CS2601 Linear and Convex Optimization
Homework 13

Due: 2022.1.3

1. Consider the following optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}} \quad & f(x) = \log(1+e^x) \\ \text{s.t.} \quad & x \geq 0 \end{aligned}$$

(a). Find the optimal solution and the optimal value.

(b). Find the dual function and the dual problem.

(c). Find the dual optimal solution and the dual optimal value. Does strong duality hold?

2. Consider the optimization problem in Problem 2 of Homework 10,

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^2} \quad & f(\boldsymbol{x}) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & (x_1-1)^2 + (x_2-1)^2 \leq 1 \\ & (x_1-1)^2 + (x_2+1)^2 \leq 1 \end{aligned}$$

(a). Find the Lagrange dual function and the dual problem.

(b). Find the dual optimal value ϕ^*. Does strong duality hold?

(c). Does Slater's condition hold? What can you conclude about the necessity of Slater's condition for strong duality?

(d). Is the dual optimal value ϕ^* attained by any dual feasible point? Is this expected, given the answer to Problem 2(b) of Homework 10?

3. Consider the following minimization problem

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^2} \quad & f(\boldsymbol{x}) = \begin{cases} x_1^3 + x_2^3, & \text{if } \boldsymbol{x} \geq \mathbf{0} \\ +\infty, & \text{otherwise} \end{cases} \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \end{aligned} \tag{P1}$$

Note the domain of f is $\operatorname{dom} f = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} \geq \mathbf{0}\}$ and the domain of the problem is $D = \operatorname{dom} f$.

(a). Since D is not the entire space, the dual function of this problem is defined by

$$\phi(\mu) = \inf_{\boldsymbol{x} \in D} \{f(\boldsymbol{x}) + \mu(1 - x_1 - x_2)\}$$

Find the explicit expression of $\phi(\mu)$.

(b). Find the dual optimal solution.

(c). What is the primal optimal value? Hint: Note f is convex on its domain.

(d). Note the primal problem (P1) is equivalent to

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathbb{R}^2} \quad & f_1(\boldsymbol{x}) = x_1^3 + x_2^3 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{P2}$$

What's the dual function of this equivalent problem (P2)? Does strong duality hold for (P2)?

Remark. Note $\operatorname{dom} f_1 = \mathbb{R}^2$ and f_1 is not convex. This problem shows that equivalent primal problems can have very different dual problems. Not all dual problems are equally useful.

4. Hard-margin SVM. Recall the primal formulation of the hard-margin SVM is

$$\begin{aligned} \min_{\boldsymbol{w}, b} \quad & \frac{1}{2}\|\boldsymbol{w}\|_2^2 \\ \text{s. t.} \quad & y_i(\boldsymbol{x}_i^T \boldsymbol{w} + b) \geq 1, \quad i = 1, 2, \ldots, n \end{aligned}$$

and the dual formulation is

$$\begin{aligned} \max_{\boldsymbol{\mu}} \quad & \boldsymbol{1}^T \boldsymbol{\mu} - \frac{1}{2}\boldsymbol{\mu}^T \boldsymbol{Q} \boldsymbol{\mu} \\ \text{s. t.} \quad & \boldsymbol{\mu}^T \boldsymbol{y} = 0 \\ & \boldsymbol{\mu} \geq \boldsymbol{0} \end{aligned}$$

where

$$Q_{ij} = y_i y_j \boldsymbol{x}_i^T \boldsymbol{x}_j = (y_i \boldsymbol{x}_i)^T (y_j \boldsymbol{x}_j).$$

The primal optimal $\boldsymbol{w}^*$ can be obtained from the dual optimal solution $\boldsymbol{\mu}^*$ by

$$\boldsymbol{w}^* = \sum_{i=1}^{n} \mu_i^* y_i \boldsymbol{x}_i$$

(a). Show that for any i with $\mu_i^* > 0$,

$$y_i(\boldsymbol{x}_i^T \boldsymbol{w}^* + b^*) = 1,$$

and hence

$$b^* = y_i - \boldsymbol{x}_i^T \boldsymbol{w}^*$$

(b). In this problem, we use projected gradient descent to solve the dual problem and then recover the primal optimal solutions $\boldsymbol{w}^*$ and b^*. Complete the implementation in `svm.py`. For the projection `proj`, use your implementation in Problem 4(b) of Homework 10. Show the output.