CS2601 Linear and Convex Optimization
Homework 3

Due: 2021.10.18

1. Suppose f is a convex function and $S \subset \operatorname{dom} f$ is a convex set. Let M be the set of global minima of f over S,

$$M = \{\boldsymbol{x}^* \in S : f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}), \forall \boldsymbol{x} \in S\}.$$

Show that M is a convex set.

2. Let f be convex. If $f(\theta\boldsymbol{x} + \bar{\theta}\boldsymbol{y}) = \theta f(\boldsymbol{x}) + \bar{\theta} f(\boldsymbol{y})$ for some $\boldsymbol{x}, \boldsymbol{y}$ and $\theta = \theta_0 \in (0, 1)$, then it holds for the same $\boldsymbol{x}, \boldsymbol{y}$ and any $\theta \in [0, 1]$.

Hint: Assume $f(\theta_1\boldsymbol{x} + \bar{\theta}_1\boldsymbol{y}) < \theta_1 f(\boldsymbol{x}) + \bar{\theta}_1 f(\boldsymbol{y})$ for some θ_1. Without loss of generality, you may assume $\theta_1 \in (0, \theta_0)$; the case $\theta_1 \in (\theta_0, 1)$ is similar. Express $\theta_0\boldsymbol{x} + \bar{\theta}_0\boldsymbol{y}$ as a convex combination of $\theta_1\boldsymbol{x} + \bar{\theta}_1\boldsymbol{y}$ and $\boldsymbol{x}$. Then deduce a contradiction.

3. Determine if the following functions are convex, concave, or neither.

(a). $f(\boldsymbol{x}) = f(x_1, x_2, x_3) = x_1^2 + x_1x_3 + x_2^2 + x_2x_3 + \frac{1}{2}x_3^2$ on $\mathbb{R}^3$

(b). $f(\boldsymbol{x}) = f(x_1, x_2) = (x_1x_2)^{-1}$ on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

(c). $f(x_1, x_2) = x_1x_2^2$ on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

(d). $f(x_1, x_2) = x_1x_2^{-1/2}$ on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

(e). $f(x_1, x_2) = x_1^{\alpha}x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

4. Suppose $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are strictly convex functions. Show that $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$ is strictly convex over $\mathbb{R}^2$, and in particular $f(x_1, x_2) = x_1^2 + x_2^4$ is strictly convex.

5. Let $f : C \subset \mathbb{R}^n \to \mathbb{R}$ be a differentiable function defined on a nonempty open convex set C. Show that f is convex if and only if

$$\langle \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \geq 0, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in C. \tag{1}$$

Hint: For the sufficiency, consider the restrictions of f to straight lines, and note that a univariate function h is increasing iff $[h(t) - h(s)](t - s) \geq 0$. You can assume the fact that the intersection of C with a straight line is an open interval when it is not empty.