- **Problem Descriptions**

We generate *N* paths of the MCMC sampling for the CIR bond prices $B_{CIR}(t), t = 1,2, \dots, T$ and contigent payoffs $\{P_{CAT}(Y_t), t = 1,2, \dots, T\}$, denote them by

$$\left\{\left({}_1B_{CIR}^{(i)},\ {}_1P_{CAT}^{(i)}\right), \left({}_2B_{CIR}^{(i)},\ {}_2P_{CAT}^{(i)}\right), \dots, \left({}_TB_{CIR}^{(i)},\ {}_TP_{CAT}^{(i)}\right), i = 1,2, \dots, N\right\}$$

Let π denote the *physical* distribution of the *N* paths of the MCMC sampling with equal probability of *1/N* on each path. We convert π into the risk-neutral version π^* with a constraint such that

$$\sum_{i=1}^{N} \pi_i^* \sum_{t=1}^{T} \exp(\delta t)\left({}_tB_{CIR}^{(i)} \cdot\ {}_tP_{CAT}^{(i)}\right) = V_o$$

where V_o is a vector of N equally probable(under the *physical* distribution, π) market prices of the bond at some time $t = 0$

$$V_o = \sum_{i=1}^{N} \pi_i \sum_{t=1}^{T}\ {}_tB_{CIR}^{(i)} \exp(\delta t)\ {}_tP_{CAT}^{(i)} \quad for i = 1, \dots, N \qquad (1)$$

where,

$$\pi_i = \frac{1}{N}$$

Based on the maximum entropy principle the risk-neutral distribution π^* should minimize the Kullback-Leibler information divergence

$$KL(\mathbb{P} \parallel \mathbb{Q}) : arg\ min \sum_{i=1}^{N} \pi_i^* ln\left(\frac{\pi_i^*}{\pi_i}\right)$$

with the following additional constraints

$$\pi^* > 0, for\ i = 1, \dots . N,$$

$$\sum_{i=1}^{N} \pi_i^* \ = 1$$

$$\sum_{i=1}^{N} \pi_i^* \sum_{t=1}^{T} \exp(\delta t)\left({}_tB_{CIR}^{(i)} \cdot\ {}_tP_{CAT}^{(i)}\right) = V_o$$

- **Lagrange Dual Function**

This KL divergence minimization problem is a convex optimization problem, and it can be solved by the method of Lagrange multipliers as follows.

$$\mathcal{L} = \sum_{i=1}^{N} \pi_i^* ln\left(\frac{\pi_i^*}{\pi_i}\right) - \gamma\left(\sum_{i=1}^{N} \pi_i^* \ - 1\right) - \lambda\left[\left\{\sum_{i=1}^{N} \pi_i^* \sum_{t=1}^{T} \exp(\delta t)\left(\ {}_tB_{CIR}^{(i)} \cdot \ {}_tP_{CAT}^{(i)}\right)\right\} - V_o\right]$$

$\gamma \in R, \lambda \in R$

Taking first order conditions with respect to π_i^*, we get

$$\frac{\partial \mathcal{L}}{\partial \pi_i^*} = ln\frac{\pi_i^*}{\pi_i} + 1 - \gamma - \lambda\left[\sum_{t=1}^{T} \exp(\delta t)\left(\ {}_tB_{CIR}^{(i)} \cdot \ {}_tP_{CAT}^{(i)}\right)\right] = 0$$

$$exp\left(ln\frac{\pi_i^*}{\pi_i} + 1 - \gamma - \lambda\left[\textstyle\sum_{t=1}^{T} \exp(\delta t)\left(\ {}_tB_{CIR}^{(i)} \cdot \ {}_tP_{CAT}^{(i)}\right)\pi_i^*\right]\right)=1$$

$$exp\left(ln\,\pi_i^* - \ln \pi_i + 1 - \gamma - \lambda\left[\sum_{t=1}^{T} \exp(\delta t)\left(\ {}_tB_{CIR}^{(i)} \cdot \ {}_tP_{CAT}^{(i)}\right)\right]\right) = 1$$

$$\pi_i^* = \pi_i\, exp\left(-1 + \gamma + \lambda\left[\sum_{t=1}^{T} \exp(\delta t)\left(\ {}_tB_{CIR}^{(i)} \cdot \ {}_tP_{CAT}^{(i)}\right)\right]\right)$$

$$\pi_i^* = \exp(-1 + \gamma)\, \pi_i exp\left(\lambda \sum_{t=1}^{T} \exp(\delta t)\left(\ {}_tB_{CIR}^{(i)} \cdot \ {}_tP_{CAT}^{(i)}\right)\right)(2)$$

Since $\sum_{i=1}^{N} \pi_i^* \ = 1$ and $\sum_{i=1}^{N} \pi_i \ = 1$, it follows that

$$\sum_{i=1}^{N} \pi_i^* = \exp(-1 + \gamma) \sum_{i=1}^{N} \pi_i \exp\left(\lambda \sum_{t=1}^{T} \exp(\delta t)\left(\ {}_tB_{CIR}^{(i)} \cdot \ {}_tP_{CAT}^{(i)}\right)\right) = 1$$

Dividing through, we get

$$\exp(-1 + \gamma) = \frac{1}{\sum_{i=1}^{N} \pi_i \exp\left(\lambda \sum_{t=1}^{T} \exp(\delta t)\left(\ {}_tB_{CIR}^{(i)} \cdot \ {}_tP_{CAT}^{(i)}\right)\right)} \quad (3)$$

Insert equation (3) in to equation (2) to get

$$\pi_i^* = \frac{\pi_i \exp\left(\lambda \sum_{t=1}^{T} \exp(\delta t)\left({}_tB_{CIR}^{(i)} \cdot {}_tP_{CAT}^{(i)}\right)\right)}{\sum_{i=1}^{N} \pi_i \exp\left(\lambda \sum_{t=1}^{T} \exp(\delta t)\left({}_tB_{CIR}^{(i)} \cdot {}_tP_{CAT}^{(i)}\right)\right)} for i = 1, \ldots, N, t = 1, \ldots, T. (4)$$

Let $\alpha_i \triangleq \sum_{t=1}^{T} \exp(\delta t)\left({}_tB_{CIR}^{(i)} \cdot {}_tP_{CAT}^{(i)}\right)$ then (4) can be simplifed as

i.e.

$$\pi_i^* = \frac{\pi_i exp(\lambda \alpha_i)}{\sum_{i=1}^{N} \pi_i \, exp\,(\lambda \alpha_i)} \quad (5)$$

Let $\Gamma = \max_{i=1,\ldots N} \lambda \alpha_i$

Then (5) is equivalent to

$$\pi_i^* = \frac{\pi_i exp(\lambda \alpha_i - \Gamma)}{\sum_{i=1}^{N} \pi_i \, exp\,(\lambda \alpha_i - \Gamma)}$$

This can be used to calculate π_i^* once λ is known, in order to avoid overflow in the numerical calculation of exp.

- **Solving the dual problem by equivalence to another minimization problem**

The followings are steps to calculate λ.

First, (1) can be simplifed as

$$\sum_{i=1}^{N} \pi_i^* \alpha_i = V_o \qquad (6)$$

Subsituting (5) into (6), it comes to

$$\sum_{i=1}^{N} \frac{\pi_i exp(\lambda \alpha_i)}{\sum_{i=1}^{N} \pi_i \, exp\,(\lambda \alpha_i)} \alpha_i = V_o$$

Remember that $\pi_i = \frac{1}{N}$, so

$$\frac{\sum_{i=1}^{N} \alpha_i exp(\lambda \alpha_i)}{\sum_{i=1}^{N} \, exp\,(\lambda \alpha_i)} = V_o \quad (7)$$

This happens to be the minimizer of the following minimization problem

$$\min_{\lambda} \sum_{i=1}^{N} ex\,p[\lambda\{\alpha_i - V_o\}] \quad (8)$$

This can be proved by the following steps:

Let $f(\lambda) = \sum_{i=1}^{N} ex\,p[\lambda\{\alpha_i - V_o\}]$

Then

$$\frac{df}{d\lambda} = \sum_{i=1}^{N} \{\alpha_i - V_o\} ex\,p[\lambda\{\alpha_i - V_o\}]$$

$$= exp[-\lambda V_o] \sum_{i=1}^{N} \{\alpha_i - V_o\} ex\,p[\lambda \alpha_i]$$

Let $\frac{df}{d\lambda} = 0$, leading to

$$exp[-\lambda V_o] \sum_{i=1}^{N} \{\alpha_i - V_o\} \exp[\lambda \alpha_i] = 0$$

i.e.

$$\sum_{i=1}^{N} \{\alpha_i - V_o\} \exp[\lambda \alpha_i] = 0$$

$$\sum_{i=1}^{N} \alpha_i \exp[\lambda \alpha_i] = V_o \sum_{i=1}^{N} \exp[\lambda \alpha_i]$$

i.e.

$$\frac{\sum_{i=1}^{N} \alpha_i \exp[\lambda \alpha_i]}{\sum_{i=1}^{N} \exp[\lambda \alpha_i]} = V_o$$

That is the same form as (7). So solving the minimization problem (8) is the idential to solving the equation (7).

The minimization problem can be solved by **scipy.optimize.minimize_scalar.**

- **Domain of δ**

The domain of δ should be depending on $\{\alpha_i\}$, this can be proved from the followings:

Note that,

$$\sum_{i=1}^{N} \pi_i^* \alpha_i = V_o$$

$$\sum_{i=1}^{N} \pi_i^* \ = 1$$

So

$$\max_{\pi_i^*} \sum_{i=1}^{N} \pi_i^* \alpha_i = max\{\alpha_1, \alpha_2, \dots, \alpha_N\}$$

Then it requires that

$$V_o \ \le max \alpha_1, \alpha_2, \dots, \alpha_N$$

Howeve, from Eq. (1.1)

$$V_o = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \exp(\delta t) \ {}_tB_{CIR}^{(i)} \ \ {}_tP_{CAT}^{(i)} \ \ for\ i = 1, \dots, N$$

So the choice of δ should satisfy that

$$\frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \exp(\delta t) \ {}_tB_{CIR}^{(i)} \ \ {}_tP_{CAT}^{(i)} \ \le max\{\alpha_1, \alpha_2, \dots, \alpha_N\}$$

Let function $f(\delta) = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \exp(\delta t) \ {}_tB_{CIR}^{(i)} \ \ {}_tP_{CAT}^{(i)}$

It is a function of δ and its is increasing function when δ is increasing

Let function $h(\delta) = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \exp(\delta t) \ {}_tB_{CIR}^{(i)} \ \ {}_tP_{CAT}^{(i)} \ - max\{\alpha_1, \alpha_2, \dots, \alpha_N\}$

So the root of $h(\delta)$, i.e. δ^* to satisify $h(\delta^*) = 0$ is the solution we need.