

# Introduction to Quantum Mechanics

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June 6, 2022

## Contents

|          |  |           |
|----------|--|-----------|
| <b>1</b> | <b>The Wave Function</b>                     | <b>2</b>  |
| <b>2</b> | <b>Time-independent Schrodinger Equation</b> | <b>3</b>  |
| 2.1      | Stationary states                            | 3         |
| 2.2      | The infinite square well                     | 4         |
| 2.3      | The harmonic oscillator                      | 4         |
| 2.3.1    | Algebraic method                             | 5         |
| 2.3.2    | Analytic method                              | 6         |
| 2.4      | The Free Particle                            | 7         |
| <b>3</b> | <b>Formalism</b>                             | <b>8</b>  |
| 3.1      | Generalized Statistical Interpretation       | 8         |
| 3.2      | Uncertainty Principle                        | 8         |
| <b>4</b> | <b>Quantum Mechanics in Three Dimensions</b> | <b>9</b>  |
| 4.1      | The schrodinger Equation                     | 9         |
| 4.1.1    | The angular Equation                         | 9         |
| 4.1.2    | The Radial Equation                          | 10        |
| 4.2      | The Hydrogen Atom                            | 10        |
| 4.3      | Angular Momentum                             | 11        |
| 4.4      | Spin   | 12        |
| <b>5</b> | <b>Misc</b>                                  | <b>12</b> |
| 5.1      | Before Schrodinger                           | 12        |
| 5.1.1    | Black Body Radiation                         | 12        |
| 5.1.2    | Photoelectric Effect                         | 12        |
| 5.1.3    | Compton effect                               | 12        |
| 5.1.4    | Bohr Model                                   | 13        |
| 5.2      | Probability current                          | 13        |
| 5.3      | Two-state Quantum System                     | 13        |
| 5.4      | Famous Experiments                           | 13        |

## Acknowledgement

These Notes contain material developed and copyright by:

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# 1 The Wave Function

What we are looking for is the **wave function**  $\Psi$ .

**Law 1.1** (Schrodinger Equation).

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi.$$

For simplicity, we always rewrite it as:

$$i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m} \partial_x^2 \Psi + V\Psi.$$

Born's statistical interpretation:

$$\int_a^b |\Psi(x, t)|^2 dx = \text{probability of finding the particle between } a \text{ and } b \text{ at time } t.$$

**Law 1.2** (Normalization).

$$\int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = 1.$$

**Proposition 1.1.** The wave function will always stay NORMALIZED.

$$\frac{d}{dt} \int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = 0.$$

**Proof.** By Schrodinger EQ.,

$$\text{LHS} = \frac{i\hbar}{2m} \left( \Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \Big|_{-\infty}^{+\infty}.$$

□

**Definition 1.1.**

$$\langle x \rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x |\Psi|^2 dx$$

and

$$\langle p \rangle \stackrel{\text{def}}{=} m \frac{d \langle x \rangle}{dt}.$$

**Theorem 1.1.**

$$\langle x \rangle = \int \Psi^*(x) \Psi dx$$

and

$$\langle p \rangle = \int \Psi^* \left( -i\hbar \frac{\partial}{\partial x} \right) \Psi dx.$$

**Remark 1.1 (Operator).** We say that the operator  $x$  represents position, and the operator  $-i\hbar \partial/\partial x$  represents momentum. Also,

$$\langle Q(x, p) \rangle = \int_{-\infty}^{\infty} \Psi^* \left[ Q(x, -i\hbar \frac{\partial}{\partial x}) \right] \Psi \, dx.$$

**Property 1.1.** Operators do **NOT**, in general, commute. For example,  $\hat{x}\hat{p} \neq \hat{p}\hat{x}$ , i.e.,

$$\exists \text{ a function } f, \text{ s.t. } (\hat{x}\hat{p})f \neq (\hat{p}\hat{x})f.$$

**Theorem 1.2 (de Broglie formula).** The wave length is related to the momentum of the particle:

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}.$$

**Theorem 1.3 (Heisenberg's uncertainty principle).**

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}.$$

## 2 Time-independent Schrodinger Equation

### 2.1 Stationary states

We look for solutions that are simple products,

$$\Psi(x, t) = \psi(x)\varphi(t).$$

**Theorem 2.1.** By the method of separation of variables,

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = E\psi$$

and

$$\varphi(t) = e^{-iEt/\hbar}.$$

The first is called the **time-independent Schrodinger equation**.

**Definition 2.1 (Hamiltonian).** In classical mechanics, the total energy (kinetic plus potential) is called Hamiltonian:

$$H(x, p) = \frac{p^2}{2m} + V(x).$$

Now we introduce **Hamiltonian operator**:

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x).$$

Thus the time-independent Schrodinger EQ. can be written

$$\hat{H}\psi = E\psi$$

which is **IMPORTANT**.

**Remark 2.1.** Intriguingly and intuitively,

$$\langle H \rangle = E.$$

Also, if the equation yields an infinite collection of solutions  $(\psi_1(x), \psi_2(x), \dots)$ , each with its associated value of the separation constant  $(E_1, E_2, \dots)$ ; thus the wave function is:

$$\Psi(x, t) = \sum_{n=1}^{+\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}.$$

**Particularly,**

$$E_n \geq 0 \text{ for all } n$$

## 2.2 The infinite square well

Suppose

$$V(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ \infty & \text{otherwise} \end{cases}.$$

**Theorem 2.2.** Inside the well, we have

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

and

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right).$$

**Property 2.1.**  $\psi_n(x)$  has some interesting and important properties:

1. They are alternately even and odd, with the respect to the center of the well.
2. They are mutually orthogonal (i.e.,  $\int \psi_m(x)^* \psi_n(x) dx = \delta_{mn}$ )  
where  $\delta_{mn}$  is **Kronecker delta**:

$$\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}.$$

3. They are complete by Dirichlet's theorem.

## 2.3 The harmonic oscillator

Let

$$V(x) = \frac{1}{2}m\omega^2 x^2.$$

Here I will introduce 2 entirely different approaches to this problem. The first is a diabolically clever algebraic technique and the second is a straitforward “brute force” solution.

### 2.3.1 Algebraic method

To begin with, let's rewrite the EQ. in a more suggestive form:

$$\frac{1}{2m} \left[ \left( -i\hbar \frac{d}{dx} \right)^2 + (m\omega x)^2 \right] \psi = E\psi.$$

The idea is to factor the term in square brackets:

$$u^2 + v^2 = (u - iv)(u + iv).$$

**Definition 2.2** (Ladder operator).

$$\hat{a}_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}} (\mp i\hat{p} + m\omega x).$$

**Definition 2.3** (Commutator). The commutator of operators  $\hat{A}$  and  $\hat{B}$  is

$$[\hat{A}, \hat{B}] \stackrel{def}{=} \hat{A}\hat{B} - \hat{B}\hat{A}.$$

**Property 2.2.**

$$[\hat{a}_-, \hat{a}_+] = 1.$$

**Theorem 2.3.** If  $\psi$  satisfies the Schrodinger's EQ. with energy  $E$ , then  $\hat{a}_+\psi$  satisfies the Schrodinger's EQ. with energy  $E + \hbar\omega$ :

$$\hat{H}\psi = E\psi \implies \hat{H}(\hat{a}_+\psi) = (E + \hbar\omega)(\hat{a}_+\psi).$$

Similarly,

$$\hat{H}\psi = E\psi \implies \hat{H}(\hat{a}_-\psi) = (E - \hbar\omega)(\hat{a}_-\psi).$$

**Proof.**

$$\hat{H} = a_+a_- + \frac{1}{2}\hbar\omega.$$

□

Here, then, is a wonderful machine for generating new solutions—if we could just find one solution. Thus, we call  $\hat{a}_+$  raising operator and  $\hat{a}_-$  lowering operator.

But what if I apply the lowering operator **repeatedly**? We will reach a state with energy less than zero. By 2.1, there is **NO** guarantee that it will be normalized.

**Proposition 2.1.** Thus, there occurs a “lowest rung”  $\psi_0$  such that

$$\hat{a}_-\psi_0 = 0.$$

**Theorem 2.4.**

$$\psi_0(x) = A_0 e^{-m\omega/2\hbar x^2}$$

and

$$E_0 = \frac{1}{2}\hbar\omega.$$

Thus we could get

$$\psi_n(x) = A_n (a_+)^n e^{-m\omega/2\hbar x^2}, \text{ with } E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

where  $A_n$  are used for normalization.

**Theorem 2.5.**  $\psi_n$  and  $\psi_{n+1}$  should satisfy:

$$\begin{cases} a_+ \psi_n = i\sqrt{(n+1)\hbar\omega} \\ a_- \psi_n = -i\sqrt{n\hbar\omega} \psi_{n-1} \end{cases}.$$

**Proof.**

$$\int_{-\infty}^{\infty} |a_+ \psi_n|^2 dx = (n+1)\hbar\omega$$

and

$$\int_{-\infty}^{\infty} |a_- \psi_n|^2 dx = n\hbar\omega.$$

□

Ultimately,

$$A_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{(-i)^n}{\sqrt{n!(\hbar\omega)^n}}.$$

**2.3.2 Analytic method**

Things look a little cleaner if we introduce the dimensionless variables

$$\xi = \sqrt{\frac{m\omega}{\hbar}}x \text{ and } K = \frac{2E}{\hbar\omega}.$$

In terms of  $\xi$  and  $K$ , the Schrodinger equation reads

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi.$$

To begin with, consider that at very large  $\xi$ ,  $\xi^2$  completely dominates over the constant  $K$ , so in this regime  $d^2\psi/d\xi^2 = \xi^2\psi$ , which means that  $\psi \implies Ae^{\xi^2/2} + Be^{-\xi^2/2}$ . Thus we let  $\psi = h(\xi)e^{-\xi^2/2}$ . Plugging  $\psi$  into Schrodinger EQ., we have

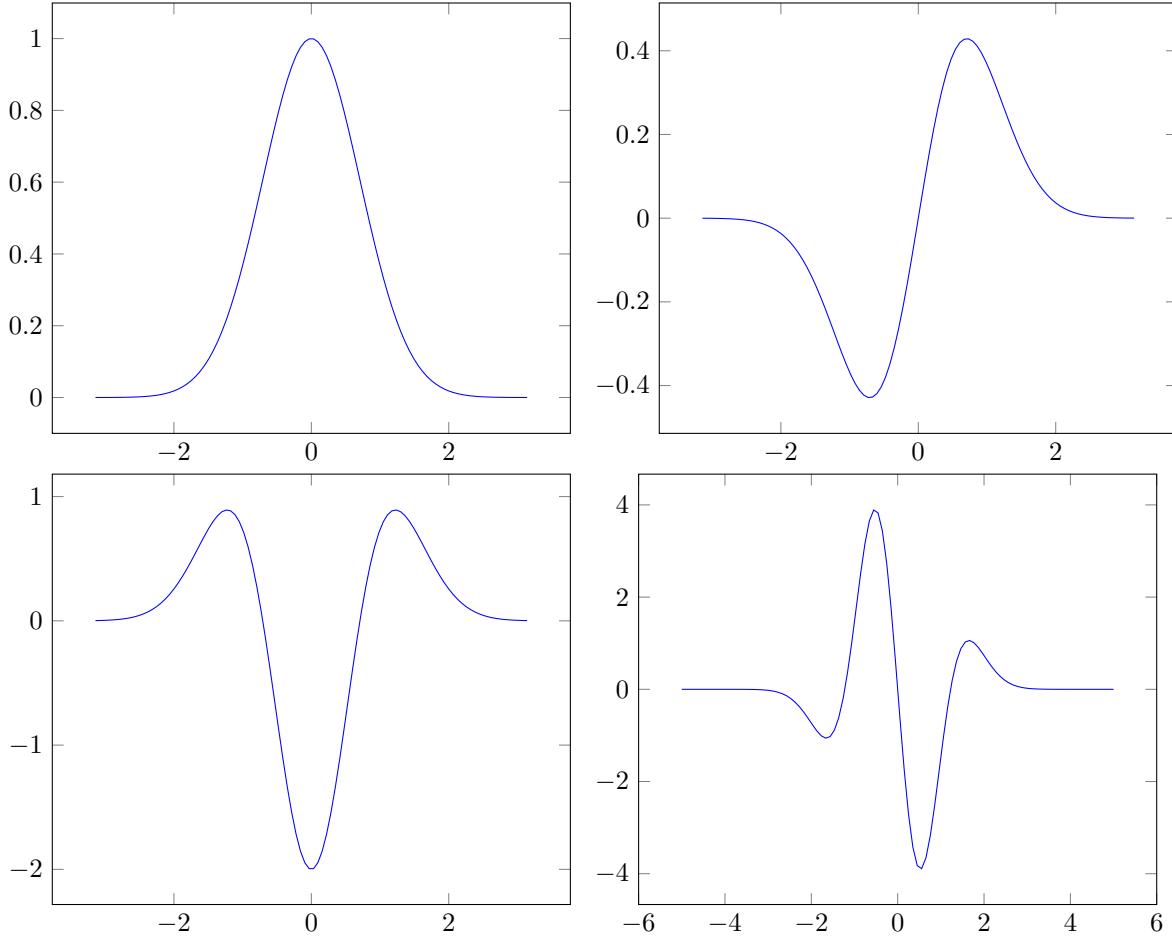
$$h(\xi) = \sum_{n=0}^{\infty} a_n \xi^n \text{ and } a_{n+2} = \frac{2n+1-K}{(n+1)(n+2)}.$$

For physically acceptable solutions (normalizable solutions), then, we must have  $K = 2n+1$ . Finally,

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

where  $H_n$  is the **Hermite polynomials**.

The first four stationary states of the harmonic oscillator are as follows.



## 2.4 The Free Particle

We turn next to what should have been the simplest case of all: the free particle. The time Schrodinger Eq. reads:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi.$$

Let  $k \equiv \sqrt{2mE/\hbar}$ , we have

$$\Psi_k(x, t) = Ae^{i(kx - \hbar k^2 t / 2m)}.$$

**Remark 2.2.** The speed of these waves is:

$$v_{\text{quantum}} = \sqrt{E/2m} = 0.5v_{\text{classical}}$$

And

$$\int_{-\infty}^{\infty} \Psi_k^*(x, t) \Psi_k(x, t) dx = +\infty,$$

which means that a free particle cannot exist in a stationart state.

**Theorem 2.6.** The general solution to the time-independent Schrodinger EQ. is still a linear combination of separable solutions:

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \hbar k^2 t / 2m)} dk.$$

Now this wave function can be normalized for appropriated  $\phi(k)$ . We call it a **wave packet**.

**Definition 2.4** (phase velocity and group velocity). For the wave function:

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \omega t)} dk.$$

We define:

$$v_{\text{phase}} = \frac{\omega}{k}, \quad v_{\text{group}} = \frac{d\omega}{dk}.$$

## 3 Formalism

### 3.1 Gerneralized Statistical Interpretation

First we assume the spectrum of the wave fuction is discrete, we have

$$\langle Q \rangle = \sum_{n'} \sum_n c_{n'}^* c_n q_n \langle f_{n'} | f_n \rangle = \sum_n |c_n|^2 q_n$$

where  $q_n$  is the eigenvalue of operator  $\hat{Q}$  and  $\Psi(x, t) = \sum_n c_n(t) f_n(x)$ . What about momentum?

$$\Phi(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x, t) dx$$

and

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Phi(p, t) dp.$$

### 3.2 Uncertainty Principle

**Theorem 3.1** (generalized uncertainty principle).

$$\sigma_A^2 \sigma_B^2 \geq \left( \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2.$$

How to interpret  $\Delta t$ ?

**Definition 3.1.**

$$\Delta t \equiv \frac{\sigma_Q}{|d\langle Q \rangle / dt|},$$

where

$$\frac{d\langle Q \rangle}{dt} = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle.$$

I recommend you to learn **Hilbert space** and **Dirac notation**.



## 4 Quantum Mechanics in Three Dimensions

### 4.1 The schrodinger Equation

The generalization oto three dimensions is straitforward.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the **Laplacian**. Also the normalization conditions reads  $\int \Psi d^3\mathbf{r} = 1$ . If  $V$  is independent of time, there will be a complete set of stationary states

$$\Psi_n(\mathbf{r}, t) = \psi_n(\mathbf{r}) e^{-iE_n t/\hbar}.$$

Now we adopt spherical coordinates

**Lemma 4.1** (Laplacian in spherical coordinates).

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} \right).$$

If  $\Psi = R(r)Y(\theta, \phi)$  and  $Y = \Theta(\theta)\Phi(\phi)$ , we could separate  $r, \theta$  and  $\phi$  into three equations with important *separation constants*.

#### 4.1.1 The angular Equation

The  $\phi$  equation is easy

$$\frac{d^2 \Phi}{d\phi^2} = -m^2 \Phi \implies \Phi = e^{im\phi}.$$

When  $\phi$  advances by  $2\pi$ , we return to the same point in space, so it is natural to require that  $\Phi(\phi+2\pi) = \Phi(\phi)$ . From this it follows that  $m$  must be an integer:

$$m = 0, \pm 1, \pm 2, \dots$$

The  $\theta$  equation reads

$$\sin \theta \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + [l(l+1) \sin^2 \theta - m^2] \Theta = 0.$$

**Lemma 4.2** (Legendre function). The solution of  $\Theta$  is

$$\Theta(\theta) = A P_l^m(\cos \theta).$$

where

$$P_l^m(x) \triangleq (-1)^m (1-x^2)^{m/2} \left( \frac{d}{dx} \right)^m P_l(x)$$

is the **associated Legendre function**, defined by the **Rodrigues formula**

$$P_l(x) \triangleq \frac{1}{2^l l!} \left( \frac{d}{dx} \right)^l (x^2 - 1)^l.$$

**Remark 4.1.** Notice that  $l$  must be a non-negative integer, for Rodrigues formula to make sense; moreover, if  $m > l$ , we will have  $P_l^m(x) = 0$ . For any given  $l$ , then there are  $2l + 1$  possible values of  $m$ :

$$l = 0, 1, 2, \dots \quad \text{and} \quad m = -l, -l + 1, \dots, l - 1, l.$$

By normalization condition

$$\int_0^\pi \int_0^{2\pi} |Y|^2 \sin \theta \, d\theta \, d\phi = 1,$$

we deduce that

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} e^{im\phi} P_l^m(\cos \theta) \quad (4.1)$$

#### 4.1.2 The Radial Equation

**Theorem 4.1** (Radial equation).

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[ V + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u = Eu$$

where  $u(r) \equiv rR(r)$ .

**Remark 4.2** (Effective potential).

$$V_{\text{eff}} = V + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}$$

and the latter term is the so-called **centrifugal potential**.

## 4.2 The Hydrogen Atom

The radial equation says:

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[ -\frac{e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u = Eu.$$

To tidy up the notation, let

$$\kappa = \frac{\sqrt{-2mE_e}}{\hbar}, \quad \rho = \kappa r \quad \text{and} \quad \rho_0 = \frac{m_e e^2}{2\pi\epsilon_0 \hbar^2 \kappa}$$

so that

$$\frac{d^2 u}{d\rho^2} = \left[ 1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2} \right] u.$$

Intuitively, ( $d^2 u / d\rho^2 = u$  when  $\rho \rightarrow +\infty$  and  $d^2 u / d\rho^2 = ul(l+1)/\rho^2$  when  $\rho \rightarrow 0$ )

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho).$$

Now we assume the solution,  $v(\rho)$ , can be expressed as a power series in  $\rho$ :

$$v(\rho) = \sum_{j=0}^{+\infty} c_j \rho^j.$$

Plug in it into the radial equation

$$c_{j+1} = \left\{ \frac{2(j+l+1) - \rho_0}{(j+1)(j+2l+2)} \right\} c_j.$$

**Theorem 4.2.** The series must terminate. I.e.,  $\exists N \in \mathbb{N}$ ,  $c_N = 0$ , which means

$$2(N + l) - \rho_0 = 0.$$

**Proof.** For large  $j$ , the recursion formula says

$$c_{j+1} \approx \frac{2}{j+1} c_j \implies c_{j+1} \approx \frac{2^j}{j!} c_0.$$

Then

$$v(\rho) = c_0 e^{2\rho} \quad \text{and} \quad u(\rho) = c_0 \rho^{l+1} e^\rho$$

which could not be **NORMALIZED**. □

**Theorem 4.3 (Bohr Formula & Radius).**

$$E_n = - \left[ \frac{m_e}{2\hbar^2} \left( \frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \quad \text{and} \quad a = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2}.$$

Finally, we obtain the spacial wave functions

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_l^m(\theta, \phi)$$

where  $R_{nl}(r) = r^{-1} \rho^{l+1} e^{-\rho} v(\rho)$  and  $Y_l^m(\theta, \phi)$  is defined by Eq 4.1.

**Remark 4.3 (Laguerre Polynomials).**

$$v(\rho) = L_{n-l-1}^{2l+1}(2\rho)$$

where

$$L_q^p(x) \triangleq (-1)^p \left( \frac{d}{dx} \right)^p L_{p+q}(x)$$

is an associated Laguerre polynomial, and

$$L_q(x) \triangleq \frac{e^x}{q!} \left( \frac{d}{dx} \right)^q (e^{-x} x^q)$$

is the  $q^{\text{th}}$  Laguerre polynomial. “**Brutally**”,

$$\psi_{nlm} = \sqrt{\left( \frac{2}{na} \right)^3 \frac{(n-l-1)!}{2n(n+l)!}} e^{-r/na} \left( \frac{2r}{na} \right)^l \left[ L_{n-l-1}^{2l+1}(2r/na) \right] Y_l^m(\theta, \phi).$$

### 4.3 Angular Momentum

By the formula  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

$$L_x = yp_z - zp_y \quad (\text{cyc}).$$

Then we deduce the fundamental commutation relations for angular momentum

$$[L_x, L_y] = i\hbar L_z \quad \text{and} \quad [L^2, L_x] = 0 \quad (\text{cyc}).$$

According to generalized uncertainty principle,

$$\sigma_{L_x} \sigma_{L_y} \geq \frac{\hbar}{2} |\langle L_z \rangle|.$$

With the help of ladder operator  $L_{\pm} = L_x \pm iL_y$ , we could obtain the eigenvalues and the eigenfunctions for angular momentum.

**Theorem 4.4** (Eigenvalues and Eigenfunctions for  $L$ ).

$$L^2 Y_l^m = l(l+1)\hbar^2 Y_l^m \quad \text{and} \quad L_z Y_l^m = m\hbar Y_l^m.$$

**Remark 4.4.** Spherical harmonics (Eq 4.1) are the eigenfunctions of  $L^2$  and  $L_z$ .

## 4.4 Spin

Similarly,

$$[S_x, S_y] = i\hbar S_z, \quad S^2 |s, m\rangle = s(s+1)\hbar^2 |s, m\rangle \quad \text{and} \quad S_z |s, m\rangle = m_s \hbar |s, m\rangle.$$

**Definition 4.1** (Quantum Numbers). Intuitively,

- $n$  ( $0, 1, 2, \dots$ ) is the **principal quantum number**; it tells you the energy of electron.
- $l$  ( $0, 1, 2, \dots, n-1$ ) is called **azimuthal quantum number** and  $m_l$  ( $0, \pm 1, \pm 2, \dots, \pm l$ ) the **magnetic quantum number**; they are related to the angular momentum of the electron.
- $s$  ( $\pm 1/2$ ) is the **spin quantum number**. And  $m_s \in \{-s, -s+1, \dots, s\}$ .

## 5 Misc

### 5.1 Before Schrodinger

First we will introduce the theories before Schrodinger Equation.

#### 5.1.1 Black Body Radiation

$$M_{\lambda}(T) = \frac{dE_{\lambda}}{d\lambda}, \quad \alpha_{\lambda}(T) = \frac{E_{\text{absorb}}}{E_{\text{in}}} \quad \text{and} \quad \frac{M_{\lambda}(T)}{\alpha_{\lambda}(T)} = M_0(\lambda, T) = \text{Const.}$$

**Law 5.1.** Stefan Boltzmann law:  $M(T) = \sigma T^4$ . Wien's displacement law:  $\lambda_m T = b$ .

#### 5.1.2 Photoelectric Effect

$$h\nu = \frac{1}{2}mv^2 + W.$$

#### 5.1.3 Compton effect

$$\Delta\lambda = \lambda - \lambda_0 = \frac{2h}{m_0 c} \sin^2 \frac{\psi}{2}.$$

### 5.1.4 Bohr Model

The quantization of angular momentum says:

$$L = mvr = n\hbar.$$

Also,

$$\frac{1}{\lambda} = R_\infty \left( \frac{1}{m^2} - \frac{1}{n^2} \right).$$

## 5.2 Probability current

**Definition 5.1** (Probability current).

$$J \triangleq -\frac{i\hbar}{2m}(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) = \frac{\hbar}{m} \text{Im}(\Psi^* \nabla \Psi).$$

Then

$$\frac{\partial \rho}{\partial t} + \nabla J = 0$$

where  $\rho = \int \Psi^* \Psi \, dx$ .

## 5.3 Two-state Quantum System

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} E_1 & A \\ A & E_2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

Then,

$$C_1 + k_\pm C_2 = \left[ C_1(0) + k_\pm C_2(0) \right] \exp \left[ -\frac{i(E_1 + k_\pm A)}{\hbar} t \right]$$

where  $k_\pm = \left( E_2 - E_1 \pm \sqrt{(E_2 - E_1)^2 + 4A^2} \right) / 2A$ .

**Lemma 5.1.**

$$C_1(t) = \frac{1}{k_+ - k_-} \left\{ k_+ \left[ C_{10} + k_- C_{20} \right] e^{-\frac{i(E_1 + k_- A)}{\hbar} t} - k_- \left[ C_{10} + k_+ C_{20} \right] e^{-\frac{i(E_1 + k_+ A)}{\hbar} t} \right\}$$

and

$$C_2(t) = \frac{1}{k_+ - k_-} \left\{ \left[ C_{10} + k_- C_{20} \right] e^{-\frac{i(E_1 + k_- A)}{\hbar} t} - \left[ C_{10} + k_+ C_{20} \right] e^{-\frac{i(E_1 + k_+ A)}{\hbar} t} \right\}$$

What if  $C_{10} = 1, C_{20} = 0$  and  $E_1 = E_2 = 0$  ?

$$C_1(t) = \frac{1}{2} \left[ e^{iAt/\hbar} + e^{-iAt/\hbar} \right] \quad \text{and} \quad C_2(t) = \frac{1}{2} \left[ e^{iAt/\hbar} - e^{-iAt/\hbar} \right]$$

which entails that

$$C_1(t) = \cos(At/\hbar) \quad \text{and} \quad C_2(t) = \sin(At/\hbar).$$

## 5.4 Famous Experiments

Milikan+Compton  
Davisson-Germer  
Zeeman  
Stern-Gerlach

Wave-particle Duality  
de Broglie Formula  
Quantization of Angular Momentum  
Electronic Spin