TIGHTNESS OF BERNOULLI LINE ENSEMBLES

ABSTRACT. Insert abstract here:

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1. INTRODUCTION AND MAIN RESULTS

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2. LINE ENSEMBLES

In this section we introduce various definitions and notation that are used throughout the paper.

2.1. **Line ensembles and the Brownian Gibbs property.** In this section we introduce the notions of a *line ensemble* and the *(partial) Brownian Gibbs property.* Our exposition in this section closely follows that of [6, Section 2] and [4, Section 2].

Given two integers $p \leq q$, we let $[\![p, q]\!]$ denote the set $\{p, p+1, \ldots, q\}$. Given an interval $\Lambda \subset \mathbb{R}$ we endow it with the subspace topology of the usual topology on $\mathbb{R}$. We let $(C(\Lambda), \mathcal{C})$ denote the space of continuous functions $f : \Lambda \to \mathbb{R}$ with the topology of uniform convergence over compacts, see [13, Chapter 7, Section 46], and Borel σ-algebra $\mathcal{C}$. Given a set $\Sigma \subset \mathbb{Z}$ we endow it with the discrete topology and denote by $\Sigma \times \Lambda$ the set of all pairs (i, x) with $i \in \Sigma$ and $x \in \Lambda$ with the product topology. We also denote by $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ the space of continuous functions on $\Sigma \times \Lambda$ with the topology of uniform convergence over compact sets and Borel σ-algebra $\mathcal{C}_\Sigma$. Typically, we will take $\Sigma = [\![1, N]\!]$ (we use the convention $\Sigma = \mathbb{N}$ if $N = \infty$) and then we write $\big(C(\Sigma \times \Lambda), \mathcal{C}_{|\Sigma|}\big)$ in place of $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$.

The following defines the notion of a line ensemble.

Definition 2.1. Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. A Σ-*indexed line ensemble* $\mathcal{L}$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that takes values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$. Intuitively, $\mathcal{L}$ is a collection of random continuous curves (sometimes referred to as *lines*), indexed by Σ, each of which maps Λ in $\mathbb{R}$. We will often slightly abuse notation and write $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$, even though it is not $\mathcal{L}$ which is such a function, but $\mathcal{L}(\omega)$ for every $\omega \in \Omega$. For $i \in \Sigma$ we write $\mathcal{L}_i(\omega) = (\mathcal{L}(\omega))(i, \cdot)$ for the curve of index i and note that the latter is a map $\mathcal{L}_i : \Omega \to C(\Lambda)$, which is $(\mathcal{C}, \mathcal{F})$−measurable. If $a, b \in \Lambda$ satisfy $a < b$ we let $\mathcal{L}_i[a, b]$ denote the restriction of $\mathcal{L}_i$ to $[a, b]$.

We will require the following result, whose proof is postponed until Section 8. In simple terms it states that the space $C(\Sigma \times \Lambda)$ where our random variables $\mathcal{L}$ take value has the structure of a complete, separable metric space.

Lemma 2.2. *Let* $\Sigma \subset \mathbb{Z}$ *and* $\Lambda \subset \mathbb{R}$ *be an interval. Suppose that* $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ *are sequences of real numbers such that* $a_n < b_n$, $[a_n, b_n] \subset \Lambda$, $a_{n+1} \leq a_n$, $b_{n+1} \geq b_n$ *and* $\cup_{n=1}^\infty [a_n, b_n] = \Lambda$. *For* $n \in \mathbb{N}$ *we let* $K_n = \Sigma_n \times [a_n, b_n]$ *where* $\Sigma_n = \Sigma \cap [\![-n, n]\!]$. *Define* $d : C(\Sigma \times \Lambda) \times C(\Sigma \times \Lambda) \to [0, \infty)$ *by*

$$(2.1) \qquad d(f, g) = \sum_{n=1}^\infty 2^{-n} \min \left\{ \sup_{(i,t) \in K_n} |f(i,t) - g(i,t)|, 1 \right\}.$$

Then d *defines a metric on* $C(\Sigma \times \Lambda)$ *and moreover the metric space topology defined by* d *is the same as the topology of uniform convergence over compact sets. Furthermore, the metric space* $(C(\Sigma \times \Lambda), d)$ *is complete and separable.*

Definition 2.3. Given a sequence $\{\mathcal{L}^n : n \in \mathbb{N}\}$ of random Σ-indexed line ensembles we say that $\mathcal{L}^n$ *converge weakly* to a line ensemble $\mathcal{L}$, and write $\mathcal{L}^n \implies \mathcal{L}$ if for any bounded continuous function $f : C(\Sigma \times \Lambda) \to \mathbb{R}$ we have that

$$\lim_{n \to \infty} \mathbb{E}\left[f(\mathcal{L}^n)\right] = \mathbb{E}\left[f(\mathcal{L})\right].$$

We also say that $\{\mathcal{L}^n : n \in \mathbb{N}\}$ is *tight* if for any $\epsilon > 0$ there exists a compact set $K \subset C(\Sigma \times \Lambda)$ such that $\mathbb{P}(\mathcal{L}^n \in K) \geq 1 - \epsilon$ for all $n \in \mathbb{N}$.

We call a line ensemble *non-intersecting* if $\mathbb{P}$-almost surely $\mathcal{L}_i(r) > \mathcal{L}_j(r)$ for all $i < j$ and $r \in \Lambda$.

We will require the following sufficient condition for tightness of a sequence of line ensembles, which extends [1, Theorem 7.3]. We give a proof in Section 8.

Lemma 2.4. *Let* $\Sigma \subset \mathbb{Z}$ *and* $\Lambda \subset \mathbb{R}$ *be an interval. Suppose that* $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ *are sequences of real numbers such that* $a_n < b_n$, $[a_n, b_n] \subset \Lambda$, $a_{n+1} \leq a_n$, $b_{n+1} \geq b_n$ *and* $\cup_{n=1}^\infty [a_n, b_n] = \Lambda$. *Then* $\{\mathcal{L}^n\}$ *is tight if and only if for every* $i \in \Sigma$ *we have*

(i) $\lim_{a \to \infty} \limsup_{n \to \infty} \mathbb{P}(|\mathcal{L}_i^n(a_0)| \geq a) = 0$;

(ii) For all $\epsilon > 0$ *and* $k \in \mathbb{N}$, $\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{\substack{x,y \in [a_k, b_k], \\ |x-y| \leq \delta}} |\mathcal{L}_i^n(x) - \mathcal{L}_i^n(y)| \geq \epsilon \right) = 0.$

We next turn to formulating the Brownian Gibbs property – we do this in Definition 2.8 after introducing some relevant notation and results. If W_t denotes a standard one-dimensional Brownian motion, then the process

$$\tilde{B}(t) = W_t - tW_1, \quad 0 \leq t \leq 1,$$

is called a *Brownian bridge (from* $\tilde{B}(0) = 0$ *to* $\tilde{B}(1) = 0$*) with diffusion parameter* 1. For brevity we call the latter object a *standard Brownian bridge*.

Given $a, b, x, y \in \mathbb{R}$ with $a < b$ we define a random variable on $(C([a, b]), \mathcal{C})$ through

$$(2.2) \qquad B(t) = (b - a)^{1/2} \cdot \tilde{B}\left(\frac{t - a}{b - a}\right) + \left(\frac{b - t}{b - a}\right) \cdot x + \left(\frac{t - a}{b - a}\right) \cdot y,$$

and refer to the law of this random variable as a *Brownian bridge (from $B(a) = x$ to $B(b) = y$) with diffusion parameter* 1. Given $k \in \mathbb{N}$ and $\vec{x}, \vec{y} \in \mathbb{R}^k$ we let $\mathbb{P}_{free}^{a,b,\vec{x},\vec{y}}$ denote the law of k independent Brownian bridges $\{B_i : [a, b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1.

We next state a couple of results about Brownian bridges from [4] for future use.

Lemma 2.5. *[4, Corollary 2.9]. Fix a continuous function $f : [0, 1] \to \mathbb{R}$ such that $f(0) > 0$ and $f(1) > 0$. Let B be a standard Brownian bridge and let $C = \{B(t) > f(t)$ for some $t \in [0, 1]\}$ (crossing) and $T = \{B(t) = f(t)$ for some $t \in [0, 1]\}$ (touching). Then $\mathbb{P}(T \cap C^c) = 0$.*

Lemma 2.6. *[4, Corollary 2.10]. Let U be an open subset of $C([0, 1])$, which contains a function f such that $f(0) = f(1) = 0$. If $B : [0, 1] \to \mathbb{R}$ is a standard Brownian bridge then $\mathbb{P}(B[0, 1] \subset U) > 0$.*

The following definition introduces the notion of an (f, g)-avoiding Brownian line ensemble, which in simple terms is a collection of k independent Brownian bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two continuous functions f and g.

Definition 2.7. Let $k \in \mathbb{N}$ and W_k° denote the open Weyl chamber in $\mathbb{R}^k$, i.e.

$$W_k^{\circ} = \{\vec{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k : x_1 > x_2 > \cdots > x_k\}$$

(in [4] the notation $\mathbb{R}_>^k$ was used for this set). Let $\vec{x}, \vec{y} \in W_k^{\circ}$, $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \to (-\infty, \infty]$ and $g : [a, b] \to [-\infty, \infty)$ be two continuous functions. The latter condition means that either $f : [a, b] \to \mathbb{R}$ is continuous or $f = \infty$ everywhere, and similarly for g. We also assume that $f(t) > g(t)$ for all $t \in [a, b]$, $f(a) > x_1$, $f(b) > y_1$ and $g(a) < x_k, g(b) < y_k$.

With the above data we define the (f, g)-*avoiding Brownian line ensemble on the interval* $[a, b]$ *with entrance data* $\vec{x}$ *and exit data* $\vec{y}$ to be the Σ-indexed line ensemble $\mathcal{Q}$ with $\Sigma = [\![1, k]\!]$ on $\Lambda = [a, b]$ and with the law of $\mathcal{Q}$ equal to $\mathbb{P}_{free}^{a,b,\vec{x},\vec{y}}$ (the law of k independent Brownian bridges $\{B_i : [a, b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$) conditioned on the event

$$E = \{f(r) > B_1(r) > B_2(r) > \cdots > B_k(r) > g(r) \text{ for all } r \in [a, b]\}.$$

It is worth pointing out that E is an open set of positive measure and so we can condition on it in the usual way – we explain this briefly in the following paragraph. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that supports k independent Brownian bridges $\{B_i : [a, b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1. Notice that we can find $\tilde{u}_1, \ldots, \tilde{u}_k \in C([0, 1])$ and $\epsilon > 0$ (depending on $\vec{x}, \vec{y}, f, g, a, b$) such that $\tilde{u}_i(0) = \tilde{u}_i(1) = 0$ for $i = 1, \ldots, k$ and such that if $\tilde{h}_1, \ldots, \tilde{h}_k \in C([0, 1])$ satisfy $\tilde{h}_i(0) = \tilde{h}_i(1) = 0$ for $i = 1, \ldots, k$ and $\sup_{t \in [0,1]} |\tilde{u}_i(t) - \tilde{h}_i(t)| < \epsilon$ then the functions

$$h_i(t) = (b - a)^{1/2} \cdot \tilde{h}_i\left(\frac{t - a}{b - a}\right) + \left(\frac{b - t}{b - a}\right) \cdot x_i + \left(\frac{t - a}{b - a}\right) \cdot y_i,$$

satisfy $f(r) > h_1(r) > \cdots > h_k(r) > g(r)$. It follows from Lemma 2.6 that

$$\mathbb{P}(E) \geq \mathbb{P}\left(\max_{1 \leq i \leq k} \sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) = \prod_{i=1}^k \mathbb{P}\left(\sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) > 0,$$

and so we can condition on the event E.

To construct a realization of $\mathcal{Q}$ we proceed as follows. For $\omega \in E$ we define

$$\mathcal{Q}(\omega)(i, r) = B_i(r)(\omega) \text{ for } i = 1, \ldots, k \text{ and } r \in [a, b].$$

Observe that for $i \in \{1, \ldots, k\}$ and an open set $U \in C([a, b])$ we have that

$$\mathcal{Q}^{-1}(\{i\} \times U) = \{B_i \in U\} \cap E \in \mathcal{F},$$

and since the sets $\{i\} \times U$ form an open basis of $C(\llbracket 1, k \rrbracket \times [a, b])$ we conclude that $\mathcal{Q}$ is $\mathcal{F}$-measurable. This implies that the law $\mathcal{Q}$ is indeed well-defined and also it is non-intersecting almost surely. Also, given measurable subsets $A_1, \ldots, A_k$ of $C([a, b])$ we have that

$$\mathbb{P}(\mathcal{Q}_i \in A_i \text{ for } i = 1, \ldots, k) = \frac{\mathbb{P}_{free}^{a,b,\vec{x},\vec{y}}(\{B_i \in A_i \text{ for } i = 1, \ldots, k\} \cap E)}{\mathbb{P}_{free}^{a,b,\vec{x},\vec{y}}(E)}.$$

We denote the probability distribution of $\mathcal{Q}$ as $\mathbb{P}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$ and write $\mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$ for the expectation with respect to this measure.

The following definition introduces the notion of the Brownian Gibbs property from [4].

Definition 2.8. Fix a set $\Sigma = \llbracket 1, N \rrbracket$ with $N \in \mathbb{N}$ or $N = \infty$ and an interval $\Lambda \subset \mathbb{R}$ and let $K = \{k_1, k_1 + 1, \ldots, k_2\} \subset \Sigma$ be finite and $a, b \in \Lambda$ with $a < b$. Set $f = \mathcal{L}_{k_1-1}$ and $g = \mathcal{L}_{k_2+1}$ with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$ and $g = -\infty$ if $k_2 + 1 \notin \Sigma$. Write $D_{K,a,b} = K \times (a, b)$ and $D_{K,a,b}^c = (\Sigma \times \Lambda) \setminus D_{K,a,b}$. A Σ-indexed line ensemble $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$ is said to have the *Brownian Gibbs property* if it is non-intersecting and

$$\text{Law}\left(\mathcal{L}|_{K \times [a,b]} \text{ conditional on } \mathcal{L}|_{D_{K,a,b}^c}\right) = \text{Law}(\mathcal{Q}),$$

where $\mathcal{Q}_i = \tilde{\mathcal{Q}}_{i-k_1+1}$ and $\tilde{\mathcal{Q}}$ is the (f, g)-avoiding Brownian line ensemble on $[a, b]$ with entrance data $(\mathcal{L}_{k_1}(a), \ldots, \mathcal{L}_{k_2}(a))$ and exit data $(\mathcal{L}_{k_1}(b), \ldots, \mathcal{L}_{k_2}(b))$ from Definition 2.7. Note that $\tilde{Q}$ is introduced because, by definition, any such (f, g)-avoiding Brownian line ensemble is indexed from 1 to $k_2 - k_1 + 1$ but we want $\mathcal{Q}$ to be indexed from k_1 to k_2.

An equivalent way to express the Brownian Gibbs property is as follows. A Σ-indexed line ensemble $\mathcal{L}$ on Λ satisfies the Brownian Gibbs property if and only if it is non-intersecting and for any finite $K = \{k_1, k_1 + 1, \ldots, k_2\} \subset \Sigma$ and $[a, b] \subset \Lambda$ and any bounded Borel-measurable function $F : C(K \times [a, b]) \to \mathbb{R}$ we have $\mathbb{P}$-almost surely

$$(2.3) \qquad \mathbb{E}\left[F\left(\mathcal{L}|_{K \times [a,b]}\right) \big| \mathcal{F}_{ext}(K \times (a, b))\right] = \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}\left[F(\tilde{\mathcal{Q}})\right],$$

where

$$\mathcal{F}_{ext}(K \times (a, b)) = \sigma\left\{\mathcal{L}_i(s) : (i, s) \in D_{K,a,b}^c\right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K \times [a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K \times [a, b]$, $\vec{x} = (\mathcal{L}_{k_1}(a), \ldots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \ldots, \mathcal{L}_{k_2}(b))$, $f = \mathcal{L}_{k_1-1}[a, b]$ (the restriction of $\mathcal{L}$ to the set $\{k_1 - 1\} \times [a, b]$) with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$, and $g = \mathcal{L}_{k_2+1}[a, b]$ with the convention that $g = -\infty$ if $k_2 + 1 \notin \Sigma$.

Remark 2.9. Let us briefly explain why equation (2.3) makes sense. Firstly, since $\Sigma \times \Lambda$ is locally compact, we know by [13, Lemma 46.4] that $\mathcal{L} \to \mathcal{L}|_{K \times [a,b]}$ is a continuous map from $C(\Sigma \times \Lambda)$ to $C(K \times [a, b])$, so that the left side of (2.3) is the conditional expectation of a bounded measurable function, and is thus well-defined. A more subtle question is why the right side of (2.3) is $\mathcal{F}_{ext}(K \times (a, b))$-measurable. This question was resolved in [6, Lemma 3.4], where it was shown that the right side is measurable with respect to the σ-algebra

$$\sigma\left\{\mathcal{L}_i(s) : i \in K \text{ and } s \in \{a, b\}, \text{ or } i \in \{k_1 - 1, k_2 + 1\} \text{ and } s \in [a, b]\right\},$$

which in particular implies the measurability with respect to $\mathcal{F}_{ext}(K \times (a, b))$.

In the present paper it is convenient for us to use the following modified version of the definition above, which we call the partial Brownian Gibbs property – it was first introduced in [6]. We explain the difference between the two definitions, and why we prefer the second one in Remark 2.12.

Definition 2.10. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and an interval $\Lambda \subset \mathbb{R}$. A Σ-indexed line ensemble $\mathcal{L}$ on Λ is said to satisfy the *partial Brownian Gibbs property* if and only if it is non-intersecting and for any finite $K = \{k_1, k_1 + 1, \ldots, k_2\} \subset \Sigma$ with $k_2 \leq N - 1$ (if $\Sigma \neq \mathbb{N}$), $[a, b] \subset \Lambda$ and any bounded Borel-measurable function $F : C(K \times [a, b]) \to \mathbb{R}$ we have $\mathbb{P}$-almost surely

$$(2.4) \qquad \mathbb{E}\left[F(\mathcal{L}|_{K \times [a,b]}) \big| \mathcal{F}_{ext}(K \times (a, b))\right] = \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}\left[F(\tilde{\mathcal{Q}})\right],$$

where we recall that $D_{K,a,b} = K \times (a, b)$ and $D_{K,a,b}^c = (\Sigma \times \Lambda) \setminus D_{K,a,b}$, and

$$\mathcal{F}_{ext}(K \times (a, b)) = \sigma\left\{\mathcal{L}_i(s) : (i, s) \in D_{K,a,b}^c\right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K \times [a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K \times [a, b]$, $\vec{x} = (\mathcal{L}_{k_1}(a), \ldots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \ldots, \mathcal{L}_{k_2}(b))$, $f = \mathcal{L}_{k_1-1}[a, b]$ with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$, and $g = \mathcal{L}_{k_2+1}[a, b]$.

Remark 2.11. Observe that if $N = 1$ then the conditions in Definition 2.10 become void. I.e., any line ensemble with one line satisfies the partial Brownian Gibbs property. Also we mention that (2.4) makes sense by the same reason that (2.3) makes sense, see Remark 2.9.

Remark 2.12. Definition 2.10 is slightly different from the Brownian Gibbs property of Definition 2.8 as we explain here. Assuming that $\Sigma = \mathbb{N}$ the two definitions are equivalent. However, if $\Sigma = \{1, \ldots, N\}$ with $1 \leq N < \infty$ then a line ensemble that satisfies the Brownian Gibbs property also satisfies the partial Brownian Gibbs property, but the reverse need not be true. Specifically, the Brownian Gibbs property allows for the possibility that $k_2 = N$ in Definition 2.10 and in this case the convention is that $g = -\infty$. As the partial Brownian Gibbs property is more general we prefer to work with it and most of the results later in this paper are formulated in terms of it rather than the usual Brownian Gibbs property.

2.2. **Bernoulli Gibbsian line ensembles.** In this section we introduce the notion of a *Bernoulli line ensemble* and the *Schur Gibbs property*. Our discussion will parallel that of [3, Section 3.1], which in turn goes back to [5, Section 2.1].

Definition 2.13. Let $\Sigma \subset \mathbb{Z}$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. Consider the set Y of functions $f : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(j, i + 1) - f(j, i) \in \{0, 1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$ and let $\mathcal{D}$ denote the discrete topology on Y. We call a function $f : [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(i + 1) - f(i) \in \{0, 1\}$ when $i \in [\![T_0, T_1 - 1]\!]$ an *up-right path* and elements in Y *collections of up-right paths*.

A Σ-*indexed Bernoulli line ensemble* $\mathfrak{L}$ on $[\![T_0, T_1]\!]$ is a random variable defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, taking values in Y such that $\mathfrak{L}$ is a $(\mathcal{B}, \mathcal{D})$-measurable function.

Remark 2.14. In [3, Section 3.1] Bernoulli line ensembles $\mathfrak{L}$ were called *discrete line ensembles* in order to distinguish them from the continuous line ensembles from Definition 2.1. In this paper we have opted to use the term Bernoulli line ensembles to emphasize the fact that the functions $f \in Y$ satisfy the property that $f(j, i+1) - f(j, i) \in \{0, 1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$. This condition essentially means that for each $j \in \Sigma$ the function $f(j, \cdot)$ can be thought of as the trajectory of a Bernoulli random walk from time T_0 to time T_1. As other types of discrete line ensembles, see e.g. [17], have appeared in the literature we have decided to modify the notation in [3, Section 3.1] so as to avoid any ambiguity.

The way we think of Bernoulli line ensembles is as random collections of up-right paths on the integer lattice, indexed by Σ (see Figure 1). Observe that one can view an up-right path L on $[\![T_0, T_1]\!]$ as a continuous curve by linearly interpolating the points $(i, L(i))$. This allows us to define $(\mathfrak{L}(\omega))(i, s)$ for non-integer $s \in [T_0, T_1]$ and to view Bernoulli line ensembles as line ensembles in the sense of Definition 2.1. In particular, we can think of $\mathfrak{L}$ as a random variable taking values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ with $\Lambda = [T_0, T_1]$. We will often slightly abuse notation and write $\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$, even

FIGURE 1. Two samples of $[\![1,3]\!]$-indexed Bernoulli line ensembles with $T_0 = 1$ and $T_1 = 8$, with the left ensemble avoiding and the right ensemble nonavoiding.

though it is not $\mathfrak{L}$ which is such a function, but rather $\mathfrak{L}(\omega)$ for each $\omega \in \Omega$. Furthermore we write $L_i = (\mathfrak{L}(\omega))(i, \cdot)$ for the index $i \in \Sigma$ path. If L is an up-right path on $[\![T_0, T_1]\!]$ and $a, b \in [\![T_0, T_1]\!]$ satisfy $a < b$ we let $L[\![a, b]\!]$ denote the resitrction of L to $[\![a, b]\!]$.

Let $t_i, z_i \in \mathbb{Z}$ for $i = 1, 2$ be given such that $t_1 < t_2$ and $0 \leq z_2 - z_1 \leq t_2 - t_1$. We denote by $\Omega(t_1, t_2, z_1, z_2)$ the collection of up-right paths that start from (t_1, z_1) and end at (t_2, z_2), by $\mathbb{P}_{Ber}^{t_1, t_2, z_1, z_2}$ the uniform distribution on $\Omega(t_1, t_2, z_1, z_2)$ and write $\mathbb{E}_{Ber}^{t_1, t_2, z_1, z_2}$ for the expectation with respect to this measure. One thinks of the distribution $\mathbb{P}_{Ber}^{t_1, t_2, z_1, z_2}$ as the law of a simple random walk with i.i.d. Bernoulli increments with parameter $p \in (0, 1)$ that starts from z_1 at time t_1 and is conditioned to end in z_2 at time t_2 – this interpretation does not depend on the choice of $p \in (0, 1)$. Notice that by our assumptions on the parameters the state space $\Omega(t_1, t_2, z_1, z_2)$ is non-empty.

Given $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$ and $\vec{x}, \vec{y} \in \mathbb{Z}^k$ we let $\mathbb{P}_{Ber}^{T_0, T_1, \vec{x}, \vec{y}}$ denote the law of k independent Bernoulli bridges $\{B_i : [\![T_0, T_1]\!] \to \mathbb{Z}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$. Equivalently, this is just k independent random up-right paths $B_i \in \Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \ldots, k$ that are uniformly distributed. This measure is well-defined provided that $\Omega(T_0, T_1, x_i, y_i)$ are non-empty for $i = 1, \ldots, k$, which holds if $T_1 - T_0 \geq y_i - x_i \geq 0$ for all $i = 1, \ldots, k$.

The following definition introduces the notion of an (f, g)-avoiding Bernoulli line ensemble, which in simple terms is a collection of k independent Bernoulli bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two functions f and g.

Definition 2.15. Let $k \in \mathbb{N}$ and $\mathfrak{W}_k$ denote the set of signatures of length k, i.e.

$$\mathfrak{W}_k = \{\vec{x} = (x_1, \ldots, x_k) \in \mathbb{Z}^k : x_1 \geq x_2 \geq \cdots \geq x_k\}.$$

Let $\vec{x}, \vec{y} \in \mathfrak{W}_k$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, and $f : [\![T_0, T_1]\!] \to (-\infty, \infty]$ and $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ be two functions.

With the above data we define the (f, g)-avoiding Bernoulli line ensemble on the interval $[\![T_0, T_1]\!]$ with entrance data $\vec{x}$ and exit data $\vec{y}$ to be the Σ-indexed Bernoulli line ensemble $\mathfrak{Q}$ with $\Sigma = [\![1, k]\!]$ on $[\![T_0, T_1]\!]$ and with the law of $\mathfrak{Q}$ equal to $\mathbb{P}_{Ber}^{T_0, T_1, \vec{x}, \vec{y}}$ (the law of k independent uniform up-right paths $\{B_i : [\![T_0, T_1]\!] \to \mathbb{R}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$) conditioned on the event

$$E = \{f(r) \geq B_1(r) \geq B_2(r) \geq \cdots \geq B_k(r) \geq g(r) \text{ for all } r \in [\![T_0, T_1]\!]\}.$$

The above definition is well-posed if there exist $B_i \in \Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \ldots, k$ that satisfy the conditions in E (i.e. if the set of such up-right paths is not empty). We will denote by $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ the set of collections of k up-right paths that satisfy the conditions in E and then the distribution on $\mathfrak{Q}$ is simply the uniform measure on $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$. We denote the probability distribution of $\mathfrak{Q}$ as $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ and write $\mathbb{E}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ for the expectation with respect to this measure. When $f = +\infty$ and $g = -\infty$, we simply write $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y}}$ and $\mathbb{E}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y}}$.

It will be useful to formulate simple conditions under which $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is non-empty and thus $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ well-defined. We accomplish this in the following lemma, whose proof is postponed until Section 8.

Lemma 2.16. *Suppose that $k \in \mathbb{N}$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. Suppose further that*
- *(1) $\vec{x}, \vec{y} \in \mathfrak{W}_k$ satisfy $T_1 - T_0 \geq y_i - x_i \geq 0$ for $i = 1, \ldots, k$*
- *(2) $f : [\![T_0, T_1]\!] \to (-\infty, \infty]$ and $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ satisfy $f(i+1) = f(i)$ or $f(i+1) = f(i) + 1$, and $g(i+1) = g(i)$ or $g(i+1) = g(i) + 1$ for $i = T_0, \ldots, T_1 - 1$*
- *(3) $f(T_0) \geq x_1, f(T_1) \geq y_1$ and $g(T_0) \leq x_k, g(T_1) \leq y_k$*

Then the set $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ from Definition 2.15 is non-empty.

The following definition introduces the notion of the Schur Gibbs property, which can be thought of a discrete analogue of the partial Brownian Gibbs property the same way that Bernoulli random walks are discrete analogues of Brownian motion.

Definition 2.17. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. A Σ-indexed Bernoulli line ensemble $\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ is said to satisfy the *Schur Gibbs property* if it is non-crossing, meaning that

$$L_j(i) \geq L_{j+1}(i) \text{ for all } j = 1, \ldots, N-1 \text{ and } i \in [\![T_0, T_1]\!],$$

and for any finite $K = \{k_1, k_1 + 1, \ldots, k_2\} \subset [\![1, N-1]\!]$ and $a, b \in [\![T_0, T_1]\!]$ with $a < b$ the following holds. Suppose that f, g are two up-right paths drawn in $\{(r, z) \in \mathbb{Z}^2 : a \leq r \leq b\}$ and $\vec{x}, \vec{y} \in \mathfrak{W}_{k_2-k_1+1}$ altogether satisfy that $\mathbb{P}(A) > 0$ where A denotes the event

$$A = \{\vec{x} = (L_{k_1}(a), \ldots, L_{k_2}(a)), \vec{y} = (L_{k_1}(b), \ldots, L_{k_2}(b)), L_{k_1-1}[\![a, b]\!] = f, L_{k_2+1}[\![a, b]\!] = g\},$$

where if $k_1 = 1$ we adopt the convention $f = \infty = L_0$. Then for any $\{B_i \in \Omega(a, b, x_i, y_i)\}_{i=1}^{k_2-k_1+1}$

$$(2.5) \qquad \mathbb{P}\left(L_{i+k_1-1}[\![a, b]\!] = B_i \text{ for } i = 1, \ldots, k_2 - k_1 + 1 | A\right) = \mathbb{P}_{avoid,Ber}^{a,b,\vec{x},\vec{y},f,g}\left(\cap_{i=1}^k \{\mathfrak{Q}_i = B_i\}\right).$$

Remark 2.18. In simple words, a Bernoulli line ensemble is said to satisfy the Schur Gibbs property if the distribution of any finite number of consecutive paths, conditioned on their end-points and the paths above and below them is simply the uniform measure on all collection of up-right paths that have the same end-points and do not cross each other or the paths above and below them.

Remark 2.19. Observe that in Definition 2.17 the index k_2 is assumed to be less than or equal to $N - 1$, so that if $N < \infty$ the N-th path is special and is not conditionally uniform. This is what makes Definition 2.17 a discrete analogue of the partial Brownian Gibbs property rather than the usual Brownian Gibbs property. Similarly to the partial Brownian Gibbs propert, see Remark 2.11, if $N = 1$ then the conditions in Definition 2.17 become void. I.e., any Bernoulli line ensemble with one line satisfies the Schur Gibbs property. Also we mention that the well-posedness of $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ in (2.5) is a consequence of Lemma 2.16 and our assumption that $\mathbb{P}(A) > 0$.

Remark 2.20. In [3] the authors studied a generalization of the Gibbs property in Definition 2.17 depending on a parameter $t \in (0, 1)$, which was called the *Hall-Littlewood Gibbs property* due to its connection to Hall-Littlewood polynomials [12]. The property in Definition 2.17 is the $t \to 0$ limit of the Hall-Littlewood Gibbs property. Since under this $t \to 0$ limit Hall-Littlewood polynomials

degenerate to Schur polynomials we have decided to call the Gibbs property in Definition 2.17 the Schur Gibbs property.

Remark 2.21. An immediate consequence of Definition 2.17 is that if $M \leq N$, we have that the induced law on $\{L_i\}_{i=1}^M$ also satisfies the Schur Gibbs property as an $\{1, \ldots, M\}$-indexed Bernoulli line ensemble on $[\![T_0, T_1]\!]$.

We end this section with the following definition of the term acceptance probability.

Definition 2.22. Assume the same notation as in Definition 2.15 and suppose that $T_1 - T_0 \geq y_i - x_i \geq 0$ for $i = 1, \ldots, k$. We define the *acceptance probability* $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$ to be the ratio

$$(2.6) \qquad Z(T_0, T_1, \vec{x}, \vec{y}, f, g) = \frac{|\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)|}{\prod_{i=1}^k |\Omega(T_0, T_1, x_i, y_i)|}.$$

Remark 2.23. The quantity $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is precisely the probability that if B_i are sampled uniformly from $\Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \ldots, k$ then the B_i satisfy the condition

$$E = \{f(r) \geq B_1(r) \geq B_2(r) \geq \cdots \geq B_k(r) \geq g(r) \text{ for all } r \in [\![T_0, T_1]\!]\}.$$

Let us explain briefly why we call this quantity an acceptance probability. One way to sample $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ is as follows. Start by sampling a sequence of i.i.d. up-right paths B_i^N uniformly from $\Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \ldots, k$ and $N \in \mathbb{N}$. For each n check if $B_1^n, \ldots, B_k^n$ satisfy the condition E and let M denote the smallest index that accomplishes this. If $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is non-empty then M is geometrically distributed with parameter $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$, and in particular M is finite almost surely and $\{B_i^M\}_{i=1}^k$ has distribution $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$. In this sampling procedure we construct a sequence of candidates $\{B_i^N\}_{i=1}^k$ for $N \in \mathbb{N}$ and reject those that fail to satisfy condition E, the first candidate that satisfies it is accepted and has law $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ and the probability that a candidate is accepted is precisely $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$, which is why we call it an acceptance probability.

2.3. **Main technical result.** In this section we present the main technical result of the paper. We start with the following technical definition.

Definition 2.24. Fix $k \in \mathbb{N}$, $\alpha, \lambda > 0$ and $p \in (0, 1)$. Suppose we are given a sequence $\{T_N\}_{N=1}^\infty$ with $T_N \in \mathbb{N}$ and that $\{\mathfrak{L}^N\}_{N=1}^\infty$, $\mathfrak{L}^N = (L_1^N, L_2^N, \ldots, L_k^N)$ is a sequence of $[\![1, k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N, T_N]\!]$. We call the sequence (α, p, λ)-*good* if

- for each $N \in \mathbb{N}$ we have that $\mathfrak{L}^N$ satisfies the Schur Gibbs property of Definition 2.17;
- there is a function $\psi : \mathbb{N} \to (0, \infty)$ such that $\lim_{N \to \infty} \psi(N) = \infty$ and for each $N \in \mathbb{N}$ we have that $T_N > \psi(N) N^\alpha$;
- there is a function $\phi : (0, \infty) \to (0, \infty)$ such that for any $\epsilon > 0$ we have

$$(2.7) \qquad \sup_{n \in \mathbb{Z}} \limsup_{N \to \infty} \mathbb{P}\left(\left|N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})\right| \geq \phi(\epsilon)\right) \leq \epsilon.$$

Remark 2.25. Let us elaborate on the meaning of Definition 2.24. In order for a sequence of $\mathfrak{L}^N$ of $[\![1, k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N, T_N]\!]$ to be (α, p, λ)-*good* we want several conditions to be satisfied. Firstly, we want for each N the Bernoulli line ensemble $\mathfrak{L}^N$ to satisfy the Schur Gibbs property. The second condition is that while the interval of definition of $\mathfrak{L}^N$ is finite for each N and given by $[\![-T_N, T_N]\!]$, we want this interval to grow at least with speed N^α. This property is quantified by the function ψ, which can be essentially thought of as an arbitrary unbounded increasing function on $\mathbb{N}$. The third condition is that we want for each $n \in \mathbb{Z}$ the sequence of random variables $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha)$ to be tight but moreover we want globally these random variables to look like the parabola $-\lambda n^2$. This statement is reflected in (2.7), which provides a certain uniform tightness of the random variables $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})$. A particular case when (2.7) is satisfied is for example if we know that for each $n \in \mathbb{Z}$

the random variables $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})$ converge to the same random variable X. In the applications that we have in mind these random variables would converge to the 1-point marginals of the Airy$_2$ process that are all given by the same Tracy-Widom distribution (since the Airy$_2$ process is stationary). Equation (2.7) is a significant relaxation of the requirement that $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})$ all converge weakly to the Tracy-Widom distribution – the convergence requirement is replaced with a mild but uniform control of all subsequential limits.

The main result of the paper is as follows.

Theorem 2.26. *Fix $k \in \mathbb{N}$ with $k \geq 2$, $\alpha, \lambda > 0$ and $p \in (0,1)$ and let $\mathfrak{L}^N = (L_1^N, L_2^N, \ldots, L_k^N)$ be an (α, p, λ)-good sequence of $[\![1, k]\!]$-indexed Bernoulli line ensembles. Set*

$$f_i^N(s) = N^{-\alpha/2}(L_i^N(sN^\alpha) - psN^\alpha + \lambda s^2 N^{\alpha/2}), \text{ for } s \in [-\psi(N), \psi(N)] \text{ and } i = 1, \ldots, k-1,$$

and extend f_i^N to $\mathbb{R}$ by setting for $i = 1, \ldots, k-1$

$$f_i^N(s) = f_i^N(-\psi(N)) \text{ for } s \leq -\psi(N) \text{ and } f_i^N(s) = f_N(\psi(N)) \text{ for } s \geq \psi(N).$$

Let $\mathbb{P}_N$ denote the law of $\{f_i^N\}_{i=1}^{k-1}$ as a $[\![1, k-1]\!]$-indexed line ensemble (i.e. as a random variable in $(C([\![1, k-1]\!] \times \mathbb{R}), \mathcal{C})$). Then the sequence $\mathbb{P}_N$ is tight. Moreover, if $\mathcal{L}^\infty = \{f_i^{infty}\}_{i=1}^{k-1}$ denotes any subsequential limit of $\mathbb{P}_N$ then $\mathcal{L}^\infty$ satisfies the Brownian Gibbs property in Definition 2.10.

Roughly, Theorem 2.26 states that if you have a sequence of $[\![1, k]\!]$-indexed Bernoulli line ensembles that satisfy the Schur Gibbs property and the top paths of these ensembles under some shift and scaling have tight one-point marginals with a non-trivial parabolic shift, then under the same shift and scaling the top $k-1$ paths of the line ensemble will be tight. The extension of f_i^N to $\mathbb{R}$ is completely arbitrary and irrelevant for the validity of Theorem 2.26 since the topology on $C([\![1, k-1]\!] \times \mathbb{R})$ is that of uniform convergence over compacts. Consequently, only the behavior of these functions on compact intervals matters in Theorem 2.26 and not what these functions do near infinity, which is where the modification happens as $\lim_{N\to\infty} \psi(N) = \infty$ by assumption. The only reason we perform the extension is to embed all Bernoulli line ensembles into the same space $(C([\![1, k-1]\!] \times \mathbb{R}), \mathcal{C})$.

We mention that the k-th up-right path in the sequence of Bernoulli line ensembles is special and Theorem 2.26 provides no tightness result for it. The reason for this stems from the Schur Gibbs property, see Definition 2.17, which assumes less information for the k-th path. In practice, one either has an infinite Bernoulli line ensemble for each N or one has a Bernoulli line ensemble with finite number of paths, which increase with N to infinity. In either of these settings one can use Theorem 2.26 to prove tightness of the full line ensemble - we will have more to say about this in Section 7.

The proof of Theorem 2.26 is presented in Section 4. In the next section we derive various properties for Bernoulli line ensembles.

3. Properties of Bernoulli line ensembles

In this section we derive several results for Bernoulli line ensembles, which will be used in the proof of Theorem 2.26 in Section 4.

3.1. **Monotone coupling lemmas.** In this section we formulate two lemmas that provide couplings of two Bernoulli line ensembles of non-intersecting Bernoulli bridges on the same interval, which depend monotonically on their boundary data. Schematic depictions of the couplings are provided in Figure 2. We postpone the proof of these lemmas until Section 8.

Lemma 3.1. *Assume the same notation as in Definition 2.15. Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, a function $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ as well as $\vec{x}, \vec{y}, \vec{x}', \vec{y}' \in \mathfrak{W}_k$. Assume that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}', \vec{y}', \infty, g)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,*

FIGURE 2. Two diagrammatic depictions of the monotone coupling Lemma 3.1 (left part) and Lemma 3.2 (right part). Red depicts the lower line ensemble and accompanying entry data, exit data, and bottom bounding curve, while blue depicts that of the higher ensemble.

which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\left(\text{resp. } \mathfrak{L}^b\right)$ under $\mathbb{P}$ is given by $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g}$ $\left(\text{resp. } \mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g}\right)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}_i^t(r) \geq \mathfrak{L}_i^b(r)$ for all $i = 1, \ldots, k$ and $r \in [\![T_0, T_1]\!]$.

Lemma 3.2. *Assume the same notation as in Definition 2.15. Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, two functions $g^t, g^b : [\![T_0, T_1]\!] \to [-\infty, \infty)$ and $\vec{x}, \vec{y} \in \mathfrak{W}_k$. We assume that $g^t(r) \geq g^b(r)$ for all $r \in [\![T_0, T_1]\!]$ and that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^t)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\left(\text{resp. } \mathfrak{L}^b\right)$ under $\mathbb{P}$ is given by $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^t}$ $\left(\text{resp. } \mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}\right)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}_i^t(r) \geq \mathfrak{L}_i^b(r)$ for all $i = 1, \ldots, k$ and $r \in [\![T_0, T_1]\!]$.*

In plain words, Lemma 3.1 states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ of non-intersecting Bernoulli bridges, bounded from below by the same function g, in such a way that if all boundary values of $\mathfrak{L}^t$ are above the respective boundary values of $\mathfrak{L}^b$, then all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the left part of Figure 2. Lemma 3.2, states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ that have the same boundary values, but the lower bound g^t of $\mathfrak{L}^t$ is above the lower bound g^b of $\mathfrak{L}^b$, in such a way that all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the right part of Figure 2.

3.2. **Properties of Bernoulli bridges.** In this section we derive several results about Bernoulli bridges, which are random up-right paths that have law $\mathbb{P}_{Ber}^{T_0,T_1,x,y}$ as in Section 2.2. Our results will rely on the two monotonicity Lemmas 3.1 and 3.2 as well as a strong coupling between Bernoulli bridges and Brownian bridges from [3] – recalled here as Theorem 3.3.

If W_t denotes a standard one-dimensional Brownian motion and $\sigma > 0$, then the process

$$B_t^\sigma = \sigma(W_t - tW_1), \quad 0 \leq t \leq 1,$$

is called a *Brownian bridge (conditioned on $B_0 = 0, B_1 = 0$) with variance σ^2*. We note that B^σ is the unique a.s. continuous Gaussian process on $[0, 1]$ with $B_0 = B_1 = 0$, $\mathbb{E}[B_t^\sigma] = 0$, and

$$(3.1) \qquad \mathbb{E}[B_r^\sigma B_s^\sigma] = \sigma^2(r \wedge s - rs - sr + sr) = \sigma^2(r \wedge s - rs).$$

With the above notation we state the strong coupling result we use.

Theorem 3.3. *Let $p \in (0, 1)$. There exist constants $0 < C, a, \alpha < \infty$ (depending on p) such that for every positive integer n, there is a probability space on which are defined a Brownian bridge B^σ with variance $\sigma^2 = p(1 - p)$ and a family of random paths $\ell^{(n,z)} \in \Omega(0, n, 0, z)$ for $z = 0, \ldots, n$ such that $\ell^{(n,z)}$ has law $\mathbb{P}_{Ber}^{0,n,0,z}$ and*

$$(3.2) \quad \mathbb{E}\left[e^{a\Delta(n,z)}\right] \le C e^{\alpha(\log n)^2} e^{|z - pn|^2/n}, \quad \text{where } \Delta(n, z) := \sup\nolimits_{0 \le t \le n}\left|\sqrt{n}B_{t/n}^\sigma + \tfrac{t}{n}z - \ell^{(n,z)}(t)\right|.$$

Remark 3.4. When $p = 1/2$ the above theorem follows (after a trivial affine shift) from [11, Theorem 6.3] and the general $p \in (0, 1)$ case was done in [3, Theorem 4.5]. We mention that a significant generalization of Theorem 3.3 for general random walk bridges has recently been proved in [7, Theorem 2.3].

We will use the following simple corollary of Theorem 3.3 in the following to compare Bernoulli bridges with Brownian bridges. We use the same notation as in the theorem.

Corollary 3.5. *Fix $p \in (0, 1)$, $\beta > 0$, and $A > 0$. Suppose $|z - pn| \le K\sqrt{n}$ for a constant $K > 0$. Then for any $\epsilon > 0$, there exists N large enough depending on p, ϵ, A, K so that for $n \ge N$,*

$$\mathbb{P}\left(\Delta(n, z) \ge An^\beta\right) < \epsilon.$$

Proof. Applying Chebyshev's inequality and (3.2) gives

$$\mathbb{P}\left(\Delta(n, z) \ge An^\beta\right) \le e^{-An^\beta}\mathbb{E}\left[e^{a\Delta(n,z)}\right] \le C\exp\left[-An^\beta + \alpha(\log n)^2 + \frac{|z - pn|^2}{n}\right]$$

$$\le C\exp\left[-An^\beta + \alpha(\log n)^2 + K\right].$$

The conclusion is now immediate. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We also state the following result regarding the distribution of the maximum of a Brownian bridge, which follows from formulas in [8, Chapter 12].

Lemma 3.6. *Fix $p \in (0, 1)$, and let B^σ be a Brownian bridge of variance $\sigma^2 = p(1 - p)$ on $[0, 1]$. Then for any $C, T > 0$ we have*

$$\mathbb{P}\left(\max_{s \in [0,T]} B_{s/T}^\sigma \ge C\right) = \exp\left(-\frac{2C^2}{p(1-p)}\right) \quad \text{and} \quad \mathbb{P}\left(\max_{s \in [0,T]} \left|B_{s/T}^\sigma\right| \ge C\right) = 2\sum_{n=1}^\infty \exp\left(-\frac{2n^2C^2}{p(1-p)}\right).$$

In particular,

$$(3.3) \qquad\qquad \mathbb{P}\left(\max_{s \in [0,T]} \left|B_{s/T}^\sigma\right| \ge C\right) \le 2\exp\left(-\frac{2C^2}{p(1-p)}\right).$$

Proof. Let B^1 be a Brownian bridge with variance 1 on $[0, 1]$. Then B_t^σ has the same distribution as σB_t^1. Hence

$$\mathbb{P}\left(\max_{s \in [0,T]} B_{s/T}^\sigma \ge C\right) = \mathbb{P}\left(\max_{t \in [0,1]} B_t^1 \ge C/\sigma\right) = e^{-2(C/\sigma)^2} = e^{-2C^2/p(1-p)}.$$

The second equality follows from [8, Proposition 12.3.3]. The other equality follows in the same way from [8, Proposition 12.3.4].

Lastly, to prove (3.3), observe that since B_t^σ has mean 0, B_t^σ and $-B_t^\sigma$ have the same distribution. It follows from the first equality above that

$$\mathbb{P}\left(\max_{s\in[0,T]} \left|B_{s/T}^\sigma\right| \geq C\right) \leq \mathbb{P}\left(\max_{s\in[0,T]} B_{s/T}^\sigma \geq C\right) + \mathbb{P}\left(\max_{s\in[0,T]} \left(-B_{s/T}^\sigma\right) \geq C\right) =$$

$$2\,\mathbb{P}\left(\max_{s\in[0,T]} B_{s/T}^\sigma \geq C\right) = 2e^{-2C^2/p(1-p)}.$$

$\square$

We state one more lemma about Brownian bridges, which allows us to decompose a bridge on $[0,1]$ into two independent bridges with Gaussian affine shifts meeting at a point in $(0,1)$.

Lemma 3.7. *Fix $p \in (0,1)$, $T > 0$, $t \in (0,T)$, and let B^σ be a Brownian bridge of variance $\sigma^2 = p(1-p)$ on $[0,1]$. Let ξ be a Gaussian random variable with mean 0 and variance*

$$\mathbb{E}[\xi^2] = \sigma^2 \frac{t}{T}\left(1 - \frac{t}{T}\right).$$

Let B^1, B^2 be two independent Brownian bridges on $[0,1]$ with variances $\sigma^2 t/T$ and $\sigma^2(T-t)/T$ respectively, also independent from B^σ. Define the process

$$\tilde{B}_{s/T} = \begin{cases} \dfrac{s}{t}\xi + B^1\left(\dfrac{s}{t}\right), & s \leq t, \\[3mm] \dfrac{T-s}{T-t}\xi + B^2\left(\dfrac{s-t}{T-t}\right), & s \geq t, \end{cases}$$

for $s \in [0,T]$. Then $\tilde{B}$ is a Brownian bridge with variance σ.

Proof. It is clear that the process $\tilde{B}$ is a.s. continuous. Since $\tilde{B}$ is built from three independent zero-centered Gaussian processes, it is itself a zero-centered Gaussian process and thus completely characterized by its covariance. Consequently, to show that $\tilde{B}$ is a Brownian bridge of variance σ^2, it suffices to show by (3.1) that if $0 \leq r \leq s \leq T$ we have

(3.4)
$$\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \sigma^2 \frac{r}{T}\left(1 - \frac{s}{T}\right).$$

First assume $s \leq t$. Using the fact that ξ and B^1 are independent with mean 0, we find

$$\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \frac{rs}{t^2}\cdot\sigma^2\frac{t}{T}\left(1 - \frac{t}{T}\right) + \sigma^2\frac{t}{T}\cdot\frac{r}{t}\left(1 - \frac{s}{t}\right)$$

$$= \sigma^2\frac{r}{T}\left(\frac{s}{t} - \frac{s}{T} + 1 - \frac{s}{t}\right) = \sigma^2\frac{r}{T}\left(1 - \frac{s}{T}\right).$$

If $r \geq t$, we compute

$$\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \frac{(T-r)(T-s)}{(T-t)^2}\cdot\sigma^2\frac{t}{T}\left(1 - \frac{t}{T}\right) + \sigma^2\frac{T-t}{T}\cdot\frac{r-t}{T-t}\left(1 - \frac{s-t}{T-t}\right)$$

$$= \frac{\sigma^2(T-s)}{T(T-t)}\left(\frac{t(T-r)}{T} + r - t\right) = \frac{\sigma^2(T-s)}{T(T-t)}\cdot\frac{r(T-t)}{T} = \sigma^2\frac{r}{T}\left(1 - \frac{s}{T}\right).$$

If $r < t < s$, then since ξ, B^1, and B^2 are all independent, we have

$$\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \frac{r}{t}\cdot\frac{T-s}{T-t}\cdot\sigma^2\frac{t(T-t)}{T^2} = \sigma^2\frac{r(T-s)}{T^2} = \sigma^2\frac{r}{T}\left(1 - \frac{s}{T}\right).$$

This proves (3.4) in all cases. $\square$

Below we list six lemmas about Bernoulli bridges. We provide a brief informal explanation of what each result says after it is stated. All six lemmas are proved in a similar fashion. For the first four lemmas one observes that the event, whose probability is being estimated, is monotone in ℓ. This allows by Lemmas 3.1 and 3.2 to replace x, y in the statements of the lemmas with the extreme values of the ranges specified in each. Once the choice of x and y is fixed one can use our strong coupling results, Theorem 3.3 and Corollary 3.5, to reduce each of the lemmas to an analogous one involving a Brownian bridge with some prescribed variance. The latter statements are then easily confirmed as one has exact formulas for Brownian bridges, such as Lemma 3.6.

Lemma 3.8. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}_{Ber}^{0,T,x,y}$. Let $M_1, M_2 \in \mathbb{R}$ be given. Then we can find $W_0 = W_0(p, M_2 - M_1) \in \mathbb{N}$ such that for $T \geq W_0$, $x \geq M_1 T^{1/2}$, $y \geq pT + M_2 T^{1/2}$ and $s \in [0,T]$ we have*

$$(3.5) \qquad \mathbb{P}_{Ber}^{0,T,x,y}\left(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right) \geq \frac{1}{3}.$$

Remark 3.9. If $M_1, M_2 = 0$ then Lemma 3.8 states that if a Bernoulli bridge ℓ is started from $(0, x)$ and terminates at (T, y), which are above the straight line of slope p, then at any given time $s \in [0, T]$ the probability that $\ell(s)$ goes a modest distance below the straight line of slope p is upper bounded by $2/3$.

Proof. Define $A = \lfloor M_1 T^{1/2} \rfloor$ and $B = \lfloor pT + M_2 T^{1/2} \rfloor$. Then since $A \leq x$ and $B \leq y$, it follows from Lemma 3.1 that there is a probability space with measure $\mathbb{P}_0$ supporting random variables L_1 and L_2, whose laws under $\mathbb{P}_0$ are $\mathbb{P}_{Ber}^{0,T,A,B}$ and $\mathbb{P}_{Ber}^{0,T,x,y}$ respectively, and $\mathbb{P}_0$-a.s. we have $L_1 \leq L_2$. Thus

$$\mathbb{P}_{Ber}^{0,T,x,y}\left(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right)$$
$$= \mathbb{P}_0\left(L_2(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right)$$
$$\geq \mathbb{P}_0\left(L_1(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right)$$
$$= \mathbb{P}_{Ber}^{0,T,A,B}\left(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right).$$

Since upright paths on $[\![0, T]\!] \times [\![A, B]\!]$ are equivalent to upright paths on $[\![0, T]\!] \times [\![0, B - A]\!]$ shifted vertically by A, the last line is equal to

$$\mathbb{P}_{Ber}^{0,T,0,B-A}\left(\ell(s) + A \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right).$$

Now we consider the coupling provided by Theorem 3.3. We have another probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random variable $\ell^{(T,B-A)}$ whose law under $\mathbb{P}$ is $\mathbb{P}_{Ber}^{0,T,0,B-A}$, and a Brownian bridge B^σ. Then

$$\mathbb{P}_{Ber}^{0,T,0,B-A}\left(\ell(s) + A \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right)$$
$$= \mathbb{P}\left(\ell^{(T,B-A)}(s) + A \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right)$$
$$= \mathbb{P}\left(\left[\ell^{(T,B-A)}(s) - \sqrt{T} B_{s/T}^\sigma - \frac{s}{T} \cdot (B - A)\right] + \sqrt{T} B_{s/T}^\sigma \geq -A - \frac{s}{T} \cdot (B - A)\right.$$
$$\left. + \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right).$$

Recalling the definitions of A and B, we can rewrite the quantity on the right hand side in the last expression and bound it by

$$\frac{T-s}{T} \cdot (M_1 T^{1/2} - A) + \frac{s}{T} \cdot (pT + M_2 T^{1/2} - B) - T^{1/4} \leq \frac{T-s}{T} + \frac{s}{T} - T^{1/4}$$
$$= -T^{1/4} + 1.$$

Thus

$$\mathbb{P}_{Ber}^{0,T,0,B-A}\left(\ell(s) + A \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right)$$
$$\geq \mathbb{P}\left(\left[\ell^{(T,B-A)}(s) - \sqrt{T}B_{s/T}^\sigma - \frac{s}{T} \cdot (B-A)\right] + \sqrt{T}B_{s/T}^\sigma \geq -T^{1/4} + 1\right)$$
$$\geq \mathbb{P}\left(\sqrt{T}B_{s/T}^\sigma \geq 0 \quad \text{and} \quad \Delta(T, B-A) < T^{1/4} - 1\right)$$
$$\geq \mathbb{P}\left(B_{s/T}^\sigma \geq 0\right) - \mathbb{P}\left(\Delta(T, B-A) \geq T^{1/4} - 1\right)$$
$$= \frac{1}{2} - \mathbb{P}\left(\Delta(T, B-A) \geq T^{1/4} - 1\right).$$

For the second inequality, we used the fact that the quantity in brackets is bounded in absolute value by $\Delta(T, B-A)$. The third inequality follows by dividing the event $\{B_{s/T}^\sigma \geq 0\}$ into cases and applying subadditivity. Since $|B - A - pT| \leq (M_2 - M_1 + 1)\sqrt{T}$, Corollary 3.5 allows us to choose W_0 large enough depending on p and $M_2 - M_1$ so that if $T \geq W_0$, then the last line is bounded above by $1/2 - 1/6 = 1/3$. This proves (3.5). $\qquad\square$

Lemma 3.10. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $y, z \in \mathbb{Z}$ such that $T \geq y, z \geq 0$, and suppose that ℓ_y, ℓ_z have distributions $\mathbb{P}_{Ber}^{0,T,0,y}$, $\mathbb{P}_{Ber}^{0,T,0,z}$ respectively. Let $M > 0$ and $\epsilon > 0$ be given. Then we can find $W_1 = W_1(M, p, \epsilon) \in \mathbb{N}$ and $A = A(M, p, \epsilon) > 0$ such that for $T \geq W_1$, $y \geq pT - MT^{1/2}$, $z \leq pT + MT^{1/2}$ we have*

(3.6)
$$\mathbb{P}_{Ber}^{0,T,0,y}\left(\inf_{s \in [0,T]} \left(\ell(s) - ps\right) \leq -AT^{1/2}\right) \leq \epsilon,$$
$$\mathbb{P}_{Ber}^{0,T,0,z}\left(\sup_{s \in [0,T]} \left(\ell(s) - ps\right) \geq AT^{1/2}\right) \leq \epsilon.$$

Remark 3.11. Roughly, Lemma 3.10 states that if a Bernoulli bridge ℓ is started from $(0,0)$ and terminates at time T not significantly lower (resp. higher) than the straight line of slope p, then the event that ℓ goes significantly below (resp. above) the straight line of slope p is very unlikely.

Proof. The two inequalities are proven in essentially the same way. We begin with the first inequality. It follows from Lemma 3.1 that

$$\mathbb{P}_{Ber}^{0,T,0,y}\left(\inf_{s \in [0,T]} \left(\ell(s) - ps\right) \leq -AT^{1/2}\right) \leq \mathbb{P}_{Ber}^{0,T,0,B}\left(\inf_{s \in [0,T]} \left(\ell(s) - ps\right) \leq -AT^{1/2}\right),$$

where $B = \lfloor pT - MT^{1/2} \rfloor$. By Theorem 3.3, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random variable $\ell^{(T,B)}$ whose law under $\mathbb{P}$ is that of ℓ, and a Brownian bridge B^σ with variance $\sigma^2 = p(1-p)$. Therefore

$$\mathbb{P}_{Ber}^{0,T,0,B}\left(\inf_{s \in [0,T]} \left(\ell(s) - ps\right) \leq -AT^{1/2}\right) = \mathbb{P}\left(\inf_{s \in [0,T]} \left(\ell^{(T,B)}(s) - ps\right) \leq -AT^{1/2}\right)$$
$$\leq \mathbb{P}\left(\inf_{s \in [0,T]} \sqrt{T}B_{s/T}^\sigma \leq -\frac{1}{2}AT^{1/2}\right) + \mathbb{P}\left(\sup_{s \in [0,T]} \left|\sqrt{T}B_{s/T}^\sigma + ps - \ell^{(T,B)}(s)\right| \geq \frac{1}{2}AT^{1/2}\right)$$

(3.7)
$$\leq \mathbb{P}\left(\max_{s \in [0,T]} B_{s/T}^\sigma \geq A/2\right) + \mathbb{P}\left(\Delta(T, B) \geq \frac{1}{2}AT^{1/2} - MT^{1/2} - 1\right).$$

For the first term in the last line, we used the fact that B^σ and $-B^\sigma$ have the same distribution. For the second term, we used the fact that

$$\sup_{s\in[0,T]} \left| ps - \frac{s}{T} \cdot B \right| \leq \sup_{s\in[0,T]} \left| ps - \frac{pT - MT^{1/2}}{T} \cdot s \right| + 1 = MT^{1/2} + 1.$$

By Lemma 3.6, the first term in (3.7) is equal to $e^{-A^2/2p(1-p)}$. If we choose $A \geq \sqrt{2p(1-p)\log(2/\epsilon)}$, then this is $\leq \epsilon/2$. If we also take $A > 2M$, then since $|B - pT| \leq (M+1)\sqrt{T}$, Corollary 3.5 gives us a W_1 large enough depending on M, p, ϵ so that the second term in (3.7) is also $< \epsilon/2$ for $T \geq W_1$. Adding the two terms gives the first inequality in (3.6).

If we replace B with $\lceil pT + MT^{1/2} \rceil$ and change signs and inequalities where appropriate, then the same argument proves the second inequality in (3.6). $\qquad\square$

Lemma 3.12. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}_{Ber}^{0,T,x,y}$. Let $M_1, M_2 > 0$ be given. Then we can find $W_2 = W_2(M_1, M_2, p) \in \mathbb{N}$ such that for $T \geq W_2$, $x \geq -M_1 T^{1/2}$, $y \geq pT - M_1 T^{1/2}$ and $t \in (0,1)$ we have*

$$(3.8) \qquad \mathbb{P}_{Ber}^{0,T,x,y}\left(\ell(tT) \geq t(M_2 T^{1/2} + pT) - T^{1/4} \right) \geq (1/2)(1 - \Phi^v(M_1 + M_2)),$$

where Φ^v is the cumulative distribution function of a Gaussian random variable with mean 0 and variance $v = pt(1-p)(1-t)$.

Remark 3.13. Lemma 3.12 states that if a Bernoulli bridge ℓ is started from $(0, x)$ and terminates at (T, y) with these points not significantly lower than the straight line of slope p, then its mid-point would lie well above the straight line of slope p at least with some quantifiably tiny probability.

Proof. By Lemma 3.1, we have

$$\mathbb{P}_{Ber}^{0,T,x,y}\left(\ell(tT) \geq t(M_2 T^{1/2} + pT) - T^{1/4} \right) \geq \mathbb{P}_{Ber}^{0,T,0,B-A}\left(\ell(tT) + A \geq t(M_2 T^{1/2} + pT) - T^{1/4} \right)$$

$$= \mathbb{P}\left(\ell^{(T,B-A)}(tT) + A \geq t(M_2 T^{1/2} + pT) - T^{1/4} \right),$$

with $A = \lfloor -M_1 T^{1/2} \rfloor$, $B = \lfloor pT - M_1 T^{1/2} \rfloor$, and $\mathbb{P}$, and $\ell^{(T,B-A)}$ provided by Theorem 3.3. If B^σ is as in the theorem, we can rewrite the expression on the second line as

$$\mathbb{P}\left([\ell^{(T,B-A)}(tT) - \sqrt{T}\, B_t^\sigma - t(B-A)] + \sqrt{T}\, B_t^\sigma \geq -A - t(B-A) + t(M_2 T^{1/2} + pT) - T^{1/4} \right).$$

We have

$$-A - t(B-A) + t(M_2 T^{1/2} + pT) - T^{1/4} \leq M_1 T^{1/2} + 1 - t(pT - 1) + t(M_2 T^{1/2} + pT) - T^{1/4}$$

$$\leq (M_1 + M_2)T^{1/2} - T^{1/4} + 2.$$

Thus the probability in question is bounded below by

$$\mathbb{P}\left([\ell^{(T,B-A)}(tT) - \sqrt{T}\, B_t^\sigma - t(B-A)] + \sqrt{T}\, B_t^\sigma \geq (M_1 + M_2)T^{1/2} - T^{1/4} + 2 \right)$$

$$\geq \mathbb{P}\left(\sqrt{T}\, B_t^\sigma \geq (M_1 + M_2)T^{1/2} \quad \text{and} \quad \Delta(T, B-A) < T^{1/4} - 2 \right)$$

$$\geq \mathbb{P}\left(B_t^\sigma \geq M_1 + M_2 \right) - \mathbb{P}\left(\Delta(T, B-A) \geq T^{1/4} - 2 \right).$$

Note that $B_t^\sigma = \sigma(W_t - tW_1)$ for a standard Brownian motion W on $[0,1]$. Thus B_t^σ is Gaussian with mean 0 and variance $\sigma^2(t - t^2) = p(1-p)t(1-t) = v$. In particular, the first term in the last line is equal to

$$1 - \Phi^v(M_1 + M_2),$$

where Φ^v is the cdf for a Gaussian random variable with mean 0 and variance v. By Corollary 3.5, since $|B - A - pT| \leq 1$, we can choose W_2 depending on M_1, M_2, and p so that the second term is less than $1/2$ the first term for $T \geq W_2$. This proves (3.8). $\qquad\square$

Lemma 3.14. *Fix* $p \in (0,1)$, $T \in \mathbb{N}$ *and* $x, y \in \mathbb{Z}$ *such that* $T \geq y - x \geq 0$, *and suppose that* ℓ *has distribution* $\mathbb{P}_{Ber}^{0,T,x,y}$. *Then we can find* $W_3 = W_3(p) \in \mathbb{N}$ *such that for* $T \geq W_3$, $x \geq T^{1/2}$, $y \geq pT + T^{1/2}$

$$(3.9) \qquad \mathbb{P}_{Ber}^{0,T,x,y}\left(\inf_{s \in [0,T]} \left(\ell(s) - ps \right) + T^{1/4} \geq 0 \right) \geq \frac{1}{2}\left(1 - \exp\left(-\frac{2}{p(1-p)} \right) \right).$$

Remark 3.15. Lemma 3.14 states that if a Bernoulli bridge ℓ is started from $(0,x)$ and terminates at (T,y) with $(0,x)$ and (T,y) well above the line of slope p then at least with some positive probability ℓ will not fall significantly below the line of slope p.

Proof. By Lemma 3.1,

$$\mathbb{P}_{Ber}^{0,T,x,y}\left(\inf_{s \in [0,T]} \left(\ell(s) - ps \right) + T^{1/4} \geq 0 \right)$$

$$\geq \mathbb{P}_{Ber}^{0,T,0,B-A}\left(\inf_{s \in [0,T]} \left(\ell(s) + A - ps \right) + T^{1/4} \geq 0 \right)$$

$$= \mathbb{P}\left(\inf_{s \in [0,T]} \left(\ell^{(T,B-A)}(s) - ps \right) \geq -T^{1/4} - A \right)$$

$$\geq \mathbb{P}\left(\inf_{s \in [0,T]} \left(\ell^{(T,B-A)}(s) - \frac{s}{T} \cdot (B - A) \right) \geq -T^{1/4} - T^{1/2} + 2 \right),$$

with $A = \lfloor T^{1/2} \rfloor$, $B = \lfloor pT + T^{1/2} \rfloor$, and $\mathbb{P}$, and $\ell^{(T,B-A)}$ as in Theorem 3.3. In the last line, we used the facts that $|A - T^{1/2}| \leq 1$ and $|p - (B-A)/T| \leq 1$. With B^σ as in the theorem, the last line is bounded below by

$$\mathbb{P}\left(\inf_{s \in [0,T]} \sqrt{T} B_{s/T}^\sigma \geq -T^{1/2} \quad \text{and} \quad \Delta(T, B-A) < T^{1/2} - 2 \right)$$

$$\geq 1 - \exp\left(-\frac{2}{p(1-p)} \right) - \mathbb{P}\left(\Delta(T, B-A) \geq T^{1/2} - 2 \right).$$

In the second line, we used Lemma 3.6. Since $|B - A - pT| \leq 1$, Corollary 3.5 allows us choose W_3 large enough depending on p so that this term is less than $\frac{1}{2}(1 - e^{-2/p(1-p)})$ for $T \geq W_3$. This implies (3.9). $\qquad\square$

We need the following definition for our next result. For a function $f \in C([a,b])$ we define its *modulus of continuity* by

$$(3.10) \qquad w(f, \delta) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|.$$

Lemma 3.16. *Fix* $p \in (0,1)$, $T \in \mathbb{N}$ *and* $y \in \mathbb{Z}$ *such that* $T \geq y \geq 0$, *and suppose that* ℓ *has distribution* $\mathbb{P}_{Ber}^{0,T,0,y}$. *For each positive* M, ϵ *and* η, *there exist a* $\delta(\epsilon, \eta, M) > 0$ *and* $W_4 = W_4(M, p, \epsilon, \eta) \in \mathbb{N}$ *such that for* $T \geq W_4$ *and* $|y - pT| \leq MT^{1/2}$ *we have*

$$(3.11) \qquad \mathbb{P}_{Ber}^{0,T,0,y}\left(w\left(f^\ell, \delta \right) \geq \epsilon \right) \leq \eta,$$

where $f^\ell(u) = T^{-1/2}\left(\ell(uT) - puT \right)$ *for* $u \in [0,1]$.

Remark 3.17. Lemma 3.16 states that if ℓ is a Bernoulli bridge that is started from $(0,0)$ and terminates at (T,y) with y close to pT (i.e. with well-behaved endpoints) then the modulus of continuity of ℓ is also well-behaved with high probability.

Proof. We have

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(w\big(f^\ell,\delta\big)\geq\epsilon\Big)=\mathbb{P}\Big(w\big(f^{\ell^{(T,y)}},\delta\big)\geq\epsilon\Big),$$

with $\mathbb{P}$, $\ell^{(T,y)}$ as in Theorem 3.3. If B^σ is the Brownian bridge provided by Theorem 3.3, then

$$\begin{aligned}
w\big(f^{\ell^{(T,y)}},\delta\big) &= T^{-1/2}\sup_{\substack{s,t\in[0,1]\\|s-t|\leq\delta}}\left|\ell^{(T,y)}(sT)-psT-\ell^{(T,y)}(tT)+ptT\right|\\
&\leq T^{-1/2}\sup_{\substack{s,t\in[0,1]\\|s-t|\leq\delta}}\Big(\left|\sqrt{T}\,B^\sigma_s+sy-psT-\sqrt{T}\,B^\sigma_t-ty+ptT\right|\\
&\qquad\qquad +\left|\sqrt{T}\,B^\sigma_s+sy-\ell^{(T,y)}(sT)\right|+\left|\sqrt{T}\,B^\sigma_t+ty-\ell^{(T,y)}(tT)\right|\Big)\\
&\leq \sup_{\substack{s,t\in[0,1]\\|s-t|\leq\delta}}\left|B^\sigma_s-B^\sigma_t+T^{-1/2}(y-pT)(s-t)\right|+2T^{-1/2}\Delta(T,y)\\
&\leq w\big(B^\sigma,\delta\big)+M\delta+2T^{-1/2}\Delta(T,y).
\end{aligned}$$

The last line follows from the assumption that $|y-pT|\leq MT^{1/2}$. Thus

$$
\begin{aligned}
(3.12)\qquad \mathbb{P}\Big(w\big(f^{\ell^{(N,y)}},\delta\big)\geq\epsilon\Big) &\leq \mathbb{P}\Big(w\big(B^\sigma,\delta\big)+M\delta+2T^{-1/2}\Delta(T,y)\geq\epsilon\Big)\\
&\leq \mathbb{P}\Big(w\big(B^\sigma,\delta\big)+M\delta\geq\epsilon/2\Big)+\mathbb{P}\Big(\Delta(T,y)\geq\epsilon\,T^{1/2}/4\Big).
\end{aligned}
$$

Corollary 3.5 gives us a W_4 large enough depending on M,p,ϵ,η so that the last expression in equation 3.12 is $\leq\eta/2$ for $T\geq W_4$. Since B^σ is a.s. uniformly continuous on the compact interval $[0,1]$, $w(B^\sigma,\delta)\to0$ as $\delta\to0$. Thus we can find $\delta_0>0$ small enough depending on ϵ,η so that $w(B^\sigma,\delta_0)<\epsilon/4$ with probability at least $1-\eta/2$. Then with $\delta=\min(\delta_0,\epsilon/4M)$, the first term is $\leq\eta/2$ as well. This implies (3.11). □

Lemma 3.18. *Fix $T\in\mathbb{N}$, $p\in(0,1)$, $C,K>0$, and $a,b\in\mathbb{Z}$ such that $\Omega(0,T,a,b)$ is nonempty. Let $\ell_{bot}\in\Omega(0,T,a,b)$. Suppose $\vec{x},\vec{y}\in\mathfrak{W}_{k-1}$, $k\geq2$, are such that $T\geq y_i-x_i\geq0$ for $1\leq i\leq k-1$. Write $\vec{z}=\vec{y}-\vec{x}$, and suppose that*

(1) $x_{k-1}+(z_{k-1}/T)s-\ell_{bot}(s)\geq C\sqrt{T}$ for all $s\in[0,T]$
(2) $x_i-x_{i+1}\geq C\sqrt{T}$ and $y_i-y_{i+1}\geq C\sqrt{T}$ for $1\leq i\leq k-2$,
(3) $|z_i-pT|\leq K\sqrt{T}$ for $1\leq i\leq k-1$, for a constant $K>0$.

Let $\mathfrak{L}=(L_1,\dots,L_{k-1})$ be a line ensemble with law $\mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}$, and let E denote the event

$$E=\{L_1(s)\geq\cdots\geq L_{k-1}(s)\geq\ell_{bot}(s),\text{for }s\in[0,T]\}.$$

Then we can find $W_5=W_5(p,C,K)$ so that for $T\geq W_5$,

$$(3.13)\qquad \mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}(E)\geq\left(1-3\sum_{n=1}^\infty e^{-n^2C^2/8p(1-p)}\right)^{k-1}.$$

In particular, if $C\geq\sqrt{8p(1-p)\log 3}$, then we have

$$(3.14)\qquad \mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}(E)\geq\left(1-3e^{-C^2/8p(1-p)}\right)^{k-1}.$$

Remark 3.19. This lemma states that if k independent Bernoulli bridges are well-separated from each other and ℓ_{bot}, then there is a positive probability that the curves will intersect neither each other nor ℓ_{bot}. We will use this result to compare curves in an avoiding Bernoulli line ensemble with free Bernoulli bridges.

Proof. Observe that condition (1) simply states that ℓ_{bot} lies a distance of at least $C\sqrt{T}$ uniformly below the line segment connecting x_{k-1} and y_{k-1}. Thus (1) and (2) imply that E occurs if each curve L_i remains within a distance of $C\sqrt{T}/2$ from the line segment connecting x_i and y_i. As in Theorem 3.3, let $\mathbb{P}_i$ be probability measures supporting random variables $\ell^{(T,z_i)}$ with laws $\mathbb{P}_{Ber}^{0,T,0,z_i}$. Then

$$\mathbb{P}_{Ber}^{0,T,\vec{x},\vec{y}}(E) \geq \mathbb{P}_{Ber}^{0,T,\vec{x},\vec{y}}\Big(\sup_{s\in[0,T]} |L_i(s) - x_i - (z_i/T)s| \leq C\sqrt{T}/2, \ 1 \leq i \leq k-1 \Big) =$$

(3.15)
$$\prod_{i=1}^{k-1} \Big[\mathbb{P}_{Ber}^{0,T,0,z_i}\Big(\sup_{s\in[0,T]} |L_i(s + rN^\alpha) - (z_i/T)s| \leq C\sqrt{T}/2 \Big) \Big] =$$

$$\prod_{i=1}^{k-1} \Big[1 - \mathbb{P}_i\Big(\sup_{s\in[0,T]} |\ell^{(T,z_i)} - (z_i/T)s| > C\sqrt{T}/2 \Big) \Big].$$

In the third line, we used the fact that $L_1, \ldots, L_{k-1}$ are independent from each other under $\mathbb{P}_{Ber}^{0,T,0,z_i}$. Let $B^{\sigma,i}$ be the Brownian bridge with variance $\sigma^2 = p(1-p)$ coupled with $\ell^{(T,z_i)}$ given by Theorem 3.3. Then we have

(3.16)
$$\mathbb{P}_i\Big(\sup_{s\in[0,T]} |\ell^{(T,z_i)}(s) - (z_i/T)s| > C\sqrt{T}/2 \Big) \leq$$
$$\mathbb{P}_i\Big(\sup_{s\in[0,T]} |\sqrt{T}B_{s/T}^\sigma| > C\sqrt{T}/4 \Big) + \mathbb{P}_i\Big(\Delta(T,z_i) > C\sqrt{T}/4 \Big).$$

By Lemma 3.6, the first term in the second line of (3.16) is equal to $2\sum_{n=1}^\infty e^{-n^2C^2/8p(1-p)}$. For the second term, condition (3) in the hypothesis and Corollary 3.5 give an upper bound of $\sum_{n=1}^\infty e^{-n^2C^2/8p(1-p)}$ for sufficiently large N depending on p, C, K, but independent of i. Adding these two terms and referring to (3.15) proves (3.13).

Now suppose $C \geq \sqrt{8p(1-p)\log 3}$. By (3.3) in Lemma 3.6, the first term in the second line of (3.16) is bounded above by bounded above by $2e^{-C^2/8p(1-p)}$. For the same choice of N as above, the second term is $< e^{-C^2/8p(1-p)}$. The assumption on C implies that $1 - 3e^{-C^2/8p(1-p)} \geq 0$, and now combining (3.16) and (3.15) proves (3.14). $\qquad\square$

3.3. **Properties of avoiding Bernoulli line ensembles.** In this section we derive several results about avoiding Bernoulli line ensembles, which are Bernoulli line ensembles with law $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ as in Definition 2.15. The lemmas we prove only involve the case when $f(r) = \infty$ for all $r \in [\![T_0, T_1]\!]$ and we denote the measure in this case by $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g}$. A $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g}$-distributed random variable will be denoted by $\mathfrak{Q} = (Q_1, \ldots, Q_k)$ where k is the number of up-right paths in the ensemble. As usual, if $g = -\infty$, we write $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y}}$. Our first result will rely on the two monotonicity Lemmas 3.1 and 3.2 as well as the strong coupling between Bernoulli bridges and Brownian bridges from Theorem 3.3, and the further results make use of the material in Section 9.

Lemma 3.20. *Fix $p \in (0,1)$, $k \in \mathbb{N}$. Let $\vec{x}, \vec{y} \in \mathfrak{W}_k$ be such that $T \geq y_i - x_i \geq 0$ for $i = 1, \ldots, k$. Then for any $M, M_1 > 0$ we can find $W_6 \in \mathbb{N}$ depending on p, k, M, M_1 such that if $T \geq W_6$, $x_k \geq -M_1\sqrt{T}$, and $y_k \geq pT - M_1\sqrt{T}$, then for any $S \subseteq [\![0,T]\!]$ we have*
(3.17)
$$\mathbb{P}_{avoid,Ber;S}^{0,T,\vec{x},\vec{y}}\Big(Q_k(T/2) - pT/2 \geq M\sqrt{T} \Big) \geq \frac{2^{k/2}\big(1 - 2e^{-4/p(1-p)}\big)^{2k}}{(\pi p(1-p))^{k/2}} \exp\Big(-\frac{2k(M + M_1 + 6)^2}{p(1-p)} \Big).$$

Proof. Define vectors $\vec{x}, \vec{y} \in \mathfrak{W}_k$ by

$$x'_i = \lfloor -M_1 \sqrt{T} \rfloor - 10(i-1)\lceil \sqrt{T} \rceil,$$
$$y'_i = \lfloor pT - M_1 \sqrt{T} \rfloor - 10(i-1)\lceil \sqrt{T} \rceil.$$

Then $x'_i \leq x_k \leq x_i$ and $y'_i \leq y_k \leq y_i$ for $1 \leq i \leq k-1$. Thus by Lemma 3.1, we have

$$\mathbb{P}^{0,T,\vec{x},\vec{y}}_{avoid,Ber;S}\left(Q_k(T/2) - pT/2 \geq M\sqrt{T}\right) \geq \mathbb{P}^{0,T,\vec{x}',\vec{y}'}_{avoid,Ber;S}\left(Q_k(T/2) - pT/2 \geq M\sqrt{T}\right).$$

Let us write $K_i = pT/2 + M\sqrt{T} + (10(k-i)-5)\lceil \sqrt{T} \rceil$ for $1 \leq i \leq k$. Note K_i is the midpoint of $pT/2 + M\sqrt{T} + 10(k-i-1)\lceil \sqrt{T} \rceil$ and $pT/2 + M\sqrt{T} + 10(k-i)\lceil \sqrt{T} \rceil$. Let E denote the event that the following conditions hold for $1 \leq i \leq k$:

(1) $\left| Q_i(T/2) - pT/2 - M\sqrt{T} - (10(k-i)-5)\lceil \sqrt{T} \rceil \right| \leq 2\lceil \sqrt{T} \rceil,$

(2) $\sup_{s \in [0,T/2]} \left| Q_i(s) - x'_i - \dfrac{K_i - x'_i}{T/2}s \right| \leq 3\sqrt{T},$

(3) $\sup_{s \in [T/2,T]} \left| Q_i(s) - K_i - \dfrac{y'_i - K_i}{T/2}(s - T/2) \right| \leq 3\sqrt{T}.$

The first condition requires in particular that $Q_k(T/2) - pT/2 \geq M\sqrt{T}$, and also that $Q_i(T/2) - Q_{i+1}(T/2) \geq 6\sqrt{T}$ for each i. The second and third conditions require that each curve Q_i remain within a distance of $3\sqrt{T}$ of the graph of the piecewise linear function on $[0,T]$ passing through the points $(0, x'_1)$, $(T/2, K_i)$, and (T, y'_i). We observe that

$$\mathbb{P}^{0,T,\vec{x}',\vec{y}'}_{avoid,Ber;S}\left(Q_k(T/2) - ptT \geq M\sqrt{T}\right) \geq \mathbb{P}^{0,T,\vec{x}',\vec{y}'}_{avoid,Ber;S}(E) \geq \mathbb{P}^{0,T,\vec{x}',\vec{y}'}_{Ber}(E).$$

The second inequality follows since the event E implies that $Q_1(s) \geq \cdots \geq Q_k(s)$ for all $s \in [\![0,T]\!]$. Write $z = y'_k - x'_k$. Then we have

(3.18)
$$\mathbb{P}^{0,T,\vec{x}',\vec{y}'}_{Ber}(E) = \left[\mathbb{P}^{0,T,0,z}_{Ber}\left(\left| \ell(T/2) - pT/2 - M\sqrt{T} - 5\lceil \sqrt{T} \rceil + x'_1 \right| \leq 2\lceil \sqrt{T} \rceil \quad \text{and} \right.\right.$$
$$\sup_{s \in [0,T/2]} \left| \ell(s) - \frac{K_1 - x'_1}{T/2}s \right| \leq 3\sqrt{T} \quad \text{and}$$
$$\left.\left. \sup_{s \in [T/2,T]} \left| \ell(s) - (K_1 - x'_1) - \frac{y'_1 - K_1}{T/2}(s - T/2) \right| \leq 3\sqrt{T} \right) \right]^k.$$

Let $\mathbb{P}$ be a probability space suporting a random variable $\ell^{(T,z)}$ with law $\mathbb{P}^{0,T,0,z}$ coupled with a Brownian bridge B^σ with variance σ^2, as in Theorem 3.3. Then the last probability in brackets is

FIGURE 3. Sketch of the argument for Lemma 3.20: We use Lemma 3.1 to lower the entry and exit data $\vec{x}, \vec{y}$ of the curves to $\vec{x}\,'$ and $\vec{y}\,'$. The event E occurs when each curve lies within the blue bounding lines shown in the figure. We then use strong coupling with Brownian bridges via Theorem 3.3 and bound the probability of the bridges remaining within the blue windows.

bounded below by

(3.19)
$$\mathbb{P}^{0,T,0,z}_{Ber}\Big(\big|\ell(T/2) - pT/2 - (M + M_1 + 5)\sqrt{T}\big| \leq 2\sqrt{T} - 5 \quad \text{and}$$

$$\sup_{s\in[0,T/2]}\Big|\ell(s) - ps - \frac{M + M_1 + 5}{\sqrt{T}/2}\,s\Big| \leq 3\sqrt{T} - 1 \quad \text{and}$$

$$\sup_{s\in[T/2,T]}\Big|\ell(s) - ps - (M + M_1 + 5)\sqrt{T} + \frac{M + M_1 + 5}{\sqrt{T}/2}(s - T/2)\Big| \leq 3\sqrt{T} - 1\Big)$$

$$\geq \mathbb{P}\Big(\big|\sqrt{T}\,B^\sigma_{1/2} - (M + M_1 + 5)\sqrt{T}\big| \leq \sqrt{T} \quad \text{and}$$

$$\sup_{s\in[0,T/2]}\Big|\sqrt{T}\,B^\sigma_{s/T} - (M + M_1 + 5)\sqrt{T}\cdot\frac{s}{T/2}\Big| \leq 2\sqrt{T} \quad \text{and}$$

$$\sup_{s\in[T/2,T]}\Big|\sqrt{T}\,B^\sigma_{s/T} - (M + M_1 + 5)\sqrt{T}\cdot\frac{T-s}{T/2}\Big| \leq 2\sqrt{T}\Big)$$

$$- \mathbb{P}\Big(\Delta(T, z) > \sqrt{T}/2\Big).$$

Note that $B_{1/2}^\sigma$ is a centered Gaussian random variable with variance $p(1-p)/4 = \sigma^2(1/2)(1-1/2)$. Writing $\xi = B_{1/2}^\sigma$, it follows from Lemma 3.7 that there exist independent Brownian bridges B^1, B^2 with variance $\sigma^2/2$ so that B_s^σ has the same law as $\frac{s}{T/2}\xi + B_{2s/T}^1$ for $s \in [0, T/2]$ and $\frac{T-s}{T/2}\xi + B_{(2s-T)/T}^2$ for $s \in [T/2, T]$. The first term in the last expression in (3.19) is thus equal to

$$\mathbb{P}\Big(|\xi - (M + M_1 + 5)| \le 1 \quad \text{and} \quad \sup_{s \in [0, T/2]} \Big| B_{s/T}^1 - (M + M_1 + 5 - \xi) \cdot \frac{s}{T/2} \Big| \le 2$$

$$\text{and} \quad \sup_{s \in [T/2, T]} \Big| B_{(2s-T)/T}^2 - (M + M_1 + 5 - \xi) \cdot \frac{T-s}{T/2} \Big| \le 2 \Big)$$

$$\ge \mathbb{P}\Big(|\xi - (M + M_1 + 5)| \le 1 \quad \text{and} \quad \sup_{s \in [0, T/2]} \big| B_{2s/T}^1 \big| \le 1 \quad \text{and} \quad \sup_{s \in [T/2, T]} \big| B_{(2s-T)/T}^2 \big| \le 1 \Big)$$

$$= \mathbb{P}\Big(|\xi - (M + M_1 + 5)| \le 1\Big) \mathbb{P}\Big(\sup_{s \in [0, T/2]} \big| B_{2s/T}^1 \big| \le 1 \Big) \mathbb{P}\Big(\sup_{s \in [0, T/2]} \big| B_{(2s-T)/T}^2 \big| \le 1 \Big)$$

$$\ge \big(1 - 2e^{-4/p(1-p)}\big)^2 \int_{M+M_1+4}^{M+M_1+6} \frac{e^{-2\xi^2/p(1-p)}}{\sqrt{\pi p(1-p)/2}} \, d\xi$$

$$\ge \frac{2\sqrt{2}\, e^{-2(M+M_1+6)^2/p(1-p)}}{\sqrt{\pi p(1-p)}} \big(1 - 2e^{-4/p(1-p)}\big)^2.$$

In the fourth line, we used the fact that ξ, B^1, and B^2 are independent, and in the second to last line, we used Lemma 3.6. Since $|z - pT| \le (M_1 + 1)\sqrt{T}$, Lemma 3.5 allows us to choose T large enough so that $\mathbb{P}(\Delta(T, z) > \sqrt{T}/2)$ is less than $1/2$ the last expression. Then in view of (3.18) and (3.19), we conclude (3.17).

$\square$

Lemma 3.21. *Fix* $p, t \in (0, 1)$, $k \in \mathbb{N}$, $\vec{a}, \vec{b} \in W_k$. *Suppose that* $\vec{x}^T = (x_1^T, \cdots, x_k^T)$ *and* $\vec{y}^T = (y_1^T, \cdots, y_k^T)$ *are two sequence of k-dimensional vectors with integer entries such that for $i \in [\![1, k]\!]$,*

$$\lim_{T \to \infty} \frac{x_i^T}{\sqrt{T}} = a_i \quad \text{and} \quad \lim_{T \to \infty} \frac{y_i^T - pT}{\sqrt{T}} = b_i$$

and define the sequence of random k-dimensional vectors Z^T by

$$Z^T = \Big(\frac{L_1(tT) - ptT}{\sqrt{T}}, \cdots, \frac{L_k(tT) - ptT}{\sqrt{T}} \Big),$$

where $(L_1, \cdots, L_k)$ *is* $\mathbb{P}_{avoid,Ber}^{0,T,\vec{x}^T,\vec{y}^T}$*-distributed. When $a_i = a_{i+1}$ or $b_j = b_{j+1}$ for any entries of $\vec{a}$ and $\vec{b}$, we write*

$$\vec{a} = (a_1, \cdots, a_k) = (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p})$$

$$\vec{b} = (b_1, \cdots, b_k) = (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q})$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_p$, $\beta_1 > \beta_2 > \cdots > \beta_q$ *and* $\sum_{i=1}^p m_i = \sum_{i=1}^q n_i = k$. *Then, the random vector Z^T converges weakly to a random variable $\hat{Z}$ with the density*

$$\rho_{\vec{a}, \vec{b}}(z_1, \cdots, z_k) = \frac{1}{Z_{\vec{a}, \vec{b}}} \cdot \varphi(\vec{a}, \vec{z}, \vec{m}) \psi(\vec{b}, \vec{z}, \vec{n}) \prod_{i=1}^k e^{-c_3(t,p)z_i^2} \mathbb{1}_{\{z_1 > \cdots > z_k\}}$$

where $\vec{m} = (m_1, \cdots, m_k)$, $\vec{n} = (n_1, \cdots, n_k)$, c_1, c_2, c_3 *are constants depending on p, t as given in Proposition 9.3 , $Z_{\vec{a}, \vec{b}}$ is a constant depending on $p, t, \vec{a}, \vec{b}$ such that $\rho_{\vec{a}, \vec{b}}(z_1, \cdots, z_k)$ integrates to 1*

over $\mathbb{R}^k$, and $\varphi(\vec{a}, \vec{z}, \vec{m})$ and $\psi(\vec{b}, \vec{z}, \vec{n})$ are determinants:

$$\varphi(\vec{a}, \vec{z}, \vec{m}) = \det \begin{bmatrix} ((c_1(t,p)z_j)^{i-1}e^{c_1(t,p)\alpha_1 z_j})_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ ((c_2(t,p)z_j)^{i-1}e^{c_1(t,p)\alpha_p z_j})_{\substack{i=1,\cdots,m_p \\ j=1,\cdots,k}} \end{bmatrix}$$

$$\psi(\vec{b}, \vec{z}, \vec{n}) = \det \begin{bmatrix} ((c_2(t,p)z_j)^{i-1}e^{c_2(t,p)\beta_1 z_j})_{\substack{i=1,\cdots,n_1 \\ j=1,\cdots,k}} \\ \vdots \\ ((c_2(t,p)z_j)^{i-1}e^{c_2(t,p)\beta_q z_j})_{\substack{i=1,\cdots,n_q \\ j=1,\cdots,k}} \end{bmatrix}$$

Proof of this lemma will be postponed until the appendix, in propositions 9.3 and 9.5

Lemma 3.22. *Fix $p, t \in (0, 1)$ and $k \in \mathbb{N}$. Suppose that $\vec{x}^T = (x_1^T, \cdots x_k^T)$ and $\vec{y}^T = (y_1^T, \cdots, y_k^T)$ is a sequence of k-dimensional vectors with integer entries such that $T \geq y_i^T - x_i^T \geq 0$ for $i \in [\![1, k]\!]$. Show that for any $M_1, M_2 > 0$ and $\epsilon > 0$ there exists $T_0 \in \mathbb{N}$ and $\delta > 0$ depending on p, k, M_1, M_2 such that if $T \geq T_0$, $\left| x_i^T \right| \leq M_1\sqrt{T}$ and $\left| y_i^T - pT \right| \leq M_2\sqrt{T}$ then*

$$\mathbb{P}_{avoid,Ber}^{0,T,\vec{x}^T,\vec{y}^T} \left(\min_{1 \leq i \leq k-1} [L_i(tT) - L_{i+1}(tT)] < \delta\sqrt{T} \right) < \epsilon$$

Proof. We prove the claim by contradiction: suppose there exists $M_1, M_2, \epsilon > 0$ such that for any $T_0 \in \mathbb{N}$ and $\delta > 0$ there exists some $T \geq T_0$ such that

$$\mathbb{P}_{avoid,Ber}^{0,T,\vec{x}^T,\vec{y}^T} \left(\min_{1 \leq i \leq k-1} [L_i(tT) - L_{i+1}(tT)] < \delta\sqrt{T} \right) \geq \epsilon$$

Therefore, we can obtain sequences $T_n, \delta_n > 0$, $T_n \uparrow \infty$, $\delta_n \downarrow 0$ such that for all n, we have

$$\mathbb{P}_{avoid,Ber}^{0,T,\vec{x}^{T_n},\vec{y}^{T_n}} \left(\min_{1 \leq i \leq k-1} \left[\frac{L_i(tT_n) - L_{i+1}(tT_n)}{\sqrt{T_n}} \right] < \delta_n \right) \geq \epsilon$$

As $\left| x_i^{T_n} \right| < M_1\sqrt{T_n}$, $\left| y_i^{T_n} - pT_n \right| \leq M_2\sqrt{T_n}$ we know that there exists some subsequence T_{n_m} such that $\lim_{m\to\infty} \frac{x^{T_{n_m}}}{\sqrt{T_{n_m}}} \to \vec{x}$ and $\lim_{m\to\infty} \frac{y^{T_{n_m}} - pT}{\sqrt{T_{n_m}}} \to \vec{y}$ by the Bolzano-Weierstrass theorem [15, Theorem 3.6], as these sequences lie in compact intervals. Denote

$$Z_i^m := \frac{L_i(tT_{n_m}) - ptT}{\sqrt{T_{n_m}}}.$$

Then, if we fix $\tilde{\delta} > 0$, because $\delta_n \downarrow 0$, there exists some M such that $m > M$ implies $\delta_m < \tilde{\delta}$. For such $m > M$, we have

$$(3.20) \qquad \epsilon \leq \liminf_{m\to\infty} \mathbb{P} \left(\min_{1 \leq i \leq k-1} Z_i^m - Z_{i+1}^m < \delta_{n_m} \right) \leq \liminf_{m\to\infty} \mathbb{P} \left(\min_{1 \leq i \leq k-1} Z_i^m - Z_{i+1}^m \leq \tilde{\delta} \right)$$

Now, if we apply the results of Lemma 3.21, we that the vector $(Z_1^m, \cdots, Z_k^m)$ converges weakly to some random vector $\hat{Z}$ with probability density function ρ as defined previously in 3.21, the definitions of which we will state in a few lines, when they are needed. Because this sequence has weak convergence we may apply portmanteau's lemma with the closed set $K = [0, \tilde{\delta}]$ to find

$$(3.21) \qquad \limsup_{m\to\infty} \mathbb{P} \left(\min_{1 \leq i \leq k-1} Z_i^m - Z_{i+1}^m \in K \right) \leq \mathbb{P} \left(\min_{1 \leq i \leq k-1} \hat{Z}_i - \hat{Z}_{i+1} \in K \right)$$

Combining (3.20) and (3.21), we arrive at the inequality

(3.22)
$$\epsilon \leq \liminf_{m \to \infty} \mathbb{P}\left(\min_{1 \leq i \leq k-1} Z_i^m - Z_{i+1}^m \leq \tilde{\delta}\right) \leq \mathbb{P}\left(\min_{1 \leq i \leq k-1} \hat{Z}_i - \hat{Z}_{i+1}\tilde{\delta}\right)$$

and so by (3.22) and countable subadditivity,

(3.23)
$$\epsilon \leq \mathbb{P}\left(\min_{1 \leq i \leq k-1} \hat{Z}_i - \hat{Z}_{i+1} \leq \tilde{\delta}\right) \leq \sum_{i=1}^{k-1} \mathbb{P}\left(\hat{Z}_i - \hat{Z}_{i+1} \leq \tilde{\delta}\right)$$

Since $\tilde{\delta}$ is arbitrary, this inequality holds for any $\tilde{\delta} > 0$. In order to pick a sufficient δ' to find a contradiction, we will now state the definition of ρ, as found in 3.21:

$$\rho(z_1, \cdots, z_k) = \frac{1}{Z_{\vec{a},\vec{b}}} \cdot \varphi(\vec{a}, \vec{z}, \vec{n}) \cdot \psi(\vec{b}, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{\frac{-z_i^2}{t(1-t)}}$$

with $\vec{z}_i \in W_k^\circ$ where we have the functions and constant

$$\varphi(\vec{a}, \vec{z}, \vec{n}) = \det \begin{bmatrix} \left[(c_1(t,p)z_j)^{i-1}e^{c_1(t,p)\alpha_1 z_j}\right]_{\substack{i=1,\ldots,m_1 \\ j=1,\ldots,k}} \\ \left[(c_1(t,p)z_j)^{i-1}e^{c_1(t,p)\alpha_2 z_j}\right]_{\substack{i=1,\ldots,m_2 \\ j=1,\ldots,k}} \\ \vdots \\ \left[(c_1(t,p)z_j)^{i-1}e^{c_1(t,p)\alpha_r z_j}\right]_{\substack{i=1,\ldots,m_r \\ j=1,\ldots,k}} \end{bmatrix}$$

$$\psi(\vec{b}, \vec{z}, \vec{n}) = \det \begin{bmatrix} \left[(c_2(t,p)z_j)^{i-1}e^{c_2(t,p)\beta_1 z_j}\right]_{\substack{i=1,\ldots,n_1 \\ j=1,\ldots,k}} \\ \left[(c_2(t,p)z_j)^{i-1}e^{c_2(t,p)\beta_2 z_j}\right]_{\substack{i=1,\ldots,n_2 \\ j=1,\ldots,k}} \\ \vdots \\ \left[(c_2(t,p)z_j)^{i-1}e^{c_2(t,p)\beta_q z_j}\right]_{\substack{i=1,\ldots,n_q \\ j=1,\ldots,k}} \end{bmatrix}$$

$$Z_{\vec{a},\vec{b}} = \int_{\mathbb{R}^k} \left(\mathbf{1}_{z_1 \geq z_2 \geq \cdots \geq z_k}(\vec{z})\right) \varphi(\vec{a}, \vec{z}, \vec{n}) \cdot \psi(\vec{b}, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{\frac{-z_i^2}{t(1-t)}} d\vec{z}$$

we find that ρ is continuous as a sum and product of continuous functions. Additionally, if for any $\chi \in [\![1, k-1]\!]$ such that $z_\chi = z_{\chi+1}$, then we have two equal columns

$$\begin{bmatrix} \left[(c_1(t,p)z_\chi)^{i-1}e^{c_1(t,p)\alpha_1 z_\chi}\right]_{i=1,\ldots,m_1} \\ \left[(c_1(t,p)z_\chi)^{i-1}e^{c_1(t,p)\alpha_2 z_\chi}\right]_{i=1,\ldots,m_2} \\ \vdots \\ \left[(c_1(t,p)z_\chi)^{i-1}e^{c_1(t,p)\alpha_r z_\chi}\right]_{i=1,\ldots,m_r} \end{bmatrix} = \begin{bmatrix} \left[(c_1(t,p)z_{\chi+1})^{i-1}e^{c_1(t,p)\alpha_1 z_{\chi+1}}\right]_{i=1,\ldots,m_1} \\ \left[(c_1(t,p)z_{\chi+1})^{i-1}e^{c_1(t,p)\alpha_2 z_{\chi+1}}\right]_{i=1,\ldots,m_2} \\ \vdots \\ \left[(c_1(t,p)z_{\chi+1})^{i-1}e^{c_1(t,p)\alpha_r z_{\chi+1}}\right]_{i=1,\ldots,m_r} \end{bmatrix}$$

which implies that the determinant in $\varphi(\vec{a}, \vec{b}, \vec{n}) = 0$, so $\rho(\vec{z}) = 0$. Hence, if we let $E_i^\delta := \{z \mid z_i - z_{i+1} \leq \delta\}$ we know by the continuity of ρ that the accumulation function

$$R_i(\delta) := \int_{E_i^\delta} \rho$$

is well defined and continuous. We can find that $R_i(0) = \int_{E_i^0} 0$ by our previous calculation since $\rho = 0$ on E_i^0 as $z_i = z_{i+1}$. By the continuity of R we know that for any ϵ there exists a $\delta_i(\epsilon)$ such that $0 < \delta < \delta_i(\epsilon)$ implies $|R_i(\delta)| < \epsilon$. Therefore, let $\tilde{\delta} = \min_{1 \leq i \leq k-1} \delta_i\left(\frac{\epsilon}{k-1}\right)$ and we find that

(3.24)
$$\mathbb{P}\left(\hat{Z}_i - \hat{Z}_{i+1} \leq \tilde{\delta}\right) = R_i(\tilde{\delta}) < \frac{\epsilon}{k-1}$$

for each integer $i \in [\![1, k-1]\!]$. Combining equations (3.23) and (3.24) yields the inequality

$$\epsilon \leq \mathbb{P}\left(\min_{1 \leq i \leq k-1} \hat{Z}_i - \hat{Z}_{i+1} \leq \delta' \right) \leq \sum_{i=1}^{k-1} \mathbb{P}\left(\hat{Z}_i - \hat{Z}_{i+1} \leq \delta' \right) < \epsilon$$

which is a contradiction. Thus we have proved the Lemma. $\square$

Remark 3.23. Lemma 3.22 provides us with some amount of separation between the lines in the ensemble at any point along the Beroulli Random Walkers as T tends to infinity. For any ϵ, it allows us to find a distance δ such that the probability that any lines get closer than δ is less than ϵ. The proof essentially passes to subsequential limits in order to use the convergence result of Lemma (3.21), and then the fact that the density ρ is continuous allows us to integrate $\rho(\vec{z})$ over areas where z_i values collide, and find that these integrals are 0. Then we may use continuity around areas where z_i values are equal to bound the probability where z_i are close.

4. Proof of Theorem 2.26

The goal of this section is to prove Theorem 2.26 and for the remainder of this section we assume that $k \in \mathbb{N}$ with $k \geq 2$, $p \in (0,1)$, $\alpha, \lambda > 0$ are all fixed and

$$\left\{ \mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N) \right\}_{N=1}^{\infty},$$

is an (α, p, λ)-good sequence of $[\![1, k]\!]$-indexed Bernoulli line ensembles as in Definition 2.24 that are all defined on a probability space with measure $\mathbb{P}$. The proof of Theorem 2.26 depends on three results – Proposition 4.1 and Lemmas 4.2 and 4.3. In these three statements we establish various properties of the sequence of line ensembles $\mathfrak{L}^N$ and the constants in those statements depend implicitly on α, p, λ, k, and the functions ϕ, ψ from Definition 2.24, which are fixed throughout. We will not list this dependence explicitly. The proof of Proposition 4.1 is given in Section 4.1 while the proofs of Lemmas 4.2 and 4.3 are in Section 5. Theorem 2.26 is proved in Section 4.2.

4.1. **Bounds on the acceptance probability.** The main result in this section is presented as Proposition 4.1 below. In order to formulate it and some of the lemmas below it will be convenient to adopt the following notation for any $r > 0$ and $m \in \mathbb{N}$

$$(4.1) \qquad\qquad\qquad\qquad t_m = \lfloor (r+m)N^\alpha \rfloor.$$

Proposition 4.1. *Let $\mathbb{P}$ be the measure from the beginning of this section. For any $\epsilon > 0$, $r > 0$ there exist $\delta = \delta(\epsilon, r) > 0$ and $N_1 = N_1(\epsilon, r)$ such that for all $N \geq N_1$ we have*

$$\mathbb{P}\left(Z\left(-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!] \right) < \delta \right) < \epsilon,$$

where $\vec{x} = (L_1^N(-t_1), \dots, L_{k-1}^N(-t_1))$, $\vec{y} = (L_1^N(t_1), \dots, L_{k-1}^N(t_1))$, $L_k^N[\![-t_1, t_1]\!]$ is the restriction of L_k^N to the set $[\![-t_1, t_1]\!]$, and Z is the acceptance probability of Definition 2.22.

The general strategy we use to prove Proposition 4.1 is inspired by the proof of [5, Proposition 6.5]. We begin by stating three key lemmas that will be required. The proofs of Lemmas 4.2 and 4.3 are postponed to Section 5 and Lemma 4.4 is proved in Section 6.

Lemma 4.2. *Let $\mathbb{P}$ be the measure from the beginning of this section. For any $\epsilon > 0$, $r > 0$ there exist $R_1 = R_1(\epsilon, r) > 0$ and $N_2 = N_2(\epsilon, r)$ such that for $N \geq N_2$*

$$\mathbb{P}\left(\sup_{s \in [-t_3, t_3]} \left[L_1^N(s) - ps \right] \geq R_1 N^{\alpha/2} \right) < \epsilon.$$

Lemma 4.3. *Let $\mathbb{P}$ be the measure from the beginning of this section. For any $\epsilon > 0$, $r > 0$ there exist $R_2 = R_2(\epsilon, r) > 0$ and $N_3 = N_3(\epsilon, r)$ such that for $N \geq N_3$*

$$\mathbb{P}\left(\inf_{s \in [-t_3, t_3]} \left[L^N_{k-1}(s) - ps\right] \leq -R_2 N^{\alpha/2}\right) < \epsilon.$$

Lemma 4.4. *Fix $k \in \mathbb{N}$, $k \geq 2$, $p \in (0,1)$, $r, \alpha, M_1, M_2 > 0$. Suppose that $\ell_{bot} : [\![-t_3, t_3]\!] \to \mathbb{R} \cup \{-\infty\}$, and $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$ are such that $|\Omega_{avoid}(-t_3, t_3, \vec{x}, \vec{y}, \infty, \ell_{bot})| \geq 1$. Suppose further that*

(1) $\sup_{s \in [-t_3, t_3]} \left[\ell_{bot}(s) - ps\right] \leq M_2(2t_3)^{1/2}$,

(2) $-pt_3 + M_1(2t_3)^{1/2} \geq x_1 \geq x_{k-1} \geq \max\left(\ell_{bot}(-t_3), -pt_3 - M_1(2t_3)^{1/2}\right)$,

(3) $pt_3 + M_1(2t_3)^{1/2} \geq y_1 \geq y_{k-1} \geq \max\left(\ell_{bot}(t_3), pt_3 - M_1(2t_3)^{1/2}\right)$.

Then there exist constants g, h and $N_4 \in \mathbb{N}$ all depending on $M_1, M_2, p, k, r, \alpha$ such that for any $\tilde{\epsilon} > 0$ and $N \geq N_4$ we have

$$(4.2) \qquad \mathbb{P}^{-t_3, t_3, \vec{x}, \vec{y}, \infty, \ell_{bot}}_{avoid, Ber}\left(Z\left(-t_1, t_1, \mathfrak{Q}(-t_1), \mathfrak{Q}(t_1), \infty, \ell_{bot}[\![-t_1, t_1]\!]\right) \leq gh\tilde{\epsilon}\right) \leq \tilde{\epsilon},$$

where Z is the acceptance probability of Definition 2.22, $\ell_{bot}[\![-t_1, t_1]\!]$ is the vector, whose coordinates match those of ℓ_{bot} on $[\![-t_1, t_1]\!]$ and $\mathfrak{Q}(a) = (Q_1(a), \ldots, Q_{k-1}(a))$ is the value of the line ensemble $\mathfrak{Q} = (Q_1, \ldots, Q_{k-1})$ whose law is $\mathbb{P}^{-t_3, t_3, \vec{x}, \vec{y}, \infty, \ell_{bot}}_{avoid, Ber}$ at location a.

Proof of Proposition 4.1. Let $\epsilon > 0$ be given. Define the event

$$E_N = \left\{L^N_{k-1}(\pm t_3) \mp pt_3 \geq -M_1(2t_3)^{1/2}\right\} \cap \left\{L^N_1(\pm t_3) \mp pt_3 \leq M_1(2t_3)^{1/2}\right\} \cap$$

$$\left\{\sup_{s \in [-t_3, t_3]} [L^N_k(s) - ps] \leq M_2(2t_3)^{1/2}\right\}.$$

In view of Lemmas 4.2 and 4.3 and the fact that $\mathbb{P}$-almost surely $L^N_1(s) \geq L^N_k(s)$ for all $s \in [-t_3, t_3]$ we can find sufficiently large M_1, M_2 and N_2 such that for $N \geq N_2$ we have

$$(4.3) \qquad\qquad\qquad \mathbb{P}(E^c_N) < \epsilon/2.$$

Let g, h, N_4 be as in Lemma 4.4 for the values M_1, M_2 as above, the values α, p, k from the beginning of the section and r as in the statement of the proposition. For $\delta = (\epsilon/2) \cdot gh$ we denote

$$V = \left\{Z\left(-t_1, t_1, \vec{x}, \vec{y}, \infty, L^N_k[\![-t_1, t_1]\!]\right) < \delta\right\}$$

and make the following deduction for $N \geq N_4$

$$\mathbb{P}(V \cap E_N) = \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{E_N} \cdot \mathbf{1}_V \Big| \sigma\left(\mathfrak{L}^N(-t_3), \mathfrak{L}^N(t_3), L^N_k[\![-t_3, t_3]\!]\right)\right]\right] =$$

$$(4.4) \quad \mathbb{E}\left[\mathbf{1}_{E_N} \cdot \mathbb{E}\left[\mathbf{1}\{Z\left(-t_1, t_1, \vec{x}, \vec{y}, \infty, L^N_k[\![-t_1, t_1]\!]\right) < \delta\} \Big| \sigma\left(\mathfrak{L}^N(-t_3), \mathfrak{L}^N(t_3), L^N_k[\![-t_3, t_3]\!]\right)\right]\right] =$$

$$\mathbb{E}\left[\mathbf{1}_{E_N} \cdot \mathbb{E}_{avoid}\left[\mathbf{1}\{Z\left(-t_1, t_1, \mathfrak{L}(-t_1), \mathfrak{L}(t_1), \infty, L^N_k[\![-t_1, t_1]\!]\right) < \delta\}\right]\right] \leq \mathbb{E}\left[\mathbf{1}_{E_N} \cdot \epsilon/2\right] \leq \epsilon/2.$$

In (4.4) we have written $\mathbb{E}_{avoid}$ in place of $\mathbb{E}^{-t_3, t_3, \mathfrak{L}^N(-t_3), \mathfrak{L}^N(t_3), \infty, L^N_k[\![-t_3, t_3]\!]}_{avoid, Ber}$ to ease the notation; in addition, we have that $\mathfrak{L}^N(a) = (L^N_1(a), \ldots, L^N_{k-1}(a))$ and $\mathfrak{L}$ on the last line is distributed according to $\mathbb{P}^{-t_3, t_3, \mathfrak{L}^N(-t_3), \mathfrak{L}^N(t_3), \infty, L^N_k[\![-t_3, t_3]\!]}_{avoid, Ber}$. We elaborate on (4.4) in the paragraph below.

The first equality in (4.4) follows from the tower property for conditional expectations. The second equality uses the definition of V and the fact that $\mathbf{1}_{E_N}$ is $\sigma\left(\mathfrak{L}^N(-t_3), \mathfrak{L}^N(t_3), L^N_k[\![-t_3, t_3]\!]\right)$-measurable and can thus be taken outside of the conditional expectation. The third equality uses the Schur Gibbs property, see Definition 2.17. The first inequality on the third line holds if $N \geq N_4$ and uses Lemma 4.4 with $\tilde{\epsilon} = \epsilon/2$ as well as the fact that on the event E_N the random variables $\mathfrak{L}^N(-t_3), \mathfrak{L}^N(t_3)$ and $L^N_k[\![-t_3, t_3]\!]$ (that play the roles of $\vec{x}, \vec{y}$ and ℓ_{bot}) satisfy the inequalities

(1) $\sup_{s\in[-t_3,t_3]} \left[L_k^N(s) - ps \right] \leq M_2(2t_3)^{1/2}$,

(2) $-pt_3 + M_1(2t_3)^{1/2} \geq L_1^N(-t_3) \geq L_{k-1}^N(-t_3) \geq \max\left(L_k^N(-t_3), -pt_3 - M_1(2t_3)^{1/2} \right)$,

(3) $pt_3 + M_1(2t_3)^{1/2} \geq L_1^N(t_3) \geq L_{k-1}^N(t_3) \geq \max\left(L_k^N(t_3), pt_3 - M_1(2t_3)^{1/2} \right)$.

The last inequality in (4.4) is trivial.

Combining (4.4) with (4.3), we see that for all $N \geq \max(N_2, N_4)$ we have

$$\mathbb{P}(V) = \mathbb{P}(V \cap E_N) + \mathbb{P}(V \cap E_N^c) \leq \epsilon/2 + \mathbb{P}(E_N^c) < \epsilon,$$

which proves the proposition. $\square$

4.2. **Proof of Theorem 2.26.** By Lemma 2.4, it suffices to verify the following two conditions for all $i \in [\![1, k-1]\!]$, $r > 0$, and $\epsilon > 0$:

$$(4.5) \qquad\qquad\qquad\qquad \lim_{a\to\infty} \limsup_{N\to\infty} \mathbb{P}(|f_i^N(0)| \geq a) = 0$$

$$(4.6) \qquad\qquad \lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}\left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} |f_i^N(x) - f_i^N(y)| \geq \epsilon \right) = 0.$$

For the sake of clarity, we will prove these conditions in several steps.

Step 1. In this step we prove (4.5). Let $\epsilon > 0$ be given. Then by Lemmas 4.2 and 4.3 we can find N_2, N_3 and R_1, R_2 such that for $N \geq \max(N_1, N_2)$

$$\mathbb{P}\left(\sup_{s\in[-t_3,t_3]} [L_1^N(s) - ps] \geq R_1 N^{\alpha/2} \right) < \epsilon/2,$$

$$\mathbb{P}\left(\inf_{s\in[-t_3,t_3]} [L_{k-1}^N(s) - ps] \leq -R_2 N^{\alpha/2} \right) < \epsilon/2.$$

In particular, if we set $R = \max(R_1, R_2)$ and utilize the fact that $L_1^N(0) \geq \cdots \geq L_{k-1}^N(0)$ we conclude that for any $i \in [\![1, k-1]\!]$ we have

$$\mathbb{P}\left(|L_i^N(0)| \geq R N^{\alpha/2}\right) \leq \mathbb{P}\left(L_1^N(0) \geq R_1 N^{\alpha/2}\right) + \mathbb{P}\left(L_{k-1}^N(0) \leq -R_2 N^{\alpha/2}\right) < \epsilon,$$

which implies (4.5).

Step 2. In this step we prove (4.6). In the sequel we fix $r, \epsilon > 0$ and $i \in [\![1, k-1]\!]$. To prove (4.6) it suffices to show that for any $\eta > 0$, there exists a $\delta > 0$ and N_0 such that $N \geq N_0$ implies

$$(4.7) \qquad\qquad\qquad \mathbb{P}\left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} |f_i^N(x) - f_i^N(y)| \geq \epsilon \right) < \eta.$$

For $\delta > 0$ we define the event

$$(4.8) \qquad A_\delta^N = \left\{ \sup_{x,y\in[-t_1,t_1],|x-y|\leq\delta N^\alpha} \left| L_i^N(x) - L_i^N(y) - p(x-y) \right| \geq \frac{3N^{\alpha/2}\epsilon}{4} \right\},$$

where we recall that $t_1 = \lfloor (r+1)N^\alpha \rfloor$ from (4.1). We claim that there exist $\delta_0 > 0$ and $N_0 \in \mathbb{N}$ such that for $\delta \in (0, \delta_0]$ and $N \geq N_0$ we have

$$(4.9) \qquad\qquad\qquad\qquad\qquad \mathbb{P}(A_\delta^N) < \eta.$$

We prove (4.9) in the steps below. Here we assume its validity and conclude the proof of (4.7).

Observe that if $\delta \in \left(0, \min\left(\delta_0, \epsilon \cdot (8\lambda r)^{-1}\right)\right)$, where λ is as in the statement of the theorem, we have the following tower of inequalities

$$
\begin{aligned}
\mathbb{P} & \left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} |f_i^N(x) - f_i^N(y)| \geq \epsilon \right) = \\
& \mathbb{P} \left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} \left| N^{-\alpha/2} \left(L_i^N(xN^\alpha) - L_i^N(yN^\alpha) \right) - p(x-y)N^{\alpha/2} + \lambda(x^2 - y^2) \right| \geq \epsilon \right) \leq \\
& \mathbb{P} \left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} N^{-\alpha/2} \left| L_i^N(xN^\alpha) - L_i^N(yN^\alpha) - p(x-y)N^\alpha \right| + 2\lambda r\delta \geq \epsilon \right) \leq \\
& \mathbb{P} \left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} \left| L_i^N(xN^\alpha) - L_i^N(yN^\alpha) - p(x-y)N^\alpha \right| \geq \frac{3N^{\alpha/2}\epsilon}{4} \right) \leq \mathbb{P}(A_\delta^N) < \eta.
\end{aligned}
$$

(4.10)

In (4.10) the first equality follows from the definition of f_i^N, and the inequality on the second line follows from the inequality $|x^2 - y^2| \leq 2r\delta$, which holds for all $x, y \in [-r, r]$ such that $|x - y| \leq \delta$. The inequality in the third line of (4.10) follows from our assumption that $\delta < \epsilon \cdot (8\lambda r)^{-1}$ and the first inequality on the last line follows from the definition of A_δ^N in (4.8), and the fact that $t_1 \geq rN^\alpha$. The last inequality follows from our assumption that $\delta < \delta_0$ and (4.9). In view of (4.10) we conclude (4.7).

Step 3. In this step we prove (4.9) and fix $\eta > 0$ in the sequel. For $\delta_1, M_1 > 0$ and $N \in \mathbb{N}$ we define the events

$$(4.11) \quad E_1 = \left\{ \max_{1\leq j\leq k-1} |L_j^N(\pm t_1) \mp pt_1| \leq M_1 N^{\alpha/2} \right\}, E_2 = \left\{ Z(-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!]) > \delta_1 \right\},$$

where we used the same notation as in Proposition 4.1 (in particular $\vec{x} = (L_1^N(-t_1), \ldots, L_{k-1}^N(-t_1))$ and $\vec{y} = (L_1^N(t_1), \ldots, L_{k-1}^N(t_1))$). Combining Lemmas 4.2, 4.3 and Proposition 4.1 we know that we can find $\delta_1 > 0$ sufficiently small, M_1 sufficiently large and $\tilde{N} \in \mathbb{N}$ such that for $N \geq \tilde{N}$ we know

$$(4.12) \qquad \qquad \mathbb{P}\left(E_1^c \cup E_2^c\right) < \eta/2.$$

We claim that we can find $\delta_0 > 0$ and $N_0 \geq \tilde{N}$ such that for $N \geq N_0$ and $\delta \in (0, \delta_0)$ we have

$$(4.13) \qquad \qquad \mathbb{P}(A_\delta^N \cap E_1 \cap E_2) < \eta/2.$$

Since

$$\mathbb{P}(A_\delta^N) = \mathbb{P}(A_\delta^N \cap E_1 \cap E_2) + \mathbb{P}(A_\delta^N \cap (E_1^c \cup E_2^c)) \leq \mathbb{P}(A_\delta^N \cap E_1 \cap E_2) + \mathbb{P}(E_1^c \cup E_2^c),$$

we see that (4.12) and (4.13) together imply (4.9).

Step 4. In this step we prove (4.13). We define the σ-algebra

$$\mathcal{F} = \sigma\left(L_k^N[\![-t_1, t_1]\!], L_1^N(\pm t_1), L_2^N(\pm t_1), \ldots, L_{k-1}^N(\pm t_1)\right).$$

Clearly $E_1, E_2 \in \mathcal{F}$, so the indicator random variables $\mathbf{1}_{E_1}$ and $\mathbf{1}_{E_2}$ are $\mathcal{F}$-measurable. It follows from the tower property of conditional expectation that

$$(4.14) \qquad \mathbb{P}\left(A_\delta^N \cap E_1 \cap E_2\right) = \mathbb{E}\left[\mathbf{1}_{A_\delta^N}\mathbf{1}_{E_1}\mathbf{1}_{E_2}\right] = \mathbb{E}\left[\mathbf{1}_{E_1}\mathbf{1}_{E_2}\mathbb{E}\left[\mathbf{1}_{A_\delta^N} \mid \mathcal{F}\right]\right].$$

By the Schur-Gibbs property (see Definition 2.17), we know that $\mathbb{P}$-almost surely

$$(4.15) \qquad \qquad \mathbb{E}\left[\mathbf{1}_{A_\delta^N} \mid \mathcal{F}\right] = \mathbb{E}_{avoid,Ber}^{-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!]}\left[\mathbf{1}_{A_\delta^N}\right].$$

We now observe that the Radon-Nikodym derivative of $\mathbb{P}_{avoid,Ber}^{-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]}$ with respect to $\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}$ is given by

$$(4.16) \qquad \frac{d\mathbb{P}_{avoid,Ber}^{-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]}}{d\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}}(Q_1,\ldots,Q_{k-1}) = \frac{\mathbf{1}_{\{Q_1 \geq \cdots \geq Q_{k-1} \geq Q_k\}}}{Z(-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!])},$$

where $\mathfrak{Q} = (Q_1,\ldots,Q_{k-1})$ is $\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}$-distributed and $Q_k = L_k^N[\![-t_1,t_1]\!]$. To see this, note that by Definition 2.15 we have for any set $A \subset \prod_{i=1}^{k-1}\Omega(-t_1,t_1,x_i,y_i)$ that

$$\mathbb{P}_{avoid,Ber}^{-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]}(A) = \frac{\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(A \cap \{Q_1 \geq \cdots \geq Q_{k-1} \geq Q_k\})}{\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(Q_1 \geq \cdots \geq Q_{k-1} \geq Q_k)} =$$

$$\frac{\mathbb{E}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}\left[\mathbf{1}_A \cdot \mathbf{1}_{\{Q_1 \geq \cdots \geq Q_{k-1} \geq Q_k\}}\right]}{Z(-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!])} = \int_A \frac{\mathbf{1}_{\{Q_1 \geq \cdots \geq Q_{k-1} \geq Q_k\}}}{Z(-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!])}\, d\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}.$$

It follows from (4.14), (4.16), and the definition of E_2 that

$$(4.17) \qquad \mathbb{P}(A_\delta^N \cap E_1 \cap E_2) = \mathbb{E}\left[\mathbf{1}_{E_1}\mathbf{1}_{E_2}\mathbb{E}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}\left[\frac{\mathbf{1}_{B_\delta^N} \cdot \mathbf{1}_{\{Q_1 \geq \cdots \geq Q_k\}}}{Z(-t_1,t_1,\vec{x},\vec{y},L_k^N[\![-t_1,t_1]\!])}\right]\right] \leq$$

$$\leq \mathbb{E}\left[\mathbf{1}_{E_1}\mathbf{1}_{E_2}\mathbb{E}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}\left[\frac{\mathbf{1}_{B_\delta^N}}{\delta_1}\right]\right] \leq \frac{1}{\delta_1}\mathbb{E}\left[\mathbf{1}_{E_1}\cdot\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(B_\delta^N)\right],$$

where

$$B_\delta^N = \left\{\sup_{x,y \in [-t_1,t_1],|x-y| \leq \delta N^\alpha}|Q_i(x) - Q_i(y) - p(x-y)| \geq \frac{3N^{\alpha/2}\epsilon}{4}\right\}.$$

Notice that under $\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}$ the law of Q_i is precisely $\mathbb{P}_{Ber}^{-t_1,t_1,x_i,y_i}$, and so we conclude that

$$(4.18) \quad \mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(B_\delta^N) = \mathbb{P}_{Ber}^{0,2t_1,0,y_i-x_i}\left(\sup_{x,y \in [0,2t_1],|x-y| \leq \delta N^\alpha}|\ell(x) - \ell(y) - p(x-y)| \geq \frac{3N^{\alpha/2}\epsilon}{4}\right),$$

where ℓ has law $\mathbb{P}_{Ber}^{0,2t_1,0,y_i-x_i}$ (note that in (4.18) we implicitly translated the path ℓ to the right by t_1 and up by $-x_i$, which does not affect the probability in question). Since on the event E_1 we know that $|y_i - x_i - 2pt_1| \leq 2M_1N^\alpha$ we conclude from Lemma 3.16 that we can find N_0 and $\delta_0 > 0$ depending on M_1, r, α such that for $N \geq N_0$ and $\delta \in (0,\delta_0)$ we have

$$(4.19) \qquad \mathbf{1}_{E_1}\cdot\mathbb{P}_{Ber}^{0,2t_1,0,y_i-x_i}\left(\sup_{x,y \in [0,2t_1],|x-y| \leq \delta N^\alpha}|\ell(x) - \ell(y) - p(x-y)| \geq \frac{3N^{\alpha/2}\epsilon}{4}\right) < \frac{\delta_1\eta}{2}.$$

Combining (4.17), (4.18) and (4.19) we conclude (4.13), and hence the theorem.

5. Bounding the max and min

In this section we prove Lemmas 4.2 and 4.3 and we assume the same notation as in the statements of these lemmas. In particular, we assume that $k \in \mathbb{N}$, $k \geq 2$, $p \in (0,1)$, $\alpha, \lambda > 0$ are all fixed and

$$\left\{\mathfrak{L}^N = (L_1^N, L_2^N, \ldots, L_k^N)\right\}_{N=1}^\infty,$$

is an (α, p, λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles as in Definition 2.24 that are all defined on a probability space with measure $\mathbb{P}$. The proof of Lemma 4.2 is given in Section 5.1 and Lemma 4.3 is proved in Section 5.2.

5.1. **Proof of Lemma 4.2.** Our proof of Lemma 4.2 is similar to that of [3, Lemma 5.2]. For clarity we split the proof into three steps. In the first step we introduce some notation that will be required in the proof of the lemma, which is presented in Steps 2 and 3.

Step 1. We write $s_4 = \lceil r + 4 \rceil N^\alpha$, $s_3 = \lfloor r + 3 \rfloor N^\alpha$, so that $s_3 \leq t_3 \leq s_4$, and assume that N is large enough so that $\psi(N)N^\alpha$ from Definition 2.24 is at least s_4. Notice that such a choice is possible by our assumption that $\mathfrak{L}^N$ is an (α, p, λ)-good sequence and in particular, we know that L_i^N are defined at $\pm s_4$ for $i \in [\![1, k]\!]$. We define events

$$E(M) = \left\{ |L_1^N(-s_4) + ps_4| > MN^{\alpha/2} \right\}, \quad F(M) = \left\{ L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} \right\},$$

$$G(M) = \left\{ \sup_{s \in [0, s_4]} \left[L_1^N(s) - ps \right] \geq (6r + 22)(2r + 10)^{1/2}(M + 1)N^{\alpha/2} \right\}.$$

If $\epsilon > 0$ is as in the statement of the lemma, we note by (2.7) that we can find M and $\tilde{N}_1$ sufficiently large so that if $N \geq \tilde{N}_1$ we have

(5.1) $$\mathbb{P}(E(M)) < \epsilon/4 \text{ and } \mathbb{P}(F(M)) < \epsilon/12.$$

In the remainder of this step we show that the event $G(M) \setminus E(M)$ can be written as a *countable disjoint union* of certain events, i.e. we show that

(5.2) $$\bigsqcup_{(a, b, s, \ell_{top}, \ell_{bot}) \in D(M)} E(a, b, s, \ell_{top}, \ell_{bot}) = G(M) \setminus E(M),$$

where the sets $E(a, b, s, \ell_{top}, \ell_{bot})$ and $D(M)$ are described below.

For $a, b, z_1, z_2, z_3 \in \mathbb{Z}$ with $z_1 \leq a$, $z_2 \leq b$, $s \in [\![0, s_4]\!]$, $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$ and $\ell_{top} \in \Omega(s, s_4, b, z_3)$ we define $E(a, b, s, \ell_{top}, \ell_{bot})$ to be the event that $L_1^N(-s_4) = a$, $L_1^N(s) = b$, L_1^N agrees with ℓ_{top} on $[\![s, s_4]\!]$, and L_2^N agrees with ℓ_{bot} on $[\![-s_4, s]\!]$. Let $D(M)$ be the set of tuples $(a, b, s, \ell_{top}, \ell_{bot})$ satisfying

   (1) $0 \leq s \leq s_4$,
   (2) $0 \leq b - a \leq s + s_4$, $|a + ps_4| \leq MN^{\alpha/2}$, and $b - ps \geq (6r + 22)(2r + 10)^{1/2}(M + 1)N^{\alpha/2}$,
   (3) $z_1 \leq a$, $z_2 \leq b$, and $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$,
   (4) $b \leq z_3 \leq b + (s_4 - s)$, and $\ell_{top} \in \Omega(s, s_4, b, z_3)$,
   (5) if $s < s' \leq s_4$, then $\ell_{top}(s') - ps' < (6r + 22)(2r + 10)^{1/2}(M + 1)N^{\alpha/2}$.

It is clear that $D(M)$ is countable. The five conditions above together imply that

$$\bigcup_{(a, b, s, \ell_{top}, \ell_{bot}) \in D(M)} E(a, b, s, \ell_{top}, \ell_{bot}) = G(M) \setminus E(M),$$

and what remains to be shown to prove (5.2) is that $E(a, b, s, \ell_{top}, \ell_{bot})$ are pairwise disjoint.

On the intersection of $E(a, b, s, \ell_{top}, \ell_{bot})$ and $E(\tilde{a}, \tilde{b}, \tilde{s}, \tilde{\ell}_{top}, \tilde{\ell}_{bot})$ we must have $\tilde{a} = L_1^N(-s_4) = a$ so that $a = \tilde{a}$. Furthermore, we have by properties (2) and (5) that $s \geq \tilde{s}$ and $\tilde{s} \geq s$ from which we conclude that $s = \tilde{s}$ and then we conclude $\tilde{b} = b$, $\ell_{top} = \tilde{\ell}_{top}$, $\ell_{bot} = \tilde{\ell}_{bot}$. In summary, if $E(a, b, s, \ell_{top}, \ell_{bot})$ and $E(\tilde{a}, \tilde{b}, \tilde{s}, \tilde{\ell}_{top}, \tilde{\ell}_{bot})$ have a non-trivial intersection then $(a, b, s, \ell_{top}, \ell_{bot}) = (\tilde{a}, \tilde{b}, \tilde{s}, \tilde{\ell}_{top}, \tilde{\ell}_{bot})$, which proves (5.2).

Step 2. In this step we prove that we can find an N_2 so that for $N \geq N_2$

(5.3) $$\mathbb{P} \left(\sup_{s \in [0, t_3]} \left[L_1^N(s) - ps \right] \geq (6r + 22)(2r + 10)^{1/2}(M + 1)N^{\alpha/2} \right) \leq \mathbb{P}(G(M)) < \epsilon/2.$$

A similar argument, which we omit, proves the same inequality with $[-t_3, 0]$ in place of $[0, t_3]$ and then the statement of the lemma holds for all $N \geq N_2$, with $R_1 = (6r + 22)(2r + 10)^{1/2}(M + 1)$.

We claim that we can find $\tilde{N}_2 \in \mathbb{N}$ sufficiently large so that if $N \geq \tilde{N}_2$ and $(a, b, s, \ell_{top}, \ell_{bot}) \in D(M)$ satisfies $\mathbb{P}(E(a, b, s, \ell_{top}, \ell_{bot})) > 0$ then we have

$$(5.4) \qquad \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) \geq \frac{1}{3}.$$

We will prove (5.4) in Step 3. For now we assume its validity and conclude the proof of (5.3).

Let $(a, b, s, \ell_{top}, \ell_{bot}) \in D(M)$ be such that $\mathbb{P}(E(a, b, s, \ell_{top}, \ell_{bot})) > 0$. By the Schur Gibbs property, see Definition 2.17, we have for any $\ell_0 \in \Omega(-s_4, s, a, b)$ that

$$(5.5) \qquad \mathbb{P}\left(L_1^N[\![-s_4, s]\!] = \ell_0 \mid E(a, b, s, \ell_{top}, \ell_{bot})\right) = \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}(\ell = \ell_0),$$

where $L_1^N[\![-s_4, s]\!]$ denotes the restriction of L_1^N to the set $[\![-s_4, s]\!]$.

Combining (5.4) and (5.5) we get for $N \geq \tilde{N}_2$

$$(5.6) \qquad \begin{aligned} &\mathbb{P}\left(L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} | E(a, b, s, \ell_{top}, \ell_{bot})\right) = \\ &\mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) \geq \frac{1}{3}. \end{aligned}$$

It follows from (5.6) that for $N \geq \tilde{N}_2$ we have

$$\epsilon/12 > \mathbb{P}(F(M)) \geq \sum_{\substack{(a,b,s,\ell_{top},\ell_{bot}) \in D(M), \\ \mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot})) > 0}} \mathbb{P}\left(F(M) \cap E(a, b, s, \ell_{top}, \ell_{bot})\right) =$$

$$(5.7) \quad \sum_{\substack{(a,b,s,\ell_{top},\ell_{bot}) \in D(M), \\ \mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot})) > 0}} \mathbb{P}\left(L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} | E(a, b, s, \ell_{top}, \ell_{bot})\right) \mathbb{P}\left(E(a, b, s, \ell_{top}, \ell_{bot})\right) \geq$$

$$\sum_{\substack{(a,b,s,\ell_{top},\ell_{bot}) \in D(M), \\ \mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot})) > 0}} \frac{1}{3} \cdot \mathbb{P}\left(E(a, b, s, \ell_{top}, \ell_{bot})\right) = \frac{1}{3} \cdot \mathbb{P}(G(M) \setminus E(M)),$$

where in the last equality we used (5.2). From (5.1) and (5.7) we have for $N \geq N_2 = \max(\tilde{N}_1, \tilde{N}_2)$ that

$$\mathbb{P}(G(M)) \leq \mathbb{P}(E(M)) + \mathbb{P}(G(M) \setminus E(M)) < \epsilon/4 + \epsilon/4,$$

which proves (5.3).

Step 3. In this step we prove (5.4) and in the sequel we let $(a, b, s, \ell_{top}, \ell_{bot}) \in D(M)$ be such that $\mathbb{P}(E(a, b, s, \ell_{top}, \ell_{bot})) > 0$. We remark that the condition $\mathbb{P}(E(a, b, s, \ell_{top}, \ell_{bot})) > 0$ implies that $\Omega_{avoid}(-s_4, s, a, b, \infty, \ell_{bot})$ is not empty. By Lemma 3.2 we know that

$$\mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) \geq \mathbb{P}_{Ber}^{-s_4,s,a,b}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right),$$

and so it suffices to show that

$$(5.8) \qquad \mathbb{P}_{Ber}^{-s_4,s,a,b}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) \geq \frac{1}{3}.$$

One directly observes that

$$(5.9) \qquad \begin{aligned} &\mathbb{P}_{Ber}^{-s_4,s,a,b}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) = \mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\left(\ell(s_4 - s_3) + a \geq -ps_3 + MN^{\alpha/2}\right) \geq \\ &\mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\left(\ell(s_4 - s_3) \geq p(s_4 - s_3) + 2MN^{\alpha/2}\right), \end{aligned}$$

where the inequality follows from the assumption in (2) that $a + ps_4 \geq -MN^{\alpha/2}$. Moreover, since $b - ps \geq (6r + 22)(2r + 10)^{1/2}(M + 1)N^{\alpha/2}$ and $a + ps_4 \leq MN^{\alpha/2}$, we have

$$b - a \geq p(s + s_4) + (6r + 21)(2r + 10)^{1/2}(M + 1)N^{\alpha/2} \geq p(s + s_4) + (6r + 21)(M + 1)(s + s_4)^{1/2}.$$

The second inequality follows since $s + s_4 \leq 2s_4 \leq (2r + 10)N^{\alpha}$.

It follows from Lemma 3.8 with $M_1 = 0$, $M_2 = (6r + 21)(M + 1)$ that for sufficiently large N

$$(5.10) \qquad \mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\left(\ell(s_4 - s_3) \geq \frac{s_4 - s_3}{s + s_4}[p(s + s_4) + M_2(s + s_4)^{1/2}] - (s + s_4)^{1/4}\right) \geq 1/3.$$

Note that $\frac{s_4 - s_3}{s + s_4} \geq \frac{N^{\alpha}}{(2r+10)N^{\alpha}} = \frac{1}{2r+10}$ and so for all $N \in \mathbb{N}$ we have

$$(5.11) \qquad \begin{aligned} &\frac{s_4 - s_3}{s + s_4}[p(s + s_4) + M_2(s + s_4)^{1/2}] - (s + s_4)^{1/4} \geq \\ &p(s_4 - s_3) + \frac{(6r + 21)(M + 1)(s + s_4)^{1/2}}{2r + 10} - (s + s_4)^{1/4} \geq p(s_4 - s_3) + 2MN^{\alpha/2}. \end{aligned}$$

Combining (5.9), (5.10) and (5.11) we conclude that we can find $\tilde{N}_2 \in \mathbb{N}$ such that if $N \geq \tilde{N}_2$ we have (5.8). This suffices for the proof.

5.2. Proof of Lemma 4.3.

We begin by proving the following important lemma, which allows us to prevent the curve L_{k-1}^N from falling too low on a large enough interval.

Lemma 5.1. *Fix $p \in (0, 1)$, $k \in \mathbb{N}$, and $\alpha, \lambda > 0$. Suppose that $\mathfrak{L}^N = (L_1^N, \dots, L_k^N)$ is a (α, p, λ)-good sequence of $[\![1, k]\!]$-indexed Bernoulli line ensembles. Then for any $r, \epsilon > 0$, there exist $R > 0$ and $N_5 \in \mathbb{N}$ such that for all $N \geq N_5$,*

$$\mathbb{P}\left(\max_{x \in [r, R]}\left(L_{k-1}^N(xN^{\alpha}) - pxN^{\alpha}\right) \leq -(\lambda R^2 + \phi(\epsilon/16))N^{\alpha/2}\right) < \epsilon.$$

The same statement holds if $[r, R]$ is replaced with $[-R, -r]$.

Remark 5.2. The key to this lemma is the parabolic shift implicit in the definition of an (α, p, λ)-good sequence. This requires the deviation of the top curve from the line of slope p to appear roughly parabolic. Using monotone coupling, we separate the curves of the ensemble so that L_1^N is nearly independent of the other curves. Then we would expect the value of L_1^N at the midpoint of r and R to be close to the midpoint of the straight line segment connecting two points of the parabola. But the parabola is convex, so for large enough R this violates the one-point tightness assumpion at $(R + r)/2$.

Proof. For clarity we split this proof into five steps. The proof of the analogous statement with $[r, R]$ replaced by $[-R, -r]$ is very similar. In particular, only trivial modifications to Steps 4 and 5 are necessary. In the first step we introduce notation that will be used in the proof of the lemma in the subsequent steps.

Step 1. Fix $r > 0$. Note that for any $R > r$,

$$\max_{r \leq x \leq R}\left(L_{k-1}^N(xN^{\alpha}) - pxN^{\alpha}\right) \geq \max_{\lceil r \rceil \leq x \leq R}\left(L_{k-1}^N(xN^{\alpha}) - pxN^{\alpha}\right).$$

Thus by replacing r with $\lceil r \rceil$, we can assume that $r \in \mathbb{Z}$. Define constants

$$(5.12) \qquad C = \sqrt{8p(1 - p)\log\frac{3}{1 - (11/12)^{1/(k-2)}}},$$

$$(5.13) \qquad R_0 = \frac{2}{\lambda}\left(Ck + \sqrt{C^2 k^2 + 2\lambda\phi(\epsilon/16)}\right) + r + 1.$$

Note that $R_0 \geq r$. We define $R = \lceil R_0 \rceil + \mathbf{1}_{\lceil R_0 \rceil + r \text{ odd}}$, so that $R \geq R_0$ and the midpoint $(R+r)/2$ are integers. In the following, we always assume N is large enough so that $\psi(N) > R$, hence L_i^N are defined at RN^α for $1 \leq i \leq k$. We may do so by the second condition in the definition of an (α, p, λ)-good sequence (see Definition 2.24). Define events

(5.14)
$$A = \left\{ L_1^N \left(\frac{R+r}{2} N^\alpha \right) - pN^\alpha \frac{R+r}{2} + \lambda \left(\frac{R+r}{2} \right)^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2} \right\},$$
$$B = \left\{ \max_{x \in [r,R]} \left(L_{k-1}^N(xN^\alpha) - pxN^\alpha \right) \leq -(\lambda R^2 + \phi(\epsilon/16))N^{\alpha/2} \right\}.$$

We will argue in the subsequent steps that we can find $N_5 \in \mathbb{N}$ so that for all $N \geq N_5$,

(5.15)
$$\mathbb{P}(B) < \epsilon,$$

which will prove the lemma. Let F denote the subset of B for which the inequalities

(5.16)
$$N^{-\alpha/2} \left| L_1^N(rN^\alpha) - prN^\alpha + \lambda r^2 N^{\alpha/2} \right| < \phi(\epsilon/16),$$
$$N^{-\alpha/2} \left| L_1^N(RN^\alpha) - pRN^\alpha + \lambda R^2 N^{\alpha/2} \right| < \phi(\epsilon/16)$$

hold. In the remainder of this step we show that F can be written as a *countable disjoint union* of the form

(5.17)
$$F = \bigsqcup_{(\vec{x}, \vec{y}, \ell_{bot}) \in D} \left(F \cap E(\vec{x}, \vec{y}, \ell_{bot}) \right),$$

with the sets $E(\vec{x}, \vec{y}, \ell_{bot})$ and D defined in the following.

For $\vec{x}, \vec{y} \in \mathfrak{W}_{k-2}$, $z_1, z_2 \in \mathbb{Z}$, and $\ell_{bot} \in \Omega(rN^\alpha, RN^\alpha, z_1, z_2)$, let $E(\vec{x}, \vec{y}, \ell_{bot})$ denote the event that $L_i^N(rN^\alpha) = x_i$ and $L_i^N(RN^\alpha) = y_i$ for $1 \leq i \leq k-2$, and L_{k-1}^N agrees with ℓ_{bot} on $[rN^\alpha, RN^\alpha]$. Let D denote the set of triples $(\vec{x}, \vec{y}, \ell_{bot})$ satisfying

(1) $0 \leq y_i - x_i \leq (R-r)N^\alpha$ for $1 \leq i \leq k-2$,
(2) $N^{-\alpha/2}|x_1 - prN^\alpha + \lambda r^2 N^{\alpha/2}| < \phi(\epsilon/16)$ and $N^{-\alpha/2}|y_1 - pRN^\alpha + \lambda R^2 N^{\alpha/2}| < \phi(\epsilon/16)$,
(3) $z_1 \leq x_{k-2}$, $z_2 \leq y_{k-2}$, and $\ell_{bot} \in \Omega(rN^\alpha, RN^\alpha, z_1, z_2)$.

Then D is countable, since there are finitely many integers x_i, y_i satisfying (1), countably many $z_1, z_2 \in \mathbb{Z}$ satisfying (3), and at most finitely many ℓ_{bot} satisfying (3) for each choice of z_1, z_2. Moreover, the $E(\vec{x}, \vec{y}, \ell_{bot})$ are pairwise disjoint, and the three conditions together show that $F = \bigcup_{(\vec{x}, \vec{y}, \ell_{bot}) \in D} (F \cap E(\vec{x}, \vec{y}, \ell_{bot}))$. This proves (5.17).

Step 2. We will argue in the following steps that we can find $\tilde{N}_0$ so that for $N \geq \tilde{N}_0$ we have

(5.18)
$$\mathbb{P}(A \mid F \cap E(\vec{x}, \vec{y}, \ell_{bot})) > 1/4$$

uniformly in $\vec{x}, \vec{y}, \ell_{bot}$. In this step, we prove (5.15) assuming this fact.

It follows from (5.17) and (5.18) that for $N \geq \tilde{N}_0$,

$$\mathbb{P}(A \mid F) = \sum_{(\vec{x}, \vec{y}, \ell_{bot}) \in D} \frac{\mathbb{P}(A \mid F \cap E(\vec{x}, \vec{y}, \ell_{bot}))\mathbb{P}(F \cap E(\vec{x}, \vec{y}, \ell_{bot}))}{\mathbb{P}(F)} \geq$$
$$\frac{1}{4} \cdot \frac{\sum_{(\vec{x}, \vec{y}, \ell_{bot}) \in D} \mathbb{P}(F \cap E(\vec{x}, \vec{y}, \ell_{bot}))}{\mathbb{P}(F)} = \frac{1}{4}.$$

From the third condition in the definition of an (α, p, λ)-good sequence, Definition 2.24, we can find $\tilde{N}_1$ so that $\mathbb{P}(A) < \epsilon/8$ for $N \geq \tilde{N}_1$. Hence

$$\mathbb{P}(F) = \frac{\mathbb{P}(A \cap F)}{\mathbb{P}(A \mid F)} \leq 4\mathbb{P}(A) < \epsilon/2.$$

Lastly, we can find $\tilde{N}_2$ so that for $N \geq \tilde{N}_2$, the two inequalities in (5.16) hold with probability $> 1 - \epsilon/2$. We conclude that

$$\mathbb{P}(B) \leq \mathbb{P}(F) + \epsilon/2 \leq \epsilon$$

for $N \geq N_5 = \max(\tilde{N}_0, \tilde{N}_1, \tilde{N}_2)$.

Step 3. We will now prove (5.18), assuming results from Steps 3 and 4 below. We first note that by Lemma 3.1, if we raise the curves at the endpoints rN^α, RN^α, then the probability of the event A will decrease. In particular, let us write

(5.19) $$\bar{x} = \lceil prN^\alpha - (\lambda r^2 - \phi(\epsilon/16))N^{\alpha/2} \rceil, \quad \bar{y} = \lceil pRN^\alpha - (\lambda R^2 - \phi(\epsilon/16))N^{\alpha/2} \rceil,$$

and $T = (R - r)N^\alpha$, and define $\vec{x}', \vec{y}' \in \mathfrak{W}_{k-2}$ by

$$x_i' = \bar{x} + (k - 1 - i)\lceil C\sqrt{T} \rceil, \quad y_i' = \bar{y} + (k - 1 - i)\lceil C\sqrt{T} \rceil.$$

Note that $x_i' \geq \bar{x} \geq x_1 \geq x_i$ for each i by condition (2) above. Furthermore, $x_i' - x_{i+1}' \geq C\sqrt{T}$. The same observations hold for y_i'. Using the Schur Gibbs property (Definition 2.17) and Lemma 3.1, we have

$$\mathbb{P}(A \mid F \cap E(\vec{x}, \vec{y}, \ell_{bot})) = \mathbb{P}_{avoid,Ber}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}, \infty, \ell_{bot}}(A \mid F) \geq \mathbb{P}_{avoid,Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}', \infty, \ell_{bot}}(A \mid F) \geq$$

(5.20) $$\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}\left(\tilde{A} \cap \{L_1 \geq \cdots \geq L_{k-2} \geq \ell_{bot}\} \mid \tilde{F}\right) \geq$$

$$\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}\left(\tilde{A} \mid \tilde{F}\right) - \left(1 - \mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}\left(L_1 \geq \cdots \geq L_{k-2} \geq \ell_{bot} \mid \tilde{F}\right)\right).$$

Here, we have written $(L_1, \ldots, L_{k-2})$ for the line ensemble with law $\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}$, and we define the events $\tilde{A}, \tilde{F}$ in the same way as A, F in Step 1 but replacing L_1^N, L_{k-1}^N with L_1, ℓ_{bot}. In Step 4, we will find $\tilde{N}_0$ so that for $N \geq \tilde{N}_0$,

(5.21) $$\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}\left(\tilde{A} \mid \tilde{F}\right) \geq 1/3,$$

(5.22) $$\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}\left(L_1 \geq \cdots \geq L_{k-2} \geq \ell_{bot} \mid \tilde{F}\right) \geq 11/12,$$

independent of $\vec{x}, \vec{y}, \ell_{bot}$. Then (5.20) implies $\mathbb{P}(A \mid F \cap E(\vec{x}, \vec{y}, \ell_{bot})) \geq 1/3 - 1/12 = 1/4$ for $N \geq \tilde{N}_0$, proving (5.18).

Step 4. In this step we prove (5.21). A very similar argument proves the same inequality if $[r, R]$ is replaced by $[-R, -r]$ in the definition of B.

Let ℓ be the random variable with law $\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, x_1', y_1'}$, and define $\tilde{A}_\ell$ in the same way as A in (5.14) but with L_1^N replaced by ℓ. Since $L_1, \ldots, L_{k-2}, \ell_{bot}$ are independent under $\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}$, and the event $\tilde{A}$ in (5.20) depends only on L_1, we have

(5.23) $$\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}\left(\tilde{A} \mid \tilde{F}\right) = \mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, x_1', y_1'}\left(\tilde{A}_\ell\right).$$

With $\bar{x}, \bar{y}$ as in (5.19), write $\bar{z} = \bar{y} - \bar{x}$. We have

(5.24)

$$\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, x_1', y_1'}\left(\tilde{A}_\ell\right) = \mathbb{P}_{Ber}^{0, T, x_1', y_1'}\left(\ell(T/2) - pN^\alpha \frac{R+r}{2} + \lambda\left(\frac{R+r}{2}\right)^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) =$$

$$\mathbb{P}_{Ber}^{0, T, \bar{x}, \bar{y}}\left(\ell(T/2) - pN^\alpha \frac{R+r}{2} + \lambda\left(\frac{R+r}{2}\right)^2 N^{\alpha/2} < -\left(\phi(\epsilon/16) + (k-2)\lceil C\sqrt{R-r} \rceil\right)N^{\alpha/2}\right) \geq$$

$$\mathbb{P}_{Ber}^{0, T, \bar{x}, \bar{y}}\left(\ell(T/2) - \frac{\bar{x}+\bar{y}}{2} < \left(\lambda\left(\frac{R^2+r^2}{2}\right) - \lambda\left(\frac{R+r}{2}\right)^2 - Ck\sqrt{R-r} - 2\phi(\epsilon/16)\right)N^{\alpha/2}\right) =$$

$$\mathbb{P}_{Ber}^{0, T, 0, \bar{z}}\left(\ell(T/2) - \bar{z}/2 < \gamma\sqrt{T}\right).$$

The inequality in the second last line follows from the definitions of $\bar{x}, \bar{y}$, and in the last line, we have shifted the domain and written γ for the constant multiplying $N^{\alpha/2}$ in the previous line, divided by $\sqrt{R-r}$. We claim that our choice of R_0 in (5.13) and the fact that $R \geq R_0$ imply that $\gamma > 0$. To see this observe that

$$\frac{R^2 + r^2}{2} - \left(\frac{R+r}{2}\right)^2 = \frac{R^2 + r^2 - 2rR}{4} = \frac{(R-r)^2}{4}$$

for fixed r, and thus since $R - r \geq R_0 - r > 1$,

$$\gamma\sqrt{R-r} = \frac{\lambda}{4}(R-r)^2 - Ck\sqrt{R-r} - 2\phi(\epsilon/16) > \frac{\lambda}{4}(R_0-r)^2 - Ck(R_0-r) - 2\phi(\epsilon/16) \geq 0.$$

The last inequality is easily checked by computing the zeros of the quadratic function in R_0.

Now by Theorem 3.3, there is a probability measure $\tilde{\mathbb{P}}$ supporting a random variable $\ell^{(T,\bar{z})}$ with law $\mathbb{P}_{Ber}^{0,T,0,\bar{z}}$ and a Brownian bridge B^σ with variance $\sigma^2 = p(1-p)$. Then the probability in the last line of (5.24) is equal to

$$\tilde{\mathbb{P}}\left(\ell^{(T,\bar{z})}(T/2) - \bar{z}/2 < \gamma\sqrt{T}\right) = \tilde{\mathbb{P}}\left(\left[\ell^{(T,\bar{z})}(T/2) - \bar{z}/2 - \sqrt{T}B_{1/2}^\sigma\right] + \sqrt{T}B_{1/2}^\sigma < \gamma\sqrt{T}\right) \geq$$

$$\tilde{\mathbb{P}}\left(\sqrt{T}B_{1/2}^\sigma < 0 \quad \text{and} \quad \Delta(T,\bar{z}) < \gamma\sqrt{T}\right) \geq \frac{1}{2} - \tilde{\mathbb{P}}\left(\Delta(T,\bar{z}) \geq \gamma\sqrt{T}\right).$$

Here, $\Delta(T,\bar{z})$ is as defined in (3.2). Observe that

(5.25)
$$\frac{|\bar{z} - pT|^2}{T} \leq \frac{(\lambda(R^2 - r^2)N^{\alpha/2} + 1)^2}{(R-r)N^\alpha} \leq 4\lambda^2(R+r)^2(R-r).$$

Thus Corollary 3.5 gives us $\tilde{N}_0$ such that $\mathbb{P}(\Delta(T,\bar{z}) \geq \gamma\sqrt{T}) < 1/6$ for $N \geq \tilde{N}_0$. This gives a lower bound on $\mathbb{P}_{Ber}^{rN^\alpha,RN^\alpha,x_1',y_1'}(A)$ of $1/2 - 1/6 = 1/3$ as desired.

Step 5. In this last step, we prove (5.22) for $N \geq \tilde{N}_0$, after possibly enlarging $\tilde{N}_0$ from Step 4. The argument with $[-R, -r]$ in place of $[r, R]$ is essentially the same.

Note that on the event $\tilde{F}$, ℓ_{bot} lies uniformly below the line segment $\bar{\ell}_{bot}$ on $[rN^\alpha, RN^\alpha]$ connecting the points $\bar{x}$ and $\bar{y}$ defined in (5.19). This segment in turn lies uniformly at a distance of at least $C\sqrt{T}$ below the segment connecting $x_{k-2}' \geq \bar{x} + C\sqrt{T}$ and $y_{k-2}' \geq \bar{y} + C\sqrt{T}$. Moreover, the endpoints of all adjacent curves are at least $C\sqrt{T}$ apart. Since $L_1, \ldots, L_{k-2}, \bar{\ell}_{bot}$ are independent of $\tilde{F}$ under $\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}$, we have

$$\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}\left(L_1 \geq \cdots \geq L_{k-2} \geq \ell_{bot} \,\big|\, \tilde{F}\right) \geq \mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}\left(L_1 \geq \cdots \geq L_{k-2} \geq \bar{\ell}_{bot}\right).$$

In view of (5.25), we see from Lemma 3.18 that we can enlarge $\tilde{N}_0$ depending on λ, r, R, C so that for $N \geq \tilde{N}_0$, the probability on right is bounded below by

$$\left(1 - 3e^{-C^2/8p(1-p)}\right)^{k-2}.$$

Our choice of C in (5.12) implies that this quantity is at least $11/12$, proving (5.22). $\qquad\square$

We now prove Lemma 4.3. We exploit Lemma 5.1 in order to find two points on intervals $[r, R]$ and $[-R, -r]$ where $L_{k-1}^N(xN^\alpha)$ cannot be too low. After separating the curves in order to treat L_{k-1}^N as a free Bernoulli bridge as in the previous argument, we employ Lemma 3.10 to bound the deviation of L_{k-1}^N below the line of slope p.

Proof. We split the proof into four steps for clarity. In the first step we introduce notation to be used in the latter steps.

Step 1. Define events

$$A = \left\{ \inf_{s \in [-t_3, t_3]} \left(L_{k-1}^N(s) - ps \right) \leq -R_2 N^{\alpha/2} \right\},$$

$$B = \left\{ \max_{x \in [r+3, R] \cup [-R, -r-3]} \left(L_{k-1}^N(xN^\alpha) - pxN^\alpha \right) > -MN^{\alpha/2} \right\}.$$

Here, we define M, R via Lemma 5.1, taking R large enough so that with $M = \lambda R^2 + \phi(\epsilon/64)$, we have

(5.26) $$\mathbb{P}(B) \geq 1 - \epsilon/2$$

for N larger than some $\tilde{N}_0$. Lastly, define the constant

(5.27) $$C = \sqrt{16p(1-p) \log \frac{3}{1 - 2^{-1/(k-1)}}}.$$

In the following steps, we will find $N_4 \in \mathbb{N}$ and $R_2 > 0$ depending on M, C, k, r, ϵ so that

(5.28) $$\mathbb{P}(A) < \epsilon$$

for $N \geq N_4$, which will prove the lemma. In particular, we specify R_2 in Step 3.

In the remainder of this step, we prove that B can be written as a *countable disjoint union* of the form

(5.29) $$B = \bigsqcup_{(a,b,\vec{x},\vec{y},\ell_{bot}) \in D} E(a, b, \vec{x}, \vec{y}, \ell_{bot}),$$

where the sets D and $E(a, b, \vec{x}, \vec{y}, \ell_{bot})$ are defined below.

For $0 < a, b \in \mathbb{Z}$, $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$, $z_1, z_2 \in \mathbb{Z}$, and $\ell_{bot} \in \Omega(-a, b, z_1, z_2)$, we define $E(a, b, \vec{x}, \vec{y}, \ell_{bot})$ to be the event that $L_i^N(-a) = x_i$ and $L_i^N(b) = y_i$ for $1 \leq i \leq k-1$, and L_k^N agrees with ℓ_{bot} on $[-a, b]$. Let D be the collection of tuples $(a, b, \vec{x}, \vec{y}, \ell_{bot})$ satisfying

(1) $a, b \in [(r+3)N^\alpha, RN^\alpha]$.
(2) $0 \leq y_i - x_i \leq b + a$, $x_{k-1} + pa > -MN^{\alpha/2}$, and $y_{k-1} - pb > -MN^{\alpha/2}$.
(3) If $c, d \in \mathbb{Z}$, $c > a$, and $d > b$, then $L_{k-1}^N(-c) + pc \leq -MN^{\alpha/2}$ and $L_{k-1}^N(d) - pd \leq -MN^{\alpha/2}$.
(4) $z_1 \leq x_{k-1}$, $z_2 \leq y_{k-1}$, and $\ell_{bot} \in \Omega(-a, b, z_1, z_2)$.

Since there are finitely many integers a, b satisfying (1), countably many x_i, y_i and z_1, z_2 satisfying (2) and (4), and finitely many L_i^N, ℓ_{bot} satisfying (3) and (4) given a, b, x_i, y_i, z_1, z_2, we see that D is countable. The four conditions together imply that $B = \bigcup_{(a,b,\vec{x},\vec{y},\ell_{bot}) \in D} E(a, b, \vec{x}, \vec{y}, \ell_{bot})$.

To see that the $E(a, b, \vec{x}, \vec{y}, \ell_{bot})$ are pairwise disjoint for $(a, b, \vec{x}, \vec{y}, \ell_{bot}) \in D$, observe that on the intersection of $E(a, b, \vec{x}, \vec{y}, \ell_{bot})$ and $E(\tilde{a}, \tilde{b}, \tilde{\vec{x}}, \tilde{\vec{y}}, \tilde{\ell}_{bot})$, conditions (2) and (3) imply that $a = \tilde{a}$ and $b = \tilde{b}$. We then conclude from the definition of $E(a, b, \vec{x}, \vec{y}, \ell_{bot})$ that $\vec{x} = \tilde{\vec{x}}$, $\vec{y} = \tilde{\vec{y}}$, and $\ell_{bot} = \tilde{\ell}_{bot}$. This proves (5.29).

Step 2. We will prove in the steps below that we can find $\tilde{N}_1$ and R_2 so that for $N \geq \tilde{N}_1$,

(5.30) $$\mathbb{P}(A \mid E(a, b, \vec{x}, \vec{y}, \ell_{bot})) < \epsilon/2$$

uniformly in $a, b, \vec{x}, \vec{y}, \ell_{bot}$. In this step, we prove (5.28) assuming this fact.

By (5.30) and (5.29), we obtain

$$\mathbb{P}(A \cap B) = \sum_{(a,b,\vec{x},\vec{y},\ell_{bot}) \in D} \mathbb{P}(A \mid E(a, b, \vec{x}, \vec{y}, \ell_{bot})) \mathbb{P}(E(a, b, \vec{x}, \vec{y}, \ell_{bot})) \leq$$

$$\frac{\epsilon}{2} \sum_{(a,b,\vec{x},\vec{y},\ell_{bot}) \in D} \mathbb{P}(E(a, b, \vec{x}, \vec{y}, \ell_{bot})) \leq \frac{\epsilon}{2}$$

for $N \geq \tilde{N}_1$. It follows from (5.26) that

$$\mathbb{P}(A) \leq \mathbb{P}(A \cap B) + \epsilon/2 < \epsilon$$

for $N \geq N_4 = \max(\tilde{N}_0, \tilde{N}_1)$.

Step 3. We next prove (5.30), assuming results from Steps 3 below. Define $\vec{x}', \vec{y}' \in \mathfrak{W}_{k-1}$ by

$$x_i' = \lfloor -pa - MN^{\alpha/2} \rfloor - (i-1)\lceil CN^{\alpha/2} \rceil,$$
$$y_i' = \lfloor pb - MN^{\alpha/2} \rfloor - (i-1)\lceil CN^{\alpha/2} \rceil.$$

Observe that by condition (2) above, $x_i' \leq -pa - MN^{\alpha/2} \leq x_{k-1} \leq x_i$, and similarly for $\vec{y}$. It follows from the Schur Gibbs property (see Definition 2.17) and Lemmas 3.1 and 3.2 that

(5.31)
$$\mathbb{P}(A \,|\, E(a, b, \vec{x}, \vec{y}, \ell_{bot})) \leq \mathbb{P}\left(\inf_{s \in [-a,b]} \left(L_{k-1}^N(s) - ps \right) \leq -R_2 N^{\alpha/2} \,\Big|\, E(a, b, \vec{x}, \vec{y}, \ell_{bot}) \right) =$$

$$\mathbb{P}_{avoid,Ber}^{-a,b,\vec{x},\vec{y},\infty,\ell_{bot}} \left(\inf_{s \in [-a,b]} (Q_{k-1}(s) - ps) \leq -R_2 N^{\alpha/2} \right) \leq$$

$$\mathbb{P}_{avoid,Ber}^{-a,b,\vec{x}',\vec{y}'} \left(\inf_{s \in [0,a+b]} (Q_{k-1}(s) - ps) \leq -R_2 N^{\alpha/2} \right) \leq$$

$$\frac{\mathbb{P}_{Ber}^{-a,b,\vec{x}',\vec{y}'} \left(\inf_{s \in [0,a+b]} \left(L_{k-1}(s) - ps \right) \leq -R_2 N^{\alpha/2} \right)}{\mathbb{P}_{Ber}^{-a,b,\vec{x}',\vec{y}'} (L_1 \geq \cdots \geq L_{k-1})}.$$

In Step 3 below, we will find $\tilde{N}_1$ and $R_2 > 0$ such that the numerator and denominator in the last line are $< \epsilon/4$ and $\geq 1/2$ for $N \geq \tilde{N}_1$. It then follows that the first expression in (5.31) is bounded above by $\epsilon/2$, proving (5.30).

Step 4. In this step, we fix $R_2 > 0$ and $\tilde{N}_1$ so that for $N \geq \tilde{N}_1$, we have

(5.32)
$$\mathbb{P}_{Ber}^{-a,b,\vec{x}',\vec{y}'} \left(\inf_{s \in [0,a+b]} \left(L_{k-1}(s) - ps \right) \leq -R_2 N^{\alpha/2} \right) < \epsilon/4,$$

(5.33)
$$\mathbb{P}_{Ber}^{-a,b,\vec{x}',\vec{y}'} (L_1 \geq \cdots \geq L_{k-1}) \geq 1/2.$$

Let us first prove (5.32). Writing $\vec{z} = \vec{y}' - \vec{x}'$, and using the fact that $L_1, \ldots, L_{k-1}$ are independent under $\mathbb{P}_{Ber}^{-a,b,\vec{x}',\vec{y}'}$, we can rewrite the left hand side of (5.32) as

(5.34)
$$\mathbb{P}_{Ber}^{0,a+b,x_{k-1}',y_{k-1}'} \left(\inf_{s \in [0,a+b]} \left(\ell(s) - p(s-a) \right) \leq -R_2 N^{\alpha/2} \right) =$$

$$\mathbb{P}_{Ber}^{0,a+b,0,z_{k-1}} \left(\inf_{s \in [0,a+b]} \left(\ell(s) - ps + pa - \lceil pa + MN^{\alpha/2} \rceil - (k-2)\lceil CN^{\alpha/2} \rceil \right) \leq -R_2 N^{\alpha/2} \right) \leq$$

$$\mathbb{P}_{Ber}^{0,a+b,0,z_{k-1}} \left(\inf_{s \in [0,a+b]} \left(\ell(s) - ps \right) \leq -(R_2 - M - Ck)N^{\alpha/2} \right).$$

In the first line, we have implicitly shifted the domain of the paths from $[-a, b]$ to $[0, a+b]$. Since $z_k \geq p(a+b)$, Lemma 3.10 allows us to find $R_2 > 0$ depending on M, C, k, r, ϵ so that this probability is $< \epsilon/4$ as long as $a + b$ is larger than some constant W_1. Since $a + b \geq 2rN^\alpha$, so it suffices to take $\tilde{N}_1 \geq (W_1/2r)^{1/\alpha}$. Thus we obtain (5.32), *independent* of $a, b, \vec{x}, \vec{y}$.

Lastly, we prove that we can enlarge $\tilde{N}_1$ so that (5.33) holds for $N \geq \tilde{N}_1$. Write $a = a'N^\alpha, b = b'N^\alpha$, and $T = a + b = (a' + b')N^\alpha$. Also let $C' = C/\sqrt{a' + b'}$ with C as in (5.27), so that

$x_i' - x_{i+1}' \geq CN^{\alpha/2} = C'\sqrt{T}$ and likewise for y_i'. Note that $|z_{k-1} - pT| \leq 1$. It follows from Lemma 3.18, applied with $\ell_{bot} = -\infty$ and C' in place of C, that for T larger than some T_0,

(5.35)
$$\mathbb{P}_{Ber}^{-a,b,\vec{x}',\vec{y}'}(L_1 \geq \cdots \geq L_{k-1}) = \mathbb{P}_{Ber}^{0,a+b,\vec{x}',\vec{y}'}(L_1 \geq \cdots \geq L_{k-1}) \geq$$
$$\left(1 - 3e^{-(C')^2/8p(1-p)}\right)^{k-1} \geq \left(1 - 3e^{-C^2/16p(1-p)R}\right)^{k-1}.$$

Here, we used the fact that $a' + b' \leq 2R$, hence $C' \geq C/\sqrt{2R}$. The constant T_0 depends in particular on C', hence possibly on $a + b$. Referring to the proofs of Lemmas 3.18 and 3.5, we see that the dependency of T_0 on C' amounts to requiring that $e^{-C'\sqrt{T_0}}$ be sufficiently small. But $C' \geq C/\sqrt{2R}$, so for this it suffices to choose T_0 depending on C and R. Moreover, $T \geq 2rN^{\alpha}$, so as long as $\tilde{N}_1 \geq (T_0/2r)^{1/\alpha}$, we have the bound in (5.35) for $N \geq \tilde{N}_1$ independent $a, b, \vec{x}, \vec{y}$. Our choice of C in (5.27) ensures that the expression on the right in (5.35) is at least $1/2$, proving (5.33). $\qquad \square$

6. LOWER BOUNDS ON THE ACCEPTANCE PROBABILITY

6.1. **Proof of Lemma 4.4.** Throughout this section we assume the same notation as in Lemma 4.4, i.e., we assume that we have fixed $k \in \mathbb{N}$, $p \in (0,1)$, $M_1, M_2 > 0$, $\ell_{bot} : [\![-t_3, t_3]\!] \to \mathbb{R} \cup \{-\infty\}$, and $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$ such that $|\Omega_{avoid}(-t_3, t_3, \vec{x}, \vec{y}, \infty, \ell_{bot})| \geq 1$. We also assume that

(1) $\sup_{s \in [-t_3, t_3]} \left[\ell_{bot}(s) - ps\right] \leq M_2(2t_3)^{1/2}$,
(2) $-pt_3 + M_1(2t_3)^{1/2} \geq x_1 \geq x_{k-1} \geq \max\left(\ell_{bot}(-t_3), -pt_3 - M_1(2t_3)^{1/2}\right)$,
(3) $pt_3 + M_1(2t_3)^{1/2} \geq y_1 \geq y_{k-1} \geq \max\left(\ell_{bot}(t_3), pt_3 - M_1(2t_3)^{1/2}\right)$.

Definition 6.1. We write $S = [\![-t_3, -t_1]\!] \cup [\![t_1, t_3]\!]$, and we let $\tilde{\ell}_{bot}$ denote the path which is equal to ℓ_{bot} on S and equal to $-\infty$ on $[\![-t_1 + 1, t_1 - 1]\!]$. Additionally, define the subset $\Omega_{a,S}(\cdot) \subseteq \Omega(\cdot)$ by

(6.1) $$\Omega_{a,S}(\cdot) := \{\mathfrak{B} \in \Omega(\cdot) \mid \forall (i,s) \in [\![1, k-2]\!] \times S, B_i(s) \geq B_{i+1}(s)\}$$

We also denote by $\mathfrak{Q} = (Q_1, \ldots, Q_{k-1})$ and $\tilde{\mathfrak{Q}} = (\tilde{Q}_1, \ldots, \tilde{Q}_{k-1})$ the $[\![1, k-1]\!]$-indexed line ensembles which are uniformly distributed on $\Omega_{avoid}(-t_3, t_3, \vec{x}, \vec{y}, \ell_{bot})$ and $\Omega_{a,S}(-t_3, t_3, \vec{x}, \vec{y}, \tilde{\ell}_{bot})$ respectively, and we let $\mathbb{P}_{\mathfrak{Q}}$ and $\mathbb{P}_{\tilde{\mathfrak{Q}}}$ denote these uniform measures.

In other words, $\tilde{\mathfrak{Q}}$ has the law of $k - 1$ independent Bernoulli bridges that have been conditioned on not-crossing each other on the set S and also staying above the graph of ℓ_{bot} but only on the intervals $[\![-t_3, -t_1]\!]$ and $[\![t_1, t_3]\!]$. The latter restriction means that the lines are allowed to cross on $[\![-t_1 + 1, t_1 - 1]\!]$, and $\tilde{Q}_{k-1}$ is allowed to dip below ℓ_{bot} on $[\![-t_1 + 1, t_1 - 1]\!]$ as well. Essentially, the line ensemble is free on $[\![-t_1 + 1, t_1 - 1]\!]$, and avoiding on S.

Lemma 6.2. *There exists $N_5 \in \mathbb{N}$ such that for $N \geq N_5$,*

(6.2) $$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(Z\left(-t_1, t_1, \tilde{\mathfrak{Q}}(-t_1), \tilde{\mathfrak{Q}}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right) \geq g\right) \geq h,$$

where the constants g and h are as in Lemma 4.4.

We will prove Lemma 6.2 in Section 6.2. In the remainder of this section, we give the proof of Lemma 4.4. The proof begins by evaluating the Radon-Nidokum derivative between $\mathbb{P}_{\mathfrak{Q}'}$ and $\mathbb{P}_{\tilde{\mathfrak{Q}'}}$. We then use this Radon-Nikodym derivative to transition between $\tilde{\mathfrak{Q}}$ in Lemma 6.2 which ignores ℓ_{bot} on $[\![-(t_1 - 1), t_1 - 1]\!]$ and $\mathfrak{Q}$ in Lemma 4.4 which avoids ℓ_{bot} everywhere. Then we perform some calculations to achieve the desired statement in Equation (4.2).

Lemma 4.4. Let us denote by $\mathbb{P}_{\mathfrak{Q}'}$ and $\mathbb{P}_{\tilde{\mathfrak{Q}'}}$ the measures on $[\![1, k-1]\!]$-indexed Bernoulli line ensembles $\mathfrak{Q}', \tilde{\mathfrak{Q}}'$ on the set S in Definition 6.1 induced by the restrictions of the measures $\mathbb{P}_{\mathfrak{Q}}$, $\mathbb{P}_{\tilde{\mathfrak{Q}}}$ to S. Also let us write $\Omega_a(\cdot)$ for $\Omega_{avoid}(\cdot)$ for simplicity, and denote by $\Omega_a(S)$ the set of elements of

$\Omega_{avoid}(-t_3, t_3, \tilde{\mathfrak{Q}}(-t_3), \tilde{\mathfrak{Q}}(t_3))$ restricted to S. We claim that the Radon-Nikodym derivative between these two restricted measures is given on elements $\mathfrak{B}$ of $\Omega_a(S)$ by

$$(6.3) \qquad \frac{d\mathbb{P}_{\mathfrak{Q}'}}{d\mathbb{P}_{\tilde{\mathfrak{Q}}'}}(\mathfrak{B}) = \frac{\mathbb{P}_{\mathfrak{Q}'}(\mathfrak{B})}{\mathbb{P}_{\tilde{\mathfrak{Q}}'}(\mathfrak{B})} = (Z')^{-1} Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right),$$

with $Z' = \mathbb{E}_{\tilde{\mathfrak{Q}}'}[Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!])]$. The first equality holds simply because the measures are discrete. To prove the second equality, observe that

$$(6.4) \qquad \begin{aligned} \mathbb{P}_{\mathfrak{Q}'}(\mathfrak{B}) &= \frac{|\Omega_a(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!])|}{|\Omega_a(-t_3, t_3, \mathfrak{Q}(-t_3), \mathfrak{Q}(t_3), \ell_{bot})|}, \\ \mathbb{P}_{\tilde{\mathfrak{Q}}'}(\mathfrak{B}) &= \frac{\prod_{i=1}^{k-1} |\Omega(-t_1, t_1, B_i(-t_1), B_i(t_1))|}{|\Omega_{a,S}(-t_3, t_3, \tilde{\mathfrak{Q}}(-t_3), \tilde{\mathfrak{Q}}(t_3), \tilde{\ell}_{bot})|} \end{aligned}$$

These identities follow from the restriction, and the fact that the measures are uniform. Then, from Definition 2.22,

$$Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}) = \frac{|\Omega_a(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!])|}{\prod_{i=1}^{k-1} |\Omega(-t_1, t_1, B_i(-t_1), B_i(t_1))|}$$

and hence

$$\begin{aligned} Z' &= \sum_{\mathfrak{B} \in \Omega_a(S)} \frac{\prod_{i=1}^{k-1} |\Omega(-t_1, t_1, B_i(-t_1), B_i(t_1))|}{|\Omega_{a,S}(-t_3, t_3, \tilde{\mathfrak{Q}}(-t_3), \tilde{\mathfrak{Q}}(t_3), \tilde{\ell}_{bot})|} \cdot \frac{|\Omega_a(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot})|}{\prod_{i=1}^{k-1} |\Omega(-t_1, t_1, B_i(-t_1), B_i(t_1))|} = \\ &\frac{\sum_{\mathfrak{B} \in \Omega_a(S)} |\Omega_a(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot})|}{|\Omega_{a,S}(-t_3, t_3, \tilde{\mathfrak{Q}}(-t_3), \tilde{\mathfrak{Q}}(t_3), \tilde{\ell}_{bot})|} = \frac{|\Omega_a(-t_3, t_3, \mathfrak{Q}(-t_3), \mathfrak{Q}(t_3), \ell_{bot})|}{|\Omega_{a,S}(-t_3, t_3, \tilde{\mathfrak{Q}}(-t_3), \tilde{\mathfrak{Q}}(t_3), \tilde{\ell}_{bot})|}. \end{aligned}$$

Comparing the above identities proves the second equality in (6.3).

Now note that $Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!])$ is a deterministic function of $((\mathfrak{B}(-t_1), \mathfrak{B}(t_1))$. In fact, the law of $((\mathfrak{B}(-t_1), \mathfrak{B}(t_1))$ under $\mathbb{P}_{\tilde{\mathfrak{Q}}'}$ is the same as that of $(\tilde{\mathfrak{Q}}(-t_1), \tilde{\mathfrak{Q}}(t_1))$ by way of the restriction. It follows from Lemma 6.2 that

$$\begin{aligned} Z' &= \mathbb{E}_{\tilde{\mathfrak{Q}}'}[Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!])] \\ &= \mathbb{E}_{\tilde{\mathfrak{Q}}}[Z(-t_1, t_1, \mathfrak{Q}(-t_1), \mathfrak{Q}(t_1), \ell_{bot}[\![-t_1, t_1]\!])] \geq gh, \end{aligned}$$

which gives us

$$(6.5) \qquad (Z')^{-1} \leq \frac{1}{gh}.$$

Similarly, the law of $(\mathfrak{B}(-t_1), \mathfrak{B}(t_1))$ under $\mathbb{P}_{\mathfrak{Q}'}$ is the same as that of $(\mathfrak{Q}(-t_1), \mathfrak{Q}(t_1))$ under $\mathbb{P}_{\mathfrak{Q}}$. Hence

$$(6.6) \qquad \begin{aligned} \mathbb{P}_{\mathfrak{Q}}\Big(Z(-t_1, t_1, \mathfrak{Q}(-t_1), \mathfrak{Q}(t_1), \ell_{bot}[\![-t_1, t_1]\!]) \leq gh\tilde{\epsilon}\Big) = \\ \mathbb{P}_{\mathfrak{Q}'}\Big(Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]) \leq gh\tilde{\epsilon}\Big). \end{aligned}$$

Now let us write $E = \{Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]) \leq gh\tilde{\epsilon}\} \subset \Omega_a(S)$. Then according to (6.3), we have

$$\mathbb{P}_{\mathfrak{Q}'}(E) = \int_{\Omega_a(S)} \mathbf{1}_E \, d\mathbb{P}_{\mathfrak{Q}'} = (Z')^{-1} \int_{\Omega_a(S)} \mathbf{1}_E \cdot Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]) \, d\mathbb{P}_{\tilde{\mathfrak{Q}}'}(\mathfrak{B}).$$

From the definition of E, the inequality (6.5), and the fact that $\mathbf{1}_E \leq 1$, it follows that

$$\mathbb{P}_{\mathfrak{Q}'}(E) \leq (Z')^{-1} \int_{\Omega_a(S)} \mathbf{1}_E \cdot gh\tilde{\epsilon} \, d\mathbb{P}_{\tilde{\mathfrak{Q}}'} \leq \frac{1}{gh} \int_{\Omega_a(S)} gh\tilde{\epsilon} \, d\mathbb{P}_{\tilde{\mathfrak{Q}}'} \leq \tilde{\epsilon}.$$

In combination with (6.6), this proves (4.2).

$\square$

6.2. Proof of Lemma 6.2.

In this section, we prove Lemma 6.2. We first state and prove two auxiliary lemmas necessary for the proof. The first lemma establishes a set of conditions under which we have the desired lower bound on the acceptance probability.

Lemma 6.3. *Let $\epsilon > 0$ and $V^{top} > 0$ be given such that $V^{top} > M_2 + 6(k-1)\epsilon$. Suppose further that $\vec{a}, \vec{b} \in \mathfrak{W}_{k-1}$ are such that*

(1) $V^{top}(2t_2)^{1/2} \geq a_1 + pt_1 \geq a_{k-1} + pt_1 \geq (M_2 + 2\epsilon)(2t_2)^{1/2}$;

(2) $V^{top}(2t_2)^{1/2} \geq b_1 - pt_1 \geq b_{k-1} - pt_1 \geq (M_2 + 2\epsilon)(2t_2)^{1/2}$;

(3) $a_i - a_{i+1} \geq 3\epsilon(2t_2)^{1/2}$ and $b_i - b_{i+1} \geq 3\epsilon(2t_2)^{1/2}$ for $i = 1, \ldots, k-2$.

Then we can find $g = g(\epsilon, V^{top}, M_2) > 0$ and $N_6 \in \mathbb{N}$ such that for all $N \geq N_6$ we have

$$(6.7) \qquad\qquad Z\big(-t_1, t_1, \vec{a}, \vec{b}, \ell_{bot}[\![-t_1, t_1]\!]\big) \geq g.$$

Proof. Observe by the rightmost inequalities in conditions (1) and (2) in the hypothesis, as well as condition (1) in Lemma 4.4, that ℓ_{bot} lies a distance of at least $2\epsilon(2t_2)^{1/2}$ uniformly below the line segment connecting a_{k-1} and b_{k-1}. Also note that (1) and (2) imply $|b_i - a_i - 2pt_1| \leq (V^{top} - M_2)(2pt_1)^{1/2}$ for each i. Lastly noting (3), we see that the conditions of Lemma 3.18 are satisfied with $C = 2\epsilon$. This implies (6.7), with

$$g = \left(1 - 3 \sum_{n=1}^{\infty} e^{-\epsilon^2 n^2 / 2p(1-p)} \right)^{k-1}.$$

$\square$

The next lemma helps us derive the lower bound h in (6.2).

Lemma 6.4. *For any $R > 0$ we can find $V_1^t, V_1^b \geq M_2 + R$, $h_1 > 0$ and $N_7 \in \mathbb{N}$ (depending on R) such that if $N \geq N_7$ we have*

$$(6.8) \qquad \mathbb{P}_{\tilde{\mathfrak{Q}}}\left((2t_3)^{1/2} V_1^t \geq \tilde{Q}_1(\pm t_2) \mp pt_2 \geq \tilde{Q}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b \right) \geq h_1.$$

Proof. Let the constant C be as in (5.12), and put

$$(6.9) \qquad\qquad V_1^b = M_1 + Ck + M_2 + R, \quad K_1 = (4r + 10)V_1^b,$$

$$(6.10) \qquad h_1 = \frac{2^{k/2-5}\big(1 - 2e^{-4/p(1-p)}\big)^{2k}}{(\pi p(1-p))^{k/2}} \exp\left(-\frac{2k(K_1 + M_1 + 6)^2}{p(1-p)} \right).$$

Note in particular that $V_1^b > M_2 + R$. We will choose $V_1^t > V_1^b$ in the below depending on h_1. We claim that for these choices of V_1^b, V_1^t, h_1, and for large enough N, we have

$$(6.11) \qquad\qquad \mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b \right) \geq 2h_1,$$

$$(6.12) \qquad\qquad \mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_1(\pm t_2) \mp pt_2 > (2t_3)^{1/2} V_1^t \right) \leq h_1.$$

Assuming the validity of the claim, we then observe that the probability in (6.8) is bounded below by $2h_1 - h_1 = h_1$, proving the lemma. We will prove (6.11) and (6.12) in three steps.

Step 1. In this step we prove (6.11) assuming results from Step 2 below. We condition on the value of $\tilde{\mathfrak{L}}$ at 0 and use the Schur Gibbs property to divide $\tilde{\mathfrak{Q}}$ into two independent line ensembles on $[-t_3, 0]$ and $[0, t_3]$. Observe by Lemma 3.2 that

$$(6.13) \qquad \mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b \right) \geq \mathbb{P}_{avoid, Ber; S}^{-t_3, t_3, \vec{x}, \vec{y}}\left(\tilde{Q}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b \right).$$

With K_1 as in (6.9), we define events

$$E_{\vec{z}} = \left\{ (\tilde{Q}_1(0), \ldots, \tilde{Q}_{k-1}(0)) = \vec{z} \right\}, \quad X = \left\{ \vec{z} \in \mathfrak{W}_{k-1} : z_{k-1} \geq K_1(2t_3)^{1/2} \text{ and } \mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}(E_{\vec{z}}) > 0 \right\},$$

and $E = \bigsqcup_{\vec{z} \in X} E_{\vec{z}}$. Note that X is non-empty if N is sufficiently large depending on M_1, C, k, M_2, R. By Lemma 3.20, we have

$$(6.14) \quad \mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}(E) \geq \mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}\left(\tilde{Q}_{k-1}(0) \geq K_1(2t_3)^{1/2}\right) \geq A \exp\left(-\frac{2k(K_1 + M_1 + 6)^2}{p(1-p)}\right)$$

for sufficiently large N, where $A = A(p, k)$ is a constant given explicitly in (3.17).

Now let $\tilde{Q}_i^1$ and $\tilde{Q}_i^2$ denote the restrictions of $\tilde{Q}_i$ to $[-t_3, 0]$ and $[0, t_3]$ respectively for $1 \leq i \leq k-1$, and write $S_1 = S \cap [\![-t_3, 0]\!]$, $S_2 = S \cap [\![0, t_3]\!]$. We observe that if $\vec{z} \in X$, then

$$(6.15) \quad \mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}\left(\tilde{Q}_{k-1}^1 = \ell_1, \tilde{Q}_{k-1}^2 = \ell_2 \mid E_{\vec{z}}\right) = \mathbb{P}^{-t_3,0,\vec{x},\vec{z}}_{avoid,Ber;S_1}(\ell_1) \cdot \mathbb{P}^{0,t_3,\vec{z},\vec{y}}_{avoid,Ber;S_2}(\ell_2).$$

In Step 2, we will prove that for large N,

$$(6.16) \quad \begin{aligned} \mathbb{P}^{-t_3,0,\vec{x},\vec{z}}_{avoid,Ber;S_1}\left(\tilde{Q}_{k-1}^1(-t_2) + pt_2 \geq (2t_3)^{1/2}V_1^b\right) &\geq \frac{1}{4}, \\ \mathbb{P}^{0,t_3,\vec{x},\vec{z}}_{avoid,Ber;S_2}\left(\tilde{Q}_{k-1}^2(t_2) - pt_{12} \geq (2t_3)^{1/2}V_1^b\right) &\geq \frac{1}{4}. \end{aligned}$$

Using (6.14), (6.15), and (6.16), we conclude that

$$\mathbb{P}^{-t_2,t_2,\vec{x},\vec{y}}_{avoid,Ber}\left(\tilde{L}_{k-1}(\pm t_{12}) \mp pt_{12} \geq (2t_2)^{1/2}V_1^b\right) \geq \frac{A}{16} \exp\left(-\frac{2k(K_1 + M_1 + 6)^2}{p(1-p)}\right).$$

In combination with (6.13), this proves (6.11) with h_1 as in (6.10).

Step 2. In this step, we prove the inequalities (6.16) from Step 1, using Lemma 3.8. Let us define vectors $\vec{x}', \vec{z}', \vec{y}'$ by

$$\begin{aligned} x_i' &= \lfloor -pt_3 - M_1(2t_3)^{1/2} \rfloor - (i-1)\lceil C(2t_3)^{1/2} \rceil, \\ z_i' &= \lfloor K_1(2t_3)^{1/2} \rfloor - (i-1)\lceil C(2t_3)^{1/2} \rceil, \\ y_i' &= \lfloor pt_3 - M_1(2t_3)^{1/2} \rfloor - (i-1)\lceil C(2t_3)^{1/2} \rceil. \end{aligned}$$

Note that $x_i' \leq x_{k-1} \leq x_i$ and $x_i' - x_{i+1}' \geq C(2t_3)^{1/2}$ for $1 \leq i \leq k-1$, and likewise for z_i', y_i'. By Lemma 3.1 we have

$$(6.17)$$
$$\mathbb{P}^{-t_3,0,\vec{x},\vec{z}}_{avoid,Ber;S_1}\left(\tilde{Q}_{k-1}^1(-t_2) + pt_2 \geq (2t_3)^{1/2}V_1^b\right) \geq \mathbb{P}^{-t_3,0,\vec{x}',\vec{z}'}_{avoid,Ber;S_1}\left(\tilde{Q}_{k-1}^1(-t_2) + pt_2 \geq (2t_3)^{1/2}V_1^b\right) \geq$$
$$\mathbb{P}^{-t_3,0,x_{k-1}',z_{k-1}'}_{Ber}\left(\ell_1(-t_2) + pt_2 \geq (2t_3)^{1/2}V_1^b\right) - \left(1 - \mathbb{P}^{-t_3,t_3,\vec{x}',\vec{z}'}_{Ber}\left(\tilde{Q}_1^1 \geq \cdots \geq \tilde{Q}_{k-1}^1\right)\right).$$

To bound the first term on the second line, first note that $x_{k-1}' \geq -pt_3 - (M_1 + C(k-1))(2t_3)^{1/2}$ and $z_{k-1}' \geq K_1(2t_3)^{1/2} - C(k-1)(2t_3)^{1/2}$ for sufficiently large N. Let us write $\tilde{x}, \tilde{z}$ for these two lower bounds. Then by Lemma 3.8, we have

$$(6.18) \quad \mathbb{P}^{-t_3,0,x_{k-1}',z_{k-1}'}_{Ber}\left(\ell_1(-t_2) \geq \frac{t_2}{t_3}\tilde{x} + \frac{t_3 - t_2}{t_3}\tilde{z} - (2t_3)^{1/4}\right) \geq \frac{1}{3}.$$

Moreover, as long as $N^\alpha > 2$, we have

$$(6.19) \quad \frac{t_3 - t_2}{t_3} \geq 1 - \frac{(r+2)N^\alpha}{(r+3)N^\alpha - 1} > 1 - \frac{r+2}{r+5/2} = \frac{1}{2r+5}.$$

It follows from our choice of V_1^b and $K_1 = 2(2r+5)V_1^b$ in (6.9), as well as (6.19), that

$$\frac{t_2}{t_3}\,\tilde{x} + \frac{t_3 - t_2}{t_3}\,\tilde{z} - (2t_3)^{1/4} = -pt_2 - C(k-1)(2t_3)^{1/2} - \frac{t_2}{t_3}M_1(2t_3)^{1/2} + \frac{t_3 - t_2}{t_3}K_1(2t_3)^{1/2} - (2t_3)^{1/4}$$

$$\geq -pt_2 - Ck(2t_3)^{1/2} - M_1(2t_3)^{1/2} + \frac{1}{2r+5}K_1(2t_3)^{1/2}$$

$$= -pt_2 + (M_1 + Ck + 2(M_2 + R))(2t_3)^{1/2}$$

$$> -pt_2 + (2t_3)^{1/2}V_1^b.$$

For the first inequality, we used the fact that $t_{12}/t_2 < 1$, and we assumed that N is sufficiently large so that $C(k-1)(2t_2)^{1/2} + (2t_2)^{1/4} \leq Ck(2t_2)^{1/2}$. Using (6.18), we conclude that

$$(6.20) \qquad \mathbb{P}_{Ber}^{-t_3,0,x'_{k-1},z'_{k-1}}\left(\ell_1(-t_2) + pt_2 \geq (2t_3)^{1/2}V_1^b\right) \geq \frac{1}{3}.$$

Since $|z'_i - x'_i - pt_2| \leq (K_1 + M_1 + 1)(2t_2)^{1/2}$, we have by Lemma 3.18 and our choice of C that the second probability in the second line of (6.17) is bounded below by

$$\left(1 - 3e^{-C^2/8p(1-p)}\right)^{k-1} \geq 11/12.$$

It follows from (6.17) and (6.20) that

$$\mathbb{P}_{avoid,Ber;S_1}^{-t_3,0,\vec{x},\vec{z}}\left(\tilde{Q}_{k-1}^1(-t_2) + pt_2 \geq (2t_3)^{1/2}V_1^b\right) \geq \frac{1}{3} - \frac{1}{12} = \frac{1}{4},$$

proving the first inequality in (6.16). The second inequality is proven similarly.

Step 3. Here we prove (6.12). Let C be as in Step 1, and define vectors $\vec{x}'', \vec{y}'' \in \mathfrak{W}_{k-1}$ by

$$x''_i = \lceil -pt_3 + M_1(2t_3)^{1/2}\rceil + (k-i)\lceil C(2t_3)^{1/2}\rceil,$$

$$y''_i = \lceil pt_3 + M_1(2t_3)^{1/2}\rceil + (k-i)\lceil C(2t_3)^{1/2}\rceil.$$

Note that $x''_i \geq x_1 \geq x_i$ and $x''_i - x''_{i+1} \geq C(2t_3)^{1/2}$, and likewise for y''_i. Moreover, $\tilde{\ell}_{bot}$ lies a distance of at least $C(2t_3)^{1/2}$ uniformly below the line segment connecting x''_{k-1} and y''_{k-1}. By Lemma 3.1 and the Schur Gibbs property, we have

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_1(\pm t_2) \mp pt_2 > (2t_3)^{1/2}V_1^t\right) \leq \mathbb{P}_{avoid,Ber;S}^{-t_3,t_3,\vec{x}'',\vec{y}'',\infty,\tilde{\ell}_{bot}}\left(\sup_{s\in[-t_3,t_3]}[\tilde{Q}_1(s) - ps] \geq (2t_3)^{1/2}V_1^t\right)$$

$$\leq \frac{\mathbb{P}_{Ber}^{-t_3,t_3,x''_1,y''_1}\left(\sup_{s\in[-t_3,t_3]}[\tilde{L}_1(s) - ps] \geq (2t_3)^{1/2}V_1^t\right)}{\mathbb{P}_{Ber}^{-t_3,t_3,\vec{x}'',\vec{y}''}\left(\tilde{Q}_1 \geq \cdots \geq \tilde{Q}_{k-1} \geq \tilde{\ell}_{bot}\right)}.$$

By Lemma 3.10, since $\min(x''_1 + pt_3, y''_1 - pt_3) \leq (M_1 + C(k-1))(2t_3)^{1/2}$, we can choose $V_1^t > V_1^b$ large enough so that the numerator is bounded above by $h_1/2$. Since $|y''_i - x''_i - 2pt_3| \leq 1$, our choice of C and Lemma 3.18 imply that the denominator is at least $11/12$. This gives an upper bound of $12/11 \cdot h_1/2 < h_1/2$ in the above, proving (6.12).

$\square$

We are now equipped to prove Lemma 6.2.

Proof. We first introduce some notation to be used in the proof. For $\vec{c}, \vec{d} \in \mathfrak{W}_{k-1}$, let us write $\Omega(\vec{c}, \vec{d}) = \Omega_{avoid}(-t_2, t_2, \vec{c}, \vec{d}, \infty, -\infty)$ and $\Omega_{a,S}(\vec{c}, \vec{d}) = \Omega_{a,S}(-t_2, t_2, \vec{c}, \vec{d}, \infty, \tilde{\ell}_{bot})$, and define events

$$E(\vec{c}, \vec{d}, V_1^t, V_1^b) = \left\{ \tilde{\mathfrak{B}} \in \Omega_{a,S}(\vec{c}, \vec{d}) \mid (2t_3)^{1/2} V_1^t \geq \tilde{B}_1(\pm t_2) \mp pt_2 \geq \tilde{B}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b \right\}$$

$$A(\vec{c}, \vec{d}, V_1^t, V_1^b) = \left\{ \tilde{\mathfrak{B}} \in \Omega_{a,S}(\vec{c}, \vec{d}) \mid \tilde{B}_1(\pm t_1) \mp pt_1 \leq (2t_3)^{1/2} V_1^t, \tilde{B}_{k-1}(\pm t_1) \pm pt_1 \geq (2t_3)^{1/2} V_1^b \right\}$$

$$B(\vec{c}, \vec{d}, \epsilon) = \left\{ \tilde{\mathfrak{B}} \in \Omega_{a,S}(\vec{c}, \vec{d}) \mid \min_{\substack{i \in [\![1,k-2]\!] \\ \varsigma \in \{-1,1\}}} \tilde{B}_i(\varsigma t_2) - \tilde{B}_{i+1}(\varsigma t_2) \geq 3\epsilon(2t_2)^{1/2} \right\}$$

$$C(\vec{c}, \vec{d}, V_1^t, V_1^b, \vec{\epsilon}) = \left\{ \tilde{\mathfrak{B}} \in \Omega_{a,S}(\vec{c}, \vec{d}) \mid \min_{\substack{i \in [\![1,k-2]\!] \\ \varsigma \in \{-1,1\}}} \tilde{B}_i(\varsigma t_1) - \tilde{B}_{i+1}(\varsigma t_1) \geq 3\epsilon(2t_2)^{1/2} \right\}$$

The event E is the condition from 6.3 and ensures that after scaling, the line ensemble is in the compact set $[V_1^b, V_1^t]$ at times $\pm t_2$; A is the same statement for times $\pm t_1$; B is the event that the ensemble is well separated at times $\pm t_2$, C is the event that the Note that

$$A(\vec{c}, \vec{d}, V_1^t, V_1^b) \cap C(\vec{c}, \vec{d}, V_1^t, V_1^b, \epsilon) \subseteq \left\{ \mathfrak{B} \in \Omega_{a,S}(\vec{c}, \vec{d}) \mid Z\left(-t_1, t_1, \vec{c}, \vec{d}, \ell_{bot}[\![-t_1, t_1]\!]\right) > g(\epsilon, V_1^t, V_1^b) \right\}$$

since the conditions on the set $A \cap C$ are precisely those required by the Lemma 6.3. Moreover, by Lemma 6.4, we have $\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(E(\vec{c}, \vec{d}, V_1^t, V_1^b)\right) > h_1$. Therefore, we know

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}(A \cap C \cap E) = \mathbb{P}_{\tilde{\mathfrak{Q}}}(A \cap C \mid E) \cdot \mathbb{P}_{\tilde{\mathfrak{Q}}}(E) \geq h_1 \cdot \mathbb{P}(A \cap C \mid E).$$

and if we prove that $\mathbb{P}_{\tilde{\mathfrak{Q}}}(A \cap C \mid E) > h_2$ for some h_2, then we have $\mathbb{P}_{\tilde{\mathfrak{Q}}}(A \cap C \cap E) \geq h_1 \cdot h_2 = h$, then we have proved Lemma 6.2. In order to prove this statement, we split the proof into several steps. $\qquad\square$

Proof. We first introduce some notation to be used in the proof. For $\vec{c}, \vec{d} \in \mathfrak{W}_{k-1}$, let us write $\Omega(\vec{c}, \vec{d}) = \Omega_{avoid}(-t_{12}, t_{12}, \vec{c}, \vec{d}, \infty, -\infty)$ and $\tilde{\Omega}(\vec{c}, \vec{d}) = \Omega_{avoid}(-t_{12}, t_{12}, \vec{c}, \vec{d}, \infty, \tilde{\ell}_{bot})$, and define events

$$\tilde{A}(\vec{c}, \vec{d}) = \{\mathfrak{L} \in \tilde{\Omega}(\vec{c}, \vec{d}) : L_{k-1}(\pm t_1) \mp pt_1 \geq (M_2 + 1)(2t_2)^{1/2}\},$$

$$\tilde{B}(\vec{c}, \vec{d}, \epsilon) = \left\{ \mathfrak{L} \in \tilde{\Omega}(\vec{c}, \vec{d}) : \min_{\substack{1 \leq i \leq k-2 \\ \varsigma \in \{-1,1\}}} \left(L_i(\varsigma t_1) - L_{i+1}(\varsigma t_1)\right) \geq 3\epsilon(2t_2)^{1/2} \right\},$$

(6.21) $\quad A(\vec{c}, \vec{d}) = \{\mathfrak{L} \in \Omega(\vec{c}, \vec{d}) : L_{k-1}(\pm t_1) \mp pt_1 \geq (M_2 + 1)(2t_2)^{1/2}\},$

$$B(\vec{c}, \vec{d}, \epsilon) = \left\{ \mathfrak{L} \in \Omega(\vec{c}, \vec{d}) : \min_{\substack{1 \leq i \leq k-2 \\ \varsigma \in \{-1,1\}}} \left(L_i(\varsigma t_1) - L_{i+1}(\varsigma t_1)\right) \geq 3\epsilon(2t_2)^{1/2} \right\},$$

$$\tilde{C}(\vec{c}, \vec{d}, V^{top}) = \{\mathfrak{L} \in \tilde{\Omega}(\vec{c}, \vec{d}) : L_1(\pm t_1) \mp pt_1 \leq V^{top}(2t_2)^{1/2}\}.$$

Here, ϵ and V^{top} are constants which we will specify later. By Lemma 6.3, for N sufficiently large we have

$$\tilde{A}(\vec{c}, \vec{d}) \cap \tilde{B}(\vec{c}, \vec{d}, \epsilon) \cap \tilde{C}(\vec{c}, \vec{d}, V^{top}) \subset \{Z(-t_1, t_1, \vec{a}, \vec{b}, \ell_{bot}[\![-t_1, t_1]\!]) > g\}$$

for some g depending on ϵ, V^{top}, M_2. Thus we will prove that probability of the event on the left under $\mathbb{P}_{\tilde{\mathfrak{L}}}$ is bounded below by $h = h_1/2$, with h_1 as in (6.10). We split the proof into several steps.

Step 1. In this step, we show that there is an $R > 0$ sufficiently large so that if $c_{k-1} + pt_{12} \geq (2t_2)^{1/2}(M_2 + R)$ and $d_{k-1} - pt_{12} \geq (2t_2)^{1/2}(M_2 + R)$, then we have

$$(6.22) \qquad \frac{|\tilde{A}(\vec{c}, \vec{d})|}{|\tilde{\Omega}(\vec{c}, \vec{d})|} \geq \frac{|A(\vec{c}, \vec{d})|}{|\Omega(\vec{c}, \vec{d})|} \geq \frac{9}{10} \quad \text{and} \quad \frac{|\tilde{\Omega}(\vec{c}, \vec{d})|}{|\Omega(\vec{c}, \vec{d})|} \geq \frac{99}{100}.$$

The first inequality follows immediately from Lemma 3.2.

For the second inequality, define the constant

$$(6.23) \qquad C = \sqrt{8p(1-p)\log \frac{3}{1 - (199/200)^{1/(k-1)}}}$$

and vectors $\vec{c}', \vec{d}' \in \mathfrak{W}_k$ by

$$c_i' = \lfloor -pt_{12} + (M_2 + R)(2t_2)^{1/2} \rfloor - (i-1)\lceil C(2t_{12})^{1/2} \rceil,$$
$$d_i' = \lfloor pt_{12} + (M_2 + R)(2t_2)^{1/2} \rfloor - (i-1)\lceil C(2t_{12})^{1/2} \rceil.$$

Then by Lemma 3.1 and the Schur Gibbs property,

$$\mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d}}(A(\vec{c}, \vec{d})) \geq \mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c}',\vec{d}'}(A(\vec{c}', \vec{d}'))$$

$$(6.24) \qquad\qquad \geq \mathbb{P}_{Ber}^{-t_{12},t_{12},c_{k-1}',d_{k-1}'}\left(\ell(\pm t_1) \mp pt_1 \geq (M_2 + 1)(2t_2)^{1/2}\right)$$

$$- \left(1 - \mathbb{P}_{Ber}^{-t_{12},t_{12},\vec{c}',\vec{d}'}(L_1 \geq \cdots \geq L_{k-1})\right).$$

By Lemma 3.18 and our choice of C, $\mathbb{P}_{Ber}^{-t_{12},t_{12},\vec{c}',\vec{d}'}(L_1 \geq \cdots \geq L_{k-1}) > 199/200 > 19/20$ for sufficiently large N. Writing $z = d_{k-1}' - c_{k-1}'$, the term in the second line of (6.24) is equal to

$$\mathbb{P}_{Ber}^{-t_{12},t_{12},0,z}\left(\ell(\pm t_1) \mp pt_1 + c_{k-1}' \geq (M_2 + 1)(2t_2)^{1/2}\right)$$

$$\geq \mathbb{P}_{Ber}^{0,2t_{12},0,z}\left(\ell(\pm t_1) \mp pt_1 \geq (-R + Ck + 1)(2t_2)^{1/2}\right)$$

$$\geq \mathbb{P}_{Ber}^{0,2t_{12},0,z}\left(\inf_{s \in [0, 2t_{12}]}(\ell(s) - ps) \geq -(R - Ck - 1)(2t_{12})^{1/2}\right).$$

In the second line, we used the estimate $c_{k-1}' \geq -pt_{12} + (M_2 + R - Ck)(2t_2)^{1/2}$. Now by Lemma 3.10, we can choose R large enough depending on C, k, M_2, p so that this probability is greater than $19/20$ for sufficiently large N. This gives a lower bound in (6.24) of $19/20 - 1/20 = 9/10$, proving the second inequality in (6.22).

We prove the third inequality in (6.22) similarly. Note that since $\ell_{bot}(s) \leq ps + M_2(2t_2)^{1/2}$ on $[-t_2, t_2]$ by assumption, we have

$$\frac{|\tilde{\Omega}(\vec{c}, \vec{d})|}{|\Omega(\vec{c}, \vec{d})|} \geq \mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d}}\left(\inf_{s \in [-t_{12}, t_{12}]}(L_{k-1}(s) - ps) \geq M_2(2t_2)^{1/2}\right)$$

$$(6.25) \qquad\qquad \geq \mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c}',\vec{d}'}\left(\inf_{s \in [-t_{12}, t_{12}]}(L_{k-1}(s) - ps) \geq M_2(2t_2)^{1/2}\right)$$

$$\geq \mathbb{P}_{Ber}^{0,2t_{12},0,z}\left(\inf_{s \in [0, 2t_{12}]}(\ell(s) - ps) \geq -(R - Ck)(2t_2)^{1/2}\right)$$

$$- \left(1 - \mathbb{P}_{Ber}^{-t_{12},t_{12},\vec{c}',\vec{d}'}(L_1 \geq \cdots \geq L_{k-1})\right).$$

We enlarge R if necessary so that the probability in the third line of (6.25) is $> 199/200$ by Lemma 3.10, and 3.18 implies as above that the expression in the last line of (6.25) is $> -1/200$. This gives us a lower bound of $199/200 - 1/200 = 99/100$ as desired.

Step 2. With R fixed from Step 1, let V_1^t, V_1^b and h_1 be as in Lemma 6.4 for this choice of R. Define the event

$$E = \{\vec{c}, \vec{d} \in \mathfrak{W}_{k-1} : (2t_2)^{1/2}V_1^t \geq \max(c_1+pt_{12}, d_1-pt_{12}) \text{ and } \min(c_{k-1}+pt_{12}, d_{k-1}-pt_{12}) \geq (2t_2)^{1/2}V_1^b\}.$$

We show in this step that there exists $V^{top} \geq M_2 + 6(k-1)$ such that for all $(\vec{c}, \vec{d}) \in E$, we have

$$(6.26) \qquad \frac{|\tilde{C}(\vec{c}, \vec{d}, V^{top})|}{|\tilde{\Omega}(\vec{c}, \vec{d})|} \geq \frac{9}{10}.$$

Let C be as in (6.23), and define $\vec{c}'', \vec{d}'' \in \mathfrak{W}_{k-1}$ by

$$c_i'' = \lceil -pt_{12} + (2t_2)^{1/2}V_1^t \rceil + (k-1-i)\lceil C(2t_{12})^{1/2} \rceil,$$
$$d_i'' = \lceil pt_{12} + (2t_2)^{1/2}V_1^t \rceil + (k-1-i)\lceil C(2t_{12})^{1/2} \rceil.$$

Then $c_i'' \geq c_1 \geq c_i$ and $c_i'' - c_{i+1}'' \geq C(2t_2)^{1/2}$ for each i, and likewise for d_i''. Furthermore, since $V_1^b \geq M_2 + R$, we see that $\tilde{\ell}_{bot}$ lies a distance of at least $R(2t_2)^{1/2}$ uniformly below the line segment connecting c_{k-1}'' and d_{k-1}''. By construction, $R > C$. By Lemma 3.1, we have

$$
\begin{aligned}
(6.27) \qquad \frac{|\tilde{C}(\vec{c}, \vec{d}, V^{top})|}{|\tilde{\Omega}(\vec{c}, \vec{d})|} &\geq \mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c}'',\vec{d}'',\infty,\tilde{\ell}_{bot}}\left(L_1(\pm t_1) \mp pt_1 \leq V^{top}(2t_2)^{1/2}\right) \\
&\geq \mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c}'',\vec{d}'',\infty,\tilde{\ell}_{bot}}\left(\sup_{s \in [-t_{12},t_{12}]}\left(L_1(s) - ps\right) \leq V^{top}(2t_2)^{1/2}\right) \\
&\geq \mathbb{P}_{Ber}^{0,2t_{12},0,z'}\left(\sup_{s \in [-t_{12},t_{12}]}\left(\ell(s) - ps\right) \leq (V^{top} - V_1^t - Ck)(2t_2)^{1/2}\right) \\
&\quad - \left(1 - \mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c}'',\vec{d}'',\infty,\tilde{\ell}_{bot}}\left(L_1 \geq \cdots \geq L_{k-1} \geq \tilde{\ell}_{bot}\right)\right).
\end{aligned}
$$

In the last line, we have used the Schur Gibbs property and written $z' = d_1'' - c_1''$. We also used the fact that $c_1'' \leq -pt_{12} + (V_1^t + Ck)(2t_2)^{1/2}$. By Lemma 3.10, we can find V^{top} large enough depending on V_1^t, C, k, p so that the probability in the third line of (6.27) is at least $19/20$ for sufficiently large N. On the other hand, the above observations regarding $\vec{c}''$, $\vec{d}''$, and $\tilde{\ell}_{bot}$, as well as the fact that $|d_1'' - c_1'' - 2pt_{12}| \leq 1$, allow us to conclude from Lemma 3.18 that the probability in the last line of (6.27) is at least $19/20$ for sufficiently large N. This gives a lower bound of $19/20 - 1/20 = 9/10$ in (6.27) as desired.

Step 3. In this step, we show that with V_1^t and V_1^b as in Step 2, there is an $\epsilon > 0$ sufficiently small such that for $(\vec{c}, \vec{d}) \in E$ we have

$$(6.28) \qquad \frac{|B(\vec{c}, \vec{d}, \epsilon)|}{|\Omega(\vec{c}, \vec{d})|} \geq \frac{9}{10} \quad \text{and} \quad \frac{|\tilde{B}(\vec{c}, \vec{d}, \epsilon)|}{|\tilde{\Omega}(\vec{c}, \vec{d})|} \geq \frac{4}{5}.$$

We prove this inequality using Lemma 3.22. In order to apply this result, we approximate t_1 in the form $s \cdot 2t_{12}$, for $s \in (0, 1)$.

Observe from (**??**) and the definitions of t_1, t_2 that the ratio $\frac{t_1}{2t_{12}}$ depends on N and satisfies the inequality

$$\frac{(r+1)N^\alpha - 1}{(2r+3)N^\alpha} \leq \frac{t_1}{2t_{12}} \leq \frac{(r+1)N^\alpha}{(2r+3)N^\alpha - 4}.$$

We put $s = \frac{r+1}{2r+3}$, and observe that $s < 1/2$. Also observe that

$$(2r+3)N^\alpha - 4 \leq 2t_{12} \leq (2r+3)N^\alpha,$$

and thus

$$(r+1)N^\alpha \geq s \cdot 2t_{12} \geq (r+1)N^\alpha - 4s \geq t_1 - 2.$$

It follows that $t_1 - 2 \leq \lfloor s \cdot 2t_{12} \rfloor \leq t_1$, from which we conclude that

$$(6.29) \qquad 0 \leq L_i(t_1) - L_i(s \cdot 2t_{12}) \leq 2.$$

Now applying Lemma 3.22 with $M_1, M_2 = \max(V_1^t, V_1^b)$, we obtain N_0 and $\delta > 0$ such that if $N \geq N_0$, then

$$\mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d}}\left(\min_{1 \leq i \leq k-1} \left(L_i(s \cdot 2t_{12}) - L_{i+1}(s \cdot 2t_{12}) \right) < \delta(2t_{12})^{1/2} \right) < \frac{1}{20}.$$

Together with (6.29) and the fact that $t_2/4 < t_1 < t_{12}$, this implies that

$$(6.30) \qquad \mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d}}\left(\min_{1 \leq i \leq k-1} \left(L_i(t_1) - L_{i+1}(t_1) \right) < (\delta/2)(2t_2)^{1/2} - 2 \right) < \frac{1}{20}$$

for $N \geq N_0$. Now we observe that if $N \geq \left(\frac{1+1/\delta}{r+2} \right)^{2/\alpha}$, then $(\delta/4)(2t_2)^{1/2} \leq (\delta/2)(2t_2)^{1/2} - 1$. Thus for all sufficiently large N, we have

$$\mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d}}\left(\min_{1 \leq i \leq k-1} \left(L_i(t_1) - L_{i+1}(t_1) \right) < (\delta/4)(2t_2)^{1/2} \right) < \frac{1}{20}.$$

A similar argument gives us a $\tilde{\delta} > 0$ such that

$$\mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d}}\left(\min_{1 \leq i \leq k-2} \left(L_i(-t_1) - L_{i+1}(-t_1) \right) < (\tilde{\delta}/4)(2t_2)^{1/2} \right) < \frac{1}{20}$$

for large enough N. Then putting $\epsilon = \min(\delta, \tilde{\delta})/12$, we find

$$\mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d}}\left(\min_{1 \leq i \leq k-2} \left(L_i(\pm t_1) - L_{i+1}(\pm t_1) \right) < 3\epsilon(2t_2)^{1/2} \right) < \frac{1}{20} + \frac{1}{20} = \frac{1}{10},$$

and the first inequality in (6.28) follows.

For the second inequality in (6.28), we observe via the second inequality in (6.22) that

$$\frac{|\tilde{B}(\vec{c},\vec{d},\epsilon)|}{|\tilde{\Omega}(\vec{c},\vec{d})|} \geq \frac{|B(\vec{c},\vec{d},\epsilon) \cap \tilde{\Omega}(\vec{c},\vec{d})|}{|\Omega(\vec{c},\vec{d})|} \geq \frac{|B(\vec{c},\vec{d},\epsilon)|}{|\Omega(\vec{c},\vec{d})|} - \frac{|\tilde{\Omega}^c(\vec{c},\vec{d})|}{|\Omega(\vec{c},\vec{d})|} \geq \frac{9}{10} - \frac{1}{100} > \frac{4}{5}.$$

Step 4. In this last step, we complete the proof of the lemma. Let $g = g(\epsilon, V^{top}, M_2)$ be as in Lemma 6.3 for the choices of ϵ, V^{top} in Steps 2 and 3, and let $h = h_1/2$, with h_1 as in Step 2. If $(\vec{c},\vec{d}) \in E$, then it follows from (6.22), (6.26), and (6.28) that

$$\frac{|\tilde{B}^c(\vec{c},\vec{d},\epsilon) \cup \tilde{A}^c(\vec{c},\vec{d}) \cup \tilde{C}^c(\vec{c},\vec{d},V^{top})|}{|\tilde{\Omega}(\vec{c},\vec{d})|} \leq \frac{|\tilde{B}^c(\vec{c},\vec{d},\epsilon)|}{|\tilde{\Omega}(\vec{c},\vec{d})|} + \frac{|\tilde{A}^c(\vec{c},\vec{d})|}{|\tilde{\Omega}(\vec{c},\vec{d})|} + \frac{|\tilde{C}^c(\vec{c},\vec{d},V^{top})|}{|\tilde{\Omega}(\vec{c},\vec{d})|}$$

$$\leq \frac{1}{5} + \frac{1}{10} + \frac{1}{10} < \frac{1}{2}.$$

In particular,

$$(6.31) \qquad \frac{|\tilde{B}(\vec{c},\vec{d},\epsilon) \cap \tilde{A}(\vec{c},\vec{d}) \cap \tilde{C}(\vec{c},\vec{d},V^{top})|}{|\tilde{\Omega}(\vec{c},\vec{d})|} \geq \frac{1}{2},$$

for all large N, independent of $\vec{c},\vec{d}$. That is, we have for all $(\vec{c},\vec{d}) \in E$ that

$$\mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d},\infty,\tilde{\ell}_{bot}}(G) \geq \frac{1}{2},$$

where G is the event that

(1) $V^{top}(2t_2)^{1/2} \geq L_1(-t_1) + pt_1 \geq L_{k-1}(-t_1) + pt_1 \geq (M_2 + 1)(2t_2)^{1/2}$;
(2) $V^{top}(2t_2)^{1/2} \geq L_1(t_1) - pt_1 \geq L_{k-1}(t_1) - pt_1 \geq (M_2 + 1)(2t_2)^{1/2}$;
(3) $L_i(-t_1) - L_{i+1}(-t_1) \geq 3\epsilon(2t_2)^{1/2}$ and $L_i(t_1) - L_{i+1}(t_1) \geq 3\epsilon(2t_2)^{1/2}$ for $i = 1, \ldots, k-2$.

Let F denote the event appearing in (6.8). Then we can write $F = \bigsqcup_{(\vec{c},\vec{d}) \in E} F(\vec{c},\vec{d})$, where $F(\vec{c},\vec{d})$ is the event that $\tilde{L}(-t_{12}) = \vec{c}$, $\tilde{L}(t_{12}) = \vec{d}$. It follows from Lemma 6.4 that

$$\mathbb{P}_{\tilde{\mathfrak{L}}}(G) \geq \mathbb{P}_{\tilde{\mathfrak{L}}}(G \cap F) = \sum_{(\vec{c},\vec{d}) \in E} \mathbb{P}_{\tilde{\mathfrak{L}}}(F(\vec{c},\vec{d}))\mathbb{P}_{\tilde{\mathfrak{L}}}(G \,|\, F(\vec{c},\vec{d})) =$$

$$\sum_{(\vec{c},\vec{d}) \in E} \mathbb{P}_{\tilde{\mathfrak{L}}}(F(\vec{c},\vec{d}))\mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d},\infty,\tilde{\ell}_{bot}}(G) \geq \frac{1}{2}\sum_{(\vec{c},\vec{d}) \in E} \mathbb{P}_{\tilde{\mathfrak{L}}}(F(\vec{c},\vec{d})) = \frac{1}{2}\mathbb{P}_{\tilde{\mathfrak{L}}}(F) \geq \frac{1}{2}h_1$$

for large N. Now Lemma 6.3 implies (6.2), completing the proof. $\square$

7. Applications to uniform lozenge tilings

8. Appendix A

8.1. **Proof of Lemma 2.2.** Observe that the sets $K_1 \subset K_2 \subset \cdots \subset \Sigma \times \Lambda$ are compact, they cover $\Sigma \times \Lambda$, and any compact subset K of $\Sigma \times \Lambda$ is contained in all K_n for sufficiently large n. To see this last fact, let π_1, π_2 denote the canonical projection maps of $\Sigma \times \Lambda$ onto Σ and Λ respectively. Since these maps are continuous, $\pi_1(K)$ and $\pi_2(K)$ are compact in Σ and Λ. This implies that $\pi_1(K)$ is finite, so it is contained in $\Sigma_{n_1} = \Sigma \cap [\![-n_1, n_1]\!]$ for some n_1. On the other hand, $\pi_2(K)$ is closed and bounded in $\mathbb{R}$, thus contained in some closed interval $[\alpha, \beta] \subseteq \Lambda$. Since $a_n \searrow a$ and $b_n \nearrow b$, we can choose n_2 large enough so that $\pi_2(K) \subseteq [\alpha, \beta] \subseteq [a_{n_2}, b_{n_2}]$. Then taking $n = \max(n_1, n_2)$, we have $K \subseteq \pi_1(K) \times \pi_2(K) \subseteq \Sigma_n \times [a_n, b_n] = K_n$.

We now split the proof into several steps.

Step 1. In this step, we show that the function d defined in the statement of the lemma is a metric. For each n and $f, g \in C(\Sigma \times \Lambda)$, we define

$$d_n(f,g) = \sup_{(i,t) \in K_n} |f(i,t) - g(i,t)|, \quad d'_n(f,g) = \min\{d_n(f,g), 1\}$$

Then we have

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} d'_n(f,g).$$

Clearly each d_n is nonnegative and satisfies the triangle inequality, and it is then easy to see that the same properties hold for d'_n. Furthermore, $d'_n \leq 1$, so d is well-defined. Observe that d is nonnegative, and if $f = g$, then each $d'_n(f,g) = 0$, so the sum $d(f,g)$ is 0. Conversely, if $f \neq g$, then since the K_n cover $\Sigma \times \Lambda$, we can choose n large enough so that K_n contains an x with $f(x) \neq g(x)$. Then $d'_n(f,g) \neq 0$, and hence $d(f,g) \neq 0$. Lastly, the triangle inequality holds for d since it holds for each d'_n.

Step 2. Now we prove that the topology τ_d on $C(\Sigma \times \Lambda)$ induced by d is the same as the topology of uniform convergence over compacts, which we denote by τ_c. Recall that τ_c is generated by the basis consisting of sets

$$B_K(f,\epsilon) = \left\{g \in C(\Sigma \times \Lambda) : \sup_{(i,t) \in K} |f(i,t) - g(i,t)| < \epsilon\right\},$$

for $K \subset \Sigma \times \Lambda$ compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and τ_d is generated by sets of the form $B_\epsilon^d(f) = \{g : d(f,g) < \epsilon\}$.

We first show that $\tau_d \subseteq \tau_c$. It suffices to prove that every set $B_\epsilon^d(f)$ is a union of sets $B_K(f,\epsilon)$. First, choose $\epsilon > 0$ and $f \in C(\Sigma \times \Lambda)$. Let $g \in B_\epsilon^d(f)$. We will find a basis element A_g of τ_c such

that $g \in A_g \subset B_\epsilon^d(f)$. Let $\delta = d(f,g) < \epsilon$, and choose n large enough so that $\sum_{k>n} 2^{-k} < \frac{\epsilon-\delta}{2}$. Define $A_g = B_{K_n}(g, \frac{\epsilon-\delta}{n})$, and suppose $h \in A_g$. Then since $K_m \subseteq K_n$ for $m \leq n$, we have

$$d(f,h) \leq d(f,g) + d(g,h) \leq \delta + \sum_{k=1}^{n} 2^{-k} d_n(g,h) + \sum_{k>n} 2^{-k} \leq \delta + \frac{\epsilon-\delta}{2} + \frac{\epsilon-\delta}{2} = \epsilon.$$

Therefore $g \in A_g \subset B_\epsilon^d(f)$. Then we can write

$$B_\epsilon^d(f) = \bigcup_{g \in B_\epsilon^d(f)} A_g,$$

a union of basis elements of τ_c.

We now prove conversely that $\tau_c \subseteq \tau_d$. Let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$. Choose n so that $K \subset K_n$, and let $g \in B_K(f, \epsilon)$ and $\delta = \sup_{x \in K} |f(x) - g(x)| < \epsilon$. If $d(g,h) < 2^{-n}(\epsilon - \delta)$, then $d_n'(g,h) \leq 2^n d(g,h) < \epsilon - \delta$, hence $d_n(g,h) < \epsilon - \delta$, assuming without loss of generality that $\epsilon \leq 1$. It follows that

$$\sup_{x \in K} |f(x) - h(x)| \leq \delta + \sup_{x \in K} |g(x) - h(x)| \leq \delta + d_n(g,h) \leq \delta + \epsilon - \delta = \epsilon.$$

Thus $g \in B_{2^{-n}(\epsilon-\delta)}^d(g) \subset B_K(f, \epsilon)$, proving that $B_K(f, \epsilon) \in \tau_d$ by the same argument as above. We conclude that $\tau_d = \tau_c$.

Step 3. In this step, we show that $(C(\Sigma \times \Lambda), d)$ is a complete metric space. Let $\{f_n\}_{n \geq 1}$ be Cauchy with respect to d. Then we claim that $\{f_n\}$ must be Cauchy with respect to d_n', on each K_n. This follows from the observation that $d_n'(f_\ell, f_m) \leq 2^n d(f_\ell, f_m)$. Thus $\{f_n\}$ is Cauchy with respect to the uniform metric on each K_n, and hence converges uniformly to a continuous limit f^{K_n} on each K_n (see [15, Theorem 7.15]). Since the pointwise limit must be unique at each $x \in \Sigma \times \Lambda$, we have $f^{K_n}(x) = f^{K_m}(x)$ if $x \in K_n \cap K_m$. Since $\bigcup K_n = \Sigma \times \Lambda$, we obtain a well-defined function f on all of $\Sigma \times \Lambda$ given by $f(x) = \lim_{n \to \infty} f^{K_n}(x)$. We have $f \in C(\Sigma \times \Lambda)$ since $f|_{K_n} = f^{K_n}$ is continuous on K_n for all n. Moreover, if $K \subset \Sigma \times \Lambda$ is compact and n is large enough so that $K \subset K_n$, then because $f_n \to f^{K_n} = f|_{K_n}$ uniformly on K_n, we have $f_n \to f^{K_n}|_K = f|_K$ uniformly on K. That is, for any $K \subset \Sigma \times \Lambda$ compact and $\epsilon > 0$, we have $f_n \in B_K(f, \epsilon)$ for all sufficiently large n. Therefore $f_n \to f$ in τ_c, and equivalently in the metric d by Step 2.

Step 4. Lastly, we prove separability, c.f. [1, Example 1.3]. For each pair of positive integers n, k, let $D_{n,k}$ be the subcollection of $C(\Sigma \times \Lambda)$ consisting of polygonal functions that are piecewise linear on $\{j\} \times I_{n,k,i}$ for each $j \in \Sigma_n$ and each subinterval

$$I_{n,k,i} = \left[a_n + \tfrac{i-1}{k}(b_n - a_n), \, a_n + \tfrac{i}{k}(b_n - a_n)\right], \quad 1 \leq i \leq k,$$

taking rational values at the endpoints of these subintervals, and extended linearly to all of $\Lambda = [a,b]$. Then $D = \bigcup_{n,k} D_{n,k}$ is countable, and we claim that it is dense in τ_c. To see this, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and choose n so that $K \subset K_n$. Since f is uniformly continuous on K_n, we can choose k large enough so that for $0 \leq i \leq k$, if $t \in I_{n,k,i}$, then

$$\left| f(j,t) - f(j, a_n + \tfrac{i}{k}(b_n - a_n)) \right| < \epsilon/2$$

for all $j \in \Sigma_n$. We then choose $g \in \bigcup_k D_{n,k}$ with $|g(j, a_n + \tfrac{i}{k}(b_n - a_n)) - f(j, a_n + \tfrac{i}{k}(b_n - a_n))| < \epsilon/2$. Then we have

$$\left| f(j,t) - g(j, a_n + \tfrac{i-1}{k}(b_n - a_n)) \right| < \epsilon \quad \text{and} \quad \left| f(j,t) - g(j, a_n + \tfrac{i}{k}(b_n - a_n)) \right| < \epsilon.$$

Since $g(j, a_n + \tfrac{i-1}{k}(b_n - a_n)) \leq g(j,t) \leq g(j, a_n + \tfrac{i}{k}(b_n - a_n))$, it follows that

$$|f(j,t) - g(j,t)| < \epsilon$$

as well. In summary,

$$\sup_{(j,t)\in K} |f(j,t) - g(j,t)| \leq \sup_{(j,t)\in K_n} |f(j,t) - g(j,t)| < \epsilon,$$

so $g \in B_K(f,\epsilon)$. This proves that D is a countable dense subset of $C(\Sigma \times \Lambda)$.

8.2. **Proof of Lemma 2.4.** We first prove two lemmas that will be used in the proof of Lemma 2.4. The first result allows us to identify the space $C(\Sigma \times \Lambda)$ with a product of copies of $C(\Lambda)$. In the following, we assume the notation of Lemma 2.4.

Lemma 8.1. *Let $\pi_i : C(\Sigma \times \Lambda) \to C(\Lambda)$, $i \in \Sigma$, be the projection maps given by $\pi_i(F)(x) = F(i,x)$ for $x \in \Lambda$. Then the π_i are continuous. Endow the space $\prod_{i\in\Sigma} C(\Lambda)$ with the product topology induced by the topology of uniform convergence over compacts on $C(\Lambda)$. Then the mapping*

$$F : C(\Sigma \times \Lambda) \longrightarrow \prod_{i\in\Sigma} C(\Lambda), \quad f \mapsto (\pi_i(f))_{i\in\Sigma}$$

is a homeomorphism.

Proof. We first prove that the π_i are continuous. Since $C(\Sigma \times \Lambda)$ is metrizable by Lemma 2.2, and by a similar argument so is $C(\Lambda)$, it suffices to assume that $f_n \to f$ in $C(\Sigma \times \Lambda)$ and show that $\pi_i(f_n) \to \pi_i(f)$ in $C(\Lambda)$. Let K be compact in Λ. Then $\{i\} \times K$ is compact in $\Sigma \times \Lambda$, and $f_n \to f$ on $\{i\} \times K$ by assumption, so we have $\pi_i(f_n)|_K = f_n|_{\{i\}\times K} \to f|_{\{i\}\times K} = \pi_i(f)|_K$ uniformly on K. Since K was arbitrary, we conclude that $\pi_i(f_n) \to \pi_i(f)$ in $C(\Lambda)$ as desired.

We now observe that F is invertible. If $(f_i)_{i\in\Sigma} \in \prod_{i\in\Sigma} C(\Lambda)$, then the function f defined by $f(i,\cdot) = f_i(\cdot)$ is in $C(\Sigma \times \Lambda)$, since Σ has the discrete topology. This gives a well-defined inverse for F. It suffices to prove that F and F^{-1} are open maps.

We first show that F sends each basis element $B_K(f,\epsilon)$ of $C(\Sigma \times \Lambda)$ to a basis element in $\prod_{i\in\Sigma} C(\Lambda)$. Note that a basis for the product topology is given by products $\prod_{i\in\Sigma} B_{K_i}(f_i,\epsilon)$, where at most finitely many of the K_i are nonempty. Here, we use the convention that $B_\varnothing(f_i,\epsilon) = C(\Lambda)$. Let π_Σ, π_Λ denote the canonical projections of $\Sigma \times \Lambda$ onto Σ, Λ. The continuity of π_Σ implies that if $K \subset \Sigma \times \Lambda$ is compact, then $\pi_\Sigma(K)$ is compact in Σ, hence finite. Observe that the set $K \cap (\{i\} \times \Lambda)$ is an intersection of two compacts sets, hence compact in $\Sigma \times \Lambda$. Therefore the sets $K_i = \pi_\Lambda(K \cap (\{i\} \times \Lambda))$ are compact in Λ for each $i \in \Sigma$ since π_Λ is continuous. We observe that $F(B_K(f,\epsilon)) = \prod_{i\in\Sigma} U_i$, where

$$U_i = B_{K_i}(\pi_i(f),\epsilon), \quad \text{if} \quad i \in \pi_\Sigma(K),$$

and $U_i = C(\Lambda)$ otherwise. Since $\pi_\Sigma(K)$ is finite and the K_i are compact, we see that $F(B_K(f,\epsilon))$ is a basis element in the product topology as claimed.

Lastly, we show that F^{-1} sends each basis element $U = \prod_{i\in\Sigma} B_{K_i}(f_i,\epsilon)$ for the product topology to a set of the form $B_K(f,\epsilon)$. We have $K_i = \varnothing$ for all but finitely many i. Write $f = F^{-1}((f_i)_{i\in\Sigma})$ and $K = \prod_{i\in\Sigma} K_i$. By Tychonoff's theorem, [13, Theorem 37.3], K is compact in $\Sigma \times \Lambda$, and

$$F^{-1}(U) = B_K(f,\epsilon).$$

$\square$

We next prove a lemma which states that a sequence of line ensembles is tight if and only if all individual curves form tight sequences.

Lemma 8.2. *Suppose that $\{\mathcal{L}^n\}_{n\geq 1}$ is a sequence of Σ-indexed line ensembles on Λ, and let $X_i^n = \pi_i(\mathcal{L}^n)$. Then the X_i^n are $C(\Lambda)$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{L}^n\}$ is tight if and only if for each $i \in \Sigma$ the sequence $\{X_i^n\}_{n\geq 1}$ is tight.*

Proof. The fact that the X_i^n are random variables follows from the continuity of the π_i in Lemma 8.1 and [9, Theorem 1.3.5]. First suppose the sequence $\{\mathcal{L}^n\}$ is tight. By Lemma 2.2, $C(\Sigma \times \Lambda)$ is a Polish space, so it follows from Prohorov's theorem, [1, Theorem 5.1], that $\{\mathcal{L}^n\}$ is relatively compact. That is, every subsequence $\{\mathcal{L}^{n_k}\}$ has a further subsequence $\{\mathcal{L}^{n_{k_\ell}}\}$ converging weakly to some $\mathcal{L}$. Then for each $i \in \Sigma$, since π_i is continuous by the above, the subsequence $\{\pi_i(\mathcal{L}^{n_{k_\ell}})\}$ of $\{\pi_i(\mathcal{L}^{n_k})\}$ converges weakly to $\pi_i(\mathcal{L})$ by the continuous mapping theorem, [9, Theorem 3.2.10]. Thus every subsequence of $\{\pi_i(\mathcal{L}^n)\}$ has a convergent subsequence. Since $C(\Lambda)$ is a Polish space by the same argument as in the proof of Lemma 2.2, Prohorov's theorem implies that each $\{\pi_i(\mathcal{L}^n)\}$ is tight.

Conversely, suppose $\{X_i^n\}$ is tight for all $i \in \Sigma$. Then given $\epsilon > 0$, we can find compact sets $K_i \subset C(\Lambda)$ such that

$$\mathbb{P}(X_i^n \notin K_i) \leq \epsilon/2^i$$

for each $i \in \Sigma$. By Tychonoff's theorem, [13, Theorem 37.3], the product $\tilde{K} = \prod_{i \in \Sigma} K_i$ is compact in $\prod_{i \in \Sigma} C(\Lambda)$. We have

$$(8.1) \qquad \mathbb{P}\big((X_i^n)_{i \in \Sigma} \notin \tilde{K}\big) \leq \sum_{i \in \Sigma} \mathbb{P}(X_i^n \notin K_i) \leq \sum_{i=1}^{\infty} \epsilon/2^i = \epsilon.$$

By Lemma 8.1, we have a homeomorphism $G : \prod_{i \in \Sigma} C(\Lambda) \to C(\Sigma \times \Lambda)$. We observe that $G((X_i^n)_{i \in \Sigma}) = \mathcal{L}^n$, and $K = G(\tilde{K})$ is compact in $C(\Sigma \times \Lambda)$. Thus $\mathcal{L}^n \in K$ if and only if $(X_i^n)_{i \in \Sigma} \in \tilde{K}$, and it follows from (8.1) that

$$\mathbb{P}(\mathcal{L}^n \in K) \geq 1 - \epsilon.$$

This proves that $\{\mathcal{L}^n\}$ is tight.

$\square$

We are now ready to prove Lemma 2.4.

Proof. Fix an $i \in \Sigma$. By Lemma 8.2, it suffices to show that the sequence $\{\mathcal{L}_i^n\}_{n \geq 1}$ of $C(\Lambda)$-valued random variables is tight. By [1, Theorem 7.3], a sequence $\{P_n\}$ of probability measures on $C[0,1]$ with the uniform topology is tight if and only if the following conditions hold:

$$\lim_{a \to \infty} \limsup_{n \to \infty} P_n(|x(0)| \geq a) = 0,$$

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P_n\Big(\sup_{|s-t| \leq \delta} |x(s) - x(t)| \geq \epsilon \Big) = 0 \quad \text{for all } \epsilon > 0.$$

By replacing $[0,1]$ with $[a_m, b_m]$ and 0 with a_0, we see that the hypotheses in the lemma imply that the sequence $\{\mathcal{L}_i^n|_{[a_m,b_m]}\}_n$ is tight for every $m \geq 1$. Let $\pi_m : C(\Lambda) \to C([a_m, b_m])$ denote the map $f \mapsto f|_{[a_m,b_m]}$. Then π_m is continuous, since $C(\Lambda)$ and $C([a_m, b_m])$ with the topologies of uniform convergence over compacts are metrizable by Lemma 2.2, and if $f_n \to f$ uniformly on compact subsets of Λ, then $f_n|_{[a_m,b_m]} \to f|_{[a_m,b_m]}$ uniformly on compact subsets of $[a_m, b_m]$. It follows from [9, Theorem 1.3.5] that $\pi_m(\mathcal{L}^n) = \mathcal{L}_i^n|_{[a_m,b_m]}$ is a $C([a_m, b_m])$-valued random variable. Tightness of the sequence implies that for any $\epsilon > 0$, we can find compact sets $K_m \subset C([a_m, b_m])$ so that

$$\mathbb{P}\big(\pi_m(\mathcal{L}_i^n) \notin K_m\big) \leq \epsilon/2^m$$

for each $m \geq 1$. Writing $K = \bigcap_{m=1}^{\infty} \pi_m^{-1}(K_m)$, it follows that

$$\mathbb{P}\big(\mathcal{L}_i^n \in K\big) \geq 1 - \sum_{m=1}^{\infty} \epsilon/2^m = 1 - \epsilon.$$

To conclude tightness of $\{\mathcal{L}_i^n\}$, it suffices to prove that $K = \bigcap_{m=1}^{\infty} \pi_m^{-1}(K_m)$ is sequentially compact in $C(\Lambda)$. We argue by diagonalization. Let $\{f_n\}$ be a sequence in K, so that $f_n|_{[a_m,b_m]} \in K_m$ for every m, n. Since K_1 is compact, there is a sequence $\{n_{1,k}\}$ of natural numbers such that the

subsequence $\{f_{n_{1,k}}|_{[a_1,b_1]}\}_k$ converges in $C([a_1, b_1])$. Since K_2 is compact, we can take a further subsequence $\{n_{2,k}\}$ of $\{n_{1,k}\}$ so that $\{f_{n_{2,k}}|_{[a_2,b_2]}\}_k$ converges in $C([a_2, b_2])$. Continuing in this manner, we obtain sequences $\{n_{1,k}\} \supseteq \{n_{2,k}\} \supseteq \cdots$ so that $\{f_{n_{m,k}}|_{[a_m,b_m]}\}_k$ converges in $C([a_m, b_m])$ for all m. Writing $n_k = n_{k,k}$, it follows that the sequence $\{f_{n_k}\}$ converges uniformly on each $[a_m, b_m]$. If K is any compact subset of $C(\Lambda)$, then $K \subset [a_m, b_m]$ for some m, and hence $\{f_{n_k}\}$ converges uniformly on K. Therefore $\{f_{n_k}\}$ is a convergent subsequence of $\{f_n\}$.

$\square$

8.3. **Proof of Lemmas 3.1 and 3.2.** We will prove the following lemma, of which the two lemmas are immediate consequences. In particular, Lemma 3.1 is the special case when $g^b = g^t$, and Lemma 3.2 is the case when $\vec{x} = \vec{x}'$ and $\vec{y} = \vec{y}'$. We argue in analogy to [6, Lemma 5.6].

Lemma 8.3. *Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, $S \subseteq [\![T_0, T_1]\!]$, and two functions $g^b, g^t : [\![T_0, T_1]\!] \to [-\infty, \infty)$ with $g^b \leq g^t$ on S. Also fix $\vec{x}, \vec{y}, \vec{x}', \vec{y}' \in \mathfrak{W}_k$ such that $x_i \leq x_i'$, $y_i \leq y_i'$ for $1 \leq i \leq k$. Assume that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b; S)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}', \vec{y}', \infty, g^t; S)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\big($resp. $\mathfrak{L}^b\big)$ under $\mathbb{P}$ is given by $\mathbb{P}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g^t}_{avoid,Ber;S}$ $\big($resp. $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber;S}\big)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}^t_i(r) \geq \mathfrak{L}^b_i(r)$ for all $i = 1, \ldots, k$ and $r \in [\![T_0, T_1]\!]$.*

Proof. Throughout the proof, we will write $\Omega_{a,S}$ to mean $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b; S)$ and $\Omega'_{a,S}$ to mean $\Omega_{avoid}(T_0, T_1, \vec{x}', \vec{y}', \infty, g^t; S)$. We split the proof into two steps.

Step 1. We first aim to construct a Markov chain $(X^n, Y^n)_{n \geq 0}$, with $X^n \in \Omega_{a,S}$, $Y^n \in \Omega'_{a,S}$, with initial distribution given by

$$X_i^0(t) = \min(x_i + t - T_0, y_i), \qquad Y_i^0(t) = \min(x_i' + t - T_0, y_i'),$$

for $t \in [\![T_0, T_1]\!]$ and $1 \leq i \leq k$. First observe that we do in fact have $X^0 \in \Omega_{a,S}$, since $X_i^0(T_0) = x_i$, $X_i^0(T_1) = y_i$, $X_i^0(t) \leq \min(x_{i-1} + t - T_0, y_{i-1}) = X_{i-1}^0(t)$, and $X_k^0(t) \geq x_i + t - T_0 \geq g^b(T_0) + t - T_0 \geq g^b(t)$. We also note here that X^0 is *maximal* on the entire space $\Omega(T_0, T_1, \vec{x}, \vec{y})$, in the sense that for any $Z \in \Omega(T_0, T_1, \vec{x}, \vec{y})$, we have $Z_i(t) \leq X_i^0(t)$ for all $t \in [\![T_0, T_1]\!]$. In particular, X^0 is maximal on $\Omega_{a,S}$. Likewise, we see that Y^0 is maximal on $\Omega'_{a,S}$.

We want the chain (X^n, Y^n) to have the following properties:

(1) $(X^n)_{n \geq 0}$ and $(Y^n)_{n \geq 0}$ are both Markov in their own filtrations,
(2) (X^n) is irreducible and aperiodic, with invariant distribution $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber;S}$,
(3) (Y^n) is irreducible and aperiodic, with invariant distribution $\mathbb{P}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g^t}_{avoid,Ber;S}$,
(4) $X_i^n \leq Y_i^n$ on $[\![T_0, T_1]\!]$ for all $n \geq 0$ and $1 \leq i \leq k$.

This will allow us to conclude convergence of X^n and Y^n to these two uniform measures.

We specify the dynamics of (X^n, Y^n) as follows. At time n, we uniformly sample a triple $(i, t, z) \in [\![1, k]\!] \times [\![T_0, T_1]\!] \times [\![x_k, y_1' - 1]\!]$. We also flip a fair coin, with $\mathbb{P}(\text{heads}) = \mathbb{P}(\text{tails}) = 1/2$. We update X^n and Y^n using the following procedure. If $j \neq i$, we leave X_j, Y_j unchanged, and for all points $s \neq t$, we set $X_i^{n+1}(s) = X_i^n(s)$. If $T_0 < t < T_1$, $X_i^n(t-1) = z$, and $X_i^n(t+1) = z+1$ (note that this implies $X_i^n(t) \in \{z, z+1\}$), we consider two cases. If $t \in S$, then we set

$$X_i^{n+1}(t) = \begin{cases} z+1, & \text{if heads}, \\ z, & \text{if tails}, \end{cases}$$

assuming this does not cause $X_i^{n+1}(t)$ to fall below $X_{i+1}^n(t)$, with the convention that $X_{k+1}^n = g^b$. If $t \notin S$, we perform the same update regardless of whether it results in a crossing. In all other cases, we leave $X_i^{n+1}(t) = X_i^n(t)$. We update Y^n using the same rule, with g^t in place of g^b.

We first observe that X^n and Y^n are in fact non-intersecting on S for all n. Note X^0 is non-intersecting, and if X^n is non-intersecting, then the only way X^{n+1} could be intersecting on S is if the update were to push $X_i^{n+1}(t)$ below $X_{i+1}^n(t)$ for some i, t with $t \in S$. But any update of this form is suppressed, so it follows by induction that $X^n \in \Omega_{a,S}$ for all n. Similarly, we see that $Y^n \in \Omega'_{a,S}$.

It is easy to see that (X^n, Y^n) is a Markov chain, since at each time n, the value of (X^{n+1}, Y^{n+1}) depends only on the current state (X^n, Y^n), and not on the time n or any of the states prior to time n. Moreover, the value of X^{n+1} depends only on the state X^n, not on Y^n, so (X^n) is a Markov chain in its own filtration. The same applies to (Y^n). This proves the property (1) above.

We now argue that (X^n) and (Y^n) are irreducible. Fix any $Z \in \Omega_{a;S}$. As observed above, we have $Z_i \leq X_i^0$ on $[\![T_0, T_1]\!]$ for all i. We argue that we can reach the state Z starting from X^0 in some finite number of steps with positive probability. Due to the maximality of X^0, we only need to move the paths downward. If we do this starting with the bottom path, then there is no danger of the paths X_i crossing on S, or of X_k crossing g^b on S. To ensure that $X_k^n = Z_k$, we successively sample triples (k, t, z) as follows. We initialize $t = T_0 + 1$. If $X_k^n(t) = Z_k(t)$, we increment t by 1. Otherwise, we have $X_k^n(t) > Z_k(t)$, so we set $z = X_k^n(t) - 1$ and flip tails. This may or may not push $X_k(t)$ downwards by 1. We then increment t and repeat this process. If t reaches $T_1 - 1$, then at the increment we reset $t = T_0 + 1$. After finitely many steps, X_k will agree with Z_k on all of $[\![T_0, T_1]\!]$. We then repeat this process for X_i^n and Z_i, with i descending. Since each of these samples and flips has positive probability, and this process terminates in finitely many steps, the probability of transitioning from X^n to Z after some number of steps is positive. The same reasoning applies to show that (Y^n) is irreducible.

To see that the chains are aperiodic, simply observe that if we sample a triple (i, T_0, z) or (i, T_1, z), then the states of both chains will be unchanged.

To see that the uniform measure $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$ on $\Omega_{a,S}$ is invariant for (X^n), fix any $\omega \in \Omega_{a,S}$. For simplicity, write μ for the uniform measure. Then for all $\tau \in \Omega_{a,S}$, we have $\mu(\tau) = 1/|\Omega_{a,S}|$. Hence

$$\sum_{\tau \in \Omega_{a,S}} \mu(\tau) \mathbb{P}(X^{n+1} = \omega \mid X^n = \tau) = \frac{1}{|\Omega_{a,S}|} \sum_{\tau \in \Omega_{a,S}} \mathbb{P}(X^{n+1} = \omega \mid X^n = \tau) =$$

$$\frac{1}{|\Omega_{a,S}|} \sum_{\tau \in \Omega_{a,S}} \mathbb{P}(X^{n+1} = \tau \mid X^n = \omega) = \frac{1}{|\Omega_{a,S}|} \cdot 1 = \mu(\omega).$$

The second equality is clear if $\tau = \omega$. Otherwise, note that $\mathbb{P}(X_{n+1} = \omega \mid X_n = \tau) \neq 0$ if and only if τ and ω differ only in one indexed path (say the ith) at one point t, where $|\tau_i(t) - \omega_i(t)| = 1$, and this condition is also equivalent to $\mathbb{P}(X^{n+1} = \tau \mid X^n = \omega) \neq 0$. If $X^n = \tau$, there is exactly one choice of triple (i, t, z) and one coin flip which will ensure $X_i^{n+1}(t) = \omega(t)$, i.e., $X^{n+1} = \omega$. Conversely, if $X^n = \omega$, there is one triple and one coin flip which will ensure $X^{n+1} = \tau$. Since the triples are sampled uniformly and the coin flips are fair, these two conditional probabilities are in fact equal. This proves (2), and an analogous argument proves (3).

Lastly, we argue that $X_i^n \leq Y_i^n$ on $[\![T_0, T_1]\!]$ for all $n \geq 0$ and $1 \leq i \leq k$. This is of course true at $n = 0$. Suppose it holds at some $n \geq 0$, and suppose that we sample a triple (i, t, z). Then the update rule can only change the values of the $X_i^n(t)$ and $Y_i^n(t)$. Notice that the values can change by at most 1, and if $Y_i^n(t) - X_i^n(t) = 1$, then the only way the ordering could be violated is if Y_i were lowered and X_i were raised at the next update. But this is impossible, since a coin flip of heads can only raise or leave fixed both curves, and tails can only lower or leave fixed both curves. Thus it suffices to assume $X_i^n(t) = Y_i^n(t)$.

There are two cases to consider that violate the ordering of $X_i^{n+1}(t)$ and $Y_i^{n+1}(t)$. Either (i) $X_i(t)$ is raised but $Y_i(t)$ is left fixed, or (ii) $Y_i(t)$ is lowered yet $X_i(t)$ is left fixed. These can only occur if the curves exhibit one of two specific shapes on $[\![t - 1, t + 1]\!]$. For $X_i(t)$ to be raised,

we must have $X_i^n(t-1) = X_i^n(t) = X_i^n(t+1) - 1$, and for $Y_i(t)$ to be lowered, we must have $Y_i^n(t-1) - 1 = Y_i^n(t) = Y_i^n(t+1)$. From the assumptions that $X_i^n(t) = Y_i^n(t)$, and $X_i^n \leq Y_i^n$, we observe that both of these requirements force the other curve to exhibit the same shape on $[\![t-1, t+1]\!]$. Then the update rule will be the same for both curves for either coin flip, proving that both (i) and (ii) are impossible.

Step 2. It follows from (2) and (3) and [14, Theorem 1.8.3] that $(X^n)_{n \geq 0}$ and $(Y^n)_{n \geq 0}$ converge weakly to $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$ and $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g^t}$ respectively. In particular, (X^n) and (Y^n) are tight, so $(X^n, Y^n)_{n \geq 0}$ is tight as well. By Prohorov's theorem, it follows that (X^n, Y^n) is relatively compact. Let (n_m) be a sequence such that (X^{n_m}, Y^{n_m}) converges weakly. Then by the Skorohod representation theorem [1, Theorem 6.7], it follows that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting random variables $\mathfrak{X}^n$, $\mathfrak{Y}^n$ and $\mathfrak{X}, \mathfrak{Y}$ taking values in $\Omega_{a,S}, \Omega'_{a,S}$ respectively, such that

(1) The law of $(\mathfrak{X}^n, \mathfrak{Y}^n)$ under $\mathbb{P}$ is the same as that of (X^n, Y^n),
(2) $\mathfrak{X}^n(\omega) \longrightarrow \mathfrak{X}(\omega)$ for all $\omega \in \Omega$,
(3) $\mathfrak{Y}^n(\omega) \longrightarrow \mathfrak{Y}(\omega)$ for all $\omega \in \Omega$.

In particular, (1) implies that $\mathfrak{X}^{n_m}$ has the same law as X^{n_m}, which converges weakly to $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$. It follows from (2) and the uniqueness of limits that $\mathfrak{X}$ has law $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$. Similarly, $\mathfrak{Y}$ has law $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g^t}$. Moreover, condition (4) in Step 1 implies that $\mathfrak{X}_i^n \leq \mathfrak{Y}_i^n$, $\mathbb{P}$-a.s., so $\mathfrak{X}_i \leq \mathfrak{Y}_i$ for $1 \leq i \leq k$, $\mathbb{P}$-a.s. Thus we can take $\mathfrak{L}^b = \mathfrak{X}$ and $\mathfrak{L}^t = \mathfrak{Y}$.

$\square$

9. Appendix B

The goal of this section is to establish the weak convergence of scaled avoiding Bernoulli line ensemble. We consider the $[\![1,k]\!]$-indexed line ensembles with distribution given by $\mathbb{P}_{avoid,Ber}^{0,T,\vec{x},\vec{y},\infty,-\infty}$ in the sense of Definition 2.15. Recall that this is just the law of k independent Bernoulli random walks that have been conditioned to start from $\vec{x} = (x_1, \ldots, x_k)$ at time 0 and and at $\vec{y} = (y_1, \cdots, y_k)$ at time T and are always ordered. Here $\vec{x}, \vec{y} \in \mathfrak{W}_k$ satisfy $T \geq y_i - x_i \geq 0$ for $i = 1, \ldots, k$. We will drop the infinities and simply write $\mathbb{P}_{avoid,Ber}^{0,T,\vec{x},\vec{y}}$ for the measure.

This section will be divided into 6 subsections. In Section 9.1, we introduce some definitions, formulate the precise statements of two main results we want to prove as Proposition 9.3 and Proposition 9.5 and introduce Lemma 9.2 and Lemma 9.4 that help to prove them. In Section 9.2, we introduce some fundamental knowledge about Skew Schur Polynomials and give the distribution of avoiding Bernoulli line ensembles at integer points using Skew Schur Polynomials as Lemma 9.9. In Section 9.3, we will prove our first main result Proposition 9.3 assuming Lemma 9.2. In Section 9.4 we introduce some notations and results about multi-indices and multivariate functions which paves the way for proof of Proposition 9.5. Section 9.5 will prove our second main result Proposition 9.5 assuming Lemma 9.4. Finally, in Section 9.6, we prove Lemma 9.2 and Lemma 9.4 and complete our proof.

9.1. **Definitions and Main Results.** We start by introducing some helpful notations.

Definition 9.1. Fix $p, t \in (0,1)$, $k \in \mathbb{N}$, $\vec{a}, \vec{b} \in \mathbb{W}_k$ are two vectors in Weyl chamber defined in Definition 2.7. Suppose that $\vec{x}^T = (x_1^T, \cdots, x_k^T)$ and $\vec{y}^T = (y_1^T, \cdots, y_k^T)$ are two sequence of k-dimensional vectors with integer entries such that

$$\lim_{T \to \infty} \frac{x_i^T}{\sqrt{T}} = a_i \text{ and } \lim_{T \to \infty} \frac{y_i^T - pT}{\sqrt{T}} = b_i$$

for $i = 1, \ldots, k$. Define the sequence of random k-dimensional vectors Z^T by

$$Z^T = \left(\frac{L_1(tT) - ptT}{\sqrt{T}}, \cdots, \frac{L_k(tT) - ptT}{\sqrt{T}} \right),$$

where $(L_1, \cdots, L_k)$ is $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$-distributed.

We introduce some constants below

(9.1)
$$c_1(p,t) = \frac{1}{p(p+1)t}, \quad c_2(p,t) = \frac{1}{p(p+1)(1-t)}, \quad c_3(p,t) = \frac{1}{2p(p+1)t(1-t)}$$

$$Z = (2\pi)^{\frac{k}{2}} (p(p+1)t(1-t))^{\frac{k}{2}} \cdot e^{c_1(t,p) \sum_{i=1}^{k} a_i^2} \cdot e^{c_2(t,p) \sum_{i=1}^{k} b_i^2} \det \left(e^{-\frac{1}{2p(p+1)}(b_i - a_j)^2} \right)^k_{i,j=1}$$

and define the function $\rho(z_1, \cdots, z_k) \equiv \rho(\vec{z})$ as the following:

(9.2)
$$\rho(z_1, \cdots, z_k) = \frac{1}{Z} \cdot \mathbb{1}_{\{z_1 > \cdots > z_k\}} \cdot \det \left[e^{c_1(t,p)a_i z_j} \right]^k_{i,j=1} \det \left[e^{c_2(t,p)b_i z_j} \right]^k_{i,j=1} \prod_{i=1}^{k} e^{-c_3(t,p)z_i^2}$$

We can prove that the function $\rho(z)$ defined in (9.2) is a density function, meaning that it is non-negative and integrates to 1 over $\mathbb{R}^k$. Since this is an important ingredient of our results, we isolate it in the following lemma and will prove it in Section 9.6.

Lemma 9.2. *The function $\rho(z_1, \cdots, z_k)$ defined in (9.2) is a probability density function.*

Now we are ready to state our first main result, which gives the limiting distribution when vectors $\vec{a}$ and $\vec{b}$ contain distinct values.

Proposition 9.3. *Assume the same notation as in the Definition 9.1. When $a_1 > \cdots > a_k$ and $b_1 > \cdots > b_k$ are all distinct, the random vector Z^T converges weakly to a continuous distribution with the density in (9.2).*

Proposition 9.3 states the result when $\vec{a}$ and $\vec{b}$ consist of distinct values. When the values in $\vec{a}$ and $\vec{b}$ start to collide, the three determinants in the density function (9.2) will vanish(one in constant Z in equation (9.1) and the other two are in the expression of equation (9.2)). In the following, we are going to formulate the result under this new situation. We will construct a modified density function and the random vector Z^T will weakly converge to this new density function.

Suppose vectors $\vec{a}$ and $\vec{b}$ cluster as the following:

(9.3)
$$\vec{a}_0 = (a_1, \cdots, a_k) = (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p})$$

$$\vec{b}_0 = (b_1, \cdots, b_k) = (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q})$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_p$, $\beta_1 > \beta_2 > \cdots > \beta_q$ and $\sum_{i=1}^{p} m_i = \sum_{i=1}^{q} n_i = k$. Denote $\vec{m} = (m_1, \cdots, m_p)$, $\vec{n} = (n_1, \cdots, n_q)$ and define two determinants $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ and $\psi(\vec{b}_0, \vec{z}, \vec{n})$ below:

(9.4)
$$\varphi(\vec{a}_0, \vec{z}, \vec{m}) = \det \begin{bmatrix} ((c_1(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_1 z_j})_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ ((c_2(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_p z_j})_{\substack{i=1,\cdots,m_p \\ j=1,\cdots,k}} \end{bmatrix}$$

$$\psi(\vec{b}_0, \vec{z}, \vec{n}) = \det \begin{bmatrix} ((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_1 z_j})_{\substack{i=1,\cdots,n_1 \\ j=1,\cdots,k}} \\ \vdots \\ ((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_q z_j})_{\substack{i=1,\cdots,n_q \\ j=1,\cdots,k}} \end{bmatrix}$$

Then define the function

$$(9.5) \qquad H(\vec{z}) = \varphi(\vec{a}_0, \vec{z}, \vec{m}) \psi(\vec{b}_0, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{-c_3(t,p)z_i^2}$$

we can prove that $H(\vec{z})$ in (9.5) is non-negative and integrable over $\mathbb{R}^k$, which is formalized in the following lemma:

Lemma 9.4. *The function $H(\vec{z})$ is non-negative and integrable over $\mathbb{R}^k$. Thus, we can define the constant $Z_0 = \int_{\mathbb{R}^k} H(\vec{z}) \mathbb{1}_{\{z \in \mathbb{W}_k\}} dz < \infty$, and define the following density function:*

$$(9.6) \qquad \rho_{\vec{a}_0, \vec{b}_0}(z_1, \cdots, z_k) = \frac{1}{Z_0} \cdot \mathbb{1}_{\{z_1 > \cdots > z_k\}} \cdot \varphi(\vec{a}_0, \vec{z}, \vec{m}) \psi(\vec{b}_0, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{-c_3(t,p)z_i^2}$$

Now we are ready to state our second main result, which gives the weak convergence of Z^T when $\vec{a}$ and $\vec{b}$ have collided values.

Proposition 9.5. *Assume the same notation as in the Definition 9.1 and suppose vectors $\vec{a}_0$, $\vec{b}_0$ has the form in (9.3). Then, the random vector Z^T converges weakly to a continuous distribution with the density in (9.6).*

9.2. **Skew Schur polynomials and distribution of avoiding Bernoulli line ensembles.** We give some definitions and elementary results regarding skew Schur polynomials first, which are mainly from [12, Chapter 1].

Definition 9.6. *Partition, Interlaced, Tableau*
   (1) A *partition* is any infinite sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r, \cdots)$ of non-negative integers in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \cdots$ and containing only finitely many non-zero terms. The non-zero λ_i are called *parts* of λ, and the sum of the parts is the *weight* of λ, denoted by $|\lambda|$.
   (2) Suppose λ and μ are two partitions, we denote $\lambda \supset \mu$ if $\lambda_i \geq \mu_i$ for all $i \in \mathbb{Z}^+$, and we can define a new partition $\lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \cdots)$.
   (3) Partitions $\lambda = (\lambda_1, \lambda_2, \cdots)$ and $\mu = (\mu_1, \mu_2, \cdots)$ are call *interlaced*, denoted by $\mu \preceq \lambda$, if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$. By definition, if $\mu \preceq \lambda$, then $\mu \subset \lambda$.
   (4) A *tableau* $T = (\lambda^0, \lambda^1, \cdots \lambda^r)$ is a sequence of partitions such that $\lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r$.

Definition 9.7. *Complete Symmetric Function*
For each $r \geq 0$ the r-th *complete symmetric function* h_r is the sum of all monomials of total degree r in the variables $x_1, x_2, \cdots$ so that

$$(9.7) \qquad h_r = \sum_{|\lambda|=r} m_\lambda$$

where $m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ is the monomial with parts of partition λ as corresponding powers. For $r < 0$, we define h_r to be zero. In particular, when $x_1 = x_2 = \cdots = x_n = 1$, $x_{n+1} = x_{n+2} = \cdots = 0$, h_r is just the number of partitions that have weight r and n nonzero parts. Thus, we have:

$$(9.8) \qquad h_r(1^n) = \binom{r+n-1}{r}$$

Next, we introduce Skew Schur Polynomial based on [12, Chapter 1, (5.4), (5.11), (5.12)].

Definition 9.8. *Skew Schur Polynomial, Jacob-Trudi Formula*
   (1) Suppose $\mu \preceq \lambda$ are two interlaced partitions, then the *skew Schur polynomial* $s_{\lambda/\mu}$ with single variable x is defined by $s_{\lambda/\mu}(x) = x^{|\lambda - \mu|}$. Otherwise, we define $s_{\lambda/\mu} = 0$.

(2) Suppose $\lambda \supset \mu$ are two partitions, define the *skew Schur polynomial* $s_{\lambda/\mu}$ with respect to variables $x_1, x_2, \cdots, x_n$ by

$$(9.9) \qquad s_{\lambda/\mu}(x_1, \cdots, x_n) = \sum_{(\nu)} \prod_{i=1}^{n} s_{\nu^i/\nu^{i-1}}(x_i) = \sum_{(\nu)} \prod_{i=1}^{n} x_i^{|\nu^i - \nu^{i-1}|}$$

summed over all sequences $(\nu) = (\nu^0, \nu^1, \cdots, \nu^n)$ of partitions, such that $\nu^0 = \mu$, $\nu^n = \lambda$ and $\nu^0 \preceq \nu^1 \preceq \cdots \preceq \nu^n$. In particular, when $x_1 = x_2 = \cdots = x_n = 1$, the skew Schur polynomial is just the number of such sequences of interlaced partitions (ν).

(3) We also have the following *Jacob-Trudi Formula* [12, Chapter 1, (5.4)] for the skew Schur polynomial:

$$(9.10) \qquad s_{\lambda/\mu} = \det \left(h_{\lambda_i - \mu_j - i + j} \right)_{1 \leq i,j \leq n}$$

where h_r is the complete symmetric function in Definition 9.7.

Based on the above preparation, we are ready to state the following lemma giving the distribution of avoiding Bernoulli line ensembles at time $\lfloor tT \rfloor$.

Lemma 9.9. *Assume the same notations as in Section 9.1, denote $m = \lfloor tT \rfloor$, $n = T - \lfloor tT \rfloor$ and $T = m + n$. Then, the avoiding Bernoulli line ensemble at time m has the following distribution:*

$$(9.11) \qquad \mathbb{P}(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k) = \frac{s_{\lambda/\mu}(1^m) \cdot s_{\kappa/\lambda}(1^n)}{s_{\kappa/\mu}(1^T)}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ are positive integers, $s_{\lambda/\mu}$ denote skew Schur polynomials and they are specialized in all parameters equal to 1. The μ partition is just the vector $\vec{x}^T$ and the κ partition should be $\vec{y}^T$.

Remark 9.10. Here we let $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ be positive integers, althought they could potentially be negative. However, we can shift all the endpoints up such that all possible λ_i are positive. Also, in the proof we treat finite dimensional vectors as partitions because as long as we add infinitely many zeros at their ends we can and make them "partitions" in Definition 9.6.

Proof. Let $\Omega(0, T, \vec{x}^T, \vec{y}^T)$ be the set of all non-intersecting Bernoulli line ensembles from $\vec{x}^T$ to $\vec{y}^T$. For each line ensemble $\mathfrak{B} \in \Omega(0, T, \vec{x}^T, \vec{y}^T)$ with $\mathfrak{B} = (B_1, ..., B_k)$, we may define $\lambda^i(\mathfrak{B}) := (B_1(i), B_2(i), ..., B_k(i))$, where $1 \leq i \leq T$ is an integer. The $\lambda^i(\mathfrak{B})$ form partitions since by the definition of avoiding Bernoulli line ensembles, we have the inequality $B_\alpha(i) \geq B_\beta(i)$ if $\alpha < \beta$. Now we claim that partitions $\lambda^i(\mathfrak{B})$, $1 \leq i \leq T$ are interlaced. By definition of Bernoulli random walk, we have $B_\alpha(i+1) = B_\alpha(i) + \xi$ and $B_{\alpha+1}(i+1) = B_{\alpha+1}(i) + \eta$, where $\xi, \eta \in \{0, 1\}$, so $B_\alpha(i+1) \geq B_\alpha(i)$. If $B_\alpha(i) \geq B_{\alpha+1}(i) + 1$, then $B_\alpha(i) \geq B_{\alpha+1}(i+1)$; If $B_\alpha(i) = B_{\alpha+1}(i)$, in order to assure $B_\alpha(i+1) \geq B_{\alpha+1}(i+1)$, ξ and η have to be 0 or 1 at the same time, or $\xi = 1$ and $\eta = 0$, all of which deduce that $B_\alpha(i) \geq B_{\alpha+1}(i+1)$. Therefore,

$$B_{\alpha+1}(i+1) \leq B_\alpha(i) \leq B_\alpha(i+1)$$

and $\lambda^1(\mathfrak{B}) \preceq \cdots \preceq \lambda^T(\mathfrak{B})$ are interlaced. Note that when $i = 0$, we get $\lambda^0(\mathfrak{B}) = \vec{x}^T$ and $\lambda^T(\mathfrak{B}) = \vec{y}^T$.

Now, let us define the set

$$TB_{\lambda/\mu}^T := \{(\lambda^0, ..., \lambda^T) \mid \lambda^0 = \mu, \lambda^T = \lambda, \lambda^i \preceq \lambda^{i+1}\}$$

Now, if we take $f : \Omega(0, T, \vec{x}^T, \vec{y}^T) \to TB_{\kappa/\mu}^T$ with $f(\mathfrak{B}) = (\lambda_0(\mathfrak{B}), \cdots \lambda_T(\mathfrak{B}))$. We find that this function is in fact a bijection.

First, to show injectivity, suppose that there are two Bernoulli line ensembles, $\mathfrak{B}, \widetilde{\mathfrak{B}} \in \Omega(0, T, \vec{x}^T, \vec{y}^T)$ such that $\mathfrak{B} \neq \widetilde{\mathfrak{B}}$. Bernoulli line ensembles are determined by their values at integer times, so this would imply that there exists some (q, r) such that $0 \leq r \leq T$, $0 \leq q \leq k$ and $B_q(r) \neq \widetilde{\mathfrak{B}}_q(r)$ where

B_q and $\widetilde{B}_q$ are components of $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$ respectively. This implies that $\lambda^r(\mathfrak{B}) \neq \lambda^r(\widetilde{\mathfrak{B}})$, so we have injectivity.

Now, surjectivity follows since for any $\bar{\lambda} = (\lambda^0, ..., \lambda^T)$ we may define $\mathfrak{B}(\bar{\lambda}) = (B_1(\bar{\lambda}), ..., B_k(\bar{\lambda}))$ where $B_r(\bar{\lambda})(i) = \lambda_r^i$ where λ_r^i is the r-*th* entry of λ^i. The restrictions on $TB_{\kappa/\mu}^T$ ensure that each $\mathfrak{B}(\bar{\lambda}) \in \Omega(0, T, \bar{x}^T, \bar{y}^T)$, and so $f(\mathfrak{B}(\bar{\lambda})) = (\lambda^0, \cdots \lambda^T)$ by the definition $\mathfrak{B}(\bar{\lambda})$.

Applying the result regarding the relationship between number of sequences of interlaced partitions and skew Schur polynomial(Definition 9.8, (2)), we have $|TB_{\lambda/\mu}^T| = s_{\lambda/\mu}(1^T)$.

Therefore, we conclude that

$$\mathbb{P}(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k) = \frac{|\Omega(0, m, \vec{x}^T, \lambda)| \cdot |\Omega(m, T, \lambda, \vec{y}^T)|}{|\Omega(0, T, \vec{x}^T, \vec{y}^T)|}$$

$$= \frac{s_{\lambda/\vec{x}^T}(1^m) \cdot s_{\vec{y}^T/\lambda}(1^n)}{s_{\vec{y}^T/\vec{x}^T}(1^T)}$$

where $\lambda = (\lambda_1, \cdots, \lambda_k)$ is a partition such that $\vec{x}^T \preceq \lambda \preceq \vec{y}^T$. □

9.3. **Proof of Proposition 9.3.** In this section, we prove Proposition 9.3 assuming the validity of Lemma 9.2. First,we show that the limiting function of probability mass function (9.11) in Lemma 9.9 is exactly the function in (9.2). We then use Lemma 9.2 to show that the limiting function is a probability density function. Finally, we prove that the random vector Z^T weakly converges to the density (9.2).

We start with finding the asymptotic behavior of skew Schur polynomials in (9.11). By the Jacob-Trudi formula (9.10), we only need to find the asymptotic formula for complete symmetric functions. The result is stated as the following lemma.

Lemma 9.11. *Assume the same notation as in Section 9.2. We then have the following asymptotic formula*

(9.12)

$$h_{\lambda_i - x_j^T + j - i}(1^{\lfloor tT \rfloor}) = \sqrt{2\pi}^{-1} \sqrt{\frac{p+1}{pt}} \sqrt{T}^{-1} \cdot Exp\Big\{ ((p+1)tT) \log((p+1)tT) - (ptT) \log(ptT)$$

$$- (tT) \log(tT) + (-i+j) \log(\frac{p+1}{p}) - \log(p+1) + (z_i - a_j)\sqrt{T} \cdot \log(\frac{p+1}{p}) - \frac{1}{2} \frac{(z_i - a_j)^2}{p(p+1)t} + o(1) \Big\}$$

where $S_1(p, t, T) = ((p+1)tT) \log((p+1)tT) - (ptT) \log(ptT) - (tT) \log(tT)$.

Proof. We first compute the first determinant in the numerator. Using the identity for complete symmetric functions([12, Example 1, Section I.2]) that $h_r(1^n) = \binom{n+r-1}{r}$, we get the resulting equation

(9.13) $$h_{\lambda_i - x_j^T + j - i}(1^{\lfloor tT \rfloor}) = \frac{(\lambda_i - x_j^T - i + j + \lfloor tT \rfloor - 1)!}{(\lambda_i - x_j^T - i + j)!(\lfloor tT \rfloor - 1)!}$$

We have the following Stirling's formula [10],

$$n! = \sqrt{2\pi n} n^n e^{-n} e^{r_n}, \text{ where } \frac{1}{12n+1} < r_n < \frac{1}{12n}$$

Write $K = \lambda_i - x_j^T - i + j + \lfloor tT \rfloor - 1$ and apply the above Stirling's formula to get

$$K! = \sqrt{2\pi} \sqrt{K} \cdot e^{K \log K - K + r_K}$$

Additionally, since $\lambda_i = \lfloor z_i \sqrt{T} + ptT \rfloor$ and $x_i^T = a_i \sqrt{T} + o(\sqrt{T})$, we get

$$K = \lambda_i - x_j^T - i + j + \lfloor tT \rfloor - 1 = (z_i - a_j)\sqrt{T} + (p+1)tT - i + j - 1 + o(\sqrt{T})$$

$$\log K = \log\left(1 + \frac{-i+j-1}{(z_i - a_j)\sqrt{T} + (p+1)tT} + o(\frac{1}{\sqrt{T}})\right) + \log((z_i - a_j)\sqrt{T} + (p+1)tT)$$

$$= \frac{-i+j-1}{(z_i - a_j)\sqrt{T} + (p+1)tT} + \log((z_i - a_j)\sqrt{T} + (p+1)tT) + o(\frac{1}{\sqrt{T}})$$

where the constant in little o notation only depends on A, p and does not depend on T. Next, we compute $K \log K$:

$$K \log K = (-i+j-1) + \left[(z_i - a_j)\sqrt{T} + (p+1)tT\right] \cdot \log\left((z_i - a_j)\sqrt{T} + (p+1)tT\right)$$

$$+ (-i+j-1)\log\left((p+1)tT\right) + o(\sqrt{T})$$

Now we further compute the term $\left[(z_i - a_j)\sqrt{T} + (p+1)tT\right] \cdot \log\left((z_i - a_j)\sqrt{T} + (p+1)tT\right)$. Notice that

$$\log\left((z_i - a_j)\sqrt{T} + (p+1)tT\right) = \log\left((p+1)tT\right) + \log\left(1 + \frac{z_i - a_j}{(p+1)t\sqrt{T}}\right)$$

$$= \log((p+1)tT) + \frac{z_i - a_j}{(p+1)t\sqrt{T}} - \frac{1}{2}\frac{(z_i - a_j)^2}{(p+1)^2 t^2 T} + o(\frac{1}{T})$$

Then,

$$\left[(z_i - a_j)\sqrt{T} + (p+1)tT\right] \cdot \log\left((z_i - a_j)\sqrt{T} + (p+1)tT\right)$$

$$= \left[(z_i - a_j)\sqrt{T} + (p+1)tT\right] \cdot \left[\log((p+1)tT) + \frac{z_i - a_j}{(p+1)t\sqrt{T}} - \frac{1}{2}\frac{(z_i - a_j)^2}{(p+1)^2 t^2 T} + o(\frac{1}{T})\right]$$

$$= ((p+1)tT)\log((p+1)tT) + (z_i - a_j)\sqrt{T} \cdot \log((p+1)tT) + (z_i - a_j)\sqrt{T} - \frac{1}{2} \cdot \frac{(z_i - a_j)^2}{(p+1)t} + o(1)$$

Therefore, we find that $(\lambda_i - x_j^T - i + j + \lfloor tT \rfloor - 1)! =$

(9.14)
$$\sqrt{2\pi}\sqrt{(p+1)tT} \cdot \text{Exp}\{(-i+j-1) + ((p+1)tT)\log((p+1)tT) + (z_i - a_j)\sqrt{T} \cdot ((p+1)tT)$$

$$+ (z_i - a_j)\sqrt{T} - \frac{1}{2} \cdot \frac{(z_i - a_j)^2}{(p+1)t} + (-i+j-1)\log((p+1)tT)$$

$$- \left((z_i - a_j)\sqrt{T} + (p+1)tT - i + j - 1\right) + o(1)\}$$

Similarly, $(\lambda_i - x_j^T - i + j)! =$

(9.15)

$$\sqrt{2\pi}\sqrt{ptT} \cdot \text{Exp}\{(-i+j) + (ptT)\log(ptT) + (z_i - a_j)\sqrt{T} \cdot \log(ptT) + (z_i - a_j)\sqrt{T}$$

$$+ \frac{1}{2} \cdot \frac{(z_i - a_j)^2}{pt} + (-i+j)\log(ptT) - ((z_i - a_j)\sqrt{T} + ptT - i + j) + o(1)\}$$

and the final term:

(9.16) $$(\lfloor tT \rfloor - 1)! = \sqrt{2\pi}\sqrt{tT} \cdot \text{Exp}\{(tT)\log(tT) - 1 - \log(tT) + o(1)\}$$

Plugging (9.14), (9.15) and (9.16) into equation (9.13) we get $h_{\lambda_i - x_j^T + j - i}(1^{\lfloor tT \rfloor}) =$

$$
\sqrt{2\pi}^{-1} \sqrt{\frac{p+1}{pt}} \sqrt{T}^{-1} \cdot \mathrm{Exp} \bigg\{ ((p+1)tT) \log((p+1)tT) - (ptT) \log(ptT) - (tT) \log(tT)
$$

$$
+ (-i+j) \log(\frac{p+1}{p}) - \log(p+1) + (z_i - a_j)\sqrt{T} \cdot \log(\frac{p+1}{p}) - \frac{1}{2} \frac{(z_i - a_j)^2}{p(p+1)t} + o(1) \bigg\}
$$

where the constant in little o notation only depends on A, p and does not depend on T.   □

9.4. **Multi-indices and multivariate functions.**

9.5. **Proof of Proposition 9.5.**

9.6. **Proof of Lemma 9.2 and Lemma 9.4.** The following lemma helps us to prove Proposition 9.3. It shows the asymptotic formula for the distribution of avoiding Bernoulli line ensembles at time $\lfloor tT \rfloor$.

Lemma 9.12. *Let $\mathbb{W}_k^o$ denote the open Weyl chamber in $\mathbb{R}^N$:*

$$
\mathbb{W}_N^o = \{(x_1, \cdots, x_k) \in \mathbb{R}^N : x_1 > x_2 > \cdots > x_k\}
$$

Fix a real number $A > 0$, $p, t \in (0,1)$, take $z = (z_1, \cdots, z_k) \in \mathbb{W}_k^o$ such that $A > z_1 > \cdots > z_k > -A$. Choose sufficiently large T_0 such that $ptT_0 - A\sqrt{T_0} \geqslant 1$, then for $T \geqslant T_0$, define $\lambda_i(T) = \lfloor z_i \sqrt{T} + ptT \rfloor \geqslant 1$ for $i = 1, \cdots, k$. Denote $P_T(z) = (\sqrt{T})^k \mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1(T), \cdots, L_k(\lfloor tT \rfloor) = \lambda_k(T))$, then we have for almost every $z \in [-A, A]^k$:

$$
\lim_{T \to \infty} P_T(z) = \rho(z_1, \cdots, z_k)
$$

and $P_T(z)$ is bounded on $[-A, A]^k$, where $\rho(z)$ is given in Proposition 9.3.

Proof. (i) First, we discuss the pointwise convergence of $P_T(z)$. By the Jacobi-Trudi formula([12, Chapter 1, (5.4)]), we conclude:

(9.17)

$$
\mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \cdots, L_k(\lfloor tT \rfloor) = \lambda_k) = \frac{\det \left(h_{\lambda_i - x_j^T + j - i}(1^{\lfloor tT \rfloor}) \right)_{i,j=1}^k \cdot \det \left(h_{y_i^T - \lambda_j + j - i}(1^{T - \lfloor tT \rfloor}) \right)_{i,j=1}^k}{\det \left(h_{y_i^T - x_j^T + j - i}(1^T) \right)_{i,j=1}^k}
$$

We write $S_1(p, t, T) = ((p+1)tT) \log((p+1)tT) - (ptT) \log(ptT) - (tT) \log(tT)$ and further calculate the determinant

$$
\det(h_{\lambda_i - x_j - i + j}(1^{\lfloor tT \rfloor}))_{i,j=1}^k = \left[(\sqrt{2\pi})^{-1} \sqrt{\frac{p+1}{pt}} \sqrt{T}^{-1} e^{S_1(p,t,T) - \log(p+1)} \right]^k
$$

$$
\cdot \det \left(e^{(-i+j) \log(\frac{p+1}{p}) + (z_i - a_j)\sqrt{T} \cdot \log(\frac{p+1}{p}) - \frac{1}{2} \frac{(z_i - a_j)^2}{p(p+1)t} + o(1)} \right)_{i,j=1}^k
$$

$$
= \left[(\sqrt{2\pi})^{-1} \sqrt{\frac{p+1}{pt}} \sqrt{T}^{-1} e^{S_1(p,t,T) - \log(p+1)} \right]^k \left(\frac{p+1}{p} \right)^{\sum_{i=1}^k (z_i - a_i) \cdot \sqrt{T}}
$$

$$
\cdot e^{-\frac{1}{2p(p+1)t} \sum_{i=1}^k (a_i^2 + z_i^2)} \cdot \det \left(e^{c_1(p,t) z_i a_j + o(1)} \right)_{i,j=1}^k
$$

where the constant $c_1(p, t) = \frac{1}{p(p+1)t}$.

We calculate the other two determinants in equation 9.17 analogously:

$$\det\left(h_{y_i-\lambda_j-i+j}(1^{T-\lfloor tT\rfloor})\right) = \left[(\sqrt{2\pi})^{-1}\sqrt{\frac{p+1}{p(1-t)}}\sqrt{T}^{-1}\cdot e^{S_2(p,t,T)-\log(p+1)}\right]^k \left(\frac{p+1}{p}\right)^{\sum_{i=1}^k (b_i-z_i)\cdot\sqrt{T}}$$

$$\cdot\, e^{-\frac{1}{2p(p+1)(1-t)}\sum_{i=1}^k (b_i^2+z_i^2)}\cdot\det\left(e^{c_2(p,t)b_iz_j+o(1)}\right)_{i,j=1}^k$$

$$\det(h_{y_i^T-x_j^T-i+j}(1^T)) = \left[(\sqrt{2\pi})^{-1}\sqrt{\frac{p+1}{p}}\sqrt{T}^{-1}\cdot e^{S_3(p,t,T)-\log(p+1)}\right]^k \left(\frac{p+1}{p}\right)^{\sum_{i=1}^k (b_i-a_i)\cdot\sqrt{T}}$$

$$\cdot\,\det\left(e^{-\frac{1}{2p(p+1)}(b_i-a_j)^2+o(1)}\right)_{i,j=1}^k$$

where the constants $S_2(p,t,T)$, $S_3(p,t,T)$, $c_2(p,t)$ are:

$$S_2(p,t,T) = ((p+1)(1-t)T)\log((p+1)tT) - (p(1-t)T)\log(ptT) - ((1-t)T)\log((1-t)T)$$

$$S_3(p,t,T) = ((p+1)T)\log((p+1)tT) - (pT)\log(pT) - T\log T, \quad c_2(p,t) = \frac{1}{p(p+1)(1-t)}$$

Notice that $S_1(p,t,T) + S_2(p,t,T) - S_3(p,t,T) = 0$. Plugging the above three determinants into equation 9.17, we get

$$\mathbb{P}(L_1(\lfloor tT\rfloor)=\lambda_1,\cdots,L_k(\lfloor tT\rfloor)=\lambda_k)$$

$$=(2\pi)^{-\frac{k}{2}}\left[\sqrt{\frac{p+1}{pt(1-t)}}\right]^k\cdot T^{-\frac{k}{2}}\cdot e^{-\frac{1}{2p(p+1)t}\sum_{i=1}^k(a_i^2+z_i^2)-\frac{1}{2p(p+1)(1-t)}\sum_{i=1}^k(b_i^2+z_i^2)}$$

$$\cdot\, e^{-k\log(p+1)}\cdot\frac{\det(e^{c_1(p,t)z_ia_j+o(1)})_{i,j=1}^k\cdot\det(e^{c_2(p,t)b_iz_j+o(1)})_{i,j=1}^k}{\det(e^{-\frac{1}{2p(p+1)}(b_i-a_j)^2+o(1)})_{i,j=1}^k}$$

(9.18)

$$=(2\pi)^{-\frac{k}{2}}\left[\sqrt{\frac{1}{p(p+1)t(1-t)}}\right]^k\cdot T^{-\frac{k}{2}}\cdot e^{-\frac{1}{2p(p+1)t}\sum_{i=1}^k a_i^2-\frac{1}{2p(p+1)(1-t)}\sum_{i=1}^k b_i^2}$$

$$\cdot\,\frac{\det\left(e^{c_1(p,t)z_ia_j}\right)_{i,j=1}^k\cdot\det\left(e^{c_2(p,t)b_iz_j}\right)_{i,j=1}^k}{\det\left(e^{-\frac{1}{2p(p+1)}(b_i-a_j)^2}\right)_{i,j=1}^k}\cdot\exp\{o(1)\}\cdot\prod_{i=1}^k e^{-c_3(t,p)z_i^2}$$

where $c_3(t,p) = \frac{1}{2p(p+1)t(1-t)}$, and the constant in little o notation depends on A,p.

In conclusion, $P_T(z) = (\sqrt{T})^k\cdot\mathbb{P}(L_1(\lfloor tT\rfloor)=\lambda_1,\cdots,L_k(\lfloor tT\rfloor)=\lambda_k)$ converges to $\rho(z_1,\ldots,z_k)$ in Proposition 9.3 as $T\to\infty$ and

$$c_1(p,t) = \frac{1}{p(p+1)t}, \quad c_2(p,t) = \frac{1}{p(p+1)(1-t)}, \quad c_3(p,t) = \frac{1}{2p(p+1)t(1-t)}$$

$$Z = (2\pi)^{\frac{k}{2}}(p(p+1)t(1-t))^{\frac{k}{2}}\cdot e^{c_1(t,p)\sum_{i=1}^k a_i^2}\cdot e^{c_2(t,p)\sum_{i=1}^k b_i^2}\det\left(e^{-\frac{1}{2p(p+1)}(b_i-a_j)^2}\right)_{i,j=1}^k$$

(ii) We now discuss boundedness. By the Equation 9.18 we just derived, $P_T(z) = \rho(z)\cdot\exp\{o(1)\}$ on the compact set $[-A,A]^k$, where the constant in little o notation only depends on A,p. Since the continuous function $\rho(z)$ is bounded on $[-A,A]^k$, and $\exp\{o(1)\}$ is uniformly bounded on $[-A,A]^k$, we conclude that $P_T(z)$ is bounded as well. $\qquad\square$

We are now ready to prove Proposition 9.3.

Proof of Proposition 9.3. In the following, we prove the weak convergence of the random vector Z^T, when $\vec{a} = (a_1,\cdots,a_k)$ and $\vec{b} = (b_1,\cdots,b_k)$ consist of distinct entries. In order to show weak

convergence, it is sufficient to show that for every open set $O \in \mathbb{R}^k$, we have:

$$\liminf_{T \to \infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O) \geqslant \int_O \rho(z_1, \cdots, z_k) dz_1 dz_2 \cdots dz_k$$

according to [9, Theorem 3.2.11]. It is also sufficient to show that for any open set $U \in \mathbb{W}_k^o$, we have:

$$(9.19) \qquad \liminf_{T \to \infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in U) \geqslant \int_U \rho(z_1, \cdots, z_k) dz_1 dz_2 \cdots dz_k$$

which implies that:

$$\liminf_{T \to \infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O) \geqslant \liminf_{T \to \infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O \cap \mathbb{W}_k^o)$$

$$\geqslant \int_{\mathbb{W}_k^o \cap O} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k = \int_O \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k$$

The second inequality uses the above result (9.19), since $\mathbb{W}_k^o \cap O$ is an open set in $\mathbb{W}_k^o$. The last equality is because $\rho(z)$ is zero outside $\mathbb{W}_k^o$. The rest of the proof will be divided into 4 steps. In Step 1, we prove that weak convergence holds on every closed rectangle. In Step 2, we prove the result 9.19 using Lemma 9.12. In Step 3, we prove that $\rho(z)$ is actually a density and conclude weak convergence.

Step 1. In this step, we establish the following result:
For any closed rectangle $R = [u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_N, v_N] \in \mathbb{W}_k^o$,

$$(9.20) \qquad \lim_{T \to \infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_R \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k$$

where $\rho(z)$ is given in Proposition 9.3.

Define $m_i^T = \lceil u_i \sqrt{T} + ptT \rceil$ and $M_i^T = \lfloor v_i \sqrt{T} + ptT \rfloor$. Then we have:

$$\mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \mathbb{P}(u_1 \leqslant Z_1^T \leqslant v_1, \ldots, u_k \leqslant Z_k^T \leqslant v_k)$$

$$= \mathbb{P}(u_i \sqrt{T} + ptT \leqslant L_i(\lfloor tT \rfloor) \leqslant v_i \sqrt{T} + ptT, i = 1, \ldots, k)$$

$$= \sum_{\lambda_1(T)=m_1^T}^{M_1^T} \cdots \sum_{\lambda_k(T)=m_k^T}^{M_k^T} \mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1(T), \ldots, L_k(\lfloor tT \rfloor) = \lambda_k(T))$$

$$= \sum_{\lambda_1(T)=m_1^T}^{M_1^T} \cdots \sum_{\lambda_k(T)=m_k^T}^{M_k^T} (\sqrt{T})^{-k} \cdot (\sqrt{T})^k \mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1(T), \ldots, L_k(\lfloor tT \rfloor) = \lambda_k(T))$$

Find sufficiently large A such that $R \subset [-A, A]^k$, for example, $A = 1 + \max_{1 \leqslant i \leqslant k} |a_i| + \max_{1 \leqslant i \leqslant k} |b_i|$. Define $f_T(z_1, \cdots, z_k)$ as a simple function on $\mathbb{R}^k$: When $(z_1, \cdots, z_k) \in R$, it takes value $(\sqrt{T})^k \cdot \mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1(T), \cdots, L_k(\lfloor tT \rfloor) = \lambda_k(T))$ if there exist $\lambda_1(T), \cdots, \lambda_k(T)$ such that $\lambda_i(T) \leqslant z_i \sqrt{T} + ptT < \lambda_i(T) + 1$; It takes value 0 otherwise, when $(z_1, \cdots, z_k) \notin R$. Since the Lebesgue measure of the set $\{z : \lambda_i(T) \leqslant z_i \sqrt{T} + ptT < \lambda_i(T) + 1, i = 1, \cdots, k\}$ is $(\sqrt{T})^{-k}$, the above probability can be further written as an integral of simple functions $f_T(z_1, \cdots, z_k)$:

$$\mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_{[-A,A]^k} f_T(z_1, \cdots, z_k) dz_1 \cdots dz_k$$

By Lemma 9.12, the function $f_T(z_1, \cdots, z_k)$ pointwise converges to $\rho(z)$ and is bounded on the compact set $[-A, A]^k$. Since the Lebesgue measure of $[-A, A]^k$ is finite, by the bounded convergence theorem we have:

$$(9.21) \qquad \lim_{T \to \infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_R \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k$$

Step 2. In this step, we prove the statement 9.19. Take any open set $U \in \mathbb{W}_k^o$. Then this set can be written as a countable union of closed rectangles with disjoint interiors: $U = \bigcup_{i=1}^{\infty} R_i$, where $R_i = [a_1^i, b_1^i] \times \cdots \times [a_k^i, b_k^i]$([16, Theorem 1.4]). Choose sufficiently small $\epsilon > 0$, and denote $R_i^{\epsilon} = [a_1^i + \epsilon, b_1^i - \epsilon] \times \cdots \times [a_k^i + \epsilon, b_k^i - \epsilon]$, then R_i^{ϵ} are disjoint. Therefore,

$$\liminf_{T \to \infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in U) \geqslant \liminf_{T \to \infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \bigcup_{i=1}^{n} R_i^{\epsilon})$$

$$= \liminf_{T \to \infty} \sum_{i=1}^{n} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R_i^{\epsilon}) = \sum_{i=1}^{n} \int_{R_i^{\epsilon}} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k$$

$$= \int_{\bigcup_{i=1}^{n} R_i^{\epsilon}} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \xrightarrow{\epsilon \downarrow 0} \int_U \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k$$

The last line uses the monotone convergence theorem. Thus, we have proved the inequality 9.19.

Step 3. In this step, we prove that $\rho(z)$ is actually a density. First, it is nonnegative because it is the limit of a sequence of probabilities. Next, we prove it integrates to 1 over $\mathbb{R}^k$. Let the open set U in Step 2 be $\mathbb{W}_k^o$ to get:

$$1 = \liminf_{T \to \infty} \mathbb{P}(Z^T \in \mathbb{W}_k^o) \geqslant \int_{\mathbb{W}_k^o} \rho(z) dz$$

On the other hand, write the open set $\mathbb{W}_k^o$ as a countable union of almost disjoint closed rectangles: $\mathbb{W}_k^o = \bigcup_{i=1}^{\infty} R_i$. Then, for any $n \in \mathbb{Z}^+$ we can find $N_n \in \mathbb{Z}^+$ such that $\mathbb{P}(Z^T \in \mathbb{W}_k^o) - \mathbb{P}(Z^T \in \bigcup_{i=1}^{N_n} R_i) \leq 1/n$, and $N_n \to \infty$ as $n \to \infty$.

$$1 = \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \mathbb{W}_k^o) \leq \mathbb{P}(Z^T \in \bigcup_{i=1}^{N_n} R_i) + 1/n \leq \sum_{i=1}^{N_n} \mathbb{P}(Z^T \in R_i) + 1/n$$

Let $T \to \infty$. Then we obtain

$$1 \leq \sum_{i=1}^{N_n} \int_{R_i} \rho(z) dz + 1/n = \int_{\bigcup_{i=1}^{N_n} R_i} \rho(z) dz + 1/n$$

due to (9.21) and the fact that $\rho(z)$ integrates to 0 over the boundary of rectangles. Finally, let $n \to \infty$. We then have $\int_{\mathbb{W}_k^o} \rho(z) dz \geq 1$ by the monotone convergence theorem and conclude that $\rho(z)$ is a density. $\qquad \square$

Next we are going to prove Proposition 9.5. Before that, we first introduce some notations and results about multivariate functions.

Suppose $\sigma = (\sigma_1, \cdots, \sigma_n)$ is a multi-index of *length* n. In our context, we require $\sigma_1, \cdots, \sigma_n$ be all non-negative integers(some of them might be equal). We define $|\sigma| = \sum_{i=1}^n \sigma_i$ as the *order* of σ. Suppose $\tau = (\tau_1, \cdots, \tau_n)$ is another multi-index of length n. We say $\tau \leqslant \sigma$ if $\tau_i \leqslant \sigma_i$ for $i = 1, \cdots, n$. We say $\tau < \sigma$ if $\tau \leqslant \sigma$ and there exists at least one index i such that $\tau_i < \sigma_i$. Then, define the partial derivative with respect to the multi-index σ:

$$D^{\sigma} f(x_1, \cdots, x_n) = \frac{\partial^{|\sigma|} f(x_1, \cdots, x_n)}{\partial x_1^{\sigma_1} \partial x_2^{\sigma_2} \cdots \partial x_n^{\sigma_n}}$$

We have the general Leibniz rule:

$$D^{\sigma}(fg) = \sum_{\tau \leqslant \sigma} \binom{\sigma}{\tau} D^{\tau} f \cdot D^{\sigma - \tau} g$$

where $\binom{\sigma}{\tau} = \frac{\sigma_1! \cdots \sigma_n!}{\tau_1! \cdots \tau_n!(\sigma_1 - \tau_1)! \cdots (\sigma_n - \tau_n)!}$.

We also have the Taylor expansion for multi-variable functions:

$$f(x_1, \cdots, x_n) = \sum_{|\sigma| \leqslant r} \frac{1}{\sigma!} D^\sigma f(\vec{x}_0)(\vec{x} - \vec{x}_0)^\sigma + R_{r+1}(\vec{x}, \vec{x}_0)$$

In the equation, $\sigma! = \sigma_1! \sigma_2! \cdots \sigma_n!$ is the factorial with respect to the multi-index σ, $\vec{x}_0 = (x_1^0, \cdots, x_n^0)$ is a constant vector at which we expands the function f, $(\vec{x} - \vec{x}_0)^\sigma$ stands for $(x_1 - x_1^0)^{\sigma_1} \cdots (x_n - x_n^0)^{\sigma_n}$, and

$$R_{r+1}(\vec{x}, \vec{x}_0) = \sum_{\sigma:|\sigma|=r+1} \frac{1}{\sigma!} D^\sigma f(\vec{x}_0 + \theta(\vec{x} - \vec{x}_0))(\vec{x} - \vec{x}_0)^\sigma$$

is the remainder, where $\theta \in (0,1)$ ([2, Theorem 3.18 & Corollary 3.19]).

We also need some knowledge about *permutation*. Suppose s_n is a permutation of n non-negative integers, for example $\{1, \cdots, n\}$, and $s_n(i)$ represents the *i-th* element in the permutation s_n. We define *the number of inversions* of s_n by $I(s_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbb{1}_{\{s_n(i)>s_n(j)\}}$. For example, the permutation $s_n = (1, \cdots, n)$ has 0 number of inversions, while the permutation $s_5 = (3, 2, 5, 1, 4)$ has number of inversions $5(2+1+2+0+0)$. Define the sign of permutation s_n by $sgn(s_n) = (-1)^{I(s_n)}$. For instance, $sgn((1, \cdots, n)) = 1$ and $sgn(s_5) = -1$ in the previous example.

Then, we introduce some notations associated with Proposition 9.5 in order to better discuss the problem. Recall that

$$\vec{a}_0 = (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p})$$

$$\vec{b}_0 = (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q})$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_p$, $\beta_1 > \beta_2 > \cdots > \beta_q$ and $\sum_{i=1}^{p} \alpha_i = \sum_{i=1}^{q} \beta_i = k$. Denote $\vec{a} = (a_1, \cdots, a_k)$, $\vec{b} = (b_1, \cdots, b_k)$. Also denote $\vec{a}^{(1)} = (a_1, \cdots, a_{m_1})$, $\vec{a}^{(2)} = (a_{m_1+1}, \cdots, a_{m_1+m_2})$, $\cdots$, $\vec{a}^{(p)} = (a_{m_1+\cdots+m_{p-1}+1}, \cdots, a_{m_1+\cdots+m_p})$ and $\vec{a} = (\vec{a}^{(1)}, \cdots, \vec{a}^{(p)})$. That is, we divide the vector $\vec{a}$ into p blocks according to the shape of $\vec{a}_0$. Similarly, we write $\vec{b} = (b^{(1)}, \cdots, b^{(q)})$ according to the shape of $\vec{b}_0$. We will keep using similar notations in the following discussion, when we need to divide the vector according to the shape of $\vec{a}_0$ and $\vec{b}_0$. Next, denote

$$f(a_1, \cdots, a_k) \equiv f(\vec{a}) = \det[e^{c_1(t,p)a_i z_j}]_{i,j=1}^{k}, \quad g(b_1, \cdots, b_k) \equiv g(\vec{b}) = \det[e^{c_2(t,p)b_i z_j}]_{i,j=1}^{k}$$

and it's not difficult to see that they are all smooth multi-variable functions with respect to corresponding vectors. In addition, $\lim_{\vec{a} \to \vec{a}_0} f(\vec{a}) = 0$ and $\lim_{\vec{b} \to \vec{b}_0} g(\vec{b}) = 0$.

The last thing before we formally prove Proposition 9.5 is to introduce the following lemmas about the non-vanishing of the determinants.

Lemma 9.13. *Suppose $\sigma_a = (\sigma_1^a, \cdots, \sigma_k^a)$ and $\sigma_b = (\sigma_1^b, \cdots, \sigma_k^b)$ are two multi-indices of length k. We divide them into p and q parts according to the shape of $\vec{a}_0$ and $\vec{b}_0$ as mentioned before:* $\sigma_a = (\sigma_a^{(1)}, \sigma_a^{(2)}, \cdots, \sigma_a^{(p)})$, $\sigma_b = (\sigma_b^{(1)}, \sigma_b^{(2)}, \cdots, \sigma_b^{(q)})$. *Denote*

$$f(a_1, \cdots, a_k) \equiv f(\vec{a}) = \det[e^{a_i z_j}]_{i,j=1}^{k}, \quad g(b_1, \cdots, b_k) \equiv g(\vec{b}) = \det[e^{b_i z_j}]_{i,j=1}^{k}$$

where we ignore the constants $c_1(t,p)$ and $c_2(t,p)$ temporarily for simplicity. Suppose S_{m_i} is the set of all permutations of $\{0, 1, \cdots, m_i - 1\}$. If $\sigma_a^{(i)} \in S_{m_i}$ for $i = 1, \cdots, p$, then

$$D^{\sigma_a} f(\vec{a}_0) = \det \begin{bmatrix} (z_j^{\sigma_i^a} e^{\alpha_1 z_j})_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ (z_j^{\sigma_i^a} e^{\alpha_p z_j})_{\substack{i=m_1+\cdots+m_{p-1}+1, \\ \cdots,m_1+\cdots+m_p \\ j=1,\cdots,k}} \end{bmatrix}$$

is non-zero for any $(z_1, \cdots, z_k)$ whose elements are distinct. Analogous result also holds for $D^{\sigma_b} g(\vec{b}_0)$.

Proof. Since $f(\vec{a})$ is actually a determinant and its i-th row only depends on the variable a_i, taking derivative of $f(\vec{a})$ with respect to a_i is taking derivative of every entries in the i-th row, and we can get the determinant above. Next, we prove that it is non-zero. WLOG, we can assume $\sigma_a^{(i)} = \{0, 1, \cdots, m_i - 1\}$, because the determinant will only change by -1 when $\sigma_a^{(i)}$ is replaced by other permutations in S_{m_i}. We claim that, the following equation with respect to z:

$$(\xi_1 + \xi_2 z + \cdots + \xi_{m_1} z^{m_i-1})e^{\alpha_1 z} + \cdots (\xi_{m_1+\cdots+m_{p-1}+1} + \cdots + \xi_k z^{m_p-1})e^{\alpha_p z} = 0$$

has at most $(k-1)$ distinct roots, where $\sum_{i=1}^p m_i = k$ and $(\xi_1, \cdots, \xi_k) \in \mathbb{R}^k$ is non-zero.

Denote the above determinant by $\det \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$. If this claim holds, we can conclude that we cannot find non-zero $(\xi_1, \cdots, \xi_k) \in \mathbb{R}^k$ such that $\xi_1 v_1 + \cdots + \xi_k v_k = 0$. Thus, the k row vectors of the determinant are linear independent and the determinant is non-zero. Then we prove the claim by induction on k.

$1°$ If $k = 2$, the equation is $(\xi_1 + \xi_2 z)e^{\alpha_1 z} = 0$ or $\xi_1 e^{\alpha_1 z} + \xi_2 e^{\alpha_2 z} = 0$, where $\xi_1, \xi_2 \in \mathbb{R}$ cannot be zero at the same time. Then, it's easy to see that the equation has at most 1 root in two scenarios.

$2°$ Suppose the claim holds for $k \leqslant n$.

$3°$ When $k = n + 1$, we have the equation

$$(\xi_1 + \xi_2 z + \cdots + \xi_{m_1} z^{m_i-1})e^{\alpha_1 z} + \cdots (\xi_{m_1+\cdots+m_{p-1}+1} + \cdots + \xi_k z^{m_p-1})e^{\alpha_p z} = 0$$

but now $\sum_{i=1}^p m_i = n + 1$. WLOG, suppose $(\xi_1, \cdots, \xi_{m_1})$ has a non-zero element and ξ_ℓ is the first non-zero element. Notice that the above equation has the same roots as the following one:

$$F(z) = (\xi_\ell z^{\ell-1} + \cdots + \xi_{m_1} z^{m_1-1}) + \cdots + (\xi_{m_1+\cdots+m_{p-1}+1} + \cdots + \xi_k z^{m_p-1})e^{(\alpha_p-\alpha_1)z} = 0$$

Assume it has at least $(n+1)$ distinct roots $\eta_1 < \eta_2 < \cdots < \eta_{n+1}$. Then $F'(z) = 0$ has at least n distinct roots $\delta_1 < \cdots < \delta_n$ such that $\eta_1 < \delta_1 < \eta_2 < \cdots < \delta_n < \eta_{n+1}$, by Rolle's Theorem. Actually, $F'(z) = (\xi_\ell(\ell-1))z^{\ell-2} + \cdots + \xi_{m_1}(m_1-1)z^{m_1-2}) + \cdots + [\xi'_{m_1+\cdots+m_{p-1}+1} + \cdots + \xi'_k z^{m_p-1})]e^{(\alpha_p-\alpha_1)z} = 0$ where $\xi'_{m_1+\cdots+m_{p-1}+1}$ and ξ'_k are coefficients that can be calculated. This equation has at most $(m_1 - 1) + m_2 + \cdots + m_p - 1 = n - 1$ roots by $2°$, which leads to a contradiction. Therefore, our claim holds and we proved Lemma 9.13. □

Remark 9.14. Denote the set $\Lambda_a = \{\sigma_a = (\sigma_a^{(1)}, \cdots, \sigma_a^{(p)}) : \sigma_a^{(i)} \in S_{m_i}, i = 1, \cdots, p\}$, and we have if $\sigma_a \in \Lambda_a$, then $D^{\sigma_a} f(\vec{a}_0)$ is non-zero. Similarly, if $\sigma_b^{(j)} \in S_{n_j}$ for $j = 1, \cdots, q$, then $D^{\sigma_b} g(\vec{b})$ is non-zero, and define $\Lambda_b = \{\sigma_b = (\sigma_b^{(1)}, \cdots, \sigma_b^{(q)}) : \sigma_b^{(j)} \in S_{n_j}, j = 1, \cdots, q\}$.

Lemma 9.15. *The smallest order of σ_a that makes the partial derivative $D^{\sigma_a} f(\vec{a}_0)$ non-zero is $u = \sum_{i=1}^p \sum_{j=0}^{m_i-1} j = \sum_{i=1}^p \frac{m_i(m_i-1)}{2}$. Similarly, $v = \sum_{j=1}^q \frac{n_j(n_j-1)}{2}$ is the smallest order of σ_b that makes $D^{\sigma_b} f(\vec{b}_0)$ non-zero.*

Proof. If the order of derivative is less than u, then there exists a $i \in \{1, \cdots, p\}$ such that $\sigma_a^{(i)}$ contains two equal elements, and the determinant $D^{\sigma_a} f(\vec{a}_0)$ would have two equal rows, thus equal to zero. If the order of derivative is u, then when $\sigma_a \in \Lambda_a$, $D^{\sigma_a} f(\vec{a}_0)$ is non-zero by Lemma 9.13. Thus, Lemma 9.15 holds. $\square$

Finally, we give the proof for Proposition 9.5.

Proof of Proposition 9.5. For clarity, the proof will be split into 3 steps. In Step 1, we use multivariate Taylor expansion to find the speed of convergence of $f(\vec{a})$ and $g(\vec{b})$ to zero. In Step 2, we construct a new density function based on Step 1, and we will prove that Z^T weakly converges to the this newly constructed density in Step 3. In Step 3, we use monotone coupling lemma to prove the weak convergence.

Step 1. In this step, we estimate the converging speed of $f(\vec{a})$. Take $\epsilon \in (0, k^{-1} \min_{1 \leqslant i \leqslant p-1} (\alpha_i - \alpha_{i+1}))$ and construct the following vectors:

$$\vec{A}_{\epsilon,+} = (\alpha_1 + m_1\epsilon, \alpha_1 + (m_1 - 1)\epsilon, \cdots, \alpha_1 + \epsilon, \cdots, \alpha_p + m_p\epsilon, \cdots, \alpha_p + \epsilon)$$
$$\vec{A}_{\epsilon,-} = (\alpha_1 - \epsilon, \alpha_1 - 2\epsilon, \cdots, \alpha_1 - m_1\epsilon, \cdots, \alpha_p - \epsilon, \cdots, \alpha_p - m_p\epsilon)$$

That is, the vector $\vec{A}_{\epsilon,+}$(resp. $\vec{A}_{\epsilon,-}$) upwardly(resp. downwardly) spreads out the vector $\vec{a}_0$ such that $\vec{A}_{\epsilon,+}$(resp. $\vec{A}_{\epsilon,-}$) has distinct elements. In addition, when $\epsilon \downarrow 0$, we have $\vec{A}_{\epsilon,\pm}$ converges to $\vec{a}_0$. The main result of this step is the following:

$$(9.22) \qquad \lim_{\epsilon \downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,\pm}) = \varphi(\vec{a}_0, \vec{z}, \vec{m})$$

where $u = \sum_{i=1}^{p} \frac{m_i(m_i-1)}{2}$ in Lemma 9.15, $\vec{m} = (m_1, \cdots, m_p)$, and $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ is a non-zero function associated with $\vec{a}_0$ and $\vec{m}$.

To prove this result, we first expand the function $f(\vec{a})$ to the order of u at $\vec{a}_0$:

$$f(\vec{a}) = \sum_{|\sigma_a| \leqslant u} \frac{D^{\sigma_a} f(\vec{a}_0)}{\sigma_a!} (\vec{a} - \vec{a}_0)^{\sigma_a} + R_{u+1}(\vec{a}, \vec{a}_0)$$

$$= \sum_{\sigma_a \in \Lambda_a} \frac{D^{\sigma_a} f(\vec{a}_0)}{\sigma_a!} (\vec{a} - \vec{a}_0)^{\sigma_a} + R_{u+1}(\vec{a}, \vec{a}_0)$$

where the $R_{u+1}(\vec{a}, \vec{a}_0) = \sum_{\sigma_a:|\sigma_a|=u+1} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0 + \theta(\vec{a} - \vec{a}_0))(\vec{a} - \vec{a}_0)^{\sigma_a}$, $\theta \in (0,1)$ is the remainder. The second equality results from Lemma 9.15, since it indicates that all the terms of order less than u are zero, and for the terms of order u, they are non-zero only when $\sigma_a \in \Lambda_a$.

Consider the first term. Denote $sgn(\sigma_a^{(i)})$ as the sign of the permutation $\sigma_a^{(i)} \in S_{m_i}$, and define the sign of σ_a by: $sgn(\sigma_a) = \prod_{i=1}^{p} sgn(\sigma_a^{(i)})$. Denote $\sigma_a^\star = (\sigma_a^{(1)\star}, \cdots, \sigma_a^{(p)\star})$, where $\sigma_a^{(i)\star} = (0, 1, \cdots, m_i - 1)$. Thus, $\sigma_a^\star$ is a special element in Λ_a and $sgn(\sigma_a^\star) = 1$ because $\sigma_a^{(1)\star}, \cdots, \sigma_a^{(p)\star}$ all have 0 number of inversions. Notice that for any $\sigma_a \in \Lambda_a$, we have $D^{\sigma_a} f(\vec{a}_0) = sgn(\sigma_a) \cdot D^{\sigma_a^\star} f(\vec{a}_0)$ by the property of determinant. Then we obtain:

$$\sum_{\sigma_a \in \Lambda_a} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0)(\vec{a} - \vec{a}_0)^{\sigma_a} = \frac{D^{\sigma_a^\star} f(\vec{a}_0)}{\prod_{i=1}^{p}(m_i - 1)!} \sum_{\sigma_a \in \Lambda_a} (\vec{a} - \vec{a}_0)^{\sigma_a} \cdot sgn(\sigma_a)$$

Notice that

$$\sum_{\sigma_a \in \Lambda_a} (\vec{a} - \vec{a}_0)^{\sigma_a} \cdot sgn(\sigma_a) = \prod_{i=1}^{p} \Big[\sum_{\sigma_a^{(i)} \in S_{m_i}} (\vec{a}^{(i)} - \vec{a}_0^{(i)})^{\sigma_a^{(i)}} \cdot sgn(\sigma_a^{(i)}) \Big]$$

$$= \prod_{i=1}^{p} \Delta_{m_i}(a_1^{(i)} - \alpha_i, a_2^{(i)} - \alpha_i, \cdots, a_{m_i}^{(i)} - \alpha_i) \equiv \prod_{i=1}^{p} \Delta_{m_i}^{a}$$

where $\Delta_n(x_1, x_2, \cdots, x_n)$ is the Vandermonde Determinant, $a_j^{(i)} = a_{m_1 + \cdots + m_{i-1} + j}$ is the j-th element of $\vec{a}^{(i)}$, and the last equality holds by the definition of determinant and Vandermonde Determinant. Now replace $\vec{a}$ with $\vec{A}_{\epsilon,+}$, we get the Vandermonde determinant $\Delta_{m_i}^{a}$ is actually $(m_i - 1)! \cdot \epsilon^{\frac{1}{2} m_i (m_i - 1)}$. Therefore, we have:

$$\sum_{\sigma_a \in \Lambda_a} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0)(\vec{a} - \vec{a}_0)^{\sigma_a} = D^{\sigma_a^\star} f(\vec{a}_0) \cdot \epsilon^u$$

Now we consider the remainder $R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0)$. Since $D^{\sigma_a} f(\vec{a})$ is a continuous function of vector $\vec{a}$, we have that the quantity $D^{\sigma_a^\star} f(\vec{a}_0 + \theta(\vec{a} - \vec{a}_0))$ can be bounded by a constant $M(\vec{a}, \vec{a}_0)$. In addition, $\sigma_a!$ only have finitely many possible outcomes when its order is $u + 1$, thus $\frac{1}{\sigma_a!}$ can be bounded by a constant $N(u)$. Also, $|(\vec{A}_{\epsilon,+} - \vec{a}_0)|^{\sigma_a} \leqslant (\max_{1 \leqslant i \leqslant p} m_i \cdot \epsilon)^{u+1}$. Therefore,

$$|R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0)| \leqslant N \cdot M \cdot (\max_{1 \leqslant i \leqslant p} m_i \cdot \epsilon)^{u+1}$$

and this indicates that $R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0)$ is $O(\epsilon^{u+1})$, where the constant in Big O notation only depends on $\vec{a}_0$, $\vec{a}$, $\vec{m}$ and u and does not depend on ϵ. Therefore, we conclude that

$$\lim_{\epsilon \downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,+}) = D^{\sigma_a^\star} f(\vec{a}_0)$$

By Lemma 9.13, $D^{\sigma_a^\star} f(\vec{a}_0)$ is non-zero. Thus, we find the limit function $\varphi(\vec{a}_0, \vec{z}, \vec{m}) = D^{\sigma_a^\star} f(\vec{a}_0)$, and its expression can be found in Lemma 9.13. Following similar procedure we can prove $\lim_{\epsilon \downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,-}) = D^{\sigma_a^\star} f(\vec{a}_0)$ also holds, and we established the equation (9.22).

We can construct vectors $\vec{B}_{\epsilon,\pm}$ analogously, which spread out from vector $\vec{b}_0$ upward and downward, and get similar results for $g(\vec{B}_{\epsilon,\pm})$ and then we have:

$$\lim_{\epsilon \downarrow 0} \epsilon^{-v} f(\vec{B}_{\epsilon,\pm}) = D^{\sigma_b^\star} g(\vec{b}_0) \equiv \psi(\vec{b}_0, \vec{z}, \vec{n})$$

where $v = \sum_{i=1}^{q} \frac{n_i(n_i - 1)}{2}$ in Lemma 9.15, $\vec{n} = (n_1, \cdots, n_q)$ and the expression of non-zero function $\psi(\vec{b}_0, \vec{z}, \vec{n})$ can be found by Lemma 9.13.

Step 2. In this step, we mainly prove the following result:

The function of $\vec{z}$

$$H(\vec{z}) = \varphi(\vec{a}_0, \vec{z}, \vec{m}) \psi(\vec{b}_0, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{-c_3(t,p) z_i^2}$$

is integrable over $\mathbb{R}^k$.

For simplicity, we ignore the constants c_1, c_2, c_3 temporarily and prove the function $H(\vec{z})$ without those constants is integrable. It's not difficult to see $H(\vec{z})$ is still integrable when adding those constants. Notice that $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ is a determinant whose expression is given in Lemma 9.13. Suppose $z_{j_1}^{i_1 - 1} e^{a_{i_1} z_{j_1}}$ is the entry that has the largest absolute value. Then

$$|\varphi(\vec{a}_0, \vec{z}, \vec{m})| \leqslant k! |z_{j_1}^{i_1 - 1} e^{a_{i_1} z_{j_1}}|^k = k! |z_{j_1}^{k(i_1 - 1)}| e^{k a_{i_1} z_{j_1}}$$

Similarly, we can find index i_2 and j_2 such that $|\varphi(\vec{b}_0, \vec{z}, \vec{n})| \leqslant k! |z_{j_2}^{k(i_2-1)}| e^{kb_{i_2} z_{j_2}}$. Then, we get

$$|H(\vec{z})| \leqslant (k!)^2 \Big[\prod_{j \neq j_1, j_2} e^{-z_j^2} |z_{j_1}^{k(i_1-1)} z_{j_2}^{k(i_2-1)}| \Big] e^{ka_{i_1} z_{j_1} - z_{j_1}^2} e^{kb_{i_2} z_{j_2} - z_{j_2}^2}$$

The right hand side is integrable over $\mathbb{R}^k$ because the exponential terms have power of some quadratic functions with negative quadratic coefficients. Thus, $H(\vec{z})$ is integrable.

Since $H(\vec{z})$ is integrable, we can define the constant $Z_{\vec{a}_0, \vec{b}_0} = \int_{\mathbb{R}^k} H(\vec{z}) \mathbb{1}_{\{z_1 > z_2 > \cdots > z_k\}} dz < \infty$ and the function

$$(9.23) \qquad \rho_{\vec{a}_0, \vec{b}_0}(z_1, \cdots, z_k) = Z_{\vec{a}_0, \vec{b}_0}^{-1} \varphi(\vec{a}_0, \vec{z}, \vec{m}) \psi(\vec{b}_0, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{-c_3(t,p)z_i^2} \mathbb{1}_{\{z_1 > z_2 > \cdots > z_k\}}$$

is a density because it's non-negative and integrates to 1 over $\mathbb{R}^k$.

Step 3. Denote $Z_{\vec{a}_0, \vec{b}_0}^T$ as the random vector Z^T associated with vectors $\vec{a}_0$ and b_0, and in this step we prove it weakly converges to the continuous distribution with the density $\rho_{\vec{a}_0, \vec{b}_0}(z)$ we just constructed in (9.23). Suppose $\mathfrak{L}_+^T$ is an avoiding Bernoulli line ensemble starting with $\vec{x}_+^T$ and ending with $\vec{y}_+^T$ and follows the distribution $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}_+^T,\vec{y}_+^T}$. The vectors $\vec{x}_+^T$ and $\vec{y}_+^T$ are two signatures of length k that satisfies the following:

(i)
$$\lim_{T \to \infty} \frac{\vec{x}_+^T}{\sqrt{T}} = \vec{A}_{\epsilon,+}, \qquad \lim_{T \to \infty} \frac{\vec{y}_+^T - pT1_k}{\sqrt{T}} = \vec{B}_{\epsilon,+}$$

(ii) $\vec{x}_+^T \geqslant \vec{x}^T$, $\vec{y}_+^T \geqslant \vec{y}^T$, which means the endpoints of the newly constructed line ensembles dominate the original ones. This can be achieved due to the limiting behavior of $\vec{x}_+^T$ and $\vec{y}_+^T$ and the construction of $\vec{A}_{\epsilon,+}$ and $\vec{B}_{\epsilon,+}$. Analogously, we construct another avoiding Bernoulli line ensemble $\mathfrak{L}_-^T$ with endpoints $\vec{x}_-^T$ and $\vec{y}_-^T$ and distribution $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}_-^T,\vec{y}_-^T}$ such that $\lim_{T \to \infty} \frac{\vec{x}_-^T}{\sqrt{T}} = \vec{A}_{\epsilon,-}$, $\lim_{T \to \infty} \frac{\vec{y}_-^T - pT1_k}{\sqrt{T}} = \vec{B}_{\epsilon,-}$, and $\vec{x}_-^T \leqslant \vec{x}^T$, $\vec{y}_-^T \leqslant \vec{y}^T$.

Since now $\vec{A}_{\epsilon,+}$, $\vec{A}_{\epsilon,-}$, $\vec{B}_{\epsilon,+}$, $\vec{B}_{\epsilon,-}$ have distinct elements, we can apply the results in Proposition 9.3 and conclude the weak convergence:

$$Z_{\vec{A}_{\epsilon,+}, \vec{B}_{\epsilon,+}}^T \Rightarrow \rho_{\epsilon,+}(z), \qquad Z_{\vec{A}_{\epsilon,-}, \vec{B}_{\epsilon,-}}^T \Rightarrow \rho_{\epsilon,-}(z)$$

where $Z_{\vec{A}_{\epsilon,+}, \vec{B}_{\epsilon,+}}^T$ and $Z_{\vec{A}_{\epsilon,-}, \vec{B}_{\epsilon,-}}^T$ are obtained by scaling the line ensembles $\mathfrak{L}_+^T$, and $\mathfrak{L}_-^T$, $\rho_{\epsilon,+}(z)$ and $\rho_{\epsilon,-}(z)$ are densities which are obtained by plugging $\vec{A}_{\epsilon,+}$, $\vec{B}_{\epsilon,+}$ and $\vec{A}_{\epsilon,-}$, $\vec{B}_{\epsilon,-}$ into the formula of $\rho(z)$ in Proposition 9.3.

In order to prove the weak convergence of $Z_{\vec{a}_0, \vec{b}_0}^T$, it is sufficient to prove for any $R = (-\infty, u_1] \times (-\infty, u_2] \times \cdots \times (-\infty, u_k]$, where $u_i \in \mathbb{R}$, we have

$$\lim_{T \to \infty} (Z_{\vec{a}_0, \vec{b}_0}^T \in R) = \int_R \rho_{\vec{a}_0, \vec{b}_0}(z) dz$$

Actually, by Lemma 3.1, we can construct a sequence of probability spaces $(\Omega_T, \mathcal{F}_T, \mathbb{P}_T)_{T \geqslant 1}$ such that for each $T \in \mathbb{Z}^+$, we have random variables $\mathfrak{L}_+^T$ and $\mathfrak{L}^T$ have law $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}_+^T,\vec{y}_+^T}$ and $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}^T,\vec{y}^T}$ under measure $\mathbb{P}_T$, respectively. Also, we have $\mathfrak{L}_+^T(i,r) \geqslant \mathfrak{L}^T(i,r)$ with probability 1, where $\mathfrak{L}_+^T(i,r)$(resp., $\mathfrak{L}^T(i,r)$) is the value of the i-th up-right path of $\mathfrak{L}_+^T$(resp., $\mathfrak{L}^T$) at $r \in [\![0,T]\!]$. Similarly, we can construct another sequence of probability spaces $(\Omega_T', \mathcal{F}_T', \mathbb{Q}_T)_{T \geqslant 1}$ such that for each $T \in \mathbb{Z}^+$, we have random variables $\mathfrak{L}_-^T$ and $\mathfrak{L}^T$ have law $\mathbb{P}_{avoid,Ber}^{0,T,\vec{x}_-^T,\vec{y}_-^T}$ and $\mathbb{P}_{avoid,Ber}^{0,T,\vec{x}^T,\vec{y}^T}$ under measure $\mathbb{Q}_T$, respectively, along with $\mathbb{Q}_T(\mathfrak{L}_-^T(i,r) \leqslant \mathfrak{L}^T(i,r), i = 1, \cdots, k, r \in [\![0,T]\!]) = 1$.

Therefore, we have that under measure $\mathbb{P}_T$ and $\mathbb{Q}_T$:

$$\mathbb{P}_T(Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} \in R) \leqslant \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R), \quad \mathbb{Q}_T(Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}} \in R) \geqslant \mathbb{Q}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R)$$

Take $liminf$ and $limsup$ on both side of the first and second inequality respectively, we get

$$(9.24) \qquad \int_R \rho_{\epsilon,+}(z)dz \leqslant \liminf_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R), \quad \int_R \rho_{\epsilon,-}(z)dz \geqslant \limsup_{T\to\infty} \mathbb{Q}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R)$$

because of the weak convergence of $Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}$ and $Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}}$. Since the distribution of $Z^T_{\vec{a}_0,\vec{b}_0}$ under measure $\mathbb{P}_T$ and $\mathbb{Q}_T$ are the same, we can combine the above two inequalities (9.24) and get

$$(9.25) \qquad \int_R \rho_{\epsilon,+}(z)dz \leqslant \liminf_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \leqslant \limsup_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \leqslant \int_R \rho_{\epsilon,-}(z)dz$$

The rest of the proof establishes the following statement:

$$(9.26) \qquad \lim_{\epsilon\downarrow 0} \int_R \rho_{\epsilon,+}(z)dz = \lim_{\epsilon\downarrow 0} \int_R \rho_{\epsilon,-}(z)dz = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$$

and thereby concluding

$$\lim_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$$

by letting $\epsilon \downarrow 0$ in the inequality (9.25), and we prove the weak convergence of $Z^T_{\vec{a}_0,\vec{b}_0}$.

To prove the statement (9.26), first notice that

$$Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} = \int_{\mathbb{R}^k} f(\vec{a},\vec{z})g(\vec{b},\vec{z}) \prod_{i=1}^k e^{-c_3(t,p)z_i^2} dz$$

$$= \int_{\mathbb{R}^k} \left[\epsilon^{u+v}\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n}) + o(\epsilon^{u+v})\right] \prod_{i=1}^k e^{-c_3(t,p)z_i^2} dz$$

$$= \epsilon^{u+v} \int_{\mathbb{R}^k} \left[\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n}) + o(1)\right] \prod_{i=1}^k e^{-c_3(t,p)z_i^2} dz$$

Then, we get

$$\lim_{\epsilon\downarrow 0} \epsilon^{-(u+v)} Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} = \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}^k} \left[\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n}) + o(1)\right] \prod_{i=1}^k e^{-c_3(t,p)z_i^2} dz = Z_{\vec{a}_0,\vec{b}_0}$$

by definition of the constant $Z_{\vec{a}_0,\vec{b}_0}$. Therefore, we conclude

$$\lim_{\epsilon\downarrow 0} \rho_{\epsilon,+}(z) = \lim_{\epsilon\downarrow 0}(\epsilon^{-(u+v)} Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}})(\epsilon^u f(\vec{a},\vec{z}))(\epsilon^v g(\vec{b},\vec{z})) = Z_{\vec{a},\vec{b}_0}\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n}) = \rho_{\vec{a}_0,\vec{b}_0}(z)$$

Since $\rho_{\epsilon,+}(z)\mathbb{1}_R dz \leqslant \rho_{\epsilon,+}(z)dz$ is bounded by an integrable function, by Dominated Convergence Theorem we have:

$$\lim_{\epsilon\downarrow 0} \int_R \rho_{\epsilon,+}(z)dz = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$$

Analogously, we can get $\lim_{\epsilon\downarrow 0}\int_R \rho_{\epsilon,-}(z)dz = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$ and we proved the statement (9.26), which completes the proof. $\square$

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