TIGHTNESS OF BERNOULLI LINE ENSEMBLES

Abstract. Insert abstract here:

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2. Line ensembles

In this section we introduce various definitions and notation that are used throughout the paper.

2.1. Line ensembles and the Brownian Gibbs property.

In this section we introduce the notions of a *line ensemble* and the *(partial) Brownian Gibbs property.* Our exposition in this section closely follows that of [6, Section 2] and [4, Section 2].

Given two integers $p \leq q$, we let $[\![p, q]\!]$ denote the set $\{p, p+1, \dots, q\}$. Given an interval $\Lambda \subset \mathbb{R}$ we endow it with the subspace topology of the usual topology on $\mathbb{R}$. We let $(C(\Lambda), \mathcal{C})$ denote the space of continuous functions $f : \Lambda \to \mathbb{R}$ with the topology of uniform convergence over compacts, see [13, Chapter 7, Section 46], and Borel σ-algebra $\mathcal{C}$. Given a set $\Sigma \subset \mathbb{Z}$ we endow it with the discrete topology and denote by $\Sigma \times \Lambda$ the set of all pairs (i, x) with $i \in \Sigma$ and $x \in \Lambda$ with the product topology. We also denote by $(C(\Sigma \times \Lambda), \mathcal{C}_{\Sigma})$ the space of continuous functions on $\Sigma \times \Lambda$ with the topology of uniform convergence over compact sets and Borel σ-algebra $\mathcal{C}_{\Sigma}$. Typically, we will take $\Sigma = [\![1, N]\!]$ (we use the convention $\Sigma = \mathbb{N}$ if $N = \infty$) and then we write $\big(C(\Sigma \times \Lambda), \mathcal{C}_{|\Sigma|}\big)$ in place of $(C(\Sigma \times \Lambda), \mathcal{C}_{\Sigma})$.

The following defines the notion of a line ensemble.

Definition 2.1. Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. A Σ-*indexed line ensemble* $\mathcal{L}$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that takes values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$. Intuitively, $\mathcal{L}$ is a collection of random continuous curves (sometimes referred to as *lines*), indexed by Σ, each of which maps Λ in $\mathbb{R}$. We will often slightly abuse notation and write $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$, even though it is not $\mathcal{L}$ which is such a function, but $\mathcal{L}(\omega)$ for every $\omega \in \Omega$. For $i \in \Sigma$ we write $\mathcal{L}_i(\omega) = (\mathcal{L}(\omega))(i, \cdot)$ for the curve of index i and note that the latter is a map $\mathcal{L}_i : \Omega \to C(\Lambda)$, which is $(\mathcal{C}, \mathcal{F})-$measurable. If $a, b \in \Lambda$ satisfy $a < b$ we let $\mathcal{L}_i[a, b]$ denote the restriction of $\mathcal{L}_i$ to $[a, b]$.

We will require the following result, whose proof is postponed until Section 8.1. In simple terms it states that the space $C(\Sigma \times \Lambda)$ where our random variables $\mathcal{L}$ take value has the structure of a complete, separable metric space.

Lemma 2.2. *Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. Suppose that $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ are sequences of real numbers such that $a_n < b_n$, $[a_n, b_n] \subset \Lambda$, $a_{n+1} \leq a_n$, $b_{n+1} \geq b_n$ and $\cup_{n=1}^\infty [a_n, b_n] = \Lambda$. For $n \in \mathbb{N}$ we let $K_n = \Sigma_n \times [a_n, b_n]$ where $\Sigma_n = \Sigma \cap [\![-n, n]\!]$. Define $d : C(\Sigma \times \Lambda) \times C(\Sigma \times \Lambda) \to [0, \infty)$ by*

$$d(f, g) = \sum_{n=1}^\infty 2^{-n} \min \left\{ \sup_{(i,t) \in K_n} |f(i, t) - g(i, t)|, 1 \right\}. \tag{2.1}$$

Then d defines a metric on $C(\Sigma \times \Lambda)$ and moreover the metric space topology defined by d is the same as the topology of uniform convergence over compact sets. Furthermore, the metric space $(C(\Sigma \times \Lambda), d)$ is complete and separable.

Definition 2.3. Given a sequence $\{\mathcal{L}^n : n \in \mathbb{N}\}$ of random Σ-indexed line ensembles we say that $\mathcal{L}^n$ *converge weakly* to a line ensemble $\mathcal{L}$, and write $\mathcal{L}^n \implies \mathcal{L}$ if for any bounded continuous function $f : C(\Sigma \times \Lambda) \to \mathbb{R}$ we have that

$$\lim_{n \to \infty} \mathbb{E}\left[f(\mathcal{L}^n)\right] = \mathbb{E}\left[f(\mathcal{L})\right].$$

We also say that $\{\mathcal{L}^n : n \in \mathbb{N}\}$ is *tight* if for any $\epsilon > 0$ there exists a compact set $K \subset C(\Sigma \times \Lambda)$ such that $\mathbb{P}(\mathcal{L}^n \in K) \geq 1 - \epsilon$ for all $n \in \mathbb{N}$.

We call a line ensemble *non-intersecting* if $\mathbb{P}$-almost surely $\mathcal{L}_i(r) > \mathcal{L}_j(r)$ for all $i < j$ and $r \in \Lambda$.

We will require the following sufficient condition for tightness of a sequence of line ensembles, which extends [1, Theorem 7.3]. We give a proof in Section 8.2.

Lemma 2.4. *Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. Suppose that $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ are sequences of real numbers such that $a_n < b_n$, $[a_n, b_n] \subset \Lambda$, $a_{n+1} \leq a_n$, $b_{n+1} \geq b_n$ and $\cup_{n=1}^\infty [a_n, b_n] = \Lambda$. Then $\{\mathcal{L}^n\}$ is tight if and only if for every $i \in \Sigma$ we have*

(i) $\lim_{a \to \infty} \limsup_{n \to \infty} \mathbb{P}(|\mathcal{L}_i^n(a_0)| \geq a) = 0$;

(ii) For all $\epsilon > 0$ and $k \in \mathbb{N}$, $\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{\substack{x, y \in [a_k, b_k], \\ |x - y| \leq \delta}} |\mathcal{L}_i^n(x) - \mathcal{L}_i^n(y)| \geq \epsilon \right) = 0.$

We next turn to formulating the Brownian Gibbs property – we do this in Definition 2.8 after introducing some relevant notation and results. If W_t denotes a standard one-dimensional Brownian motion, then the process

$$\tilde{B}(t) = W_t - tW_1, \quad 0 \leq t \leq 1,$$

is called a *Brownian bridge (from $\tilde{B}(0) = 0$ to $\tilde{B}(1) = 0$) with diffusion parameter* 1. For brevity we call the latter object a *standard Brownian bridge.*

Given $a, b, x, y \in \mathbb{R}$ with $a < b$ we define a random variable on $(C([a, b]), \mathcal{C})$ through

$$B(t) = (b - a)^{1/2} \cdot \tilde{B}\left(\frac{t - a}{b - a}\right) + \left(\frac{b - t}{b - a}\right) \cdot x + \left(\frac{t - a}{b - a}\right) \cdot y, \tag{2.2}$$

and refer to the law of this random variable as a *Brownian bridge (from $B(a) = x$ to $B(b) = y$) with diffusion parameter* 1. Given $k \in \mathbb{N}$ and $\vec{x}, \vec{y} \in \mathbb{R}^k$ we let $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}$ denote the law of k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1.

We next state a couple of results about Brownian bridges from [4] for future use.

Lemma 2.5. *[4, Corollary 2.9]. Fix a continuous function $f : [0,1] \to \mathbb{R}$ such that $f(0) > 0$ and $f(1) > 0$. Let B be a standard Brownian bridge and let $C = \{B(t) > f(t)$ for some $t \in [0,1]\}$ (crossing) and $T = \{B(t) = f(t)$ for some $t \in [0,1]\}$ (touching). Then $\mathbb{P}(T \cap C^c) = 0$.*

Lemma 2.6. *[4, Corollary 2.10]. Let U be an open subset of $C([0,1])$, which contains a function f such that $f(0) = f(1) = 0$. If $B : [0,1] \to \mathbb{R}$ is a standard Brownian bridge then $\mathbb{P}(B[0,1] \subset U) > 0$.*

The following definition introduces the notion of an (f,g)-avoiding Brownian line ensemble, which in simple terms is a collection of k independent Brownian bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two continuous functions f and g.

Definition 2.7. Let $k \in \mathbb{N}$ and W_k° denote the open Weyl chamber in $\mathbb{R}^k$, i.e.

$$W_k^\circ = \{\vec{x} = (x_1, \dots, x_k) \in \mathbb{R}^k : x_1 > x_2 > \cdots > x_k\}.$$

(In [4] the notation $\mathbb{R}^k_>$ was used for this set.) Let $\vec{x}, \vec{y} \in W_k^\circ$, $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to (-\infty, \infty]$ and $g : [a,b] \to [-\infty, \infty)$ be two continuous functions. The latter condition means that either $f : [a,b] \to \mathbb{R}$ is continuous or $f = \infty$ everywhere, and similarly for g. We also assume that $f(t) > g(t)$ for all $t \in [a,b]$, $f(a) > x_1, f(b) > y_1$ and $g(a) < x_k, g(b) < y_k$.

With the above data we define the *(f,g)-avoiding Brownian line ensemble on the interval* $[a,b]$ *with entrance data $\vec{x}$ and exit data $\vec{y}$* to be the Σ-indexed line ensemble $\mathcal{Q}$ with $\Sigma = [\![1,k]\!]$ on $\Lambda = [a,b]$ and with the law of $\mathcal{Q}$ equal to $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}$ (the law of k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$) conditioned on the event

$$E = \{f(r) > B_1(r) > B_2(r) > \cdots > B_k(r) > g(r) \text{ for all } r \in [a,b]\}.$$

It is worth pointing out that E is an open set of positive measure and so we can condition on it in the usual way – we explain this briefly in the following paragraph. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that supports k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1. Notice that we can find $\tilde{u}_1, \dots, \tilde{u}_k \in C([0,1])$ and $\epsilon > 0$ (depending on $\vec{x}, \vec{y}, f, g, a, b$) such that $\tilde{u}_i(0) = \tilde{u}_i(1) = 0$ for $i = 1, \dots, k$ and such that if $\tilde{h}_1, \dots, \tilde{h}_k \in C([0,1])$ satisfy $\tilde{h}_i(0) = \tilde{h}_i(1) = 0$ for $i = 1, \dots, k$ and $\sup_{t \in [0,1]} |\tilde{u}_i(t) - \tilde{h}_i(t)| < \epsilon$ then the functions

$$h_i(t) = (b-a)^{1/2} \cdot \tilde{h}_i\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot x_i + \left(\frac{t-a}{b-a}\right) \cdot y_i,$$

satisfy $f(r) > h_1(r) > \cdots > h_k(r) > g(r)$. It follows from Lemma 2.6 that

$$\mathbb{P}(E) \geq \mathbb{P}\left(\max_{1 \leq i \leq k} \sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) = \prod_{i=1}^k \mathbb{P}\left(\sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) > 0,$$

and so we can condition on the event E.

To construct a realization of $\mathcal{Q}$ we proceed as follows. For $\omega \in E$ we define

$$\mathcal{Q}(\omega)(i,r) = B_i(r)(\omega) \text{ for } i = 1, \dots, k \text{ and } r \in [a,b].$$

Observe that for $i \in \{1, \dots, k\}$ and an open set $U \in C([a,b])$ we have that

$$\mathcal{Q}^{-1}(\{i\} \times U) = \{B_i \in U\} \cap E \ \in \ \mathcal{F},$$

and since the sets $\{i\}\times U$ form an open basis of $C([\![1,k]\!]\times[a,b])$ we conclude that $\mathcal{Q}$ is $\mathcal{F}$-measurable. This implies that the law $\mathcal{Q}$ is indeed well-defined and also it is non-intersecting almost surely. Also, given measurable subsets $A_1,\dots,A_k$ of $C([a,b])$ we have that

$$\mathbb{P}(\mathcal{Q}_i \in A_i \text{ for } i=1,\dots,k) = \frac{\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}\left(\{B_i \in A_i \text{ for } i=1,\dots,k\}\cap E\right)}{\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}(E)}.$$

We denote the probability distribution of $\mathcal{Q}$ as $\mathbb{P}^{a,b,\vec{x},\vec{y},f,g}_{avoid}$ and write $\mathbb{E}^{a,b,\vec{x},\vec{y},f,g}_{avoid}$ for the expectation with respect to this measure.

The following definition introduces the notion of the Brownian Gibbs property from [4].

Definition 2.8. Fix a set $\Sigma=[\![1,N]\!]$ with $N\in\mathbb{N}$ or $N=\infty$ and an interval $\Lambda\subset\mathbb{R}$ and let $K=\{k_1,k_1+1,\dots,k_2\}\subset\Sigma$ be finite and $a,b\in\Lambda$ with $a<b$. Set $f=\mathcal{L}_{k_1-1}$ and $g=\mathcal{L}_{k_2+1}$ with the convention that $f=\infty$ if $k_1-1\notin\Sigma$ and $g=-\infty$ if $k_2+1\notin\Sigma$. Write $D_{K,a,b}=K\times(a,b)$ and $D^c_{K,a,b}=(\Sigma\times\Lambda)\setminus D_{K,a,b}$. A Σ-indexed line ensemble $\mathcal{L}:\Sigma\times\Lambda\to\mathbb{R}$ is said to have the *Brownian Gibbs property* if it is non-intersecting and

$$\mathrm{Law}\left(\mathcal{L}|_{K\times[a,b]} \text{ conditional on } \mathcal{L}|_{D^c_{K,a,b}}\right) = \mathrm{Law}\left(\mathcal{Q}\right),$$

where $\mathcal{Q}_i=\tilde{\mathcal{Q}}_{i-k_1+1}$ and $\tilde{\mathcal{Q}}$ is the (f,g)-avoiding Brownian line ensemble on $[a,b]$ with entrance data $(\mathcal{L}_{k_1}(a),\dots,\mathcal{L}_{k_2}(a))$ and exit data $(\mathcal{L}_{k_1}(b),\dots,\mathcal{L}_{k_2}(b))$ from Definition 2.7. Note that $\tilde{Q}$ is introduced because, by definition, any such (f,g)-avoiding Brownian line ensemble is indexed from 1 to k_2-k_1+1 but we want $\mathcal{Q}$ to be indexed from k_1 to k_2.

An equivalent way to express the Brownian Gibbs property is as follows. A Σ-indexed line ensemble $\mathcal{L}$ on Λ satisfies the Brownian Gibbs property if and only if it is non-intersecting and for any finite $K=\{k_1,k_1+1,\dots,k_2\}\subset\Sigma$ and $[a,b]\subset\Lambda$ and any bounded Borel-measurable function $F:C(K\times[a,b])\to\mathbb{R}$ we have $\mathbb{P}$-almost surely

$$\mathbb{E}\left[F\left(\mathcal{L}|_{K\times[a,b]}\right)\big|\mathcal{F}_{ext}(K\times(a,b))\right] = \mathbb{E}^{a,b,\vec{x},\vec{y},f,g}_{avoid}\big[F(\tilde{\mathcal{Q}})\big], \tag{2.3}$$

where

$$\mathcal{F}_{ext}(K\times(a,b)) = \sigma\left\{\mathcal{L}_i(s):(i,s)\in D^c_{K,a,b}\right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K\times[a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K\times[a,b]$, $\vec{x}=(\mathcal{L}_{k_1}(a),\dots,\mathcal{L}_{k_2}(a))$, $\vec{y}=(\mathcal{L}_{k_1}(b),\dots,\mathcal{L}_{k_2}(b))$, $f=\mathcal{L}_{k_1-1}[a,b]$ (the restriction of $\mathcal{L}$ to the set $\{k_1-1\}\times[a,b]$) with the convention that $f=\infty$ if $k_1-1\notin\Sigma$, and $g=\mathcal{L}_{k_2+1}[a,b]$ with the convention that $g=-\infty$ if $k_2+1\notin\Sigma$.

Remark 2.9. Let us briefly explain why equation (2.3) makes sense. Firstly, since $\Sigma\times\Lambda$ is locally compact, we know by [13, Lemma 46.4] that $\mathcal{L}\to\mathcal{L}|_{K\times[a,b]}$ is a continuous map from $C(\Sigma\times\Lambda)$ to $C(K\times[a,b])$, so that the left side of (2.3) is the conditional expectation of a bounded measurable function, and is thus well-defined. A more subtle question is why the right side of (2.3) is $\mathcal{F}_{ext}(K\times(a,b))$-measurable. This question was resolved in [6, Lemma 3.4], where it was shown that the right side is measurable with respect to the σ-algebra

$$\sigma\left\{\mathcal{L}_i(s):\ i\in K \text{ and } s\in\{a,b\}, \text{ or } i\in\{k_1-1,k_2+1\} \text{ and } s\in[a,b]\right\},$$

which in particular implies the measurability with respect to $\mathcal{F}_{ext}(K\times(a,b))$.

In the present paper it is convenient for us to use the following modified version of the definition above, which we call the partial Brownian Gibbs property – it was first introduced in [6]. We explain the difference between the two definitions, and why we prefer the second one in Remark 2.12.

Definition 2.10. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and an interval $\Lambda \subset \mathbb{R}$. A Σ-indexed line ensemble $\mathcal{L}$ on Λ is said to satisfy the *partial Brownian Gibbs property* if and only if it is non-intersecting and for any finite $K = \{k_1, k_1 + 1, \dots, k_2\} \subset \Sigma$ with $k_2 \leq N - 1$ (if $\Sigma \neq \mathbb{N}$), $[a, b] \subset \Lambda$ and any bounded Borel-measurable function $F : C(K \times [a, b]) \to \mathbb{R}$ we have $\mathbb{P}$-almost surely

$$\mathbb{E}\left[F(\mathcal{L}|_{K\times[a,b]})\middle|\mathcal{F}_{ext}(K \times (a, b))\right] = \mathbb{E}^{a,b,\vec{x},\vec{y},f,g}_{avoid}\left[F(\tilde{\mathcal{Q}})\right], \tag{2.4}$$

where we recall that $D_{K,a,b} = K \times (a, b)$ and $D^c_{K,a,b} = (\Sigma \times \Lambda) \setminus D_{K,a,b}$, and

$$\mathcal{F}_{ext}(K \times (a, b)) = \sigma\left\{\mathcal{L}_i(s) : (i, s) \in D^c_{K,a,b}\right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K\times[a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K \times [a, b]$, $\vec{x} = (\mathcal{L}_{k_1}(a), \dots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \dots, \mathcal{L}_{k_2}(b))$, $f = \mathcal{L}_{k_1-1}[a, b]$ with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$, and $g = \mathcal{L}_{k_2+1}[a, b]$.

Remark 2.11. Observe that if $N = 1$ then the conditions in Definition 2.10 become void, i.e., any line ensemble with one line satisfies the partial Brownian Gibbs property. Also we mention that (2.4) makes sense by the same reason that (2.3) makes sense, see Remark 2.9.

Remark 2.12. Definition 2.10 is slightly different from the Brownian Gibbs property of Definition 2.8 as we explain here. Assuming that $\Sigma = \mathbb{N}$ the two definitions are equivalent. However, if $\Sigma = \{1, \dots, N\}$ with $1 \leq N < \infty$ then a line ensemble that satisfies the Brownian Gibbs property also satisfies the partial Brownian Gibbs property, but the reverse need not be true. Specifically, the Brownian Gibbs property allows for the possibility that $k_2 = N$ in Definition 2.10 and in this case the convention is that $g = -\infty$. As the partial Brownian Gibbs property is more general we prefer to work with it and most of the results later in this paper are formulated in terms of it rather than the usual Brownian Gibbs property.

2.2. **Bernoulli Gibbsian line ensembles.** In this section we introduce the notion of a *Bernoulli line ensemble* and the *Schur Gibbs property*. Our discussion will parallel that of [3, Section 3.1], which in turn goes back to [5, Section 2.1].

Definition 2.13. Let $\Sigma \subset \mathbb{Z}$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. Consider the set Y of functions $f : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(j, i + 1) - f(j, i) \in \{0, 1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$ and let $\mathcal{D}$ denote the discrete topology on Y. We call a function $f : [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(i + 1) - f(i) \in \{0, 1\}$ when $i \in [\![T_0, T_1 - 1]\!]$ an *up-right path* and elements in Y *collections of up-right paths*.

A Σ-*indexed Bernoulli line ensemble* $\mathfrak{L}$ *on* $[\![T_0, T_1]\!]$ is a random variable defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, taking values in Y such that $\mathfrak{L}$ is a $(\mathcal{B}, \mathcal{D})$-measurable function.

Remark 2.14. In [3, Section 3.1] Bernoulli line ensembles $\mathfrak{L}$ were called *discrete line ensembles* in order to distinguish them from the continuous line ensembles from Definition 2.1. In this paper we have opted to use the term Bernoulli line ensembles to emphasize the fact that the functions $f \in Y$ satisfy the property that $f(j, i+1) - f(j, i) \in \{0, 1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$. This condition essentially means that for each $j \in \Sigma$ the function $f(j, \cdot)$ can be thought of as the trajectory of a Bernoulli random walk from time T_0 to time T_1. As other types of discrete line ensembles, see e.g. [18], have appeared in the literature we have decided to modify the notation in [3, Section 3.1] so as to avoid any ambiguity.

The way we think of Bernoulli line ensembles is as random collections of up-right paths on the integer lattice, indexed by Σ (see Figure 1). Observe that one can view an up-right path L on $[\![T_0, T_1]\!]$ as a continuous curve by linearly interpolating the points $(i, L(i))$. This allows us to define $(\mathfrak{L}(\omega))(i, s)$ for non-integer $s \in [T_0, T_1]$ and to view Bernoulli line ensembles as line ensembles in the sense of Definition 2.1. In particular, we can think of $\mathfrak{L}$ as a random variable taking values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ with $\Lambda = [T_0, T_1]$. We will often slightly abuse notation and write $\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$, even

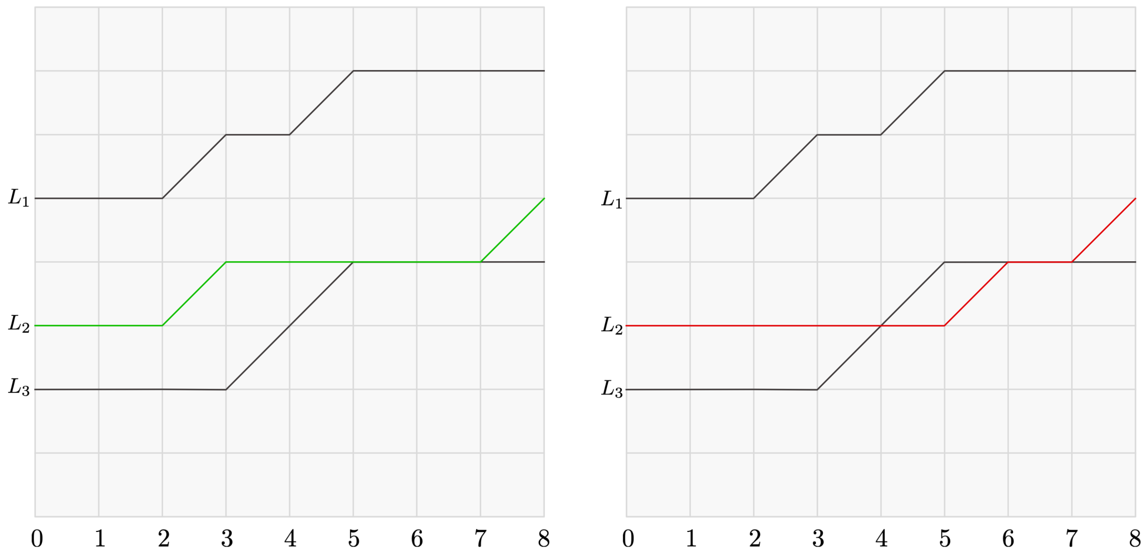

FIGURE 1. Two samples of $[\![1,3]\!]$-indexed Bernoulli line ensembles with $T_0 = 1$ and $T_1 = 8$, with the left ensemble avoiding and the right ensemble nonavoiding.

though it is not $\mathfrak{L}$ which is such a function, but rather $\mathfrak{L}(\omega)$ for each $\omega \in \Omega$. Furthermore we write $L_i = (\mathfrak{L}(\omega))(i,\cdot)$ for the index $i \in \Sigma$ path. If L is an up-right path on $[\![T_0,T_1]\!]$ and $a,b \in [\![T_0,T_1]\!]$ satisfy $a < b$ we let $L[\![a,b]\!]$ denote the resitrction of L to $[\![a,b]\!]$.

Let $t_i, z_i \in \mathbb{Z}$ for $i = 1,2$ be given such that $t_1 < t_2$ and $0 \leq z_2 - z_1 \leq t_2 - t_1$. We denote by $\Omega(t_1,t_2,z_1,z_2)$ the collection of up-right paths that start from (t_1,z_1) and end at (t_2,z_2), by $\mathbb{P}_{Ber}^{t_1,t_2,z_1,z_2}$ the uniform distribution on $\Omega(t_1,t_2,z_1,z_2)$ and write $\mathbb{E}_{Ber}^{t_1,t_2,z_1,z_2}$ for the expectation with respect to this measure. One thinks of the distribution $\mathbb{P}_{Ber}^{t_1,t_2,z_1,z_2}$ as the law of a simple random walk with i.i.d. Bernoulli increments with parameter $p \in (0,1)$ that starts from z_1 at time t_1 and is conditioned to end in z_2 at time t_2 – this interpretation does not depend on the choice of $p \in (0,1)$. Notice that by our assumptions on the parameters the state space $\Omega(t_1,t_2,z_1,z_2)$ is non-empty.

Given $k \in \mathbb{N}$, $T_0,T_1 \in \mathbb{Z}$ with $T_0 < T_1$ and $\vec{x},\vec{y} \in \mathbb{Z}^k$ we let $\mathbb{P}_{Ber}^{T_0,T_1,\vec{x},\vec{y}}$ denote the law of k independent Bernoulli bridges $\{B_i : [\![T_0,T_1]\!] \to \mathbb{Z}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$. Equivalently, this is just k independent random up-right paths $B_i \in \Omega(T_0,T_1,x_i,y_i)$ for $i = 1,\dots,k$ that are uniformly distributed. This measure is well-defined provided that $\Omega(T_0,T_1,x_i,y_i)$ are non-empty for $i = 1,\dots,k$, which holds if $T_1 - T_0 \geq y_i - x_i \geq 0$ for all $i = 1,\dots,k$.

The following definition introduces the notion of an (f,g)-avoiding Bernoulli line ensemble, which in simple terms is a collection of k independent Bernoulli bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two functions f and g.

Definition 2.15. Let $k \in \mathbb{N}$ and $\mathfrak{W}_k$ denote the set of signatures of length k, i.e.

$$\mathfrak{W}_k = \{\vec{x} = (x_1,\dots,x_k) \in \mathbb{Z}^k : x_1 \geq x_2 \geq \cdots \geq x_k\}.$$

Let $\vec{x},\vec{y} \in \mathfrak{W}_k$, $T_0,T_1 \in \mathbb{Z}$ with $T_0 < T_1$, $S \subseteq [\![T_0,T_1]\!]$, and $f : [\![T_0,T_1]\!] \to (-\infty,\infty]$ and $g : [\![T_0,T_1]\!] \to [-\infty,\infty)$ be two functions.

With the above data we define the *$(f,g;S)$-avoiding Bernoulli line ensemble on the interval $[\![T_0,T_1]\!]$ with entrance data $\vec{x}$ and exit data $\vec{y}$* to be the Σ-indexed Bernoulli line ensemble $\mathfrak{Q}$ with $\Sigma = [\![1,k]\!]$ on $[\![T_0,T_1]\!]$ and with the law of $\mathfrak{Q}$ equal to $\mathbb{P}_{Ber}^{T_0,T_1,\vec{x},\vec{y}}$ (the law of k independent uniform up-right paths $\{B_i : [\![T_0,T_1]\!] \to \mathbb{R}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$) conditioned on the event

$$E_S = \{f(r) \geq B_1(r) \geq B_2(r) \geq \cdots \geq B_k(r) \geq g(r) \text{ for all } r \in S\}.$$

The above definition is well-posed if there exist $B_i \in \Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ that satisfy the conditions in E_S (i.e. if the set of such up-right paths is not empty). We will denote by $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g; S)$ the set of collections of k up-right paths that satisfy the conditions in E_S and then the distribution on $\mathfrak{Q}$ is simply the uniform measure on $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g; S)$. We denote the probability distribution of $\mathfrak{Q}$ as $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber;S}$ and write $\mathbb{E}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber;S}$ for the expectation with respect to this measure. If $S = [\![T_0, T_1]\!]$, we write $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$, $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$, and $\mathbb{E}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$. If $f = +\infty$ and $g = -\infty$, we write $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y})$, $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y}}_{avoid,Ber}$, and $\mathbb{E}^{T_0,T_1,\vec{x},\vec{y}}_{avoid,Ber}$.

It will be useful to formulate simple conditions under which $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is non-empty and thus $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ well-defined. Note that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g; S) \supseteq \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ for any $S \subseteq [\![T_0, T_1]\!]$, so $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber;S}$ is also well-defined in this case. We accomplish this in the following lemma, whose proof is postponed until Section 8.3.

Lemma 2.16. *Suppose that $k \in \mathbb{N}$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. Suppose further that*

(1) $\vec{x}, \vec{y} \in \mathfrak{W}_k$ satisfy $T_1 - T_0 \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$,
(2) $f : [\![T_0, T_1]\!] \to (-\infty, \infty]$ and $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ satisfy $f(i+1) = f(i)$ or $f(i+1) = f(i) + 1$, and $g(i+1) = g(i)$ or $g(i+1) = g(i) + 1$ for $i = T_0, \dots, T_1 - 1$,
(3) $f(T_0) \geq x_1, f(T_1) \geq y_1$ and $g(T_0) \leq x_k, g(T_1) \leq y_k$.

Then the set $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ from Definition 2.15 is non-empty.

The following definition introduces the notion of the Schur Gibbs property, which can be thought of a discrete analogue of the partial Brownian Gibbs property the same way that Bernoulli random walks are discrete analogues of Brownian motion.

Definition 2.17. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. A Σ-indexed Bernoulli line ensemble $\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ is said to satisfy the *Schur Gibbs property* if it is non-crossing, meaning that

$$L_j(i) \geq L_{j+1}(i) \text{ for all } j = 1, \dots, N-1 \text{ and } i \in [\![T_0, T_1]\!],$$

and for any finite $K = \{k_1, k_1 + 1, \dots, k_2\} \subset [\![1, N-1]\!]$ and $a, b \in [\![T_0, T_1]\!]$ with $a < b$ the following holds. Suppose that f, g are two up-right paths drawn in $\{(r, z) \in \mathbb{Z}^2 : a \leq r \leq b\}$ and $\vec{x}, \vec{y} \in \mathfrak{W}_k$ with $k = k_2 - k_1 + 1$ altogether satisfy that $\mathbb{P}(A) > 0$ where A denotes the event

$$A = \{\vec{x} = (L_{k_1}(a), \dots, L_{k_2}(a)), \vec{y} = (L_{k_1}(b), \dots, L_{k_2}(b)), L_{k_1-1}[\![a, b]\!] = f, L_{k_2+1}[\![a, b]\!] = g\},$$

where if $k_1 = 1$ we adopt the convention $f = \infty = L_0$. Then writing $k = k_2 - k_1 + 1$, we have for any $\{B_i \in \Omega(a, b, x_i, y_i)\}_{i=1}^k$ that

$$\mathbb{P}\left(L_{i+k_1-1}[\![a, b]\!] = B_i \text{ for } i = 1, \dots, k \,|\, A\right) = \mathbb{P}^{a,b,\vec{x},\vec{y},f,g}_{avoid,Ber}\left(\cap_{i=1}^k \{\mathfrak{Q}_i = B_i\}\right). \tag{2.5}$$

Remark 2.18. In simple words, a Bernoulli line ensemble is said to satisfy the Schur Gibbs property if the distribution of any finite number of consecutive paths, conditioned on their end-points and the paths above and below them is simply the uniform measure on all collection of up-right paths that have the same end-points and do not cross each other or the paths above and below them.

Remark 2.19. Observe that in Definition 2.17 the index k_2 is assumed to be less than or equal to $N - 1$, so that if $N < \infty$ the N-th path is special and is not conditionally uniform. This is what makes Definition 2.17 a discrete analogue of the partial Brownian Gibbs property rather than the usual Brownian Gibbs property. Similarly to the partial Brownian Gibbs property, see Remark 2.11, if $N = 1$ then the conditions in Definition 2.17 become void, i.e., any Bernoulli line ensemble with one line satisfies the Schur Gibbs property. Also we mention that the well-posedness of $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ in (2.5) is a consequence of Lemma 2.16 and our assumption that $\mathbb{P}(A) > 0$.

Remark 2.20. In [3] the authors studied a generalization of the Gibbs property in Definition 2.17 depending on a parameter $t \in (0,1)$, which was called the *Hall-Littlewood Gibbs property* due to its connection to Hall-Littlewood polynomials [12]. The property in Definition 2.17 is the $t \to 0$ limit of the Hall-Littlewood Gibbs property. Since under this $t \to 0$ limit Hall-Littlewood polynomials degenerate to Schur polynomials we have decided to call the Gibbs property in Definition 2.17 the Schur Gibbs property.

Remark 2.21. An immediate consequence of Definition 2.17 is that if $M \leq N$, we have that the induced law on $\{L_i\}_{i=1}^M$ also satisfies the Schur Gibbs property as an $\{1,\dots,M\}$-indexed Bernoulli line ensemble on $[\![T_0,T_1]\!]$.

We end this section with the following definition of the term acceptance probability.

Definition 2.22. Assume the same notation as in Definition 2.15 and suppose that $T_1 - T_0 \geq y_i - x_i \geq 0$ for $i = 1,\dots,k$. We define the *acceptance probability* $Z(T_0,T_1,\vec{x},\vec{y},f,g)$ to be the ratio

$$Z(T_0,T_1,\vec{x},\vec{y},f,g) = \frac{|\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},f,g)|}{\prod_{i=1}^k |\Omega(T_0,T_1,x_i,y_i)|}. \tag{2.6}$$

Remark 2.23. The quantity $Z(T_0,T_1,\vec{x},\vec{y},f,g)$ is precisely the probability that if B_i are sampled uniformly from $\Omega(T_0,T_1,x_i,y_i)$ for $i=1,\dots,k$ then the B_i satisfy the condition

$$E = \{f(r) \geq B_1(r) \geq B_2(r) \geq \cdots \geq B_k(r) \geq g(r) \text{ for all } r \in [\![T_0,T_1]\!]\}.$$

Let us explain briefly why we call this quantity an acceptance probability. One way to sample $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ is as follows. Start by sampling a sequence of i.i.d. up-right paths B_i^N uniformly from $\Omega(T_0,T_1,x_i,y_i)$ for $i=1,\dots,k$ and $N \in \mathbb{N}$. For each n check if $B_1^n,\dots,B_k^n$ satisfy the condition E and let M denote the smallest index that accomplishes this. If $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},f,g)$ is non-empty then M is geometrically distributed with parameter $Z(T_0,T_1,\vec{x},\vec{y},f,g)$, and in particular M is finite almost surely and $\{B_i^M\}_{i=1}^k$ has distribution $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$. In this sampling procedure we construct a sequence of candidates $\{B_i^N\}_{i=1}^k$ for $N \in \mathbb{N}$ and reject those that fail to satisfy condition E, the first candidate that satisfies it is accepted and has law $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ and the probability that a candidate is accepted is precisely $Z(T_0,T_1,\vec{x},\vec{y},f,g)$, which is why we call it an acceptance probability.

2.3. **Main technical result.** In this section we present the main technical result of the paper. We start with the following technical definition.

Definition 2.24. Fix $k \in \mathbb{N}$, $\alpha,\lambda > 0$ and $p \in (0,1)$. Suppose we are given a sequence $\{T_N\}_{N=1}^\infty$ with $T_N \in \mathbb{N}$ and that $\{\mathfrak{L}^N\}_{N=1}^\infty$, $\mathfrak{L}^N = (L_1^N, L_2^N,\dots,L_k^N)$ is a sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N,T_N]\!]$. We call the sequence (α,p,λ)-*good* if

- for each $N \in \mathbb{N}$ we have that $\mathfrak{L}^N$ satisfies the Schur Gibbs property of Definition 2.17;
- there is a function $\psi : \mathbb{N} \to (0,\infty)$ such that $\lim_{N\to\infty} \psi(N) = \infty$ and for each $N \in \mathbb{N}$ we have that $T_N > \psi(N)N^\alpha$;
- there is a function $\phi : (0,\infty) \to (0,\infty)$ such that for any $\epsilon > 0$ we have

$$\sup_{n\in\mathbb{Z}} \limsup_{N\to\infty} \mathbb{P}\left(\left|N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})\right| \geq \phi(\epsilon)\right) \leq \epsilon. \tag{2.7}$$

Remark 2.25. Let us elaborate on the meaning of Definition 2.24. In order for a sequence of $\mathfrak{L}^N$ of $[\![1,k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N,T_N]\!]$ to be (α,p,λ)-*good* we want several conditions to be satisfied. Firstly, we want for each N the Bernoulli line ensemble $\mathfrak{L}^N$ to satisfy the Schur Gibbs property. The second condition is that while the interval of definition of $\mathfrak{L}^N$ is finite for each N and given by $[\![-T_N,T_N]\!]$, we want this interval to grow at least with speed N^α. This property is quantified by the function ψ, which can be essentially thought of as an arbitrary unbounded increasing function on $\mathbb{N}$. The third condition is that we want for each $n \in \mathbb{Z}$

the sequence of random variables $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha)$ to be tight but moreover we want globally these random variables to look like the parabola $-\lambda n^2$. This statement is reflected in (2.7), which provides a certain uniform tightness of the random variables $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})$. A particular case when (2.7) is satisfied is for example if we know that for each $n \in \mathbb{Z}$ the random variables $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})$ converge to the same random variable X. In the applications that we have in mind these random variables would converge to the 1-point marginals of the Airy$_2$ process that are all given by the same Tracy-Widom distribution (since the Airy$_2$ process is stationary). Equation (2.7) is a significant relaxation of the requirement that $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})$ all converge weakly to the Tracy-Widom distribution – the convergence requirement is replaced with a mild but uniform control of all subsequential limits.

The main result of the paper is as follows.

Theorem 2.26. *Fix $k \in \mathbb{N}$ with $k \geq 2$, $\alpha, \lambda > 0$ and $p \in (0,1)$ and let $\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)$ be an (α, p, λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles. Set*

$$f_i^N(s) = N^{-\alpha/2}(L_i^N(sN^\alpha) - psN^\alpha + \lambda s^2 N^{\alpha/2}), \text{ for } s \in [-\psi(N), \psi(N)] \text{ and } i = 1, \dots, k-1,$$

and extend f_i^N to $\mathbb{R}$ by setting for $i = 1, \dots, k-1$

$$f_i^N(s) = f_i^N(-\psi(N)) \text{ for } s \leq -\psi(N) \text{ and } f_i^N(s) = f_N(\psi(N)) \text{ for } s \geq \psi(N).$$

Let $\mathbb{P}_N$ denote the law of $\{f_i^N\}_{i=1}^{k-1}$ as a $[\![1,k-1]\!]$-indexed line ensemble (i.e. as a random variable in $(C([\![1,k-1]\!] \times \mathbb{R}), \mathcal{C})$), and let $\tilde{\mathbb{P}}_N$ denote the law of $\{(f_i^N - \lambda s^2)/\sqrt{p(1-p)}\}_{i=1}^{k-1}$. Then we have

- *(i) The sequence $\mathbb{P}_N$ is tight;*
- *(ii) Any subsequential limit $\mathcal{L}^\infty = \{f_i^\infty\}_{i=1}^{k-1}$ of $\tilde{\mathbb{P}}_N$ satisfies the partial Brownian Gibbs property of Definition 2.10.*

Roughly, Theorem 2.26 (i) states that if we have a sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles that satisfy the Schur Gibbs property and the top paths of these ensembles under some shift and scaling have tight one-point marginals with a non-trivial parabolic shift, then under the same shift and scaling the top $k-1$ paths of the line ensemble will be tight. The extension of f_i^N to $\mathbb{R}$ is completely arbitrary and irrelevant for the validity of Theorem 2.26 since the topology on $C([\![1,k-1]\!] \times \mathbb{R})$ is that of uniform convergence over compacts. Consequently, only the behavior of these functions on compact intervals matters in Theorem 2.26 and not what these functions do near infinity, which is where the modification happens as $\lim_{N\to\infty} \psi(N) = \infty$ by assumption. The only reason we perform the extension is to embed all Bernoulli line ensembles into the same space $(C([\![1,k-1]\!] \times \mathbb{R}), \mathcal{C})$.

We mention that the k-th up-right path in the sequence of Bernoulli line ensembles is special and Theorem 2.26 provides no tightness result for it. The reason for this stems from the Schur Gibbs property, see Definition 2.17, which assumes less information for the k-th path. In practice, one either has an infinite Bernoulli line ensemble for each N or one has a Bernoulli line ensemble with finite number of paths, which increase with N to infinity. In either of these settings one can use Theorem 2.26 to prove tightness of the full line ensemble - we will have more to say about this in Section 7.

The proof of Theorem 2.26 is presented in Section 4. In the next section we derive various properties for Bernoulli line ensembles.

3. Properties of Bernoulli line ensembles

In this section we derive several results for Bernoulli line ensembles, which will be used in the proof of Theorem 2.26 in Section 4.

3.1. **Monotone coupling lemmas.** In this section we formulate two lemmas that provide couplings of two Bernoulli line ensembles of non-intersecting Bernoulli bridges on the same interval, which depend monotonically on their boundary data. Schematic depictions of the couplings are provided in Figure 2. We postpone the proof of these lemmas until Section 8.

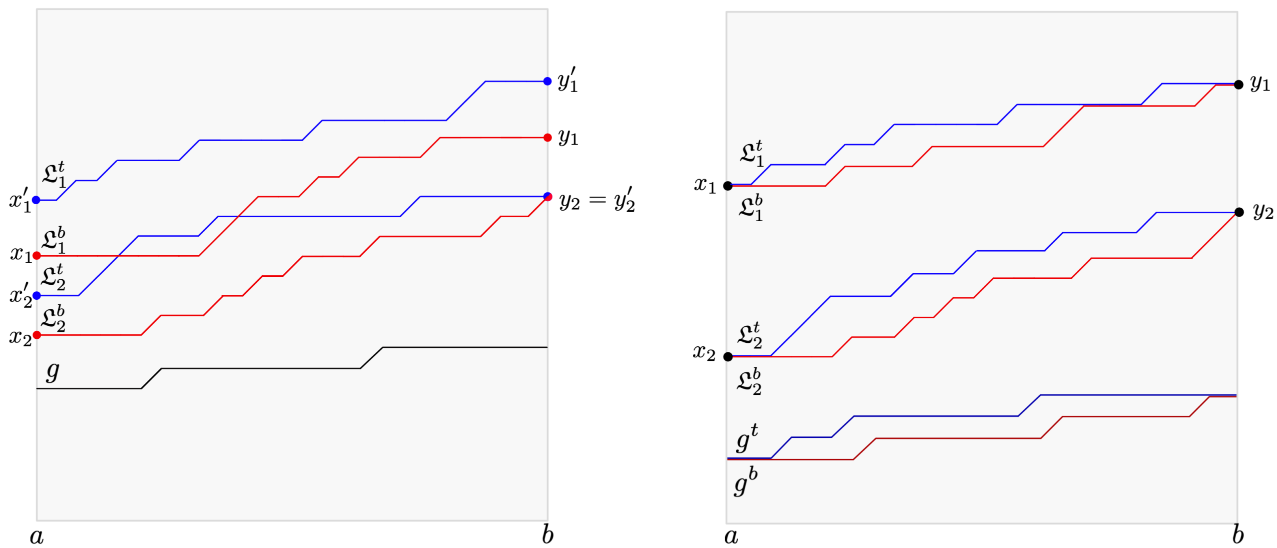

FIGURE 2. Two diagrammatic depictions of the monotone coupling Lemma 3.1 (left part) and Lemma 3.2 (right part). Red depicts the lower line ensemble and accompanying entry data, exit data, and bottom bounding curve, while blue depicts that of the higher ensemble.

Lemma 3.1. *Assume the same notation as in Definition 2.15. Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, $S \subseteq [\![T_0, T_1]\!]$, a function $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ as well as $\vec{x}, \vec{y}, \vec{x}\,', \vec{y}\,' \in \mathfrak{W}_k$. Assume that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g; S)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}', \vec{y}', \infty, g; S)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\big($resp. $\mathfrak{L}^b\big)$ under $\mathbb{P}$ is given by $\mathbb{P}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g}_{avoid,Ber;S}$ $\big($resp. $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber;S}\big)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}_i^t(r) \geq \mathfrak{L}_i^b(r)$ for all $i = 1, \dots, k$ and $r \in [\![T_0, T_1]\!]$.*

Lemma 3.2. *Assume the same notation as in Definition 2.15. Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, $S \subseteq [\![T_0, T_1]\!]$, two functions $g^t, g^b : [\![T_0, T_1]\!] \to [-\infty, \infty)$ and $\vec{x}, \vec{y} \in \mathfrak{W}_k$. We assume that $g^t(r) \geq g^b(r)$ for all $r \in [\![T_0, T_1]\!]$ and that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^t; S)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b; S)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\big($resp. $\mathfrak{L}^b\big)$ under $\mathbb{P}$ is given by $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^t}_{avoid,Ber;S}$ $\big($resp. $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber;S}\big)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}_i^t(r) \geq \mathfrak{L}_i^b(r)$ for all $i = 1, \dots, k$ and $r \in [\![T_0, T_1]\!]$.*

In plain words, Lemma 3.1 states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ of non-intersecting Bernoulli bridges, bounded from below by the same function g, in such a way that if all boundary values of $\mathfrak{L}^t$ are above the respective boundary values of $\mathfrak{L}^b$, then all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the left part of Figure 2. Lemma 3.2, states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ that have the same boundary values, but the lower bound g^t of $\mathfrak{L}^t$ is above the lower bound g^b of $\mathfrak{L}^b$, in such a way that all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the right part of Figure 2.

3.2. **Properties of Bernoulli and Brownian bridges.** In this section we derive several results about Bernoulli bridges, which are random up-right paths that have law $\mathbb{P}_{Ber}^{T_0,T_1,x,y}$ as in Section 2.2, as well as Brownian bridges with law $\mathbb{P}_{free}^{T_0,T_1,x,y}$ as in Section 2.1. Our results will rely on the two monotonicity Lemmas 3.1 and 3.2 as well as a strong coupling between Bernoulli bridges and Brownian bridges from [3] – recalled here as Theorem 3.3.

If W_t denotes a standard one-dimensional Brownian motion and $\sigma > 0$, then the process

$$B_t^\sigma = \sigma(W_t - tW_1), \quad 0 \le t \le 1,$$

is called a *Brownian bridge (conditioned on $B_0 = 0, B_1 = 0$) with variance σ^2*. We note that B^σ is the unique a.s. continuous Gaussian process on $[0,1]$ with $B_0 = B_1 = 0$, $\mathbb{E}[B_t^\sigma] = 0$, and

$$\mathbb{E}[B_r^\sigma B_s^\sigma] = \sigma^2(r \wedge s - rs - sr + sr) = \sigma^2(r \wedge s - rs). \tag{3.1}$$

With the above notation we state the strong coupling result we use.

Theorem 3.3. *Let $p \in (0,1)$. There exist constants $0 < C, a, \alpha < \infty$ (depending on p) such that for every positive integer n, there is a probability space on which are defined a Brownian bridge B^σ with variance $\sigma^2 = p(1-p)$ and a family of random paths $\ell^{(n,z)} \in \Omega(0,n,0,z)$ for $z = 0, \dots, n$ such that $\ell^{(n,z)}$ has law $\mathbb{P}_{Ber}^{0,n,0,z}$ and*

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \le Ce^{\alpha(\log n)^2} e^{|z-pn|^2/n}, \text{ where } \Delta(n,z) := \sup_{0\le t\le n}\left|\sqrt{n}B_{t/n}^\sigma + \tfrac{t}{n}z - \ell^{(n,z)}(t)\right|. \tag{3.2}$$

Remark 3.4. When $p = 1/2$ the above theorem follows (after a trivial affine shift) from [11, Theorem 6.3] and the general $p \in (0,1)$ case was done in [3, Theorem 4.5]. We mention that a significant generalization of Theorem 3.3 for general random walk bridges has recently been proved in [7, Theorem 2.3].

We will use the following simple corollary of Theorem 3.3 to compare Bernoulli bridges with Brownian bridges. We use the same notation as in the theorem.

Corollary 3.5. *Fix $p \in (0,1)$, $\beta > 0$, and $A > 0$. Suppose $|z - pn| \le K\sqrt{n}$ for a constant $K > 0$. Then for any $\epsilon > 0$, there exists N large enough depending on p, ϵ, A, K so that for $n \ge N$,*

$$\mathbb{P}\Big(\Delta(n,z) \ge An^\beta\Big) < \epsilon.$$

Proof. Applying Chebyshev's inequality and (3.2) gives

$$\begin{aligned}\mathbb{P}\Big(\Delta(n,z) \ge An^\beta\Big) &\le e^{-An^\beta}\mathbb{E}\Big[e^{a\Delta(n,z)}\Big] \le C\exp\Big[-An^\beta + \alpha(\log n)^2 + \frac{|z-pn|^2}{n}\Big] \\ &\le C\exp\Big[-An^\beta + \alpha(\log n)^2 + K\Big].\end{aligned}$$

The conclusion is now immediate. □

We also state the following result regarding the distribution of the maximum of a Brownian bridge, which follows from formulas in [8, Section 12.3].

Lemma 3.6. *Fix $p \in (0,1)$, and let B^σ be a Brownian bridge of variance $\sigma^2 = p(1-p)$ on $[0,1]$. Then for any $C, T > 0$ we have*

$$\begin{aligned}\mathbb{P}\left(\max_{s\in[0,T]} B_{s/T}^\sigma \ge C\right) &= \exp\left(-\frac{2C^2}{p(1-p)}\right), \\ \mathbb{P}\left(\max_{s\in[0,T]} \left|B_{s/T}^\sigma\right| \ge C\right) &= 2\sum_{n=1}^\infty (-1)^{n-1}\exp\left(-\frac{2n^2C^2}{p(1-p)}\right).\end{aligned} \tag{3.3}$$

In particular,

$$\mathbb{P}\left(\max_{s\in[0,T]} \left|B_{s/T}^\sigma\right| \ge C\right) \le 2\exp\left(-\frac{2C^2}{p(1-p)}\right). \tag{3.4}$$

Proof. Let B^1 be a Brownian bridge with variance 1 on $[0,1]$. Then B^σ_t has the same distribution as σB^1_t. Hence

$$\mathbb{P}\left(\max_{s\in[0,T]} B^\sigma_{s/T} \geq C\right) = \mathbb{P}\left(\max_{t\in[0,1]} B^1_t \geq C/\sigma\right) = e^{-2(C/\sigma)^2} = e^{-2C^2/p(1-p)}.$$

The second equality follows from [8, Proposition 12.3.3]. This proves the first equality in (3.3). Similarly, using [8, Proposition 12.3.4] we find

$$\mathbb{P}\left(\max_{s\in[0,T]} \left|B^\sigma_{s/T}\right| \geq C\right) = \mathbb{P}\left(\max_{t\in[0,1]} \left|B^1_t\right| \geq C/\sigma\right) = 2\sum_{n=1}^{\infty}(-1)^{n-1}e^{-2n^2C^2/\sigma^2},$$

proving the second inequality in (3.3).

Lastly to prove (3.4), observe that since B^σ_t has mean 0, B^σ_t and $-B^\sigma_t$ have the same distribution. It follows from the first equality above that

$$\mathbb{P}\left(\max_{s\in[0,T]} \left|B^\sigma_{s/T}\right| \geq C\right) \leq \mathbb{P}\left(\max_{s\in[0,T]} B^\sigma_{s/T} \geq C\right) + \mathbb{P}\left(\max_{s\in[0,T]} \big(-B^\sigma_{s/T}\big) \geq C\right) =$$
$$2\,\mathbb{P}\left(\max_{s\in[0,T]} B^\sigma_{s/T} \geq C\right) = 2e^{-2C^2/p(1-p)}.$$

□

We state one more lemma about Brownian bridges, which allows us to decompose a bridge on $[0,1]$ into two independent bridges with Gaussian affine shifts meeting at a point in $(0,1)$.

Lemma 3.7. *Fix $p\in(0,1)$, $T>0$, $t\in(0,T)$, and let B^σ be a Brownian bridge of variance $\sigma^2=p(1-p)$ on $[0,1]$. Let ξ be a Gaussian random variable with mean 0 and variance*

$$\mathbb{E}[\xi^2] = \sigma^2\frac{t}{T}\left(1-\frac{t}{T}\right).$$

Let B^1,B^2 be two independent Brownian bridges on $[0,1]$ with variances $\sigma^2 t/T$ and $\sigma^2(T-t)/T$ respectively, also independent from B^σ. Define the process

$$\tilde{B}_{s/T} = \begin{cases} \dfrac{s}{t}\,\xi + B^1\Big(\dfrac{s}{t}\Big), & s\leq t,\\[2mm] \dfrac{T-s}{T-t}\,\xi + B^2\Big(\dfrac{s-t}{T-t}\Big), & s\geq t,\end{cases}$$

for $s\in[0,T]$. Then $\tilde{B}$ is a Brownian bridge with variance σ.

Proof. It is clear that the process $\tilde{B}$ is a.s. continuous. Since $\tilde{B}$ is built from three independent zero-centered Gaussian processes, it is itself a zero-centered Gaussian process and thus completely characterized by its covariance. Consequently, to show that $\tilde{B}$ is a Brownian bridge of variance σ^2, it suffices to show by (3.1) that if $0\leq r\leq s\leq T$ we have

$$\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \sigma^2\frac{r}{T}\Big(1-\frac{s}{T}\Big). \tag{3.5}$$

First assume $s\leq t$ Using the fact that ξ and $B^1_\cdot$ are independent with mean 0, we find

$$\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \frac{rs}{t^2}\cdot\sigma^2\frac{t}{T}\Big(1-\frac{t}{T}\Big) + \sigma^2\frac{t}{T}\cdot\frac{r}{t}\Big(1-\frac{s}{t}\Big) =$$
$$\sigma^2\frac{r}{T}\Big(\frac{s}{t}-\frac{s}{T}+1-\frac{s}{t}\Big) = \sigma^2\frac{r}{T}\Big(1-\frac{s}{T}\Big).$$

If $r \geq t$, we compute

$$\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \frac{(T-r)(T-s)}{(T-t)^2} \cdot \sigma^2 \frac{t}{T}\Big(1 - \frac{t}{T}\Big) + \sigma^2 \frac{T-t}{T} \cdot \frac{r-t}{T-t}\Big(1 - \frac{s-t}{T-t}\Big) =$$
$$\frac{\sigma^2(T-s)}{T(T-t)}\left(\frac{t(T-r)}{T} + r - t\right) = \frac{\sigma^2(T-s)}{T(T-t)} \cdot \frac{r(T-t)}{T} = \sigma^2 \frac{r}{T}\Big(1 - \frac{s}{T}\Big).$$

If $r < t < s$, then since ξ, $B^1_\cdot$, and $B^2_\cdot$ are all independent, we have

$$\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \frac{r}{t} \cdot \frac{T-s}{T-t} \cdot \sigma^2 \frac{t(T-t)}{T^2} = \sigma^2 \frac{r(T-s)}{T^2} = \sigma^2 \frac{r}{T}\Big(1 - \frac{s}{T}\Big).$$

This proves (3.5) in all cases. ☐

Below we list four lemmas about Bernoulli bridges. We provide a brief informal explanation of what each result says after it is stated. All six lemmas are proved in a similar fashion. For the first two lemmas one observes that the event whose probability is being estimated is monotone in ℓ. This allows us by Lemmas 3.1 and 3.2 to replace x, y in the statements of the lemmas with the extreme values of the ranges specified in each. Once the choice of x and y is fixed one can use our strong coupling results, Theorem 3.3 and Corollary 3.5, to reduce each of the lemmas to an analogous one involving a Brownian bridge with some prescribed variance. The latter statements are then easily confirmed as one has exact formulas for Brownian bridges, such as Lemma 3.6.

Lemma 3.8. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,x,y}_{Ber}$. Let $M_1, M_2 \in \mathbb{R}$ be given. Then we can find $W_0 = W_0(p, M_2 - M_1) \in \mathbb{N}$ such that for $T \geq W_0$, $x \geq M_1 T^{1/2}$, $y \geq pT + M_2 T^{1/2}$ and $s \in [0,T]$ we have*

$$\mathbb{P}^{0,T,x,y}_{Ber}\left(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\right) \geq \frac{1}{3}. \tag{3.6}$$

Remark 3.9. If $M_1, M_2 = 0$ then Lemma 3.8 states that if a Bernoulli bridge ℓ is started from $(0,x)$ and terminates at (T,y), which are above the straight line of slope p, then at any given time $s \in [0,T]$ the probability that $\ell(s)$ goes a modest distance below the straight line of slope p is upper bounded by $2/3$.

Proof. Define $A = \lfloor M_1 T^{1/2} \rfloor$ and $B = \lfloor pT + M_2 T^{1/2} \rfloor$. Then since $A \leq x$ and $B \leq y$, it follows from Lemma 3.1 that there is a probability space with measure $\mathbb{P}_0$ supporting random variables L_1 and L_2, whose laws under $\mathbb{P}_0$ are $\mathbb{P}^{0,T,A,B}_{Ber}$ and $\mathbb{P}^{0,T,x,y}_{Ber}$ respectively, and $\mathbb{P}_0$-a.s. we have $L_1 \leq L_2$. Thus

$$\begin{aligned}
&\mathbb{P}^{0,T,x,y}_{Ber}\left(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\right) = \\
&\mathbb{P}_0\left(L_2(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\right) \geq \\
&\mathbb{P}_0\left(L_1(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\right) = \\
&\mathbb{P}^{0,T,A,B}_{Ber}\left(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\right).
\end{aligned} \tag{3.7}$$

Since the uniform distribution on upright paths on $[\![0,T]\!] \times [\![A,B]\!]$ is the same as that on upright paths on $[\![0,T]\!] \times [\![0,B-A]\!]$ shifted vertically by A, the last line of (3.7) is equal to

$$\mathbb{P}^{0,T,0,B-A}_{Ber}\left(\ell(s) + A \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\right).$$

Now we employ the coupling provided by Theorem 3.3. We have another probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random variable $\ell^{(T,B-A)}$ whose law under $\mathbb{P}$ is $\mathbb{P}_{Ber}^{0,T,0,B-A}$ as well as a Brownian bridge B^{σ} coupled with $\ell^{(T,B-A)}$. We have

$$
\begin{aligned}
&\mathbb{P}_{Ber}^{0,T,0,B-A}\left(\ell(s) + A \geq \frac{T-s}{T}\cdot M_1 T^{1/2} + \frac{s}{T}\cdot\left(pT + M_2T^{1/2}\right) - T^{1/4}\right) = \\
&\mathbb{P}\left(\ell^{(T,B-A)}(s) + A \geq \frac{T-s}{T}\cdot M_1 T^{1/2} + \frac{s}{T}\cdot\left(pT + M_2T^{1/2}\right) - T^{1/4}\right) = \\
&\mathbb{P}\Bigg(\left[\ell^{(T,B-A)}(s) - \sqrt{T}B^{\sigma}_{s/T} - \frac{s}{T}\cdot(B-A)\right] + \sqrt{T}B^{\sigma}_{s/T} \geq \\
&\qquad - A - \frac{s}{T}\cdot(B-A) + \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\left(pT + M_2T^{1/2}\right) - T^{1/4}\Bigg).
\end{aligned}
\tag{3.8}
$$

Recalling the definitions of A and B, we can rewrite the quantity in the last line of (3.8) and bound by

$$
\begin{aligned}
&\frac{T-s}{T}\cdot\left(M_1T^{1/2} - A\right) + \frac{s}{T}\cdot\left(pT + M_2T^{1/2} - B\right) - T^{1/4} \leq \\
&\frac{T-s}{T} + \frac{s}{T} - T^{1/4} = -T^{1/4} + 1.
\end{aligned}
$$

Thus the last line of (3.7) is bounded below by

$$
\begin{aligned}
&\mathbb{P}\left(\left[\ell^{(T,B-A)}(s) - \sqrt{T}B^{\sigma}_{s/T} - \frac{s}{T}\cdot(B-A)\right] + \sqrt{T}B^{\sigma}_{s/T} \geq -T^{1/4} + 1\right) \geq \\
&\mathbb{P}\left(\sqrt{T}B^{\sigma}_{s/T} \geq 0 \quad \text{and} \quad \Delta(T, B-A) < T^{1/4} - 1\right) \geq \\
&\mathbb{P}\left(B^{\sigma}_{s/T} \geq 0\right) - \mathbb{P}\left(\Delta(T, B-A) \geq T^{1/4} - 1\right) = \\
&\frac{1}{2} - \mathbb{P}\left(\Delta(T, B-A) \geq T^{1/4} - 1\right).
\end{aligned}
\tag{3.9}
$$

For the first inequality, we used the fact that the quantity in brackets is bounded in absolute value by $\Delta(T, B-A)$. The second inequality follows by dividing the event $\{B^{\sigma}_{s/T} \geq 0\}$ into cases and applying subadditivity. Since $|B - A - pT| \leq (M_2 - M_1 + 1)\sqrt{T}$, Corollary 3.5 allows us to choose W_0 large enough depending on p and $M_2 - M_1$ so that if $T \geq W_0$, then the last line of (3.9) is bounded above by $1/2 - 1/6 = 1/3$. In combination with (3.7) this proves (3.6). ☐

Lemma 3.10. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $y, z \in \mathbb{Z}$ such that $T \geq y, z \geq 0$, and suppose that ℓ_y, ℓ_z have distributions $\mathbb{P}_{Ber}^{0,T,0,y}$, $\mathbb{P}_{Ber}^{0,T,0,z}$ respectively. Let $M > 0$ and $\epsilon > 0$ be given. Then we can find $W_1 = W_1(M, p, \epsilon) \in \mathbb{N}$ and $A = A(M, p, \epsilon) > 0$ such that for $T \geq W_1$, $y \geq pT - MT^{1/2}$, $z \leq pT + MT^{1/2}$ we have*

$$
\begin{aligned}
&\mathbb{P}_{Ber}^{0,T,0,y}\left(\inf_{s\in[0,T]}\left[\ell_y(s) - ps\right] \leq -AT^{1/2}\right) \leq \epsilon, \\
&\mathbb{P}_{Ber}^{0,T,0,z}\left(\sup_{s\in[0,T]}\left[\ell_z(s) - ps\right] \geq AT^{1/2}\right) \leq \epsilon.
\end{aligned}
\tag{3.10}
$$

Remark 3.11. Roughly, Lemma 3.10 states that if a Bernoulli bridge ℓ is started from $(0,0)$ and terminates at time T not significantly lower (resp. higher) than the straight line of slope p, then the event that ℓ goes significantly below (resp. above) the straight line of slope p is very unlikely.

Proof. The two inequalities are proven in essentially the same way. We begin with the first inequality. If $B = \lfloor pT - MT^{1/2} \rfloor$ then it follows from Lemma 3.1 that

$$\mathbb{P}_{Ber}^{0,T,0,y}\left(\inf_{s\in[0,T]}\big[\ell_y(s) - ps\big] \le -AT^{1/2}\right) \le \mathbb{P}_{Ber}^{0,T,0,B}\left(\inf_{s\in[0,T]}\big[\ell(s) - ps\big] \le -AT^{1/2}\right), \tag{3.11}$$

where ℓ has law $\mathbb{P}_{Ber}^{0,T,0,B}$. By Theorem 3.3, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random variable $\ell^{(T,B)}$ whose law under $\mathbb{P}$ is also $\mathbb{P}_{Ber}^{0,T,0,B}$, and a Brownian bridge B^σ with variance $\sigma^2 = p(1-p)$. Therefore

$$\begin{aligned}
&\mathbb{P}_{Ber}^{0,T,0,B}\left(\inf_{s\in[0,T]}\big[\ell(s) - ps\big] \le -AT^{1/2}\right) = \mathbb{P}\left(\inf_{s\in[0,T]}\big[\ell^{(T,B)}(s) - ps\big] \le -AT^{1/2}\right) \le \\
&\mathbb{P}\left(\inf_{s\in[0,T]}\sqrt{T}B^\sigma_{s/T} \le -\frac{1}{2}AT^{1/2}\right) + \mathbb{P}\left(\sup_{s\in[0,T]}\left|\sqrt{T}B^\sigma_{s/T} + ps - \ell^{(T,B)}(s)\right| \ge \frac{1}{2}AT^{1/2}\right) \le \\
&\mathbb{P}\left(\max_{s\in[0,T]} B^\sigma_{s/T} \ge A/2\right) + \mathbb{P}\left(\Delta(T,B) \ge \frac{1}{2}AT^{1/2} - MT^{1/2} - 1\right).
\end{aligned} \tag{3.12}$$

For the first term in the last line, we used the fact that B^σ and $-B^\sigma$ have the same distribution. For the second term, we used the fact that

$$\sup_{s\in[0,T]}\left|ps - \frac{s}{T}\cdot B\right| \le \sup_{s\in[0,T]}\left|ps - \frac{pT - MT^{1/2}}{T}\cdot s\right| + 1 = MT^{1/2} + 1.$$

By Lemma 3.6, the first term in the last line of (3.12) is equal to $e^{-A^2/2p(1-p)}$. If we choose $A \ge \sqrt{2p(1-p)\log(2/\epsilon)}$, then this is $\le \epsilon/2$. If we also take $A > 2M$, then since $|B - pT| \le (M+1)\sqrt{T}$, Corollary 3.5 gives us a W_1 large enough depending on M, p, ϵ so that the second term in the last line of (3.12) is also $< \epsilon/2$ for $T \ge W_1$. Adding the two terms and using (3.11) gives the first inequality in (3.10).

If we replace B with $\lceil pT + MT^{1/2} \rceil$ and change signs and inequalities where appropriate, then the same argument proves the second inequality in (3.10). ☐

We need the following definition for our next result. For a function $f \in C([a,b])$ we define its *modulus of continuity* for $\delta > 0$ by

$$w(f,\delta) = \sup_{\substack{x,y\in[a,b] \\ |x-y|\le\delta}} |f(x) - f(y)|. \tag{3.13}$$

Lemma 3.12. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $y \in \mathbb{Z}$ such that $T \ge y \ge 0$, and suppose that ℓ has distribution $\mathbb{P}_{Ber}^{0,T,0,y}$. For each positive M, ϵ and η, there exist a $\delta(\epsilon,\eta,M) > 0$ and $W_2 = W_2(M,p,\epsilon,\eta) \in \mathbb{N}$ such that for $T \ge W_2$ and $|y - pT| \le MT^{1/2}$ we have*

$$\mathbb{P}_{Ber}^{0,T,0,y}\left(w\big(f^\ell,\delta\big) \ge \epsilon\right) \le \eta, \tag{3.14}$$

where $f^\ell(u) = T^{-1/2}\big(\ell(uT) - puT\big)$ for $u \in [0,1]$.

Remark 3.13. Lemma 3.12 states that if ℓ is a Bernoulli bridge that is started from $(0,0)$ and terminates at (T,y) with y close to pT (i.e. with well-behaved endpoints) then the modulus of continuity of ℓ is also well-behaved with high probability.

Proof. By Theorem 3.3, we have a probability measure $\mathbb{P}$ supporting a random variable $\ell^{(T,y)}$ with law $\mathbb{P}_{Ber}^{0,T,0,y}$ as well as a Brownian bridge B^σ with variance $\sigma^2 = p(1-p)$. We have

$$\mathbb{P}_{Ber}^{0,T,0,y}\Big(w\big(f^\ell,\delta\big) \ge \epsilon\Big) = \mathbb{P}\Big(w\big(f^{\ell^{(T,y)}},\delta\big) \ge \epsilon\Big), \tag{3.15}$$

and

$$\begin{aligned} & w\big(f^{\ell^{(T,y)}},\delta\big) = T^{-1/2} \sup_{s,t\in[0,1],\,|s-t|\le\delta} \Big|\ell^{(T,y)}(sT) - psT - \ell^{(T,y)}(tT) + ptT\Big| \le \\ & T^{-1/2} \sup_{s,t\in[0,1],\,|s-t|\le\delta} \bigg(\Big|\sqrt{T}\,B^\sigma_s + sy - psT - \sqrt{T}\,B^\sigma_t - ty + ptT\Big| + \\ & \qquad \Big|\sqrt{T}\,B^\sigma_s + sy - \ell^{(T,y)}(sT)\Big| + \Big|\sqrt{T}\,B^\sigma_t + ty - \ell^{(T,y)}(tT)\Big| \bigg) \le \\ & \sup_{s,t\in[0,1],\,|s-t|\le\delta} \Big|B^\sigma_s - B^\sigma_t + T^{-1/2}(y-pT)(s-t)\Big| + 2T^{-1/2}\Delta(T,y) \le \\ & w\big(B^\sigma,\delta\big) + M\delta + 2T^{-1/2}\Delta(T,y). \end{aligned} \tag{3.16}$$

The last line follows from the assumption that $|y - pT| \le MT^{1/2}$. Now (3.15) and (3.16) together imply that

$$\begin{aligned} & \mathbb{P}^{0,T,0,y}_{Ber}\Big(w\big(f^\ell,\delta\big) \ge \epsilon\Big) \le \mathbb{P}\Big(w\big(B^\sigma,\delta\big) + M\delta + 2T^{-1/2}\Delta(T,y) \ge \epsilon\Big) \le \\ & \mathbb{P}\left(w\big(B^\sigma,\delta\big) + M\delta \ge \epsilon/2\right) + \mathbb{P}\Big(\Delta(T,y) \ge \epsilon\, T^{1/2}/4\Big). \end{aligned} \tag{3.17}$$

Corollary 3.5 gives us a W_2 large enough depending on M, p, ϵ, η so that the second term in the second line of 3.17 is $\le \eta/2$ for $T \ge W_2$. Since B^σ is a.s. uniformly continuous on the compact interval $[0,1]$, $w(B^\sigma,\delta) \to 0$ as $\delta \to 0$. Thus we can find $\delta_0 > 0$ small enough depending on ϵ, η so that $w(B^\sigma, \delta_0) < \epsilon/4$ with probability at least $1 - \eta/2$. Then with $\delta = \min(\delta_0, \epsilon/4M)$, the first term in the second line of (3.17) is $\le \eta/2$ as well. This implies (3.14). $\square$

Lemma 3.14. *Fix $T \in \mathbb{N}$, $p \in (0,1)$, $C, K > 0$, and $a, b \in \mathbb{Z}$ such that $\Omega(0,T,a,b)$ is nonempty. Let $\ell_{bot} \in \Omega(0,T,a,b)$. Suppose $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$, $k \ge 2$, are such that $T \ge y_i - x_i \ge 0$ for $1 \le i \le k-1$. Write $\vec{z} = \vec{y} - \vec{x}$, and suppose that*

(1) $x_{k-1} + (z_{k-1}/T)s - \ell_{bot}(s) \ge C\sqrt{T}$ for all $s \in [0,T]$
(2) $x_i - x_{i+1} \ge C\sqrt{T}$ and $y_i - y_{i+1} \ge C\sqrt{T}$ for $1 \le i \le k-2$,
(3) $|z_i - pT| \le K\sqrt{T}$ for $1 \le i \le k-1$, for a constant $K > 0$.

Let $\mathfrak{L} = (L_1, \dots, L_{k-1})$ be a line ensemble with law $\mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}$, and let E denote the event

$$E = \{L_1(s) \ge \cdots \ge L_{k-1}(s) \ge \ell_{bot}(s) \text{ for } s \in [0,T]\}.$$

Then we can find $W_3 = W_3(p, C, K)$ so that for $T \ge W_3$,

$$\mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}(E) \ge \left(\frac{1}{2} - \sum_{n=1}^{\infty}(-1)^{n-1}e^{-n^2C^2/8p(1-p)}\right)^{k-1}. \tag{3.18}$$

Moreover if $C \ge \sqrt{8p(1-p)\log 3}$, then for $T \ge W_3$ we have

$$\mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}(E) \ge \Big(1 - 3e^{-C^2/8p(1-p)}\Big)^{k-1}. \tag{3.19}$$

Remark 3.15. This lemma states that if k independent Bernoulli bridges are well-separated from each other and ℓ_{bot}, then there is a positive probability that the curves will intersect neither each other nor ℓ_{bot}. We will use this result to compare curves in an avoiding Bernoulli line ensemble with free Bernoulli bridges.

Proof. Observe that condition (1) simply states that ℓ_{bot} lies a distance of at least $C\sqrt{T}$ uniformly below the line segment connecting x_{k-1} and y_{k-1}. Thus (1) and (2) imply that E occurs if each curve L_i remains within a distance of $C\sqrt{T}/2$ from the line segment connecting x_i and y_i. As in

Theorem 3.3, let $\mathbb{P}_i$ be probability measures supporting random variables $\ell^{(T,z_i)}$ with laws $\mathbb{P}^{0,T,0,z_i}_{Ber}$. Then

$$
\begin{aligned}
&\mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}(E) \geq \mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}\left(\sup_{s\in[0,T]} \big|L_i(s) - x_i - (z_i/T)s\big| \leq C\sqrt{T}/2,\ 1 \leq i \leq k-1 \right) = \\
&\prod_{i=1}^{k-1} \left[\mathbb{P}^{0,T,0,z_i}_{Ber}\left(\sup_{s\in[0,T]} \big|L_i(s+rN^\alpha) - (z_i/T)s\big| \leq C\sqrt{T}/2 \right) \right] = \\
&\prod_{i=1}^{k-1} \left[1 - \mathbb{P}_i\left(\sup_{s\in[0,T]} \big|\ell^{(T,z_i)} - (z_i/T)s\big| > C\sqrt{T}/2 \right) \right].
\end{aligned}
\tag{3.20}
$$

In the third line, we used the fact that $L_1, \dots, L_{k-1}$ are independent from each other under $\mathbb{P}^{0,T,0,z_i}_{Ber}$. Let $B^{\sigma,i}$ be the Brownian bridge with variance $\sigma^2 = p(1-p)$ coupled with $\ell^{(T,z_i)}$ given by Theorem 3.3. Then we have

$$
\begin{aligned}
&\mathbb{P}_i\left(\sup_{s\in[0,T]} \big|\ell^{(T,z_i)}(s) - (z_i/T)s\big| > C\sqrt{T}/2 \right) \leq \\
&\mathbb{P}_i\left(\sup_{s\in[0,T]} |\sqrt{T}B^{\sigma}_{s/T}| > C\sqrt{T}/4 \right) + \mathbb{P}_i\left(\Delta(T,z_i) > C\sqrt{T}/4 \right).
\end{aligned}
\tag{3.21}
$$

By Lemma 3.6, the first term in the second line of (3.21) is equal to $2\sum_{n=1}^\infty (-1)^{n-1} e^{-n^2C^2/8p(1-p)}$. Moreover, condition (3) in the hypothesis and Corollary 3.5 allow us to find W_3 depending on p, C, K but not on i so that the last probability in (3.21) is bounded above by $\frac{1}{2} - \sum_{n=1}^\infty (-1)^{n-1} e^{-n^2C^2/8p(1-p)}$ for $T \geq W_3$. Adding these two terms and referring to (3.20) proves (3.18).

Now suppose $C \geq \sqrt{8p(1-p)\log 3}$. By (3.4) in Lemma 3.6, the first term in the second line of (3.21) is bounded above by bounded above by $2e^{-C^2/8p(1-p)}$. After possibly enlarging W_3 from above, the second term is $< e^{-C^2/8p(1-p)}$ for $T \geq W_3$. The assumption on C implies that $1 - 3e^{-C^2/8p(1-p)} \geq 0$, and now combining (3.21) and (3.20) proves (3.19). ☐

3.3. **Properties of avoiding Bernoulli line ensembles.** In this section we derive two results about avoiding Bernoulli line ensembles, which are Bernoulli line ensembles with law $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber;S}$ as in Definition 2.15. The lemmas we prove only involve the case when $f(r) = \infty$ for all $r \in [\![T_0, T_1]\!]$ and we denote the measure in this case by $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber;S}$. A $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber;S}$-distributed random variable will be denoted by $\mathfrak{Q} = (Q_1, \dots, Q_k)$ where k is the number of up-right paths in the ensemble. As usual, if $g = -\infty$, we write $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y}}_{avoid,Ber;S}$. Our first result will rely on the two monotonicity Lemmas 3.1 and 3.2 as well as the strong coupling between Bernoulli bridges and Brownian bridges from Theorem 3.3, and the further results make use of the material in Section 9.

Lemma 3.16. *Fix $p \in (0,1)$, $k \in \mathbb{N}$. Let $\vec{x}, \vec{y} \in \mathfrak{W}_k$ be such that $T \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$. Then for any $M, M_1 > 0$ we can find $W_4 \in \mathbb{N}$ depending on p, k, M, M_1 such that if $T \geq W_4$, $x_k \geq -M_1\sqrt{T}$, and $y_k \geq pT - M_1\sqrt{T}$, then for any $S \subseteq [\![0, T]\!]$ we have*

$$
\mathbb{P}^{0,T,\vec{x},\vec{y}}_{avoid,Ber;S}\left(Q_k(T/2) - pT/2 \geq M\sqrt{T} \right) \geq \frac{2^{k/2}\left(1 - 2e^{-4/p(1-p)}\right)^{2k}}{(\pi p(1-p))^{k/2}} \exp\left(-\frac{2k(M+M_1+6)^2}{p(1-p)} \right).
\tag{3.22}
$$

Proof. Define vectors $\vec{x}, \vec{y} \in \mathfrak{W}_k$ by

$$
\begin{aligned}
x_i' &= \lfloor -M_1\sqrt{T} \rfloor - 10(i-1)\lceil \sqrt{T} \rceil, \\
y_i' &= \lfloor pT - M_1\sqrt{T} \rfloor - 10(i-1)\lceil \sqrt{T} \rceil.
\end{aligned}
$$

Then $x_i' \leq x_k \leq x_i$ and $y_i' \leq y_k \leq y_i$ for $1 \leq i \leq k-1$. Thus by Lemma 3.1, we have

$$\mathbb{P}^{0,T,\vec{x},\vec{y}}_{avoid,Ber;S}\Big(Q_k(T/2) - pT/2 \geq M\sqrt{T}\Big) \geq \mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{avoid,Ber;S}\Big(Q_k(T/2) - pT/2 \geq M\sqrt{T}\Big).$$

Let us write $K_i = pT/2 + M\sqrt{T} + (10(k-i)-5)\lceil\sqrt{T}\rceil$ for $1 \leq i \leq k$. Note K_i is the midpoint of $pT/2 + M\sqrt{T} + 10(k-i-1)\lceil\sqrt{T}\rceil$ and $pT/2 + M\sqrt{T} + 10(k-i)\lceil\sqrt{T}\rceil$. Let E denote the event that the following conditions hold for $1 \leq i \leq k$:

(1) $\Big|Q_i(T/2) - pT/2 - M\sqrt{T} - (10(k-i)-5)\lceil\sqrt{T}\rceil\Big| \leq 2\lceil\sqrt{T}\rceil$,

(2) $\sup_{s\in[0,T/2]}\Big|Q_i(s) - x_i' - \dfrac{K_i - x_i'}{T/2}\,s\Big| \leq 3\sqrt{T}$,

(3) $\sup_{s\in[T/2,T]}\Big|Q_i(s) - K_i - \dfrac{y_i' - K_i}{T/2}(s - T/2)\Big| \leq 3\sqrt{T}$.

The first condition implies in particular that $Q_k(T/2) - pT/2 \geq M\sqrt{T}$, and also that $Q_i(T/2) - Q_{i+1}(T/2) \geq 6\sqrt{T}$ for each i. The second and third conditions require that each curve Q_i remain within a distance of $3\sqrt{T}$ of the graph of the piecewise linear function on $[0,T]$ passing through the points $(0, x_1')$, $(T/2, K_i)$, and (T, y_i'). We observe that

$$\mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{avoid,Ber;S}\left(Q_k(T/2) - ptT \geq M\sqrt{T}\right) \geq \mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{avoid,Ber;S}(E) \geq \mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{Ber}(E).$$

The second inequality follows since the event E implies that $Q_1(s) \geq \cdots \geq Q_k(s)$ for all $s \in [\![0,T]\!]$. Write $z = y_k' - x_k'$. Then we have

$$\begin{aligned}\mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{Ber}(E) = \Bigg[\mathbb{P}^{0,T,0,z}_{Ber}\Bigg(&\Big|\ell(T/2) - pT/2 - M\sqrt{T} - 5\lceil\sqrt{T}\rceil + x_1'\Big| \leq 2\lceil\sqrt{T}\rceil \quad \text{and} \\ &\sup_{s\in[0,T/2]}\Big|\ell(s) - \frac{K_1 - x_1'}{T/2}\,s\Big| \leq 3\sqrt{T} \quad \text{and} \\ &\sup_{s\in[T/2,T]}\Big|\ell(s) - (K_1 - x_1') - \frac{y_1' - K_1}{T/2}(s - T/2)\Big| \leq 3\sqrt{T}\Bigg)\Bigg]^k.\end{aligned} \tag{3.23}$$

Let $\mathbb{P}$ be a probability space supporting a random variable $\ell^{(T,z)}$ with law $\mathbb{P}^{0,T,0,z}$ coupled with a Brownian bridge B^σ with variance σ^2, as in Theorem 3.3. Then the expression in (3.23) is bounded below by

$$\begin{aligned}&\mathbb{P}^{0,T,0,z}_{Ber}\Bigg(\Big|\ell(T/2) - pT/2 - (M + M_1 + 5)\sqrt{T}\Big| \leq 2\sqrt{T} - 5 \quad \text{and} \\ &\sup_{s\in[0,T/2]}\Big|\ell(s) - ps - \frac{M + M_1 + 5}{\sqrt{T}/2}\,s\Big| \leq 3\sqrt{T} - 1 \quad \text{and} \\ &\sup_{s\in[T/2,T]}\Big|\ell(s) - ps - (M + M_1 + 5)\sqrt{T} + \frac{M + M_1 + 5}{\sqrt{T}/2}(s - T/2)\Big| \leq 3\sqrt{T} - 1\Bigg) \geq \\ &\mathbb{P}\Bigg(\Big|\sqrt{T}\,B^\sigma_{1/2} - (M + M_1 + 5)\sqrt{T}\Big| \leq \sqrt{T} \quad \text{and} \\ &\sup_{s\in[0,T/2]}\Big|\sqrt{T}\,B^\sigma_{s/T} - (M + M_1 + 5)\sqrt{T}\cdot\frac{s}{T/2}\Big| \leq 2\sqrt{T} \quad \text{and} \\ &\sup_{s\in[T/2,T]}\Big|\sqrt{T}\,B^\sigma_{s/T} - (M + M_1 + 5)\sqrt{T}\cdot\frac{T - s}{T/2}\Big| \leq 2\sqrt{T}\Bigg) - \mathbb{P}\left(\Delta(T,z) > \sqrt{T}/2\right).\end{aligned} \tag{3.24}$$

Note that $B^\sigma_{1/2}$ is a centered Gaussian random variable with variance $p(1-p)/4 = \sigma^2(1/2)(1-1/2)$. Writing $\xi = B^\sigma_{1/2}$, it follows from Lemma 3.7 that there exist independent Brownian bridges B^1, B^2

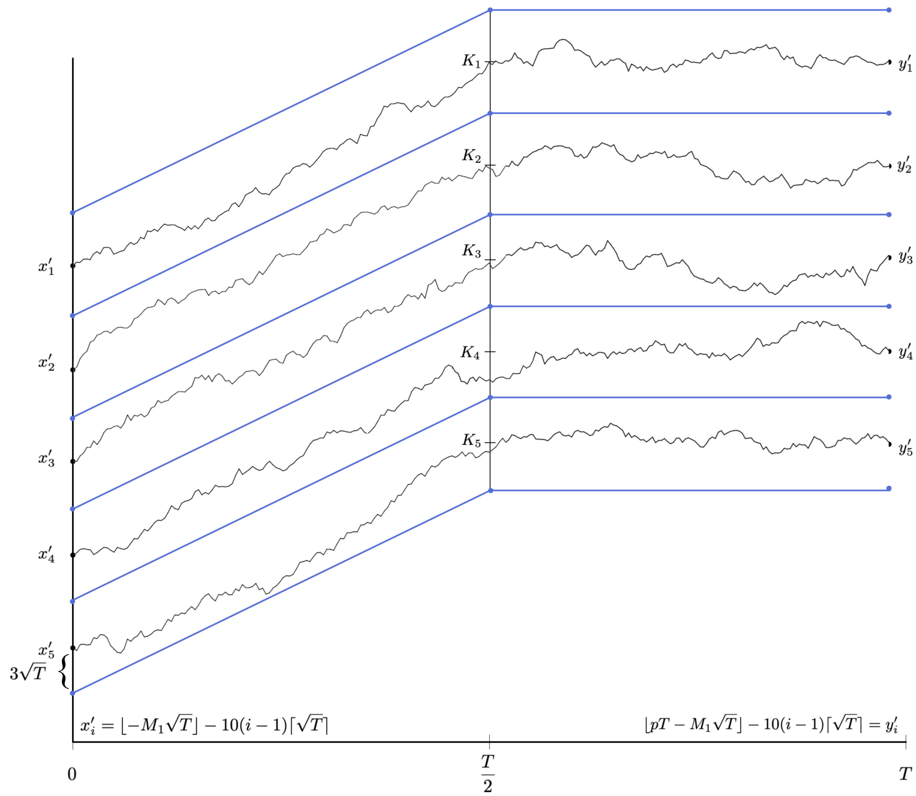

FIGURE 3. Sketch of the argument for Lemma 3.16: We use Lemma 3.1 to lower the entry and exit data $\vec{x}, \vec{y}$ of the curves to $\vec{x}\,'$ and $\vec{y}\,'$. The event E occurs when each curve lies within the blue bounding lines shown in the figure. We then use strong coupling with Brownian bridges via Theorem 3.3 and bound the probability of the bridges remaining within the blue windows.

with variance $\sigma^2/2$ so that B_s^σ has the same law as $\frac{s}{T/2}\xi + B^1_{2s/T}$ for $s \in [0, T/2]$ and $\frac{T-s}{T/2}\xi + B^2_{(2s-T)/T}$

for $s \in [T/2, T]$. The first term in the last expression in (3.24) is thus equal to

(3.25)
$$
\begin{aligned}
&\mathbb{P}\bigg(|\xi - (M + M_1 + 5)| \leq 1 \quad \text{and} \sup_{s \in [0, T/2]} \left| B^1_{s/T} - (M + M_1 + 5 - \xi) \cdot \frac{s}{T/2} \right| \leq 2 \quad \text{and} \\
&\qquad \sup_{s \in [T/2, T]} \left| B^2_{(2s-T)/T} - (M + M_1 + 5 - \xi) \cdot \frac{T - s}{T/2} \right| \leq 2 \bigg) \geq \\
&\mathbb{P}\bigg(|\xi - (M + M_1 + 5)| \leq 1 \quad \text{and} \sup_{s \in [0, T/2]} \left| B^1_{2s/T} \right| \leq 1 \qquad \text{and} \sup_{s \in [T/2, T]} \left| B^2_{(2s-T)/T} \right| \leq 1 \bigg) = \\
&\mathbb{P}\Big(|\xi - (M + M_1 + 5)| \leq 1 \Big) \cdot \mathbb{P}\bigg(\sup_{s \in [0, T/2]} \left| B^1_{2s/T} \right| \leq 1 \bigg) \cdot \mathbb{P}\bigg(\sup_{s \in [0, T/2]} \left| B^2_{(2s-T)/T} \right| \leq 1 \bigg) \geq \\
&\Big(1 - 2e^{-4/p(1-p)} \Big)^2 \int_{M+M_1+4}^{M+M_1+6} \frac{e^{-2\xi^2/p(1-p)}}{\sqrt{\pi p(1-p)/2}} \, d\xi \geq \\
&\frac{2\sqrt{2}\, e^{-2(M+M_1+6)^2/p(1-p)}}{\sqrt{\pi p(1-p)}} \left(1 - 2e^{-4/p(1-p)} \right)^2.
\end{aligned}
$$

In the fourth line, we used the fact that ξ, $B^1_\cdot$, and $B^2_\cdot$ are independent, and in the second to last line we used Lemma 3.6. Since $|z - pT| \leq (M_1 + 1)\sqrt{T}$, Lemma 3.5 allows us to choose T large enough so that $\mathbb{P}(\Delta(T, z) > \sqrt{T}/2)$ is less than $1/2$ the expression in the last line of (3.25). Then in view of (3.23) and (3.24), we conclude (3.22).

□

We now state an important weak convergence result, whose proof occupies Section 9. (See Propositions 9.2 and 9.3.)

Proposition 3.17. *Fix $p, t \in (0, 1)$, $k \in \mathbb{N}$, $\vec{a}, \vec{b} \in \mathfrak{W}_k$. Suppose that $\vec{x}^T = (x_1^T, \ldots, x_k^T)$ and $\vec{y}^T = (y_1^T, \ldots, y_k^T)$ are two sequences in $\mathfrak{W}_k$ such that for $i \in [\![1, k]\!]$,*

$$
\lim_{T \to \infty} \frac{x_i^T}{\sqrt{T}} = a_i \quad \text{and} \quad \lim_{T \to \infty} \frac{y_i^T - pT}{\sqrt{T}} = b_i.
$$

Let $(Q_1^T, \ldots, Q_k^T)$ have law $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$, and define the sequence $\{Z^T\}$ of random k-dimensional vectors by

$$
Z^T = \left(\frac{Q_1^T(tT) - ptT}{\sqrt{T}}, \ldots, \frac{Q_k^T(tT) - ptT}{\sqrt{T}} \right).
$$

Then as $T \to \infty$, Z^T converges weakly to a random vector $\hat{Z}$ on $\mathbb{R}^k$ with a probability density ρ supported on W_k°.

The convergence result in Lemma 3.17 allows us to prove the following lemma, which roughly states that if the entrance and exit data of a sequence of avoiding Bernoulli line ensembles remain in compact windows, then with high probability the curves of the ensemble will remain separated from one another at each point by some small positive distance on scale $\sqrt{T}$.

Lemma 3.18. *Fix $p, t \in (0, 1)$ and $k \in \mathbb{N}$. Suppose that $\vec{x}^{\,T} = (x_1^T, \ldots, x_k^T)$, $\vec{y}^{\,T} = (y_1^T, \ldots, y_k^T)$ are elements of $\mathfrak{W}_k$ such that $T \geq y_i^T - x_i^T \geq 0$ for $i \in [\![1, k]\!]$. Then for any $M_1, M_2 > 0$ and $\epsilon > 0$ there exists $W_5 \in \mathbb{N}$ and $\delta > 0$ depending on p, k, M_1, M_2 such that if $T \geq W_5$, $|x_i^T| \leq M_1\sqrt{T}$ and $|y_i^T - pT| \leq M_2\sqrt{T}$, then*

$$
\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber} \left(\min_{1 \leq i \leq k-1} \left[Q_i(tT) - Q_{i+1}(tT) \right] < \delta\sqrt{T} \right) < \epsilon.
$$

Proof. We prove the claim by contradiction. Suppose there exist $M_1, M_2, \epsilon > 0$ such that for any $W_5 \in \mathbb{N}$ and $\delta > 0$ there exists some $T \geq W_5$ with

$$\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}\left(\min_{1\leq i\leq k-1}\left[Q_i(tT) - Q_{i+1}(tT)\right] < \delta\sqrt{T}\right) \geq \epsilon.$$

Then we can obtain sequences T_n, $\delta_n > 0$, $T_n \nearrow \infty$, $\delta_n \searrow 0$, such that for all n we have

$$\mathbb{P}^{0,T,\vec{x}^{T_n},\vec{y}^{T_n}}_{avoid,Ber}\left(\min_{1\leq i\leq k-1}\left[\frac{Q_i(tT_n) - Q_{i+1}(tT_n)}{\sqrt{T_n}}\right] < \delta_n\right) \geq \epsilon.$$

Since $|x_i^{T_n}| < M_1\sqrt{T_n}$ and $|y_i^{T_n} - pT_n| \leq M_2\sqrt{T_n}$ for $1 \leq i \leq k$, the sequences $\{\vec{x}^{T_n}/\sqrt{T_n}\}$, $\{(\vec{y}^{T_n} - pT_n)/\sqrt{T_n}\}$ are bounded in $\mathbb{R}^k$. It follows that there exist $\vec{x}, \vec{y} \in \mathbb{R}^n$ and a subsequence $\{T_{n_m}\}$ such that

$$\frac{\vec{x}^{T_{n_m}}}{\sqrt{T_{n_m}}} \longrightarrow \vec{x}, \quad \frac{\vec{y}^{T_{n_m}} - pT_{n_m}}{\sqrt{T_{n_m}}} \longrightarrow \vec{y}$$

as $m \to \infty$ (see [15, Theorem 3.6]). Denote

$$Z_i^m = \frac{Q_i(tT_{n_m}) - ptT_{n_m}}{\sqrt{T_{n_m}}}.$$

Fix $\delta > 0$ and choose M large enough so that if $m \geq M$ then $\delta_m < \delta$. Then for $m \geq M$ we have

$$\epsilon \leq \liminf_{m\to\infty}\mathbb{P}\left(\min_{1\leq i\leq k-1}\left[Z_i^m - Z_{i+1}^m\right] < \delta_{n_m}\right) \leq \liminf_{m\to\infty}\mathbb{P}\left(\min_{1\leq i\leq k-1}\left[Z_i^m - Z_{i+1}^m\right] \leq \delta\right). \tag{3.26}$$

Now by Lemma 3.17, $(Z_1^m, \dots, Z_k^m)$ converges weakly to a random vector $\hat{Z}$ on $\mathbb{R}^k$ with a density ρ. It follows from the portmanteau theorem [9, Theorem 3.2.11] applied with the closed set $K = [0, \delta]$ that

$$\limsup_{m\to\infty}\mathbb{P}\left(\min_{1\leq i\leq k-1}\left[Z_i^m - Z_{i+1}^m\right] \in K\right) \leq \mathbb{P}\left(\min_{1\leq i\leq k-1}\left[\hat{Z}_i - \hat{Z}_{i+1}\right] \in K\right). \tag{3.27}$$

Combining (3.26) and (3.27), we obtain

$$\epsilon \leq \mathbb{P}\left(0 \leq \min_{1\leq i\leq k-1}\left[\hat{Z}_i - \hat{Z}_{i+1}\right] \leq \delta\right) \leq \sum_{i=1}^{k-1}\mathbb{P}\left(0 \leq \hat{Z}_i - \hat{Z}_{i+1} \leq \delta\right). \tag{3.28}$$

To conclude the proof, we find a δ for which (3.28) cannot hold. For $\tilde{\eta} \geq 0$ put

$$E_i^{\tilde{\eta}} = \{\vec{z} \in \mathbb{R}^k : 0 \leq z_i - z_{i+1} \leq \tilde{\eta}\}.$$

For each $i \in [\![1, k-1]\!]$ and $\eta > 0$, we have

$$\mathbb{P}\left(0 \leq \hat{Z}_i - \hat{Z}_{i+1} \leq \eta\right) = \int_{\mathbb{R}^k}\rho \cdot \mathbf{1}_{E_i^\eta}\, dz_1 \cdots dz_k. \tag{3.29}$$

Clearly $\rho \cdot \mathbf{1}_{E_i^\eta} \to \rho \cdot \mathbf{1}_{E_i^0}$ pointwise as $\eta \to 0$, and $E_i^0 = \{\vec{z} \in \mathbb{R}^k : z_i = z_{i+1}\}$ has Lebesgue measure 0. Thus $\rho \cdot \mathbf{1}_{E_i^\eta} \to 0$ a.e. as $\eta \to 0$. Since $|\rho \cdot \mathbf{1}_{E_i^\eta}| \leq \rho$ and ρ is integrable, the dominated convergence theorem [16, Theorem 1.34] and (3.29) imply that

$$\mathbb{P}\left(0 \leq \hat{Z}_i - \hat{Z}_{i+1} \leq \eta\right) \longrightarrow 0$$

as $\eta \to 0$. Thus for each $i \in [\![1, k-1]\!]$ and $\epsilon > 0$ we can find an $\eta_i > 0$ such that $0 < \eta \leq \eta_i$ implies $\mathbb{P}(0 \leq \hat{Z}_i - \hat{Z}_{i+1} \leq \eta) < \epsilon/(k-1)$. Putting $\delta = \min_{1\leq i\leq k-1}\eta_i$ we find that

$$\sum_{i=1}^{k-1}\mathbb{P}\left(0 \leq \hat{Z}_i - \hat{Z}_{i+1} \leq \delta\right) < \epsilon,$$

contradicting (3.28) for this choice of δ.

□

4. Proof of Theorem 2.26

The goal of this section is to prove Theorem 2.26. Throughout this section, we assume that we have fixed $k \in \mathbb{N}$ with $k \geq 2$, $p \in (0,1)$, $\alpha, \lambda > 0$, and

$$\{\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)\}_{N=1}^{\infty}$$

an (α, p, λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles as in Definition 2.24, all defined on a probability space with measure $\mathbb{P}$. The proof of Theorem 2.26 depends on three results – Proposition 4.1 and Lemmas 4.2 and 4.3. In these three statements we establish various properties of the sequence of line ensembles $\mathfrak{L}^N$. The constants in these statements depend implicitly on α, p, λ, k, and the functions ϕ, ψ from Definition 2.24, which are fixed throughout. We will not list these dependencies explicitly. The proof of Proposition 4.1 is given in Section 4.1 while the proofs of Lemmas 4.2 and 4.3 are in Section 5. Theorem 2.26 (i) and (ii) are proved in Sections 4.2 and 4.3 respectively.

4.1. **Bounds on the acceptance probability.** The main result in this section is presented as Proposition 4.1 below. In order to formulate it and some of the lemmas below, it will be convenient to adopt the following notation for any $r > 0$ and $m \in \mathbb{N}$:

$$t_m = \lfloor (r+m)N^{\alpha} \rfloor. \tag{4.1}$$

Proposition 4.1. *Let $\mathbb{P}$ be the measure from the beginning of this section. For any $\epsilon > 0$, $r > 0$ there exist $\delta = \delta(\epsilon, r) > 0$ and $N_1 = N_1(\epsilon, r)$ such that for all $N \geq N_1$ we have*

$$\mathbb{P}\Big(Z\big(-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!]\big) < \delta\Big) < \epsilon,$$

where $\vec{x} = (L_1^N(-t_1), \dots, L_{k-1}^N(-t_1))$, $\vec{y} = (L_1^N(t_1), \dots, L_{k-1}^N(t_1))$, $L_k^N[\![-t_1, t_1]\!]$ is the restriction of L_k^N to the set $[\![-t_1, t_1]\!]$, and Z is the acceptance probability of Definition 2.22.

The general strategy we use to prove Proposition 4.1 is inspired by the proof of [5, Proposition 6.5]. We begin by stating three key lemmas that will be required. The proofs of Lemmas 4.2 and 4.3 are postponed to Section 5 and Lemma 4.4 is proved in Section 6.

Lemma 4.2. *Let $\mathbb{P}$ be the measure from the beginning of this section. For any $\epsilon > 0$, $r > 0$ there exist $R_1 = R_1(\epsilon, r) > 0$ and $N_2 = N_2(\epsilon, r)$ such that for $N \geq N_2$*

$$\mathbb{P}\left(\sup_{s \in [-t_3, t_3]} \big[L_1^N(s) - ps\big] \geq R_1 N^{\alpha/2}\right) < \epsilon.$$

Lemma 4.3. *Let $\mathbb{P}$ be the measure from the beginning of this section. For any $\epsilon > 0$, $r > 0$ there exist $R_2 = R_2(\epsilon, r) > 0$ and $N_3 = N_3(\epsilon, r)$ such that for $N \geq N_3$*

$$\mathbb{P}\left(\inf_{s \in [-t_3, t_3]} \big[L_{k-1}^N(s) - ps\big] \leq -R_2 N^{\alpha/2}\right) < \epsilon.$$

Lemma 4.4. *Fix $k \in \mathbb{N}$, $k \geq 2$, $p \in (0,1)$, $r, \alpha, M_1, M_2 > 0$. Suppose that $\ell_{bot} : [\![-t_3, t_3]\!] \to \mathbb{R} \cup \{-\infty\}$, and $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$ are such that $|\Omega_{avoid}(-t_3, t_3, \vec{x}, \vec{y}, \infty, \ell_{bot})| \geq 1$. Suppose further that*

(1) $\sup_{s \in [-t_3, t_3]} \big[\ell_{bot}(s) - ps\big] \leq M_2 (2t_3)^{1/2}$,
(2) $-pt_3 + M_1(2t_3)^{1/2} \geq x_1 \geq x_{k-1} \geq \max\big(\ell_{bot}(-t_3), -pt_3 - M_1(2t_3)^{1/2}\big)$,
(3) $pt_3 + M_1(2t_3)^{1/2} \geq y_1 \geq y_{k-1} \geq \max\big(\ell_{bot}(t_3), pt_3 - M_1(2t_3)^{1/2}\big)$.

Then there exist constants g, h and $N_4 \in \mathbb{N}$ all depending on $M_1, M_2, p, k, r, \alpha$ such that for any $\tilde{\epsilon} > 0$ and $N \geq N_4$ we have

$$\mathbb{P}_{avoid,Ber}^{-t_3,t_3,\vec{x},\vec{y},\infty,\ell_{bot}}\Big(Z\big(-t_1,t_1,\mathfrak{Q}(-t_1),\mathfrak{Q}(t_1),\infty,\ell_{bot}[\![-t_1,t_1]\!]\big) \leq gh\tilde{\epsilon}\Big) \leq \tilde{\epsilon}, \tag{4.2}$$

where Z is the acceptance probability of Definition 2.22, $\ell_{bot}[\![-t_1,t_1]\!]$ is the vector, whose coordinates match those of ℓ_{bot} on $[\![-t_1,t_1]\!]$ and $\mathfrak{Q}(a) = (Q_1(a),\ldots,Q_{k-1}(a))$ is the value of the line ensemble $\mathfrak{Q} = (Q_1,\ldots,Q_{k-1})$ whose law is $\mathbb{P}_{avoid,Ber}^{-t_3,t_3,\vec{x},\vec{y},\infty,\ell_{bot}}$ at location a.

Proof of Proposition 4.1. Let $\epsilon > 0$ be given. Define the event

$$E_N = \Big\{L_{k-1}^N(\pm t_3) \mp pt_3 \geq -M_1(2t_3)^{1/2}\Big\} \cap \Big\{L_1^N(\pm t_3) \mp pt_3 \leq M_1(2t_3)^{1/2}\Big\} \cap \left\{\sup_{s\in[-t_3,t_3]}[L_k^N(s) - ps] \leq M_2(2t_3)^{1/2}\right\}.$$

In view of Lemmas 4.2 and 4.3 and the fact that $\mathbb{P}$-almost surely $L_1^N(s) \geq L_k^N(s)$ for all $s \in [-t_3,t_3]$ we can find sufficiently large M_1, M_2 and N_2 such that for $N \geq N_2$ we have

$$\mathbb{P}(E_N^c) < \epsilon/2. \tag{4.3}$$

Let g, h, N_4 be as in Lemma 4.4 for the values M_1, M_2 as above, the values α, p, k from the beginning of the section and r as in the statement of the proposition. For $\delta = (\epsilon/2)\cdot gh$ we denote

$$V = \Big\{Z\big(-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]\big) < \delta\Big\}$$

and make the following deduction for $N \geq N_4$

$$\begin{aligned}
&\mathbb{P}\big(V \cap E_N\big) = \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{E_N}\cdot\mathbf{1}_V\Big|\sigma\big(\mathfrak{L}^N(-t_3),\mathfrak{L}^N(t_3),L_k^N[\![-t_3,t_3]\!]\big)\Big]\Big] = \\
&\mathbb{E}\Big[\mathbf{1}_{E_N}\cdot\mathbb{E}\Big[\mathbf{1}\{Z\big(-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]\big) < \delta\}\Big|\sigma\big(\mathfrak{L}^N(-t_3),\mathfrak{L}^N(t_3),L_k^N[\![-t_3,t_3]\!]\big)\Big]\Big] = \\
&\mathbb{E}\left[\mathbf{1}_{E_N}\cdot\mathbb{E}_{avoid}\left[\mathbf{1}\{Z\big(-t_1,t_1,\mathfrak{L}(-t_1),\mathfrak{L}(t_1),\infty,L_k^N[\![-t_1,t_1]\!]\big) < \delta\}\right]\right] \leq \mathbb{E}\left[\mathbf{1}_{E_N}\cdot\epsilon/2\right] \leq \epsilon/2.
\end{aligned} \tag{4.4}$$

In (4.4) we have written $\mathbb{E}_{avoid}$ in place of $\mathbb{E}_{avoid,Ber}^{-t_3,t_3,\mathfrak{L}^N(-t_3),\mathfrak{L}^N(t_3),\infty,L_k^N[\![-t_3,t_3]\!]}$ to ease the notation; in addition, we have that $\mathfrak{L}^N(a) = (L_1^N(a),\ldots,L_{k-1}^N(a))$ and $\mathfrak{L}$ on the last line is distributed according to $\mathbb{P}_{avoid,Ber}^{-t_3,t_3,\mathfrak{L}^N(-t_3),\mathfrak{L}^N(t_3),\infty,L_k^N[\![-t_3,t_3]\!]}$. We elaborate on (4.4) in the paragraph below.

The first equality in (4.4) follows from the tower property for conditional expectations. The second equality uses the definition of V and the fact that $\mathbf{1}_{E_N}$ is $\sigma\big(\mathfrak{L}^N(-t_3),\mathfrak{L}^N(t_3),L_k^N[\![-t_3,t_3]\!]\big)$-measurable and can thus be taken outside of the conditional expectation. The third equality uses the Schur Gibbs property, see Definition 2.17. The first inequality on the third line holds if $N \geq N_4$ and uses Lemma 4.4 with $\tilde{\epsilon} = \epsilon/2$ as well as the fact that on the event E_N the random variables $\mathfrak{L}^N(-t_3), \mathfrak{L}^N(t_3)$ and $L_k^N[\![-t_3,t_3]\!]$ (that play the roles of $\vec{x}, \vec{y}$ and ℓ_{bot}) satisfy the inequalities

1. $\sup_{s\in[-t_3,t_3]}\left[L_k^N(s) - ps\right] \leq M_2(2t_3)^{1/2}$,
2. $-pt_3 + M_1(2t_3)^{1/2} \geq L_1^N(-t_3) \geq L_{k-1}^N(-t_3) \geq \max\left(L_k^N(-t_3), -pt_3 - M_1(2t_3)^{1/2}\right)$,
3. $pt_3 + M_1(2t_3)^{1/2} \geq L_1^N(t_3) \geq L_{k-1}^N(t_3) \geq \max\left(L_k^N(t_3), pt_3 - M_1(2t_3)^{1/2}\right)$.

The last inequality in (4.4) is trivial.

Combining (4.4) with (4.3), we see that for all $N \geq \max(N_2, N_4)$ we have

$$\mathbb{P}(V) = \mathbb{P}(V \cap E_N) + \mathbb{P}(V \cap E_N^c) \leq \epsilon/2 + \mathbb{P}(E_N^c) < \epsilon,$$

which proves the proposition. □

4.2. **Proof of Theorem 2.26 (i).** By Lemma 2.4, it suffices to verify the following two conditions for all $i \in [\![1, k-1]\!]$, $r > 0$, and $\epsilon > 0$:

$$\lim_{a \to \infty} \limsup_{N \to \infty} \mathbb{P}(|f_i^N(0)| \geq a) = 0 \tag{4.5}$$

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}\left(\sup_{x,y \in [-r,r], |x-y| \leq \delta} |f_i^N(x) - f_i^N(y)| \geq \epsilon \right) = 0. \tag{4.6}$$

For the sake of clarity, we will prove these conditions in several steps.

Step 1. In this step we prove (4.5). Let $\epsilon > 0$ be given. Then by Lemmas 4.2 and 4.3 we can find N_2, N_3 and R_1, R_2 such that for $N \geq \max(N_1, N_2)$

$$\mathbb{P}\left(\sup_{s \in [-t_3, t_3]} [L_1^N(s) - ps] \geq R_1 N^{\alpha/2} \right) < \epsilon/2,$$
$$\mathbb{P}\left(\inf_{s \in [-t_3, t_3]} [L_{k-1}^N(s) - ps] \leq -R_2 N^{\alpha/2} \right) < \epsilon/2.$$

In particular, if we set $R = \max(R_1, R_2)$ and utilize the fact that $L_1^N(0) \geq \cdots \geq L_{k-1}^N(0)$ we conclude that for any $i \in [\![1, k-1]\!]$ we have

$$\mathbb{P}\big(|L_i^N(0)| \geq R N^{\alpha/2}\big) \leq \mathbb{P}\big(L_1^N(0) \geq R_1 N^{\alpha/2}\big) + \mathbb{P}\big(L_{k-1}^N(0) \leq -R_2 N^{\alpha/2}\big) < \epsilon,$$

which implies (4.5).

Step 2. In this step we prove (4.6). In the sequel we fix $r, \epsilon > 0$ and $i \in [\![1, k-1]\!]$. To prove (4.6) it suffices to show that for any $\eta > 0$, there exists a $\delta > 0$ and N_0 such that $N \geq N_0$ implies

$$\mathbb{P}\left(\sup_{x,y \in [-r,r], |x-y| \leq \delta} |f_i^N(x) - f_i^N(y)| \geq \epsilon \right) < \eta. \tag{4.7}$$

For $\delta > 0$ we define the event

$$A_\delta^N = \left\{ \sup_{x,y \in [-t_1, t_1], |x-y| \leq \delta N^\alpha} \big| L_i^N(x) - L_i^N(y) - p(x-y) \big| \geq \frac{3 N^{\alpha/2} \epsilon}{4} \right\}, \tag{4.8}$$

where we recall that $t_1 = \lfloor (r+1) N^\alpha \rfloor$ from (4.1). We claim that there exist $\delta_0 > 0$ and $N_0 \in \mathbb{N}$ such that for $\delta \in (0, \delta_0]$ and $N \geq N_0$ we have

$$\mathbb{P}(A_\delta^N) < \eta. \tag{4.9}$$

We prove (4.9) in the steps below. Here we assume its validity and conclude the proof of (4.7).

Observe that if $\delta \in \left(0, \min\left(\delta_0, \epsilon \cdot (8\lambda r)^{-1}\right)\right)$, where λ is as in the statement of the theorem, we have the following tower of inequalities

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} |f_i^N(x) - f_i^N(y)| \geq \epsilon\right) = \\
&\mathbb{P}\left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} \left|N^{-\alpha/2}\left(L_i^N(xN^\alpha) - L_i^N(yN^\alpha)\right) - p(x-y)N^{\alpha/2} + \lambda(x^2-y^2)\right| \geq \epsilon\right) \leq \\
&\mathbb{P}\left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} N^{-\alpha/2}\left|L_i^N(xN^\alpha) - L_i^N(yN^\alpha) - p(x-y)N^{\alpha}\right| + 2\lambda r\delta \geq \epsilon\right) \leq \\
&\mathbb{P}\left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} \left|L_i^N(xN^\alpha) - L_i^N(yN^\alpha) - p(x-y)N^{\alpha}\right| \geq \frac{3N^{\alpha/2}\epsilon}{4}\right) \leq \mathbb{P}(A_\delta^N) < \eta.
\end{aligned} \tag{4.10}
$$

In (4.10) the first equality follows from the definition of f_i^N, and the inequality on the second line follows from the inequality $|x^2 - y^2| \leq 2r\delta$, which holds for all $x, y \in [-r, r]$ such that $|x - y| \leq \delta$. The inequality in the third line of (4.10) follows from our assumption that $\delta < \epsilon \cdot (8\lambda r)^{-1}$ and the first inequality on the last line follows from the definition of A_δ^N in (4.8), and the fact that $t_1 \geq rN^\alpha$. The last inequality follows from our assumption that $\delta < \delta_0$ and (4.9). In view of (4.10) we conclude (4.7).

Step 3. In this step we prove (4.9) and fix $\eta > 0$ in the sequel. For $\delta_1, M_1 > 0$ and $N \in \mathbb{N}$ we define the events

$$
E_1 = \left\{\max_{1\leq j\leq k-1} \left|L_j^N(\pm t_1) \mp pt_1\right| \leq M_1 N^{\alpha/2}\right\}, E_2 = \left\{Z(-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!]) > \delta_1\right\}, \tag{4.11}
$$

where we used the same notation as in Proposition 4.1 (in particular $\vec{x} = (L_1^N(-t_1), \dots, L_{k-1}^N(-t_1))$ and $\vec{y} = (L_1^N(t_1), \dots, L_{k-1}^N(t_1)))$. Combining Lemmas 4.2, 4.3 and Proposition 4.1 we know that we can find $\delta_1 > 0$ sufficiently small, M_1 sufficiently large and $\tilde{N} \in \mathbb{N}$ such that for $N \geq \tilde{N}$ we know

$$
\mathbb{P}\left(E_1^c \cup E_2^c\right) < \eta/2. \tag{4.12}
$$

We claim that we can find $\delta_0 > 0$ and $N_0 \geq \tilde{N}$ such that for $N \geq N_0$ and $\delta \in (0, \delta_0)$ we have

$$
\mathbb{P}(A_\delta^N \cap E_1 \cap E_2) < \eta/2. \tag{4.13}
$$

Since

$$
\mathbb{P}(A_\delta^N) = \mathbb{P}(A_\delta^N \cap E_1 \cap E_2) + \mathbb{P}(A_\delta^N \cap (E_1^c \cup E_2^c)) \leq \mathbb{P}(A_\delta^N \cap E_1 \cap E_2) + \mathbb{P}\left(E_1^c \cup E_2^c\right),
$$

we see that (4.12) and (4.13) together imply (4.9).

Step 4. In this step we prove (4.13). We define the σ-algebra

$$
\mathcal{F} = \sigma\left(L_k^N[\![-t_1, t_1]\!], L_1^N(\pm t_1), L_2^N(\pm t_1), \dots, L_{k-1}^N(\pm t_1)\right).
$$

Clearly $E_1, E_2 \in \mathcal{F}$, so the indicator random variables $\mathbf{1}_{E_1}$ and $\mathbf{1}_{E_2}$ are $\mathcal{F}$-measurable. It follows from the tower property of conditional expectation that

$$
\mathbb{P}\left(A_\delta^N \cap E_1 \cap E_2\right) = \mathbb{E}\left[\mathbf{1}_{A_\delta^N} \mathbf{1}_{E_1} \mathbf{1}_{E_2}\right] = \mathbb{E}\left[\mathbf{1}_{E_1} \mathbf{1}_{E_2} \mathbb{E}\left[\mathbf{1}_{A_\delta^N} \mid \mathcal{F}\right]\right]. \tag{4.14}
$$

By the Schur-Gibbs property (see Definition 2.17), we know that $\mathbb{P}$-almost surely

$$
\mathbb{E}\left[\mathbf{1}_{A_\delta^N} \mid \mathcal{F}\right] = \mathbb{E}_{avoid,Ber}^{-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]}\left[\mathbf{1}_{A_\delta^N}\right]. \tag{4.15}
$$

We now observe that the Radon-Nikodym derivative of $\mathbb{P}_{avoid,Ber}^{-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]}$ with respect to $\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}$ is given by

$$\frac{d\mathbb{P}_{avoid,Ber}^{-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]}(Q_1,\dots,Q_{k-1})}{d\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}} = \frac{\mathbf{1}_{\{Q_1\geq\cdots\geq Q_{k-1}\geq Q_k\}}}{Z(-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!])}, \tag{4.16}$$

where $\mathfrak{Q} = (Q_1,\dots,Q_{k-1})$ is $\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}$-distributed and $Q_k = L_k^N[\![-t_1,t_1]\!]$. To see this, note that by Definition 2.15 we have for any set $A \subset \prod_{i=1}^{k-1}\Omega(-t_1,t_1,x_i,y_i)$ that

$$\mathbb{P}_{avoid,Ber}^{-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]}(A) = \frac{\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(A\cap\{Q_1\geq\cdots\geq Q_{k-1}\geq Q_k\})}{\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(Q_1\geq\cdots\geq Q_{k-1}\geq Q_k)} =$$
$$\frac{\mathbb{E}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}\left[\mathbf{1}_A\cdot\mathbf{1}_{\{Q_1\geq\cdots\geq Q_{k-1}\geq Q_k\}}\right]}{Z(-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!])} = \int_A \frac{\mathbf{1}_{\{Q_1\geq\cdots\geq Q_{k-1}\geq Q_k\}}}{Z(-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!])}\,d\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}.$$

It follows from (4.14), (4.16), and the definition of E_2 that

$$\begin{aligned}\mathbb{P}(A_\delta^N\cap E_1\cap E_2) &= \mathbb{E}\left[\mathbf{1}_{E_1}\,\mathbf{1}_{E_2}\,\mathbb{E}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}\left[\frac{\mathbf{1}_{B_\delta^N}\cdot\mathbf{1}_{\{Q_1\geq\cdots\geq Q_k\}}}{Z(-t_1,t_1,\vec{x},\vec{y},L_k^N[\![-t_1,t_1]\!])}\right]\right]\leq\\ &\leq \mathbb{E}\left[\mathbf{1}_{E_1}\,\mathbf{1}_{E_2}\,\mathbb{E}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}\left[\frac{\mathbf{1}_{B_\delta^N}}{\delta_1}\right]\right] \leq \frac{1}{\delta_1}\mathbb{E}\left[\mathbf{1}_{E_1}\cdot\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(B_\delta^N)\right],\end{aligned} \tag{4.17}$$

where

$$B_\delta^N = \left\{\sup_{x,y\in[-t_1,t_1],|x-y|\leq\delta N^\alpha}|Q_i(x)-Q_i(y)-p(x-y)|\geq\frac{3N^{\alpha/2}\epsilon}{4}\right\}.$$

Notice that under $\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}$ the law of Q_i is precisely $\mathbb{P}_{Ber}^{-t_1,t_1,x_i,y_i}$, and so we conclude that

$$\mathbb{P}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(B_\delta^N) = \mathbb{P}_{Ber}^{0,2t_1,0,y_i-x_i}\left(\sup_{x,y\in[0,2t_1],|x-y|\leq\delta N^\alpha}|\ell(x)-\ell(y)-p(x-y)|\geq\frac{3N^{\alpha/2}\epsilon}{4}\right), \tag{4.18}$$

where ℓ has law $\mathbb{P}_{Ber}^{0,2t_1,0,y_i-x_i}$ (note that in (4.18) we implicitly translated the path ℓ to the right by t_1 and up by $-x_i$, which does not affect the probability in question). Since on the event E_1 we know that $|y_i-x_i-2pt_1|\leq 2M_1N^\alpha$ we conclude from Lemma 3.12 that we can find N_0 and $\delta_0>0$ depending on M_1,r,α such that for $N\geq N_0$ and $\delta\in(0,\delta_0)$ we have

$$\mathbf{1}_{E_1}\cdot\mathbb{P}_{Ber}^{0,2t_1,0,y_i-x_i}\left(\sup_{x,y\in[0,2t_1],|x-y|\leq\delta N^\alpha}|\ell(x)-\ell(y)-p(x-y)|\geq\frac{3N^{\alpha/2}\epsilon}{4}\right)<\frac{\delta_1\eta}{2}. \tag{4.19}$$

Combining (4.17), (4.18) and (4.19) we conclude (4.13), and hence statement (i) of the theorem.

4.3. **Proof of Theorem 2.26 (ii).** In this section we fix a subsequential limit $\mathcal{L}^\infty = (f_1^\infty,\dots,f_{k-1}^\infty)$ of the sequence $\tilde{\mathbb{P}}_N$ as in the statement of Theorem 2.26, and we prove that $\mathcal{L}^\infty$ possesses the partial Brownian Gibbs property. Our approach is similar to that in [6, Sections 5.1 and 5.2]. We first give a definition of measures on scaled free and avoiding Bernoulli random walks. These measures will appear when we apply the Schur Gibbs property to the scaled line ensembles f^N.

Definition 4.5. Let $a,b\in N^{-\alpha}\mathbb{Z}$ with $a<b$ and $x,y\in N^{-\alpha/2}\mathbb{Z}$ satisfy $0\leq y-x\leq(b-a)N^{\alpha/2}$. Let $\ell^{(T,z)}$ denote a random variable with law $\mathbb{P}_{Ber}^{0,T,0,z}$ as in Definition 2.15. We define $\mathbb{P}_{free,N}^{a,b,x,y}$ to be the law of the $C([a,b])$-valued random variable Y given by

$$Y(t) = \frac{x+N^{-\alpha/2}\left[\ell_{(t-a)N^\alpha}^{((b-a)N^\alpha,\,(y-x)N^{\alpha/2}))}-ptN^\alpha\right]}{\sqrt{p(1-p)}},\quad t\in[a,b].$$

Now for $i \in [\![1, k]\!]$, let $\ell^{(N,z),i}$ denote iid random variables with laws $\mathbb{P}^{0,N,0,z}_{Ber}$. Let $\vec{x}, \vec{y} \in (N^{-\alpha/2}\mathbb{Z})^k$ satisfy $0 \le y_i - x_i \le (b-a)N^{\alpha/2}$ for $i \in [\![1, k]\!]$. We define the $[\![1, k]\!]$-indexed line ensemble $\mathcal{Y}^N$ by

$$\mathcal{Y}^N_i(t) = \frac{x_i + N^{-\alpha/2}\left[\ell^{((b-a)N^\alpha,\,(y_i-x_i)N^{\alpha/2})),i}_{(t-a)N^\alpha} - ptN^\alpha\right]}{\sqrt{p(1-p)}}, \quad i \in [\![1, k]\!], t \in [a, b].$$

We let $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free,N}$ denote the law of $\mathcal{Y}^N$. Suppose $\vec{x}, \vec{y} \in (N^{-\alpha/2}\mathbb{Z})^k \cap W^\circ_k$ and $f : [a, b] \to (-\infty, \infty]$, $g : [a, b] \to [-\infty, \infty)$ are continuous functions. We define the probability measure $\mathbb{P}^{a,b,\vec{x},\vec{y},f,g}_{avoid,N}$ to be $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free,N}$ conditioned on the event

$$E = \{f(r) \ge \mathcal{Y}^N_1(r) \ge \cdots \ge \mathcal{Y}^N_k(r) \ge g(r) \text{ for } r \in [a, b]\}.$$

This measure is well-defined if E is nonempty.

Next, we state two lemmas whose proofs we give in Section 8.4. The first lemma proves weak convergence of the scaled avoiding random walk measures in Definition 4.5. It states roughly that if the data of these measures converge, then the measures converge weakly to the law of avoiding Brownian bridges with the limiting data.

Lemma 4.6. *Fix $k \in \mathbb{N}$ and $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a-1, b+1] \to (-\infty, \infty]$, $g : [a-1, b+1] \to [-\infty, \infty)$ be continuous functions such that $f(t) > g(t)$ for all $t \in [a-1, b+1]$. Let $\vec{x}, \vec{y} \in W^\circ_k$ be such that $f(a) > x_1$, $f(b) > y_1$, $g(a) < x_k$, and $g(b) < y_k$. Let $a_N = \lfloor aN^\alpha \rfloor N^{-\alpha}$ and $b_N = \lceil bN^\alpha \rceil N^{-\alpha}$, and let $f_N : [a-1, b+1] \to (-\infty, \infty]$ and $g_N : [a-1, b+1] \to [-\infty, \infty)$ be continuous functions such that $f_N \to f$ and $g_N \to g$ uniformly on $[a-1, b+1]$. Lastly, let $\vec{x}^{\,N}, \vec{y}^{\,N} \in (N^{-\alpha/2}\mathbb{Z})^k \cap W^\circ_k$, write $\tilde{x}^N_i = (x^N_i - pa_N N^{\alpha/2})/\sqrt{p(1-p)}$, $\tilde{y}^N_i = (y^N_i - pb_N N^{\alpha/2})/\sqrt{p(1-p)}$, and suppose that $\tilde{x}^N_i \to x_i$ and $\tilde{y}^N_i \to y_i$ as $N \to \infty$ for each $i \in [\![1, k]\!]$. Then there exists $N_0 \in \mathbb{N}$ so that $\mathbb{P}^{a_N,b_N,\vec{x}^{\,N},\vec{y}^{\,N},f_N,g_N}_{avoid,N}$ is well-defined for $N \ge N_0$. Moreover, if $\mathcal{Y}^N$ have laws $\mathbb{P}^{a_N,b_N,\vec{x}^{\,N},\vec{y}^{\,N},f_N,g_N}_{avoid,N}$ and $\mathcal{Z}^N = \mathcal{Y}^N|_{\Sigma\times[a,b]}$, then the law of $\mathcal{Z}^N$ converges weakly to $\mathbb{P}^{a,b,\vec{x},\vec{y},f,g}_{avoid}$ as $N \to \infty$.*

The next lemma shows that at any given point, the values of the $k-1$ curves in $\mathcal{L}^\infty$ are each distinct, so that Lemma 4.6 may be applied.

Lemma 4.7. *For any $s \in \mathbb{R}$, we have $\mathcal{L}^\infty(s) = (f^\infty_1(s), \ldots, f^\infty_{k-1}(s)) \in W^\circ_{k-1}$, $\mathbb{P}$-a.s.*

Using these two lemmas, we now give the proof of Theorem 2.26 (ii).

Proof. We will write $\Sigma = [\![1, k]\!]$. Let us write $\mathcal{Y}^N = (Y^N_1, \ldots, Y^N_{k-1})$ with $Y^N_i(s) = N^{-\alpha/2}(L^N_i(sN^\alpha) - psN^\alpha)/\sqrt{p(1-p)}$. We may assume without loss of generality that $\mathcal{Y}^N \implies \mathcal{L}^\infty$ as $N \to \infty$. Fix a set $K = [\![k_1, k_2]\!] \subseteq [\![1, k-2]\!]$ and $a, b \in \mathbb{R}$ with $a < b$. We also fix a bounded Borel-measurable function $F : C(K \times [a, b]) \to \mathbb{R}$. It suffices to prove that $\mathbb{P}$-a.s.,

$$\mathbb{E}[F(\mathcal{L}^\infty|_{K\times[a,b]}) \,|\, \mathcal{F}_{ext}(K \times (a, b))] = \mathbb{E}^{a,b,\vec{x},\vec{y},f,g}_{avoid}[F(\mathcal{Q})], \tag{4.20}$$

where $\vec{x} = (f^\infty_{k_1}(a), \ldots, f^\infty_{k_2}(a))$, $\vec{y} = (f^\infty_{k_1}(b), \ldots, f^\infty_{k_2}(b))$, $f = f^\infty_{k_1-1}$ (with $f^\infty_0 = +\infty$), $g = f^\infty_{k_2+1}$, and the σ-algebra $\mathcal{F}_{ext}(K \times (a, b))$ is as in Definition 2.8. We prove (4.20) in two steps.

Step 1. Fix $m \in \mathbb{N}$, $n_1, \ldots, n_m \in \Sigma$, $t_1, \ldots, t_m \in \mathbb{R}$, and $h_1, \ldots, h_m : \mathbb{R} \to \mathbb{R}$ bounded continuous functions. Define $S = \{i \in [\![1, m]\!] : n_i \in K, t_i \in [a, b]\}$. In this step we prove that

$$\mathbb{E}\left[\prod_{i=1}^m h_i(f^\infty_{n_i}(t_i))\right] = \mathbb{E}\left[\prod_{s\notin S} h_s(f^\infty_{n_s}(t_s)) \cdot \mathbb{E}^{a,b,\vec{x},\vec{y},f,g}_{avoid}\left[\prod_{s\in S} h_s(Q_{n_s}(t_s))\right]\right], \tag{4.21}$$

where Q denotes a random variable with law $\mathbb{P}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$. By assumption, we have

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{i=1}^m h_i(Y_{n_i}^N(t_i))\right] = \mathbb{E}\left[\prod_{i=1}^m h_i(f_{n_i}^\infty(t_i))\right]. \tag{4.22}$$

We define the sequences $a_N = \lfloor aN^\alpha \rfloor N^{-\alpha}$, $b_N = \lceil bN^\alpha \rceil N^{-\alpha}$, $\vec{x}^{\,N} = (L_{k_1}^N(a_N), \dots, L_{k_2}^N(a_N))$, $\vec{y}^{\,N} = (L_{k_1}^N(b_N), \dots, L_{k_2}^N(b_N))$, $f_N = Y_{k_1-1}^N$ (where $Y_0 = +\infty$), $g_N = Y_{k_2+1}^N$. Since $a_N \to a$, $b_N \to b$, we may choose N_0 sufficiently large so that if $N \geq N_0$, then $t_s < a_N$ or $t_s > b_N$ for all $s \notin S$ with $n_s \in K$. Since the line ensemble $(L_1^N, \dots, L_{k-1}^N)$ in the definition of $\mathcal{Y}^N$ satisfies the Schur Gibbs property (see Definition 2.17), we see from Definition 4.5 that the law of $\mathcal{Y}^N|_{K\times[a,b]}$ conditioned on the σ-algebra $\mathcal{F} = \sigma\left(Y_{k_1-1}^N, Y_{k_2+1}^N, Y_{k_1}^N(a_N), Y_{k_1}^N(b_N), \dots, Y_{k_2}^N(a_N), Y_{k_2}^N(b_N)\right)$ is precisely $\mathbb{P}_{avoid,N}^{a_N,b_N,\vec{x}^{\,N},\vec{y}^{\,N},f_N,g_N}$. Therefore, writing Z^N for a random variable with this law, we have

$$\mathbb{E}\left[\prod_{i=1}^m h_i(Y_{n_i}^N(t_i))\right] = \mathbb{E}\left[\prod_{s\notin S} h_s(Y_{n_s}^N(t_s)) \cdot \mathbb{E}_{avoid,N}^{a_N,b_N,\vec{x}^{\,N},\vec{y}^{\,N},f_N,g_N}\left[\prod_{s\in S} h_s(Z_{n_s-k_1+1}^N(t_s))\right]\right]. \tag{4.23}$$

Now by Lemma 4.7, we have $\mathbb{P}$-a.s. that $\vec{x}, \vec{y} \in W_{k_2-k_1+1}^\circ$, where we recall that $\vec{x} = \mathcal{L}^\infty(a)$, $\vec{y} = \mathcal{L}^\infty(b)$. By the Skorohod representation theorem, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting random variables with the laws of $\mathcal{Y}^N$, $\mathcal{L}^\infty$ (which we denote by the same symbols), such that $\mathcal{Y}^N \to \mathcal{L}^\infty$ uniformly on compact sets at every point of Ω. In particular, $f_N \to f = f_{k_2+1}^\infty$ and $g_N \to g = f_{k_1-1}^\infty$ uniformly on $[a-1, b+1] \supseteq [a_N, b_N]$, and $(x_i{}^N - pa_N N^{\alpha/2})/\sqrt{p(1-p)} \to \vec{x}$, $(y_i{}^N - pb_N N^{\alpha/2})/\sqrt{p(1-p)} \to \vec{y}$ for $i \in [\![1, k-1]\!]$. It follows from Lemma 4.6 that

$$\lim_{N\to\infty} \mathbb{E}_{avoid,N}^{a_N,b_N,\vec{x}^{\,N},\vec{y}^{\,N},f_N,g_N}\left[\prod_{s\in S} h_s(Z_{n_s-k_1+1}^N(t_s))\right] = \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}\left[\prod_{s\in S} h_s(Q_{n_s}(t_s))\right]. \tag{4.24}$$

Lastly, the continuity of the h_i implies that

$$\lim_{N\to\infty} \prod_{s\notin S} h_s(Y_{n_s}^N(t_s)) = \prod_{s\notin S} h_s(f_{n_s}^\infty(t_s)). \tag{4.25}$$

Combining (4.22), (4.23), (4.24), and (4.25) and applying the bounded convergence theorem proves (4.21).

Step 2. In this step we prove (4.20) as a consequence of (4.21). For $n \in \mathbb{N}$ we define piecewise linear functions

$$\chi_n(x,r) = \begin{cases} 0, & x > r + 1/n, \\ 1 - n(x-r), & x \in [r, r+1/n], \\ 1, & x < r. \end{cases}$$

We fix $m_1, m_2 \in \mathbb{N}$, $n_1^1, \dots, n_{m_1}^1, n_1^2, \dots, n_{m_2}^2 \in \Sigma$, $t_1^1, \dots, t_{m_1}^1, t_1^2, \dots, t_{m_2}^2 \in \mathbb{R}$, such that $(n_i^1, t_i^1) \notin K \times [a,b]$ and $(n_i^2, t_i^2) \in K \times [a,b]$ for all i. Then (4.21) implies that

$$\mathbb{E}\left[\prod_{i=1}^{m_1} \chi_n(f_{n_i^1}^\infty(t_i^1), a_i) \prod_{i=1}^{m_2} \chi_n(f_{n_i^2}^\infty(t_i^2), b_i)\right] = \mathbb{E}\left[\prod_{i=1}^{m_1} \chi_n(f_{n_i^1}^\infty(t_i^1), a_i) \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}\left[\prod_{i=1}^{m_2} \chi_n(Q_{n_i^2}(t_i^2), b_i)\right]\right].$$

Letting $n \to \infty$, we have $\chi_n(x,r) \to \chi(x,r) = \mathbf{1}_{x\leq r}$, and the bounded convergence theorem implies that

$$\mathbb{E}\left[\prod_{i=1}^{m_1} \chi(f_{n_i^1}^\infty(t_i^1), a_i) \prod_{i=1}^{m_2} \chi(f_{n_i^2}^\infty(t_i^2), b_i)\right] = \mathbb{E}\left[\prod_{i=1}^{m_1} \chi(f_{n_i^1}^\infty(t_i^1), a_i) \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}\left[\prod_{i=1}^{m_2} \chi(Q_{n_i^2}(t_i^2), b_i)\right]\right].$$

Let $\mathcal{H}$ denote the space of bounded Borel measurable functions $H : C(K \times [a,b]) \to \mathbb{R}$ satisfying

$$\mathbb{E}\left[\prod_{i=1}^{m_1} \chi(f^{\infty}_{n_i^1}(t_i^1), a_i) H(\mathcal{L}^{\infty}|_{K\times[a,b]})\right] = \mathbb{E}\left[\prod_{i=1}^{m_1} \chi(f^{\infty}_{n_i^1}(t_i^1), a_i) \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}\left[H(\mathcal{Q})\right]\right]. \tag{4.26}$$

The above shows that $\mathcal{H}$ contains all functions $\mathbf{1}_A$ for sets A contained in the π-system $\mathcal{A}$ consisting of sets of the form

$$\{h \in C(K \times [a,b]) : h(n_i^2, t_i^2) \le b_i \text{ for } i \in [\![1, m_2]\!]\}.$$

We note that $\mathcal{H}$ is closed under linear combinations simply by linearity of expectation, and if $H_n \in \mathcal{H}$ are nonnegative bounded measurable functions converging monotonically to a bounded function H, then $H \in \mathcal{H}$ by the monotone convergence theorem. Thus by the monotone class theorem [9, Theorem 5.2.2], $\mathcal{H}$ contains all bounded $\sigma(\mathcal{A})$-measurable functions. Since the finite dimensional sets in $\mathcal{A}$ generate the full Borel σ-algebra $\mathcal{C}_K$ (see for instance [6, Lemma 3.1]), we have in particular that $F \in \mathcal{H}$.

Now let $\mathcal{B}$ denote the collection of sets $B \in \mathcal{F}_{ext}(K \times (a,b))$ such that

$$\mathbb{E}[\mathbf{1}_B \cdot F(\mathcal{L}^{\infty}|_{K\times[a,b]})] = \mathbb{E}[\mathbf{1}_B \cdot \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}[F(\mathcal{Q})]]. \tag{4.27}$$

We observe that $\mathcal{B}$ is a λ-system. Indeed, since (4.26) holds for $H = F$, taking $a_i, b_i \to \infty$ and applying the bounded convergence theorem shows that (4.27) holds with $\mathbf{1}_B = 1$. Thus if $B \in \mathcal{B}$ then $B^c \in \mathcal{B}$ since $\mathbf{1}_{B^c} = 1 - \mathbf{1}_B$. If $B_i \in \mathcal{B}$, $i \in \mathbb{N}$, are pairwise disjoint and $B = \bigcup_i B_i$, then $\mathbf{1}_B = \sum_i \mathbf{1}_{B_i}$, and it follows from the monotone convergence theorem that $B \in \mathcal{B}$. Moreover, (4.26) with $H = F$ implies that $\mathcal{B}$ contains the π-system P of sets of the form

$$\{h \in C(\Sigma \times \mathbb{R}) : h(n_i, t_i) \le a_i \text{ for } i \in [\![1, m_1]\!], \text{ where } (n_i, t_i) \notin K \times (a,b)\}.$$

By the π-λ theorem [9, Theorem 2.1.6] it follows that $\mathcal{B}$ contains $\sigma(P) = \mathcal{F}_{ext}(K \times (a,b))$. Thus (4.27) holds for all $B \in \mathcal{F}_{ext}(K \times (a,b))$. It is proven in [6, Lemma 3.4] that $\mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}[F(\mathcal{Q})]$ is an $\mathcal{F}_{ext}(K \times (a,b))$-measurable function. Therefore (4.20) follows from (4.27) by the definition of conditional expectation.

□

5. Bounding the max and min

In this section we prove Lemmas 4.2 and 4.3 and we assume the same notation as in the statements of these lemmas. In particular, we assume that $k \in \mathbb{N}$, $k \ge 2$, $p \in (0,1)$, $\alpha, \lambda > 0$ are all fixed and

$$\left\{\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)\right\}_{N=1}^{\infty},$$

is an (α, p, λ)-good sequence of $[\![1, k]\!]$-indexed Bernoulli line ensembles as in Definition 2.24 that are all defined on a probability space with measure $\mathbb{P}$. The proof of Lemma 4.2 is given in Section 5.1 and Lemma 4.3 is proved in Section 5.2.

5.1. **Proof of Lemma 4.2.**

Our proof of Lemma 4.2 is similar to that of [3, Lemma 5.2]. For clarity we split the proof into three steps. In the first step we introduce some notation that will be required in the proof of the lemma, which is presented in Steps 2 and 3.

Step 1. We write $s_4 = \lceil r + 4 \rceil N^{\alpha}$, $s_3 = \lfloor r + 3 \rfloor N^{\alpha}$, so that $s_3 \le t_3 \le s_4$, and assume that N is large enough so that $\psi(N)N^{\alpha}$ from Definition 2.24 is at least s_4. Notice that such a choice is possible by our assumption that $\mathfrak{L}^N$ is an (α, p, λ)-good sequence and in particular, we know that L_i^N are defined at $\pm s_4$ for $i \in [\![1, k]\!]$. We define events

$$E(M) = \left\{\left|L_1^N(-s_4) + ps_4\right| > MN^{\alpha/2}\right\}, \quad F(M) = \left\{L_1^N(-s_3) > -ps_3 + MN^{\alpha/2}\right\},$$

$$G(M) = \left\{ \sup_{s \in [0, s_4]} \left[L_1^N(s) - ps \right] \geq (6r+22)(2r+10)^{1/2}(M+1)N^{\alpha/2} \right\}.$$

If $\epsilon > 0$ is as in the statement of the lemma, we note by (2.7) that we can find M and $\tilde{N}_1$ sufficiently large so that if $N \geq \tilde{N}_1$ we have

$$\mathbb{P}(E(M)) < \epsilon/4 \text{ and } \mathbb{P}(F(M)) < \epsilon/12. \tag{5.1}$$

In the remainder of this step we show that the event $G(M) \setminus E(M)$ can be written as a *countable disjoint union* of certain events, i.e. we show that

$$\bigsqcup_{(a,b,s,\ell_{top},\ell_{bot}) \in D(M)} E(a,b,s,\ell_{top},\ell_{bot}) = G(M) \setminus E(M), \tag{5.2}$$

where the sets $E(a,b,s,\ell_{top},\ell_{bot})$ and $D(M)$ are described below.

For $a, b, z_1, z_2, z_3 \in \mathbb{Z}$ with $z_1 \leq a$, $z_2 \leq b$, $s \in [\![0, s_4]\!]$, $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$ and $\ell_{top} \in \Omega(s, s_4, b, z_3)$ we define $E(a,b,s,\ell_{top},\ell_{bot})$ to be the event that $L_1^N(-s_4) = a$, $L_1^N(s) = b$, L_1^N agrees with ℓ_{top} on $[\![s, s_4]\!]$, and L_2^N agrees with ℓ_{bot} on $[\![-s_4, s]\!]$. Let $D(M)$ be the set of tuples $(a,b,s,\ell_{top},\ell_{bot})$ satisfying

(1) $0 \leq s \leq s_4$,
(2) $0 \leq b - a \leq s + s_4$, $|a + ps_4| \leq MN^{\alpha/2}$, and $b - ps \geq (6r+22)(2r+10)^{1/2}(M+1)N^{\alpha/2}$,
(3) $z_1 \leq a$, $z_2 \leq b$, and $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$,
(4) $b \leq z_3 \leq b + (s_4 - s)$, and $\ell_{top} \in \Omega(s, s_4, b, z_3)$,
(5) if $s < s' \leq s_4$, then $\ell_{top}(s') - ps' < (6r+22)(2r+10)^{1/2}(M+1)N^{\alpha/2}$.

It is clear that $D(M)$ is countable. The five conditions above together imply that

$$\bigcup_{(a,b,s,\ell_{top},\ell_{bot}) \in D(M)} E(a,b,s,\ell_{top},\ell_{bot}) = G(M) \setminus E(M),$$

and what remains to be shown to prove (5.2) is that $E(a,b,s,\ell_{top},\ell_{bot})$ are pairwise disjoint.

On the intersection of $E(a,b,s,\ell_{top},\ell_{bot})$ and $E(\tilde{a},\tilde{b},\tilde{s},\tilde{\ell}_{top},\tilde{\ell}_{bot})$ we must have $\tilde{a} = L_1^N(-s_4) = a$ so that $a = \tilde{a}$. Furthermore, we have by properties (2) and (5) that $s \geq \tilde{s}$ and $\tilde{s} \geq s$ from which we conclude that $s = \tilde{s}$ and then we conclude $\tilde{b} = b$, $\ell_{top} = \tilde{\ell}_{top}$, $\ell_{bot} = \tilde{\ell}_{bot}$. In summary, if $E(a,b,s,\ell_{top},\ell_{bot})$ and $E(\tilde{a},\tilde{b},\tilde{s},\tilde{\ell}_{top},\tilde{\ell}_{bot})$ have a non-trivial intersection then $(a,b,s,\ell_{top},\ell_{bot}) = (\tilde{a},\tilde{b},\tilde{s},\tilde{\ell}_{top},\tilde{\ell}_{bot})$, which proves (5.2).

Step 2. In this step we prove that we can find an N_2 so that for $N \geq N_2$

$$\mathbb{P}\left(\sup_{s \in [0, t_3]} \left[L_1^N(s) - ps \right] \geq (6r+22)(2r+10)^{1/2}(M+1)N^{\alpha/2} \right) \leq \mathbb{P}(G(M)) < \epsilon/2. \tag{5.3}$$

A similar argument, which we omit, proves the same inequality with $[-t_3, 0]$ in place of $[0, t_3]$ and then the statement of the lemma holds for all $N \geq N_2$, with $R_1 = (6r+22)(2r+10)^{1/2}(M+1)$.

We claim that we can find $\tilde{N}_2 \in \mathbb{N}$ sufficiently large so that if $N \geq \tilde{N}_2$ and $(a,b,s,\ell_{top},\ell_{bot}) \in D(M)$ satisfies $\mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot})) > 0$ then we have

$$\mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}} \left(\ell(-s_3) > -ps_3 + MN^{\alpha/2} \right) \geq \frac{1}{3}. \tag{5.4}$$

We will prove (5.4) in Step 3. For now we assume its validity and conclude the proof of (5.3).

Let $(a,b,s,\ell_{top},\ell_{bot}) \in D(M)$ be such that $\mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot})) > 0$. By the Schur Gibbs property, see Definition 2.17, we have for any $\ell_0 \in \Omega(-s_4, s, a, b)$ that

$$\mathbb{P}\left(L_1^N[\![-s_4, s]\!] = \ell_0 \,|\, E(a,b,s,\ell_{top},\ell_{bot}) \right) = \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}(\ell = \ell_0), \tag{5.5}$$

where $L_1^N[\![-s_4, s]\!]$ denotes the restriction of L_1^N to the set $[\![-s_4, s]\!]$.

Combining (5.4) and (5.5) we get for $N \geq \tilde{N}_2$

$$\begin{aligned}\mathbb{P}\left(L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} | E(a,b,s,\ell_{top},\ell_{bot})\right) = \\ \mathbb{P}^{-s_4,s,a,b,\infty,\ell_{bot}}_{avoid,Ber}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) \geq \frac{1}{3}.\end{aligned} \tag{5.6}$$

It follows from (5.6) that for $N \geq \tilde{N}_2$ we have

$$\begin{aligned}\epsilon/12 > \mathbb{P}(F(M)) \geq \sum_{\substack{(a,b,s,\ell_{top},\ell_{bot}) \in D(M), \\ \mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot}))>0}} \mathbb{P}\left(F(M) \cap E(a,b,s,\ell_{top},\ell_{bot})\right) = \\ \sum_{\substack{(a,b,s,\ell_{top},\ell_{bot}) \in D(M), \\ \mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot}))>0}} \mathbb{P}\left(L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} | E(a,b,s,\ell_{top},\ell_{bot})\right) \mathbb{P}\left(E(a,b,s,\ell_{top},\ell_{bot})\right) \geq \\ \sum_{\substack{(a,b,s,\ell_{top},\ell_{bot}) \in D(M), \\ \mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot}))>0}} \frac{1}{3} \cdot \mathbb{P}\left(E(a,b,s,\ell_{top},\ell_{bot})\right) = \frac{1}{3} \cdot \mathbb{P}(G(M) \setminus E(M)),\end{aligned} \tag{5.7}$$

where in the last equality we used (5.2). From (5.1) and (5.7) we have for $N \geq N_2 = \max(\tilde{N}_1, \tilde{N}_2)$

$$\mathbb{P}(G(M)) \leq \mathbb{P}(E(M)) + \mathbb{P}(G(M) \setminus E(M)) < \epsilon/4 + \epsilon/4,$$

which proves (5.3).

Step 3. In this step we prove (5.4) and in the sequel we let $(a,b,s,\ell_{top},\ell_{bot}) \in D(M)$ be such that $\mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot})) > 0$. We remark that the condition $\mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot})) > 0$ implies that $\Omega_{avoid}(-s_4, s, a, b, \infty, \ell_{bot})$ is not empty. By Lemma 3.2 we know that

$$\mathbb{P}^{-s_4,s,a,b,\infty,\ell_{bot}}_{avoid,Ber}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) \geq \mathbb{P}^{-s_4,s,a,b}_{Ber}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right),$$

and so it suffices to show that

$$\mathbb{P}^{-s_4,s,a,b}_{Ber}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) \geq \frac{1}{3}. \tag{5.8}$$

One directly observes that

$$\begin{aligned}\mathbb{P}^{-s_4,s,a,b}_{Ber}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) = \mathbb{P}^{0,s+s_4,0,b-a}_{Ber}\left(\ell(s_4 - s_3) + a \geq -ps_3 + MN^{\alpha/2}\right) \geq \\ \mathbb{P}^{0,s+s_4,0,b-a}_{Ber}\left(\ell(s_4 - s_3) \geq p(s_4 - s_3) + 2MN^{\alpha/2}\right),\end{aligned} \tag{5.9}$$

where the inequality follows from the assumption in (2) that $a + ps_4 \geq -MN^{\alpha/2}$. Moreover, since $b - ps \geq (6r+22)(2r+10)^{1/2}(M+1)N^{\alpha/2}$ and $a + ps_4 \leq MN^{\alpha/2}$, we have

$$b - a \geq p(s+s_4) + (6r+21)(2r+10)^{1/2}(M+1)N^{\alpha/2} \geq p(s+s_4) + (6r+21)(M+1)(s+s_4)^{1/2}.$$

The second inequality follows since $s + s_4 \leq 2s_4 \leq (2r+10)N^{\alpha}$.

It follows from Lemma 3.8 with $M_1 = 0$, $M_2 = (6r+21)(M+1)$ that for sufficiently large N

$$\mathbb{P}^{0,s+s_4,0,b-a}_{Ber}\left(\ell(s_4 - s_3) \geq \frac{s_4 - s_3}{s + s_4}[p(s+s_4) + M_2(s+s_4)^{1/2}] - (s+s_4)^{1/4}\right) \geq 1/3. \tag{5.10}$$

Note that $\frac{s_4 - s_3}{s+s_4} \geq \frac{N^{\alpha}}{(2r+10)N^{\alpha}} = \frac{1}{2r+10}$ and so for all $N \in \mathbb{N}$ we have

$$\begin{aligned}\frac{s_4 - s_3}{s + s_4}[p(s+s_4) + M_2(s+s_4)^{1/2}] - (s+s_4)^{1/4} \geq \\ p(s_4 - s_3) + \frac{(6r+21)(M+1)(s+s_4)^{1/2}}{2r+10} - (s+s_4)^{1/4} \geq p(s_4 - s_3) + 2MN^{\alpha/2}.\end{aligned} \tag{5.11}$$

Combining (5.9), (5.10) and (5.11) we conclude that we can find $\tilde{N}_2 \in \mathbb{N}$ such that if $N \geq \tilde{N}_2$ we have (5.8). This suffices for the proof.

5.2. **Proof of Lemma 4.3.** We begin by proving the following important lemma, which shows that it is unlikely that the curve L_{k-1}^N falls uniformly very low on a large interval.

Lemma 5.1. *Under the same conditions as in Lemma 4.3 the following holds. For any $r, \epsilon > 0$ there exist $R > 0$ and $N_5 \in \mathbb{N}$ such that for all $N \geq N_5$*

$$\mathbb{P}\left(\sup_{x\in[r,R]}\left(L_{k-1}^N(xN^\alpha) - pxN^\alpha\right) \leq -(\lambda R^2 + \phi(\epsilon/16) + 1)N^{\alpha/2}\right) < \epsilon, \tag{5.12}$$

where λ, ϕ are as in the definition of an (α, p, λ)-good sequence of line ensembles, see Definition 2.24. The same statement holds if $[r, R]$ is replaced with $[-R, -r]$ and the constants N_5, R depend on ϵ, r as well as the parameters α, p, λ, k and the functions ϕ, ψ from Definition 2.24.

Proof. Before we go into the proof we give an informal description of the main ideas. The key to this lemma is the parabolic shift implicit in the definition of an (α, p, λ)-good sequence. This shift requires that the deviation of the top curve L_1^N from the line of slope p to appear roughly parabolic. On the event in equation (5.12) we have that the $(k-1)$-th curve dips very low uniformly on the interval $[r, R]$ and we will argue that on this event the top $k-2$ curves essentially do not feel the presence of the $(k-1)$-th curve. After a careful analysis using the monotone coupling lemmas from Section 3.1 we will see that the latter statement implies that the curve L_1^N behaves like a free bridge between its end-points that have been slighly raised. Consequently, we would expect the midpoint $L_1^N(N^\alpha(R+r)/2)$ to be close (on scale $N^{\alpha/2}$) to $[L_1^N(rN^\alpha) + L_1^N(RN^\alpha)]/2$. However, with high probability $[L_1^N(rN^\alpha) + L_1^N(RN^\alpha)]/2$ lies much lower than the inverted parabola $-\lambda(R+r)^2N^{\alpha/2}/4$ (due to the concavity of the latter), and so it is very unlikely for $L_1^N(N^\alpha(R+r)/2)$ to be near it by our assumption. The latter would imply that the event in (5.12) is itself unlikely, since conditional on it an unlikely event suddenly became likely.

We proceed to fill in the details of the above sketch of the proof in the following steps. In total there are six steps and we will only prove the statement of the lemma for the interval $[r, R]$, since the argument for $[-R, -r]$ is very similar.

Step 1. We begin by specifying the choice of R in the statement of the lemma, fixing some notation and making a few simplifying assumptions.

Fix $r, \epsilon > 0$ as in the statement of the lemma. Note that for any $R > r$,

$$\sup_{r\leq x\leq R}\left(L_{k-1}^N(xN^\alpha) - pxN^\alpha\right) \geq \sup_{\lceil r\rceil\leq x\leq R}\left(L_{k-1}^N(xN^\alpha) - pxN^\alpha\right).$$

Thus by replacing r with $\lceil r \rceil$, we can assume that $r \in \mathbb{Z}$, which we do in the sequel. Notice that by our assumption that $\mathfrak{L}^N$ is (α, p, λ)-good we know that (5.12) holds trivially if $k = 2$ (with the right side of (5.12) being any number greater than $\epsilon/16$ and in particular ϵ) and so in the sequel we assume that $k \geq 3$.

Define constants

$$C = \sqrt{8p(1-p)\log\frac{3}{1-(11/12)^{1/(k-2)}}}, \tag{5.13}$$

and $R_0 > r$ sufficiently large so that for $R \geq R_0$ and $N \in \mathbb{N}$ we have

$$\frac{\lambda(R-r)^2}{4} \geq 2\phi(\epsilon/16) + 2 + k\lceil C\lceil RN^\alpha\rceil - \lfloor rN^\alpha\rfloor\rceil N^{-\alpha/2}. \tag{5.14}$$

We define $R = \lceil R_0 \rceil + \mathbf{1}_{\lceil R_0\rceil + r \text{ odd}}$, so that $R \geq R_0$ and the midpoint $(R+r)/2$ are integers. This specifies our choice of R and for convenience we denote $m = (R+r)/2$.

In the following, we always assume N is large enough so that $\psi(N) > R$, hence L_i^N are defined at RN^α for $1 \le i \le k$. We may do so by the second condition in the definition of an (α, p, λ)-good sequence (see Definition 2.24).

With the choice of R as above we define the events

$$A = \left\{ L_1^N(mN^\alpha) - pmN^\alpha + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16) N^{\alpha/2} \right\},$$
$$B = \left\{ \sup_{x \in [r,R]} \left(L_{k-1}^N(xN^\alpha) - pxN^\alpha \right) \le -[\lambda R^2 + \phi(\epsilon/16) + 1] N^{\alpha/2} \right\}. \tag{5.15}$$

The goal of the lemma is to prove that we can find $N_5 \in \mathbb{N}$ so that for all $N \ge N_5$

$$\mathbb{P}(B) < \epsilon, \tag{5.16}$$

which we accomplish in the steps below.

Step 2. In this step we introduce some notation that will be used throughout the next steps. Let $\gamma = \lfloor rN^\alpha \rfloor$ and $\Gamma = \lceil RN^\alpha \rceil$. We also define the event

$$F = \left\{ \sup_{s \in \{\gamma, \Gamma\}} \left| L_1^N(s) - ps + \lambda s^2 N^{-\alpha/2} \right| < [\phi(\epsilon/16) + 1] N^{\alpha/2} \right\}. \tag{5.17}$$

In the remainder of this step we show that $F \cap B$ can be written as a *countable disjoint union*

$$F \cap B = \bigsqcup_{(\vec{x}, \vec{y}, \ell_{bot}) \in D} E(\vec{x}, \vec{y}, \ell_{bot}), \tag{5.18}$$

where the sets $E(\vec{x}, \vec{y}, \ell_{bot})$ and D are defined below.

For $\vec{x}, \vec{y} \in \mathfrak{W}_{k-2}$, $z_1, z_2 \in \mathbb{Z}$, and $\ell_{bot} \in \Omega(\gamma, \Gamma, z_1, z_2)$, let $E(\vec{x}, \vec{y}, \ell_{bot})$ denote the event that $L_i^N(\gamma) = x_i$ and $L_i^N(\Gamma) = y_i$ for $1 \le i \le k-2$, and L_{k-1}^N agrees with ℓ_{bot} on $[\gamma, \Gamma]$. Let D denote the set of triples $(\vec{x}, \vec{y}, \ell_{bot})$ satisfying

(1) $0 \le y_i - x_i \le \Gamma - \gamma$ for $1 \le i \le k-2$,
(2) $|x_1 - p\gamma + \lambda \gamma^2 N^{-3\alpha/2}| < \phi(\epsilon/16) N^{\alpha/2}$ and $|y_1 - p\Gamma + \lambda \Gamma^2 N^{-3\alpha/2}| < \phi(\epsilon/16) N^{\alpha/2}$,
(3) $z_1 \le x_{k-2}$, $z_2 \le y_{k-2}$, and $\ell_{bot} \in \Omega(\gamma, \Gamma, z_1, z_2)$,
(4) $\sup_{x \in [r,R]} [\ell_{bot}(xN^\alpha) - pxN^\alpha] \le -[\lambda R^2 + \phi(\epsilon/16) + 1] N^{\alpha/2}$.

It is clear that D is countable, the events $E(\vec{x}, \vec{y}, \ell_{bot})$ are pairwise disjoint for different elements in D and (5.18) is satisfied.

Step 3. We claim that we can find $\tilde{N}_0$ so that for $N \ge \tilde{N}_0$ we have

$$\mathbb{P}(A | E(\vec{x}, \vec{y}, \ell_{bot})) \ge 1/4 \tag{5.19}$$

for all $(\vec{x}, \vec{y}, \ell_{bot}) \in D$ such that $\mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot})) > 0$. We will prove (5.19) in the steps below. In this step we assume its validity and conclude the proof of (5.16).

It follows from (5.18) and (5.19) that for $N \ge \tilde{N}_0$ and $\mathbb{P}(F \cap B) > 0$ we have

$$\mathbb{P}(A | F \cap B) = \sum_{(\vec{x}, \vec{y}, \ell_{bot}) \in D, \mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot}))) > 0} \frac{\mathbb{P}(A | E(\vec{x}, \vec{y}, \ell_{bot}) \mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot}))}{\mathbb{P}(F \cap B)} \ge$$
$$\frac{1}{4} \cdot \frac{\sum_{(\vec{x}, \vec{y}, \ell_{bot}) \in D, \mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot}))) > 0} \mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot}))}{\mathbb{P}(F \cap B)} = \frac{1}{4}.$$

From the third condition in the definition of an (α, p, λ)-good sequence, see Definition 2.24, we can find $\tilde{N}_1$ so that $\mathbb{P}(A) < \epsilon/8$ for $N \geq \tilde{N}_1$. Hence if $N \geq \max(\tilde{N}_1, \tilde{N}_2)$ and $\mathbb{P}(F \cap B) > 0$ we have

$$\mathbb{P}(F \cap B) = \frac{\mathbb{P}(A \cap F \cap B)}{\mathbb{P}(A | F \cap B)} \leq 4\mathbb{P}(A) < \epsilon/2. \tag{5.20}$$

Lastly, by the same condition in Definition 2.24 we can find $\tilde{N}_2$ so that for $N \geq \tilde{N}_2$ we have

$$\mathbb{P}(F^c) = 2 \cdot \epsilon/8 = \epsilon/4. \tag{5.21}$$

In deriving (5.21) we used the fact that $|L_1^N(\gamma) - L_1^N(rN^\alpha)| \leq 1$, $|L_1^N(\Gamma) - L_1^N(RN^\alpha)| \leq 1$ and $p \in [0,1]$. Combining (5.20) and (5.21) we conclude that if $N \geq N_5 = \max(\tilde{N}_0, \tilde{N}_1, \tilde{N}_2)$

$$\mathbb{P}(B) \leq \mathbb{P}(F \cap B) + \mathbb{P}(F^c) \leq \epsilon/2 + \epsilon/4 < \epsilon,$$

which proves (5.16).

Step 4. In this step we prove (5.19). We define $\vec{x}\,', \vec{y}\,' \in \mathfrak{W}_{k-2}$ through

$$\begin{aligned} &x_i' = \overline{x} + (k-1-i)\lceil C\sqrt{T}\,\rceil, \quad y_i' = \overline{y} + (k-1-i)\lceil C\sqrt{T}\,\rceil \text{ for } i = 1, \dots, k-2 \text{ with} \\ &\overline{x} = \lceil p\gamma - \lambda\gamma^2 N^{-3\alpha/2} + [\phi(\epsilon/16) + 1]N^{\alpha/2}\rceil, \ \overline{y} = \lceil p\Gamma - \lambda\Gamma^2 N^{-3\alpha/2} + [\phi(\epsilon/16) + 1]N^{\alpha/2}\rceil, \end{aligned} \tag{5.22}$$

where C is as in (5.13) and $T = \Gamma - \gamma$. Note that for any $(\vec{x}, \vec{y}, \ell_{bot}) \in D$ we have

$$x_i' \geq \overline{x} \geq x_1 \geq x_i \text{ and } y_i' \geq \overline{y} \geq y_1 \geq y_i$$

for each $i = 1, \dots, k-2$. Furthermore,

$$x_i' - x_{i+1}' \geq C\sqrt{T} \text{ and } y_i' - y_{i+1}' \geq C\sqrt{T}$$

for all $i = 1, \dots, k-2$ with the convention $x_{k-1}' = \overline{x}$ and $y_{k-1}' = \overline{y}$.

We claim that we can find $\tilde{N}_0$ so that for all $N \geq \tilde{N}_0$ and $(\vec{x}, \vec{y}, \ell_{bot}) \in D$ such that $\mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot})) > 0$ we have $\prod_{i=1}^{k-2} |\Omega(\gamma, \Gamma, x_i', y_i')| \geq |\Omega_{avoid}(\gamma, \Gamma, \vec{x}\,', \vec{y}\,', \infty, \ell_{bot})| \geq 1$ and moreover we have

$$\mathbb{P}_{Ber}^{\gamma, \Gamma, \vec{x}\,', \vec{y}\,'}\left(Q_1(mN^\alpha) - pmN^\alpha + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) \geq 1/3, \tag{5.23}$$

$$\mathbb{P}_{Ber}^{\gamma, \Gamma, \vec{x}\,', \vec{y}\,'}\left(Q_1 \geq \cdots \geq Q_{k-1}\right) \geq 11/12, \tag{5.24}$$

where $\mathfrak{Q} = (Q_1, \dots, Q_{k-2})$ is $\mathbb{P}_{Ber}^{\gamma, \Gamma, \vec{x}\,', \vec{y}\,'}$-distributed and we used the convention that $Q_{k-1} = \ell_{bot}$. We prove (5.23) and (5.24) in the steps below. In this step we assume their validity and conclude the proof of (5.19).

Observe that for any $(\vec{x}, \vec{y}, \ell_{bot}) \in D$ such that $\mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot})) > 0$ we the following tower of inequalities provided that $N \geq \tilde{N}_0$

$$\begin{aligned} &\mathbb{P}(A | E(\vec{x}, \vec{y}, \ell_{bot})) = \mathbb{P}_{avoid, Ber}^{\gamma, \Gamma, \vec{x}, \vec{y}, \infty, \ell_{bot}}\left(Q_1(mN^\alpha) - pmN^\alpha + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) \geq \\ &\mathbb{P}_{avoid, Ber}^{\gamma, \Gamma, \vec{x}\,', \vec{y}\,', \infty, \ell_{bot}}\left(Q_1(mN^\alpha) - pmN^\alpha + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) = \\ &\frac{\mathbb{P}_{Ber}^{\gamma, \Gamma, \vec{x}\,', \vec{y}\,'}\left(\{Q_1(mN^\alpha) - pmN^\alpha + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\} \cap \{Q_1 \geq \cdots \geq Q_{k-1}\}\right)}{\mathbb{P}_{Ber}^{\gamma, \Gamma, \vec{x}\,', \vec{y}\,'}\left(Q_1 \geq \cdots \geq Q_{k-1}\right)}. \end{aligned} \tag{5.25}$$

Let us elaborate on (5.25) briefly. The condition that $\mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot})) > 0$ is required to ensure that the probabilities on the first line of (5.25) are well-defined and $N \geq \tilde{N}_0$ ensures that all other probabilities are also well-defined. The equality on the first line of (5.25) follows from the definition of A and the Schur Gibbs property, see Definition 2.17, and $\mathfrak{Q} = (Q_1, \dots, Q_{k-2})$ is $\mathbb{P}_{avoid, Ber}^{\gamma, \Gamma, \vec{x}, \vec{y}, \infty, \ell_{bot}}$-distributed. The inequality in the first line of (5.25) follows from Lemma 3.1, while the equality in the second line follows from Definition 2.15, and now $\mathfrak{Q} = (Q_1, \dots, Q_{k-2})$ is $\mathbb{P}^{\gamma,\Gamma,\vec{x}',\vec{y}'}_{Ber}$-distributed with the convention that $Q_{k-1} = \ell_{bot}$.

Combining (5.23), (5.24) and (5.25) we conclude that

$$\mathbb{P}(A|E(\vec{x},\vec{y},\ell_{bot})) \geq 1/3 - 1/12 = 1/4,$$

which proves (5.19).

Step 5. In this step we prove (5.23). We observe that since $\mathbb{P}(E(\vec{x},\vec{y},\ell_{bot})) > 0$ we know that $|\Omega_{avoid}(\gamma,\Gamma,\vec{x},\vec{y},\infty,\ell_{bot})| \geq 1$ and then we conclude from Lemma 2.16 that there exist $\hat{N}_1 \in \mathbb{N}$ such that for $N \geq \hat{N}_1$ we have $|\Omega_{avoid}(\gamma,\Gamma,\vec{x}',\vec{y}',\infty,\ell_{bot})| \geq 1$.

Below ℓ will be used for a generic random variable with law $\mathbb{P}^{\cdot,\cdot,\cdot,\cdot}_{Ber}$, where the boundary data changes from line to line. With $\overline{x},\overline{y}$ as in (5.22), write $\overline{z} = \overline{y} - \overline{x}$ and recall that $T = \Gamma - \gamma$. Then

$$\begin{aligned}
&\mathbb{P}^{\gamma,\Gamma,x_1',y_1'}_{Ber}\left(\ell\left(mN^{\alpha}\right) - pmN^{\alpha} + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) = \\
&\mathbb{P}^{0,T,x_1',y_1'}_{Ber}\left(\ell(T/2) - pmN^{\alpha} + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) = \\
&\mathbb{P}^{0,T,\overline{x},\overline{y}}_{Ber}\left(\ell(T/2) - pmN^{\alpha} + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2} - (k-2)\lceil C\sqrt{T}\rceil\right) \geq \\
&\mathbb{P}^{0,T,\overline{x},\overline{y}}_{Ber}\left(\ell(T/2) - \frac{\overline{x}+\overline{y}}{2} < \lambda\left(\frac{\gamma^2+\Gamma^2}{2N^{3\alpha/2}}\right) - [2\phi(\epsilon/16) + 1 + \lambda m^2]N^{\alpha/2} - k\lceil C\sqrt{T}\rceil\right) = \\
&\mathbb{P}^{0,T,0,\overline{z}}_{Ber}\left(\ell(T/2) - \overline{z}/2 < \lambda\left(\frac{\gamma^2+\Gamma^2}{2N^{3\alpha/2}}\right) - [2\phi(\epsilon/16) + 1 + \lambda m^2]N^{\alpha/2} - k\lceil C\sqrt{T}\rceil\right).
\end{aligned} \tag{5.26}$$

The equalities in (5.26) follow from shifting the boundary data of the curve ℓ, while the inequality on the third line follows from the definition of $\overline{x},\overline{y}$ as in (5.22).

From our choice of R in Step 1 and the definition of γ,Γ we know that

$$\lambda\frac{\gamma^2+\Gamma^2}{2N^{2\alpha}} - \lambda m^2 \geq \lambda\frac{(R-r)^2}{4} - \frac{r\lambda}{N^{\alpha}} \geq 2\phi(\epsilon/16) + 2 + k\lceil C\sqrt{T}\rceil N^{-\alpha/2} - \frac{r\lambda}{N^{\alpha}}.$$

.

The last inequality and (5.26) imply

$$\begin{aligned}
&\mathbb{P}^{\gamma,\Gamma,x_1',y_1'}_{Ber}\left(\ell\left(mN^{\alpha}\right) - pmN^{\alpha} + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) \geq \\
&\mathbb{P}^{0,T,0,\overline{z}}_{Ber}\left(\ell(T/2) - \overline{z}/2 < N^{\alpha/2} - r\lambda N^{-\alpha/2}\right).
\end{aligned} \tag{5.27}$$

Let $\tilde{\mathbb{P}}$ be the probability measure on the space afforded by Theorem 3.3, supporting a random variable $\ell^{(T,\overline{z})}$ with law $\mathbb{P}^{0,T,0,\overline{z}}_{Ber}$ and a Brownian bridge B^{σ} with variance $\sigma^2 = p(1-p)$. Then the probability in the last line of (5.26) is equal to

$$\begin{aligned}
&\mathbb{P}^{0,T,0,\overline{z}}_{Ber}\left(\ell(T/2) - \overline{z}/2 < N^{\alpha/2} - r\lambda N^{-\alpha/2}\right) = \tilde{\mathbb{P}}\left(\ell^{(T,\overline{z})}(T/2) - \overline{z}/2 < N^{\alpha/2} - r\lambda N^{-\alpha/2}\right) \geq \\
&\tilde{\mathbb{P}}\left(\sqrt{T}B^{\sigma}_{1/2} < 0 \text{ and } \Delta(T,\overline{z}) < N^{\alpha/2} - r\lambda N^{-\alpha/2}\right) \geq \frac{1}{2} - \tilde{\mathbb{P}}\left(\Delta(T,\overline{z}) \geq N^{\alpha/2} - r\lambda N^{-\alpha/2}\right),
\end{aligned} \tag{5.28}$$

where we recall that $\Delta(T,\overline{z})$ is as in (3.2). Since as $N \to \infty$ we have

$$T \sim (R-r)N^{\alpha} \text{ and } \frac{|\overline{z} - pT|^2}{T} \sim (R+r),$$

we conclude from Lemma 3.5 that there exists $\hat{N}_2 \in \mathbb{N}$ such that if $N \geq \max(\hat{N}_1,\hat{N}_2)$ we have

$$\tilde{\mathbb{P}}\left(\Delta(T,\overline{z}) \geq N^{\alpha/2} - r\lambda N^{-\alpha/2}\right) \leq \frac{1}{6}. \tag{5.29}$$

Combining (5.27), (5.28) and (5.29) we obtain (5.23).

Step 6. In this last step, we prove (5.24). Let $\overline{\ell}_{bot}$ be the straight segment connecting $\overline{x}$ and $\overline{y}$, defined in (5.22). By construction, we have that there is $\hat{N}_3 \in \mathbb{N}$ such that if $N \geq \hat{N}_3$ we have for any $(\vec{x}, \vec{y}, \ell_{bot}) \in D$ that ℓ_{bot} lies uniformly below the line segment $\overline{\ell}_{bot}$, which in turn lies at least $C\sqrt{T}$ below the straight segment connecting x'_{k-2} and y'_{k-2}. If $\hat{N}_1$ is as in Step 5 we conclude from Lemma 3.14 that there exists $\hat{N}_4 \in \mathbb{N}$ such that if $N \geq \max(\hat{N}_1, \hat{N}_3, \hat{N}_4)$ and $\mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot})) > 0$

$$\mathbb{P}^{\gamma,\Gamma,\vec{x}',\vec{y}'}_{Ber}\left(Q_1 \geq \cdots \geq Q_{k-1}\right) \geq \left(1 - 3e^{-C^2/8p(1-p)}\right)^{k-2} = \frac{11}{12}. \tag{5.30}$$

where the condition that $N \geq \hat{N}_1$ is included to ensure that the probability $\mathbb{P}^{\gamma,\Gamma,\vec{x}',\vec{y}'}_{Ber}$ is well-defined. In deriving (5.30) we also used (5.13), which implies

$$C = \sqrt{8p(1-p)\log\frac{3}{1-(11/12)^{1/(k-2)}}} \geq \sqrt{8p(1-p)\log 3}.$$

We see that (5.30) implies (5.24), which concludes the proof of the lemma. □

In the remainder of this section we use Lemma 5.1 to prove Lemma 4.3.

Proof. (Lemma 4.3) For clarity we split the proof into five steps.

Step 1. In this step we specify the choice of R_2 in the statement of the lemma and introduce some notation that will be used in the proof of the lemma, which is given in Steps 2,3 and 4 below. Throughout we fix $r, \epsilon > 0$. Define the constant

$$C_1 = \sqrt{16p(1-p)\log\frac{3}{1-2^{-1/(k-1)}}}. \tag{5.31}$$

Let $R > r+3$, $M > 0$ and $\tilde{N}_1 \in \mathbb{N}$ be such that for $N \geq \tilde{N}_1$ we have that the event

$$B = \left\{ \sup_{x \in [r+3,R] \cup [-R,-r-3]} \left(L^N_{k-1}(xN^\alpha) - pxN^\alpha\right) \geq -MN^{\alpha/2} \right\} \tag{5.32}$$

satisfies

$$\mathbb{P}(B) \geq 1 - \epsilon/2. \tag{5.33}$$

Such a choice of $R, M, \tilde{N}_1$ is possible by Lemma 5.1.

Let us set

$$s_1^- = \lceil -R \cdot N^\alpha \rceil, \ \ s_2^- = \lfloor -(r+3) \cdot N^\alpha \rfloor, \ \ s_1^+ = \lceil (r+3) \cdot N^\alpha \rceil, \ \ s_2^+ = \lfloor R \cdot N^\alpha \rfloor,$$

and for $a \in [\![s_1^-, s_2^-]\!]$ and $b \in [\![s_1^+, s_2^+]\!]$ we define $\vec{x}', \vec{y}' \in \mathfrak{W}_{k-1}$ by

$$\begin{aligned} x'_i &= \lfloor pa - MN^{\alpha/2} \rfloor - (i-1)\lceil C_1 N^{\alpha/2} \rceil, \\ y'_i &= \lfloor pb - MN^{\alpha/2} \rfloor - (i-1)\lceil C_1 N^{\alpha/2} \rceil, \end{aligned} \tag{5.34}$$

for $i = 1, \dots, k-1$. We will write $\vec{z} = \vec{y}' - \vec{x}'$, and we note that $z_{k-1} \geq p(b-a) - 1$ and also $2RN^\alpha \geq b - a \geq 2(r+3)N^\alpha$. The latter and Lemma 3.10 imply that there exists $R_2 > 0$ and $\tilde{N}_2 \in \mathbb{N}$ such that if $N \geq \tilde{N}_2$ we have

$$\mathbb{P}^{0,b-a,0,z_{k-1}}_{Ber}\left(\inf_{s \in [0,b-a]}\left(\ell(s) - ps\right) \leq -(R_2 - M - C_1 k)N^{\alpha/2}\right) < \epsilon/4. \tag{5.35}$$

This fixes our choice of R_2 in the statement of the lemma.

With the above choice of R_2 we define the event

$$A = \left\{ \inf_{s \in [-t_3, t_3]} \left[L_{k-1}^N(s) - ps \right] \leq -R_2 N^{\alpha/2} \right\}, \tag{5.36}$$

and then to prove the lemma it suffices to show that there exists $N_4 \in \mathbb{N}$ such that for $N \geq N_4$

$$\mathbb{P}(A) < \epsilon \tag{5.37}$$

Step 2. In this step, we prove that the event B from (5.32) can be written as a *countable disjoint union* of the form

$$B = \bigsqcup_{(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+) \in D} E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+), \tag{5.38}$$

where the set D and events $E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)$ are defined below.

For $a \in [\![s_1^-, s_2^-]\!]$ and $b \in [\![s_1^+, s_2^+]\!]$, $\vec{x},\vec{y} \in \mathfrak{W}_{k-1}$, $z_1, z_2, z_1^-, z_2^+ \in \mathbb{Z}$, $\ell_{bot} \in \Omega(a,b,z_1,z_2)$, $\ell_{top}^- \in \Omega(s_1^-, a, z_1^-, x_{k-1})$, $\ell_{top}^+ \in \Omega(b, s_2^+, y_{k-1}, z_2^+)$ we define $E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)$ to be the event that $L_i^N(a) = x_i$ and $L_i^N(b) = y_i$ for $1 \leq i \leq k-1$, and L_k^N agrees with ℓ_{bot} on $[\![a,b]\!]$, L_{k-1}^N agrees with ℓ_{top}^- on $[\![s_1^-, a]\!]$ and with ℓ_{top}^+ on $[\![b, s_2^+]\!]$.

We also let D be the collection of tuples $(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)$ satisfying:

(1) $a \in [\![s_1^-, s_2^-]\!]$, $b \in [\![s_1^+, s_2^+]\!]$;
(2) $\vec{x},\vec{y} \in \mathfrak{W}_{k-1}$, $0 \leq y_i - x_i \leq b - a$, $x_{k-1} - pa > -MN^{\alpha/2}$, and $y_{k-1} - pb > -MN^{\alpha/2}$;
(3) if $c \in [\![s_1^-, s_2^-]\!] \cap (-\infty, a)$ then $\ell_{top}^-(c) - pc \leq -MN^{\alpha/2}$;
(4) if $d \in [\![s_1^+, s_2^+]\!] \cap (b, \infty)$ then $\ell_{top}^+(d) - pd \geq -MN^{\alpha/2}$;
(5) $z_1 \leq x_{k-1}$, $z_2 \leq y_{k-1}$, and $\ell_{bot} \in \Omega(a,b,z_1,z_2)$.

It is clear that D is countable, and that

$$B = \bigcup_{(a,b,\vec{x},\vec{y},\ell_{bot}) \in D} E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+),$$

so to prove (5.38) it suffices to show that the events $E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)$ are pairwise disjoint. Observe that on the intersection of $E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)$ and $E(\tilde{a},\tilde{b},\tilde{\vec{x}},\tilde{\vec{y}},\tilde{\ell}_{bot},\tilde{\ell}_{top}^-,\tilde{\ell}_{top}^+)$, conditions (2) and (3) imply that $a = \tilde{a}$, while conditions (2) and (4) that $b = \tilde{b}$. Afterwards, we conclude that $\vec{x} = \tilde{\vec{x}}$, $\vec{y} = \tilde{\vec{y}}$, $\ell_{bot} = \tilde{\ell}_{bot}$, $\ell_{top}^- = \tilde{\ell}_{top}^-$ and $\ell_{top}^+ = \tilde{\ell}_{top}^+$, confirming (5.38).

Step 3. In this step we prove (5.37). We claim that we can find $\tilde{N}_3 \in \mathbb{N}$ such that if $N \geq \tilde{N}_3$ and $(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+) \in D$ is such that $\mathbb{P}\left(E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)\right) > 0$ we have

$$\mathbb{P}(A \,|\, E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)) < \epsilon/2. \tag{5.39}$$

We will prove (5.39) in the steps below. Here we assume its validity and conclude the proof of (5.37).

If $N \geq \max(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3)$ we have in view of (5.38) and (5.39) that

$$\mathbb{P}(A) \leq \mathbb{P}(A \cap B) + \mathbb{P}(B^c) = \mathbb{P}(B^c) + \sum_{\substack{(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+) \in D \\ \mathbb{P}\left(E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)\right) > 0}} \mathbb{P}(A | E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)) \times$$

$$\mathbb{P}(E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)) \leq \mathbb{P}(B^c) + \frac{\epsilon}{2} \sum_{\substack{(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+) \in D \\ \mathbb{P}\left(E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)\right) > 0}} \mathbb{P}(E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)) =$$

$$\mathbb{P}(B^c) + \frac{\epsilon}{2} \cdot \mathbb{P}(B) < \epsilon,$$

where in the last inequality we used (5.33). The above inequality clearly implies (5.37).

Step 4. In this step we prove (5.39). We claim that there exists $\tilde{N}_4 \in \mathbb{N}$ such that if $N \geq \tilde{N}_4$, $a \in [\![s_1^-, s_2^-]\!]$ and $b \in [\![s_1^+, s_2^+]\!]$ we have that $\prod_{i=1}^{k-1} |\Omega(a,b,x_i',y_i')| \geq 1$ and

$$\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}(Q_1 \geq \cdots \geq Q_{k-1}) \geq \frac{1}{2}, \tag{5.40}$$

where $\mathfrak{Q} = (Q_1, \dots, Q_{k-1})$ is $\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}$-distributed and we recall that $\vec{x}'$, $\vec{y}'$ were defined in (5.34). We will prove (5.40) in Step 5 below. Here we assume its validity and conclude the proof of (5.39).

Observe that by condition (2) in Step 2, we have that $x_i' \leq pa - MN^{\alpha/2} \leq x_{k-1} \leq x_i$, and similarly $y_i' \leq pb - MN^{\alpha/2} \leq y_{k-1} \leq y_i$ for $i = 1, \dots, k-1$. From this observation we conclude that if $N \geq \tilde{N}_4$ is sufficiently large and $(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell_{top}^-, \ell_{top}^+) \in D$ is such that $\mathbb{P}\left(E(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell_{top}^-, \ell_{top}^+)\right) > 0$ we have

$$\begin{aligned}
&\mathbb{P}(A | E(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell_{top}^-, \ell_{top}^+)) \leq \\
&\mathbb{P}\left(\inf_{s\in[a,b]} \left(L_{k-1}^N(s) - ps\right) \leq -R_2 N^{\alpha/2} \,\middle|\, E(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell_{top}^-, \ell_{top}^+)\right) = \\
&\mathbb{P}_{avoid,Ber}^{a,b,\vec{x},\vec{y},\infty,\ell_{bot}}\left(\inf_{s\in[a,b]} (Q_{k-1}(s) - ps) \leq -R_2 N^{\alpha/2}\right) \leq \\
&\mathbb{P}_{avoid,Ber}^{a,b,\vec{x}',\vec{y}'}\left(\inf_{s\in[a,b]} \left(Q_{k-1}(s) - ps\right) \leq -R_2 N^{\alpha/2}\right) = \\
&\frac{\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}\left(\{\inf_{s\in[a,b]}\left(Q_{k-1}(s) - ps\right) \leq -R_2 N^{\alpha/2}\} \cap \{Q_1 \geq \cdots \geq Q_{k-1}\}\right)}{\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}(Q_1 \geq \cdots \geq Q_{k-1})} \leq \\
&\frac{\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}\left(\inf_{s\in[a,b]}\left(Q_{k-1}(s) - ps\right) \leq -R_2 N^{\alpha/2}\right)}{\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}(Q_1 \geq \cdots \geq Q_{k-1})}.
\end{aligned} \tag{5.41}$$

Let us elaborate on (5.41) briefly. The first inequality in (5.41) follows from the definition of A and the fact that $a \leq -t_3$ while $b \geq t_3$ by construction. The condition $\mathbb{P}\left(E(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell_{top}^-, \ell_{top}^+)\right) > 0$ ensures that the first three probabilities in (5.41) are all well-defined. The equality on the second line follows from the Schur Gibbs property and the inequality on the third line follows from Lemmas 3.1 and 3.2 since $x_i' \leq x_i$ and $y_i' \leq y_i$ by construction. To ensure that the probability in the fourth line is well-defined (and hence Lemmas 3.1 and 3.2 are applicable) it suffices to assume that $N \geq \tilde{N}_4$, in view of Lemma 2.16. The equality on the fourth line follows from the definition of $\mathbb{P}_{avoid,Ber}^{a,b,\vec{x}',\vec{y}'}$, see Definition 2.15 and the last inequality is trivial.

By our choice of R_2, see (5.35), we know that there is $\tilde{N}_5 \in \mathbb{N}$ such that if $N \geq \tilde{N}_5$

$$\begin{aligned}
&\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}\left(\inf_{s\in[a,b]} \left(Q_{k-1}(s) - ps\right) \leq -R_2 N^{\alpha/2}\right) = \\
&\mathbb{P}_{Ber}^{0,b-a,0,z_{k-1}}\left(\inf_{s\in[0,b-a]} \left(\ell(s) - ps\right) \leq -R_2 N^{\alpha/2} - x_{k-1}'\right) \leq \\
&\mathbb{P}_{Ber}^{0,b-a,0,z_{k-1}}\left(\inf_{s\in[0,b-a]} \left(\ell(s) - ps\right) \leq -(R_2 - M - C_1 k) N^{\alpha/2}\right) < \epsilon/4.
\end{aligned} \tag{5.42}$$

Combining (5.40), (5.41) and (5.41) we conclude that for $N \geq \tilde{N}_3 = \max(\tilde{N}_4, \tilde{N}_5)$ we have

$$\mathbb{P}(A | E(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell_{top}^-, \ell_{top}^+)) < 2 \cdot \epsilon/4 = \epsilon/2,$$

which implies (5.39).

Step 5. In this final step we prove (5.40).

Lastly, we prove that we can enlarge $\tilde{N}_1$ so that (5.45) holds for $N \geq \tilde{N}_1$. Write $a = a'N^\alpha, b = b'N^\alpha$, and $T = a + b = (a' + b')N^\alpha$. Also let $C' = C/\sqrt{a' + b'}$ with C as in (5.31), so that $x'_i - x'_{i+1} \geq CN^{\alpha/2} = C'\sqrt{T}$ and likewise for y'_i. Note that $|z_{k-1} - pT| \leq 1$. It follows from Lemma 3.14, applied with $\ell_{bot} = -\infty$ and C' in place of C, that for T larger than some T_0,

$$\begin{aligned} &\mathbb{P}_{Ber}^{-a,b,\vec{x}\,',\vec{y}\,'}(L_1 \geq \cdots \geq L_{k-1}) = \mathbb{P}_{Ber}^{0,a+b,\vec{x}\,',\vec{y}\,'}(L_1 \geq \cdots \geq L_{k-1}) \geq \\ &\Big(1 - 3e^{-(C')^2/8p(1-p)}\Big)^{k-1} \geq \Big(1 - 3e^{-C^2/16p(1-p)R}\Big)^{k-1}. \end{aligned} \tag{5.43}$$

Here, we used the fact that $a' + b' \leq 2R$, hence $C' \geq C/\sqrt{2R}$. The constant T_0 depends in particular on C', hence possibly on $a + b$. Referring to the proofs of Lemmas 3.14 and 3.5, we see that the dependency of T_0 on C' amounts to requiring that $e^{-C'\sqrt{T_0}}$ be sufficiently small. But $C' \geq C/\sqrt{2R}$, so for this it suffices to choose T_0 depending on C and R. Moreover, $T \geq 2rN^\alpha$, so as long as $\tilde{N}_1 \geq (T_0/2r)^{1/\alpha}$, we have the bound in (5.43) for $N \geq \tilde{N}_1$ independent $a, b, \vec{x}, \vec{y}$. Our choice of C in (5.31) ensures that the expression on the right in (5.43) is at least $1/2$, proving (5.45).

In this step, we fix $R_2 > 0$ and $\tilde{N}_1$ so that for $N \geq \tilde{N}_1$, we have

$$\mathbb{P}_{Ber}^{-a,b,\vec{x}\,',\vec{y}\,'}\left(\inf_{s\in[0,a+b]}\big(L_{k-1}(s) - ps\big) \leq -R_2N^{\alpha/2}\right) < \epsilon/4, \tag{5.44}$$

$$\mathbb{P}_{Ber}^{-a,b,\vec{x}\,',\vec{y}\,'}(L_1 \geq \cdots \geq L_{k-1}) \geq 1/2. \tag{5.45}$$

Let us first prove (5.44). Writing $\vec{z} = \vec{y}\,' - \vec{x}\,'$, and using the fact that $L_1, \ldots, L_{k-1}$ are independent under $\mathbb{P}_{Ber}^{-a,b,\vec{x}\,',\vec{y}\,'}$, we can rewrite the left hand side of (5.44) as

$$\begin{aligned} &\mathbb{P}_{Ber}^{0,a+b,x'_{k-1},y'_{k-1}}\left(\inf_{s\in[0,a+b]}\big(\ell(s) - p(s-a)\big) \leq -R_2N^{\alpha/2}\right) = \\ &\mathbb{P}_{Ber}^{0,a+b,0,z_{k-1}}\left(\inf_{s\in[0,a+b]}\Big(\ell(s) - ps + pa - \lceil pa + MN^{\alpha/2}\rceil - (k-2)\lceil CN^{\alpha/2}\rceil\Big) \leq -R_2N^{\alpha/2}\right) \leq \\ &\mathbb{P}_{Ber}^{0,a+b,0,z_{k-1}}\left(\inf_{s\in[0,a+b]}\big(\ell(s) - ps\big) \leq -(R_2 - M - Ck)N^{\alpha/2}\right). \end{aligned} \tag{5.46}$$

□

6. Lower bounds on the acceptance probability

6.1. **Proof of Lemma 4.4.** Throughout this section we assume the same notation as in Lemma 4.4, i.e., we assume that we have fixed $k \in \mathbb{N}$, $p \in (0,1)$, $M_1, M_2 > 0$, $\ell_{bot} : [\![-t_3, t_3]\!] \to \mathbb{R} \cup \{-\infty\}$, and $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$ such that $|\Omega_{avoid}(-t_3, t_3, \vec{x}, \vec{y}, \infty, \ell_{bot})| \geq 1$. We also assume that

(1) $\sup_{s\in[-t_3,t_3]}\big[\ell_{bot}(s) - ps\big] \leq M_2(2t_3)^{1/2}$,
(2) $-pt_3 + M_1(2t_3)^{1/2} \geq x_1 \geq x_{k-1} \geq \max\big(\ell_{bot}(-t_3), -pt_3 - M_1(2t_3)^{1/2}\big)$,
(3) $pt_3 + M_1(2t_3)^{1/2} \geq y_1 \geq y_{k-1} \geq \max\big(\ell_{bot}(t_3), pt_3 - M_1(2t_3)^{1/2}\big)$.

Definition 6.1. We write $S = [\![-t_3, -t_1]\!] \cup [\![t_1, t_3]\!]$, and we denote by $\mathfrak{Q} = (Q_1, \ldots, Q_{k-1})$ and $\tilde{\mathfrak{Q}} = (\tilde{Q}_1, \ldots, \tilde{Q}_{k-1})$ the $[\![1, k-1]\!]$-indexed line ensembles which are uniformly distributed on $\Omega_{avoid}(-t_3, t_3, \vec{x}, \vec{y}, \ell_{bot})$ and $\Omega_{avoid}(-t_3, t_3, \vec{x}, \vec{y}, \ell_{bot}; S)$ respectively. We let $\mathbb{P}_{\mathfrak{Q}}$ and $\mathbb{P}_{\tilde{\mathfrak{Q}}}$ denote these uniform measures.

In other words, $\tilde{\mathfrak{Q}}$ has the law of $k-1$ independent Bernoulli bridges that have been conditioned on not-crossing each other on the set S and also staying above the graph of ℓ_{bot} but only on the intervals $[\![-t_3,-t_1]\!]$ and $[\![t_1,t_3]\!]$. The latter restriction means that the lines are allowed to cross on $[\![-t_1+1,t_1-1]\!]$, and $\tilde{Q}_{k-1}$ is allowed to dip below ℓ_{bot} on $[\![-t_1+1,t_1-1]\!]$ as well. Essentially, the line ensemble is free on $[\![-t_1+1,t_1-1]\!]$, and avoiding on S.

Lemma 6.2. *There exists $N_5 \in \mathbb{N}$ such that for $N \geq N_5$,*

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\Big(Z\big(-t_1,t_1,\tilde{\mathfrak{Q}}(-t_1),\tilde{\mathfrak{Q}}(t_1),\ell_{bot}[\![-t_1,t_1]\!]\big)\geq g\Big)\geq h, \tag{6.1}$$

where the constants g and h are as in Lemma 4.4.

We will prove Lemma 6.2 in Section 6.2. In the remainder of this section, we give the proof of Lemma 4.4. The proof begins by evaluating the Radon-Nidokum derivative between $\mathbb{P}_{\mathfrak{Q}'}$ and $\mathbb{P}_{\tilde{\mathfrak{Q}}'}$. We then use this Radon-Nikodym derivative to transition between $\tilde{\mathfrak{Q}}$ in Lemma 6.2 which ignores ℓ_{bot} on $[\![-(t_1-1),t_1-1]\!]$ and $\mathfrak{Q}$ in Lemma 4.4 which avoids ℓ_{bot} everywhere. Then we perform some calculations to achieve the desired statement in Equation (4.2).

Lemma 4.4. Let us denote by $\mathbb{P}_{\mathfrak{Q}'}$ and $\mathbb{P}_{\tilde{\mathfrak{Q}}'}$ the measures on $[\![1,k-1]\!]$-indexed Bernoulli line ensembles $\mathfrak{Q}'$, $\tilde{\mathfrak{Q}}'$ on the set S in Definition 6.1 induced by the restrictions of the measures $\mathbb{P}_{\mathfrak{Q}}$, $\mathbb{P}_{\tilde{\mathfrak{Q}}}$ to S. Also let us write $\Omega_a(\cdot)$ for $\Omega_{avoid}(\cdot)$ for simplicity, and denote by $\Omega_a(S)$ the set of elements of $\Omega_{avoid}(-t_3,t_3,\tilde{\mathfrak{Q}}(-t_3),\tilde{\mathfrak{Q}}(t_3))$ restricted to S. We claim that the Radon-Nikodym derivative between these two restricted measures is given on elements $\mathfrak{B}$ of $\Omega_a(S)$ by

$$\frac{d\mathbb{P}_{\mathfrak{Q}'}}{d\mathbb{P}_{\tilde{\mathfrak{Q}}'}}(\mathfrak{B})=\frac{\mathbb{P}_{\mathfrak{Q}'}(\mathfrak{B})}{\mathbb{P}_{\tilde{\mathfrak{Q}}'}(\mathfrak{B})}=(Z')^{-1}Z\left(-t_1,t_1,\mathfrak{B}\left(-t_1\right),\mathfrak{B}\left(t_1\right),\ell_{bot}[\![-t_1,t_1]\!]\right), \tag{6.2}$$

with $Z'=\mathbb{E}_{\tilde{\mathfrak{Q}}'}\left[Z\left(-t_1,t_1,\mathfrak{B}(-t_1),\mathfrak{B}(t_1),\ell_{bot}[\![-t_1,t_1]\!]\right)\right]$. The first equality holds simply because the measures are discrete. To prove the second equality, observe that

$$\begin{aligned}\mathbb{P}_{\mathfrak{Q}'}(\mathfrak{B})&=\frac{|\Omega_a(-t_1,t_1,\mathfrak{B}(-t_1),\mathfrak{B}(t_1),\ell_{bot}[\![-t_1,t_1]\!])|}{|\Omega_a(-t_3,t_3,\mathfrak{Q}(-t_3),\mathfrak{Q}(t_3),\ell_{bot})|},\\ \mathbb{P}_{\tilde{\mathfrak{Q}}'}(\mathfrak{B})&=\frac{\prod_{i=1}^{k-1}|\Omega(-t_1,t_1,B_i(-t_1),B_i(t_1))|}{|\Omega_a(-t_3,t_3,\tilde{\mathfrak{Q}}(-t_3),\tilde{\mathfrak{Q}}(t_3),\ell_{bot};S)|}\end{aligned} \tag{6.3}$$

These identities follow from the restriction, and the fact that the measures are uniform. Then, from Definition 2.22,

$$Z(-t_1,t_1,\mathfrak{B}(-t_1),\mathfrak{B}(t_1),\ell_{bot})=\frac{|\Omega_a(-t_1,t_1,\mathfrak{B}(-t_1),\mathfrak{B}(t_1),\ell_{bot}[\![-t_1,t_1]\!])|}{\prod_{i=1}^{k-1}|\Omega(-t_1,t_1,B_i(-t_1),B_i(t_1))|}$$

and hence

$$\begin{aligned}Z'&=\sum_{\mathfrak{B}\in\Omega_a(S)}\frac{\prod_{i=1}^{k-1}|\Omega(-t_1,t_1,B_i(-t_1),B_i(t_1))|}{|\Omega_a(-t_3,t_3,\tilde{\mathfrak{Q}}(-t_3),\tilde{\mathfrak{Q}}(t_3),\ell_{bot};S)|}\cdot\frac{|\Omega_a(-t_1,t_1,\mathfrak{B}(-t_1),\mathfrak{B}(t_1),\ell_{bot})|}{\prod_{i=1}^{k-1}|\Omega(-t_1,t_1,B_i(-t_1),B_i(t_1))|}=\\ &\frac{\sum_{\mathfrak{B}\in\Omega_a(S)}|\Omega_a(-t_1,t_1,\mathfrak{B}(-t_1),\mathfrak{B}(t_1),\ell_{bot})|}{|\Omega_a(-t_3,t_3,\tilde{\mathfrak{Q}}(-t_3),\tilde{\mathfrak{Q}}(t_3),\ell_{bot};S)|}=\frac{|\Omega_a(-t_3,t_3,\mathfrak{Q}(-t_3),\mathfrak{Q}(t_3),\ell_{bot})|}{|\Omega_a(-t_3,t_3,\tilde{\mathfrak{Q}}(-t_3),\tilde{\mathfrak{Q}}(t_3),\ell_{bot};S)|}.\end{aligned}$$

Comparing the above identities proves the second equality in (6.2).

Now note that $Z\left(-t_1,t_1,\mathfrak{B}(-t_1),\mathfrak{B}(t_1),\ell_{bot}[\![-t_1,t_1]\!]\right)$ is a deterministic function of $((\mathfrak{B}(-t_1),\mathfrak{B}(t_1))$. In fact, the law of $((\mathfrak{B}(-t_1),\mathfrak{B}(t_1))$ under $\mathbb{P}_{\tilde{\mathfrak{Q}}'}$ is the same as that of $\big(\tilde{\mathfrak{Q}}(-t_1),\tilde{\mathfrak{Q}}(t_1)\big)$ by way of the restriction. It follows from Lemma 6.2 that

$$\begin{aligned}Z'&=\mathbb{E}_{\tilde{\mathfrak{Q}}'}\left[Z\left(-t_1,t_1,\mathfrak{B}(-t_1),\mathfrak{B}(t_1),\ell_{bot}[\![-t_1,t_1]\!]\right)\right]\\ &=\mathbb{E}_{\tilde{\mathfrak{Q}}}\left[Z\left(-t_1,t_1,\mathfrak{Q}(-t_1),\mathfrak{Q}(t_1),\ell_{bot}[\![-t_1,t_1]\!]\right)\right]\geq gh,\end{aligned}$$

which gives us

$$(Z')^{-1} \leq \frac{1}{gh}. \tag{6.4}$$

Similarly, the law of $(\mathfrak{B}(-t_1), \mathfrak{B}(t_1))$ under $\mathbb{P}_{\mathfrak{Q}'}$ is the same as that of $(\mathfrak{Q}(-t_1), \mathfrak{Q}(t_1))$ under $\mathbb{P}_{\mathfrak{Q}}$. Hence

$$\begin{aligned} &\mathbb{P}_{\mathfrak{Q}}\Big(Z(-t_1, t_1, \mathfrak{Q}(-t_1), \mathfrak{Q}(t_1), \ell_{bot}[\![-t_1, t_1]\!]) \leq gh\tilde{\epsilon}\Big) = \\ &\qquad \mathbb{P}_{\mathfrak{Q}'}\Big(Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right) \leq gh\tilde{\epsilon}\Big). \end{aligned} \tag{6.5}$$

Now let us write $E = \{Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right) \leq gh\tilde{\epsilon}\} \subset \Omega_a(S)$. Then according to (6.2), we have

$$\mathbb{P}_{\mathfrak{Q}'}(E) = \int_{\Omega_a(S)} \mathbf{1}_E \, d\mathbb{P}_{\mathfrak{Q}'} = (Z')^{-1} \int_{\Omega_a(S)} \mathbf{1}_E \cdot Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right) d\mathbb{P}_{\tilde{\mathfrak{Q}}'}(\mathfrak{B}).$$

From the definition of E, the inequality (6.4), and the fact that $\mathbf{1}_E \leq 1$, it follows that

$$\mathbb{P}_{\mathfrak{Q}'}(E) \leq (Z')^{-1} \int_{\Omega_a(S)} \mathbf{1}_E \cdot gh\tilde{\epsilon} \, d\mathbb{P}_{\tilde{\mathfrak{Q}}'} \leq \frac{1}{gh} \int_{\Omega_a(S)} gh\tilde{\epsilon} \, d\mathbb{P}_{\tilde{\mathfrak{Q}}'} \leq \tilde{\epsilon}.$$

In combination with (6.5), this proves (4.2).

□

6.2. **Proof of Lemma 6.2.** In this section, we prove Lemma 6.2. We first state and prove two auxiliary lemmas necessary for the proof. The first lemma establishes a set of conditions under which we have the desired lower bound on the acceptance probability.

Lemma 6.3. *Let $\epsilon > 0$ and $V^{top} > 0$ be given such that $V^{top} > M_2 + 6(k-1)\epsilon$. Suppose further that $\vec{a}, \vec{b} \in \mathfrak{W}_{k-1}$ are such that*

(1) $V^{top}(2t_3)^{1/2} \geq a_1 + pt_1 \geq a_{k-1} + pt_1 \geq (M_2 + 2\epsilon)(2t_3)^{1/2}$;
(2) $V^{top}(2t_3)^{1/2} \geq b_1 - pt_1 \geq b_{k-1} - pt_1 \geq (M_2 + 2\epsilon)(2t_3)^{1/2}$;
(3) $a_i - a_{i+1} \geq 3\epsilon(2t_3)^{1/2}$ and $b_i - b_{i+1} \geq 3\epsilon(2t_3)^{1/2}$ for $i = 1, \dots, k-2$.

Then we can find $g = g(\epsilon, V^{top}, M_2) > 0$ and $N_6 \in \mathbb{N}$ such that for all $N \geq N_6$ we have

$$Z\big(-t_1, t_1, \vec{a}, \vec{b}, \ell_{bot}[\![-t_1, t_1]\!]\big) \geq g. \tag{6.6}$$

Proof. Observe by the rightmost inequalities in conditions (1) and (2) in the hypothesis, as well as condition (1) in Lemma 4.4, that ℓ_{bot} lies a distance of at least $2\epsilon(2t_3)^{1/2} \geq 2\epsilon(2t_1)^{1/2}$ uniformly below the line segment connecting a_{k-1} and b_{k-1}. Also note that (1) and (2) imply $|b_i - a_i - 2pt_1| \leq (V^{top} - M_2 - 2\epsilon)(2t_3)^{1/2}$ for each i. Lastly noting (3), we see that the conditions of Lemma 3.14 are satisfied with $C = 2\epsilon$. This implies (6.6), with

$$g = \left(\frac{1}{2} - \sum_{n=1}^{\infty} (-1)^{n-1} e^{-\epsilon^2 n^2/2p(1-p)}\right)^{k-1}.$$

□

The next lemma helps us derive the lower bound h in (6.1).

Lemma 6.4. *For any $R > 0$ we can find $V_1^t, V_1^b \geq M_2 + R$, $h_1 > 0$ and $N_7 \in \mathbb{N}$ (depending on R) such that if $N \geq N_7$ we have*

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left((2t_3)^{1/2} V_1^t \geq \tilde{Q}_1(\pm t_2) \mp pt_2 \geq \tilde{Q}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq h_1. \tag{6.7}$$

Proof. We first define the constants V_1^b and h_1, as well as two other constants C and K_1 to be used in the proof. We put

$$\begin{aligned} C &= \sqrt{8p(1-p)\log\frac{3}{1-(11/12)^{1/(k-2)}}}, \\ V_1^b &= M_1 + Ck + M_2 + R, \qquad K_1 = (4r+10)V_1^b, \\ h_1 &= \frac{2^{k/2-5}\left(1-2e^{-4/p(1-p)}\right)^{2k}}{(\pi p(1-p))^{k/2}} \exp\left(-\frac{2k(K_1+M_1+6)^2}{p(1-p)}\right). \end{aligned} \tag{6.8}$$

Note in particular that $V_1^b > M_2 + R$. We will fix $V_1^t > V_1^b$ in Step 3 below depending on h_1. We will prove in the following steps that for these choices of V_1^b, V_1^t, h_1, we can find N_7 so that for $N \geq N_7$ we have

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq 2h_1, \tag{6.9}$$

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_1(\pm t_2) \mp pt_2 > (2t_3)^{1/2} V_1^t\right) \leq h_1. \tag{6.10}$$

Assuming the validity of the claim, we then observe that the probability in (6.7) is bounded below by $2h_1 - h_1 = h_1$, proving the lemma. We will prove (6.9) and (6.10) in three steps.

Step 1. In this step we prove that there exists N_7 so that (6.9) holds for $N \geq N_7$, assuming results from Step 2 below. We condition on the value of $\tilde{\mathfrak{Q}}$ at 0 and use the Schur Gibbs property to divide $\tilde{\mathfrak{Q}}$ into two independent line ensembles on $[-t_3, 0]$ and $[0, t_3]$. Observe by Lemma 3.2 that

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq \mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}\left(\tilde{Q}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b\right). \tag{6.11}$$

With K_1 as in (6.8), we define events

$$E_{\vec{z}} = \left\{\left(\tilde{Q}_1(0), \dots, \tilde{Q}_{k-1}(0)\right) = \vec{z}\right\}, \quad X = \left\{\vec{z} \in \mathfrak{W}_{k-1} : z_{k-1} \geq K_1(2t_3)^{1/2} \text{ and } \mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}(E_{\vec{z}}) > 0\right\},$$

and $E = \bigsqcup_{\vec{z} \in X} E_{\vec{z}}$. By Lemma 2.16, we can choose $\tilde{N}_0$ large enough depending on M_1, C, k, M_2, R so that X is non-empty for $N \geq \tilde{N}_0$. By Lemma 3.16 we can find $\tilde{N}_1$ so that

$$\mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}(E) \geq \mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}\left(\tilde{Q}_{k-1}(0) \geq K_1(2t_3)^{1/2}\right) \geq A\exp\left(-\frac{2k(K_1+M_1+6)^2}{p(1-p)}\right) \tag{6.12}$$

for $N \geq \tilde{N}_1$, where $A = A(p,k)$ is a constant given explicitly in (3.22).

Now let $\tilde{Q}_i^1$ and $\tilde{Q}_i^2$ denote the restrictions of $\tilde{Q}_i$ to $[-t_3, 0]$ and $[0, t_3]$ respectively for $1 \leq i \leq k-1$, and write $S_1 = S \cap [\![-t_3, 0]\!]$, $S_2 = S \cap [\![0, t_3]\!]$. We observe that if $\vec{z} \in X$, then

$$\mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}\left(\tilde{Q}^1_{k-1} = \ell_1, \tilde{Q}^2_{k-1} = \ell_2 \,|\, E_{\vec{z}}\right) = \mathbb{P}^{-t_3,0,\vec{x},\vec{z}}_{avoid,Ber;S_1}(\ell_1) \cdot \mathbb{P}^{0,t_3,\vec{z},\vec{y}}_{avoid,Ber;S_2}(\ell_2). \tag{6.13}$$

In Step 2, we will find $\tilde{N}_2$ so that for $N \geq \tilde{N}_2$ we have

$$\begin{aligned} &\mathbb{P}^{-t_3,0,\vec{x},\vec{z}}_{avoid,Ber;S_1}\left(\tilde{Q}^1_{k-1}(-t_2) + pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq \frac{1}{4}, \\ &\mathbb{P}^{0,t_3,\vec{x},\vec{z}}_{avoid,Ber;S_2}\left(\tilde{Q}^2_{k-1}(t_2) - pt_{12} \geq (2t_3)^{1/2} V_1^b\right) \geq \frac{1}{4}. \end{aligned} \tag{6.14}$$

Using (6.12), (6.13), and (6.14), we conclude that

$$\mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}\left(\tilde{Q}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq \frac{A}{16}\exp\left(-\frac{2k(K_1+M_1+6)^2}{p(1-p)}\right)$$

for $N \geq N_7 = \max(\tilde{N}_0, \tilde{N}_1, \tilde{N}_2)$. In combination with (6.11), this proves (6.9) with $h_1 = A/16$ as in (6.8).

Step 2. In this step, we prove the inequalities in (6.14) from Step 1, using Lemma 3.8. Let us define vectors $\vec{x}\,', \vec{z}\,', \vec{y}\,'$ by

$$\begin{aligned} x_i' &= \lfloor -pt_3 - M_1(2t_3)^{1/2} \rfloor - (i-1)\lceil C(2t_3)^{1/2} \rceil, \\ z_i' &= \lfloor K_1(2t_3)^{1/2} \rfloor - (i-1)\lceil C(2t_3)^{1/2} \rceil, \\ y_i' &= \lfloor pt_3 - M_1(2t_3)^{1/2} \rfloor - (i-1)\lceil C(2t_3)^{1/2} \rceil. \end{aligned}$$

Note that $x_i' \leq x_{k-1} \leq x_i$ and $x_i' - x_{i+1}' \geq C(2t_3)^{1/2}$ for $1 \leq i \leq k-1$, and likewise for z_i', y_i'. By Lemma 3.1 we have

(6.15)
$$\mathbb{P}^{-t_3,0,\vec{x},\vec{z}}_{avoid,Ber;S_1}\left(\tilde{Q}^1_{k-1}(-t_2) + pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq \mathbb{P}^{-t_3,0,\vec{x}\,',\vec{z}\,'}_{avoid,Ber;S_1}\left(\tilde{Q}^1_{k-1}(-t_2) + pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq$$
$$\mathbb{P}^{-t_3,0,x_{k-1}',z_{k-1}'}_{Ber}\left(\ell_1(-t_2) + pt_2 \geq (2t_3)^{1/2} V_1^b\right) - \left(1 - \mathbb{P}^{-t_3,t_3,\vec{x}\,',\vec{z}\,'}_{Ber}\left(\tilde{Q}^1_1 \geq \cdots \geq \tilde{Q}^1_{k-1}\right)\right).$$

To bound the first term on the second line, first note that $x_{k-1}' \geq -pt_3 - (M_1 + C(k-1))(2t_3)^{1/2}$ and $z_{k-1}' \geq K_1(2t_3)^{1/2} - C(k-1)(2t_3)^{1/2}$ for sufficiently large N. Let us write $\tilde{x}, \tilde{z}$ for these two lower bounds. Then by Lemma 3.8, we have an $\tilde{N}_3$ so that for $N \geq \tilde{N}_3$,

$$\mathbb{P}^{-t_3,0,x_{k-1}',z_{k-1}'}_{Ber}\left(\ell_1(-t_2) \geq \frac{t_2}{t_3}\,\tilde{x} + \frac{t_3 - t_2}{t_3}\,\tilde{z} - (2t_3)^{1/4}\right) \geq \frac{1}{3}. \tag{6.16}$$

Moreover, as long as $\tilde{N}_3^\alpha > 2$, we have for $N \geq \tilde{N}_3^\alpha$ that

$$\frac{t_3 - t_2}{t_3} \geq 1 - \frac{(r+2)N^\alpha}{(r+3)N^\alpha - 1} > 1 - \frac{r+2}{r+5/2} = \frac{1}{2r+5}. \tag{6.17}$$

It follows from our choice of V_1^b and $K_1 = 2(2r+5)V_1^b$ in (6.8), as well as (6.17), that

$$\frac{t_2}{t_3}\,\tilde{x} + \frac{t_3 - t_2}{t_3}\,\tilde{z} - (2t_3)^{1/4} = -pt_2 - C(k-1)(2t_3)^{1/2} - \frac{t_2}{t_3}\,M_1(2t_3)^{1/2} + \frac{t_3 - t_2}{t_3}\,K_1(2t_3)^{1/2} - (2t_3)^{1/4} \geq$$
$$-pt_2 - Ck(2t_3)^{1/2} - M_1(2t_3)^{1/2} + \frac{1}{2r+5}\,K_1(2t_3)^{1/2} = -pt_2 + (M_1 + Ck + 2(M_2 + R))(2t_3)^{1/2} >$$
$$-pt_2 + (2t_3)^{1/2} V_1^b.$$

For the first inequality, we used the fact that $t_2/t_3 < 1$, and we assumed that $\tilde{N}_3$ is sufficiently large so that $C(k-1)(2t_3)^{1/2} + (2t_3)^{1/4} \leq Ck(2t_3)^{1/2}$ for $N \geq \tilde{N}_3$. Using (6.16), we conclude for $N \geq \tilde{N}_3$ that

$$\mathbb{P}^{-t_3,0,x_{k-1}',z_{k-1}'}_{Ber}\left(\ell_1(-t_2) + pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq \frac{1}{3}. \tag{6.18}$$

Since $|z_i' - x_i' - pt_2| \leq (K_1 + M_1 + 1)(2t_2)^{1/2}$, we have by Lemma 3.14 and our choice of C that the second probability in the second line of (6.15) is bounded below by

$$\left(1 - 3e^{-C^2/8p(1-p)}\right)^{k-1} \geq 11/12$$

for N larger than some $\tilde{N}_4$. It follows from (6.15) and (6.18) that for $N \geq \tilde{N}_2 = \max(\tilde{N}_3, \tilde{N}_4)$,

$$\mathbb{P}^{-t_3,0,\vec{x},\vec{z}}_{avoid,Ber;S_1}\left(\tilde{Q}^1_{k-1}(-t_2) + pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq \frac{1}{3} - \frac{1}{12} = \frac{1}{4},$$

proving the first inequality in (6.14). The second inequality is proven similarly.

Step 3. In this last step, we fix V_1^t and prove that we can enlarge N_7 from Step 1 so that (6.10) holds for $N \geq N_7$. Let C be as in (6.8), and define vectors $\vec{x}\,'', \vec{y}\,'' \in \mathfrak{W}_{k-1}$ by

$$x_i'' = \lceil -pt_3 + M_1(2t_3)^{1/2} \rceil + (k-i)\lceil C(2t_3)^{1/2} \rceil,$$
$$y_i'' = \lceil pt_3 + M_1(2t_3)^{1/2} \rceil + (k-i)\lceil C(2t_3)^{1/2} \rceil.$$

Note that $x_i'' \geq x_1 \geq x_i$ and $x_i'' - x_{i+1}'' \geq C(2t_3)^{1/2}$, and likewise for y_i''. Moreover, ℓ_{bot} lies a distance of at least $C(2t_3)^{1/2}$ uniformly below the line segment connecting x_{k-1}'' and y_{k-1}''. By Lemma 3.1 we have

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_1(\pm t_2) \mp pt_2 > (2t_3)^{1/2}V_1^t\right) \leq \mathbb{P}^{-t_3,t_3,\vec{x}\,'',\vec{y}\,'',\infty,\ell_{bot}}_{avoid,Ber;S}\left(\sup_{s\in[-t_3,t_3]}\left[\tilde{Q}_1(s) - ps\right] \geq (2t_3)^{1/2}V_1^t\right) \leq$$
$$\frac{\mathbb{P}^{-t_3,t_3,x_1'',y_1''}_{Ber}\left(\sup_{s\in[-t_3,t_3]}\left[\tilde{L}_1(s) - ps\right] \geq (2t_3)^{1/2}V_1^t\right)}{\mathbb{P}^{-t_3,t_3,\vec{x}\,'',\vec{y}\,''}_{Ber}\left(\tilde{L}_1 \geq \cdots \geq \tilde{L}_{k-1} \geq \ell_{bot}\right)}.$$

In the numerator in the second line, we used the fact that the curves $\tilde{L}_1, \dots, \tilde{L}_{k-1}$ are independent under $\mathbb{P}^{-t_3,t_3,x_1'',y_1''}_{Ber}$, and the event in the parentheses depends only on $\tilde{L}_1$. By Lemma 3.10, since $\min(x_1'' + pt_3,\, y_1'' - pt_3) \leq (M_1 + C(k-1))(2t_3)^{1/2}$, we can choose $V_1^t > V_1^b$ as well as $\tilde{N}_5$ large enough so that the numerator is bounded above by $h_1/2$ for $N \geq \tilde{N}_5$. Since $|y_i'' - x_i'' - 2pt_3| \leq 1$, our choice of C and Lemma 3.14 give a $\tilde{N}_6$ so that the denominator is at least $11/12$ for $N \geq \tilde{N}_6$. This gives an upper bound of $12/11 \cdot h_1/2 < h_1/2$ in the above as long as $N_7 \geq \max(\tilde{N}_5, \tilde{N}_6)$, proving (6.10).

□

We are now equipped to prove Lemma 6.2. Let us put

$$t_{12} = \left\lfloor \frac{t_1 + t_2}{2} \right\rfloor. \tag{6.19}$$

Proof. We first introduce some notation to be used in the proof. Let S be as in Definition 6.1. For $\vec{c}, \vec{d} \in \mathfrak{W}_{k-1}$, let us write $\tilde{S} = [\![-t_2, -t_1]\!] \cup [\![t_1, t_2]\!]$, $\tilde{\Omega}(\vec{c}, \vec{d}) = \Omega_{avoid}(-t_2, t_2, \vec{c}, \vec{d}, \infty, \ell_{bot}; \tilde{S})$. For $s \in \tilde{S}$ we define events

$$\begin{aligned}
A(\vec{c}, \vec{d}, s) &= \left\{\tilde{\mathfrak{Q}} \in \tilde{\Omega}(\vec{c}, \vec{d}) : \tilde{Q}_{k-1}(\pm s) \mp ps \geq (M_2 + 1)(2t_3)^{1/2}\right\}, \\
B(\vec{c}, \vec{d}, V^{top}, s) &= \left\{\tilde{\mathfrak{Q}} \in \tilde{\Omega}(\vec{c}, \vec{d}) : \tilde{Q}_1(\pm s) \mp ps \leq V^{top}(2t_3)^{1/2}\right\}, \\
C(\vec{c}, \vec{d}, \epsilon, s) &= \left\{\tilde{\mathfrak{Q}} \in \tilde{\Omega}(\vec{c}, \vec{d}) : \min_{1\leq i\leq k-2,\, \varsigma\in\{-1,1\}}\left[\tilde{Q}_i(\varsigma s) - \tilde{Q}_{i+1}(\varsigma s)\right] \geq 3\epsilon(2t_3)^{1/2}\right\}, \\
D(\vec{c}, \vec{d}, V^{top}, \epsilon, s) &= A(\vec{c}, \vec{d}, s) \cap B(\vec{c}, \vec{d}, V^{top}, s) \cap C(\vec{c}, \vec{d}, \epsilon, s).
\end{aligned} \tag{6.20}$$

Here, ϵ and V^{top} are constants which we will specify later. By Lemma 6.3, for all $(\vec{c}, \vec{d})$ and N sufficiently large we have

$$D(\vec{c}, \vec{d}, V^{top}, \epsilon, s) \subset \{Z\left(-t_1, t_1, \mathfrak{Q}(-t_1), \mathfrak{Q}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right) > g\}$$

for some g depending on ϵ, V^{top}, M_2. Thus we will prove that probability of the event on the left under the uniform measure on $\tilde{\Omega}(\vec{c}, \vec{d})$ is bounded below by $h = h_1/2$, with h_1 as in (6.8). We split the proof into several steps.

Step 1. In this step, we show that there exist $R > 0$ and $\tilde{N}_0$ sufficiently large so that if $c_{k-1}+pt_2 \geq (2t_3)^{1/2}(M_2+R)$ and $d_{k-1} - pt_2 \geq (2t_3)^{1/2}(M_2+R)$, then for all $s \in \tilde{S}$ and $N \geq \tilde{N}_0$ we have

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\big(A(\vec{c},\vec{d},s)\big) \geq \frac{19}{20} \quad \text{and} \quad \mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\left(Q_{k-1}|_{\tilde{S}} \geq \ell_{bot}|_{\tilde{S}}\right) \geq \frac{99}{100}. \tag{6.21}$$

Let us begin with the first inequality. We observe via Lemma 3.2 that

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\big(A(\vec{c},\vec{d},s)\big) \geq \mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\big(A(\vec{c},\vec{d},s)\big). \tag{6.22}$$

Now define the constant

$$C = \sqrt{8p(1-p)\log\frac{3}{1-(199/200)^{1/(k-1)}}} \tag{6.23}$$

and vectors $\vec{c}\,', \vec{d}\,' \in \mathfrak{W}_k$ by

$$c_i' = \lfloor -pt_2 + (M_2+R)(2t_3)^{1/2}\rfloor - (i-1)\lceil C(2t_2)^{1/2}\rceil,$$
$$d_i' = \lfloor pt_2 + (M_2+R)(2t_3)^{1/2}\rfloor - (i-1)\lceil C(2t_2)^{1/2}\rceil.$$

Then by Lemma 3.1 we have

$$\begin{aligned}
&\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\big(A(\vec{c},\vec{d},s)\big) \geq \mathbb{P}^{-t_2,t_2,\vec{c}\,',\vec{d}\,'}_{avoid,Ber;\tilde{S}}(A(\vec{c}\,',\vec{d}\,',s)) \geq \\
&\mathbb{P}^{-t_2,t_2,c_{k-1}',d_{k-1}'}_{Ber}\left(\inf_{s\in\tilde{S}}\big[\ell(s)-ps\big] \geq (M_2+1)(2t_3)^{1/2}\right) - \\
&\qquad\left(1-\mathbb{P}^{-t_2,t_2,\vec{c}\,',\vec{d}\,'}_{Ber}\left(L_1 \geq \cdots \geq L_{k-1}\right)\right).
\end{aligned} \tag{6.24}$$

By Lemma 3.14 and our choice of C, we can find $\tilde{N}_0$ so that $\mathbb{P}^{-t_2,t_2,\vec{c}\,',\vec{d}\,'}_{Ber}(L_1 \geq \cdots \geq L_{k-1}) > 199/200 > 39/40$ for $N \geq \tilde{N}_0$. Writing $z = d_{k-1}' - c_{k-1}'$, the term in the second line of (6.24) is equal to

$$\begin{aligned}
&\mathbb{P}^{-t_2,t_2,0,z}_{Ber}\left(\inf_{s\in\tilde{S}}\big[\ell(s)+c_{k-1}'-ps\big] \geq (M_2+1)(2t_3)^{1/2}\right) \geq \\
&\mathbb{P}^{0,2t_2,0,z}_{Ber}\left(\inf_{s\in[0,2t_2]}\big[\ell(s)-ps\big] \geq (-R+Ck+1)(2t_3)^{1/2}\right).
\end{aligned}$$

In the second line, we used the estimate $c_{k-1}' \geq -pt_2 + (M_2+R-Ck)(2t_3)^{1/2}$. Now by Lemma 3.10, we can choose R large enough depending on C, k, M_2, p so that this probability is greater than 39/40 for N greater than some $\tilde{N}_1$. This gives a lower bound in (6.24) of $39/40 - 1/40 = 19/20$ for $N \geq \max(\tilde{N}_0, \tilde{N}_1)$, and in combination with (6.22) this proves the first inequality in (6.21).

We prove the second inequality in (6.21) similarly. Note that since $\ell_{bot}(s) \leq ps + M_2(2t_3)^{1/2}$ on $[-t_3, t_3]$ by assumption, we have

$$\begin{aligned}
&\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\left(\tilde{Q}_{k-1}|_{\tilde{S}} \geq \ell_{bot}|_{\tilde{S}}\right) \geq \mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\left(\inf_{s\in[-t_2,t_2]}\big[Q_{k-1}(s)-ps\big] \geq M_2(2t_3)^{1/2}\right) \geq \\
&\mathbb{P}^{-t_2,t_2,\vec{c}\,',\vec{d}\,'}_{avoid,Ber;\tilde{S}}\left(\inf_{s\in[-t_2,t_2]}\big[\tilde{Q}_{k-1}(s)-ps\big] \geq M_2(2t_3)^{1/2}\right) \geq \\
&\mathbb{P}^{0,2t_2,0,z}_{Ber}\left(\inf_{s\in[0,2t_2]}\big[\ell(s)-ps\big] \geq -(R-Ck)(2t_3)^{1/2}\right) - \\
&\qquad\left(1-\mathbb{P}^{-t_2,t_2,\vec{c}\,',\vec{d}\,'}_{Ber}(\tilde{L}_1 \geq \cdots \geq \tilde{L}_{k-1})\right).
\end{aligned} \tag{6.25}$$

We enlarge R if necessary so that the probability in the third line of (6.25) is $> 199/200$ for $N \geq \tilde{N}_2$ by Lemma 3.10, and 3.14 implies as above that the expression in the last line of (6.25) is $> -1/200$

for $N \geq \tilde{N}_3$. This gives us a lower bound of $199/200 - 1/200 = 99/100$ for $N \geq \tilde{N}_0 = \max(\tilde{N}_2, \tilde{N}_3)$ as desired. This proves the two inequalities in (6.21) for $N \geq \bar{N}_0 = \max(\tilde{N}_0, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3)$.

Step 2. With R fixed from Step 1, let V_1^t, V_1^b, and h_1 be as in Lemma 6.4 for this choice of R. Define the event

$$E = \big\{\vec{c}, \vec{d} \in \mathfrak{W}_{k-1} : (2t_3)^{1/2} V_1^t \geq \max(c_1 + pt_2 d_1 - pt_2) \text{ and } \min(c_{k-1} + pt_2, d_{k-1} - pt_2) \geq (2t_3)^{1/2} V_1^b\big\}. \tag{6.26}$$

We show in this step that there exists $V^{top} \geq M_2 + 6(k-1)$ and $\bar{N}_1$ such that for all $(\vec{c}, \vec{d}) \in E$, $s \in \tilde{S}$, and $N \geq \bar{N}_1$ we have

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\big(B(\vec{c}, \vec{d}, V^{top}, s)\big) \geq \frac{19}{20}. \tag{6.27}$$

Let C be as in (6.23), and define $\vec{c}\,'', \vec{d}\,'' \in \mathfrak{W}_{k-1}$ by

$$c_i'' = \lceil -pt_2 + (2t_3)^{1/2} V_1^t \rceil + (k-1-i)\lceil C(2t_2)^{1/2}\rceil,$$
$$d_i'' = \lceil pt_2 + (2t_3)^{1/2} V_1^t \rceil + (k-1-i)\lceil C(2t_2)^{1/2}\rceil.$$

Then $c_i'' \geq c_1 \geq c_i$ and $c_i'' - c_{i+1}'' \geq C(2t_2)^{1/2}$ for each i, and likewise for d_i''. Furthermore, since $V_1^b \geq M_2 + R$, we see that ℓ_{bot} lies a distance of at least $R(2t_3)^{1/2}$ uniformly below the line segment connecting c_{k-1}'' and d_{k-1}''. By construction, $R > C$. By Lemma 3.1, the left hand side of (6.27) is bounded below by

$$\begin{aligned}
&\mathbb{P}^{-t_2,t_2,\vec{c}\,'',\vec{d}\,'',\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\left(\sup_{s\in\tilde{S}}\big[\tilde{Q}_1(s) - ps\big] \leq V^{top}(2t_3)^{1/2}\right) \geq \\
&\mathbb{P}^{0,2t_2,0,z'}_{Ber}\left(\sup_{s\in[-t_2,t_2]}\big[\ell(s) - ps\big] \leq (V^{top} - V_1^t - Ck)(2t_3)^{1/2}\right) - \\
&\left(1 - \mathbb{P}^{-t_2,t_2,\vec{c}\,'',\vec{d}\,'',\infty,\ell_{bot}}_{Ber}\left(L_1 \geq \cdots \geq L_{k-1} \geq \ell_{bot}\right)\right).
\end{aligned} \tag{6.28}$$

In the last line, we have written $z' = d_1'' - c_1''$, and we used the fact that $c_1'' \leq -pt_2 + (V_1^t + Ck)(2t_3)^{1/2}$. By Lemma 3.10, we can find V^{top} large enough depending on V_1^t, C, k, p so that the probability in the third line of (6.28) is at least $39/40$ for $N \geq \tilde{N}_4$. On the other hand, the above observations regarding $\vec{c}\,''$, $\vec{d}\,''$, and ℓ_{bot}, as well as the fact that $|d_1'' - c_1'' - 2pt_2| \leq 1$, allow us to conclude from Lemma 3.14 that the probability in the last line of (6.28) is at least $39/40$ for $N \geq \tilde{N}_5$. This gives a lower bound of $39/40 - 1/40 = 19/20$ in (6.28) for $\bar{N}_1 = \max(\tilde{N}_4, \tilde{N}_5)$ as desired.

Step 3. In this step, we show that with V_1^t and V_1^b as in Step 2, there exist $\epsilon > 0$ sufficiently small and $\bar{N}_2$ such that for $(\vec{c}, \vec{d}) \in E$ and $N \geq \bar{N}_2$, we have

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\big(D(\vec{c}, \vec{d}, V^{top}, \epsilon, t_{12})\big) \geq \frac{1}{2}. \tag{6.29}$$

We claim that this follows if we find $\tilde{N}_6$ so that for $N \geq \tilde{N}_6$,

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\big(C(\vec{c}, \vec{d}, \epsilon, t_{12}) \,|\, A(\vec{c}, \vec{d}, t_1) \cap B(\vec{c}, \vec{d}, V^{top}, t_1)\big) \geq \frac{9}{10}. \tag{6.30}$$

To see this, note that (6.21) and (6.27) imply that for $N \geq \max(\bar{N}_0, \bar{N}_1)$,

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\big(A(\vec{c}, \vec{d}, t_1) \cap B(\vec{c}, \vec{d}, V^{top}, t_1)\big) \geq \frac{19}{20} - \frac{1}{20} - \frac{1}{100} > \frac{4}{5},$$

and then (6.30) and the second inequality in (6.21) imply that for $N \geq \bar{N}_2 = \max(\bar{N}_0, \bar{N}_1, \tilde{N}_6)$,

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\big(A(\vec{c},\vec{d},t_1) \cap B(\vec{c},\vec{d},V^{top},t_1) \cap C(\vec{c},\vec{d},\epsilon,t_{12})\big) > \frac{9}{10}\cdot\frac{4}{5} - \frac{1}{100} > \frac{17}{25}.$$

Then using (6.21) and (6.27) once again and recalling the definition of $D(\vec{c},\vec{d},V^{top},\epsilon,t_{12})$ gives a lower bound on the probability in (6.29) of $17/25 - 1/10 > 14/25 > 1/2$ for $N \geq \bar{N}_2$ as desired.

In the remainder of this step, we verify (6.30). Observe that $A(\vec{c},\vec{d},t_1) \cap B(\vec{c},\vec{d},V^{top},t_1)$ can be written as a countable disjoint union:

$$A(\vec{c},\vec{d},t_1) \cap B(\vec{c},\vec{d},V^{top},t_1) = \bigsqcup_{(\vec{a},\vec{b})\in I} F(\vec{a},\vec{b}). \tag{6.31}$$

Here, for $\vec{a},\vec{b} \in \mathfrak{W}_{k-1}$, $F(\vec{a},\vec{b})$ is the event that $\mathfrak{Q}(-t_1) = \vec{a}$ and $\mathfrak{Q}(t_1) = \vec{b}$, and I is the collection of pairs $(\vec{a},\vec{b})$ satisfying

(1) $0 \leq \min(a_i - c_i,\, d_i - b_i) \leq t_2 - t_1$ and $0 \leq b_i - a_i \leq 2t_1$ for $1 \leq i \leq k-1$,
(2) $\min(a_{k-1} + pt_1,\, b_{k-1} - pt_1) \geq (M_2 + 1)(2t_3)^{1/2}$,
(3) $\max(a_1 + pt_1,\, b_1 - pt_1) \leq V^{top}(2t_3)^{1/2}$.

Now let $\mathfrak{Q}^1 = (Q^1_1, \dots, Q^1_{k-1})$ and $\mathfrak{Q}^2 = (Q^2_2, \dots, Q^2_{k-1})$ denote the restrictions of $\tilde{\mathfrak{Q}}$ to $[\![-t_2,-t_1]\!]$ and $[\![t_1,t_2]\!]$ respectively. Then we observe that

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\left(\mathfrak{Q}^1 = \mathfrak{B}^1, \mathfrak{Q}^2 = \mathfrak{B}^2 \,\middle|\, F(\vec{a},\vec{b})\right) = \mathbb{P}^{-t_2,-t_1,\vec{c},\vec{a}}_{avoid,Ber}\left(\mathfrak{Q}^1 = \mathfrak{B}^1\right) \cdot \mathbb{P}^{t_1,t_2,\vec{b},\vec{d}}_{avoid,Ber}\left(\mathfrak{Q}^2 = \mathfrak{B}^2\right). \tag{6.32}$$

We now fix $(\vec{a},\vec{b})$ and argue that we can choose $\epsilon > 0$ small enough and $\tilde{N}_7$ so that for $N \geq \tilde{N}_7$,

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\left(C(\vec{c},\vec{d},\epsilon,t_{12}) \,\middle|\, F(\vec{a},\vec{b})\right) \geq \frac{9}{10}. \tag{6.33}$$

We then choose $\tilde{N}_8$ via Lemma 2.16 so that the set $\tilde{I} = \{(\vec{a},\vec{b}) \in I : \mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}(F(\vec{a},\vec{b})) > 0\}$ is nonempty. Then using (6.33) and summing over $\tilde{I}$ proves (6.30) for $N \geq \tilde{N}_6 = \max(\tilde{N}_7, \tilde{N}_8)$.

To prove (6.33), we first show that we can find $\delta > 0$ and $\tilde{N}_7$ so that

$$\mathbb{P}^{-t_2,-t_1,\vec{c},\vec{a}}_{avoid,Ber}\left(\max_{1\leq i\leq k-2}\left[Q^1_i(-t_{12}) - Q^1_{i+1}(-t_{12})\right] \geq \delta(2t_3)^{1/2}\right) \geq \frac{3}{\sqrt{10}} \tag{6.34}$$

for $N \geq \tilde{N}_7$. We prove this inequality using Lemma 3.18. In order to apply this result, we first observe that since $|-t_{12} + \frac{1}{2}(t_1+t_2)| \leq 1$ by (6.19), we have

$$0 \leq Q^1_i(-t_{12}) - Q^1_i(-\tfrac{1}{2}(t_1+t_2)) \leq 1. \tag{6.35}$$

Now applying Lemma 3.18 with $M_1 = V^t_1$, $M_2 = V^{top}$, we obtain $\tilde{N}_7$ and $\delta > 0$ such that if $N \geq \tilde{N}_7$, then

$$\mathbb{P}^{-t_2,-t_1,\vec{c},\vec{a}}_{avoid,Ber}\left(\min_{1\leq i\leq k-1}\left[Q^1_i(-\tfrac{1}{2}(t_1+t_2)) - Q^1_{i+1}(-\tfrac{1}{2}(t_1+t_2))\right] < \delta(t_2-t_1)^{1/2}\right) < 1 - \frac{3}{\sqrt{10}}.$$

Together with (6.35) and the fact that $t_3/4 < t_2 - t_1$, this implies that

$$\mathbb{P}^{-t_2,-t_1,\vec{c},\vec{a}}_{avoid,Ber}\left(\min_{1\leq i\leq k-1}\left[Q^1_i(-t_{12}) - Q^1_{i+1}(-t_{12})\right] < (\delta/2)(2t_3)^{1/2} - 1\right) < 1 - \frac{3}{\sqrt{10}} \tag{6.36}$$

for $N \geq \tilde{N}_7$. Now we observe that as long as $\tilde{N}_7^\alpha \geq \frac{1+8/\delta^2}{r+3}$, then $(\delta/4)(2t_3)^{1/2} \leq (\delta/2)(2t_2)^{1/2} - 1$ for $N \geq \tilde{N}_7$. This implies (6.34). A similar argument gives us a $\tilde{\delta} > 0$ such that

$$\mathbb{P}^{-t_2,-t_1,\vec{c},\vec{a}}_{avoid,Ber}\left(\min_{1\leq i\leq k-1}\left[Q_i(-t_{12}) - Q_{i+1}(-t_{12})\right] < (\tilde{\delta}/4)(2t_3)^{1/2}\right) < 1 - \frac{3}{\sqrt{10}}$$

for $N \geq \tilde{N}_7$. Then putting $\epsilon = \min(\delta, \tilde{\delta})/12$ and using (6.32), we obtain (6.33) for $N \geq \tilde{N}_7$.

Step 4. In this step, we find $\bar{N}_3$ so that

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\big(D(\vec{c},\vec{d},V^{top},\epsilon,t_1)\big) \geq \frac{1}{2}\left(\frac{1}{2} - \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\epsilon^2 n^2/2p(1-p)}\right)^{k-1} \tag{6.37}$$

for $N \geq \bar{N}_3$. We will find $\tilde{N}_9$ so that for $N \geq \tilde{N}_9$,
(6.38)

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\left(D(\vec{c},\vec{d},V^{top},\epsilon,t_1)\,\middle|\,D(\vec{c},\vec{d},V^{top},\epsilon,t_{12})\right) \geq \left(\frac{1}{2} - \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\epsilon^2 n^2/2p(1-p)}\right)^{k-1}.$$

Then (6.29) implies (6.37) for $N \geq \bar{N}_3 = \max(\bar{N}_2, \tilde{N}_9)$.

To prove (6.38) we first observe that we can write

$$D(\vec{c},\vec{d},V^{top},\epsilon,t_{12}) = \bigsqcup_{(\vec{a},\vec{b})\in J} G(\vec{a},\vec{b}). \tag{6.39}$$

Here, for $\vec{a},\vec{b} \in \mathfrak{W}_{k-1}$, $G(\vec{a},\vec{b})$ is the event that $\mathfrak{Q}(-t_{12}) = \vec{a}$ and $\mathfrak{Q}(t_{12}) = \vec{b}$, and J is the collection of $(\vec{a},\vec{b})$ satisfying

(1) $0 \leq \min(a_i - c_i,\, d_i - b_i) \leq t_2 - t_{12}$ and $0 \leq b_i - a_i \leq 2t_{12}$ for $1 \leq i \leq k-1$,
(2) $\min(a_{k-1} + pt_1,\, b_{k-1} - pt_1) \geq (M_2+1)(2t_3)^{1/2}$,
(3) $\max(a_1 + pt_1,\, b_1 - pt_1) \leq V^{top}(2t_3)^{1/2}$,
(4) $\min(a_i - a_{i+1},\, b_i - b_{i+1}) \geq 3\epsilon(2t_3)^{1/2}$ for $1 \leq i \leq k-2$.

Now let $\tilde{D}(V^{top},\epsilon,t_1)$ be the set consisting of elements of $D(\vec{c},\vec{d},V^{top},\epsilon,t_1)$ restricted to $[\![-t_{12},t_{12}]\!]$. Then we have

$$\begin{aligned}\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}&\left(D(\vec{c},\vec{d},V^{top},\epsilon,t_1)\,\middle|\,G(\vec{a},\vec{b})\right) = \mathbb{P}^{-t_{12},t_{12},\vec{a},\vec{b},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\left(\tilde{D}(V^{top},\epsilon,t_1)\right) \geq \\ \mathbb{P}^{-t_{12},t_{12},\vec{a},\vec{b}}_{Ber}&\left(\tilde{D}(V^{top},\epsilon,t_1) \cap \{L_1 \geq \cdots \geq L_{k-1} \geq \ell_{bot}\}\right).\end{aligned} \tag{6.40}$$

For any $(\vec{a},\vec{b}) \in J$, we observe that the event in the second line of (6.40) occurs as long as each curve L_i remains within a distance of $\epsilon(2t_3)^{1/2}$ from the straight line segment connecting a_i and b_i on $[-t_{12},t_{12}]$, for $1 \leq i \leq k-2$. By the argument in the proof of Lemma 3.14, we can find $\tilde{N}_9$ so that the probability of this event is bounded below by the expression on the right in (6.38) for $N \geq \tilde{N}_9$. If $\tilde{N}_9$ is large enough so that the set $\tilde{J}$ of $(\vec{a},\vec{b}) \in J$ such that $\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}(G(\vec{a},\vec{b})) > 0$ is nonempty, then using (6.40) and (6.39) and summing over $\tilde{J}$ implies (6.38).

Step 5. In this last step, we complete the proof of the lemma, fixing the constants g and h as well as N_5. Let $g = g(\epsilon, V^{top}, M_2)$ be as in Lemma 6.3 for the choices of ϵ, V^{top} in Steps 2 and 3, let

$$h = \frac{h_1}{2}\left(\frac{1}{2} - \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\epsilon^2 n^2/2p(1-p)}\right)^{k-1}$$

with h_1 as in Step 2, and let $N_5 = \max(\bar{N}_0, \bar{N}_1, \bar{N}_2, \bar{N}_3, N_7)$, with N_7 as in Lemma 6.4. In the following we assume that $N \geq N_7$. By (6.37) we have that if $(\vec{c},\vec{d}) \in E$, then

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}(H) \geq \frac{h}{h_1},$$

where H is the event that

(1) $V^{top}(2t_3)^{1/2} \geq \tilde{Q}_1(-t_1) + pt_1 \geq \tilde{Q}_{k-1}(-t_1) + pt_1 \geq (M_2+1)(2t_2)^{1/2}$,

(2) $V^{top}(2t_3)^{1/2} \geq \tilde{Q}_1(t_1) - pt_1 \geq \tilde{Q}_{k-1}(t_1) - pt_1 \geq (M_2 + 1)(2t_3)^{1/2}$,
(3) $\tilde{Q}_i(-t_1) - \tilde{Q}_{i+1}(-t_1) \geq 3\epsilon(2t_2)^{1/2}$ and $\tilde{Q}_i(t_1) - \tilde{Q}_{i+1}(t_1) \geq 3\epsilon(2t_2)^{1/2}$ for $i = 1, \dots, k-2$.

Let Y denote the event appearing in (6.7). Then we can write $Y = \bigsqcup_{(\vec{c},\vec{d})\in E} Y(\vec{c},\vec{d})$, where $Y(\vec{c},\vec{d})$ is the event that $\tilde{\mathfrak{Q}}(-t_2) = \vec{c}$, $\tilde{\mathfrak{Q}}(t_2) = \vec{d}$, and E is defined in Step 2. It follows from Lemma 6.4 that $\mathbb{P}_{\tilde{\mathfrak{Q}}}(Y) \geq h_1$. We conclude from the definition of $\mathbb{P}_{\tilde{\mathfrak{Q}}}$ that

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}(H) \geq \mathbb{P}_{\tilde{\mathfrak{Q}}}(H \cap Y) = \sum_{(\vec{c},\vec{d})\in E} \mathbb{P}_{\tilde{\mathfrak{Q}}}(Y(\vec{c},\vec{d})) \cdot \mathbb{P}_{\tilde{\mathfrak{Q}}}(H \,|\, Y(\vec{c},\vec{d})) =$$

$$\sum_{(\vec{c},\vec{d})\in E} \mathbb{P}_{\tilde{\mathfrak{Q}}}(Y(\vec{c},\vec{d})) \cdot \mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}(H) \geq \frac{h}{h_1} \sum_{(\vec{c},\vec{d})\in E} \mathbb{P}_{\tilde{\mathfrak{Q}}}(Y(\vec{c},\vec{d})) = \frac{h}{h_1} \mathbb{P}_{\tilde{\mathfrak{Q}}}(Y) \geq h.$$

Now Lemma 6.3 implies (6.1), completing the proof. □

7. Applications to uniform lozenge tilings

8. Appendix A

8.1. **Proof of Lemma 2.2.** Observe that the sets $K_1 \subset K_2 \subset \cdots \subset \Sigma \times \Lambda$ are compact, they cover $\Sigma \times \Lambda$, and any compact subset K of $\Sigma \times \Lambda$ is contained in all K_n for sufficiently large n. To see this last fact, let π_1, π_2 denote the canonical projection maps of $\Sigma \times \Lambda$ onto Σ and Λ respectively. Since these maps are continuous, $\pi_1(K)$ and $\pi_2(K)$ are compact in Σ and Λ. This implies that $\pi_1(K)$ is finite, so it is contained in $\Sigma_{n_1} = \Sigma \cap [\![-n_1, n_1]\!]$ for some n_1. On the other hand, $\pi_2(K)$ is closed and bounded in $\mathbb{R}$, thus contained in some closed interval $[\alpha, \beta] \subseteq \Lambda$. Since $a_n \searrow a$ and $b_n \nearrow b$, we can choose n_2 large enough so that $\pi_2(K) \subseteq [\alpha, \beta] \subseteq [a_{n_2}, b_{n_2}]$. Then taking $n = \max(n_1, n_2)$, we have $K \subseteq \pi_1(K) \times \pi_2(K) \subseteq \Sigma_n \times [a_n, b_n] = K_n$.

We now split the proof into several steps.

Step 1. In this step, we show that the function d defined in the statement of the lemma is a metric. For each n and $f, g \in C(\Sigma \times \Lambda)$, we define

$$d_n(f,g) = \sup_{(i,t)\in K_n} |f(i,t) - g(i,t)|, \quad d_n'(f,g) = \min\{d_n(f,g), 1\}$$

Then we have

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} d_n'(f,g).$$

Clearly each d_n is nonnegative and satisfies the triangle inequality, and it is then easy to see that the same properties hold for d_n'. Furthermore, $d_n' \leq 1$, so d is well-defined. Observe that d is nonnegative, and if $f = g$, then each $d_n'(f,g) = 0$, so the sum $d(f,g)$ is 0. Conversely, if $f \neq g$, then since the K_n cover $\Sigma \times \Lambda$, we can choose n large enough so that K_n contains an x with $f(x) \neq g(x)$. Then $d_n'(f,g) \neq 0$, and hence $d(f,g) \neq 0$. Lastly, the triangle inequality holds for d since it holds for each d_n'.

Step 2. Now we prove that the topology τ_d on $C(\Sigma \times \Lambda)$ induced by d is the same as the topology of uniform convergence over compacts, which we denote by τ_c. Recall that τ_c is generated by the basis consisting of sets

$$B_K(f,\epsilon) = \Big\{ g \in C(\Sigma \times \Lambda) : \sup_{(i,t)\in K} |f(i,t) - g(i,t)| < \epsilon \Big\},$$

for $K \subset \Sigma \times \Lambda$ compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and τ_d is generated by sets of the form $B_\epsilon^d(f) = \{g : d(f,g) < \epsilon\}$.

We first show that $\tau_d \subseteq \tau_c$. It suffices to prove that every set $B^d_\epsilon(f)$ is a union of sets $B_K(f,\epsilon)$. First, choose $\epsilon > 0$ and $f \in C(\Sigma \times \Lambda)$. Let $g \in B^d_\epsilon(f)$. We will find a basis element A_g of τ_c such that $g \in A_g \subset B^d_\epsilon(f)$. Let $\delta = d(f,g) < \epsilon$, and choose n large enough so that $\sum_{k>n} 2^{-k} < \frac{\epsilon-\delta}{2}$. Define $A_g = B_{K_n}(g, \frac{\epsilon-\delta}{n})$, and suppose $h \in A_g$. Then since $K_m \subseteq K_n$ for $m \leq n$, we have

$$d(f,h) \leq d(f,g) + d(g,h) \leq \delta + \sum_{k=1}^{n} 2^{-k} d_n(g,h) + \sum_{k>n} 2^{-k} \leq \delta + \frac{\epsilon-\delta}{2} + \frac{\epsilon-\delta}{2} = \epsilon.$$

Therefore $g \in A_g \subset B^d_\epsilon(f)$. Then we can write

$$B^d_\epsilon(f) = \bigcup_{g \in B^d_\epsilon(f)} A_g,$$

a union of basis elements of τ_c.

We now prove conversely that $\tau_c \subseteq \tau_d$. Let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$. Choose n so that $K \subset K_n$, and let $g \in B_K(f,\epsilon)$ and $\delta = \sup_{x\in K} |f(x) - g(x)| < \epsilon$. If $d(g,h) < 2^{-n}(\epsilon - \delta)$, then $d'_n(g,h) \leq 2^n d(g,h) < \epsilon - \delta$, hence $d_n(g,h) < \epsilon - \delta$, assuming without loss of generality that $\epsilon \leq 1$. It follows that

$$\sup_{x\in K} |f(x) - h(x)| \leq \delta + \sup_{x \in K} |g(x) - h(x)| \leq \delta + d_n(g,h) \leq \delta + \epsilon - \delta = \epsilon.$$

Thus $g \in B^d_{2^{-n}(\epsilon-\delta)}(g) \subset B_K(f,\epsilon)$, proving that $B_K(f,\epsilon) \in \tau_d$ by the same argument as above. We conclude that $\tau_d = \tau_c$.

Step 3. In this step, we show that $(C(\Sigma\times\Lambda), d)$ is a complete metric space. Let $\{f_n\}_{n\geq 1}$ be Cauchy with respect to d. Then we claim that $\{f_n\}$ must be Cauchy with respect to d'_n, on each K_n. This follows from the observation that $d'_n(f_\ell, f_m) \leq 2^n d(f_\ell, f_m)$. Thus $\{f_n\}$ is Cauchy with respect to the uniform metric on each K_n, and hence converges uniformly to a continuous limit f^{K_n} on each K_n (see [15, Theorem 7.15]). Since the pointwise limit must be unique at each $x \in \Sigma \times \Lambda$, we have $f^{K_n}(x) = f^{K_m}(x)$ if $x \in K_n \cap K_m$. Since $\bigcup K_n = \Sigma \times \Lambda$, we obtain a well-defined function f on all of $\Sigma \times \Lambda$ given by $f(x) = \lim_{n\to\infty} f^{K_n}(x)$. We have $f \in C(\Sigma \times \Lambda)$ since $f|_{K_n} = f^{K_n}$ is continuous on K_n for all n. Moreover, if $K \subset \Sigma \times \Lambda$ is compact and n is large enough so that $K \subset K_n$, then because $f_n \to f^{K_n} = f|_{K_n}$ uniformly on K_n, we have $f_n \to f^{K_n}|_K = f|_K$ uniformly on K. That is, for any $K \subset \Sigma \times \Lambda$ compact and $\epsilon > 0$, we have $f_n \in B_K(f,\epsilon)$ for all sufficiently large n. Therefore $f_n \to f$ in τ_c, and equivalently in the metric d by Step 2.

Step 4. Lastly, we prove separability, c.f. [1, Example 1.3]. For each pair of positive integers n,k, let $D_{n,k}$ be the subcollection of $C(\Sigma\times\Lambda)$ consisting of polygonal functions that are piecewise linear on $\{j\} \times I_{n,k,i}$ for each $j \in \Sigma_n$ and each subinterval

$$I_{n,k,i} = \left[a_n + \tfrac{i-1}{k}(b_n - a_n),\, a_n + \tfrac{i}{k}(b_n - a_n)\right], \quad 1 \leq i \leq k,$$

taking rational values at the endpoints of these subintervals, and extended linearly to all of $\Lambda = [a,b]$. Then $D = \bigcup_{n,k} D_{n,k}$ is countable, and we claim that it is dense in τ_c. To see this, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma\times\Lambda)$, and $\epsilon > 0$, and choose n so that $K \subset K_n$. Since f is uniformly continuous on K_n, we can choose k large enough so that for $0 \leq i \leq k$, if $t \in I_{n,k,i}$, then

$$\left|f(j,t) - f(j, a_n + \tfrac{i}{k}(b_n - a_n))\right| < \epsilon/2$$

for all $j \in \Sigma_n$. We then choose $g \in \bigcup_k D_{n,k}$ with $|g(j, a_n + \frac{i}{k}(b_n - a_n)) - f(j, a_n + \frac{i}{k}(b_n - a_n))| < \epsilon/2$. Then we have

$$\left|f(j,t) - g(j, a_n + \tfrac{i-1}{k}(b_n - a_n))\right| < \epsilon \quad \text{and} \quad \left|f(j,t) - g(j, a_n + \tfrac{i}{k}(b_n - a_n))\right| < \epsilon.$$

Since $g(j, a_n + \frac{i-1}{k}(b_n - a_n)) \leq g(j,t) \leq g(j, a_n + \frac{i}{k}(b_n - a_n))$, it follows that

$$|f(j,t) - g(j,t)| < \epsilon$$

as well. In summary,

$$\sup_{(j,t)\in K} |f(j,t) - g(j,t)| \leq \sup_{(j,t)\in K_n} |f(j,t) - g(j,t)| < \epsilon,$$

so $g \in B_K(f, \epsilon)$. This proves that D is a countable dense subset of $C(\Sigma \times \Lambda)$.

8.2. **Proof of Lemma 2.4.** We first prove two lemmas that will be used in the proof of Lemma 2.4. The first result allows us to identify the space $C(\Sigma \times \Lambda)$ with a product of copies of $C(\Lambda)$. In the following, we assume the notation of Lemma 2.4.

Lemma 8.1. *Let $\pi_i : C(\Sigma \times \Lambda) \to C(\Lambda)$, $i \in \Sigma$, be the projection maps given by $\pi_i(F)(x) = F(i, x)$ for $x \in \Lambda$. Then the π_i are continuous. Endow the space $\prod_{i\in\Sigma} C(\Lambda)$ with the product topology induced by the topology of uniform convergence over compacts on $C(\Lambda)$. Then the mapping*

$$F : C(\Sigma \times \Lambda) \longrightarrow \prod_{i\in\Sigma} C(\Lambda), \quad f \mapsto (\pi_i(f))_{i\in\Sigma}$$

is a homeomorphism.

Proof. We first prove that the π_i are continuous. Since $C(\Sigma \times \Lambda)$ is metrizable by Lemma 2.2, and by a similar argument so is $C(\Lambda)$, it suffices to assume that $f_n \to f$ in $C(\Sigma \times \Lambda)$ and show that $\pi_i(f_n) \to \pi_i(f)$ in $C(\Lambda)$. Let K be compact in Λ. Then $\{i\} \times K$ is compact in $\Sigma \times \Lambda$, and $f_n \to f$ on $\{i\} \times K$ by assumption, so we have $\pi_i(f_n)|_K = f_n|_{\{i\}\times K} \to f|_{\{i\}\times K} = \pi_i(f)|_K$ uniformly on K. Since K was arbitrary, we conclude that $\pi_i(f_n) \to \pi_i(f)$ in $C(\Lambda)$ as desired.

We now observe that F is invertible. If $(f_i)_{i\in\Sigma} \in \prod_{i\in\Sigma} C(\Lambda)$, then the function f defined by $f(i, \cdot) = f_i(\cdot)$ is in $C(\Sigma \times \Lambda)$, since Σ has the discrete topology. This gives a well-defined inverse for F. It suffices to prove that F and F^{-1} are open maps.

We first show that F sends each basis element $B_K(f, \epsilon)$ of $C(\Sigma \times \Lambda)$ to a basis element in $\prod_{i\in\Sigma} C(\Lambda)$. Note that a basis for the product topology is given by products $\prod_{i\in\Sigma} B_{K_i}(f_i, \epsilon)$, where at most finitely many of the K_i are nonempty. Here, we use the convention that $B_\varnothing(f_i, \epsilon) = C(\Lambda)$. Let π_Σ, π_Λ denote the canonical projections of $\Sigma \times \Lambda$ onto Σ, Λ. The continuity of π_Σ implies that if $K \subset \Sigma \times \Lambda$ is compact, then $\pi_\Sigma(K)$ is compact in Σ, hence finite. Observe that the set $K \cap (\{i\} \times \Lambda)$ is an intersection of two compacts sets, hence compact in $\Sigma \times \Lambda$. Therefore the sets $K_i = \pi_\Lambda(K \cap (\{i\} \times \Lambda))$ are compact in Λ for each $i \in \Sigma$ since π_Λ is continuous. We observe that $F(B_K(f, \epsilon)) = \prod_{i\in\Sigma} U_i$, where

$$U_i = B_{K_i}(\pi_i(f), \epsilon), \quad \text{if} \quad i \in \pi_\Sigma(K),$$

and $U_i = C(\Lambda)$ otherwise. Since $\pi_\Sigma(K)$ is finite and the K_i are compact, we see that $F(B_K(f, \epsilon))$ is a basis element in the product topology as claimed.

Lastly, we show that F^{-1} sends each basis element $U = \prod_{i\in\Sigma} B_{K_i}(f_i, \epsilon)$ for the product topology to a set of the form $B_K(f, \epsilon)$. We have $K_i = \varnothing$ for all but finitely many i. Write $f = F^{-1}((f_i)_{i\in\Sigma})$ and $K = \prod_{i\in\Sigma} K_i$. By Tychonoff's theorem, [13, Theorem 37.3], K is compact in $\Sigma \times \Lambda$, and

$$F^{-1}(U) = B_K(f, \epsilon).$$

□

We next prove a lemma which states that a sequence of line ensembles is tight if and only if all individual curves form tight sequences.

Lemma 8.2. *Suppose that $\{\mathcal{L}^n\}_{n\geq 1}$ is a sequence of Σ-indexed line ensembles on Λ, and let $X_i^n = \pi_i(\mathcal{L}^n)$. Then the X_i^n are $C(\Lambda)$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{L}^n\}$ is tight if and only if for each $i \in \Sigma$ the sequence $\{X_i^n\}_{n\geq 1}$ is tight.*

Proof. The fact that the X_i^n are random variables follows from the continuity of the π_i in Lemma 8.1 and [9, Theorem 1.3.5]. First suppose the sequence $\{\mathcal{L}^n\}$ is tight. By Lemma 2.2, $C(\Sigma \times \Lambda)$ is a Polish space, so it follows from Prohorov's theorem, [1, Theorem 5.1], that $\{\mathcal{L}^n\}$ is relatively compact. That is, every subsequence $\{\mathcal{L}^{n_k}\}$ has a further subsequence $\{\mathcal{L}^{n_{k_\ell}}\}$ converging weakly to some $\mathcal{L}$. Then for each $i \in \Sigma$, since π_i is continuous by the above, the subsequence $\{\pi_i(\mathcal{L}^{n_{k_\ell}})\}$ of $\{\pi_i(\mathcal{L}^{n_k})\}$ converges weakly to $\pi_i(\mathcal{L})$ by the continuous mapping theorem, [9, Theorem 3.2.10]. Thus every subsequence of $\{\pi_i(\mathcal{L}^n)\}$ has a convergent subsequence. Since $C(\Lambda)$ is a Polish space by the same argument as in the proof of Lemma 2.2, Prohorov's theorem implies that each $\{\pi_i(\mathcal{L}^n)\}$ is tight.

Conversely, suppose $\{X_i^n\}$ is tight for all $i \in \Sigma$. Then given $\epsilon > 0$, we can find compact sets $K_i \subset C(\Lambda)$ such that

$$\mathbb{P}(X_i^n \notin K_i) \leq \epsilon/2^i$$

for each $i \in \Sigma$. By Tychonoff's theorem, [13, Theorem 37.3], the product $\tilde{K} = \prod_{i\in\Sigma} K_i$ is compact in $\prod_{i\in\Sigma} C(\Lambda)$. We have

$$\mathbb{P}\big((X_i^n)_{i\in\Sigma} \notin \tilde{K}\big) \leq \sum_{i\in\Sigma} \mathbb{P}(X_i^n \notin K_i) \leq \sum_{i=1}^{\infty} \epsilon/2^i = \epsilon. \tag{8.1}$$

By Lemma 8.1, we have a homeomorphism $G : \prod_{i\in\Sigma} C(\Lambda) \to C(\Sigma \times \Lambda)$. We observe that $G((X_i^n)_{i\in\Sigma}) = \mathcal{L}^n$, and $K = G(\tilde{K})$ is compact in $C(\Sigma\times\Lambda)$. Thus $\mathcal{L}^n \in K$ if and only if $(X_i^n)_{i\in\Sigma} \in \tilde{K}$, and it follows from (8.1) that

$$\mathbb{P}(\mathcal{L}^n \in K) \geq 1 - \epsilon.$$

This proves that $\{\mathcal{L}^n\}$ is tight. □

We are now ready to prove Lemma 2.4.

Proof. Fix an $i \in \Sigma$. By Lemma 8.2, it suffices to show that the sequence $\{\mathcal{L}_i^n\}_{n\geq 1}$ of $C(\Lambda)$-valued random variables is tight. By [1, Theorem 7.3], a sequence $\{P_n\}$ of probability measures on $C[0,1]$ with the uniform topology is tight if and only if the following conditions hold:

$$\lim_{a\to\infty} \limsup_{n\to\infty} P_n(|x(0)| \geq a) = 0,$$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P_n\Big(\sup_{|s-t|\leq\delta} |x(s) - x(t)| \geq \epsilon\Big) = 0 \quad \text{for all } \epsilon > 0.$$

By replacing $[0,1]$ with $[a_m, b_m]$ and 0 with a_0, we see that the hypotheses in the lemma imply that the sequence $\{\mathcal{L}_i^n|_{[a_m,b_m]}\}_n$ is tight for every $m \geq 1$. Let $\pi_m : C(\Lambda) \to C([a_m, b_m])$ denote the map $f \mapsto f|_{[a_m,b_m]}$. Then π_m is continuous, since $C(\Lambda)$ and $C([a_m, b_m])$ with the topologies of uniform convergence over compacts are metrizable by Lemma 2.2, and if $f_n \to f$ uniformly on compact subsets of Λ, then $f_n|_{[a_m,b_m]} \to f|_{[a_m,b_m]}$ uniformly on compact subsets of $[a_m, b_m]$. It follows from [9, Theorem 1.3.5] that $\pi_m(\mathcal{L}^n) = \mathcal{L}_i^n|_{[a_m,b_m]}$ is a $C([a_m, b_m])$-valued random variable. Tightness of the sequence implies that for any $\epsilon > 0$, we can find compact sets $K_m \subset C([a_m, b_m])$ so that

$$\mathbb{P}\big(\pi_m(\mathcal{L}_i^n) \notin K_m\big) \leq \epsilon/2^m$$

for each $m \geq 1$. Writing $K = \bigcap_{m=1}^{\infty} \pi_m^{-1}(K_m)$, it follows that

$$\mathbb{P}\big(\mathcal{L}_i^n \in K\big) \geq 1 - \sum_{m=1}^{\infty} \epsilon/2^m = 1 - \epsilon.$$

To conclude tightness of $\{\mathcal{L}_i^n\}$, it suffices to prove that $K = \bigcap_{m=1}^{\infty} \pi_m^{-1}(K_m)$ is sequentially compact in $C(\Lambda)$. We argue by diagonalization. Let $\{f_n\}$ be a sequence in K, so that $f_n|_{[a_m,b_m]} \in K_m$ for every m, n. Since K_1 is compact, there is a sequence $\{n_{1,k}\}$ of natural numbers such that the subsequence $\{f_{n_{1,k}}|_{[a_1,b_1]}\}_k$ converges in $C([a_1,b_1])$. Since K_2 is compact, we can take a further subsequence $\{n_{2,k}\}$ of $\{n_{1,k}\}$ so that $\{f_{n_{2,k}}|_{[a_2,b_2]}\}_k$ converges in $C([a_2,b_2])$. Continuing in this manner, we obtain sequences $\{n_{1,k}\} \supseteq \{n_{2,k}\} \supseteq \cdots$ so that $\{f_{n_{m,k}}|_{[a_m,b_m]}\}_k$ converges in $C([a_m,b_m])$ for all m. Writing $n_k = n_{k,k}$, it follows that the sequence $\{f_{n_k}\}$ converges uniformly on each $[a_m,b_m]$. If K is any compact subset of $C(\Lambda)$, then $K \subset [a_m,b_m]$ for some m, and hence $\{f_{n_k}\}$ converges uniformly on K. Therefore $\{f_{n_k}\}$ is a convergent subsequence of $\{f_n\}$.

□

8.3. **Proof of Lemma 2.16.**

Proof. We will construct a candidate $\mathfrak{B}$ of $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},f,g)$ with the conditions of 2.16 assumed. Construct the ensemble $\mathfrak{B}$ in the following manner. Denote $B_0 = f$ and $B_{k+1} = g$ with $x_0 = f(T_0)$ and $y_0 = f(T_1)$. By Condition (3) of Lemma 2.16 we know $x_0 \geq x_1$ and $y_0 \geq y_1$. Then let $B_j(T_0) = x_j$ for all $j \in [\![1,k]\!]$ and then for all $i \in [\![T_0,T_1-1]\!]$ we have

$$B_j(i+1) = \begin{cases} B_j(i)+1 & \text{if } B_j(i)+1 \leq \min\{B_{j-1}(i+1), y_j\} \\ B_j(i) & \text{Else.} \end{cases} \tag{8.2}$$

This definition is well-defined, since we may find B_1 depending solely on the predetermined f, and then inductively find B_j since B_{j-1} has been determined by the previous curves in $\mathfrak{B}$.

In order to verify that this candidate ensemble $\mathfrak{B}$ is an element of $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},f,g)$, three properties must be ensured:

$$\begin{aligned} &\text{(a) } \mathfrak{B}(T_0) = \vec{x} \text{ and } \mathfrak{B}(T_1) = \vec{y} \\ &\text{(b) } f(i) \geq B_1(i) \geq \cdots \geq B_k(i) \geq g(i) \text{ for all } i \in [\![T_0,T_1]\!] \\ &\text{(c) } B_j(i+1) - B_j(i) \in \{0,1\} \text{ for all } i \in [\![T_0,T_1-1]\!] \text{ and } j \in [\![1,k]\!] \end{aligned} \tag{8.3}$$

Property (c) follows directly from Definition 8.2, since $B_j(i+1) = B_j(i)$ or $B_j(i+1) = B_j(i)+1$, and hence $B_j(i+1) - B_j(i) \in \{0,1\}$. The remainder of the proof will be broken up into two steps, the first step proving property (a), and the second proving Property (b).

Step 1:
We know by definition that $\mathfrak{B}(T_0) = \vec{x}$, and we claim that $\mathfrak{B}(T_1) = \vec{y}$. We will show this claim inductively on j: We trivially know the claim is true for $j = 0$, since $y_0 = f(T_1)$ is given. Then suppose that $B_j(T_1) = x_j$ holds upto $j = n-1$. First, we know by definition that $B_n(i+1) = B_n(i)$ if either $B_n(i) = y_n$ or $B_n(i)+1 > B_n(i+1)$. Suppose that for some $i_0 \in [\![T_0,T_1]\!]$ we have $B_n(i_0) = B_n(i_0+1)$.

If $B_n(i_0) = B_n(i_0+1)$ because $B_n(i_0) = y_n$, then $B_n(T_1) = y_n$ since

$$y_n = B_n(i_0) = B_n(i_0+1) = \cdots = B_n(T_1)$$

and then the claim is true, namely that $B_n(T_1) = y_n$, and so induction holds.

Then, for the other case, when $B_n(i_0) = B_n(i_0+1)$ because $B_n(i_0)+1 > B_{n-1}(i_0+1)$, we first need to prove that $B_j(i) \leq B_{j-1}(i)$ for $j \in [\![1,k]\!]$ for any $i \in [\![T_0,T_1]\!]$. We know this is true for T_0 since $x_0 \geq x_1 \geq \cdots \geq x_k$. Then, inductively we know that if $B_j(i) \leq B_{j-1}(i)$, $B_j(i+1) = B_j(i)$ or $B_j(i)+1$. In the first case, $B_j(i+1) = B_j(i) \leq B_{j-1}(i) \leq B_j(i)$ by property (3) of 8.3. Then, if $B_j(i+1) = B_j(i)+1$ implies $B_j(i+1) \leq B_j(i+1)$ by equation 8.2. Hence, we know that for $i \in [\![T_0,T_1]\!]$

$$f(i) \geq B_1(i) \geq \cdots \geq B_k(i) \tag{8.4}$$

Therefore, we know that $B_n(i_0) = B_n(i_0+1)$ and $B_n(i_0)+1 > B_{n-1}(i_0+1)$, which implies $B_n(i) = B_{n-1}(i)$. This implies that if we denote i_1 as the least i such that $B_{n-1}(i_1) = y_n$ then

$$B_n(i) = B_{n-1}(i) \text{ for all } i \in [\![i_0,i_1]\!]$$

We know that there exists such an $i_1 \in [\![T_0, T_1]\!]$ because where i_1 is the first i such that $B_{n-1}(i_1) = y_n$, since if $B_{n-1}(i+1) = B_{n-1}(i)$ then $B_n(i)+1 = B_{n-1}(i)+1 > B_{n-1}(i+1)$ by 8.2, the definition of $\mathfrak{B}$. Therefore we know $B_n(i+1) = B_n(i) = B_{n-1}(i) = B_{n-1}(i+1)$.

If $B_{n-1}(i+1) = B_n(i)+1$ then $B_n(i)+1 \leq B_{n-1}(i+1)$ by 8.2 so $B_n(i+1) = B_{n-1}(i+1)$ therefore inductively until B_n cannot increase above y_n, we know $B_n(i) = B_{n-1}(i)$. Because we know that there is some i_1 such that $B_{n-1}(i_1) = y_n$, and hence $B_n(i_1) = y_n$ we get $B_n(T_1) = y_n$ and the claim that $B_n(T_1) = y_n$ is true if there exists some i_0 such that $B_n(i_0) = B_n(i_0+1)$

Finally, assume that there exists no such i_0 that $B_n(i_0) = B_n(i_0+1)$. Then conversely $B_n(i)+1 \leq B_n(i+1)$ for all i, then we know that $B_n(i+1) = B_n(i)+1$ for all i unless $B_n(i) = y_n$ by 8.2. Therefore, until $B_n(i) = y_n$, we have $B_n(i+s) = B_n(i)+s$ hence $B_n(T_0+y_n-x_n) = B_n(T_0)+y_n-x_n = y_n$. By the inequality in condition (1) of 2.16, we have the following inequalities:

$$\begin{aligned} T_1 - T_0 &\geq y_n - x_n \geq 0 \\ T_0 &\leq T_0 + y_n - x_n \\ T_1 &\geq T_0 + y_n - x_n \end{aligned} \tag{8.5}$$

so $T_0 + y_n - x_n \in [\![T_0, T_1]\!]$ and so $B_n(T_0+y_n-x_n)B_n(T_1) = y_n$. This means whether or not i_0 exists, the induction holds and therefore we know that for all j we have $B_j(T_1) = y_j$, so we know that $\mathfrak{B}(T_0) = \vec{x}$ and $\mathfrak{B}(T_1) = \vec{y}$ which concludes Step 1, proving Property (a) of 8.3.

Step 2: Now all that is left to verify avoidance, or Property (b) of 8.3. In equation 8.4, we already found that

$$f(i) \geq B_1(i) \geq \cdots \geq B_k(i)$$

so we must only prove that $B_k(i) \geq g(i)$ for all i. Suppose that $g(i) > B_k(i)$ for some $i \in [\![T_0, T_1]\!]$. Since $g(T_0) < B_k(T_0) = x_k$ by Condition (3) in Lemma 2.16, we know that there exists some point i_0 such that $g(i_0) = B_k(i_0)$ and $g(i_0+1) > B_k(i_0+1)$. In particular, since g and B_k can each only increase by 1, this implies $B_k(i_0) = B_k(i_0+1)$. This implies either $B_k(i_0) = y_k$ or $B_k(i_0)+1 > B_{k-1}(i_0+1)$. If $B_k(i_0) = y_k$ then since $g(i_0+1) \leq y_k$ by Condition (3) of Lemma 2.16, there is a contradiction.

Therefore, it must be the case that $B_k(i)+1 > B_{k-1}(i+1)$. Then we find that for any $j \in [\![1, k]\!]$ we know that $B_j(i)+1 > B_{j-1}(i+1)$ implies $B_j(i_0) = B_{j-1}(i_0)$ since $B_j(i_0) = B_j(i_0+1)$ and $B_{j-1}(i_0) \geq B_j(i_0)$. This can be applied to each j to find that $g(i_0) = f(i_0)$ and $g(i_0+1) > f(i_0+1)$, which we assumed not to be the case. Therefore, we know that $g \leq B_k$ and so we have proven property (3), implying that if the three conditions in the statement of Lemma 2.16 are met then we know $\mathfrak{B} \in \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ and so $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is non-empty. $\square$

8.4. **Proof of Lemmas 4.6 and 4.7.** We first prove Lemma 4.6. We will use the following lemma, which proves an analogous convergence result for a single rescaled Bernoulli random walk.

Lemma 8.3. *Let $x, y, a, b \in \mathbb{R}$ with $a < b$, and let $a_N, b_N \in N^{-\alpha}\mathbb{Z}$, $x^N, y^N \in N^{-\alpha/2}\mathbb{Z}$ be sequences with $a_N \leq a$, $b_N \geq b$, and $|y^N - x^N| \leq (b_N - a_N)N^{\alpha/2}$. Suppose $a_N \to a$, $b_N \to b$. Write $\tilde{x}^N = (x^N - pa_N N^{\alpha/2})/\sqrt{p(1-p)}$, $\tilde{y}^N = (y^N - pb_N N^{\alpha/2})/\sqrt{p(1-p)}$, and assume $\tilde{x}^N \to x$, $\tilde{y}^N \to y$ as $N \to \infty$. Let Y^N be a sequence of random variables with laws $\mathbb{P}^{a_N, b_N, x^N, y^N}_{free,N}$, and let $Z^N = Y^N|_{[a,b]}$. Then the law of Z^N converges weakly to $\mathbb{P}^{a,b,x,y}_{free}$ as $N \to \infty$.*

Proof. Let us write $z^N = (y^N - x^N)N^{\alpha/2}$ and $T_N = (b_N - a_N)N^{\alpha}$. Let $\tilde{B}$ be a standard Brownian bridge on $[0, 1]$, and define random variables B^N, B taking values in $C([a_N, b_N])$, $C([a, b])$

respectively via

$$B^N(t) = \sqrt{b_N - a_N} \cdot \tilde{B}\left(\frac{t - a_N}{b_N - a_N}\right) + \frac{t - a_N}{b_N - a_N} \cdot \tilde{y}^N + \frac{b_N - t}{b_N - a_N} \cdot \tilde{x}^N,$$

$$B(t) = \sqrt{b - a} \cdot \tilde{B}\left(\frac{t - a}{b - a}\right) + \frac{t - a}{b - a} \cdot y + \frac{b - t}{b - a} \cdot x.$$

We observe that B has law $\mathbb{P}^{a,b,x,y}_{free}$ and $B^N \implies B$ as $N \to \infty$. By [1, Theorem 3.1], to show that $Z^N \implies B$, it suffices to find a sequence of probability spaces supporting Y^N, B^N so that

$$\rho(B^N, Y^N) = \sup_{t \in [a_N, b_N]} |B^N(t) - Y^N(t)| \implies 0 \quad \text{as} \quad N \to \infty. \tag{8.6}$$

It follows from Theorem 3.3 that for each $N \in \mathbb{N}$ there is a probability space supporting B^N and Y^N, as well as constants $C, a', \alpha' > 0$, such that

$$\mathbb{E}\left[e^{a'\Delta(N, x^N, y^N)}\right] \le C e^{\alpha' \log N} e^{|z^N - pT_N|^2 / N^\alpha}, \tag{8.7}$$

where $\Delta(N, x^N, y^N) = \sqrt{p(1-p)}\, N^{\alpha/2} \rho(B^N, Y^N)$. Since $(z^N - pT_N)N^{-\alpha/2} \to \sqrt{p(1-p)}\,(y - x)$ by assumption, there exist $N_0 \in \mathbb{N}$ and $A > 0$ so that $|z - pT_N| \le AN^{\alpha/2}$ for $N \ge N_0$. Then for $\epsilon > 0$ and $N \ge N_0$, Chebyshev's inequality and (8.7) give

$$\mathbb{P}(\rho(B^N, Y^N) > \epsilon) \le C e^{-a'\epsilon\sqrt{p(1-p)}\, N^{\alpha/2}} e^{\alpha' \log N} e^{A^2}.$$

The right hand side tends to 0 as $N \to \infty$, implying (8.6). □

We now give the proof of Lemma 4.6.

Proof. We prove the two statements of the lemma in two steps.

Step 1. In this step we fix $N_0 \in \mathbb{N}$ so that $\mathbb{P}^{a_N, b_N, \vec{x}^N, \vec{y}^N, f_N, g_N}_{avoid, N}$ is well-defined for $N \ge N_0$. Observe that we can choose $\epsilon > 0$ and continuous functions $h_1, \dots, h_k : [a, b] \to \mathbb{R}$ depending on $a, b, \vec{x}, \vec{y}, f, g$ with $h_i(a) = x_i$, $h_i(b) = y_i$ for $i \in [\![1, k]\!]$, such that if $u_i : [a, b] \to \mathbb{R}$ are continuous functions with $\rho(u_i, h_i) = \sup_{x \in [a,b]} |u_i(x) - h_i(x)| < \epsilon$, then

$$f(x) - \epsilon > u_1(x) + \epsilon > u_1(x) - \epsilon > \cdots > u_k(x) + \epsilon > u_k(x) - \epsilon > g(x) + \epsilon \tag{8.8}$$

for all $x \in [a, b]$. By Lemma 2.6, we have

$$\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}(\rho(\mathcal{Q}_i, h_i) > \epsilon \text{ for } i \in [\![1, k]\!]) > 0. \tag{8.9}$$

Since $y_i^N - x_i^N - p(b_N - a_N)N^{\alpha/2} \to \sqrt{p(1-p)}\,(y_i - x_i)$ as $N \to \infty$ for $i \in [\![1, k]\!]$ and $p < 1$, we can find $N_1 \in \mathbb{N}$ so that for $N \ge N_1$, $|y_i^N - x_i^N| \le (b_N - a_N)N^{\alpha/2}$. It follows from Lemma 8.3 that if $\mathcal{Y}^N$ have laws $\mathbb{P}^{a_N, b_N, \vec{x}^N, \vec{y}^N}_{free, N}$ for $N \ge N_1$ and $\mathcal{Z}^N = \mathcal{Y}^N|_{\Sigma \times [a,b]}$, then the law of $\mathcal{Z}^N$ converges weakly to $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{avoid}$. In view of (8.9) we can then find N_2 so that if $N \ge \max(N_1, N_2)$ then

$$\mathbb{P}^{a_N, b_N, \vec{x}^N, \vec{y}^N}_{free, N}(\rho(\mathcal{Q}_i, h_i) > \epsilon \text{ for } i \in [\![1, k]\!]) > 0.$$

We now choose N_3 so that $\sup_{x \in [a-1, b+1]} |f(x) - f_N(x)| < \epsilon/4$ and $\sup_{x \in [a-1, b+1]} |g(x) - g_N(x)| < \epsilon/4$. If $f = \infty$ (resp. $g = -\infty$), we interpret this to mean that $f_N = \infty$ (resp. $g_N = -\infty$). We take N_4 large enough so that if $N \ge N_4$ and $|x - y| \le N^{-\alpha/2}$ then $|f(x) - f(y)| < \epsilon/4$ and $|g(x) - g(y)| < \epsilon/4$. Lastly, we choose N_5 so that $N_5^{-\alpha} < \epsilon/4$. Then for $N \ge N_0 = \max(N_1, N_2, N_3, N_4, N_5)$, we have

$$\{\rho(\mathcal{Q}_i, h_i) > \epsilon \text{ for } i \in [\![1, k]\!]\} \subset \{f_N \ge \mathcal{Y}_1^N \ge \cdots \ge \mathcal{Y}_k^N \ge g_N \text{ on } [a_N, b_N]\}.$$

By (8.9), this implies that $\mathbb{P}^{a_N, b_N, \vec{x}^N, \vec{y}^N, f_N, g_N}_{avoid, N}$ is well-defined.

Step 2. In this step we prove that $\mathcal{Z}^N \implies \mathbb{P}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$, with $\mathcal{Z}^N$ defined in the statement of the lemma. We write $\Sigma = [\![1,k]\!]$, $\Lambda = [a,b]$, and $\Lambda_N = [a_N, b_N]$. It suffices to show that for any bounded continuous function $F : C(\Sigma \times \Lambda) \to \mathbb{R}$ we have

$$\lim_{N\to\infty} \mathbb{E}[F(\mathcal{Z}^N)] = \mathbb{E}[F(\mathcal{Q})], \tag{8.10}$$

where $\mathcal{Q}$ has law $\mathbb{P}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$.

We define the functions $H_{f,g} : C(\Sigma \times \Lambda) \to \mathbb{R}$ and $H_{f,g}^N : C(\Sigma \times \Lambda_N) \to \mathbb{R}$ by

$$H_{f,g}(\mathcal{L}) = \mathbf{1}\{f > \mathcal{L}_1 > \cdots > \mathcal{L}_k > g \text{ on } \Lambda\},$$
$$H_{f,g}^N(\mathcal{L}^N) = \mathbf{1}\{f \geq \mathcal{L}_1^N \geq \cdots \geq \mathcal{L}_k^N \geq g \text{ on } \Lambda_N\}.$$

Then we observe that for $N \geq N_0$,

$$\mathbb{E}[F(\mathcal{Z}^N)] = \frac{\mathbb{E}[F(\mathcal{L}^N|_{\Sigma\times[a,b]})H_{f,g}^N(\mathcal{L}^N)]}{\mathbb{E}[H_{f,g}^N(\mathcal{L}^N)]}, \tag{8.11}$$

where $\mathcal{L}^N$ has law $\mathbb{P}_{free,N}^{a_N,b_N,\vec{x}^N,\vec{y}^N}$. By our choice of N_0 in Step 1, the denominator in (8.11) is positive for all $N \geq N_0$. Similarly, we have

$$\mathbb{E}[F(\mathcal{Q})] = \frac{\mathbb{E}[F(\mathcal{L})H_{f,g}(\mathcal{L})]}{\mathbb{E}[H_{f,g}(\mathcal{L})]}, \tag{8.12}$$

where $\mathcal{L}$ has law $\mathbb{P}_{free}^{a,b,\vec{x},\vec{y}}$. From (8.11) and (8.12), we see that to prove (8.10) it suffices to show that for any bounded continuous function $F : C(\Sigma \times \Lambda) \to \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{E}[F(\mathcal{L}^N|_{\Sigma\times[a,b]})H_{f,g}^N(\mathcal{L}^N)] = \mathbb{E}[F(\mathcal{L})H_{f,g}(\mathcal{L})]. \tag{8.13}$$

By Lemma 8.3, $\mathcal{L}^N|_{\Sigma\times[a,b]} \implies \mathcal{L}$ as $N \to \infty$. Since $C(\Sigma \times \Lambda)$ is separable, the Skorohod representation theorem [1, Theorem 6.7] gives a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting $C(\Sigma \times \Lambda_N)$-valued random variables $\mathcal{L}^N$ with laws $\mathbb{P}_{free,N}^{a_N,b_N,\vec{x}^N,\vec{y}^N}$ and a $C(\Sigma \times \Lambda)$-valued random variable $\mathcal{L}$ with law $\mathbb{P}_{free}^{a,b,\vec{x},\vec{y}}$ such that $\mathcal{L}^N|_{\Sigma\times[a,b]} \to \mathcal{L}$ uniformly on compact sets, pointwise on Ω. Here we rely on the fact that a_N, b_N are respectively the largest element of $N^{-\alpha}\mathbb{Z}$ less than a and the smallest element greater than b, so that $\mathcal{L}^N|_{\Sigma\times[a,b]}$ uniquely determines $\mathcal{L}^N$ on $[a_N, b_N]$.

Define the events

$$E_1 = \{\omega : f > \mathcal{L}_1(\omega) > \cdots > \mathcal{L}_k(\omega) > g \text{ on } [a,b]\},$$
$$E_2 = \{\omega : \mathcal{L}_i(\omega)(r) < \mathcal{L}_{i+1}(\omega)(r) \text{ for some } i \in [\![0,k]\!] \text{ and } r \in [a,b]\},$$

where in the definition of E_2 we use the convention $\mathcal{L}_0 = f$, $\mathcal{L}_{k+1} = g$. The continuity of F implies that $F(\mathcal{L}^N|_{\Sigma\times[a,b]})H_{f_N,g_N}^N(\mathcal{L}^N) \to F(\mathcal{L})$ on the event E_1, and $F(\mathcal{L}^N|_{\Sigma\times[a,b]})H_{f_N,g_N}^N(\mathcal{L}^N) \to 0$ on the event E_2. By Lemma 2.5 we have $\mathbb{P}(E_1 \cup E_2) = 1$, so $\mathbb{P}$-a.s. we have $F(\mathcal{L}^N|_{\Sigma\times[a,b]})H_{f_N,g_N}^N(\mathcal{L}^N) \to F(\mathcal{L})H_{f,g}(\mathcal{L})$. The bounded convergence theorem then implies (8.13), completing the proof of (8.10). ☐

We now state two lemmas about Brownian bridges which will be used in the proof of Lemma 4.7. The first lemma shows that a Brownian bridge started at 0 almost surely becomes negative somewhere on its domain.

Lemma 8.4. *Fix any $T > 0$ and $y \in \mathbb{R}$, and let Q denote a random variable with law $\mathbb{P}_{free}^{0,T,0,y}$. Define the event $C = \{\inf_{s\in[0,T]} Q(s) < 0\}$. Then $\mathbb{P}_{free}^{0,T,0,y}(C) = 1$.*

Proof. Let B denote a standard Brownian bridge on $[0,1]$, and let

$$\tilde{B}_s = B_{s/T} + \frac{sy}{T}, \quad \text{for } s \in [0,T].$$

Then $\tilde{B}$ has the law of Q. Consider the stopping time $\tau = \inf\{s > 0 : \tilde{B}_s < 0\}$. We will argue that $\tau = 0$ a.s, which implies the conclusion of the lemma since $\{\tau = 0\} \subset C$. We observe that since $\tilde{B}$ is a.s. continuous and $\mathbb{Q}$ is dense in $\mathbb{R}$,

$$\{\tau = 0\} = \bigcap_{\epsilon>0} \bigcup_{s\in(0,\epsilon)\cap\mathbb{Q}} \{\tilde{B}_s < 0\} \in \bigcap_{\epsilon>0} \sigma(\tilde{B}_s : s < \epsilon).$$

Here, $\sigma(\tilde{B}_s : s < \epsilon)$ denotes the σ-algebra generated by $\tilde{B}_s$ for $s < \epsilon$. We used the fact that for a fixed ϵ, each set $\{\tilde{B}_s < 0\}$ for $s \in (0,\epsilon) \cap \mathbb{Q}$ is contained in this σ-algebra, and thus so is their countable union. It follows from Blumenthal's 0-1 law [9, Theorem 7.2.3] that $\mathbb{P}(\tau = 0) \in \{0,1\}$. To complete the proof, it suffices to show that $\mathbb{P}(\tau = 0) > 0$. By (3.1), $B_{s/T}$ is distributed normally with mean 0 and variance $\sigma^2 = (s/T)(1 - s/T)$. We observe that for any $s \in (0,T)$,

$$\mathbb{P}(\tau \le s) \ge \mathbb{P}(B_{s/T} < -sy/T) = \mathbb{P}\left(\sigma\mathcal{N}(0,1) > (s/T)y\right) = \mathbb{P}\left(\mathcal{N}(0,1) > y\sqrt{s/(T-s)}\right).$$

As $s \to 0$, the probability on the right tends to $\mathbb{P}(\mathcal{N}(0,1) > 0) = 1/2$. Since $\{\tau = 0\} = \bigcap_{n=1}^{\infty}\{\tau \le 1/n\}$ and $\{\tau \le 1/(n+1)\} \subset \{\tau \le 1/n\}$, we conclude that

$$\mathbb{P}(\tau = 0) = \lim_{n\to\infty} \mathbb{P}(\tau \le 1/n) \ge 1/2.$$

Therefore $\mathbb{P}(\tau = 0) = 1$. ∎

The second lemma shows that a difference of two independent Brownian bridges is another Brownian bridge.

Lemma 8.5. *Let $a, b, x_1, y_1, x_2, y_2, \in \mathbb{R}$ with $a < b$. Let $B_1(t)$, $B_2(t)$ be independent Brownian bridges from on $[a,b]$ from x_1 to y_1 and from x_2 to y_2 respectively, as defined in 2.2. If $B(t) = B_1(t) - B_2(t)$ for $t \in [a,b]$, then B is itself a Brownian bridge on $[a,b]$.*

Proof. By definition, for $i = 1,2$ we have

$$B_i(t) = (b-a)^{1/2} \cdot \tilde{B}_i\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot x_i + \left(\frac{t-a}{b-a}\right) \cdot y_i,$$

with $\tilde{B}_i(t) = W_t^i - tW_1^i$ for independent Brownian motions W^1 and W^2. We have

$$B_1(t) - B_2(t) = (b-a)^{1/2} \cdot (\tilde{B}_1 - \tilde{B}_2)\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot (x_1 - x_2) + \left(\frac{t-a}{b-a}\right) \cdot (y_1 - y_2). \tag{8.14}$$

Note that the process $\tilde{B}_1 - \tilde{B}_2$ is a linear combination of continuous Gaussian mean 0 processes, so it is a continuous Gaussian mean 0 process, and is thus characterized by its covariance. Since $\tilde{B}_1(\cdot)$ and $\tilde{B}_2(\cdot)$ are both Gaussian with mean 0 and the same covariance, their difference $\tilde{B}_1(\cdot) - \tilde{B}_2(\cdot)$ is also Gaussian with the same mean and covariance. This implies that $\tilde{B}_1 - \tilde{B}_2$ is itself a Brownian bridge $\tilde{B}$ on $[a,b]$, and hence equation 8.14 can be rewritten

$$B_1(t) - B_2(t) = (b-a)^{1/2} \cdot \tilde{B}\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot (x_1 - x_2) + \left(\frac{t-a}{b-a}\right) \cdot (y_1 - y_2).$$

This is a Brownian bridge on $[a,b]$ from $x_1 - x_2$ to $y_1 - y_2$. ∎

To conclude this section, we prove Lemma 4.7.

Proof. Without loss of generality we may assume that $\mathcal{L}^N$ is the weak limit of $(f^N - \lambda s^2)/\sqrt{p(1-p)}$ as $N \to \infty$. By the Skorohod representation theorem, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting random variables $\mathcal{X}^N$ and $\mathcal{X}$ with the laws of f^N and f^∞ respectively, such that $\mathcal{X}^N \to \mathcal{X}$ uniformly on compact sets as $N \to \infty$, pointwise on all of Ω. In particular, $\mathcal{X}^N(s) \to \mathcal{X}(s)$. We have $f_i^N(s) = N^{-\alpha/2}(L_i^N(sN^\alpha) - psN^\alpha) + \lambda s^2$, so $\mathcal{X}_i^N(s) = N^{-\alpha/2}(\mathcal{L}_i^N(sN^\alpha) - psN^\alpha)/\sqrt{p(1-p)}$, where $\mathcal{L}^N$ has the law of L^N.

Suppose that $\mathcal{X}_i(s) = \mathcal{X}_{i+1}(s)$ for some $i \in [\![1, k-2]\!]$. Then we have $\mathcal{X}_i^N(s) - \mathcal{X}_{i+1}^N(s) \to 0$, i.e., $N^{-\alpha/2}(\mathcal{L}_i^N(sN^\alpha) - \mathcal{L}_{i+1}^N(sN^\alpha)) \to 0$ as $N \to \infty$. Let us write $a = \lfloor sN^\alpha \rfloor N^{-\alpha}$, $b = \lceil (s+2)N^\alpha \rceil N^{-\alpha}$ and $x^N = \mathcal{L}_i^N(aN^\alpha) - \mathcal{L}_{i+1}^N(aN^\alpha)$, $y^N = \mathcal{L}_i^N(bN^\alpha) - \mathcal{L}_{i+1}^N(bN^\alpha)$. Then $N^{-\alpha/2}x^N \to 0$. If Q_i, Q_{i+1} are independent Bernoulli bridges with laws $\mathbb{P}_{Ber}^{a,b,\mathcal{L}_i^N(aN^\alpha),\mathcal{L}_i^N(bN^\alpha)}$ and $\mathbb{P}_{Ber}^{a,b,\mathcal{L}_{i+1}^N(aN^\alpha),\mathcal{L}_{i+1}^N(bN^\alpha)}$, then $\ell = Q_i - Q_{i+1}$ is a random walk bridge taking values in $\{-1, 0, 1\}$, from (a, x^N) to (b, y^N). Let us denote the law of $N^{-\alpha/2}\ell/\sqrt{p(1-p)}$ by $\mathbb{P}_{diff}^{a,b,x^N,y^N}$.

By Lemma 8.3, $(x^N + N^{-\alpha/2}Q_{i+1} - ptN^\alpha)/\sqrt{p(1-p)}$ and $(x^N + N^{-\alpha/2}Q_i - ptN^\alpha)/\sqrt{p(1-p)}$ converge weakly to the law of two Brownian bridges B^1 and B^2 respectively, and hence their difference $N^{-\alpha/2}\ell/\sqrt{p(1-p)}$ converges weakly to the difference of two independent Brownian bridges, $B^1 - B^2$. By Lemma 8.5, this difference is itself a Brownian bridge B on $[s, s+2]$ from 0 to y, i.e., B has law $\mathbb{P}_{free}^{s,s+2,0,y}$. Therefore $\mathbb{P}_{diff}^{a,b,x^N,y^N}$ converges weakly to $\mathbb{P}_{free}^{s,s+2,0,y}$. With probability one, $\min_{t \in [s,s+2]} B_t < 0$ by Lemma 8.4. Thus given $\delta > 0$, we can choose N large enough so that the probability of $N^{-\alpha/2}\ell/\sqrt{p(1-p)}$, or equivalently ℓ, remaining above 0 on $[a, b]$ is less than δ. Thus for large enough N we have

$$\begin{aligned} &\mathbb{P}\left(f_i^\infty(s) = f_{i+1}^\infty(s)\right) \leq \mathbb{P}\left(\mathbb{P}_{diff}^{a,b,x^N,y^N}\left(\sup_{s \in [a,b]} \ell(s) \geq 0\right) < \delta\right) \leq \\ &\mathbb{P}\left(Z(a, b, \mathcal{L}^N(aN^\alpha), \mathcal{L}^N(bN^\alpha), \infty, \mathcal{L}_k^N) < \delta\right). \end{aligned} \tag{8.15}$$

Here, Z denotes the acceptance probability of Definition 2.22. This is the probability that $k-1$ independent Bernoulli bridges $Q_1, \dots, Q_{k-1}$ on $[a, b]$ with entrance and exit data $\mathcal{L}^N(a)$ and $\mathcal{L}^N(b)$ do not cross one another or $\mathcal{L}_k^N$. The last inequality follows because ℓ has the law of the difference of Q_i and Q_{i+1}, and the acceptance probability is bounded above by the probability that Q_i and Q_{i+1} do not cross, i.e., that $Q_i - Q_{i+1} \geq 0$. By Proposition 4.1, given $\epsilon > 0$ we can choose δ so that the probability on the right in (8.15) is $< \epsilon$. We conclude that

$$\mathbb{P}\left(f_i^\infty(s) = f_{i+1}^\infty(s)\right) = 0.$$

□

8.5. **Proof of Lemmas 3.1 and 3.2.** We will prove the following lemma, of which the two lemmas are immediate consequences. In particular, Lemma 3.1 is the special case when $g^b = g^t$, and Lemma 3.2 is the case when $\vec{x} = \vec{x}\,'$ and $\vec{y} = \vec{y}\,'$. We argue in analogy to [6, Lemma 5.6].

Lemma 8.6. *Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, $S \subseteq [\![T_0, T_1]\!]$, and two functions $g^b, g^t : [\![T_0, T_1]\!] \to [-\infty, \infty)$ with $g^b \leq g^t$ on S. Also fix $\vec{x}, \vec{y}, \vec{x}\,', \vec{y}\,' \in \mathfrak{W}_k$ such that $x_i \leq x_i'$, $y_i \leq y_i'$ for $1 \leq i \leq k$. Assume that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b; S)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}\,', \vec{y}\,', \infty, g^t; S)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ (resp. $\mathfrak{L}^b$) under $\mathbb{P}$ is given by $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}$ (resp. $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$) and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}_i^t(r) \geq \mathfrak{L}_i^b(r)$ for all $i = 1, \dots, k$ and $r \in [\![T_0, T_1]\!]$.*

Proof. Throughout the proof, we will write $\Omega_{a,S}$ to mean $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b; S)$ and $\Omega'_{a,S}$ to mean $\Omega_{avoid}(T_0, T_1, \vec{x}\,', \vec{y}\,', \infty, g^t; S)$. We split the proof into two steps.

Step 1. We first aim to construct a Markov chain $(X^n, Y^n)_{n\geq 0}$, with $X^n \in \Omega_{a,S}$, $Y^n \in \Omega'_{a,S}$, with initial distribution given by

$$X_i^0(t) = \min(x_i + t - T_0,\, y_i), \qquad Y_i^0(t) = \min(x_i' + t - T_0,\, y_i'),$$

for $t \in [\![T_0, T_1]\!]$ and $1 \leq i \leq k$. First observe that we do in fact have $X^0 \in \Omega_{a,S}$, since $X_i^0(T_0) = x_i$, $X_i^0(T_1) = y_i$, $X_i^0(t) \leq \min(x_{i-1}+t-T_0,\, y_{i-1}) = X_{i-1}^0(t)$, and $X_k^0(t) \geq x_i + t - T_0 \geq g^b(T_0)+t-T_0 \geq g^b(t)$. We also note here that X^0 is *maximal* on the entire space $\Omega(T_0, T_1, \vec{x}, \vec{y})$, in the sense that for any $Z \in \Omega(T_0, T_1, \vec{x}, \vec{y})$, we have $Z_i(t) \leq X_i^0(t)$ for all $t \in [\![T_0, T_1]\!]$. In particular, X^0 is maximal on $\Omega_{a,S}$. Likewise, we see that Y^0 is maximal on $\Omega'_{a,S}$.

We want the chain (X^n, Y^n) to have the following properties:

(1) $(X^n)_{n\geq 0}$ and $(Y^n)_{n\geq 0}$ are both Markov in their own filtrations,
(2) (X^n) is irreducible and aperiodic, with invariant distribution $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber;S}$,
(3) (Y^n) is irreducible and aperiodic, with invariant distribution $\mathbb{P}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}_{avoid,Ber;S}$,
(4) $X_i^n \leq Y_i^n$ on $[\![T_0, T_1]\!]$ for all $n \geq 0$ and $1 \leq i \leq k$.

This will allow us to conclude convergence of X^n and Y^n to these two uniform measures.

We specify the dynamics of (X^n, Y^n) as follows. At time n, we uniformly sample a triple $(i, t, z) \in [\![1, k]\!] \times [\![T_0, T_1]\!] \times [\![x_k, y_1' - 1]\!]$. We also flip a fair coin, with $\mathbb{P}(\text{heads}) = \mathbb{P}(\text{tails}) = 1/2$. We update X^n and Y^n using the following procedure. If $j \neq i$, we leave X_j, Y_j unchanged, and for all points $s \neq t$, we set $X_i^{n+1}(s) = X_i^n(s)$. If $T_0 < t < T_1$, $X_i^n(t-1) = z$, and $X_i^n(t+1) = z+1$ (note that this implies $X_i^n(t) \in \{z, z+1\}$), we consider two cases. If $t \in S$, then we set

$$X_i^{n+1}(t) = \begin{cases} z+1, & \text{if heads,} \\ z, & \text{if tails,} \end{cases}$$

assuming this does not cause $X_i^{n+1}(t)$ to fall below $X_{i+1}^n(t)$, with the convention that $X_{k+1}^n = g^b$. If $t \notin S$, we perform the same update regardless of whether it results in a crossing. In all other cases, we leave $X_i^{n+1}(t) = X_i^n(t)$. We update Y^n using the same rule, with g^t in place of g^b.

We first observe that X^n and Y^n are in fact non-intersecting on S for all n. Note X^0 is non-intersecting, and if X^n is non-intersecting, then the only way X^{n+1} could be intersecting on S is if the update were to push $X_i^{n+1}(t)$ below $X_{i+1}^n(t)$ for some i, t with $t \in S$. But any update of this form is suppressed, so it follows by induction that $X^n \in \Omega_{a,S}$ for all n. Similarly, we see that $Y^n \in \Omega'_{a,S}$.

It is easy to see that (X^n, Y^n) is a Markov chain, since at each time n, the value of (X^{n+1}, Y^{n+1}) depends only on the current state (X^n, Y^n), and not on the time n or any of the states prior to time n. Moreover, the value of X^{n+1} depends only on the state X^n, not on Y^n, so (X^n) is a Markov chain in its own filtration. The same applies to (Y^n). This proves the property (1) above.

We now argue that (X^n) and (Y^n) are irreducible. Fix any $Z \in \Omega_{a;S}$. As observed above, we have $Z_i \leq X_i^0$ on $[\![T_0, T_1]\!]$ for all i. We argue that we can reach the state Z starting from X^0 in some finite number of steps with positive probability. Due to the maximality of X^0, we only need to move the paths downward. If we do this starting with the bottom path, then there is no danger of the paths X_i crossing on S, or of X_k crossing g^b on S. To ensure that $X_k^n = Z_k$, we successively sample triples (k, t, z) as follows. We initialize $t = T_0 + 1$. If $X_k^n(t) = Z_k(t)$, we increment t by 1. Otherwise, we have $X_k^n(t) > Z_k(t)$, so we set $z = X_k^n(t) - 1$ and flip tails. This may or may not push $X_k(t)$ downwards by 1. We then increment t and repeat this process. If t reaches $T_1 - 1$, then at the increment we reset $t = T_0 + 1$. After finitely many steps, X_k will agree with Z_k on all of $[\![T_0, T_1]\!]$. We then repeat this process for X_i^n and Z_i, with i descending. Since each of these samples and flips has positive probability, and this process terminates in finitely many steps, the probability of transitioning from X^n to Z after some number of steps is positive. The same reasoning applies to show that (Y^n) is irreducible.

To see that the chains are aperiodic, simply observe that if we sample a triple (i, T_0, z) or (i, T_1, z), then the states of both chains will be unchanged.

To see that the uniform measure $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber;S}$ on $\Omega_{a,S}$ is invariant for (X^n), fix any $\omega \in \Omega_{a,S}$. For simplicity, write μ for the uniform measure. Then for all $\tau \in \Omega_{a,S}$, we have $\mu(\tau) = 1/|\Omega_{a,S}|$. Hence

$$\sum_{\tau\in\Omega_{a,S}} \mu(\tau)\mathbb{P}(X^{n+1} = \omega \,|\, X^n = \tau) = \frac{1}{|\Omega_{a,S}|}\sum_{\tau\in\Omega_{a,S}} \mathbb{P}(X^{n+1} = \omega \,|\, X^n = \tau) =$$
$$\frac{1}{|\Omega_{a,S}|}\sum_{\tau\in\Omega_{a,S}} \mathbb{P}(X^{n+1} = \tau \,|\, X^n = \omega) = \frac{1}{|\Omega_{a,S}|}\cdot 1 = \mu(\omega).$$

The second equality is clear if $\tau = \omega$. Otherwise, note that $\mathbb{P}(X_{n+1} = \omega \,|\, X_n = \tau) \neq 0$ if and only if τ and ω differ only in one indexed path (say the ith) at one point t, where $|\tau_i(t) - \omega_i(t)| = 1$, and this condition is also equivalent to $\mathbb{P}(X^{n+1} = \tau \,|\, X^n = \omega) \neq 0$. If $X^n = \tau$, there is exactly one choice of triple (i, t, z) and one coin flip which will ensure $X_i^{n+1}(t) = \omega(t)$, i.e., $X^{n+1} = \omega$. Conversely, if $X^n = \omega$, there is one triple and one coin flip which will ensure $X^{n+1} = \tau$. Since the triples are sampled uniformly and the coin flips are fair, these two conditional probabilities are in fact equal. This proves (2), and an analogous argument proves (3).

Lastly, we argue that $X_i^n \le Y_i^n$ on $[\![T_0, T_1]\!]$ for all $n \ge 0$ and $1 \le i \le k$. This is of course true at $n = 0$. Suppose it holds at some $n \ge 0$, and suppose that we sample a triple (i, t, z). Then the update rule can only change the values of the $X_i^n(t)$ and $Y_i^n(t)$. Notice that the values can change by at most 1, and if $Y_i^n(t) - X_i^n(t) = 1$, then the only way the ordering could be violated is if Y_i were lowered and X_i were raised at the next update. But this is impossible, since a coin flip of heads can only raise or leave fixed both curves, and tails can only lower or leave fixed both curves. Thus it suffices to assume $X_i^n(t) = Y_i^n(t)$.

There are two cases to consider that violate the ordering of $X_i^{n+1}(t)$ and $Y_i^{n+1}(t)$. Either (i) $X_i(t)$ is raised but $Y_i(t)$ is left fixed, or (ii) $Y_i(t)$ is lowered yet $X_i(t)$ is left fixed. These can only occur if the curves exhibit one of two specific shapes on $[\![t-1, t+1]\!]$. For $X_i(t)$ to be raised, we must have $X_i^n(t-1) = X_i^n(t) = X_i^n(t+1) - 1$, and for $Y_i(t)$ to be lowered, we must have $Y_i^n(t-1) - 1 = Y_i^n(t) = Y_i^n(t+1)$. From the assumptions that $X_i^n(t) = Y_i^n(t)$, and $X_i^n \le Y_i^n$, we observe that both of these requirements force the other curve to exhibit the same shape on $[\![t-1, t+1]\!]$. Then the update rule will be the same for both curves for either coin flip, proving that both (i) and (ii) are impossible.

Step 2. It follows from (2) and (3) and [14, Theorem 1.8.3] that $(X^n)_{n\ge0}$ and $(Y^n)_{n\ge0}$ converge weakly to $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber;S}$ and $\mathbb{P}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}_{avoid,Ber;S}$ respectively. In particular, (X^n) and (Y^n) are tight, so $(X^n, Y^n)_{n\ge0}$ is tight as well. By Prohorov's theorem, it follows that (X^n, Y^n) is relatively compact. Let (n_m) be a sequence such that (X^{n_m}, Y^{n_m}) converges weakly. Then by the Skorohod representation theorem [1, Theorem 6.7], it follows that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting random variables $\mathfrak{X}^n$, $\mathfrak{Y}^n$ and $\mathfrak{X}, \mathfrak{Y}$ taking values in $\Omega_{a,S}, \Omega'_{a,S}$ respectively, such that

(1) The law of $(\mathfrak{X}^n, \mathfrak{Y}^n)$ under $\mathbb{P}$ is the same as that of (X^n, Y^n),
(2) $\mathfrak{X}^n(\omega) \longrightarrow \mathfrak{X}(\omega)$ for all $\omega \in \Omega$,
(3) $\mathfrak{Y}^n(\omega) \longrightarrow \mathfrak{Y}(\omega)$ for all $\omega \in \Omega$.

In particular, (1) implies that $\mathfrak{X}^{n_m}$ has the same law as X^{n_m}, which converges weakly to $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber;S}$. It follows from (2) and the uniqueness of limits that $\mathfrak{X}$ has law $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber;S}$. Similarly, $\mathfrak{Y}$ has law $\mathbb{P}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}_{avoid,Ber;S}$. Moreover, condition (4) in Step 1 implies that $\mathfrak{X}_i^n \le \mathfrak{Y}_i^n$, $\mathbb{P}$-a.s., so $\mathfrak{X}_i \le \mathfrak{Y}_i$ for $1 \le i \le k$, $\mathbb{P}$-a.s. Thus we can take $\mathfrak{L}^b = \mathfrak{X}$ and $\mathfrak{L}^t = \mathfrak{Y}$.

$\square$

9. Appendix B

The goal of this section is to establish the weak convergence of scaled avoiding Bernoulli line ensemble. We consider the $[\![1,k]\!]$-indexed line ensembles with distribution given by $\mathbb{P}^{0,T,\vec{x},\vec{y},\infty,-\infty}_{avoid,Ber}$ in the sense of Definition 2.15. Recall that this is just the law of k independent Bernoulli random walks that have been conditioned to start from $\vec{x}=(x_1,\dots,x_k)$ at time 0 and and at $\vec{y}=(y_1,\cdots,y_k)$ at time T and are always ordered. Here $\vec{x}$, $\vec{y}\in\mathfrak{W}_k$ satisfy $T\ge y_i-x_i\ge 0$ for $i=1,\dots,k$. We will drop the infinities and simply write $\mathbb{P}^{0,T,\vec{x},\vec{y}}_{avoid,Ber}$ for the measure.

This section will be divided into 5 subsections. In Section 9.1, we introduce some definitions and formulate the precise statements of two main results we want to prove as Proposition 9.2 and Proposition 9.3. In Section 9.2, we introduce some fundamental knowledge about Skew Schur Polynomials and give the distribution of avoiding Bernoulli line ensembles at integer times through Skew Schur Polynomials as Lemma 9.8. In Section 9.3, we will prove our first main result Proposition 9.2. In Section 9.4 we introduce some notations and results about multi-indices and multivariate functions which paves the way for proof of Proposition 9.3. Section 9.5 will prove our second main result Proposition 9.3.

9.1. **Definitions and Main Results.** We start by introducing some helpful notations.

Definition 9.1. Fix $p,t\in(0,1)$, $k\in\mathbb{N}$, $\vec{a}$, $\vec{b}\in\mathbb{W}_k$ are two vectors in Weyl chamber defined in Definition 2.7. Suppose that $\vec{x}^T=(x_1^T,\cdots,x_k^T)$ and $\vec{y}^T=(y_1^T,\cdots,y_k^T)$ are two sequences of k-dimensional vectors in $\mathfrak{W}_k$ such that

$$\lim_{T\to\infty}\frac{x_i^T}{\sqrt{T}}=a_i \text{ and } \lim_{T\to\infty}\frac{y_i^T-pT}{\sqrt{T}}=b_i$$

for $i=1,\dots,k$. Define the sequence of random k-dimensional vectors Z^T by

$$Z^T=\Big(\frac{L_1(tT)-ptT}{\sqrt{T}},\cdots,\frac{L_k(tT)-ptT}{\sqrt{T}}\Big) \tag{9.1}$$

where $(L_1,\cdots,L_k)$ is $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$-distributed.

We also introduce some constants below

$$\begin{aligned} &c_1(p,t)=\frac{1}{p(1-p)t},\quad c_2(p,t)=\frac{1}{p(1-p)(1-t)},\quad c_3(p,t)=\frac{1}{2p(1-p)t(1-t)}\\ &Z=(2\pi)^{\frac{k}{2}}(p(1-p)t(1-t))^{\frac{k}{2}}\cdot e^{\frac{c_1(t,p)}{2}\sum_{i=1}^k a_i^2}\cdot e^{\frac{c_2(t,p)}{2}\sum_{i=1}^k b_i^2}\det\left[e^{-\frac{1}{2p(1-p)}(b_i-a_j)^2}\right]_{i,j=1}^k \end{aligned} \tag{9.2}$$

and define the function $\rho(z_1,\cdots,z_k)\equiv\rho(\vec{z})$ as the following:

$$\rho(z_1,\cdots,z_k)=\frac{1}{Z}\cdot\mathbf{1}_{\{z_1>\cdots>z_k\}}\cdot\det\left[e^{c_1(t,p)a_iz_j}\right]_{i,j=1}^k\det\left[e^{c_2(t,p)b_iz_j}\right]_{i,j=1}^k\prod_{i=1}^k e^{-c_3(t,p)z_i^2} \tag{9.3}$$

We will prove that the function $\rho(z)$ defined in (9.3) is a probability density function, meaning that it is non-negative and integrates to 1 over $\mathbb{R}^k$. Since this is an important ingredient of our results, we isolate it as Lemma 9.10 and will prove it in Section 9.3. For now, we assume that $\rho(\vec{z})$ in (9.3) is a density so that we can state our first main result in the following, which gives the limiting distribution of Z^T when vectors $\vec{a}$ and $\vec{b}$ contain distinct values.

Proposition 9.2. *Assume the same notation as in the Definition 9.1. When $a_1>\cdots>a_k$ and $b_1>\cdots>b_k$ are all distinct, the random vector Z^T converges weakly to a continuous distribution with the density in (9.3).*

Proposition 9.2 states the result when $\vec{a}$ and $\vec{b}$ consist of distinct values. When the values in $\vec{a}$ and $\vec{b}$ start to collide, the three determinants in the density function (9.3) will vanish(one in constant Z in equation (9.2) and the other two are in the expression of equation (9.3)). In the following, we are going to formulate the result under this new situation. We will construct a modified density function and the random vector Z^T will weakly converge to this new density function.

Suppose vectors $\vec{a}$ and $\vec{b}$ cluster as the following:

$$\begin{aligned} \vec{a} &= (a_1, \cdots, a_k) = (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p}) \\ \vec{b} &= (b_1, \cdots, b_k) = (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q}) \end{aligned} \tag{9.4}$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_p$, $\beta_1 > \beta_2 > \cdots > \beta_q$ and $\sum_{i=1}^p m_i = \sum_{i=1}^q n_i = k$. Denote $\vec{m} = (m_1, \cdots, m_p)$, $\vec{n} = (n_1, \cdots, n_q)$ and define two determinants $\varphi(\vec{a}, \vec{z}, \vec{m})$ and $\psi(\vec{b}, \vec{z}, \vec{n})$ below:

$$\begin{aligned} \varphi(\vec{a}, \vec{z}, \vec{m}) &= \det \begin{bmatrix} \big((c_1(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_1 z_j}\big)_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ \big((c_1(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_p z_j}\big)_{\substack{i=1,\cdots,m_p \\ j=1,\cdots,k}} \end{bmatrix} \\ \psi(\vec{b}, \vec{z}, \vec{n}) &= \det \begin{bmatrix} \big((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_1 z_j}\big)_{\substack{i=1,\cdots,n_1 \\ j=1,\cdots,k}} \\ \vdots \\ \big((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_q z_j}\big)_{\substack{i=1,\cdots,n_q \\ j=1,\cdots,k}} \end{bmatrix} \end{aligned} \tag{9.5}$$

Then define the function

$$H(\vec{z}) = \varphi(\vec{a}, \vec{z}, \vec{m})\psi(\vec{b}, \vec{z}, \vec{n}) \prod_{i=1}^k e^{-c_3(t,p)z_i^2} \tag{9.6}$$

we can prove that $H(\vec{z})$ in (9.6) is non-negative and integrable over $\mathbb{R}^k$, so that we can multiply it with the normalizing constant $Z_c = \int_{\mathbb{R}^k} H(z) \cdot \mathbf{1}_{\{z_1 > \cdots > z_k\}} dz < \infty$ (the subscript c is for "collide") and make it a probability density function:

$$\rho_c(z_1, \cdots, z_k) = \frac{1}{Z_c} \cdot \mathbf{1}_{\{z_1 > \cdots > z_k\}} \cdot \varphi(\vec{a}, \vec{z}, \vec{m})\psi(\vec{b}, \vec{z}, \vec{n}) \prod_{i=1}^k e^{-c_3(t,p)z_i^2} \tag{9.7}$$

Now we are ready to state our second main result, which gives the weak convergence of Z^T when $\vec{a}$ and $\vec{b}$ have collided values.

Proposition 9.3. *Assume the same notation as in the Definition 9.1 and suppose vectors $\vec{a}$, $\vec{b}$ has the form in (9.4). Then, the random vector Z^T converges weakly to a continuous distribution with density in (9.7).*

9.2. **Skew Schur polynomials and distribution of avoiding Bernoulli line ensembles.** First, We give some definitions and elementary results regarding skew Schur polynomials, which are mainly based on [12, Chapter 1].

Definition 9.4. *Partition, Interlaced, Conjugate*

(1) A *partition* is an infinite sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r, \cdots)$ of non-negative integers in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \cdots$ and containing only finitely many non-zero terms. The non-zero λ_i are called *parts* of λ, the number of parts is called the *length* of the partition λ, denoted by $l(\lambda)$, and the sum of the parts is the *weight* of λ, denoted by $|\lambda|$.

(2) Suppose λ and μ are two partitions, we denote $\lambda \supset \mu$ if $\lambda_i \geq \mu_i$ for all $i \in \mathbb{Z}^+$, and we can define a new partition $\lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \cdots)$.
(3) Partitions $\lambda = (\lambda_1, \lambda_2, \cdots)$ and $\mu = (\mu_1, \mu_2, \cdots)$ are call *interlaced*, denoted by $\mu \preceq \lambda$, if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$.
(4) The *conjugate* of a partition λ is the partition λ' such that

$$\lambda'_i = \max_{j \geq 1} \{j : \lambda_j \geq i\}$$

In particular, $\lambda'_1 = l(\lambda)$, $\lambda_1 = l(\lambda')$ and notice that $\lambda'' = \lambda$. For example, the conjugate of (5441) is (43331).

According to Definition 9.4, we directly get that if $\mu \subset \lambda$ then $l(\lambda) \geq l(\mu)$ and $l(\lambda') \geq l(\mu')$. Also, $\mu \preceq \lambda$ implies $\mu \subset \lambda$. We can also derive the following corollary that is not very immediate.

Corollary 9.5. *If $\mu \preceq \lambda$ are interlaced, then $\lambda'_i - \mu'_i = 0$ or 1 for every $i \geq 1$,.*

Proof. By definition, $\lambda'_i = \max\{j : \lambda_j \geq i\}$ and $\mu'_i = \max\{j : \mu_j \geq i\}$. Since $\mu \preceq \lambda$ are interlaced, we have $\lambda_j \geq \mu_j \geq \lambda_{j+1}$ for every $j \geq 1$, where the first inequality $\lambda_j \geq \mu_j$ directly implies $\lambda'_i \geq \mu'_i$. Suppose there exists an i such that $\lambda'_i - \mu'_i \geq 2$. Then, by definition of μ'_i and λ'_i we have $\lambda_{\lambda'_i} \geq i$ and $\mu_{\mu'_i+1} < i$. When $\lambda'_i - \mu'_i \geq 2$, we have $\lambda_{\mu'_i+2} \geq \lambda_{\lambda'_i} \geq i > \mu_{\mu'_i+1}$, which contradicts the fact that $\mu \preceq \lambda$ are interlaced. Therefore, we conclude that $\lambda'_i - \mu'_i$ can only be 0 or 1. □

Definition 9.6. *Elementary Symmetric Function*
For each integer $r \geq 0$, the r-th *elementary symmetric function* e_r is the sum of all products of r distinct variables x_i, so that $e_0 = 1$ and

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \tag{9.8}$$

for $r \geq 1$. For $r < 0$, we define e_r to be zero. In particular, when $x_1 = x_2 = \cdots = x_n = 1$, $x_{n+1} = x_{n+2} = \cdots = 0$, e_r is just the binomial coefficient when $0 \leq r \leq n$:

$$e_r(1^n) = \binom{n}{r}$$

and $e_r = 0$ when $r > n$.

Next, we introduce Skew Schur Polynomial based on [12, Chapter 1, (5.5), (5.11), (5.12)].

Definition 9.7. *Skew Schur Polynomial, Jacob-Trudi Formula*

(1) Suppose $\mu \subset \lambda$ are partitions. If $\mu \preceq \lambda$ are interlaced, then the *skew Schur polynomial* $s_{\lambda/\mu}$ with single variable x is defined by $s_{\lambda/\mu}(x) = x^{|\lambda - \mu|}$. Otherwise, we define $s_{\lambda/\mu}(x) = 0$.
(2) Suppose $\mu \subset \lambda$ are two partitions, define the *skew Schur polynomial* $s_{\lambda/\mu}$ with respect to variables $x_1, x_2, \cdots, x_n$ by

$$s_{\lambda/\mu}(x_1, \cdots, x_n) = \sum_{(\nu)} \prod_{i=1}^{n} s_{\nu^i/\nu^{i-1}}(x_i) = \sum_{(\nu)} \prod_{i=1}^{n} x_i^{|\nu^i - \nu^{i-1}|} \tag{9.9}$$

summed over all sequences $(\nu) = (\nu^0, \nu^1, \cdots, \nu^n)$ of partitions such that $\nu^0 = \mu$, $\nu^n = \lambda$ and $\nu^0 \preceq \nu^1 \preceq \cdots \preceq \nu^n$. In particular, when $x_1 = x_2 = \cdots = x_n = 1$, the skew Schur polynomial is just the number of such sequences of interlaced partitions (ν). This definition also implies the following *branching relation* of skew Schur polynomials:

$$s_{\kappa/\mu} = \sum_{\lambda} s_{\kappa/\lambda} \cdot s_{\lambda/\mu} \tag{9.10}$$

(3) We also have the following *Jacob-Trudi Formula* [12, Chapter 1, (5.5)] for the skew Schur polynomial:

$$s_{\lambda/\mu} = \det\left(e_{\lambda'_i - \mu'_j - i + j}\right)_{1\le i,j\le m} \tag{9.11}$$

where $m \ge l(\lambda')$, and e_r is the elementary symmetric function in Definition 9.6.

Based on the above preparation, we are ready to state the following lemma giving the distribution of avoiding Bernoulli line ensembles at time $\lfloor tT \rfloor$.

Lemma 9.8. *Assume the same notations as in Section 9.1, denote $m = \lfloor tT \rfloor$, $n = T - \lfloor tT \rfloor$. Then, the avoiding Bernoulli line ensemble at time m has the following distribution:*

$$\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k) = \frac{s_{\lambda'/\mu'}(1^m) \cdot s_{\kappa'/\lambda'}(1^n)}{s_{\kappa'/\mu'}(1^T)} \tag{9.12}$$

where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k$ are positive integers, $s_{\lambda/\mu}$ denote skew Schur polynomials and they are specialized in all parameters equal to 1. The μ partition is just the vector $\vec{x}^T$ and the κ partition should be $\vec{y}^T$.

Remark 9.9. Here we let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k$ be positive integers, although they could potentially be negative. However, we can shift all the endpoints up such that all possible λ_i are positive. Also, in the proof we treat finite dimensional vectors as partitions because as long as we add infinitely many zeros at their ends we can and make them "partitions" in Definition 9.4.

Proof. Let $\Omega(0, T, \vec{x}^T, \vec{y}^T)$ be the set of all avoiding Bernoulli line ensembles from $\vec{x}^T$ to $\vec{y}^T$ and define the set

$$TB^T_{\lambda/\mu} := \{(\lambda^0, ..., \lambda^T) \mid \lambda^0 = \mu, \lambda^T = \lambda, \lambda^i \preceq \lambda^{i+1} \text{ for } i = 0, \cdots, T-1\}$$

From the result regarding the relationship between number of sequences of interlaced partitions and skew Schur polynomials (Definition 9.7, (2)), we have $|TB^T_{\lambda/\mu}| = s_{\lambda/\mu}(1^T)$. In the rest of the proof, we want to establish the fact that there is a bijection between $\Omega(0, T, \vec{x}^T, \vec{y}^T)$ and $TB^T_{\kappa'/\mu'}$ so that we have $|\Omega(0, T, \vec{x}^T, \vec{y}^T)| = s_{\kappa'/\mu'}$. Similarly, we get $|\Omega(0, m, \vec{x}^T, \lambda)| = s_{\lambda'/\mu'}$ and $|\Omega(m, T, \lambda, \vec{y}^T)| = s_{\kappa'/\lambda'}$. Then, since $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$ puts uniform measure on the set $\Omega(0, T, \vec{x}^T, \vec{y}^T)$, we conclude

$$\begin{aligned}\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k) =& \frac{|\Omega(0, m, \vec{x}^T, \lambda)| \cdot |\Omega(m, T, \lambda, \vec{y}^T)|}{|\Omega(0, T, \vec{x}^T, \vec{y}^T)|} \\ =& \frac{s_{\lambda'/\mu'}(1^m) \cdot s_{\kappa'/\lambda'}(1^n)}{s_{\kappa'/\mu'}(1^T)}\end{aligned}$$

where $\lambda = (\lambda_1, \cdots, \lambda_k)$ is a partition such that $\vec{x}^T \subset \lambda \subset \vec{y}^T$, thus finishing the proof.

Now we prove that there exists a bijection $f : \Omega(0, T, \vec{x}^T, \vec{y}^T) \to TB^T_{\kappa'/\mu'}$. For each line ensemble $\mathfrak{L} \in \Omega(0, T, \vec{x}^T, \vec{y}^T)$ with $\mathfrak{L} = (L_1, \cdots, L_k)$, we define a k-dimensional vector $\lambda^i(\mathfrak{L}) := (\lambda^i_1, \lambda^i_2, ..., \lambda^i_k)$, where $0 \le i \le T$ is an integer and $\lambda^i_\alpha = L_\alpha(i)$. In the following discussion we drop $\mathfrak{L}$ and briefly write λ^i. We claim that their conjugates $(\lambda^i)'$ form interlaced partitions:

$$(\lambda^0)' \preceq (\lambda^1)' \preceq \cdots \preceq (\lambda^T)' \tag{9.13}$$

where $\left(\lambda^0\right)' = \mu'$ and $\left(\lambda^T\right)' = \kappa'$. We first explain that $(\lambda^i)'$ is a partition for every $i = 0, \cdots, T$. Actually, it simply follows from that $(\lambda^i_\alpha)' = \max\{j : \lambda^i_j \ge \alpha\} \ge \max\{j : \lambda^i_j \ge \alpha + 1\} = (\lambda^i_{\alpha+1})'$. Now we prove (9.13), which requires us to show

$$(\lambda^{i+1}_\alpha)' \ge (\lambda^i_\alpha)' \ge (\lambda^{i+1}_{\alpha+1})', \text{ for every } i = 0, \cdots, T-1 \text{ and } \alpha = 1, \cdots, k-1$$

By the definition of Bernoulli random walk, we have $\lambda_j^{i+1} \geq \lambda_j^i \geq \lambda_j^{i+1} - 1$. Therefore, we have

$$\max\{j : \lambda_j^{i+1} \geq \alpha\} \geq \max\{j : \lambda_j^i \geq \alpha\} \geq \max\{j : \lambda_j^{i+1} \geq \alpha + 1\}$$

and this is exactly (9.13). Therefore, we have defined a function $f : \Omega(0, T, \bar{x}^T, \bar{y}^T) \to TB^T_{\kappa'/\mu'}$ by

$$f(\mathfrak{L}) = \left((\lambda^0)', \cdots, (\lambda^T)'\right) \tag{9.14}$$

Next, we prove the function f is in fact a bijection. First, to show injectivity, suppose that there are two Bernoulli line ensembles, $\mathfrak{L}, \widetilde{\mathfrak{L}} \in \Omega(0, T, \bar{x}^T, \bar{y}^T)$ such that $\mathfrak{L} \neq \widetilde{\mathfrak{L}}$. Bernoulli line ensembles are determined by their values at integer times, so this would imply that there exists some (q, r) such that $0 \leq r \leq T$, $0 \leq q \leq k$ and $L_q(r) \neq \widetilde{L}_q(r)$ where L_q and $\widetilde{L}_q$ are components of $\mathfrak{L}$ and $\widetilde{\mathfrak{L}}$ respectively. This implies that $(\lambda^r(\mathfrak{L}))' \neq \left(\lambda^r(\widetilde{\mathfrak{L}})\right)'$, so we have injectivity.

Now, we prove surjectivity. For any sequence of interlaced partitions $\bar{\lambda} = (\lambda^0, \cdots, \lambda^T)$ satisfying $(\lambda^0)' = \vec{x}^T$ and $(\lambda^T)' = \vec{y}^T$, we claim that $(\lambda^0)'$, $(\lambda^1)', \cdots, (\lambda^T)'$ consist of an avoiding Bernoulli line ensemble in $\Omega(0, T, \vec{x}^T, \vec{y}^T)$ by letting $L_\alpha(i) = (\lambda_\alpha^i)'$. Applying Corollary 9.5, we have $L_\alpha(i + 1) - L_\alpha(i) = (\lambda_\alpha^{i+1})' - (\lambda_\alpha^i)'$ can only be 0 or 1, thus $L_\alpha(i)$, $0 \leq i \leq T$ is a Bernoulli random walk for every $1 \leq \alpha \leq k$. In addition, by using $\lambda_\alpha^i \geq \lambda_{\alpha+1}^i$ we get $(\lambda_\alpha^i)' \geq (\lambda_{\alpha+1}^i)'$, which indicates that k Bernoulli random walks avoid each other. Therefore, we proved the surjectivity and complete the proof. □

By Jacob-Trudi formula (9.11) and Lemma 9.8, we further get

$$\mathbb{P}(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k) = \frac{\det\left[e_{\lambda_i - \mu_j + j - i}(1^m)\right]_{i,j=1}^k \cdot \det\left[e_{\kappa_i - \lambda_j + j - i}(1^n)\right]_{i,j=1}^k}{\det\left[e_{\kappa_i - \mu_j + j - i}(1^T)\right]_{i,j=1}^k} \tag{9.15}$$

where $\mu_i = \vec{x}_i^T$ and $\kappa_i = \vec{y}_i^T$.

9.3. **Proof of Proposition 9.2.** In this section, we first prove prove that the function in (9.3) is a density and then prove the weak convergence result in Proposition 9.2. The fact that (9.3) is a density is formulated in the following lemma.

Lemma 9.10. *Assume the same notations as in Section 9.1. Denote the function*

$$\widetilde{\rho}(z_1, \cdots, z_k) = \mathbf{1}_{\{z_1 > z_2 > \cdots > z_k\}} \det\left[e^{c_1(t,p)a_i z_j}\right]_{i,j=1}^k \cdot \det\left[e^{c_2(t,p)b_i z_j}\right]_{i,j=1}^k \cdot \prod_{i=1}^k e^{-c_3(t,p)z_i^2} \tag{9.16}$$

Then $\widetilde{\rho}(z_1, \cdots, z_k) \geq 0$ *for all* $\vec{z} = (z_1, \cdots, z_k) \in \mathbb{R}^k$ *and* $\widetilde{\rho}(z_1, \cdots, z_k) > 0$ *if* $z_1 > z_2 > \cdots > z_k$. *Moreover, the function* $\widetilde{\rho}$ *is integrable on* $\mathbb{R}^k$ *and we have*

$$\int_{\mathbb{R}^k} \widetilde{\rho}(z_1, \cdots, z_k) dz_1 \cdots dz_k = Z \tag{9.17}$$

where the constant Z *is defined in (9.2), thus implying the function* $\rho(\vec{z})$ *in (9.3) is a density.*

To prove Lemma 9.10, we are going to find the asymptotic formula of the probability mass function (9.15) and its relationship with function $\rho(z)$ in (9.3). By Jacob-Trudi formula (9.11), we only need to find the asymptotic formula for elementary symmetric functions $e_{\lambda_i - x_j^T + j - i}(1^m)$, $e_{y_i^T - \lambda_j + j - i}(1^n)$ and $e_{y_i^T - x_j^T + j - i}(1^T)$. By the definition of random vector Z^T in (9.1), we find that

$$\left\{Z_1^T = z_1, \cdots, Z_k^T = z_k\right\} \equiv \{L_1(tT) = \lambda_1, \cdots, L_k(tT) = \lambda_k\} \tag{9.18}$$

where $\lambda_i = z_i\sqrt{T} + ptT$ are integers for $i = 1, \cdots, k$. In addition, $x_i^T = a_i\sqrt{T} + o\left(\sqrt{T}\right)$ and $y_i^T = b_i\sqrt{T} + pT + o\left(\sqrt{T}\right)$ by Definition 9.1. Therefore, we have

$$\begin{aligned} \lambda_i - x_j^T + j - i &= pm + (z_i - a_j)\sqrt{T} + o\left(T^{1/2}\right), \\ y_i^T - \lambda_j + j - i &= pn + (b_i - z_j)\sqrt{T} + o\left(T^{1/2}\right), \\ y_i^T - x_j^T + j - i &= pT + (b_i - a_j)\sqrt{T} + o\left(T^{1/2}\right) \end{aligned} \tag{9.19}$$

Thus, we only need to consider the elementary symmetric functions in the form $e_N(1^n)$, where $N = pn + x\sqrt{n}$ and $x \in [-R, R]$ is bounded. In this case, we have the following lemma giving the asymptotic behavior of $e_N(1^n)$.

Lemma 9.11. *Suppose that $p \in (0,1)$ and $R > 0$ are given. Suppose that $x \in [-R, R]$ and $N = pn + \sqrt{n}x$ is an integer. Then*

$$\begin{aligned} e_N(1^n) = (\sqrt{2\pi})^{-1} \cdot \exp\left(-\frac{x^2}{2(1-p)p}\right) \cdot \exp\left(N\log\left(\frac{1-p}{p}\right)\right) \cdot \exp\left(O(n^{-1/2})\right) \\ \cdot \exp\left(-n\log(1-p) - (1/2)\log n - (1/2)\log\left(p(1-p)\right)\right) \end{aligned} \tag{9.20}$$

where the constant in the big O notation depends on p and R alone. Moreover, there exist positive constants $C, c > 0$ depending on p alone such that for all large enough $n \in \mathbb{N}$ and $N \in [0, n]$,

$$e_N(1^n) \leq C \cdot \exp\left(N\log\frac{1-p}{p} - n\log(1-p) - (1/2)\log n\right) \cdot \exp\left(-cn^{-1}(N - pn)^2\right). \tag{9.21}$$

Remark 9.12. Notice that when $R > 0$ is fixed, $N \in [pn - R\sqrt{n}, pn + R\sqrt{n}]$. However, we specify the range of N by $[0, n]$. First, it is because when $N < 0$ or $N > n$ the elementary symmetric function $e_N(1^n)$ would be zero by Definition 9.6 and the situation becomes trivial. Second, when n is sufficiently large, the interval $[0, n]$ will cover $[pn - R\sqrt{n}, pn + R\sqrt{n}]$, so it's sufficient to consider the case when $N \in [0, n]$.

Proof of Lemma 9.11. For clarity the proof is split into several steps.
Step 1. In this step we prove (9.20). Using the formula for elementary symmetric function (9.6), we obtain

$$e_N(1^n) = \frac{n!}{N!(n-N)!} \tag{9.22}$$

We have the following Stirling's formula [10] that for $n \geq 1$

$$n! = \sqrt{2\pi n}n^n e^{-n} e^{r_n}, \text{ where } \frac{1}{12n+1} < r_n < \frac{1}{12n} \tag{9.23}$$

Applying the Stirling's formula to equation (9.22) implies that

$$\begin{aligned} e_N(1^n) &= \frac{\exp\left((n+1/2)\log n - (N+1/2)\log N - (n-N+1/2)\log(n-N) + O\left(n^{-1}\right)\right)}{\sqrt{2\pi}} \\ &= (\sqrt{2\pi})^{-1} \cdot \exp\left((n+1/2)\log n - (N+1/2)\log\frac{N}{pn} - (n-N+1/2)\log\frac{n-N}{(1-p)n}\right) \\ &\cdot \exp\left(-(N+1/2)\log(pn) - (n-N+1/2)\log((1-p)n) + O\left(n^{-1}\right)\right). \end{aligned} \tag{9.24}$$

Denote $\Delta = \sqrt{n}x = O\left(n^{-1/2}\right)$, and we now use the Taylor expansion of the logarithm and the expression for N to get

$$\log\frac{N}{pn} = \log\left(1 + \frac{\Delta}{pn}\right) = \frac{\Delta}{pn} - \frac{1}{2}\frac{\Delta^2}{p^2n^2} + O\left(n^{-3/2}\right)$$

Analogously, we have

$$\log \frac{n-N}{(1-p)n} = \log\left(1 - \frac{\Delta}{(1-p)n}\right) = -\frac{\Delta}{(1-p)n} - \frac{1}{2}\frac{\Delta^2}{(1-p)^2 n^2} + O\left(n^{-3/2}\right)$$

Plugging the two equations above to equation (9.24) we get

$$\begin{aligned} e_N(1^n) = (\sqrt{2\pi})^{-1} \cdot \exp\left(-(N+1/2)\left[\frac{\Delta}{pn} - \frac{1}{2}\frac{\Delta^2}{p^2n^2} + O\left(n^{-3/2}\right)\right]\right) \\ \cdot \exp\left(-(n-N+1/2)\left[-\frac{\Delta}{(1-p)n} - \frac{1}{2}\frac{\Delta^2}{(1-p)^2n^2} + O\left(n^{-3/2}\right)\right]\right) \\ \cdot \exp\left((n+1/2)\log n - (N+1/2)\log(pn) - (n-N+1/2)\log((1-p)n) + O\left(n^{-1}\right)\right) \end{aligned} \tag{9.25}$$

We next observe that

$$\begin{aligned} &-\frac{\Delta(N+1/2)}{pn} + \frac{(n-N+1/2)\Delta}{(1-p)n} = -\frac{\Delta^2}{p(1-p)n} + O\left(n^{-1/2}\right) \\ &\frac{\Delta^2(N+1/2)}{2n^2p^2} + \frac{\Delta^2(n-N+1/2)}{2(1-p)^2n^2} = \frac{\Delta^2}{2p(1-p)n} + O\left(n^{-1/2}\right) \\ &(n+1/2)\log n - (N+1/2)\log(pn) - (n-N+1/2)\log((1-p)n) = \\ &N\log\frac{1-p}{p} - \frac{1}{2}\log p(1-p) - \frac{1}{2}\log n - n\log(1-p) \end{aligned} \tag{9.26}$$

Plugging (9.26) into (9.25) we arrive at (9.20).

Step 2. In this step we prove (9.21). If $N = 0$ or n we know that $e_N(1^n) = 1$ and then (9.21) is easily seen to hold with $C = 1$ and any $c \in (0, \min(-\log p, -\log(1-p)))$. Thus it suffices to consider the case when $N \in [1, n-1]$ and in the sequel we also assume that $n \geq 2$.

Combining (9.22) and (9.23) we conclude that

$$e_N(1^n) \leq \exp\left((n+1/2)\log n - (N+1/2)\log N - (n-N+1/2)\log(n-N)\right) \tag{9.27}$$

From (9.27) we get for all large enough n that

$$\begin{aligned} &\phi_n := \log\left[e_N(1^n)\cdot\exp\left(-N\log((1-p)/p) + n\log(1-p) + (1/2)\log n\right)\right] \\ &\leq (n+\tfrac{1}{2})\log n - (N+\tfrac{1}{2})\log N - (n-N+\tfrac{1}{2})\log(n-N) - N\log\frac{1-p}{p} + n\log(1-p) + \frac{1}{2}\log n \\ &= (n+1/2)\log n - (N+1/2)\log\frac{N}{pn} - (N+1/2)\log(pn) - (n-N+1/2)\log\frac{n-N}{(1-p)n} \\ &- (n-N+1/2)\log((1-p)n) - N\log\frac{1-p}{p} + n\log(1-p) + (1/2)\log n \\ &= -(N+\tfrac{1}{2})\log\frac{N}{pn} - (n-N+\tfrac{1}{2})\log\frac{n-N}{(1-p)n} - \frac{1}{2}\log\left(p(1-p)\right) \\ &= -(pn+\Delta+1/2)\log\left(1+\frac{\Delta}{pn}\right) - ((1-p)n-\Delta+1/2)\log\left(1-\frac{\Delta}{(1-p)n}\right) - \frac{1}{2}\log\left(p(1-p)\right) \\ &\leq C_1 + \psi_n(\Delta) \end{aligned}$$

where $C_1 > 0$ is sufficiently large depending on p alone and

$$\psi_n(s) = -(pn+s+1/2)\log\left(1+\frac{s}{pn}\right) - ((1-p)n-s+1/2)\log\left(1-\frac{s}{(1-p)n}\right) \tag{9.28}$$

where $s \in [-pn+1, (1-p)n-1]$. We claim that we can find positive constants $C_2 > 0$ and $c > 0$ such that for all n sufficiently large and $s \in [-pn+1, (1-p)n-1]$ we have

$$\psi_n(s) \leq C_2 - cn^{-1}s^2 \tag{9.29}$$

We prove (9.29) in Step 3 below. For now we assume its validity and conclude the proof of (9.21). In view of $\phi_n \leq C_1 + \psi_n(s)$ and (9.29) we know that

$$e_N(1^n) \leq \exp\left(C_1 + C_2 + N\log((1-p)/p) - n\log(1-p) - (1/2)\log n\right) \cdot \exp(-cn^{-1}(N-pn)^2),$$

which proves (9.21) with $C = e^{C_1+C_2}$.

Step 3. In this step we prove (9.29) in the case $s \in [0, n]$. A direct computation gives

$$\begin{aligned} \psi_n'(s) &= -\log\left(1 + \frac{t}{pn}\right) + \log\left(1 - \frac{t}{(1-p)n}\right) + \frac{1}{2}\cdot\frac{1}{pn+t} + \frac{1}{2}\cdot\frac{1}{(1-p)n-t} \\ \psi_n''(s) &= \frac{(n+1)\cdot s^2 + (2p-1)n(n+1)\cdot s + p(p-1)n^2(n+1) + (1/2)n^2}{(pn+s)^2((1-p)n-s)^2} \end{aligned} \tag{9.30}$$

Notice that the numerator of $\psi_n''(s)$ is a quadratic function and its minimum is at $x_{min} = -\frac{(2p-1)n(n+1)}{2(n+1)} = (-p+1/2)n$, which is the midpoint of the interval $[-pn+1, (1-p)n-1]$. Thus, the numerator reaches its maximum at either of the two endpoints of the interval $[-pn+1, (1-p)n-1]$. The denominator is the square of a parabola that reaches its minimum also at the endpoints of the interval $[-pn+1, (1-p)n-1]$. Therefore, we conclude that

$$\begin{aligned} &\psi_n''(s) \leq \psi_n''(-pn+1) = \psi_n''((1-p)n-1) = \frac{-\frac{1}{2}n^2+1}{(n-1)^2} = -\frac{1}{2} - \frac{1}{n-1} + \frac{1}{2}\cdot\frac{1}{(n-1)^2} \\ &\leq -\frac{1}{2}\cdot\frac{1}{n-1} \leq -\frac{1}{2n} = -2cn^{-1} \end{aligned} \tag{9.31}$$

where $c = 1/4$. Next, we prove (9.21) under two cases when $s \in [-pn+1, 0]$ and $s \in [0, (1-p)n-1]$, respectively.

1° When $s \in [-pn+1, 0]$, by the fundamental theorem of calculus and (9.31) we get

$$\psi_n'(s) = \psi_n'(0) - \int_s^0 \psi_n''(y)dy \geq \psi_n'(0) - (-s)(-2cn^{-1}) = \frac{2p-1}{2p(1-p)n} - 2cn^{-1}s,$$

and a second application of the same argument yields for $s \in [-pn+1, 0]$

$$\psi_n(s) = \psi_n(0) - \int_s^0 \psi_n'(y)dy \leq -\int_s^0 \left(\frac{2p-1}{2p(1-p)n} - 2cn^{-1}y\right)dy = \frac{(2p-1)s}{2p(1-p)n} - cn^{-1}s^2,$$

When $p \leq 1/2$, $\frac{(2p-1)s}{2p(1-p)n} \leq \frac{(2p-1)pn}{2p(1-p)n} = \frac{1-2p}{2(1-p)}$, so (9.29) gets proved with $C_2 = \frac{1-2p}{2(1-p)}$. When $p > 1/2$, (9.29) gets proved $C_2 = 0$.

2° When $s \in [0, (1-p)n-1]$, similarly using the fundamental theorem of calculus and (9.31) we get

$$\psi_n'(s) = \psi_n'(0) + \int_0^s \psi_n''(y)dy \leq= \frac{2p-1}{2p(1-p)n} - 2cn^{-1}s,$$

and a second application of the same argument yields for $s \in [0, (1-p)n-1]$

$$\psi_n(s) = \psi_n(0) + \int_0^s \psi_n'(y)dy \leq \frac{(2p-1)s}{2p(1-p)n} - cn^{-1}s^2,$$

When $p \geq 1/2$, $\frac{(2p-1)s}{2p(1-p)n} \leq \frac{(2p-1)(1-p)n}{2p(1-p)n} = \frac{2p-1}{2p}$, so (9.29) gets proved with $C_2 = \frac{2p-1}{2p}$. When $p < 1/2$, (9.29) gets proved $C_2 = 0$. Combining cases 1° and 2° we complete the proof. □

Based on Lemma 9.11, we introduce the following lemma computing quantities $A_\lambda(T)$ and $B_\lambda(T)$ which help us to find the asymptotic behavior of probability mass function (9.15) and its relationship with $\rho(z)$.

Lemma 9.13. *Assume the same notation as in Section 9.1 and Section 9.2. Fix $\vec{z} \in \mathbb{R}^k$ such that $z_1 > \cdots > z_k$. Suppose that $T_0 \in \mathbb{N}$ is sufficiently large so that for $T \geq T_0$ we have*

$$z_k\sqrt{T} + ptT \geq a_1\sqrt{T} + k + 1 \text{ and } b_k\sqrt{T} + pT \geq z_1\sqrt{T} + ptT + k + 1,$$

which ensures that $\lambda_i - x_j^T + j - i$ and $y_i^T - \lambda_j + j - i$ in (9.19) are positive. Then, for a signature λ of length k we define

$$A_\lambda(T) = s_{\lambda'/\mu'}(1^m) \cdot s_{\kappa'/\lambda'}(1^n), \text{ where } m = \lfloor tT \rfloor,\ n = T - m,\ \mu = \vec{x}^T,\ \kappa = \vec{y}^T \tag{9.32}$$

$$\begin{aligned} B_\lambda(T) &= (\sqrt{2\pi})^k \cdot \exp\left(kT\log(1-p) + k\log T + (k/2)\log(p(1-p))\right) \\ &\cdot \exp\left(-\log\left(\frac{1-p}{p}\right)\sum_{i=1}^k (y_i^T - x_i^T)\right) \cdot A_\lambda(T) \end{aligned} \tag{9.33}$$

We claim that

$$\lim_{T\to\infty} B_\lambda(T) = \widetilde{\rho}(z_1, \cdots, z_k) \cdot (2\pi p(1-p)t(1-t))^{-\frac{k}{2}} \cdot \prod_{i=1}^k \exp\left(-\frac{c_1(t,p)a_i^2 + c_2(t,p)b_i^2}{2}\right) \tag{9.34}$$

Proof. From the Jacob-Trudi formula for skew Schur polynomials (9.11) and Lemma 9.11 we have

$$\begin{aligned} s_{\lambda'/\mu'}(1^m) &= \det\left[\exp\left(-\frac{(\lambda_i - x_j^T + j - i - pm)^2}{2(1-p)pm}\right)\exp\left(O\left(T^{-1/2}\right)\right)\right] \cdot (\sqrt{2\pi})^{-k} \cdot \\ &\exp\left(-km\log(1-p) - (k/2)\log m - (k/2)\log(p(1-p)) + \log\left(\frac{1-p}{p}\right)\sum_{i=1}^k(\lambda_i - x_i^T)\right) \end{aligned} \tag{9.35}$$

$$\begin{aligned} s_{\kappa'/\lambda'}(1^n) &= \det\left[\exp\left(-\frac{(y_i^T - \lambda_j + j - i - pn)^2}{2(1-p)pn}\right)\exp\left(O\left(T^{-1/2}\right)\right)\right] \cdot (\sqrt{2\pi})^{-k} \cdot \\ &\exp\left(-kn\log(1-p) - (k/2)\log n - (k/2)\log(p(1-p)) + \log\left(\frac{1-p}{p}\right)\sum_{i=1}^k(y_i^T - \lambda_i)\right) \end{aligned} \tag{9.36}$$

$$\begin{aligned} s_{\kappa'/\mu'}(1^T) &= \det\left[\exp\left(-\frac{(y_i^T - x_j^T + j - i - pT)^2}{2(1-p)pT}\right)\exp\left(O\left(T^{-1/2}\right)\right)\right] \cdot (\sqrt{2\pi})^{-k} \cdot \\ &\exp\left(-kT\log(1-p) - (k/2)\log T - (k/2)\log(p(1-p)) + \log\left(\frac{1-p}{p}\right)\sum_{i=1}^k(y_i^T - x_i^T)\right) \end{aligned} \tag{9.37}$$

where the constants in the big O notation are uniform as z_i vary over compact subsets of $\mathbb{R}$. Combining (9.36), (9.35) and (9.19) we see that

$$\begin{aligned} B_\lambda(T) &= (2\pi)^{-k/2} \cdot \exp(-(k/2)\log(p(1-p)) - (k/2)\log(t(1-t)) + O(T^{-1})) \\ &\cdot \det\left[\exp\left(-\frac{(z_i - a_j)^2}{2p(1-p)t} + O(T^{-1/2})\right)\right] \cdot \det\left[\exp\left(-\frac{(b_i - z_j)^2}{2p(1-p)(1-t)} + O(T^{-1/2})\right)\right] \end{aligned} \tag{9.38}$$

Taking the limit $T \to \infty$ in (9.38), and noticing the identities

$$\det\left[\exp\left(-\frac{(z_i - a_j)^2}{2p(1-p)t}\right)\right] = \det\left[e^{c_1(t,p)a_iz_j}\right]_{i,j=1}^k \cdot \prod_{i=1}^k \exp\left(-\frac{c_1(t,p)}{2}(a_i^2 + z_i^2)\right), \text{ and}$$

$$\det\left[\exp\left(-\frac{(b_i - z_j)^2}{2p(1-p)(1-t)}\right)\right] = \det\left[e^{c_2(t,p)b_iz_j}\right]_{i,j=1}^k \cdot \prod_{i=1}^k \exp\left(-\frac{c_2(t,p)}{2}(b_i^2 + z_i^2)\right)$$

we get (9.34). □

The following corollary of Lemma 9.13 gives the connection between the probability mass function in (9.8) and the probability density function in (9.3).

Corollary 9.14. *Assume the same notation as in Lemma 9.8. Fix $R > 0$, take any $(z_1, \cdots, z_k) \in [-R,R]^k \cap \mathbb{W}_k^o$ such that $\lambda_i = z_i\sqrt{T} + ptT$ are integers for $i = 1, \cdots, k$. Define function $h_T(z)$ on $\mathbb{R}^k$:*

$$h_T(z) = \mathbf{1}_{\{[-R,R]^k \cap \mathbb{W}_k^o\}}(z) \cdot (\sqrt{T})^k \cdot \mathbb{P}(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k)$$

Then, we have

$$\lim_{T\to\infty} h_T(z) = \rho(z_1, \cdots, z_k) \tag{9.39}$$

where $\rho(z_1, \cdots, z_k)$ is defined in (9.3). Moreover, $h_T(z)$ is uniformly bounded on the compact set $[-R,R]^k$.

Proof. Plugging (9.35), (9.36) and (9.37) into (9.12) we get

$$\begin{aligned} &T^{k/2} \cdot \mathbb{P}(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k) \\ &= Z_T \cdot \frac{\det\left[\exp\left(-\frac{(z_i-a_j)^2}{2p(1-p)t}\right)\right] \cdot \det\left[\exp\left(-\frac{(b_i-z_j)^2}{2p(1-p)(1-t)}\right)\right]}{\det\left[\exp\left(-\frac{(b_i-a_j)^2}{2p(1-p)}\right)\right]} \cdot \exp(o(1)) \end{aligned} \tag{9.40}$$

where

$$\begin{aligned} Z_T &= (\sqrt{2\pi})^{-k} \exp\left(-km\log(1-p) - (k/2)\log m - (k/2)\log(p(1-p))\right) \\ &\cdot \exp\left(-kn\log(1-p) - (k/2)\log n - (k/2)\log(p(1-p))\right) \\ &\cdot \exp\left(kT\log(1-p) + k\log T + (k/2)\log(p(1-p))\right) \\ &= (\sqrt{2\pi})^{-k} \exp\left(-(k/2)\log(p(1-p)t(1-t)\right) = (2\pi p(1-p)t(1-t))^{-k/2} \end{aligned} \tag{9.41}$$

Plugging (9.41) into (9.40) we conclude (9.39) and at the meantime, we have

$$h_T(z_1, \cdots, z_k) = \frac{\det\left[\exp\left(-\frac{(z_i-a_j)^2}{2p(1-p)t}\right)\right] \cdot \det\left[\exp\left(-\frac{(b_i-z_j)^2}{2p(1-p)(1-t)}\right)\right]}{(2\pi p(1-p)t(1-t))^{k/2} \det\left[\exp\left(-\frac{(b_i-a_j)^2}{2p(1-p)}\right)\right]} \cdot \exp(o(1)) \tag{9.42}$$

Notice that the determinants in (9.42) are continuous function of z, so they are all bounded on the compact set $[-R,R]^k$. Plus, $o(1)$ is uniformly bounded on $[-R,R]^k$. Therefore, $h_T(z)$ is bounded over $[-R,R]^k$. □

Before proving Lemma 9.10, we need to introduce another result regarding the non-vanishing of determinant, which will be used in the proof of Lemma 9.10.

Lemma 9.15. *Suppose the vector $\vec{m} = (m_1, \cdots, m_p)$ satisfies $k = \sum_{i=1}^p m_i$, and $\alpha_1 > \alpha_2 > \cdots > \alpha_p$. Then the following determinant*

$$U = \det \begin{bmatrix} (z_j^{i-1} e^{\alpha_1 z_j})_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ (z_j^{i-1} e^{\alpha_p z_j})_{\substack{i=1,\cdots,m_p \\ j=1,\cdots,k}} \end{bmatrix}$$

is non-zero for any $(z_1, \cdots, z_k)$ whose elements are distinct.

Proof. We claim that, the following equation with respect to $\vec{z}$:

$$(\xi_1 + \xi_2 z + \cdots + \xi_{m_1} z^{m_i - 1}) e^{\alpha_1 z} + \cdots (\xi_{m_1 + \cdots + m_{p-1} + 1} + \cdots + \xi_k z^{m_p - 1}) e^{\alpha_p z} = 0$$

has at most $(k-1)$ distinct roots, where $(\xi_1, \cdots, \xi_k) \in \mathbb{R}^k$ is non-zero.

Denote the above determinant by $\det \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$. If this claim holds, we can conclude that we cannot find non-zero $(\xi_1, \cdots, \xi_k) \in \mathbb{R}^k$ such that $\xi_1 v_1 + \cdots + \xi_k v_k = 0$. Thus, the k row vectors of the determinant are linear independent and the determinant is non-zero. Then we prove the claim by induction on k.

1° If $k = 2$, the equation is $(\xi_1 + \xi_2 z) e^{\alpha_1 z} = 0$ or $\xi_1 e^{\alpha_1 z} + \xi_2 e^{\alpha_2 z} = 0$, where $\xi_1, \xi_2 \in \mathbb{R}$ cannot be zero at the same time. Then, it's easy to see that the equation has at most 1 root in two scenarios.

2° Suppose the claim holds for $k \le n$.

3° When $k = n + 1$, we have the equation

$$(\xi_1 + \xi_2 z + \cdots + \xi_{m_1} z^{m_i - 1}) e^{\alpha_1 z} + \cdots (\xi_{m_1 + \cdots + m_{p-1} + 1} + \cdots + \xi_k z^{m_p - 1}) e^{\alpha_p z} = 0$$

but now $\sum_{i=1}^{p} m_i = n + 1$. WLOG, suppose $(\xi_1, \cdots, \xi_{m_1})$ has a non-zero element and ξ_ℓ is the first non-zero element. Notice that the above equation has the same roots as the following one:

$$F(z) = (\xi_\ell z^{\ell - 1} + \cdots + \xi_{m_1} z^{m_1 - 1}) + \cdots + (\xi_{m_1 + \cdots + m_{p-1} + 1} + \cdots + \xi_k z^{m_p - 1}) e^{(\alpha_p - \alpha_1) z} = 0$$

Assume it has at least $(n+1)$ distinct roots $\eta_1 < \eta_2 < \cdots < \eta_{n+1}$. Then $F'(z) = 0$ has at least n distinct roots $\delta_1 < \cdots < \delta_n$ such that $\eta_1 < \delta_1 < \eta_2 < \cdots < \delta_n < \eta_{n+1}$, by Rolle's Theorem. Actually, $F'(z) = (\xi_\ell (\ell - 1)) z^{\ell - 2} + \cdots + \xi_{m_1} (m_1 - 1) z^{m_1 - 2}) + \cdots + (\xi'_{m_1 + \cdots + m_{p-1} + 1} + \cdots + \xi'_k z^{m_p - 1}) e^{(\alpha_p - \alpha_1) z} = 0$ where ξ'_i, $i = m_1 + 1, \cdots, k$ are coefficients that can be calculated. This equation has at most $(m_1 - 1) + m_2 + \cdots + m_p - 1 = n - 1$ roots by 2°, which leads to a contradiction. Therefore, our claim holds and we proved Lemma 9.15. ☐

Now, we are ready to prove Lemma 9.10.

Proof of Lemma 9.10. For clarity we split the proof into several steps.

Step 1. In this step we show that $\widetilde{\rho}(z_1, \cdots, z_k) \ge 0$ and $\widetilde{\rho}(z_1, \cdots, z_k) > 0$ if $z_1 > z_2 > \cdots > z_k$. Because of the indicator function in $\widetilde{\rho}(z_1, \cdots, z_k)$, we know $\widetilde{\rho}(z_1, \cdots, z_k) = 0$ unless $z_1 > \cdots > z_k$. Therefore, it suffices to show that

$$\widetilde{\rho}(z_1, \cdots, z_k) > 0 \text{ if } z_1 > \cdots > z_k \tag{9.43}$$

Choose T_0 as we did in Lemma 9.13 and assume $T \ge T_0$. By definition of $B_\lambda(T)$ we know $B_\lambda(T) \ge 0$ for all $T \ge T_0$, which implies $\widetilde{\rho}(z_1, \cdots, z_k) \ge 0$ combined with (9.34). Also, by Lemma 9.15 we know that $\widetilde{\rho}(z_1, \cdots, z_k) \ne 0$ so (9.43) holds.

Step 2. In this step we prove that $\widetilde{\rho}(z_1, \cdots, z_k)$ is integrable. Using the formula

$$\det [A_{i,j}]_{i,j=1}^{k} = \sum_{\sigma \in S_k} (-1)^\sigma \cdot \prod_{i=1}^{k} A_{i, \sigma(i)}$$

and the triangle inequality we see that

$$\begin{aligned} \left| \det \left[e^{c_1(t,p) a_i z_j} \right]_{i,j=1}^{k} \right| &\le \sum_{\sigma \in S_k} \prod_{j=1}^{k} e^{c_1(t,p) a_{\sigma(j)} z_j} \le \sum_{\sigma \in S_k} \prod_{j=1}^{k} e^{c_1(t,p) \left(\sum_{i=1}^{k} |a_i| \right) \cdot |z_j|} \\ &\le (k!) \prod_{i=1}^{k} e^{C_1 |z_j|}, \text{ where } C_1 = \sum_{i=1}^{k} c_1(t,p) |a_i| \end{aligned} \tag{9.44}$$

Analogously, define the constant $C_2 = \sum_{i=1}^{k} c_2(t,p)|b_i|$ and we have

$$\left|\det\left[e^{c_2(t,p)b_i z_j}\right]_{i,j=1}^{k}\right| \le (k!)\prod_{i=1}^{k} e^{C_2|z_j|} \tag{9.45}$$

Plugging (9.44) and (9.45) into the expression of $\widetilde{\rho}$ we have

$$|\widetilde{\rho}(z_1,\cdots,z_k)| \le (k!)^2\cdot\prod_{i=1}^{k} e^{C|z_i|-c_3(t,p)z_i^2} \tag{9.46}$$

where $C = C_1 + C_2$. Since the right side of (9.46) is integrable (because of the square in the exponential) we conclude that $\widetilde{\rho}$ is also integrable by domination.

Step 3. In this step, we prove (9.17) and conclude Lemma 9.10. Using the branching relations for skew Schur polynomials (9.10) we know that

$$\begin{aligned}\sum_{\lambda\in\mathfrak{W}_k}\frac{B_\lambda(T)}{T^{k/2}} &= (\sqrt{2\pi})^k\cdot\exp(kT\log(1-p)+(k/2)\log T+(k/2)\log p(1-p))\\ &\cdot\exp\left(-\log\left(\frac{1-p}{p}\right)\sum_{i=1}^{k}(y_i^T-x_i^T)\right)\cdot s_{\kappa'/\mu'}(1^T)\end{aligned} \tag{9.47}$$

Plugging (9.19) and (9.37) into (9.47) we conclude

$$\lim_{T\to\infty}\sum_{\lambda\in\mathfrak{W}_k}\frac{B_\lambda(T)}{T^{k/2}} = \det\left[e^{-\frac{1}{2p(1-p)}(b_i-a_j)^2}\right]_{i,j=1}^{k} \tag{9.48}$$

For a signature $\lambda\in\mathfrak{W}_k$ and $T\in\mathbb{N}$ we define $Q_\lambda(T)$ to be the cube $[\lambda_1 T^{-1/2}-pt\sqrt{T},(\lambda_1+1)T^{-1/2}-pt\sqrt{T}]\times\cdots\times[\lambda_k T^{-1/2}-pt\sqrt{T},(\lambda_k+1)T^{-1/2}-pt\sqrt{T}]$ with Lebesgue measure $T^{-k/2}$. In addition, we define the simple function f_T through

$$f_T(z) = \sum_{\lambda\in\mathfrak{W}_k} B_\lambda(T)\cdot\mathbf{1}_{Q_\lambda(T)}(z)\cdot\mathbf{1}_{\mathbb{W}_k^o}(z) \tag{9.49}$$

and observe that

$$\sum_{\lambda\in\mathfrak{W}_k}\frac{B_\lambda(T)}{T^{k/2}} = \int_{\mathbb{R}^k} f_T(z)dz \tag{9.50}$$

where dz represents the usual Lebesgue measure on $\mathbb{R}^k$.

In view of (9.34) we know that for almost every $z=(z_1,\cdots,z_k)\in\mathbb{R}^k$ we have

$$\lim_{T\to\infty} f_T(z) = \widetilde{\rho}(z)\cdot(2\pi p(1-p)t(1-t))^{-\frac{k}{2}}\cdot\prod_{i=1}^{k}\exp\left(-\frac{c_1(t,p)a_i^2+c_2(t,p)b_i^2}{2}\right). \tag{9.51}$$

We claim that there exists a non-negative integrable function g on $\mathbb{R}^k$ such that if T is large enough

$$|f_T(z_1,\cdots,z_k)| \le |g(z_1,\cdots,z_k)| \tag{9.52}$$

We will prove (9.52) in Step 4 below. For now we assume its validity and conclude the proof of (9.17).

From (9.51) and the dominated convergence theorem with dominating function g as in (9.52) we know that

$$\lim_{T\to\infty}\int_{\mathbb{R}^k} f_T(z)dz = \int_{\mathbb{R}^k}\widetilde{\rho}(z)dz\cdot(2\pi p(1-p)t(1-t))^{-\frac{k}{2}}\cdot\prod_{i=1}^{k}\exp\left(-\frac{c_1(t,p)a_i^2+c_2(t,p)b_i^2}{2}\right) \tag{9.53}$$

Combining (9.53), (9.50) and (9.48) we conclude that

$$\det\left[e^{-\frac{1}{2p(1-p)}(b_i-a_j)^2}\right]_{i,j=1}^k = \int_{\mathbb{R}^k}\widetilde{\rho}(z)dz\cdot(2\pi p(1-p)t(1-t))^{-\frac{k}{2}}\cdot\prod_{i=1}^k e^{-\frac{c_1(t,p)a_i^2+c_2(t,p)b_i^2}{2}}. \tag{9.54}$$

which clearly establishes (9.17).

Step 4. In this step we demonstrate an integrable function g that satisfies (9.52). Let us fix $\lambda\in\mathfrak{W}_k$. If $\lambda_i\geq x_i^T+m+1$ or $\lambda_i<\mu_i$ for some $i\in\{1,2,\cdots,k\}$ we know that $s_{\lambda'/\mu'}(1^m)=0$ because there is no avoiding Bernoulli ensembles starting with μ and ending with λ. Similarly, if $y_i^T\geq\lambda_i+n+1$ or $y_i^T<\lambda_i$ for some $i\in\{1,2,\cdots,k\}$, we have $s_{\kappa'/\lambda'}(1^n)=0$. We conclude that $B_\lambda(T)=0$ unless

$$m\geq\lambda_i-x_i^T\geq 0 \text{ and } n\geq y_i^T-\lambda_i\geq 0 \text{ for all } i\in\{1,\cdots,k\}$$

which implies that for all large enough T we have

$$B_\lambda(T)=0,\text{ unless } |\lambda_i-x_j^T+j-i|\leq(1+p)m \text{ and } |y_i^T-\lambda_j+j-i|\leq(1+p)n \tag{9.55}$$

for all $i,j\in\{1,\cdots,k\}$. This is because if there exist i,j such that $(1+p)m<|\lambda_i-x_j^T+j-i|$, then we have

$$(1+p)m<|\lambda_i-x_j^T+j-i|\leq\lambda_1-x_k^T+k-1=(\lambda_1-\lambda_k)+(\lambda_k-x_k^T)+k-1$$

When T is sufficiently large, the above inequality implies $\lambda_k-x_k^T>m$ so that $B_\lambda(T)=0$, and similar result holds for $y_i^T-\lambda_j+j-i$, which justifies (9.55). Using the definition of $A_\lambda(T)$ and $B_\lambda(T)$ we know that

$$\begin{aligned}&B_\lambda(T)=C_T\cdot\det[E(\lambda_i-x_j^T+j-i,m)]_{i,j=1}^k\cdot\det[E(y_i^T-\lambda_j+j-i,n)]_{i,j=1}^k,\text{ where}\\&E(N,n)=e_N(1^n)\cdot\exp\left(-N\log\left(\frac{1-p}{p}\right)+n\log(1-p)+(1/2)\log n\right),\text{ and}\\&C_T=(\sqrt{2\pi})^k(p(1-p))^{k/2}\cdot\exp(k\log T-(k/2)\log n-(k/2)\log m).\end{aligned} \tag{9.56}$$

Notice that C_T is uniformly bounded for all T large enough, because

$$k\log T-\frac{k}{2}\log n-\frac{k}{2}\log m=\frac{k}{2}\log\left(\frac{T^2}{\lfloor tT\rfloor\cdot(T-\lfloor tT\rfloor)}\right)=-\frac{k}{2}\log(t(1-t))+O\left(T^{-1}\right) \tag{9.57}$$

and $O\left(T^{-1}\right)$ is uniformly bounded.

In view of (9.21) we know that we can find constants C_1, $c_1>0$ such that for all large enough T and $N_1\in[0,m]$ and $N_2\in[0,n]$ we have

$$E(N_1,m)\leq C_1\exp(-c_1m^{-1}(N_1-pm)^2) \text{ and } E(N_2,n)\leq C_1\exp(-c_1n^{-1}(N_2-pn)^2) \tag{9.58}$$

Observing that $e_r(1^n)=0$ for $r>n$ or $r<0$, we know that (9.58) also holds for all $N_1\in[-(1+p)m,(1+p)m]$ and $N_2\in[-(1+p)n,(1+p)n]$. Combining (9.55), (9.56) and (9.58) we see that for all $\lambda\in\mathfrak{W}_k$ and T sufficiently large

$$\begin{aligned}0\leq B_\lambda(T)\leq\widetilde{C}\sum_{\sigma\in S_k}\sum_{\tau\in S_k}&\mathbf{1}\{|\lambda_i-x_j^T+j-i|\leq(1+p)m\}\cdot\mathbf{1}\{|y_i^T-\lambda_j+j-i|\leq(1+p)n\}\\&\cdot\exp\left(-\widetilde{c}T^{-1}\left[(\lambda_i-\sqrt{T}a_{\sigma(i)}-ptT)^2+(\sqrt{T}b_i-\lambda_{\tau(i)}+ptT)^2\right]\right)\end{aligned} \tag{9.59}$$

where $\widetilde{c}$, $\widetilde{C}>0$ depend on p,t,k but not on T provided that it is sufficiently large.

In particular, we see that if $z\in\mathbb{R}^k$ then either $z\notin Q_\lambda(T)$ for any $\lambda\in\mathfrak{W}_k$ in which case $f_T(z)=0$ or $z\in Q_\lambda(T)$ for some $\lambda\in\mathfrak{W}_k$ in which case (9.59) and (9.19) imply

$$0\leq f_T(z)\leq C\sum_{\sigma\in S_k}\sum_{\tau\in S_k}\exp\left(-c((z_i-a_{\sigma(i)})^2+(b_i-z_{\tau(i)})^2)\right) \tag{9.60}$$

where C, $c > 0$ depend on p, t, k but not on T provided that it is sufficiently large. We finally see that (9.46) holds with g being equal to the right side of (9.60), which is clearly integrable. $\square$

Now we are ready to prove Proposition 9.2.

Proof of Proposition 9.2. In the following, we prove the weak convergence of the random vector Z^T, when $\vec{a} = (a_1, \cdots, a_k)$ and $\vec{b} = (b_1, \cdots, b_k)$ consist of distinct entries. In order to show weak convergence, it is sufficient to show that for every open set $O \in \mathbb{R}^k$, we have:

$$\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O) \geq \int_O \rho(z_1, \cdots, z_k) dz_1 dz_2 \cdots dz_k$$

according to [9, Theorem 3.2.11]. It is also sufficient to show that for any open set $U \in \mathbb{W}_k^o$, we have:

$$\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in U) \geq \int_U \rho(z_1, \cdots, z_k) dz_1 dz_2 \cdots dz_k \tag{9.61}$$

which implies that:

$$\begin{aligned}
&\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O) \geq \liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O \cap \mathbb{W}_k^o) \\
&\geq \int_{\mathbb{W}_k^o \cap O} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k = \int_O \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k
\end{aligned}$$

The second inequality uses the above result (9.61), since $\mathbb{W}_k^o \cap O$ is an open set in $\mathbb{W}_k^o$. The last equality is because $\rho(z)$ is zero outside $\mathbb{W}_k^o$. The rest of the proof will be divided into 2 steps. In Step 1, we prove that weak convergence holds on every closed rectangle. In Step 2, we prove the inequality (9.61) by writing open set as countable union of almost disjoint rectangles.

Step 1. In this step, we establish the following result:
For any closed rectangle $R = [u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_k, v_k] \in \mathbb{W}_k^o$,

$$\lim_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_R \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \tag{9.62}$$

where $\rho(z)$ is given in Proposition 9.2.

Define $m_i^T = \lceil u_i\sqrt{T} + ptT \rceil$ and $M_i^T = \lfloor v_i\sqrt{T} + ptT \rfloor$. Then we have:

$$\begin{aligned}
&\mathbb{P}\left((Z_1^T, \cdots, Z_k^T) \in R\right) = \mathbb{P}\left(u_1 \leq Z_1^T \leq v_1, \ldots, u_k \leq Z_k^T \leq v_k\right) \\
&= \mathbb{P}\left(u_i\sqrt{T} + ptT \leq L_i(\lfloor tT \rfloor) \leq v_i\sqrt{T} + ptT, i = 1, \ldots, k\right) \\
&= \sum_{\lambda_1 = m_1^T}^{M_1^T} \cdots \sum_{\lambda_k = m_k^T}^{M_k^T} \mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \ldots, L_k(\lfloor tT \rfloor) = \lambda_k) \\
&= \sum_{\lambda_1 = m_1^T}^{M_1^T} \cdots \sum_{\lambda_k = m_k^T}^{M_k^T} (\sqrt{T})^{-k} \cdot (\sqrt{T})^k \mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \ldots, L_k(\lfloor tT \rfloor) = \lambda_k)
\end{aligned}$$

Find sufficiently large A such that $R \subset [-A, A]^k$, for example, $A = 1 + \max_{1\leq i\leq k} |a_i| + \max_{1\leq i\leq k} |b_i|$. Define $h_T(z_1, \cdots, z_k)$ as a simple function on $\mathbb{R}^k$: When $(z_1, \cdots, z_k) \in R$, it takes value $(\sqrt{T})^k \cdot \mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \cdots, L_k(\lfloor tT \rfloor) = \lambda_k)$ if there exist integers $\lambda_1 \geq \cdots \geq \lambda_k$ such that $\lambda_i \leq z_i\sqrt{T} + ptT < \lambda_i + 1$; It takes value 0 otherwise, when $(z_1, \cdots, z_k) \notin R$. Since the Lebesgue measure of the set $\{z : \lambda_i \leq z_i\sqrt{T} + ptT < \lambda_i + 1, i = 1, \cdots, k\} = \left[\lambda_1 T^{-1/2} - pt\sqrt{T}, (\lambda_1 + 1)T^{-1/2} - pt\sqrt{T}\right) \times$

$\cdots \times \left[\lambda_k T^{-1/2} - pt\sqrt{T}, (\lambda_k+1)T^{-1/2} - pt\sqrt{T}\right)$ is $(\sqrt{T})^{-k}$, the above probability can be further written as an integral of simple functions $h_T(z_1, \cdots, z_k)$:

$$\mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_{[-A,A]^k} h_T(z_1, \cdots, z_k) dz_1 \cdots dz_k$$

By Corollary 9.14, the function $h_T(z_1, \cdots, z_k)$ pointwise converges to $\rho(z)$ and is bounded on the compact set $[-A, A]^k$. Since the Lebesgue measure of $[-A, A]^k$ is finite, by bounded convergence theorem we have:

$$\lim_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_R \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \tag{9.63}$$

Step 2. In this step, we prove the statement (9.61). Take any open set $U \in \mathbb{W}_k^o$ and it can be written as a countable union of closed rectangles with disjoint interiors: $U = \bigcup_{i=1}^{\infty} R_i$, where $R_i = [a_1^i, b_1^i] \times \cdots \times [a_k^i, b_k^i]$([17, Theorem 1.4]). Choose sufficiently small $\epsilon > 0$, and denote $R_i^\epsilon = [a_1^i + \epsilon, b_1^i - \epsilon] \times \cdots \times [a_k^i + \epsilon, b_k^i - \epsilon]$, then R_i^ϵ are disjoint. Therefore,

$$\begin{aligned}
&\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in U) \geq \liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \bigcup_{i=1}^n R_i^\epsilon) \\
&= \liminf_{T\to\infty} \sum_{i=1}^n \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R_i^\epsilon) = \sum_{i=1}^n \int_{R_i^\epsilon} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \\
&= \int_{\bigcup_{i=1}^n R_i^\epsilon} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \xrightarrow{\epsilon\downarrow 0,\ n\uparrow\infty} \int_U \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k
\end{aligned}$$

The last line uses the monotone convergence theorem since the when we let $\epsilon \downarrow 0$ and $n \uparrow \infty$ the indicator function $\mathbf{1}_{\bigcup_{i=1}^n R_i^\epsilon}$ is monotonically increasing, and converges to $\mathbf{1}_U$. Thus, we have proved the inequality (9.61). By Lemma 9.10, $\rho(z)$ is a probability density function, thus implying the weak convergence of Z^T. ☐

9.4. **Multi-indices and Multivariate Taylor Expansion.** In this section, we introduce some notations and results about multivariate functions and permutations.

Suppose $\sigma = (\sigma_1, \cdots, \sigma_n)$ is a multi-index of *length* n. In our context, we require $\sigma_1, \cdots, \sigma_n$ be all non-negative integers(some of them might be equal). We define $|\sigma| = \sum_{i=1}^n \sigma_i$ as the *order* of σ. Suppose $\tau = (\tau_1, \cdots, \tau_n)$ is another multi-index of length n. We say $\tau \leq \sigma$ if $\tau_i \leq \sigma_i$ for $i = 1, \cdots, n$. We say $\tau < \sigma$ if $\tau \leq \sigma$ and there exists at least one index i such that $\tau_i < \sigma_i$. Then, define the partial derivative with respect to the multi-index σ:

$$D^\sigma f(x_1, \cdots, x_n) = \frac{\partial^{|\sigma|} f(x_1, \cdots, x_n)}{\partial x_1^{\sigma_1} \partial x_2^{\sigma_2} \cdots \partial x_n^{\sigma_n}}$$

We have the general Leibniz rule:

$$D^\sigma (fg) = \sum_{\tau \leq \sigma} \binom{\sigma}{\tau} D^\tau f \cdot D^{\sigma-\tau} g$$

where $\binom{\sigma}{\tau} = \frac{\sigma_1! \cdots \sigma_n!}{\tau_1! \cdots \tau_n! (\sigma_1 - \tau_1)! \cdots (\sigma_n - \tau_n)!}$.
We also have the Taylor expansion for multi-variable functions:

$$f(x_1, \cdots, x_n) = \sum_{|\sigma| \leq r} \frac{1}{\sigma!} D^\sigma f(\vec{x}_0)(\vec{x} - \vec{x}_0)^\sigma + R_{r+1}(\vec{x}, \vec{x}_0) \tag{9.64}$$

In the equation, $\sigma! = \sigma_1!\sigma_2! \cdots \sigma_n!$ is the factorial with respect to the multi-index σ, $\vec{x}_0 = (x_1^0, \cdots, x_n^0)$ is a constant vector at which we expands the function f, $(\vec{x} - \vec{x}_0)^\sigma$ stands for $(x_1 -$

$x_1^0)^{\sigma_1} \cdots (x_n - x_n^0)^{\sigma_n}$, and

$$R_{r+1}(\vec{x}, \vec{x}_0) = \sum_{\sigma:|\sigma|=r+1} \frac{1}{\sigma!} D^\sigma f(\vec{x}_0 + \theta(\vec{x} - \vec{x}_0))(\vec{x} - \vec{x}_0)^\sigma$$

is the remainder, where $\theta \in (0,1)$([2, Theorem 3.18 & Corollary 3.19]).

We also need some knowledge about *permutation*. Suppose s_n is a permutation of n non-negative integers, for example $\{1, \cdots, n\}$, and $s_n(i)$ represents the *i-th* element in the permutation s_n. We define *the number of inversions* of s_n by $I(s_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbf{1}_{\{s_n(i) > s_n(j)\}}$. For example, the permutation $s_n = (1, \cdots, n)$ has 0 number of inversions, while the permutation $s_5 = (3, 2, 5, 1, 4)$ has number of inversions 5(2+1+2+0+0). Define the sign of permutation s_n by $sgn(s_n) = (-1)^{I(s_n)}$. For instance, $sgn((1, \cdots, n)) = 1$ and $sgn(s_5) = -1$ in the previous example.

9.5. **Proof of Proposition 9.3.** Based on the notation in Section 9.4, we are going to prove Proposition 9.3 in this section. We assume vectors $\vec{a}_0$, $\vec{b}_0$ cluster in the way described in (9.4):

$$\begin{aligned} \vec{a}_0 &= (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p}) \\ \vec{b}_0 &= (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q}) \end{aligned} \tag{9.65}$$

and

$$\lim_{T\to\infty} \frac{\vec{x}^T}{\sqrt{T}} = \vec{a}_0, \ \lim_{T\to\infty} \frac{\vec{y}^T - pT1_k}{\sqrt{T}} = \vec{b}_0$$

Let $\vec{a} = (a_1, \cdots, a_k)$ and $\vec{b} = (b_1, \cdots, b_k)$ denote two vectors in $\mathbb{W}_k^o$ so they contain distinct elements. We also denote $\vec{a}^{(1)} = (a_1, \cdots, a_{m_1})$, $\cdots$, $\vec{a}^{(p)} = (a_{m_1+\cdots+m_{p-1}+1}, \cdots, a_{m_1+\cdots+m_p})$ and $\vec{a} = (\vec{a}^{(1)}, \cdots, \vec{a}^{(p)})$. That is, we divide the vector $\vec{a}$ into p blocks according to the shape of $\vec{a}_0$. Similarly, we write $\vec{b} = (b^{(1)}, \cdots, b^{(q)})$ according to the shape of $\vec{b}_0$. We will keep using similar notations in the following discussion, when we need to divide the vector according to the shape of $\vec{a}_0$ and $\vec{b}_0$. Next, denote

$$f(\vec{a}, \vec{z}) = \det[e^{c_1(t,p)a_i z_j}]_{i,j=1}^k, \ g(\vec{b}, \vec{z}) = \det[e^{c_2(t,p)b_i z_j}]_{i,j=1}^k \tag{9.66}$$

and it's not difficult to see that they are all smooth multi-variable functions with respect to corresponding vectors because of the exponentials. In addition, $\lim_{\vec{a}\to\vec{a}_0} f(\vec{a}, \vec{z}) = 0$ and $\lim_{\vec{b}\to\vec{b}_0} g(\vec{b}, \vec{z}) = 0$. However, when we taking proper derivatives with respect to $\vec{a}$ and $\vec{b}$, we can get a non-zero derivative. The following lemma gives the minimal order of derivatives such that $D^\sigma f(\vec{a}_0, \vec{z})$ and $D^\sigma g(\vec{b}_0, \vec{z})$ are non-zero, where $f(\vec{a}, \vec{z})$ and $g(\vec{b}, \vec{z})$ are defined in (9.66).

Lemma 9.16. *Assume the same notations as in (9.65) and $\vec{z} \in \mathbb{W}_k^o$. Then, the smallest order of σ_a that makes the derivative $D^{\sigma_a} f(\vec{a}_0, \vec{z})$ non-zero is $u = \sum_{i=1}^p \frac{m_i(m_i-1)}{2}$. Similarly, $v = \sum_{j=1}^q \frac{n_j(n_j-1)}{2}$ is the smallest order of σ_b that makes $D^{\sigma_b} f(\vec{b}_0, \vec{z})$ non-zero.*

Proof. If the order of derivative is less than u, then there exists an $i \in \{1, \cdots, p\}$ such that $\sigma_a^{(i)}$ contains two equal elements $< m_i - 1$, and the determinant $D^{\sigma_a} f(\vec{a}_0, \vec{z})$ would have two equal rows, thus equal to zero. Suppose s_n is the set of all permutations of $\{0, 1, \cdots, n-1\}$. Then, if $\sigma_a = \left(\sigma_a^{(1)}, \cdots, \sigma_a^{(p)}\right)$ and $\sigma_a^{(i)} \in s_{m_i}$, $D^{\sigma_a} f(\vec{a}_0, \vec{z})$ is non-zero by Lemma 9.15. In this case, the order of σ_a is $\sum_{i=1}^p \sum_{j=1}^{m_i-1} j = \sum_{i=1}^p \frac{m_i(m_i-1)}{2} = u$. Analogous result also holds for $D^{\sigma_b} g(\vec{b}_0, \vec{z})$ and we conclude Lemma 9.16. $\square$

Remark 9.17. Denote the set

$$\begin{aligned}\Lambda_a &= \{\sigma_a = \left(\sigma_a^{(1)}, \cdots, \sigma_a^{(p)}\right) : \sigma_a^{(i)} \in S_{m_i},\ i = 1, \cdots, p\} \\ \Lambda_b &= \{\sigma_b = \left(\sigma_b^{(1)}, \cdots, \sigma_b^{(p)}\right) : \sigma_b^{(i)} \in S_{n_i},\ i = 1, \cdots, q\}\end{aligned} \tag{9.67}$$

Then we have that $\sigma_a \in \Lambda_a$ and $\sigma_b \in \Lambda_b$ imply $D^{\sigma_a} f(\vec{a}_0, \vec{z})$ and $D^{\sigma_b} f(\vec{b}_0, \vec{z})$ are non-zero.

Finally, we give the proof for Proposition 9.3.

Proof of Proposition 9.3. For clarity, the proof will be split into 3 steps. In Step 1, we use multi-variate Taylor expansion to find the speed of convergence of $f(\vec{a}, \vec{z})$ and $g(\vec{b}, \vec{z})$ to zero, when $\vec{a} \to \vec{a}_0$ and $\vec{b} \to \vec{b}_0$. In Step 2, we construct a new density function based on Step 1, and we will prove that Z^T weakly converges to the this newly constructed density in Step 3. In Step 3, we use monotone coupling lemma to "squeeze" the probability and prove the weak convergence.
Step 1. In this step, we find the converging speed of $f(\vec{a}, \vec{z})$ when $\vec{a} \to \vec{a}_0$. Take $\epsilon \in (0, k^{-1} \min_{1 \le i \le p-1} (\alpha_i - \alpha_{i+1}))$ and construct the following vectors:

$$\begin{aligned}\vec{A}_{\epsilon,+} &= (\alpha_1 + m_1\epsilon, \alpha_1 + (m_1 - 1)\epsilon, \cdots, \alpha_1 + \epsilon, \cdots, \alpha_p + m_p\epsilon, \cdots, \alpha_p + \epsilon) \\ \vec{A}_{\epsilon,-} &= (\alpha_1 - \epsilon, \alpha_1 - 2\epsilon, \cdots, \alpha_1 - m_1\epsilon, \cdots, \alpha_p - \epsilon, \cdots, \alpha_p - m_p\epsilon)\end{aligned}$$

That is, the vector $\vec{A}_{\epsilon,+}$(resp. $\vec{A}_{\epsilon,-}$) upwardly(resp. downwardly) spreads out the vector $\vec{a}_0$ such that $\vec{A}_{\epsilon,+}$(resp. $\vec{A}_{\epsilon,-}$) has distinct elements. By the choice of ϵ, the elements of $\vec{A}_{\epsilon,+}$ and $\vec{A}_{\epsilon,-}$ are strictly ordered. In addition, when $\epsilon \downarrow 0$, the vector $\vec{A}_{\epsilon,\pm}$ converges to $\vec{a}_0$. The main result of this step is the following:

$$\lim_{\epsilon \downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,\pm}, \vec{z}) = \varphi(\vec{a}_0, \vec{z}, \vec{m}) \tag{9.68}$$

where $u = \sum_{i=1}^p \frac{m_i(m_i-1)}{2}$ in Lemma 9.16, $\vec{m} = (m_1, \cdots, m_p)$, and $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ defined in (9.5). Additionally, $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ is non-zero because of Lemma 9.15.

To prove this result, we first keep $\vec{z}$ fixed and expand the function $f(\vec{a}, \vec{z})$ to the order of u at $\vec{a}_0$ using multi-variate Taylor expansion (9.64):

$$\begin{aligned}f(\vec{a}, \vec{z}) &= \sum_{|\sigma_a| \le u} \frac{D^{\sigma_a} f(\vec{a}_0, \vec{z})}{\sigma_a!} (\vec{a} - \vec{a}_0)^{\sigma_a} + R_{u+1}(\vec{a}, \vec{a}_0, \vec{z}) \\ &= \sum_{\sigma_a \in \Lambda_a} \frac{D^{\sigma_a} f(\vec{a}_0, \vec{z})}{\sigma_a!} (\vec{a} - \vec{a}_0)^{\sigma_a} + R_{u+1}(\vec{a}, \vec{a}_0, \vec{z})\end{aligned} \tag{9.69}$$

where

$$R_{u+1}(\vec{a}, \vec{a}_0, \vec{z}) = \sum_{\sigma_a : |\sigma_a| = u+1} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0 + \theta(\vec{a} - \vec{a}_0), \vec{z})(\vec{a} - \vec{a}_0)^{\sigma_a},\ \theta \in (0, 1) \tag{9.70}$$

is the remainder and Λ_a is defined in remark 9.17. The second equality in (9.69) results from Lemma 9.16, since it indicates that all the terms of order less than u are zero, and for the terms of order u, they are non-zero only when $\sigma_a \in \Lambda_a$.

Consider the first term in the second line of (9.69). Denote $sgn(\sigma_a^{(i)})$ as the sign of the permutation $\sigma_a^{(i)} \in S_{m_i}$, and define the sign of σ_a by: $sgn(\sigma_a) = \prod_{i=1}^p sgn(\sigma_a^{(i)})$. Denote $\sigma_a^\star = \left(\sigma_a^{(1)\star}, \cdots, \sigma_a^{(p)\star}\right)$, where $\sigma_a^{(i)\star} = (0, 1, \cdots, m_i - 1)$. Thus, $\sigma_a^\star$ is a special element in Λ_a and $sgn(\sigma_a^\star) = 1$ because all of

$\sigma_a^{(1)\star}, \cdots, \sigma_a^{(p)\star}$ have 0 number of inversions. Notice that for any $\sigma_a \in \Lambda_a$, we have $D^{\sigma_a} f(\vec{a}_0, \vec{z}) = sgn(\sigma_a) \cdot D^{\sigma_a^\star} f(\vec{a}_0, \vec{z})$ by the property of determinant. Then we obtain:

$$\sum_{\sigma_a \in \Lambda_a} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0)(\vec{a} - \vec{a}_0)^{\sigma_a} = \frac{D^{\sigma_a^\star} f(\vec{a}_0)}{\prod_{i=1}^{p} (m_i - 1)!} \sum_{\sigma_a \in \Lambda_a} (\vec{a} - \vec{a}_0)^{\sigma_a} \cdot sgn(\sigma_a)$$

Notice that

$$\begin{aligned}\sum_{\sigma_a \in \Lambda_a} (\vec{a} - \vec{a}_0)^{\sigma_a} \cdot sgn(\sigma_a) &= \prod_{i=1}^{p} \left(\sum_{\sigma_a^{(i)} \in S_{m_i}} (\vec{a}^{(i)} - \vec{a}_0^{(i)})^{\sigma_a^{(i)}} \cdot sgn(\sigma_a^{(i)}) \right) \\ &= \prod_{i=1}^{p} \Delta_{m_i}(a_1^{(i)} - \alpha_i, a_2^{(i)} - \alpha_i, \cdots, a_{m_i}^{(i)} - \alpha_i) \equiv \prod_{i=1}^{p} \Delta_{m_i}^a\end{aligned}$$

where $\Delta_n(x_1, x_2, \cdots, x_n)$ is the Vandermonde Determinant, $a_j^{(i)} = a_{m_1 + \cdots + m_{i-1} + j}$ is the j-th element of $\vec{a}^{(i)}$, and the last line holds by the expansion formula of determinant and definition of Vandermonde Determinant. Now replace $\vec{a}$ with $\vec{A}_{\epsilon,+}$, we get the Vandermonde determinant $\Delta_{m_i}^a$ is actually $(m_i - 1)! \cdot \epsilon^{\frac{1}{2} m_i (m_i - 1)}$. Therefore, we have:

$$\sum_{\sigma_a \in \Lambda_a} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0, \vec{z})(\vec{a} - \vec{a}_0)^{\sigma_a} = D^{\sigma_a^\star} f(\vec{a}_0, \vec{z}) \cdot \epsilon^u \tag{9.71}$$

Since the i-th row of determinant $f(\vec{a}, \vec{z})$ only depends on one variable a_i if we fix $\vec{z}$, taking derivative of $f(\vec{a}, \vec{z})$ with respect to a_i is actually taking derivatives of entries in the i-th row of $f(\vec{a}, \vec{z})$ and let other rows stay unchanged. Therefore, we observe that $D^{\sigma_a^\star} f(\vec{a}_0, \vec{z})$ is exactly the determinant $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ defined in (9.5) and by Lemma 9.15, $D^{\sigma_a^\star} f(\vec{a}_0, \vec{z})$ is non-zero.

Next, we consider the remainder $R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0, \vec{z})$ in (9.70). Suppose σ_a is a permutation of order $u+1$. First notice that the terms in the sum in the remainder is non-zero only when there exist an $i \in \{1, \cdots, p\}$ such that $\sigma_a^{(i)} = (0, 1, \cdots, m_i - 2, m_i)$ and for $j \neq i$, $\sigma_a^{(j)} \in S_{m_j}$. Otherwise, the terms are zero because the determinant $D^{\sigma_a} f(\vec{a}_0, \vec{z})$ will have at least two equal lines. Therefore, there are only finitely many non-zero terms in the sum, and we denote the number of non-zero terms by N, which only depends on $\vec{m}$. Second, we observe that $\sigma_a!$ only has finitely many possible outcomes when its order is $u+1$, thus $\frac{1}{\sigma_a!}$ can be bounded by a constant M only depending on $\vec{m}$. Third, by the construction of $\vec{A}_{\epsilon,+}$ we have

$$|(\vec{A}_{\epsilon,+} - \vec{a}_0)|^{\sigma_a} \leq (\max_{1 \leq i \leq p} m_i \cdot \epsilon)^{u+1} \tag{9.72}$$

for every σ_a such that $\sigma_a = u+1$. Finally, denote vector $\vec{A}_\theta = \vec{a}_0 + \theta(\vec{A}_{\epsilon,+} - \vec{a}_0) = (A_{1,\theta}, \cdots, A_{k,\theta})$. Following similar approach as in (9.44), combined with the form of $D^{\sigma_a} f(\vec{a}_0, \vec{z})$, we have

$$\begin{aligned}\left| D^{\sigma_a} f(\vec{A}_\theta, \vec{z}) \right| &\leq (k!) \left((\max_{1 \leq i \leq p} |z_i|)^{u+1} \right) \prod_{j=1}^{k} e^{c_1(t,p)\left(\sum_{i=1}^{k} |A_{i,\theta}|\right)|z_j|} \\ &\leq (k!)(|z_1| + |z_k|)^{u+1} \prod_{j=1}^{k} e^{c_1(t,p) \cdot k \cdot (\max_{1 \leq i \leq p} m_i) \cdot \epsilon \cdot |z_j|} \qquad (9.73) \\ &\leq (k!)(|z_1| + |z_k|)^{u+1} \prod_{j=1}^{k} e^{C_1 |z_j|}\end{aligned}$$

when $\epsilon < 1$, and the constant $C_1 = c_1(t,p) \cdot k \cdot (\max_{1 \leq i \leq p} m_i)$.

Combining (9.73), (9.72) and the fact that $\frac{1}{\sigma_a!}$ is bounded by a constant $M(\vec{m})$, we have

$$|R_{u+1}(\vec{A}_{\epsilon,+},\vec{a}_0,\vec{z})| \le N\cdot M\cdot (k!)(|z_1|+|z_k|)^{u+1}\exp\left(C_1\sum_{j=1}^{k}|z_j|\right)(\max_{1\le i\le p} m_i\cdot\epsilon)^{u+1} \tag{9.74}$$

which indicates that $R_{u+1}(\vec{A}_{\epsilon,+},\vec{a}_0,\vec{z})$ is $O(\epsilon^{u+1})$, where the constant in big O notation only depends on $\vec{a}_0$, $\vec{A}_{\epsilon,+}$ and $\vec{m}$ and does not depend on ϵ. Therefore, we conclude from (9.69), (9.71) and the fact that $R_{u+1}(\vec{A}_{\epsilon,+},\vec{a}_0,\vec{z})$ is $o(\epsilon^u)$ that:

$$\lim_{\epsilon\downarrow 0}\epsilon^{-u}f(\vec{A}_{\epsilon,+},\vec{z}) = D^{\sigma_a^\star}f(\vec{a}_0,\vec{z})$$

Analogously, we can prove $\lim_{\epsilon\downarrow 0}\epsilon^{-u}f(\vec{A}_{\epsilon,-},\vec{z}) = D^{\sigma_a^\star}f(\vec{a}_0,\vec{z})$ also holds and we complete the proof of (9.68). We can construct vectors $\vec{B}_{\epsilon,\pm}$ similarly, which spread out from vector $\vec{b}_0$ upward and downward, and get similar results for $g(\vec{B}_{\epsilon,\pm},\vec{z})$:

$$\lim_{\epsilon\downarrow 0}\epsilon^{-v}f(\vec{B}_{\epsilon,\pm},\vec{z}) = D^{\sigma_b^\star}g(\vec{b}_0,\vec{z}) \equiv \psi(\vec{b}_0,\vec{z},\vec{n}) \tag{9.75}$$

where $v=\sum_{i=1}^{q}\frac{n_i(n_i-1)}{2}$ in Lemma 9.16, $\vec{n}=(n_1,\cdots,n_q)$ and the non-zero function $\psi(\vec{b}_0,\vec{z},\vec{n})$ is defined in (9.5).

Step 2. In this step, we mainly prove the following result:

$$\text{The function } H(\vec{z}) = \varphi(\vec{a}_0,\vec{z},\vec{m})\cdot\psi(\vec{b}_0,\vec{z},\vec{n})\cdot\prod_{i=1}^{k}e^{-c_3(t,p)z_i^2} \text{ is integrable over } \mathbb{R}^k. \tag{9.76}$$

Notice that $\varphi(\vec{a}_0,\vec{z},\vec{m})$ and $\psi(\vec{b}_0,\vec{z},\vec{n})$ are two determinants whose expression are given in (9.5), and they are positive when $\vec{z}\in\mathbb{W}_k^o$ because of (9.68) and (9.75). Following similar approach as in (9.73) we can find

$$\begin{aligned} &\varphi(\vec{a}_0,\vec{z},\vec{m}) \le (k!)\,(|z_1|+|z_k|)^u\prod_{j=1}^{k}e^{c_1(t,p)\cdot\left(\sum_{i=1}^{p}|\alpha_i|m_i\right)|z_j|},\\ &\psi(\vec{b}_0,\vec{z},\vec{n}) \le (k!)\,(|z_1|+|z_k|)^v\prod_{j=1}^{k}e^{c_2(t,p)\cdot\left(\sum_{i=1}^{q}|\beta_i|n_i\right)|z_j|}, \end{aligned} \tag{9.77}$$

when $z_1>z_2>\cdots>z_k$. Therefore,

$$H(\vec{z}) \le (k!)^2\cdot(|z_1|+|z_k|)^{u+v}\cdot\prod_{i=1}^{k}e^{C|z_i|-c_3(t,p)\cdot z_i^2} \tag{9.78}$$

where $C=c_1(t,p)\cdot\sum_{i=1}^{p}|\alpha_i|m_i + c_2(t,p)\cdot\sum_{i=1}^{q}|\beta_i|n_i$. The right hand side is integrable over $\mathbb{R}^k$ because of the quadratic terms in the exponential. Thus, $H(\vec{z})$ is integrable and we can define the constant $Z_c=\int_{\mathbb{R}^k}H(z)\mathbf{1}_{\{z_1>z_2>\cdots>z_k\}}dz<\infty$ and the function

$$\rho_c(z_1,\cdots,z_k) = Z_c^{-1}\cdot\mathbf{1}_{\{z_1>z_2>\cdots>z_k\}}\cdot\varphi(\vec{a}_0,\vec{z},\vec{m})\cdot\psi(\vec{b}_0,\vec{z},\vec{n})\cdot\prod_{i=1}^{k}e^{-c_3(t,p)z_i^2} \tag{9.79}$$

is a density because it's non-negative and integrates to 1 over $\mathbb{R}^k$.

Step 3. Denote $Z^T_{\vec{a}_0,\vec{b}_0}$ as the random vector Z^T in Definition 9.1 associated with vectors $\vec{a}_0$ and $\vec{b}_0$, and in this step we prove it weakly converges to the continuous distribution with density $\rho_c(z)$ we just constructed in (9.79). Suppose $\mathfrak{L}_+^T$ is an avoiding Bernoulli line ensemble starting with $\vec{x}_{+}^{T} = \left(x_{+,1}^{T}, \cdots, x_{+,k}^{T}\right)$ and ending with $\vec{y}_{+}^{T} = \left(y_{+,1}^{T}, \cdots, y_{+,k}^{T}\right)$ and follows the distribution $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}_{+}^{T},\vec{y}_{+}^{T}}$. The vectors $\vec{x}_{+}^{T}$ and $\vec{y}_{+}^{T}$ are two signatures of length k that satisfies the following:

(1) Let 1_k denote the vector $(1, 1, \cdots, 1)$ of length k, then

$$\lim_{T\to\infty} \frac{\vec{x}_{+}^{T}}{\sqrt{T}} = \vec{A}_{\epsilon,+}, \quad \lim_{T\to\infty} \frac{\vec{y}_{+}^{T} - pT1_k}{\sqrt{T}} = \vec{B}_{\epsilon,+} \tag{9.80}$$

(2) $x_{+,i}^{T} \geq x_{i}^{T}$, $y_{+,i}^{T} \geq y_{i}^{T}$, for $i = 1, \cdots, k$, which means the endpoints of the newly constructed line ensembles dominate the original ones.

This can be achieved due to the limiting behavior of $\vec{x}_{+}^{T}$ and $\vec{y}_{+}^{T}$ and the fact that $\vec{A}_{\epsilon,+}$ and $\vec{B}_{\epsilon,+}$ dominate $\vec{a}_0$ and $\vec{b}_0$. Analogously, we construct another avoiding Bernoulli line ensemble $\mathfrak{L}_{-}^{T}$ with endpoints $\vec{x}_{-}^{T}$ and $\vec{y}_{-}^{T}$ and distribution $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}_{-}^{T},\vec{y}_{-}^{T}}$ such that $\lim_{T\to\infty} \frac{\vec{x}_{-}^{T}}{\sqrt{T}} = \vec{A}_{\epsilon,-}$, $\lim_{T\to\infty} \frac{\vec{y}_{-}^{T} - pT1_k}{\sqrt{T}} = \vec{B}_{\epsilon,-}$, and $x_{-,i}^{T} \leq x_{i}^{T}$, $y_{-,i}^{T} \leq y_{i}^{T}$ for $i = 1, \cdots, k$.

Since now $\vec{A}_{\epsilon,+}$, $\vec{A}_{\epsilon,-}$, $\vec{B}_{\epsilon,+}$, $\vec{B}_{\epsilon,-}$ have distinct elements, we can apply the results in Proposition 9.2 and conclude the weak convergence:

$$Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}^{T} \Rightarrow \rho_{\epsilon,+}(z), \quad Z_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}}^{T} \Rightarrow \rho_{\epsilon,-}(z)$$

where $Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}^{T}$ and $Z_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}}^{T}$ are obtained by scaling the line ensembles $\mathfrak{L}_{+}^{T}$ and $\mathfrak{L}_{-}^{T}$ through Definition 9.1, $\rho_{\epsilon,+}(z)$ and $\rho_{\epsilon,-}(z)$ are densities which are obtained by plugging $\vec{A}_{\epsilon,+}$, $\vec{B}_{\epsilon,+}$ and $\vec{A}_{\epsilon,-}$, $\vec{B}_{\epsilon,-}$ into the formula of $\rho(z)$ in (9.3).

In order to prove the weak convergence of $Z_{\vec{a}_0,\vec{b}_0}^{T}$, it is sufficient to prove for any $R = (-\infty, u_1] \times (-\infty, u_2] \times \cdots \times (-\infty, u_k]$, where $u_i \in \mathbb{R}$, we have

$$\lim_{T\to\infty} \mathbb{P}(Z_{\vec{a}_0,\vec{b}_0}^{T} \in R) = \int_R \rho_c(z)dz \tag{9.81}$$

Actually, by Lemma 3.1, we can construct a sequence of probability spaces $(\Omega_T, \mathcal{F}_T, \mathbb{P}_T)_{T\geq 1}$ such that for each $T \in \mathbb{Z}^{+}$, we have random variables $\mathfrak{L}_{+}^{T}$ and $\mathfrak{L}^{T}$ having law $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}_{+}^{T},\vec{y}_{+}^{T}}$, and $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}^{T},\vec{y}^{T}}$ under measure $\mathbb{P}_T$, respectively. Also, we have $\mathfrak{L}_{+}^{T}(i,r) \geq \mathfrak{L}^{T}(i,r)$ with probability 1, where $\mathfrak{L}_{+}^{T}(i,r)$(resp., $\mathfrak{L}^{T}(i,r)$) is the value of the i-th up-right path of $\mathfrak{L}_{+}^{T}$(resp., $\mathfrak{L}^{T}$) at $r \in [\![0, T]\!]$. Similarly, we can construct another sequence of probability spaces $(\Omega_T', \mathcal{F}_T', \mathbb{Q}_T)_{T\geq 1}$ such that for each $T \in \mathbb{Z}^{+}$, we have random variables $\mathfrak{L}_{-}^{T}$ and $\mathfrak{L}^{T}$ have law $\mathbb{P}_{avoid,Ber}^{0,T,\vec{x}_{-}^{T},\vec{y}_{-}^{T}}$, and $\mathbb{P}_{avoid,Ber}^{0,T,\vec{x}^{T},\vec{y}^{T}}$ under measure $\mathbb{Q}_T$, respectively, along with $\mathbb{Q}_T\left(\mathfrak{L}_{-}^{T}(i,r) \leq \mathfrak{L}^{T}(i,r), i = 1, \cdots, k, r \in [\![0, T]\!]\right) = 1$.

Therefore, we have that under measure $\mathbb{P}_T$ and $\mathbb{Q}_T$:

$$\mathbb{P}_T(Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}^{T} \in R) \leq \mathbb{P}_T(Z_{\vec{a}_0,\vec{b}_0}^{T} \in R), \quad \mathbb{Q}_T(Z_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}}^{T} \in R) \geq \mathbb{Q}_T(Z_{\vec{a}_0,\vec{b}_0}^{T} \in R) \tag{9.82}$$

Take limit-infimum and limit-supremum on both sides of the first and second inequality in (9.82) respectively, we get

$$\int_R \rho_{\epsilon,+}(z)dz \leq \liminf_{T\to\infty} \mathbb{P}_T(Z_{\vec{a}_0,\vec{b}_0}^{T} \in R), \quad \int_R \rho_{\epsilon,-}(z)dz \geq \limsup_{T\to\infty} \mathbb{Q}_T(Z_{\vec{a}_0,\vec{b}_0}^{T} \in R) \tag{9.83}$$

because of the weak convergence of $Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}^{T}$ and $Z_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}}^{T}$. Since the distributions of $Z_{\vec{a}_0,\vec{b}_0}^{T}$ under measure $\mathbb{P}_T$ and $\mathbb{Q}_T$ are the same, we can combine the above two inequalities in (9.83) and get

$$\int_R \rho_{\epsilon,+}(z)dz \leq \liminf_{T\to\infty} \mathbb{P}_T(Z_{\vec{a}_0,\vec{b}_0}^{T} \in R) \leq \limsup_{T\to\infty} \mathbb{P}_T(Z_{\vec{a}_0,\vec{b}_0}^{T} \in R) \leq \int_R \rho_{\epsilon,-}(z)dz \tag{9.84}$$

The rest of the proof establishes the following statement:

$$\lim_{\epsilon\downarrow 0}\int_R \rho_{\epsilon,+}(z)dz = \lim_{\epsilon\downarrow 0}\int_R \rho_{\epsilon,-}(z)dz = \int_R \rho_c(z)dz \tag{9.85}$$

and thereby concluding

$$\lim_{T\to\infty}\mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0}\in R) = \int_R \rho_c(z)dz$$

by letting $\epsilon\downarrow 0$ in the inequality (9.84), and we prove the weak convergence of $Z^T_{\vec{a}_0,\vec{b}_0}$.

The rest of the proof intends to establish (9.85). By (9.69), (9.71) and (9.74), we have when $\epsilon<1$:

$$\epsilon^{-u} f(\vec{A}_{\epsilon,+},\vec{z}) \le \varphi(\vec{a}_0,\vec{z},\vec{m}) + \widetilde{C}_1\cdot(|z_1|+|z_k|)^{u+1}\cdot e^{C_1\cdot\sum_{j=1}^k |z_j|} \equiv F(\vec{z}) \tag{9.86}$$

where the constants $\widetilde{C}_1 = N(\vec{m})\cdot M(\vec{m})\cdot(k!)\cdot(\max_{1\le i\le p} m_i)$, $C_1 = c_1(t,p)\cdot k\cdot(\max_{1\le i\le p} m_i)$ only depend on $\vec{m}$. Analogously, we can find constants $\widetilde{C}_2$ and C_2 only depending on $\vec{n}$ such that

$$\epsilon^{-v} f(\vec{B}_{\epsilon,+},\vec{z}) \le \varphi(\vec{b}_0,\vec{z},\vec{n}) + \widetilde{C}_2\cdot(|z_1|+|z_k|)^{v+1}\cdot e^{C_2\cdot\sum_{j=1}^k |z_j|} \equiv G(\vec{z}) \tag{9.87}$$

Therefore, we obtain

$$\epsilon^{-(u+v)} f(\vec{A}_{\epsilon,+},\vec{z}) g(\vec{B}_{\epsilon,+},\vec{z}) \prod_{i=1}^k e^{-c_3(t,p)z_i^2} \le F(\vec{z})G(\vec{z})\prod_{i=1}^k e^{-c_3(t,p)z_i^2} \tag{9.88}$$

and the right hand side of (9.88) is integrable because of the quadratic terms in the exponential. Let $Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}$ be the normalizing constant in the density (9.3) when $\vec{a}$ and $\vec{b}$ equal to $\vec{A}_{\epsilon,+}$ and $\vec{B}_{\epsilon,+}$. Then, we have

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{-(u+v)} Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} &= \lim_{\epsilon\downarrow 0}\int_{\mathbb{W}_k^o}\left(\epsilon^{-u} f(\vec{A}_{\epsilon,+},\vec{z})\right)\left(\epsilon^{-v} g(\vec{B}_{\epsilon,+},\vec{z})\right)\prod_{i=1}^k e^{-c_3(t,p)z_i^2}dz \\ &= \int_{\mathbb{W}_k^o}\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n})\prod_{i=1}^k e^{-c_3(t,p)z_i^2}dz = Z_c\end{aligned} \tag{9.89}$$

where the second equality uses dominated convergence theorem with the dominating function being the right hand side of (9.88) as well as results (9.68) and (9.75), and the last equality is due to (9.79) which gives the definition of Z_c. Therefore, we conclude

$$\begin{aligned}&\lim_{\epsilon\downarrow 0} Z^{-1}_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}\cdot f(\vec{a},\vec{z})\cdot g(\vec{b},\vec{z}) = \lim_{\epsilon\downarrow 0}\left(\epsilon^{u+v} Z^{-1}_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}\right)\cdot\left(\epsilon^{-u} f(\vec{a},\vec{z})\right)\cdot\left(\epsilon^{-v} g(\vec{b},\vec{z})\right) \\ &= Z_c^{-1}\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n})\end{aligned} \tag{9.90}$$

which implies $\rho_{\epsilon,+}(z)$ pointwise converges to $\rho_c(z)$ when $\epsilon\downarrow 0$. Since $\rho_{\epsilon,+}(z)\mathbf{1}_R \le \rho_{\epsilon,+}(z)$ is bounded by an integrable function, by Dominated Convergence Theorem we have:

$$\lim_{\epsilon\downarrow 0}\int_R \rho_{\epsilon,+}(z)dz = \int_R \rho_c(z)dz$$

Analogously, we can get $\lim_{\epsilon\downarrow 0}\int_R \rho_{\epsilon,-}(z)dz = \int_R \rho_c(z)dz$ and we proved the statement (9.85), which completes the proof. $\square$

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