Asymptotics of Bernoulli Line Ensembles

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Let $\{X_i\}$ be a sequence of i.i.d. random variables, s.t. $\mathbb{E}[X_1] = \mu$, $Var(X_1^2) = \sigma^2$. Let $S_n = \sum_{i=1}^n X_i$:

- **Law of Large Numbers:** $\frac{S_n}{n} \to \mu$ as $n \to \infty$ almost surely
- **Central Limit Theorem:** $\frac{S_n - n\mu}{\sqrt{n}} \to N(0, \sigma^2)$ as $n \to \infty$
- **Donsker's Theorem:** Let $S(x) = S_k$ if $x = k$ and linearly interpolate for $x \in [0, n]$ Let $\mu = 0$ and $\sigma = 1$. Then $\frac{S(n\cdot)}{\sqrt{n}} \in C([0,1])$ and $\frac{S(n\cdot)}{\sqrt{n}} \to B(\cdot)$, where B denotes a standard Brownian Motion.

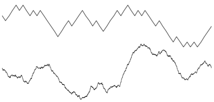


Figure: An example of a Bernoulli random walk and a Brownian Motion

Consider again Bernoulli random walks and Brownian Motion. We now increase the number of (non-intersecting) walkers:

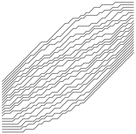


Figure: Multiple Avoiding Bernoulli Random Walks

When dealing with a family of avoiding Brownian Motions, we speak of Dyson Brownian Motion:

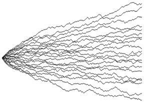


Figure: Dyson Brownian Motion

Airy Line Ensemble

As $N \to \infty$, the rescaled walks converge in distribution, uniformly over compact sets of $\mathbb{N} \times \mathbb{R}$, to the Airy line ensemble, $\mathcal{A}$, and the top curve converges to Airy process, $\mathcal{A}_1$.

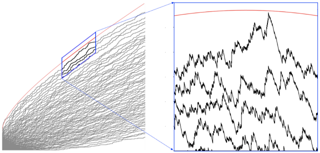


Figure: Multiple Dyson Brownian walks

Increasing the number of paths pushes us outside of the Gaussian universality class and into Kardar-Parisi-Zhang (KPZ) universality class.

Open Question

Show that any random walks with generic initial conditions convergence to the Airy line ensemble.

Two sufficient conditions:

1. Finite dimensional distribution convergence
2. Tightness, or the existence of weak subsequential limits.

We focused on tightness, which requires a maximum, minimum, and conditions on the Modulus of Continuity

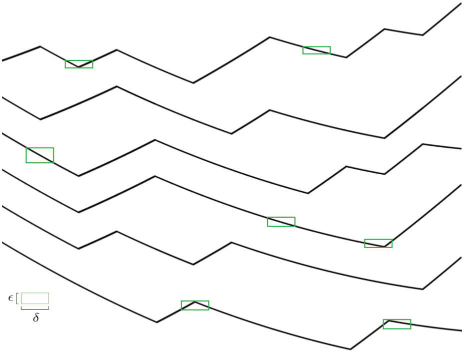


Figure: The Modulus of Continuity

Our Result

Theorem

With L_1^N being the top curve in a Bernoulli Line Ensemble $p \in (0,1)$, and $\lambda, \alpha > 0$, if for all $n \in \mathbb{Z}$,

$$\lim_{N\to\infty} P(L_1^N(nN^\alpha) - nN^\alpha p + \lambda n^2 N^{\alpha/2} \leq N^{\alpha/2}x) \to F_{TW}(x)$$

then the Line Ensemble is tight.

If the one-point marginal probabilities at integer times weakly converge to the Tracy Widom distribution then the Line Ensemble is tight.

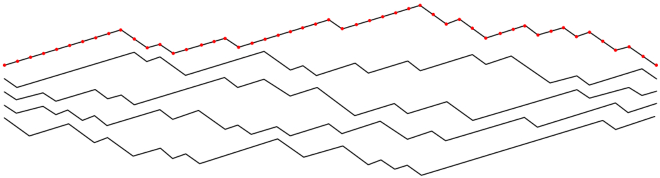


Figure: Integer time points of top line

[Duavergne, Nica, & Virag, 2019] - tightness assuming finite dimensional convergence to the Airy Line Ensemble.
We achieve the same result with much less restrictive assumptions

[Unsure of Image Choice]

History of the line ensembles

Arguments in this paper are inspired by

1. *Brownian Gibbs property for Airy line ensembles* and *KPZ line ensemble*[Corwin-Hammond '11, '13], which address the issues of continuous line ensembles
2. *Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood line ensembles* [Corwin-Dimitrov '17], which consider similar questions in a discrete setting

Problem Description

Recall that to show tightness, we want to control

1. min
2. max
3. modulus of continuity of the line ensembles

We claim that for the **top** curve of our line ensemble to have a **parabolic shift**, the **bottom** curve cannot dip too low, i.e. for any $r, \epsilon > 0$, there exist $R, M > 0$ such that for N large enough,

$$P(\max_{[r,R]} L_k(sN^\alpha) - psN^\alpha \leq -MN^\alpha) < \epsilon$$

(perhaps insert a picture)

Proof (mention monotone coupling lemmas somewhere) - say MC with picture 2min

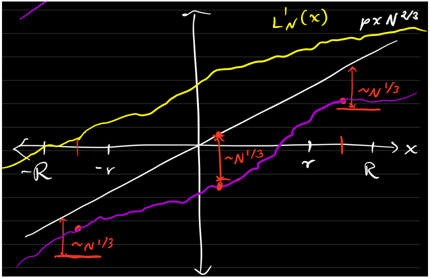
Proof (mention strong coupling somewhere) - say SC with picture L = Bernoulli bridge B is a Brownian bridge with variance. There is a probability space such that $P(sup|L-B| \geq k(\log N)^2) < \epsilon$. This is a comparison that allows for example to compare the modulus of continuity of the two. [Dimitrov-Wu '19] 2 min

Lemma (——)

For any $r, \epsilon > 0$, there exists $R > r$ and a constant $M > 0$ so that for large N,

$$\mathbb{P}\Big(\max_{x \in [r,R]} \big(L_k^N(xN^{2/3}) - pxN^{2/3}\big) \leq -MN^{1/3} \Big) < \epsilon.$$

The same is true of the maximum on $[-R, -r]$.



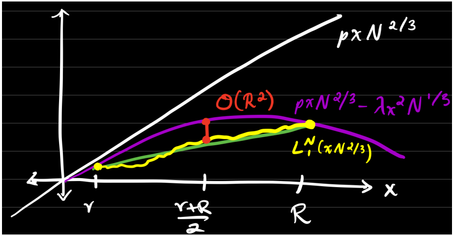
- Couple with a Brownian bridge: if "pinned" at two points $> r$ and $-r$, it cannot be low on scale $N^{1/3}$ on $[-r, r]$.

Proving the pinning lemma

- Recall our assumption:

$$\mathbb{P}\Big(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \le xN^{1/3}\Big) \underset{N\to\infty}{\longrightarrow} F_{TW}(x).$$

- The top curve looks like a parabola with an affine shift on large scales.



- Two curves: if L_2^N is low on $[r, R]$, L_1^N looks like a free Brownian bridge.

$$\lambda\left(\frac{R^2+r^2}{2}\right) - \lambda\left(\frac{R+r}{2}\right)^2 = \lambda\frac{R^2+r^2}{4} - \frac{\lambda rR}{2} = O(R^2).$$

- For large R, the top curve would be far from the parabola at the midpoint!