# Homework 9

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## Problem 1

$$\begin{aligned}\min_{x_1,x_2} \quad & f(x_1,x_2) = \frac{1}{2}x_1^2 + x_1x_2 + x_2^2 - x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 = 1\end{aligned}$$

(a)

$$\begin{aligned}f(x_1,x_2) =& \frac{1}{2}x_1^2 + x_1x_2 + x_2^2 - x_1 - 3x_2 \\ =& \frac{1}{2}(1-2x_2)^2 + (1-2x_2)x_2 + x_2^2 - (1-2x_2) - 3x_2 \\ =& x_2^2 - 2x_2 - \frac{3}{4} \\ =& (x_2-1)^2 - \frac{7}{4}\end{aligned}$$

Then we have $x_1^* = -1, x_2^* = 1$.

(b) Lagrange equations are:

$$\begin{aligned}x_1^* + 2x_2^* = 1 \\ x_1^* + x_2^* - 1 + \lambda^* = 0 \\ x_1^* + 2x_2^* - 3 + 2\lambda^* = 0\end{aligned}$$

Thus $x_1^* = -1, x_2^* = 1, \lambda^* = 1$.

## Problem 2

$$\begin{aligned}\min_{\boldsymbol{x}} \quad & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{g}^T\boldsymbol{x} + c \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\end{aligned}$$

(a) The Lagrange condition is: $(\boldsymbol{\lambda} \in \mathbb{R}^k)$

$$\begin{aligned}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{g} - \boldsymbol{A}^T\boldsymbol{\lambda} = \boldsymbol{0}\end{aligned}$$

(b)

**Lemma 1.** *Let* $\boldsymbol{Q} \succ \boldsymbol{O} \in \mathbb{R}^{n\times n}$, $\boldsymbol{A} \in \mathbb{R}^{k\times n}$ *and* $\operatorname{rank} \boldsymbol{A} = k$, *then* $\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T \succ \boldsymbol{O}$.

*Proof.* $\forall \boldsymbol{x} \in \mathbb{R}^k$:

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T\boldsymbol{x} = (\boldsymbol{A}^T\boldsymbol{x})^T\boldsymbol{Q}^{-1}(\boldsymbol{A}^T\boldsymbol{x}) > 0$$

Thus $\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T \succ \boldsymbol{O}$. □

By Lagrange condition: $\boldsymbol{x} + \boldsymbol{Q}^{-1}\boldsymbol{g} - \boldsymbol{Q}^{-1}\boldsymbol{A}^T\boldsymbol{\lambda} = 0$. Then we have $\boldsymbol{b} + \boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{g} - \boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T\boldsymbol{\lambda}$, which means that $\boldsymbol{\lambda} = (\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T)^{-1}\big(\boldsymbol{b} + \boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{g}\big)$.
Thus

$$\boldsymbol{x} = \boldsymbol{Q}^{-1}\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T)^{-1}\big(\boldsymbol{b} + \boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{g}\big) - \boldsymbol{Q}^{-1}\boldsymbol{g}$$

(c) Pluging $\boldsymbol{Q} = \boldsymbol{E}, \boldsymbol{g} = -\boldsymbol{x}_0, c = \dfrac{1}{2}\boldsymbol{x}_0^T\boldsymbol{x}$ into (b):

$$
\begin{aligned}
\boldsymbol{x} &= \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_0) + \boldsymbol{x}_0 \\
&\xlongequal{\boldsymbol{x}_0=\boldsymbol{0}} \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{b}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\boldsymbol{x} &= \boldsymbol{\omega}(\boldsymbol{\omega}^T\boldsymbol{\omega})^{-1}\big(b - \boldsymbol{\omega}^T\boldsymbol{x}_0\big) + \boldsymbol{x}_0 \\
d =& \|\boldsymbol{x} - \boldsymbol{x}_0\| \\
=& \big\|\boldsymbol{\omega}(\boldsymbol{\omega}^T\boldsymbol{\omega})^{-1}\big(b - \boldsymbol{\omega}^T\boldsymbol{x}_0\big)\big\| \\
=& \frac{\big\|\boldsymbol{\omega}\big(b - \boldsymbol{\omega}^T\boldsymbol{x}_0\big)\big\|}{\|\boldsymbol{\omega}^T\boldsymbol{\omega}\|} \\
=& \frac{\big\|b - \boldsymbol{\omega}^T\boldsymbol{x}_0\big\|}{\|\boldsymbol{\omega}\|}
\end{aligned}
$$

# Problem 3

Lagrange condition is :

$$
\begin{aligned}
x_1^2 + 4x_2^2 &= 1 \\
x_1 - 8\lambda x_2 &= 0 \\
x_2 - 2\lambda x_1 &= 0
\end{aligned}
$$

Then we have $x_1^2 = 4x_2^2 \Rightarrow x_1 = \pm\frac{\sqrt{2}}{2}, x_2 = \pm\frac{\sqrt{2}}{4}$.

Thus the minimum of $x_1x_2$ is $-\dfrac{1}{4}$ with $\begin{cases} x_1 = -\frac{\sqrt{2}}{2} \\ x_2 = \frac{\sqrt{2}}{4} \end{cases}$ or $\begin{cases} x_1 = \frac{\sqrt{2}}{2} \\ x_2 = -\frac{\sqrt{2}}{4} \end{cases}$.

# Problem 4

(a) The Lagrange condition is:

$$
\begin{aligned}
\|\boldsymbol{x}\|_2^2 &= 1 \\
\boldsymbol{A}\boldsymbol{x} + 2\lambda'\boldsymbol{x} &= 0
\end{aligned}
$$

which means that $\boldsymbol{x}^*$ is a eigenvector. Then $\boldsymbol{x}^{*T}\boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{x}^{*T}\lambda\boldsymbol{x}^* = \lambda$.

Thus $\min_{\boldsymbol{x}} \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \min_i \lambda_i = \lambda_1$.

(b) i) Lagrange condition is: $(\exists\lambda_1, \lambda_2 \in \mathbb{R})$

$$
\begin{aligned}
\|\boldsymbol{x}\|_2^2 &= 1 \\
\boldsymbol{v}^T\boldsymbol{x} &= 0 \\
\boldsymbol{A}\boldsymbol{x} + 2\lambda_1\boldsymbol{x} + \lambda_2\boldsymbol{v}_1 &= 0
\end{aligned}
$$

Then $\exists c_0 = -2\lambda_1, c_1 = -\lambda_2,\ \boldsymbol{A}\boldsymbol{x}^* = c_0\boldsymbol{x}^* + c_1\boldsymbol{v}_1$.

ii) $\boldsymbol{v}_1^T\boldsymbol{A}\boldsymbol{x}^* = c_0\boldsymbol{v}^T\boldsymbol{x}^* + c_1\boldsymbol{v}_1^T\boldsymbol{v}_1$ where LHS $= (\boldsymbol{A}^T\boldsymbol{v}_1)^T\boldsymbol{x}^* = \lambda_1\boldsymbol{v}_1^T\boldsymbol{x}^* = 0$ and RHS $= c_1\|\boldsymbol{v}_1\|^2$. Then we attain $c_1 = 0$.

iii)

**Lemma 2.** *An eigenvector orthogonal to $\boldsymbol{v}_1$ must be associated to one of the eigenvalues $\lambda_2, \ldots, \lambda_n$.*

*Proof.* Suppose that $\boldsymbol{v}_i$ is associated to $\lambda_i$ and $\boldsymbol{v}_j$ is associated to $\lambda_j$. Then $\boldsymbol{v}_i^T\boldsymbol{A}\boldsymbol{v}_j = \lambda_i\boldsymbol{v}_i^T\boldsymbol{v}_j = \lambda_j\boldsymbol{v}_i^T\boldsymbol{v}_j \Rightarrow \boldsymbol{v}_i^T\boldsymbol{v}_j = 0$. Thus $\operatorname{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \mathbb{R}^n$. □

Finally we have $\min_{\boldsymbol{v}_1^T\boldsymbol{x}=0} \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \min_{i=2,\ldots,n} \lambda_i\|\boldsymbol{x}^*\|_2^2 = \min_{i=2,\ldots,n} \lambda_i = \lambda_2$. Like (a) we have $\boldsymbol{x}^*$ is associated to $\lambda_2$.