1 Garch Model

Notation

- X_t: rate/price of a risk factor
- r_t: return of X_t over a horizon (e.g. 1 day or 10 days):

$$r_t = \mathcal{T}(X_t) - \mathcal{T}(X_{t-1}) \tag{1}$$

for a transformation function $\mathcal{T}$, either $\mathcal{T}(x) = x$ or $\mathcal{T}(x) = \ln(x)$.

Model Specification GARCH(1,1) process:

$$\begin{aligned} r_t &= \mu - \frac{\xi}{2}\sigma_t^2 + \sigma_t\epsilon_t & (2) \\ \sigma_t^2 &= \sigma_\infty^2(1-\beta-\gamma) + \beta\sigma_{t-1}^2 + \gamma\sigma_{t-1}^2\epsilon_t^2 & (3) \end{aligned}$$

where

$$\epsilon_t \sim N(0,1), \quad \text{i.i.d.}, \tag{4}$$

$$\beta \geq 0, \quad \gamma \geq 0, \quad \beta + \gamma = 1, \tag{5}$$

and

$$\xi = \begin{cases} 0 & \text{for} \quad \mathcal{T}(x) = x \\ 1 & \text{for} \quad \mathcal{T}(x) = \ln(x) \end{cases} \tag{6}$$

Observations The GARCH volatility σ_t is mean-reverting:

$$\sigma_t^2 - \sigma_{t-1}^2 = (1-\beta-\gamma)(\sigma_\infty^2 - \sigma_{t-1}^2) + \gamma\sigma_{t-1}^2(\epsilon_t^2 - 1) \tag{7}$$

Expected spot variance at future time:

$$\begin{aligned} \bar{\sigma}_t^2(\tau) &:= E\left[\sigma_{t+\tau}^2 | \mathcal{F}_t\right] \\ &= \sigma_\infty^2 + (\beta+\gamma)^{\tau-1}(\sigma_{t+1}^2 - \sigma_\infty^2) & (8) \end{aligned}$$

2 Application: Manufacturing (Implied) Volatility Curve

Expected spot variance over tenor τ:

$$\begin{aligned} \bar{\nu}_t^2(\tau) &:= \frac{1}{\tau}\sum_{n=1}^{\tau} \bar{\sigma}_t^2(n) \\ &= \sigma_\infty^2 + \frac{1-(\beta+\gamma)^\tau}{1-(\beta+\gamma)} \frac{\sigma_{t+1}^2 - \sigma_\infty^2}{\tau} & (9) \end{aligned}$$

Proxy the ATM implied volatility at tenor τ on date t by $\bar{\nu}_t(\tau)$.