have:

$$\begin{aligned} \max_{\boldsymbol{\mu}} \quad & \boldsymbol{h}^T \boldsymbol{\mu} \\ \text{s.t.} \quad & \boldsymbol{G}^T \boldsymbol{\mu} = \boldsymbol{c} \\ & \boldsymbol{\mu} \geq \boldsymbol{0} \end{aligned}$$

where

$$\boldsymbol{G} = \begin{bmatrix} -1 & -1 \\ 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{h} = \begin{bmatrix} -6 \\ -8 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{c} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

(b) Let $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix}$. Then the symmetric dual LP is

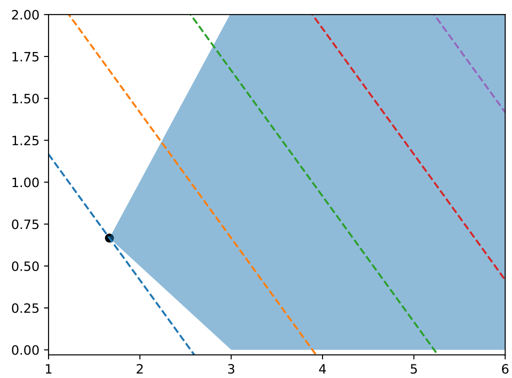
$$\begin{aligned} \max_{\boldsymbol{\mu}} \quad & -6\mu_1 - 8\mu_2 \\ \text{s.t.} \quad & \mu_1 - \mu_2 \geq 1 \\ & \mu_1 + 2\mu_2 \geq 3 \\ & \mu_1, \mu_2 \geq 0 \end{aligned}$$

(c) The dual optimal solution and value:

$$\mu_1 = \frac{5}{3}, \ \mu_2 = \frac{2}{3}, \ f'^* = -\frac{46}{3}$$

The primal optimal solution and value:

$$x_1 = \frac{4}{3}, \ x_2 = \frac{14}{3}, \ f^* = -\frac{46}{3}$$



(d) The output:

```
iteration 0:  [4.  1.  2.  3.]
iteration 1:  [2.1602764 0.54793489 0.61234151 0.25614619]
iteration 2:  [1.72344885 0.64915465 0.0742942 0.02175816]
iteration 3:  [1.67237818 0.66488395 0.00749423 0.00214608]
iteration 4:  [1.66723807e+00 6.66488125e-01 7.49943600e-04 2.14317864e-04]
iteration 5:  [1.66672380e+00 6.66648812e-01 7.49918806e-05 2.14267532e-05]
iteration 6:  [1.66667238e+00 6.66664881e-01 7.49923738e-06 2.14264427e-06]
iteration 7:  [1.66666723e+00 6.66666490e-01 7.42466015e-07 2.12133296e-07]
iteration 8:  [1.66666672e+00 6.66666651e-01 6.66555489e-08 1.90444589e-08]
iteration 9:  [1.66666667e+00 6.66666666e-01 9.38122456e-10 2.68017088e-10]
dual optimal value:  -15.333333335834917
```