Homework 2

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Problem 1

Let

$$f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}.$$

If C is convex, then $\forall \boldsymbol{x}, \boldsymbol{y} \in f^{-1}(C)$ (i.e., $f(\boldsymbol{x}), f(\boldsymbol{y}) \in C$), $\forall \theta \in (0,1)$, considering that:

$$\begin{aligned} f\big(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y}\big) &= \theta \boldsymbol{A}\boldsymbol{x} + \theta \boldsymbol{b} + (1-\theta)\boldsymbol{A}\boldsymbol{y} + (1-\theta)\boldsymbol{b} \\ &= \theta f(\boldsymbol{x}) + (1-\theta) f(\boldsymbol{y}) \\ &\in C \end{aligned}$$

Note that: the last line of the equation uses the property of convex set. Then $(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y} \in f^{-1}(C)$.

Problem 2

Trivially, C is a nonempty set.

$\forall \boldsymbol{x}, \boldsymbol{y} \in C$, $\exists \boldsymbol{x}_1, \boldsymbol{y}_1 \in C_1 \wedge \boldsymbol{x}_2, \boldsymbol{y}_2 \in C_2$, $s.t.$ $\boldsymbol{x}_1 - \boldsymbol{x}_2 = \boldsymbol{x} \wedge \boldsymbol{y}_1 - \boldsymbol{y}_2 = \boldsymbol{y}$.

Then $\forall \theta in (0,1)$, we have:

$$\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y} = \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{y}_1 - \theta \boldsymbol{x}_2 - (1-\theta)\boldsymbol{y}_2$$

By C_1, C_2 is convex, $\exists \boldsymbol{z}_1 \in C_1 \wedge \boldsymbol{z}_2 \in C_2$, $\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y} = \boldsymbol{z}_1 - \boldsymbol{z}_2 \in C$, which means that C is convex.

Trivially, $\boldsymbol{0} \in C$. (If not, then $\exists \boldsymbol{x} \in C_1, C_2$, which means $C_1 \cap C_2 \neq \varnothing$)

Problem 3

(a) $\forall \boldsymbol{x}_0, \boldsymbol{y}_0 \in int\ C$, $\exists\ \varepsilon > 0$,

$$\big\{\boldsymbol{x} |\ (\boldsymbol{x} - \boldsymbol{x}_0)^2 \leq \varepsilon^2\big\} \subset C \wedge \big\{\boldsymbol{y} |\ (\boldsymbol{y} - \boldsymbol{y}_0)^2 \leq \varepsilon^2\big\} \subset C.$$

For the sake of simplicity, let X denotes $\big\{\boldsymbol{x} |\ (\boldsymbol{x} - \boldsymbol{x}_0)^2 \leq \varepsilon^2\big\}$, Y denotes $\big\{\boldsymbol{y} |\ (\boldsymbol{y} - \boldsymbol{y}_0)^2 \leq \varepsilon^2\big\}$ and $\boldsymbol{z}_0$ denotes $\theta \boldsymbol{x}_0 + (1-\theta)\boldsymbol{y}_0$.

Then $\forall \boldsymbol{z} \in \big\{\boldsymbol{x} | (\boldsymbol{x} - \boldsymbol{z}_0)^2 \leq \varepsilon^2\big\}$, $\exists \boldsymbol{z} - \boldsymbol{z}_0 + \boldsymbol{x}_0 \in X \wedge \boldsymbol{z} - \boldsymbol{z}_0 + \boldsymbol{y}_0 \in Y$, we have

$$\boldsymbol{z} = \theta(\boldsymbol{z} - \boldsymbol{z}_0 + \boldsymbol{x}_0) + (1-\theta)(\boldsymbol{z} - \boldsymbol{z}_0 + \boldsymbol{y}_0).$$

Thus, $\boldsymbol{z} \in C$, which means that $\theta \boldsymbol{x}_0 + (1-\theta)\boldsymbol{y}_0$ is a interior point of C.

Therefore, $\theta \boldsymbol{x}_0 + (1-\theta)\boldsymbol{y}_0 \in int\ C$ and $int\ C$ is convex.

(b) $\forall \boldsymbol{x}_0, \boldsymbol{y}_0 \in \bar{C}$, we have 2 infinite sequence $\{\boldsymbol{x}_n\}, \{\boldsymbol{y}_n\}$, such that

$$\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}_0 \wedge \lim_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{y}_0,$$

and

$$\boldsymbol{x}_n, \boldsymbol{y}_n \in C \ (\forall n).$$

Thus we have: $\forall \theta \in (0,1)$,

$$\lim_{n\to\infty} (\theta \boldsymbol{x}_n + (1-\theta)\boldsymbol{y}_n) = \theta \boldsymbol{x}_0 + (1-\theta)\boldsymbol{y}_0$$

where $\forall n$, $\theta \boldsymbol{x}_n + (1-\theta)\boldsymbol{y}_n \in C$. Thus $\theta \boldsymbol{x}_0 + (1-\theta)\boldsymbol{y}_0 \in \bar{C}$, which means that $\bar{C}$ is convex.

Problem 4

(a) Let $\boldsymbol{\theta}_{im} \equiv (\theta_{i1}, \theta_{i2}, \ldots, \theta_{im})^T$, $\boldsymbol{X}_m \equiv (\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}, \ldots, \boldsymbol{x}_{im})^T$ and $\boldsymbol{Y}_n \equiv (\boldsymbol{y}_{i1}, \boldsymbol{y}_{i2}, \ldots, \boldsymbol{y}_{in})^T$.
$\forall \boldsymbol{x}, \boldsymbol{y} \in C$, $\exists \boldsymbol{\theta}_{1m}, \boldsymbol{\theta}_{2n}, \boldsymbol{X}_m, \boldsymbol{Y}_n$, such that

$$\begin{cases} \forall k \in \{1,2,\ldots,m\} : \theta_{1k} \geq 0 \wedge \boldsymbol{x}_{1k} \in S \\ \boldsymbol{x} = \boldsymbol{\theta}_{1m}^T \boldsymbol{X}_m \\ \forall l \in \{1,2,\ldots,n\} : \theta_{2k} \geq 0 \wedge \boldsymbol{y}_{2k} \in S \\ \boldsymbol{y} = \boldsymbol{\theta}_{2n}^T \boldsymbol{Y}_n \\ \sum_{k=1}^m \theta_{1k} = \sum_{l=1}^n \theta_{2l} = 1 \end{cases}$$

Then $\forall \theta \in (0,1)$:

$$\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y} = \begin{bmatrix} \theta \boldsymbol{\theta}_{1m}^T & (1-\theta)\boldsymbol{\theta}_{2n}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_m \\ \boldsymbol{Y}_n \end{bmatrix}$$

Let $\boldsymbol{\theta}_{3,m+n}$ denote $\begin{bmatrix} \theta \boldsymbol{\theta}_{1m}^T & (1-\theta)\boldsymbol{\theta}_{2n}^T \end{bmatrix}$ and $\boldsymbol{Z}$ denote $\begin{bmatrix} \boldsymbol{X}_m \\ \boldsymbol{Y}_n \end{bmatrix}$.
Trivially, for every row vector $\boldsymbol{z}_i$ in $\boldsymbol{Z}$: $\boldsymbol{z}_i \in S$ and for every element t in $\boldsymbol{\theta}_{3,m+n}$: $t \geq 0$.
Also,

$$sum(\boldsymbol{\theta}_{3,m+n}) = \theta \sum_{k=1}^{m} \theta_{1k} + (1-\theta) \sum_{l=1}^{n} \theta_{2l} = 1$$

Thus $\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y} \in C$ and C is convex.

(b) The defination of *conv S* is the smallest convex set containing S.
$\forall \boldsymbol{x} \in C$, by the defination of C, we have $\boldsymbol{x} = \sum_{i=1}^m \theta_i \boldsymbol{x}_i$, where $\forall i \in \{1,2,\ldots,m\} : \theta_i > 0 \wedge \sum_{i=1}^m \theta_i = 1$
Then considering that

$$\begin{aligned} \boldsymbol{x} &= \theta_1 \boldsymbol{x}_1 + (1-\theta_1) \sum_{i=2}^{m} \frac{\theta_i}{1-\theta_1} \boldsymbol{x}_i \\ &= \theta_1 \boldsymbol{x}_1 + (1-\theta_1) \left[\frac{\theta_2}{1-\theta_1} \boldsymbol{x}_2 + \sum_{i=3}^{m} \frac{\theta_i}{1 - \frac{\theta_2}{1-\theta_1}} \boldsymbol{x}_i \right] \\ &\ldots \end{aligned}$$

The above equation demonstrates the decomposition of $\boldsymbol{x}$. To prove $\boldsymbol{x} \in convS$, we could to prove $\sum_{i=2}^m \frac{\theta_i}{1-\theta_1} \boldsymbol{x}_i \in convS$ where $\sum_{i=2}^m \frac{\theta_i}{1-\theta_1} = 1$. By finite steps we could only to prove $\boldsymbol{x}_m \in convS$.
Thus we have $\boldsymbol{x} \in convS$. Then $C \subset convS$.

Obviously $S \subset C$. By C is convex we have $convS \subset C$.

Therefore $C = convS$.

Problem 5

The defination of **Vonoroi region** is

$$V = \{\boldsymbol{x} \in \mathbb{R} : ||\boldsymbol{x} - \boldsymbol{x}_0||_2 \leq ||\boldsymbol{x} - \boldsymbol{x}_i||_2, i = 1, 2, \ldots, K\}.$$

Let $V_i = \{\boldsymbol{x} \in \mathbb{R} : ||\boldsymbol{x} - \boldsymbol{x}_0||_2 \leq ||\boldsymbol{x} - \boldsymbol{x}_i||_2\}$. Then $V = \bigcap_{i=1}^{K} V_i$.

Lemma 1 $||\boldsymbol{x} - \boldsymbol{x}_0||_2 \leq ||\boldsymbol{x} - \boldsymbol{x}_i||_2$ *if and only if* $\boldsymbol{A}_i(\boldsymbol{x} - \dfrac{\boldsymbol{x}_0 + \boldsymbol{x}_i}{2}) \leq 0$, *where* $\boldsymbol{A}_i = (\boldsymbol{x}_i - \boldsymbol{x}_0)^T$.

Proof 1

$$\begin{aligned}
&||\boldsymbol{x} - \boldsymbol{x}_0||_2 \leq ||\boldsymbol{x} - \boldsymbol{x}_i||_2 \\
\iff &(\boldsymbol{x} - \boldsymbol{x}_0)^T(\boldsymbol{x} - \boldsymbol{x}_0) \leq (\boldsymbol{x} - \boldsymbol{x}_i)^T(\boldsymbol{x} - \boldsymbol{x}_i) \\
\iff &2(\boldsymbol{x}_i - \boldsymbol{x}_0)^T\boldsymbol{x} + \boldsymbol{x}_0^T\boldsymbol{x}_0 - \boldsymbol{x}_i^T\boldsymbol{x}_i \leq 0 \\
\iff &2(\boldsymbol{x}_i - \boldsymbol{x}_0)^T\boldsymbol{x} - (\boldsymbol{x}_i - \boldsymbol{x}_0)^T(\boldsymbol{x}_i + \boldsymbol{x}_0) \leq 0 \\
\iff &(\boldsymbol{x}_i - \boldsymbol{x}_0)^T(\boldsymbol{x} - (\boldsymbol{x}_i + \boldsymbol{x}_0)/2) \leq 0
\end{aligned}$$

Q.E.D.

Thus, $\forall \boldsymbol{x}_i$, $\exists \boldsymbol{A}_i \in \mathbb{R}^{1\times n}, b_i \in \mathbb{R}$, *s.t.* $V_i = \{\boldsymbol{x} : \boldsymbol{A}_i\boldsymbol{x} \leq b_i\}$. Then let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \\ \vdots \\ \boldsymbol{A}_n \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

Therefore, $V = \bigcap_{i=1}^{K} V_i = \{\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$