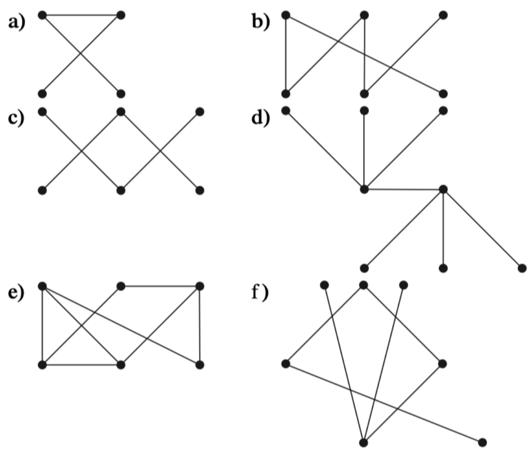
Exercise Sheet 15

Discrete Mathematics, 2020.11.17

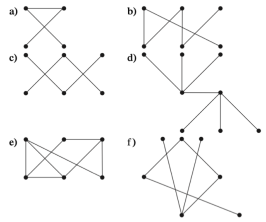
1. ([R], Page 755, Exercise 2(a)(b)(c)(d)(e)(f)) Which of these graphs are trees?



2. Suppose $G = (V, E)$ is an undirected graph with $|V| = n \geq 1$ and $|E| = n - 1$. Prove that if G is connected, then G is a tree.

3. Suppose $G = (V, E)$ is an undirected graph with $|V| = n \geq 1$ and $|E| = n - 1$. Prove that if G has no simple circuit, then G is a tree.

1. ([R], Page 755, Exercise 2(a)(b)(c)(d)(e)(f)) Which of these graphs are trees?



(a) Yes (b) Yes (c) No. Not connected

(d) Yes (e) No. Have circuits (f) Yes

2. Suppose $G = (V, E)$ is an undirected graph with $|V| = n \geq 1$ and $|E| = n - 1$. Prove that if G is connected, then G is a tree.

If G is connected.

Proof by Mathematical Induction:

$n=1$: Obviously G is a tree.

I.H.: when $n=k$, $|V|=k \wedge |E|=k-1 \wedge G$ is connected $\rightarrow G$ is a tree

Let $n=k+1$: $|V|=k+1 \wedge |E|=k \wedge G$ is connected

$\exists u \in V$, $\deg(u)=1$. Assume that u is the endpoint of e.

(If not, then $\forall u \in V$, $\deg(u) \geq 2$. Then $\sum_{u \in V} \deg(u) = 2|E| = 2(n-1) \geq 2n$ contradiction!)

Delete u and e. Then we got a new graph $G'=(V',E') \wedge V'=V \setminus \{u\} \wedge E'=E \setminus \{e\}$.

$\rightarrow |V'|=k \wedge |E'|=k-1 \wedge G'$ is connected

$\rightarrow G'$ is a tree

$\rightarrow G'$ has no simple circuit

Then we have G has no simple circuit.

(If not, simple circuit does not pass through $u \Rightarrow G'$ has a simple circuit. contradiction!)

So G is connected and undirected with no simple circuit.

Thus G is a tree.

3. Suppose $G = (V, E)$ is an undirected graph with $|V| = n \geq 1$ and $|E| = n - 1$. Prove that if G has no simple circuit, then G is a tree.

If G has no simple circuit $\wedge |V|=n \wedge |E|=n-1$

Assume that G's connected components are: $G_1, G_2, \cdots, G_k$ where $G_i=(V_i, E_i) \wedge k \geq 1$

Then we have $\forall i \in \{1, \cdots, k\}$. G_i is connected and undirected with no simple circuit.

$\Rightarrow G_i$ is a tree.

Then $|V_i| = |E_i| + 1$

$|V| = \sum_{i=1}^{k} |V_i| = \sum_{i=1}^{k} |E_i| + k = |E| + k = |E| + 1$

$\Rightarrow k=1$, which means G only has one connected component.

$\Rightarrow G$ is connected

$\Rightarrow G$ is connected and undirected with no simple circuit.

$\Rightarrow G$ is a tree.