Homework 13

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Problem 1

(a) By the fact that $f(x)$ is monotone increasing, the optimal solution is $x = 0$ and the optimal value f^* is $\log 2$.

(b) The dual function is

$$\phi(\mu) = \inf_x [f(x) - \mu x] = \inf_x \left[\log \frac{1+e^x}{e^{\mu x}}\right] = \log \left[\inf_{t>0} \left(\frac{1+t}{t^\mu}\right)\right]$$

$$= \begin{cases} -\infty & \text{if} \quad \mu > 1 \\ -(1-\mu)\log(1-\mu) - \mu \log \mu & \text{if} \quad 0 \le \mu \le 1 \\ -\infty & \text{if} \quad \mu < 0 \end{cases}$$

The dual problem is

$$\begin{aligned} \max_\mu \quad & \phi(x) = \begin{cases} -(1-\mu)\log(1-\mu) - \mu\log\mu & \text{if} \quad 0 \le \mu \le 1 \\ -\infty & \text{otherwise} \end{cases} \\ \text{s.t.} \quad & \mu \ge 0 \end{aligned}$$

(c)

$$\phi'(\mu) = \log\left(\frac{1-\mu}{\mu}\right)$$

Thus the dual optimal solution is $\mu = 1/2$ and $\phi^* = \log 2$. The strong duality holds.

Problem 2

(a) The Lagrange dual function is

$$\phi(\mu_1, \mu_2) = \inf_x \left[(\mu_1 + \mu_2 + 1)x_1^2 - 2(\mu_1 + \mu_2)x_1 + (\mu_1 + \mu_2 + 1)x_2^2 - 2(\mu_1 - \mu_2)x_2 + \mu_1 + \mu_2\right]$$

$$= \begin{cases} \mu_1 + \mu_2 - 2\,(\mu_1^2 + \mu_2^2)/(\mu_1 + \mu_2 + 1) & \text{if} \quad \mu_1 + \mu_2 + 1 \ge 0 \\ -\infty & \text{if} \quad \mu_1 + \mu_2 + 1 < 0 \end{cases}$$

Thus the dual problem is

$$\begin{aligned} \max_{\mu_1,\mu_2} \quad & \phi(\mu_1, \mu_2) = \begin{cases} \mu_1 + \mu_2 - 2\,(\mu_1^2 + \mu_2^2)/(\mu_1 + \mu_2 + 1) & \text{if} \quad \mu_1 + \mu_2 + 1 \ge 0 \\ -\infty & \text{if} \quad \mu_1 + \mu_2 + 1 < 0 \end{cases} \\ \text{s.t.} \quad & \mu_1, \mu_2 \ge 0 \end{aligned}$$

(b)

$$\phi(\mu_1, \mu_2) = -\frac{(\mu_1 - \mu_2)^2}{\mu_1 + \mu_2 + 1} + \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 + 1} \le \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 + 1}$$

Thus $\phi^* = 1$. The strong duality holds.

(c) Slaters condition does not hold (no point in int$D$ is feasible). Thus Slaters condition is not the necessary condition for strong duality.

(d) $\phi(\mu_1, \mu_2) = \phi^* \iff \mu_1 = \mu_2 \to +\infty$. The dual optimal value ϕ^* is not attained by any dual feasible point. This is expected because $\boldsymbol{x}^*$ is not regular.

Problem 3

(a)

$$\begin{aligned}\phi(\mu) &= \inf_{x_1,x_2\geq 0}\left[x_1^3+x_2^3+\mu(1-x_1-x_2)\right]\\ &= \begin{cases}\mu-\frac{4}{3\sqrt{3}}\mu^{3/2} & \text{if} \quad \mu\geq 0\\ \mu & \text{if} \quad \mu<0\end{cases}\end{aligned}$$

(b) $\phi(\mu)$ reaches the maximum at the point $\mu = 3/4$. Thus the dual optimal value is $\phi^* = \frac{1}{4}$.

(c) We only need to consider the cases where $x_1, x_2 \geq 0$.

$$f(\boldsymbol{x}) = x_1^3 + x_2^3 \geq 2\left(\frac{x_1+x_2}{2}\right)^3 \geq \frac{1}{4}$$

Finally, $f(1/2\ ,\ 1/2) = 1/4 = f^*$

(d) The dual function is

$$\phi(\mu_1,\mu_2,\mu_3) = \inf_{x_1,x_2}\left[x_1^3+x_2^3+\mu_1(1-x_1-x_2)-\mu_2x_1-\mu_3x_2\right] = -\infty$$

Strong duality does not hold for (P2).

Problem 4

(a) By Slater's condition, $f(\boldsymbol{\omega}^*, b^*) = f^* = \phi^* = \phi(\boldsymbol{\mu}^*)$. By KKT condition, if $\mu_i \neq 0$, then $1 - y_i(\boldsymbol{x}_i^T\boldsymbol{\omega} + b) = 0$.

Thus we have $b^* = y_i - \boldsymbol{x}_i^T\boldsymbol{\omega}^*$ $(y_i^2 = 1)$.

(b) The output is

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primal optimal:
w = [-1.09090895 1.45454542]
b = -0.09090925328321615

dual optimal:
mu = [1.65289246e+00 -0.00000000e+00 -0.00000000e+00 -0.00000000e+00
-0.00000000e+00 -0.00000000e+00 -0.00000000e+00 1.65289239e+00
-0.00000000e+00 -0.00000000e+00 7.15316505e-08 -0.00000000e+00
-0.00000000e+00]
```