TIGHTNESS OF BERNOULLI LINE ENSEMBLES

Abstract. Insert abstract here:

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2. Line ensembles

In this section we introduce various definitions and notation that are used throughout the paper.

2.1. **Line ensembles and the Brownian Gibbs property.** In this section we introduce the notions of a *line ensemble* and the *(partial) Brownian Gibbs property*. Our exposition in this section closely follows that of [6, Section 2] and [4, Section 2].

Given two integers $p \leq q$, we let $[\![p, q]\!]$ denote the set $\{p, p+1, \dots, q\}$. Given an interval $\Lambda \subset \mathbb{R}$ we endow it with the subspace topology of the usual topology on $\mathbb{R}$. We let $(C(\Lambda), \mathcal{C})$ denote the space of continuous functions $f : \Lambda \to \mathbb{R}$ with the topology of uniform convergence over compacts, see [12, Chapter 7, Section 46], and Borel σ-algebra $\mathcal{C}$. Given a set $\Sigma \subset \mathbb{Z}$ we endow it with the discrete topology and denote by $\Sigma \times \Lambda$ the set of all pairs (i, x) with $i \in \Sigma$ and $x \in \Lambda$ with the product topology. We also denote by $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ the space of continuous functions on $\Sigma \times \Lambda$ with the topology of uniform convergence over compact sets and Borel σ-algebra $\mathcal{C}_\Sigma$. Typically, we will take $\Sigma = [\![1, N]\!]$ (we use the convention $\Sigma = \mathbb{N}$ if $N = \infty$) and then we write $\big(C(\Sigma \times \Lambda), \mathcal{C}_{|\Sigma|}\big)$ in place of $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$.

The following defines the notion of a line ensemble.

Definition 2.1. Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. A Σ-*indexed line ensemble* $\mathcal{L}$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that takes values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$. Intuitively, $\mathcal{L}$ is a collection of random continuous curves (sometimes referred to as *lines*), indexed by Σ, each of which maps Λ in $\mathbb{R}$. We will often slightly abuse notation and write $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$, even though it is not $\mathcal{L}$ which is such a function, but $\mathcal{L}(\omega)$ for every $\omega \in \Omega$. For $i \in \Sigma$ we write $\mathcal{L}_i(\omega) = (\mathcal{L}(\omega))(i, \cdot)$ for the curve of index i and note that the latter is a map $\mathcal{L}_i : \Omega \to C(\Lambda)$, which is $(\mathcal{C}, \mathcal{F})-$measurable.

We will require the following result, whose proof is postponed until Section [Appendix]. In simple terms it states that the space $C(\Sigma \times \Lambda)$ where our random variables $\mathcal{L}$ take value has the structure of a complete, separable metric space. We say that a collection $(K_n)_{n\geq 1}$ of compact subsets $K_n \subset \Sigma \times \Lambda$ is a *compact exhaustion* if $K_n \subseteq K_{n+1}$ for all $n \geq 1$, $\bigcup_n K_n = \Sigma \times \Lambda$, and every compact subset of $\Sigma \times \Lambda$ is contained in some K_n.

Lemma 2.2. *Let $(K_n)_{n\geq 1}$ be a compact exhaustion of $\Sigma \times \Lambda$. Define $d : C(\Sigma \times \Lambda) \times C(\Sigma \times \Lambda) \to [0, \infty)$ by*

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \min \Big\{ \sup_{(i,t)\in K_n} |f(i, t) - g(i, t)|,\, 1 \Big\}. \tag{2.1}$$

Then d defines a metric on $C(\Sigma \times \Lambda)$ and moreover the metric space topology defined by d is the same as the topology of uniform convergence over compact sets. Furthermore, the metric space $(C(\Sigma \times \Lambda), d)$ is complete and separable.

Definition 2.3. Given a sequence $\{\mathcal{L}^n : n \in \mathbb{N}\}$ of random Σ-indexed line ensembles we say that $\mathcal{L}^n$ *converge weakly* to a line ensemble $\mathcal{L}$, and write $\mathcal{L}^n \implies \mathcal{L}$ if for any bounded continuous function $f : C(\Sigma \times \Lambda) \to \mathbb{R}$ we have that

$$\lim_{n\to\infty} \mathbb{E}\left[f(\mathcal{L}^n)\right] = \mathbb{E}\left[f(\mathcal{L})\right].$$

We also say that $\{\mathcal{L}^n : n \in \mathbb{N}\}$ is *tight* if for any $\epsilon > 0$ there exists a compact set $K \subset C(\Sigma \times \Lambda)$ such that $\mathbb{P}(\mathcal{L}^n \in K) \geq 1 - \epsilon$ for all $n \in \mathbb{N}$.

We call a line ensemble *non-intersecting* if $\mathbb{P}$-almost surely $\mathcal{L}_i(r) > \mathcal{L}_j(r)$ for all $i < j$ and $r \in \Lambda$.

We will require the following sufficient condition for tightness of a sequence of line ensembles, which extends [1, Theorem 7.3]. We give a proof in Section 6.

Lemma 2.4. *Let $\Lambda \subset \mathbb{R}$ be an interval and $\Sigma = [\![1, N]\!]$. Let $([a_k, b_k])_{k\geq 1}$ be a compact exhaustion of Λ, and let $a_0 \in [a_k, b_k]$ for all k. Then $(\mathcal{L}^n)$ is tight if and only if for every $i \in \Sigma$ and $k \geq 1$, we have*

(i)

$$\lim_{a\to\infty} \limsup_{n\to\infty} \mathbb{P}(|\mathcal{L}^n_i(a_0)| \geq a) = 0.$$

(ii) For all $\epsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}\Bigg(\sup_{\substack{x,y\in[a_k,b_k],\\ |x-y|\leq\delta}} |\mathcal{L}^n_i(x) - \mathcal{L}^n_i(y)| \geq \epsilon \Bigg) = 0.$$

We next turn to formulating the Brownian Gibbs property – we do this in Definition 2.8 after introducing some relevant notation and results. If W_t denotes a standard one-dimensional Brownian motion, then the process

$$\tilde{B}(t) = W_t - tW_1, \quad 0 \leq t \leq 1,$$

is called a *Brownian bridge (from $\tilde{B}(0) = 0$ to $\tilde{B}(1) = 0$) with diffusion parameter* 1. For brevity we call the latter object a *standard Brownian bridge.*

Given $a, b, x, y \in \mathbb{R}$ with $a < b$ we define a random variable on $(C([a, b]), \mathcal{C})$ through

$$B(t) = (b - a)^{1/2} \cdot \tilde{B}\left(\frac{t - a}{b - a}\right) + \left(\frac{b - t}{b - a}\right) \cdot x + \left(\frac{t - a}{b - a}\right) \cdot y, \tag{2.2}$$

and refer to the law of this random variable as a *Brownian bridge (from $B(a) = x$ to $B(b) = y$) with diffusion parameter* 1. Given $k \in \mathbb{N}$ and $\vec{x}, \vec{y} \in \mathbb{R}^k$ we let $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}$ denote the law of k independent Brownian bridges $\{B_i : [a, b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1.

We next state a couple of results about Brownian bridges from [4] for future use.

Lemma 2.5. *[4, Corollary 2.9]. Fix a continuous function $f : [0,1] \to \mathbb{R}$ such that $f(0) > 0$ and $f(1) > 0$. Let B be a standard Brownian bridge and let $C = \{B(t) > f(t)$ for some $t \in [0,1]\}$ (crossing) and $T = \{B(t) = f(t)$ for some $t \in [0,1]\}$ (touching). Then $\mathbb{P}(T \cap C^c) = 0$.*

Lemma 2.6. *[4, Corollary 2.10]. Let U be an open subset of $C([0,1])$, which contains a function f such that $f(0) = f(1) = 0$. If $B : [0,1] \to \mathbb{R}$ is a standard Brownian bridge then $\mathbb{P}(B[0,1] \subset U) > 0$.*

The following definition introduces the notion of an (f,g)-avoiding Brownian line ensemble, which in simple terms is a collection of k independent Brownian bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two continuous functions f and g.

Definition 2.7. Let $k \in \mathbb{N}$ and W_k° denote the open Weyl chamber in $\mathbb{R}^k$, i.e.

$$W_k^\circ = \{\vec{x} = (x_1, \dots, x_k) \in \mathbb{R}^k : x_1 > x_2 > \cdots > x_k\}$$

(in [4] the notation $\mathbb{R}^k_>$ was used for this set). Let $\vec{x}, \vec{y} \in W_k^\circ$, $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to (-\infty, \infty]$ and $g : [a,b] \to [-\infty, \infty)$ be two continuous functions. The latter condition means that either $f : [a,b] \to \mathbb{R}$ is continuous or $f = \infty$ everywhere, and similarly for g. We also assume that $f(t) > g(t)$ for all $t \in [a,b]$, $f(a) > x_1, f(b) > y_1$ and $g(a) < x_k, g(b) < y_k$.

With the above data we define the (f,g)-*avoiding Brownian line ensemble on the interval* $[a,b]$ *with entrance data* $\vec{x}$ *and exit data* $\vec{y}$ to be the Σ-indexed line ensemble $\mathcal{Q}$ with $\Sigma = [\![1,k]\!]$ on $\Lambda = [a,b]$ and with the law of $\mathcal{Q}$ equal to $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}$ (the law of k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$) conditioned on the event

$$E = \{f(r) > B_1(r) > B_2(r) > \cdots > B_k(r) > g(r) \text{ for all } r \in [a,b]\}.$$

It is worth pointing out that E is an open set of positive measure and so we can condition on it in the usual way – we explain this briefly in the following paragraph. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that supports k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1. Notice that we can find $\tilde{u}_1, \dots, \tilde{u}_k \in C([0,1])$ and $\epsilon > 0$ (depending on $\vec{x}, \vec{y}, f, g, a, b$) such that $\tilde{u}_i(0) = \tilde{u}_i(1) = 0$ for $i = 1, \dots, k$ and such that if $\tilde{h}_1, \dots, \tilde{h}_k \in C([0,1])$ satisfy $\tilde{h}_i(0) = \tilde{h}_i(1) = 0$ for $i = 1, \dots, k$ and $\sup_{t \in [0,1]} |\tilde{u}_i(t) - \tilde{h}_i(t)| < \epsilon$ then the functions

$$h_i(t) = (b-a)^{1/2} \cdot \tilde{h}_i\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot x_i + \left(\frac{t-a}{b-a}\right) \cdot y_i,$$

satisfy $f(r) > h_1(r) > \cdots > h_k(r) > g(r)$. It follows from Lemma 2.6 that

$$\mathbb{P}(E) \geq \mathbb{P}\left(\max_{1 \leq i \leq k} \sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) = \prod_{i=1}^k \mathbb{P}\left(\sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) > 0,$$

and so we can condition on the event E.

To construct a realization of $\mathcal{Q}$ we proceed as follows. For $\omega \in E$ we define

$$\mathcal{Q}(\omega)(i,r) = B_i(r)(\omega) \text{ for } i = 1, \dots, k \text{ and } r \in [a,b].$$

Observe that for $i \in \{1, \dots, k\}$ and an open set $U \in C([a,b])$ we have that

$$\mathcal{Q}^{-1}(\{i\} \times U) = \{B_i \in U\} \cap E \ \in \ \mathcal{F},$$

and since the sets $\{i\} \times U$ form an open basis of $C([\![1,k]\!] \times [a,b])$ we conclude that $\mathcal{Q}$ is $\mathcal{F}$-measurable. This implies that the law $\mathcal{Q}$ is indeed well-defined and also it is non-intersecting almost surely. Also, given measurable subsets $A_1, \dots, A_k$ of $C([a,b])$ we have that

$$\mathbb{P}(\mathcal{Q}_i \in A_i \text{ for } i = 1, \dots, k) = \frac{\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}\left(\{B_i \in A_i \text{ for } i = 1, \dots, k\} \cap E\right)}{\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}(E)}.$$

We denote the probability distribution of $\mathcal{Q}$ as $\mathbb{P}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$ and write $\mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$ for the expectation with respect to this measure.

The following definition introduces the notion of the Brownian Gibbs property from [4].

Definition 2.8. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and an interval $\Lambda \subset \mathbb{R}$ and let $K = \{k_1, k_1 + 1, \dots, k_2\} \subset \Sigma$ be finite and $a, b \in \Lambda$ with $a < b$. Set $f = \mathcal{L}_{k_1-1}$ and $g = \mathcal{L}_{k_2+1}$ with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$ and $g = -\infty$ if $k_2 + 1 \notin \Sigma$. Write $D_{K,a,b} = K \times (a, b)$ and $D^c_{K,a,b} = (\Sigma \times \Lambda) \setminus D_{K,a,b}$. A Σ-indexed line ensemble $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$ is said to have the *Brownian Gibbs property* if it is non-intersecting and

$$\mathrm{Law}\left(\mathcal{L}|_{K\times[a,b]} \text{ conditional on } \mathcal{L}|_{D^c_{K,a,b}}\right) = \mathrm{Law}\left(\mathcal{Q}\right),$$

where $\mathcal{Q}_i = \tilde{\mathcal{Q}}_{i-k_1+1}$ and $\tilde{\mathcal{Q}}$ is the (f, g)-avoiding Brownian line ensemble on $[a, b]$ with entrance data $(\mathcal{L}_{k_1}(a), \dots, \mathcal{L}_{k_2}(a))$ and exit data $(\mathcal{L}_{k_1}(b), \dots, \mathcal{L}_{k_2}(b))$ from Definition 2.7. Note that $\tilde{Q}$ is introduced because, by definition, any such (f, g)-avoiding Brownian line ensemble is indexed from 1 to $k_2 - k_1 + 1$ but we want $\mathcal{Q}$ to be indexed from k_1 to k_2.

A more precise way to express the Brownian Gibbs property is as follows. A Σ-indexed line ensemble $\mathcal{L}$ on Λ satisfies the Brownian Gibbs property if and only if it is non-intersecting and for any finite $K = \{k_1, k_1 + 1, \dots, k_2\} \subset \Sigma$ and $[a, b] \subset \Lambda$ and any bounded Borel-measurable function $F : C(K \times [a, b]) \to \mathbb{R}$ we have $\mathbb{P}$-almost surely

$$\mathbb{E}\left[F\left(\mathcal{L}|_{K\times[a,b]}\right)\big|\mathcal{F}_{ext}(K \times (a, b))\right] = \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}\big[F(\tilde{\mathcal{Q}})\big], \tag{2.3}$$

where

$$\mathcal{F}_{ext}(K \times (a, b)) = \sigma\left\{\mathcal{L}_i(s) : (i, s) \in D^c_{K,a,b}\right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K\times[a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K \times [a, b]$, $\vec{x} = (\mathcal{L}_{k_1}(a), \dots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \dots, \mathcal{L}_{k_2}(b))$, $f = \mathcal{L}_{k_1-1}[a, b]$ (the restriction of $\mathcal{L}$ to the set $\{k_1 - 1\} \times [a, b]$) with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$, and $g = \mathcal{L}_{k_2+1}[a, b]$ with the convention that $g = -\infty$ if $k_2 + 1 \notin \Sigma$.

Remark 2.9. Let us briefly explain why equation (2.3) makes sense. Firstly, since $\Sigma \times \Lambda$ is locally compact, we know by [12, Lemma 46.4] that $\mathcal{L} \to \mathcal{L}|_{K\times[a,b]}$ is a continuous map from $C(\Sigma \times \Lambda)$ to $C(K \times [a, b])$, so that the left side of (2.3) is the conditional expectation of a bounded measurable function, and is thus well-defined. A more subtle question is why the right side of (2.3) is $\mathcal{F}_{ext}(K \times (a, b))$-measurable. This question was resolved in [6, Lemma 3.4], where it was shown that the right side is measurable with respect to the σ-algebra

$$\sigma\left\{\mathcal{L}_i(s) :\ i \in K \text{ and } s \in \{a, b\}, \text{ or } i \in \{k_1 - 1, k_2 + 1\} \text{ and } s \in [a, b]\right\},$$

which in particular implies the measurability with respect to $\mathcal{F}_{ext}(K \times (a, b))$.

In the present paper it is convenient for us to use the following modified version of the definition above, which we call the partial Brownian Gibbs property – it was first introduced in [6]. We explain the difference between the two definitions, and why we prefer the second one in Remark 2.12.

Definition 2.10. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and an interval $\Lambda \subset \mathbb{R}$. A Σ-indexed line ensemble $\mathcal{L}$ on Λ is said to satisfy the *partial Brownian Gibbs property* if and only if it is non-intersecting and for any finite $K = \{k_1, k_1 + 1, \dots, k_2\} \subset \Sigma$ with $k_2 \leq N - 1$ (if $\Sigma \neq \mathbb{N}$), $[a, b] \subset \Lambda$ and any bounded Borel-measurable function $F : C(K \times [a, b]) \to \mathbb{R}$ we have $\mathbb{P}$-almost surely

$$\mathbb{E}\left[F(\mathcal{L}|_{K\times[a,b]})\big|\mathcal{F}_{ext}(K \times (a, b))\right] = \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}\big[F(\tilde{\mathcal{Q}})\big], \tag{2.4}$$

where we recall that $D_{K,a,b} = K \times (a, b)$ and $D^c_{K,a,b} = (\Sigma \times \Lambda) \setminus D_{K,a,b}$, and

$$\mathcal{F}_{ext}(K \times (a, b)) = \sigma\left\{\mathcal{L}_i(s) : (i, s) \in D^c_{K,a,b}\right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K\times[a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K \times [a,b]$, $\vec{x} = (\mathcal{L}_{k_1}(a), \dots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \dots, \mathcal{L}_{k_2}(b))$, $f = \mathcal{L}_{k_1-1}[a,b]$ with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$, and $g = \mathcal{L}_{k_2+1}[a,b]$.

Remark 2.11. Observe that if $N = 1$ then the conditions in Definition 2.10 become void. I.e., any line ensemble with one line satisfies the partial Brownian Gibbs property. Also we mention that (2.4) makes sense by the same reason that (2.3) makes sense, see Remark 2.9.

Remark 2.12. Definition 2.10 is slightly different from the Brownian Gibbs property of Definition 2.8 as we explain here. Assuming that $\Sigma = \mathbb{N}$ the two definitions are equivalent. However, if $\Sigma = \{1, \dots, N\}$ with $1 \leq N < \infty$ then a line ensemble that satisfies the Brownian Gibbs property also satisfies the partial Brownian Gibbs property, but the reverse need not be true. Specifically, the Brownian Gibbs property allows for the possibility that $k_2 = N$ in Definition 2.10 and in this case the convention is that $g = -\infty$. As the partial Brownian Gibbs property is more general we prefer to work with it and most of the results later in this paper are formulated in terms of it rather than the usual Brownian Gibbs property.

2.2. **Bernoulli Gibbsian line ensembles.** In this section we introduce the notion of a *Bernoulli line ensemble* and the *Schur Gibbs property*. Our discussion will parallel that of [3, Section 3.1], which in turn goes back to [5, Section 2.1].

Definition 2.13. Let $\Sigma \subset \mathbb{Z}$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. Consider the set Y of functions $f : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(j, i+1) - f(j,i) \in \{0,1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$ and let $\mathcal{D}$ denote the discrete topology on Y. We call a function $f : [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(i+1) - f(i) \in \{0,1\}$ when $i \in [\![T_0, T_1 - 1]\!]$ an *up-right path* and elements in Y *collections of up-right paths*.

A Σ-*indexed Bernoulli line ensemble* $\mathfrak{L}$ *on* $[\![T_0, T_1]\!]$ is a random variable defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, taking values in Y such that $\mathfrak{L}$ is a $(\mathcal{B}, \mathcal{D})$-measurable function.

Remark 2.14. In [3, Section 3.1] Bernoulli line ensembles $\mathfrak{L}$ were called *discrete line ensembles* in order to distinguish them from the continuous line ensembles from Definition 2.1. In this paper we have opted to use the term Bernoulli line ensembles to emphasize the fact that the functions $f \in Y$ satisfy the property that $f(j, i+1) - f(j,i) \in \{0,1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$. This condition essentially means that for each $j \in \Sigma$ the function $f(j, \cdot)$ can be thought of as the trajectory of a Bernoulli random walk from time T_0 to time T_1. As other types of discrete line ensembles, see e.g. [15], have appeared in the literature we have decided to modify the notation in [3, Section 3.1] so as to avoid any ambiguity.

The way we think of Bernoulli line ensembles is as random collections of up-right paths on the integer lattice, indexed by Σ (see Figure 1). Observe that one can view an up-right path L on $[\![T_0, T_1]\!]$ as a continuous curve by linearly interpolating the points $(i, L(i))$. This allows us to define $(\mathfrak{L}(\omega))(i,s)$ for non-integer $s \in [T_0, T_1]$ and to view Bernoulli line ensembles as line ensembles in the sense of Definition 2.1. In particular, we can think of $\mathfrak{L}$ as a random variable taking values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ with $\Lambda = [T_0, T_1]$. We will often slightly abuse notation and write $\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$, even though it is not $\mathfrak{L}$ which is such a function, but rather $\mathfrak{L}(\omega)$ for each $\omega \in \Omega$. Furthermore we write $L_i = (\mathfrak{L}(\omega))(i, \cdot)$ for the index $i \in \Sigma$ path. If L is an up-right path on $[\![T_0, T_1]\!]$ and $a, b \in [\![T_0, T_1]\!]$ satisfy $a < b$ we let $L[\![a,b]\!]$ denote the resitrction of L to $[\![a,b]\!]$.

Let $t_i, z_i \in \mathbb{Z}$ for $i = 1,2$ be given such that $t_1 < t_2$ and $0 \leq z_2 - z_1 \leq t_2 - t_1$. We denote by $\Omega(t_1, t_2, z_1, z_2)$ the collection of up-right paths that start from (t_1, z_1) and end at (t_2, z_2), by $\mathbb{P}_{Ber}^{t_1,t_2,z_1,z_2}$ the uniform distribution on $\Omega(t_1, t_2, z_1, z_2)$ and write $\mathbb{E}_{Ber}^{t_1,t_2,z_1,z_2}$ for the expectation with respect to this measure. One thinks of the distribution $\mathbb{P}_{Ber}^{t_1,t_2,z_1,z_2}$ as the law of a simple random walk with i.i.d. Bernoulli increments with parameter $p \in (0,1)$ that starts from z_1 at time t_1 and is

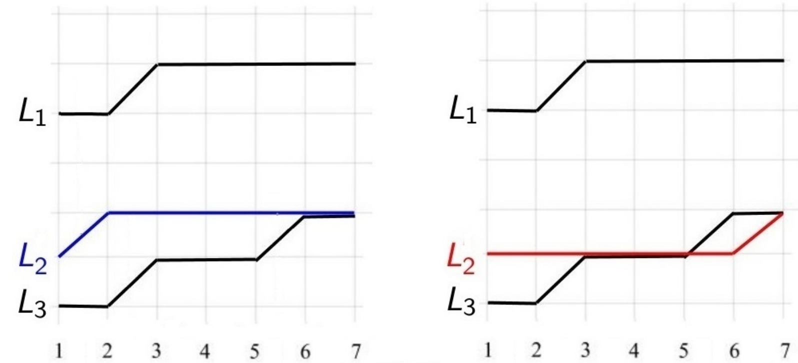

FIGURE 1. Two samples of $\{1, 2, 3\}$-indexed Bernoulli line ensembles with $T_0 = 1$ and $T_1 = 7$.

conditioned to end in z_2 at time t_2 – this interpretation does not depend on the choice of $p \in (0, 1)$. Notice that by our assumptions on the parameters the state space $\Omega(t_1, t_2, z_1, z_2)$ is non-empty.

Given $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$ and $\vec{x}, \vec{y} \in \mathbb{Z}^k$ we let $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y}}_{Ber}$ denote the law of k independent Bernoulli bridges $\{B_i : [\![T_0, T_1]\!] \to \mathbb{Z}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$. Equivalently, this is just k independent random up-right paths $B_i \in \Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ that are uniformly distributed. This measure is well-defined provided that $\Omega(T_0, T_1, x_i, y_i)$ are non-empty for $i = 1, \dots, k$, which holds if $T_1 - T_0 \geq y_i - x_i \geq 0$ for all $i = 1, \dots, k$.

The following definition introduces the notion of an (f, g)-avoiding Bernoulli line ensemble, which in simple terms is a collection of k independent Bernoulli bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two functions f and g.

Definition 2.15. Let $k \in \mathbb{N}$ and $\mathfrak{W}_k$ denote the set of signatures of length k, i.e.

$$\mathfrak{W}_k = \{\vec{x} = (x_1, \dots, x_k) \in \mathbb{Z}^k : x_1 \geq x_2 \geq \cdots \geq x_k\}.$$

Let $\vec{x}, \vec{y} \in \mathfrak{W}_k$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, and $f : [\![T_0, T_1]\!] \to (-\infty, \infty]$ and $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ be two functions.

With the above data we define the *(f, g)-avoiding Bernoulli line ensemble on the interval $[\![T_0, T_1]\!]$ with entrance data $\vec{x}$ and exit data $\vec{y}$* to be the Σ-indexed Bernoulli line ensemble $\mathfrak{Q}$ with $\Sigma = [\![1, k]\!]$ on $[\![T_0, T_1]\!]$ and with the law of $\mathfrak{Q}$ equal to $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y}}_{Ber}$ (the law of k independent uniform up-right paths $\{B_i : [\![T_0, T_1]\!] \to \mathbb{R}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$) conditioned on the event

$$E = \{f(r) \geq B_1(r) \geq B_2(r) \geq \cdots \geq B_k(r) \geq g(r) \text{ for all } r \in [\![T_0, T_1]\!]\}.$$

The above definition is well-posed if there exist $B_i \in \Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ that satisfy the conditions in E (i.e. if the set of such up-right paths is not empty). We will denote by $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ the set of collections of k up-right paths that satisfy the conditions in E and then the distribution on $\mathfrak{Q}$ is simply the uniform measure on $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$. We denote the probability distribution of $\mathfrak{Q}$ as $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ and write $\mathbb{E}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ for the expectation with respect to this measure. When $f = +\infty$ and $g = -\infty$, we simply write $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y}}_{avoid,Ber}$ and $\mathbb{E}^{T_0,T_1,\vec{x},\vec{y}}_{avoid,Ber}$.

It will be useful to formulate simple conditions under which $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is non-empty and thus $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ well-defined. We accomplish this in the following lemma, whose proof is postponed until Section [Appendix].

Lemma 2.16. *Suppose that $k \in \mathbb{N}$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. Suppose further that $\vec{x}, \vec{y} \in \mathfrak{W}_k$ satisfy $T_1 - T_0 \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$. Suppose further that $f : [\![T_0, T_1]\!] \to (-\infty, \infty]$ and $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ satisfy $f(i + 1) = f(i)$ or $f(i + 1) = f(i) + 1$, and $g(i + 1) = g(i)$*

or $g(i+1) = g(i) + 1$ for $i = T_0, \dots, T_1 - 1$. Finally, suppose that $f(T_0) \geq x_1, f(T_1) \geq y_1$ and $g(T_0) \leq x_k, g(T_1) \leq y_k$. Then the set $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ from Definition 2.15 is non-empty.

The following definition introduces the notion of the Schur Gibbs property, which can be thought of a discrete analogue of the partial Brownian Gibbs property the same way that Bernoulli random walks are discrete analogues of Brownian motion.

Definition 2.17. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. A Σ-indexed Bernoulli line ensemble $\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ is said to satisfy the *Schur Gibbs property* if it is non-crossing, meaning that

$$L_j(i) \geq L_{j+1}(i) \text{ for all } j = 1, \dots, N-1 \text{ and } i \in [\![T_0, T_1]\!],$$

and for any finite $K = \{k_1, k_1 + 1, \dots, k_2\} \subset [\![1, N-1]\!]$ and $a, b \in [\![T_0, T_1]\!]$ with $a < b$ the following holds. Suppose that f, g are two up-right paths drawn in $\{(r, z) \in \mathbb{Z}^2 : a \leq r \leq b\}$ and $\vec{x}, \vec{y} \in \mathfrak{W}_{k_2 - k_1 + 1}$ altogether satisfy that $\mathbb{P}(A) > 0$ where A denotes the event

$$A = \{\vec{x} = (L_{k_1}(a), \dots, L_{k_2}(a)), \vec{y} = (L_{k_1}(b), \dots, L_{k_2}(b)), L_{k_1 - 1}[\![a, b]\!] = f, L_{k_2+1}[\![a, b]\!] = g\},$$

where if $k_1 = 1$ we adopt the convention $f = \infty = L_0$. Then for any $\{B_i \in \Omega(a, b, x_i, y_i)\}_{i=1}^{k_2 - k_1 + 1}$

$$\mathbb{P}\left(L_{i+k_1-1}[\![a, b]\!] = B_i \text{ for } i = 1, \dots, k_2 - k_1 + 1 | A\right) = \mathbb{P}^{a,b,\vec{x},\vec{y},f,g}_{avoid,Ber}\left(\cap_{i=1}^{k}\{\mathfrak{L}_i = B_i\}\right). \tag{2.5}$$

Remark 2.18. In simple words, a Bernoulli line ensemble is said to satisfy the Schur Gibbs property if the distribution of any finite number of consecutive paths, conditioned on their end-points and the paths above and below them is simply the uniform measure on all collection of up-right paths that have the same end-points and do not cross each other or the paths above and below them.

Remark 2.19. Observe that in Definition 2.17 the index k_2 is assumed to be less than or equal to $N-1$, so that if $N < \infty$ the N-th path is special and is not conditionally uniform. This is what makes Definition 2.17 a discrete analogue of the partial Brownian Gibbs property rather than the usual Brownian Gibbs property. Similarly to the partial Brownian Gibbs propert, see Remark 2.11, if $N = 1$ then the conditions in Definition 2.17 become void. I.e., any Bernoulli line ensemble with one line satisfies the Schur Gibbs property. Also we mention that the well-posedness of $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ in (2.5) is a consequence of Lemma 2.16 and our assumption that $\mathbb{P}(A) > 0$.

Remark 2.20. In [3] the authors studied a generalization of the Gibbs property in Definition 2.17 depending on a parameter $t \in (0, 1)$, which was called the *Hall-Littlewood Gibbs property* due to its connection to Hall-Littlewood polynomials [11]. The property in Definition 2.17 is the $t \to 0$ limit of the Hall-Littlewood Gibbs property. Since under this $t \to 0$ limit Hall-Littlewood polynomials degenerate to Schur polynomials we have decided to call the Gibbs property in Definition 2.17 the Schur Gibbs property.

Remark 2.21. An immediate consequence of Definition 2.17 is that if $M \leq N$, we have that the induced law on $\{L_i\}_{i=1}^{M}$ also satisfies the Schur Gibbs property as an $\{1, \dots, M\}$-indexed Bernoulli line ensemble on $[\![T_0, T_1]\!]$.

We end this section with the following definition of the term acceptance probability.

Definition 2.22. Assume the same notation as in Definition 2.15 and suppose that $T_1 - T_0 \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$. We define the *acceptance probability* $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$ to be the ratio

$$Z(T_0, T_1, \vec{x}, \vec{y}, f, g) = \frac{|\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)|}{\prod_{i=1}^{k} |\Omega(T_0, T_1, x_i, y_i)|}. \tag{2.6}$$

Remark 2.23. The quantity $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is precisely the probability that if B_i are sampled uniformly from $\Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ then the B_i satisfy the condition

$$E = \{f(r) \geq B_1(r) \geq B_2(r) \geq \cdots \geq B_k(r) \geq g(r) \text{ for all } r \in [\![T_0, T_1]\!]\}.$$

Let us explain briefly why we call this quantity an acceptance probability. One way to sample $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ is as follows. Start by sampling a sequence of i.i.d. up-right paths B_i^N uniformly from $\Omega(T_0,T_1,x_i,y_i)$ for $i=1,\dots,k$ and $N\in\mathbb{N}$. For each n check if $B_1^n,\dots,B_k^n$ satisfy the condition E and let M denote the smallest index that accomplishes this. If $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},f,g)$ is non-empty then M is geometrically distributed with parameter $Z(T_0,T_1,\vec{x},\vec{y},f,g)$, and in particular M is finite almost surely and $\{B_i^M\}_{i=1}^k$ has distribution $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$. In this sampling procedure we construct a sequence of candidates $\{B_i^N\}_{i=1}^k$ for $N\in\mathbb{N}$ and reject those that fail to satisfy condition E, the first candidate that satisfies it is accepted and has law $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ and the probability that a candidate is accepted is precisely $Z(T_0,T_1,\vec{x},\vec{y},f,g)$, which is why we call it an acceptance probability.

2.3. **Main result.** In this section we present the main result of the paper. We start with the following technical definition.

Definition 2.24. Fix $k\in\mathbb{N}$, $\alpha,\lambda>0$ and $p\in(0,1)$. Suppose we are given a sequence $\{T_N\}_{N=1}^\infty$ with $T_N\in\mathbb{N}$ and that $\{\mathfrak{L}^N\}_{N=1}^\infty$, $\mathfrak{L}^N=(L_1^N,L_2^N,\dots,L_k^N)$ is a sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N,T_N]\!]$. We call the sequence (α,p,λ)-*good* if

- for each $N\in\mathbb{N}$ we have that $\mathfrak{L}^N$ satisfies the Schur Gibbs property of Definition 2.17;
- there is a function $\psi:\mathbb{N}\to(0,\infty)$ such that $\lim_{N\to\infty}\psi(N)=\infty$ and for each $N\in\mathbb{N}$ we have that $T_N>\psi(N)N^\alpha$;
- there is a function $\phi:(0,\infty)\to(0,\infty)$ such that for any $\epsilon>0$ we have

$$\sup_{n\in\mathbb{Z}}\limsup_{N\to\infty}\mathbb{P}\left(\left|N^{-\alpha/2}(L_1^N(nN^\alpha)-pnN^\alpha+\lambda n^2N^{\alpha/2})\right|\geq\phi(\epsilon)\right)\leq\epsilon. \tag{2.7}$$

Remark 2.25. Let us elaborate on the meaning of Definition 2.24. In order for a sequence of $\mathfrak{L}^N$ of $[\![1,k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N,T_N]\!]$ to be (α,p,λ)-*good* we want several conditions to be satisfied. Firstly, we want for each N the Bernoulli line ensemble $\mathfrak{L}^N$ to satisfy the Schur Gibbs property. The second condition is that while the interval of definition of $\mathfrak{L}^N$ is finite for each N and given by $[\![-T_N,T_N]\!]$, we want this interval to grow at least with speed N^α. This property is quantified by the function ψ, which can be essentially thought of as an arbitrary unbounded increasing function on $\mathbb{N}$. The third condition is that we want for each $n\in\mathbb{Z}$ the sequence of random variables $N^{-\alpha/2}(L_1^N(nN^\alpha)-pnN^\alpha)$ to be tight but moreover we want globally these random variables to look like the parabola $-\lambda n^2$. This statement is reflected in (2.7), which provides a certain uniform tightness of the random variables $N^{-\alpha/2}(L_1^N(nN^\alpha)-pnN^\alpha+\lambda n^2N^{\alpha/2})$. A particular case when (2.7) is satisfied is for example if we know that for each $n\in\mathbb{Z}$ the random variables $N^{-\alpha/2}(L_1^N(nN^\alpha)-pnN^\alpha+\lambda n^2N^{\alpha/2})$ converge to the same random variable X. In the applications that we have in mind these random variables would converge to the 1-point marginals of the Airy$_2$ process that are all given by the same Tracy-Widom distribution (since the Airy$_2$ process is stationary). Equation (2.7) is a significant relaxation of the requirement that $N^{-\alpha/2}(L_1^N(nN^\alpha)-pnN^\alpha+\lambda n^2N^{\alpha/2})$ all converge weakly to the Tracy-Widom distribution – the convergence requirement is replaced with a mild but uniform control of all subsequential limits.

The main result of the paper is as follows.

Theorem 2.26. *Fix $k\in\mathbb{N}$ with $k\geq 2$, $\alpha,\lambda>0$ and $p\in(0,1)$ and let $\mathfrak{L}^N=(L_1^N,L_2^N,\dots,L_k^N)$ be an (α,p,λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles. Set*

$$f_i^N(s)=N^{-\alpha/2}(L_i^N(sN^\alpha)-psN^\alpha+\lambda s^2N^{\alpha/2}),\ \textit{for } s\in[-\psi(N),\psi(N)] \textit{ and } i=1,\dots,k-1,$$

and extend f_i^N to $\mathbb{R}$ by setting for $i=1,\dots,k-1$

$$f_i^N(s)=f_i^N(-\psi(N)) \textit{ for } s\leq-\psi(N) \textit{ and } f_i^N(s)=f_N(\psi(N)) \textit{ for } s\geq\psi(N).$$

Let $\mathbb{P}_N$ denote the law of $\{f_i^N\}_{i=1}^{k-1}$ as a $[\![1,k-1]\!]$-indexed line ensemble (i.e. as a random variable in $(C([\![1,k-1]\!]\times\mathbb{R}),\mathcal{C})$). Then the sequence $\mathbb{P}_N$ is tight.

Roughly, Theorem 2.26 states that if you have a sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles that satisfy the Schur Gibbs property and the top paths of these ensembles under some shift and scaling have tight one-point marginals with a non-trivial parabolic shift, then under the same shift and scaling the top $k-1$ paths of the line ensemble will be tight. The extension of f_i^N to $\mathbb{R}$ is completely arbitrary and irrelevant for the validity of Theorem 2.26 since the topology on $C([\![1,k-1]\!]\times\mathbb{R})$ is that of uniform convergence over compacts. Consequently, only the behavior of these functions on compact intervals matters in Theorem 2.26 and not what these functions do near infinity, which is where the modification happens as $\lim_{N\to\infty}\psi(N)=\infty$ by assumption. The only reason we perform the extension is to embed all Bernoulli line ensembles into the same space $(C([\![1,k-1]\!]\times\mathbb{R}),\mathcal{C})$.

We mention that the k-th up-right path in the sequence of Bernoulli line ensembles is special and Theorem 2.26 provides no tightness result for it. The reason for this stems from the Schur Gibbs property, see Definition 2.17, which assumes less information for the k-th path. In practice, one either has an infinite Bernoulli line ensemble for each N or one has a Bernoulli line ensemble with finite number of paths, which increase with N to infinity. In either of these settings one can use Theorem 2.26 to prove tightness of the full line ensemble - we will have more to say about this in Section [Applications].

The proof of Theorem 2.26 is presented in Section 4. In the next section we derive various properties for Bernoulli line ensembles.

3. Properties of Bernoulli line ensembles

In this section we derive several results for non-intersecting Bernoulli bridges, which will be used in the proof of Theorem 2.26 in Section 4.

3.1. Monotone coupling lemmas.

In this section we formulate two lemmas that provide couplings of two Bernoulli line ensembles of non-intersecting Bernoulli bridges on the same interval, which depend monotonically on their boundary data. Schematic depictions of the couplings are provided in Figure 2. We postpone the proof of these lemmas until Section [Appendix].

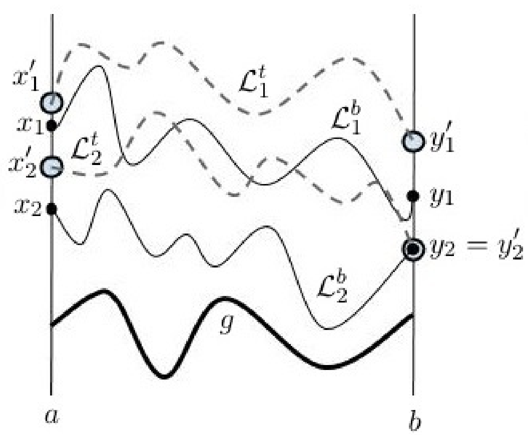

Figure 2. Two diagrammatic depictions of the monotone coupling Lemma 3.1 (left part) and Lemma 3.2 (right part).

Lemma 3.1. *Assume the same notation as in Definition 2.15. Fix $k\in\mathbb{N}$, $T_0,T_1\in\mathbb{Z}$ with $T_0<T_1$, a function $g:[\![T_0,T_1]\!]\to[-\infty,\infty)$ as well as $\vec{x},\vec{y},\vec{x}\,',\vec{y}\,'\in\mathfrak{W}_k$. Assume that $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},\infty,g)$ and $\Omega_{avoid}(T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g)$ are both non-empty. Then there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$, which supports two $[\![1,k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0,T_1]\!]$ such that the law*

of $\mathfrak{L}^t$ $\big($*resp.* $\mathfrak{L}^b\big)$ *under* $\mathbb{P}$ *is given by* $\mathbb{P}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g}_{avoid,Ber}$ $\big($*resp.* $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber}\big)$ *and such that* $\mathbb{P}$*-almost surely we have* $\mathfrak{L}^t_i(r) \geq \mathfrak{L}^b_i(r)$ *for all* $i = 1, \dots, k$ *and* $r \in [\![T_0, T_1]\!]$.

Lemma 3.2. *Assume the same notation as in Definition 2.15. Fix* $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ *with* $T_0 < T_1$, *two functions* $g^t, g^b : [\![T_0, T_1]\!] \to [-\infty, \infty)$ *and* $\vec{x}, \vec{y} \in \mathfrak{W}_k$. *We assume that* $g^t(r) \geq g^b(r)$ *for all* $r \in [\![T_0, T_1]\!]$ *and that* $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^t)$ *and* $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$ *are both non-empty. Then there exists a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *which supports two* $[\![1, k]\!]$*-indexed Bernoulli line ensembles* $\mathfrak{L}^t$ *and* $\mathfrak{L}^b$ *on* $[\![T_0, T_1]\!]$ *such that the law of* $\mathfrak{L}^t$ $\big($*resp.* $\mathfrak{L}^b\big)$ *under* $\mathbb{P}$ *is given by* $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^t}_{avoid,Ber}$ $\big($*resp.* $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber}\big)$ *and such that* $\mathbb{P}$*-almost surely we have* $\mathfrak{L}^t_i(r) \geq \mathfrak{L}^b_i(r)$ *for all* $i = 1, \dots, k$ *and* $r \in [\![T_0, T_1]\!]$.

In plain words, Lemma 3.1 states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ of non-intersecting Bernoulli bridges, bounded from below by the same function g, in such a way that if all boundary values of $\mathfrak{L}^t$ are above the respective boundary values of $\mathfrak{L}^b$, then all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the left part of Figure 2. Lemma 3.2, states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ that have the same boundary values, but the lower bound g^t of $\mathfrak{L}^t$ is above the lower bound g^b of $\mathfrak{L}^b$, in such a way that all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the right part of Figure 2.

3.2. **Properties of Bernoulli bridges.**

In this section we derive several results about Bernoulli bridges, which are random up-right paths that have law $\mathbb{P}^{T_0,T_1,x,y}_{Ber}$ as in Section 2.2. Our results will rely on the two monotonicity Lemmas 3.1 and 3.2 as well as a strong coupling between Bernoulli bridges and Brownian bridges from [3] – recalled here as Theorem 3.3.

If W_t denotes a standard one-dimensional Brownian motion and $\sigma > 0$, then the process

$$B^\sigma_t = \sigma(W_t - tW_1), \quad 0 \leq t \leq 1,$$

is called a *Brownian bridge (conditioned on* $B_0 = 0, B_1 = 0$*) with variance* σ^2. With the above notation we state the strong coupling result we use.

Theorem 3.3. *Let* $p \in (0, 1)$. *There exist constants* $0 < C, a, \alpha < \infty$ *(depending on* p*) such that for every positive integer* n, *there is a probability space on which are defined a Brownian bridge* B^σ *with variance* $\sigma^2 = p(1 - p)$ *and a family of random paths* $\ell^{(n,z)} \in \Omega(0, n, 0, z)$ *for* $z = 0, \dots, n$ *such that* $\ell^{(n,z)}$ *has law* $\mathbb{P}^{0,n,0,z}_{Ber}$ *and*

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq Ce^{\alpha(\log n)^2}e^{|z-pn|^2/n}, \text{ where } \Delta(n,z) := \sup_{0\leq t\leq n}\left|\sqrt{n}B^\sigma_{t/n} + \tfrac{t}{n}z - \ell^{(n,z)}(t)\right|. \tag{3.1}$$

Remark 3.4. When $p = 1/2$ the above theorem follows (after a trivial affine shift) from [10, Theorem 6.3] and the general $p \in (0, 1)$ case was done in [3, Theorem 4.5]. We mention that a significant generalization of Theorem 3.3 for general random walk bridges has recently been proved in [7, Theorem 2.3].

We will use the following simple corollary of Theorem 3.3 in the following to compare Bernoulli bridges with Brownian bridges. We use the same notation as in the theorem.

Corollary 3.5. *Fix* $p \in (0, 1)$, $\beta > 0$, *and* $A > 0$. *Suppose* $|z - pn| \leq K\sqrt{n}$ *for a constant* $K > 0$. *Then for any* $\epsilon > 0$, *there exists* N *large enough depending on* p, ϵ, A, K *so that for* $n \geq N$,

$$\mathbb{P}\Big(\Delta(n, z) \geq An^\beta\Big) < \epsilon.$$

Proof. Applying Chebyshev's inequality and (3.1) gives

$$\mathbb{P}\Big(\Delta(n,z) \geq An^{\beta}\Big) \leq e^{-An^{\beta}}\mathbb{E}\Big[e^{a\Delta(n,z)}\Big] \leq C\exp\Big[-An^{\beta} + \alpha(\log n)^2 + \frac{|z-pn|^2}{n}\Big]$$
$$\leq C\exp\Big[-An^{\beta} + \alpha(\log n)^2 + K\Big].$$

The conclusion is now immediate. □

We also state the following result regarding the distribution of the maximum of a Brownian bridge, which follows immediately from Equation (3.40) in [9, Chapter 4].

Lemma 3.6. *Fix $p \in (0,1)$, and let B^{σ} be a Brownian bridge of variance $\sigma^2 = p(1-p)$ on $[0,1]$. Then for any $C, T > 0$,*

$$\mathbb{P}\Big(\max_{s\in[0,T]} B^{\sigma}_{s/T} \geq C\Big) = \exp\left(-\frac{2C^2}{p(1-p)}\right).$$

In particular, given $\epsilon > 0$, we can choose C large enough depending on p so that this probability does not exceed ϵ.

Proof. Let B^1 be a standard Brownian bridge with variance 1 on $[0,1]$. Then $\sqrt{T}\,B^1_{s/T}$ is a standard Brownian bridge on $[0,T]$, and $\sigma B^1_{s/T}$ has the same distribution as $B^{\sigma}_{s/T}$. Hence

$$\mathbb{P}\Big(\max_{s\in[0,T]} B^{\sigma}_{s/T} \geq C\Big) = \mathbb{P}\Big(\max_{s\in[0,T]} \sqrt{T}\,B^1_{s/T} \geq C\sqrt{T}/\sigma\Big) = e^{-2(C\sqrt{T}/\sigma)^2/T} = e^{-2C^2/p(1-p)}.$$

The second inequality follows from [9, Chapter 4, (3.40)]. The second conclusion is immediate. □

Below we list five lemmas about Bernoulli bridges. We provide a brief informal explanation of what each result says after it is stated. All five lemmas are proved in a similar fashion. For the first four lemmas one observes that the event, whose probability is being estimated, is monotone in ℓ. This allows by Lemmas 3.1 and 3.2 to replace x, y in the statements of the lemmas with the extreme values of the ranges specified in each. Once the choice of x and y is fixed one can use our strong coupling results, Theorem 3.3 and Corollary 3.5, to reduce each of the lemmas to an analogous one involving a Brownian bridge with some prescribed variance. The latter statements are then easily confirmed as one has exact formulas for Brownian bridges, such as Lemma 3.6.

Lemma 3.7. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,x,y}_{Ber}$. Let $M_1, M_2 \in \mathbb{R}$ be given. Then we can find $W_0 = W_0(p, M_2 - M_1) \in \mathbb{N}$ such that for $T \geq W_0$, $x \geq M_1T^{1/2}$, $y \geq pT + M_2T^{1/2}$ and $s \in [0,T]$ we have*

$$\mathbb{P}^{0,T,x,y}_{Ber}\Big(\ell(s) \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\big(pT + M_2T^{1/2}\big) - T^{1/4}\Big) \geq \frac{1}{3}. \tag{3.2}$$

Remark 3.8. If $M_1, M_2 = 0$ then Lemma 3.7 states that if a Bernoulli bridge ℓ is started from $(0,x)$ and terminates at (T,y), which are above the straight line of slope p, then at any given time $s \in [0,T]$ the probability that $\ell(s)$ goes a modest distance below the straight line of slope p is upper bounded by 2/3.

Proof. Define $A = \lfloor M_1T^{1/2}\rfloor$ and $B = \lfloor pT + M_2T^{1/2}\rfloor$. Then since $A \leq x$ and $B \leq y$, it follows from Lemma 3.1 that there is a probability space with measure $\mathbb{P}_0$ supporting random variables L_1 and L_2, whose laws under $\mathbb{P}_0$ are $\mathbb{P}^{0,T,A,B}_{Ber}$ and $\mathbb{P}^{0,T,x,y}_{Ber}$ respectively, and $\mathbb{P}_0$-a.s. we have $L_1 \leq L_2$.

Thus

$$
\begin{aligned}
&\mathbb{P}^{0,T,x,y}_{Ber}\Big(\ell(s) \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\big(pT + M_2T^{1/2}\big) - T^{1/4}\Big)\\
&= \mathbb{P}_0\Big(L_2(s) \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\big(pT + M_2T^{1/2}\big) - T^{1/4}\Big)\\
&\geq \mathbb{P}_0\Big(L_1(s) \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\big(pT + M_2T^{1/2}\big) - T^{1/4}\Big)\\
&= \mathbb{P}^{0,T,A,B}_{Ber}\Big(\ell(s) \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\big(pT + M_2T^{1/2}\big) - T^{1/4}\Big).
\end{aligned}
$$

Since upright paths on $[\![0,T]\!]\times[\![A,B]\!]$ are equivalent to upright paths on $[\![0,T]\!]\times[\![0,B-A]\!]$ shifted vertically by A, the last line is equal to

$$
\mathbb{P}^{0,T,0,B-A}_{Ber}\Big(\ell(s) + A \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\big(pT + M_2T^{1/2}\big) - T^{1/4}\Big).
$$

Now we consider the coupling provided by Theorem 3.3. We have another probability space $(\Omega,\mathcal{F},\mathbb{P})$ supporting a random variable $\ell^{(T,B-A)}$ whose law under $\mathbb{P}$ is $\mathbb{P}^{0,T,0,B-A}_{Ber}$, and a Brownian bridge B^σ. Then

$$
\begin{aligned}
&\mathbb{P}^{0,T,0,B-A}_{Ber}\Big(\ell(s) + A \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\big(pT + M_2T^{1/2}\big) - T^{1/4}\Big)\\
&= \mathbb{P}\Big(\ell^{(T,B-A)}(s) + A \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\big(pT + M_2T^{1/2}\big) - T^{1/4}\Big)\\
&= \mathbb{P}\Big(\Big[\ell^{(T,B-A)}(s) - \sqrt{T}B^\sigma_{s/T} - \frac{s}{T}\cdot(B-A)\Big] + \sqrt{T}B^\sigma_{s/T} \geq -A - \frac{s}{T}\cdot(B-A)\\
&\qquad\qquad + \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\big(pT + M_2T^{1/2}\big) - T^{1/4}\Big).
\end{aligned}
$$

Recalling the definitions of A and B, we can rewrite the quantity on the right hand side in the last expression and bound it by

$$
\begin{aligned}
\frac{T-s}{T}\cdot(M_1T^{1/2} - A) + \frac{s}{T}\cdot(pT + M_2T^{1/2} - B) - T^{1/4} &\leq \frac{T-s}{T} + \frac{s}{T} - T^{1/4}\\
&= -T^{1/4} + 1.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathbb{P}^{0,T,0,B-A}_{Ber}\Big(\ell(s) + A \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\big(pT + M_2T^{1/2}\big) - T^{1/4}\Big)\\
&\geq \mathbb{P}\Big(\Big[\ell^{(T,B-A)}(s) - \sqrt{T}B^\sigma_{s/T} - \frac{s}{T}\cdot(B-A)\Big] + \sqrt{T}B^\sigma_{s/T} \geq -T^{1/4} + 1\Big)\\
&\geq \mathbb{P}\Big(\sqrt{T}B^\sigma_{s/T} \geq 0 \quad\text{and}\quad \Delta(T,B-A) < T^{1/4} - 1\Big)\\
&\geq \mathbb{P}\left(B^\sigma_{s/T} \geq 0\right) - \mathbb{P}\left(\Delta(T,B-A) \geq T^{1/4} - 1\right)\\
&= \frac{1}{2} - \mathbb{P}\left(\Delta(T,B-A) \geq T^{1/4} - 1\right).
\end{aligned}
$$

For the second inequality, we used the fact that the quantity in brackets is bounded in absolute value by $\Delta(T,B-A)$. The third inequality follows by dividing the event $\{B^\sigma_{s/T} \geq 0\}$ into cases and applying subadditivity. Since $|B - A - pT| \leq (M_2 - M_1 + 1)\sqrt{T}$, Corollary 3.5 allows us to choose W_0 large enough depending on p and $M_2 - M_1$ so that if $T \geq W_0$, then the last line is bounded above by $1/2 - 1/6 = 1/3$. This proves (3.2). □

Lemma 3.9. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $y \in \mathbb{Z}$ such that $T \geq y \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,0,y}_{Ber}$. Let $M > 0$ and $\epsilon > 0$ be given. Then we can find $W_1 = W_1(M,p,\epsilon) \in \mathbb{N}$ and $A = A(M,p,\epsilon) > 0$ such that for $T \geq W_1$, $y \geq pT - MT^{1/2}$ we have*

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(\inf_{s\in[0,T]}\big(\ell(s) - ps\big) \leq -AT^{1/2}\Big) \leq \epsilon. \tag{3.3}$$

Remark 3.10. Roughly, Lemma 3.9 states that if a Bernoulli bridge ℓ is started from $(0,0)$ and terminates at (T,y) with (T,y) not significantly lower than the straight line of slope p, then the event that ℓ goes significantly below the straight line of slope p is very unlikely.

Proof. As in the previous proof, it follows from Lemma 3.1 that

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(\inf_{s\in[0,T]}\big(\ell(s) - ps\big) \leq -AT^{1/2}\Big) \leq \mathbb{P}^{0,T,0,B}_{Ber}\Big(\inf_{s\in[0,T]}\big(\ell(s) - ps\big) \leq -AT^{1/2}\Big),$$

where $B = \lfloor pT - MT^{1/2}\rfloor$. By Theorem 3.3, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random variable $\ell^{(T,B)}$ whose law under $\mathbb{P}$ is that of ℓ, and a Brownian bridge B^σ with variance $\sigma^2 = p(1-p)$. Therefore

$$\begin{aligned}
\mathbb{P}^{0,T,0,B}_{Ber}\Big(\inf_{s\in[0,T]}\big(\ell(s) - ps\big) \leq -AT^{1/2}\Big) &= \mathbb{P}\Big(\inf_{s\in[0,T]}\big(\ell^{(T,B)}(s) - ps\big) \leq -AT^{1/2}\Big) \\
\leq \mathbb{P}\Big(\inf_{s\in[0,T]} \sqrt{T}B^\sigma_{s/T} \leq -\frac{1}{2}AT^{1/2}\Big) &+ \mathbb{P}\Big(\sup_{s\in[0,T]}\Big|\sqrt{T}B^\sigma_{s/T} + ps - \ell^{(T,B)}(s)\Big| \geq \frac{1}{2}AT^{1/2}\Big) \\
\leq \mathbb{P}\Big(\max_{s\in[0,T]} B^\sigma_{s/T} \geq A/2\Big) &+ \mathbb{P}\Big(\Delta(T,B) \geq \frac{1}{2}AT^{1/2} - MT^{1/2} - 1\Big). \qquad (3.4)
\end{aligned}$$

For the first term in the last line, we used the fact that B^σ and $-B^\sigma$ have the same distribution. For the second term, we used the fact that

$$\sup_{s\in[0,T]}\Big|ps - \frac{s}{T}\cdot B\Big| \leq \sup_{s\in[0,T]}\Big|ps - \frac{pT - MT^{1/2}}{T}\cdot s\Big| + 1 = MT^{1/2} + 1.$$

By Lemma 3.6, we can take A large enough depending on p, ϵ so that the first term in (3.4) is $< \epsilon/2$. If we also take $A > 2M$, then since $|B - pT| \leq (M+1)\sqrt{T}$, Corollary 3.5 gives us a W_1 large enough depending on M, p, ϵ so that the second term is also $< \epsilon/2$ for $T \geq W_1$. Adding the two terms gives (3.3). ☐

Lemma 3.11. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,x,y}_{Ber}$. Let $M_1, M_2 > 0$ be given. Then we can find $W_2 = W_2(M_1, M_2, p) \in \mathbb{N}$ such that for $T \geq W_2$, $x \geq -M_1T^{1/2}$, $y \geq pT - M_1T^{1/2}$ we have*

$$\mathbb{P}^{0,T,x,y}_{Ber}\left(\ell(T/2) \geq \frac{M_2T^{1/2} + pT}{2} - T^{1/4}\right) \geq (1/2)(1 - \Phi^v(M_1 + M_2)), \tag{3.5}$$

where Φ^v is the cumulative distribution function of a Gaussian random variable with mean 0 and variance $v = p(1-p)/4$.

Remark 3.12. Lemma 3.11 states that if a Bernoulli bridge ℓ is started from $(0,x)$ and terminates at (T,y) with these points not significantly lower than the straight line of slope p, then its mid-point would lie well above the straight line of slope p at least with some quantifiably tiny probability.

Proof. By Lemma 3.1, we have

$$\begin{aligned}
\mathbb{P}^{0,T,x,y}_{Ber}\left(\ell(T/2) \geq \frac{M_2T^{1/2} + pT}{2} - T^{1/4}\right) &\geq \mathbb{P}^{0,T,0,B-A}_{Ber}\left(\ell(T/2) + A \geq \frac{M_2T^{1/2} + pT}{2} - T^{1/4}\right) \\
&= \mathbb{P}\left(\ell^{(T,B-A)}(T/2) + A \geq \frac{M_2T^{1/2} + pT}{2} - T^{1/4}\right),
\end{aligned}$$

with $A = \lfloor -M_1 T^{1/2} \rfloor$, $B = \lfloor pT - M_1 T^{1/2} \rfloor$, and $\mathbb{P}$, and $\ell^{(T,B-A)}$ provided by Theorem 3.3. If B^σ is as in the theorem, we can rewrite the expression on the second line as

$$\mathbb{P}\left(\left[\ell^{(T,B-A)}(T/2) - \sqrt{T}\, B^\sigma_{1/2} - \frac{B-A}{2} \right] + \sqrt{T}\, B^\sigma_{1/2} \geq -A - \frac{B-A}{2} + \frac{M_2 T^{1/2} + pT}{2} - T^{1/4} \right).$$

We have

$$\begin{aligned} -A - \frac{B-A}{2} + \frac{M_2 T^{1/2} + pT}{2} - T^{1/4} &\leq M_1 T^{1/2} + 1 - \frac{pT-1}{2} + \frac{M_2 T^{1/2} + pT}{2} - T^{1/4} \\ &\leq (M_1 + M_2) T^{1/2} - T^{1/4} + 2. \end{aligned}$$

Thus the probability in question is bounded below by

$$\begin{aligned} &\mathbb{P}\left(\left[\ell^{(T,B-A)}(T/2) - \sqrt{T}\, B^\sigma_{1/2} - \frac{B-A}{2} \right] + \sqrt{T}\, B^\sigma_{1/2} \geq (M_1 + M_2) T^{1/2} - T^{1/4} + 2 \right) \\ &\geq \mathbb{P}\left(\sqrt{T}\, B^\sigma_{1/2} \geq (M_1 + M_2) T^{1/2} \quad \text{and} \quad \Delta(T, B-A) < T^{1/4} - 2 \right) \\ &\geq \mathbb{P}\big(B^\sigma_{1/2} \geq M_1 + M_2 \big) - \mathbb{P}\big(\Delta(T, B-A) \geq T^{1/4} - 2 \big). \end{aligned}$$

Note that $B^\sigma_{1/2} = \sigma(W_{1/2} - \frac{1}{2} W_1)$ for a standard Brownian motion W on $[0,1]$. Thus $B^\sigma_{1/2}$ is Gaussian with mean 0 and variance $\sigma^2(1/2 - (1/2)^2) = \sigma^2/4$. In particular, the first term in the last line is equal to

$$1 - \Phi^v(M_1 + M_2),$$

where Φ^v is the cdf for a Gaussian random variable with mean 0 and variance $v = \sigma^2/4 = p(1-p)/4$. By Corollary 3.5, since $|B - A - pT| \leq 1$, we can choose W_2 depending on M_1, M_2, and p so that the second term is less than 1/2 the first term for $T \geq W_2$. This proves (3.5). ☐

Lemma 3.13. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,x,y}_{Ber}$. Then we can find $W_3 = W_3(p) \in \mathbb{N}$ such that for $T \geq W_3$, $x \geq T^{1/2}$, $y \geq pT + T^{1/2}$*

$$\mathbb{P}^{0,T,x,y}_{Ber}\Big(\inf_{s \in [0,T]} \big(\ell(s) - ps\big) + T^{1/4} \geq 0 \Big) \geq \frac{1}{2}\left(1 - \exp\left(-\frac{2}{p(1-p)} \right) \right). \tag{3.6}$$

Remark 3.14. Lemma 3.13 states that if a Bernoulli bridge ℓ is started from $(0, x)$ and terminates at (T, y) with $(0, x)$ and (T, y) well above the line of slope p then at least with some positive probability ℓ will not fall significantly below the line of slope p.

Proof. By Lemma 3.1,

$$\begin{aligned} &\mathbb{P}^{0,T,x,y}_{Ber}\Big(\inf_{s \in [0,T]} \big(\ell(s) - ps\big) + T^{1/4} \geq 0 \Big) \\ &\geq \mathbb{P}^{0,T,0,B-A}_{Ber}\Big(\inf_{s \in [0,T]} \big(\ell(s) + A - ps\big) + T^{1/4} \geq 0 \Big) \\ &= \mathbb{P}\Big(\inf_{s \in [0,T]} \big(\ell^{(T,B-A)}(s) - ps\big) \geq -T^{1/4} - A \Big) \\ &\geq \mathbb{P}\Big(\inf_{s \in [0,T]} \big(\ell^{(T,B-A)}(s) - \frac{s}{T} \cdot (B-A)\big) \geq -T^{1/4} - T^{1/2} + 2 \Big), \end{aligned}$$

with $A = \lfloor T^{1/2} \rfloor$, $B = \lfloor pT + T^{1/2} \rfloor$, and $\mathbb{P}$, and $\ell^{(T,B-A)}$ provided by Theorem 3.3. In the last line, we used the facts that $|A - T^{1/2}| \leq 1$ and $|p - (B-A)/T| \leq 1$. With B^σ as in the theorem, the last line is bounded below by

$$\mathbb{P}\Big(\inf_{s\in[0,T]} \sqrt{T}\, B^{\sigma}_{s/T} \geq -T^{1/2} \quad \text{and} \quad \Delta(T, B-A) < T^{1/2} - 2 \Big)$$
$$\geq 1 - \exp\left(-\frac{2}{p(1-p)}\right) - \mathbb{P}\Big(\Delta(T, B-A) \geq T^{1/2} - 2\Big).$$

In the second line, we used Lemma 3.6. Since $|B - A - pT| \leq 1$, Corollary 3.5 allows us choose W_3 large enough depending on p so that this term is less than $\frac{1}{2}(1 - e^{-2/p(1-p)})$ for $T \geq W_3$. This implies (3.6). ☐

We need the following definition for our next result. For a function $f \in C[a,b]$ we define its *modulus of continuity* by

$$w(f, \delta) = \sup_{\substack{x,y\in[a,b]\\ |x-y|\leq\delta}} |f(x) - f(y)|. \tag{3.7}$$

Lemma 3.15. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $y \in \mathbb{Z}$ such that $T \geq y \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,0,y}_{Ber}$. For each positive M, ϵ and η, there exist a $\delta(\epsilon,\eta,M) > 0$ and $W_4 = W_4(M,p,\epsilon,\eta) \in \mathbb{N}$ such that for $T \geq W_4$ and $|y - pT| \leq MT^{1/2}$ we have*

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(w\big(f^{\ell}, \delta\big) \geq \epsilon \Big) \leq \eta, \tag{3.8}$$

where $f^{\ell}(u) = T^{-1/2}\big(\ell(uT) - puT\big)$ for $u \in [0,1]$.

Remark 3.16. Lemma 3.15 states that if ℓ is a Bernoulli bridge that is started from $(0,0)$ and terminates at (T,y) with y close to pT (i.e. with well-behaved endpoints) then the modulus of continuity of ℓ is also well-behaved with high probability.

Proof. We have

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(w\big(f^{\ell}, \delta\big) \geq \epsilon \Big) = \mathbb{P}\Big(w\big(f^{\ell^{(T,y)}}, \delta\big) \geq \epsilon \Big),$$

with $\mathbb{P}$, $f^{\ell^{(T,y)}}$ as in Theorem 3.3. If B^{σ} is the Brownian bridge provided by Theorem 3.3, then

$$\begin{aligned}
w\big(f^{\ell^{(T,y)}}, \delta\big) &= T^{-1/2} \sup_{\substack{s,t\in[0,1]\\ |s-t|\leq\delta}} \Big| \ell^{(T,y)}(sT) - psT - \ell^{(T,y)}(tT) + ptT \Big| \\
&\leq T^{-1/2} \sup_{\substack{s,t\in[0,1]\\ |s-t|\leq\delta}} \Big(\big| \sqrt{T}\, B^{\sigma}_s + sy - psT - \sqrt{T}\, B^{\sigma}_t - ty + ptT \big| \\
&\qquad + \big| \sqrt{T}\, B^{\sigma}_s + sy - \ell^{(T,y)}(sT) \big| + \big| \sqrt{T}\, B^{\sigma}_t + ty - \ell^{(T,y)}(tT) \big| \Big) \\
&\leq \sup_{\substack{s,t\in[0,1]\\ |s-t|\leq\delta}} \Big| B^{\sigma}_s - B^{\sigma}_t + T^{-1/2}(y - pT)(s-t) \Big| + 2T^{-1/2}\Delta(T,y) \\
&\leq w\big(B^{\sigma}, \delta\big) + M\delta + 2T^{-1/2}\Delta(T,y).
\end{aligned}$$

The last line follows from the assumption that $|y - pT| \leq MT^{1/2}$. Thus

$$\begin{aligned}
\mathbb{P}\Big(w\big(f^{\ell^{(N,y)}}, \delta\big) \geq \epsilon \Big) &\leq \mathbb{P}\Big(w\big(B^{\sigma}, \delta\big) + M\delta + 2T^{-1/2}\Delta(T,y) \geq \epsilon \Big) \\
&\leq \mathbb{P}\Big(w\big(B^{\sigma}, \delta\big) + M\delta \geq \epsilon/2 \Big) + \mathbb{P}\Big(\Delta(T,y) \geq \epsilon\, T^{1/2}/4 \Big).
\end{aligned}$$

Corollary 3.5 gives us a W_4 large enough depending on M, p, ϵ, η so that the last term term is $\leq \eta/2$ for $T \geq W_4$. Since B^{σ} is a.s. uniformly continuous on the compact interval $[0,1]$, $w(B^{\sigma}, \delta) \to 0$ as $\delta \to 0$. Thus we can find $\delta_0 > 0$ small enough depending on ϵ, η so that $w(B^{\sigma}, \delta_0) < \epsilon/4$ with probability at least $1-\eta/2$. Then with $\delta = \min(\delta_0, \epsilon/4M)$, the first term is $\leq \eta/2$ as well. This implies (3.8). $\square$

3.3. **Properties of avoiding Bernoulli line ensembles.** In this section we derive several results about avoiding Bernoulli line ensembles, which are Bernoulli line ensembles with law $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ as in Definition 2.15. The lemmas we prove only involve the case when $f(r)=\infty$ for all $r \in [\![T_0,T_1]\!]$ and we denote the measure in this case by $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber}$. A $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber}$-distributed random variable will be denoted by $\mathfrak{Q} = (Q_1,\dots,Q_k)$ where k is the number of up-right paths in the ensemble. Our results will rely on the two monotonicity Lemmas 3.1 and 3.2 as well as the strong coupling between Bernoulli bridges and Brownian bridges from Theorem 3.3.

Lemma 3.17. *Fix $k, T \in \mathbb{N}$, $p \in (0,1)$. Define the constant*

$$C = \sqrt{8p(1-p)\log\frac{3}{1-(1-\epsilon)^{1/k}}}.$$

Suppose $\vec{x},\vec{y} \in \mathfrak{W}_k$ are such that for $1 \leq i \leq k$,

(1) $T \geq y_i - x_i \geq 0$,
(2) $x_i = x_1 - (i-1)\lceil C\sqrt{T}\rceil$ and $y_i = y_1 - (i-1)\lceil C\sqrt{T}\rceil$,
(3) $|y_1 - x_1 - pT| \leq K\sqrt{T}$ for a constant $K>0$.

Let $\mathfrak{L} = (L_1,\dots,L_k)$ be a line ensemble with law $\mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}$, and let E denote the event that $L_1(s) \geq \cdots \geq L_k(s)$ for $s \in [0,T]$. Then we can find $W_5 = W_5(p,k)$ so that for $T \geq W_5$,

$$\mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}(E) \geq \left(1-3e^{-C^2/8p(1-p)}\right)^k$$

Lemma 3.18. *Fix $k, T \in \mathbb{N}$, $p \in (0,1)$, and $\vec{x},\vec{y} \in \mathfrak{W}_k$ such that $T \geq y_i - x_i \geq 0$ for $i=1,\dots,k$. Suppose that $g : [\![0,T]\!] \to [-\infty,\infty)$ is such that $g(0) \leq x_k$, $g(T) \leq y_k$ and $g(i+1) = g(i)$ or $g(i+1) = g(i)+1$ for $i = 0,\dots,T-1$. Suppose that ℓ has distribution $\mathbb{P}^{0,T,\vec{x},\vec{y},\infty,g}_{avoid,Ber}$ as in Definition 2.15 (notice this distribtuion is well-defined by Lemma 2.16). Let $M_1, M_2, M_3 > 0$ be given. Then we can find $W_5 = W_5(M_1,M_2,M_3,p) \in \mathbb{N}$ such that for $T \geq W_5$, $x_1 \leq M_1 T^{1/2}$, $y_1 \leq pT - M_2 T^{1/2}$ and g satisfying $g(r) \leq p\cdot r - M_3 T^{1/2}$ for $r \in [\![0,T]\!]$ we have*

$$\mathbb{P}^{0,T,\vec{x},\vec{y},\infty,g}_{avoid,Ber}\left(Q_1(T/2) \leq k(M_1+1)T^{1/2} + \frac{(M_1-M_2)T^{1/2}+pT}{2}\right) \geq [Something]. \tag{3.9}$$

Proof. $\square$

4. Proof of Theorem 2.26

The goal of this section is to prove Theorem 2.26 and for the remainder we assume that $k \in \mathbb{N}$ with $k \geq 2$, $p \in (0,1)$, $\alpha, \lambda > 0$ are all fixed and

$$\left\{\mathfrak{L}^N = (L^N_1, L^N_2, \dots, L^N_k)\right\}_{N=1}^{\infty}, \tag{4.1}$$

is an (α,p,λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles as in Definition 2.24 that are all defined on a probability space with measure $\mathbb{P}$. The main technical result we will require is contained in Proposition 4.1 below and its proof is the content of Section 4.1. The proof of Theorem 2.26 is given in Section 4.2.

4.1. **Bounds on the acceptance probability.** The main result in this section is presented as Proposition 4.1 below. In order to formulate it and some of the lemmas below it will be convenient to adopt the following notation for any $r>0$:

$$t_1 = \lfloor (r+1)N^{\alpha}\rfloor, \quad t_2 = \lfloor (r+2)N^{\alpha}\rfloor, \quad \text{and } t_3 = \lfloor (r+3)N^{\alpha}\rfloor. \tag{4.2}$$

Proposition 4.1. *For any $\epsilon > 0$, $r > 0$ and any (α, p, λ)-good sequence of Bernoulli line ensembles $\{\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)\}_{N=1}^{\infty}$ there exist $\delta > 0$ and N_1 (both depending on ϵ, r as well as α, p, λ and the functions ϕ, ψ in Definition 2.24) such that for all $N \geq N_1$ we have*

$$\mathbb{P}\Big(Z\big(-t_1, t_1, \vec{x}, \vec{y}, L_k[\![-t_1, t_1]\!]\big) < \delta\Big) < \epsilon,$$

where $\vec{x} = (L_1^N(-t_1), \dots, L_{k-1}^N(-t_1))$, $\vec{y} = (L_1^N(t_1), \dots, L_{k-1}^N(t_1))$, $L_k[\![-t_1, t_1]\!]$ is the restriction of L_k^N to the set $[\![-t_1, t_1]\!]$, and Z is the acceptance probability of Definition 2.22. $\mathbb{P}$ is the measure on a probability space that supports $\{\mathfrak{L}^N\}_{N=1}^{\infty}$.

The general strategy we use to prove Proposition 4.1 is inspired by the proof of Proposition 6.5 in [5]. We begin by stating three key lemmas that will be required. Their proofs are postponed to Section 5. All constants in the statements below will depend implicitly on α, r, p, λ, and the functions ϕ, ψ from Definition 2.24, which are fixed throughout. We will not list this dependence explicitly.

Lemma 4.2 controls the upward deviation of the top curve $L_1^N(s)$ from the line ps in the scale $N^{\alpha/2}$.

Lemma 4.2. *For each $\epsilon > 0$ there exist $R_1 = R_1(\epsilon) > 0$ and $N_2 = N_2(\epsilon)$ such that for $N \geq N_2$*

$$\mathbb{P}\Big(\sup_{s \in [-t_3, t_3]} \big(L_1^N(s) - ps\big) \geq R_1 N^{\alpha/2}\Big) < \epsilon.$$

Lemma 4.3 controls the downward deviation of the bottom curve $L_k^N(s)$ from the line ps in the scale $N^{\alpha/2}$.

Lemma 4.3. *For each $\epsilon > 0$ there exist $R_2 = R_2(\epsilon) > 0$ and $N_3 = N_3(\epsilon)$ such that for $N \geq N_3$*

$$\mathbb{P}\Big(\inf_{s \in [-t_2, t_2]} \big(L_k^N(s) - ps\big) \leq -R_2 N^{\alpha/2}\Big) < \epsilon.$$

Lemma 4.4. *Fix $k \in \mathbb{N}$, $p \in (0,1)$, $M_1, M_2 > 0$. Suppose that $\ell_{bot} : [\![-t_2, t_2]\!] \to \mathbb{R} \cup \{-\infty\}$, and $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$ are such that $|\Omega_{avoid}(-t_2, t_2, \vec{x}, \vec{y}, \infty, \ell_{bot})| \geq 1$. Suppose further that*

(1) $\sup_{s \in [-t_2, t_2]} \big(\ell_{bot}(s) - ps\big) \leq M_2 (2t_2)^{1/2}$,

(2) $-pt_2 + M_1(2t_2)^{1/2} \geq x_1 \geq x_{k-1} \geq \max\big(\ell_{bot}(-t_2), -pt_2 - M_1(2t_2)^{1/2}\big)$,

(3) $pt_2 + M_1(2t_2)^{1/2} \geq y_1 \geq y_{k-1} \geq \max\big(\ell_{bot}(t_2), pt_2 - M_1(2t_2)^{1/2}\big)$.

Define the constants g and h (depending on M_1, M_2, p, k, r) via

$$g = \cdots \text{ and } h = \cdots .$$

Then, there exists $N_4 = N_4(M_1, M_2, k) \in \mathbb{N}$ such that for any $\tilde{\epsilon} > 0$ and $N \geq N_4$ we have

$$\mathbb{P}_{avoid,Ber}^{-t_2, t_2, \vec{x}, \vec{y}, \infty, \ell_{bot}}\Big(Z\big(-t_1, t_1, Q(-t_1), Q(t_1), \ell_{bot}[\![-t_1, t_1]\!]\big) \leq gh\tilde{\epsilon}\Big) \leq \tilde{\epsilon}, \tag{4.3}$$

where $\ell_{bot}[\![-t_1, t_1]\!]$ is the vector, whose coordinates match those of ℓ_{bot} on $[\![-t_1, t_1]\!]$ and $Q(a) = (Q_1(a), \dots, Q_{k-1}(a))$ is the value of the line ensemble Q whose law is $\mathbb{P}_{avoid,Ber}^{-t_2, t_2, \vec{x}, \vec{y}, \infty, \ell_{bot}}$ at location a.

Almost proof of Proposition 4.1. Let $\epsilon > 0$ be given. Define the event

$$E_N = \Big\{L_{k-1}^N(\pm t_2) \mp pt_2 \geq -M_1(2t_2)^{1/2}\Big\} \cap \Big\{\sup_{s \in [-t_2, t_2]} \big(L_k^N(s) - ps\big) \leq M_2(2t_2)^{1/2}\Big\},$$

where M_1 and M_2 are sufficiently large so that for all large N we have $\mathbb{P}(E_N^c) < \epsilon/2$. The existence of such M_1 and M_2 is assured from Lemmas 4.2 and 4.3.

Let $\delta = (\epsilon/2) \cdot gh$, where g, h are as in Lemma 4.4 for the values M_1, M_2 as above and r as in the statement of the proposition. We denote

$$V = \Big\{Z\big(-t_1, t_1, \vec{x}, \vec{y}, L_k[\![-t_1, t_1]\!]\big) < \delta\Big\}$$

and make the following deduction

$$\mathbb{P}\big(V \cap E_N\big) = \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{E_N} \cdot \mathbf{1}_V \Big| \mathcal{F}_{ext}\big(\{1,\dots,k-1\} \times [\![-t_2+1, t_2-1]\!]\big)\Big]\Big] =$$

$$\mathbb{E}\Big[\mathbf{1}_{E_N} \cdot \mathbb{E}\Big[\mathbf{1}\{Z\big(-t_1, t_1, \vec{x}, \vec{y}, L_k[\![-t_1, t_1]\!]\big) < \delta\}\Big| \mathcal{F}_{ext}\big(\{1,\dots,k-1\} \times [\![-t_2+1, t_2-1]\!]\big)\Big]\Big] =$$

$$\mathbb{E}\left[\mathbf{1}_{E_N} \cdot \mathbb{E}_{avoid,Ber}^{-t_2,t_2,L^N(-t_2),L^N(t_2),\infty,L_k^N[\![-t_2,t_2]\!]}\left[\mathbf{1}\{Z\big(-t_1, t_1, \ell(-t_1), \ell(t_1), L_k^N[\![-t_1, t_1]\!]\big) < \delta\}\right]\right] \leq$$

$$\mathbb{E}\left[\mathbf{1}_{E_N} \cdot \epsilon/2\right] \leq \epsilon/2.$$

The first equality follows from the tower property for conditional expectations. The second equality uses the fact that $\mathbf{1}_{E_N}$ is $\mathcal{F}_{ext}\big(\{1\} \times [\![-t_2+1, t_2-1]\!]$-measurable and can thus be taken outside of the conditional expectation as well as the definition of V. The third equality uses the Schur Gibbs property. The inequality on the third line uses Lemma 4.4 with $\tilde{\epsilon} = \epsilon/2$ as well as the fact that on the event E_N^c the random variables $L^N(-t_2), L^N(t_2)$ and $L_k^N[\![-t_2, t_2]\!]$ (that play the roles of $\vec{x}, \vec{y}$ and ℓ_{bot}) satisfy the inequalities

$$L_{k-1}^N(-t_2) \geq -pt_2 - M_1(2t_2)^{1/2}, L_{k-1}^N(t_2) \geq pt_2 - M_1(2t_2)^{1/2}, \sup_{s\in[-t_2,t_2]} \big(L_k^N(s) - ps\big) \leq M_2(2t_2)^{1/2}.$$

The last inequality is trivial.

Combining the above inequality with $\mathbb{P}(E_N^c) < \epsilon/2$, we see that for all large N we have

$$\mathbb{P}\left(V\right) = \mathbb{P}(V \cap E_N) + \mathbb{P}(V \cap E_N^c) \leq \epsilon/2 + \mathbb{P}(E_N^c) < \epsilon.$$

□

4.2. **Proof of Theorem 2.26.** By Lemma 2.4, it suffices to verify the following two conditions for all $1 \leq i \leq k-1$, $R > 0$, and $\epsilon > 0$:

$$\lim_{a\to\infty} \limsup_{N\to\infty} \mathbb{P}(|f_i^N(0)| \geq a) = 0 \tag{4.4}$$

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\leq\delta}} |f_i^N(x) - f_i^N(y)| \geq \epsilon\Bigg) = 0. \tag{4.5}$$

For the sake of clarity, we will prove these conditions in three steps.

Step 1. We first prove condition (4.4), making use of Lemmas 4.2 and 4.3 in order to obtain upper and lower bounds for the top and bottom curves respectively, thus bounding all curves.

Fix $\epsilon > 0$. We show that there exists an $a > 0$ and N' depending on ϵ, such that $N > N'$ implies

$$\mathbb{P}(|f_i^N(0)| \geq a) = \mathbb{P}(|L_i^N(0)| \geq aN^{\alpha/2}) < \epsilon.$$

By Lemmas 4.2 and 4.3, there exist $R_1 := R_1(\epsilon/2) > 0$, $R_2 := R_2(\epsilon/2) > 0$ and $N_2 := N_2(\epsilon/2), N_3 := N_3(\epsilon/2)$ such that

$$N \geq N_2 \text{ implies } \mathbb{P}\Big(\sup_{s\in[-t_3,t_3]} \big(L_1^N(s) - ps\big) \geq R_1 N^{\alpha/2}\Big) < \epsilon/2,$$

$$N \geq N_3 \text{ implies } \mathbb{P}\Big(\inf_{s\in[-t_2,t_2]} \big(L_k^N(s) - ps\big) \leq -R_2 N^{\alpha/2}\Big) < \epsilon/2.$$

In particular, taking $s = 0$, we find that for $N \geq N' := N_2 \vee N_3$,

$$\mathbb{P}\big(L_1^N(0) \geq R_1 N^{\alpha/2}\big) \leq \mathbb{P}\Big(\sup_{s\in[-t_3,t_3]} \big(L_1^N(s) - ps\big) \geq R_1 N^{\alpha/2}\Big) < \epsilon/2,$$

$$\mathbb{P}\big(L_k^N(0) \leq -R_2 N^{\alpha/2}\big) \leq \mathbb{P}\Big(\inf_{s\in[-t_2,t_2]} \big(L_k^N(s) - ps\big) \leq -R_2 N^{\alpha/2}\Big) < \epsilon/2.$$

Letting $a = R_1 \vee R_2$ and noting that $L_1^N(0) > L_2^N(0) > \ldots > L_k^N(0)$, we find that for $1 \le i \le k$ and $N \ge N'$,

$$\mathbb{P}\big(|L_i^N(0)| \ge aN^{\alpha/2}\big) \le \mathbb{P}\big(L_1^N(0) \ge R_1 N^{\alpha/2}\big) + \mathbb{P}\big(L_k^N(0) \le -R_2 N^{\alpha/2}\big) < \epsilon.$$

This proves (4.4).

Step 2. This step is set up for proving condition (4.5). We will establish sets to size bias the condition in condition (4.5) for a fixed i. First, we will reorganize the statement of condition (4.5) in order to get it in terms of Bernoulli random walks, instead of f_i^N and then we will establish large events E_1 and E_2 in order so that if we prove (4.5) on the intersection of these two sets, it is equivalent to showing it on the entire probability space.

We must show that for all $\epsilon, \eta > 0$ and $R > 0$, there exists a δ and N_0 such that $N > N_0$ implies

$$\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\le\delta}} |f_i^N(x) - f_i^N(y)| \ge \epsilon\Bigg) < \eta.$$

We rewrite the left hand side as

$$\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\le\delta}} \Big|N^{-\alpha/2}\left(L_i^N(xN^\alpha) - L_i^N(yN^\alpha)\right) - p(x-y)N^{\alpha/2} + \lambda(x^2-y^2)\Big| \ge \epsilon\Bigg). \tag{4.6}$$

Given that $|x-y| < \delta$ and $x, y \in [-R, R]$, we know that $|x+y| \le 2R$ and $|x-y| < \delta$, hence $|x^2 - y^2| \le 2R\delta$. Thus if we take $\delta < \frac{\epsilon}{8\lambda R}$, then the last probability is bounded below by

$$\begin{aligned}
&\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\le\delta}} N^{-\alpha/2}\big|L_i^N(xN^\alpha) - L_i^N(yN^\alpha) - p(x-y)N^\alpha\big| + 2\lambda R\delta \ge \epsilon\Bigg)\\
&\le \mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\le\delta}} \big|L_i^N(xN^\alpha) - L_i^N(yN^\alpha) - p(x-y)N^\alpha\big| \ge \frac{3N^{\alpha/2}\epsilon}{4}\Bigg)\\
&= \mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R_N,R_N],\\ |x-y|\le\delta N^\alpha}} \big|L_i^N(x) - L_i^N(y) - p(x-y)\big| \ge \frac{3N^{\alpha/2}\epsilon}{4}\Bigg).
\end{aligned}$$

Where $R_N := \lfloor RN^\alpha \rfloor$We will take the event in this final equation and define:

$$A_\delta = \Bigg\{\sup_{\substack{x,y\in[-R_N,R_N],\\ |x-y|\le\delta N^\alpha}} \big|L_i^N(x) - L_i^N(y) - p(x-y)\big| \ge \frac{3N^{\alpha/2}\epsilon}{4}\Bigg\} \tag{4.7}$$

Define events

$$\begin{aligned}
E_1 &= \left\{\max_{1\le j\le i}|f_j(\pm R)| \le M_1\right\},\\
E_2 &= \left\{Z(-R_N, R_N, \vec{x}, \vec{y}, \infty, L_{i+1}^N[-R_N, R_N]) > \delta_1\right\}.
\end{aligned}$$

Here, $\vec{x} = (L_1^N(-R_N), \ldots, L_i(-R_N))$ and $\vec{y} = (L_1^N(R_N), \ldots, L_i(R_N))$. We argue that E_1, E_2 have high probability for appropriately chosen M_1, δ_1, and it then suffices to bound the probability of A_δ on these events.

Firstly, we observe that $L_j^N(\pm R_N) > L_{j+1}^N(\pm R_N)$, so $f_j^N(\pm R) > f_{j+1}^N(\pm R)$ as well. Thus

$$\begin{aligned} E_1^c &= \{f_1(\pm R) > M_1\} \cup \{f_i(\pm R) < -M_1\} \\ &= \left\{ \left(L_1^N(\pm R_N) \mp pR_N\right) > (M_1 - \lambda R^2)N^{\alpha/2} \right\} \\ &\quad \cup \left\{ \left(L_i^N(\pm R_N) \mp pR_N\right) < -(\lambda R^2 + M_1)N^{\alpha/2} \right\}. \end{aligned}$$

Now take $r > R$. Then in particular $R_N \le t_3$, so we have

$$\mathbb{P}\left(L_1^N(\pm R_N) \mp pR_N > (M_1 - \lambda R^2)N^{\alpha/2}\right) \le \mathbb{P}\Big(\sup_{s\in[-t_3,t_3]} L_1^N(s) - ps > (M_1 - \lambda R^2)N^{\alpha/2}\Big).$$

By Lemma 4.2, we find that if $M_1 > R_1(\eta/8) + \lambda R^2$ and $N > N_1(\eta/8)$, then this probability is less than $\eta/8$. Next, we have

$$\begin{aligned} \mathbb{P}\left(L_i^N(\pm R_N) \mp pR_N < -(\lambda R^2 + M_1)N^{\alpha/2}\right) &\le \mathbb{P}\left(L_i^N(\pm R_N) \mp pR_N < -M_1 N^{\alpha/2}\right) \\ &\le \mathbb{P}\Big(\inf_{s\in[-t_2,t_2]} \left(L_i^N(s) - ps\right) < -M_1 N^{\alpha/2}\Big), \end{aligned}$$

and this last probability is $< \eta/8$ for $M_1 \ge R_2(\eta/8)$ and $N > N_2(\eta/8)$ by Lemma 4.3. Therefore taking $M_1 = \max\{R_1(\eta/8) + \lambda R^2, R_2(\eta/8)\}$, we find

$$\mathbb{P}(E_1^c) < \frac{\eta}{4}.$$

Now by Proposition 4.1 with $r = R - 1$, there exist $\delta_1(\eta/4)$ and $N_1(\eta/4)$ such that $N \ge N_1$ implies

$$\mathbb{P}\left(E_2^c\right) < \frac{\eta}{4}.$$

In summary, for $N > \hat{N} := \max\{N_1(\eta/4), N_2(\eta/8), N_3(\eta/8)\}$,

$$\mathbb{P}(A_\delta) = \mathbb{P}(A_\delta \cap E_1 \cap E_2) + \mathbb{P}(A_\delta \cap (E_1^c \cup E_2^c)) \le \mathbb{P}(A_\delta \cap E_1 \cap E_2) + \frac{\eta}{2}. \tag{4.8}$$

Step 3. In this step, we will prove that condition 4.5 holds, using the results of the previous step, namely the inequality 4.8. We will do so by passing from Bernoulli avoidiing random walks to random walks using a Radon-Nikodym derivative, and properties of the condtional expectation.

We define a σ-algebra

$$\mathcal{F} = \sigma\left(L_{i+1}^N, L_1^N(\pm R_N), L_2^N(\pm R_N), \dots, L_i^N(\pm R_N)\right).$$

Clearly $E_1, E_2 \in \mathcal{F}$, so the indicator random variables $\mathbf{1}_{E_1}$ and $\mathbf{1}_{E_2}$ are $\mathcal{F}$-measurable. It follows from the tower property of conditional expectation that

$$\mathbb{P}\left(A_\delta \cap E_1 \cap E_2\right) = \mathbb{E}\left[\mathbf{1}_{A_\delta}\, \mathbf{1}_{E_1}\, \mathbf{1}_{E_2}\right] = \mathbb{E}\left[\mathbf{1}_{E_1}\, \mathbf{1}_{E_2}\, \mathbb{E}[\mathbf{1}_{A_\delta} \mid \mathcal{F}]\right]. \tag{4.9}$$

By the Schur-Gibbs property (see Definition 2.17),

$$\mathbb{E}\left[\mathbf{1}_{A_\delta} \mid \mathcal{F}\right] = \mathbb{E}_{avoid,Ber}^{-R_N,R_N,\vec{x},\vec{y},\infty,L_{i+1}^N}\left[\mathbf{1}_{A_\delta}\right].$$

We now observe that the Radon-Nikodym derivative of $\mathbb{P}_{avoid,Ber}^{-R_N,R_N,\vec{x},\vec{y},\infty,L_{m+1}^N}$ with respect to $\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}$ [with $\vec{x},\vec{y}$ defined as in the definition of E_2] is given by

$$\frac{d\mathbb{P}_{avoid,Ber}^{-R_N,R_N,\vec{x},\vec{y},\infty,L_{i+1}^N}}{d\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}} = \frac{\mathbf{1}_{\{L_1 \ge \cdots \ge L_{i+1}\}}}{Z(-R_N, R_N, \vec{x}, \vec{y}, L_{i+1}^N)}. \tag{4.10}$$

To see this, note that for any event A,

$$\mathbb{P}_{avoid,Ber}^{-R_N,R_N,\vec{x},\vec{y},\infty,L_{i+1}^N}(A) = \frac{\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}(A \cap \{L_1 \geq \cdots \geq L_{i+1}\})}{\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}(L_1 \geq \cdots \geq L_{i+1})}$$
$$= \frac{\mathbb{E}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}\left[\mathbf{1}_A \mathbf{1}_{\{L_1 \geq \cdots \geq L_{i+1}\}}\right]}{Z(-R_N,R_N,\vec{x},\vec{y},L_{i+1}^N)} = \int_A \frac{\mathbf{1}_{\{L_1 \geq \cdots \geq L_{i+1}\}}}{Z(-R_N,R_N,\vec{x},\vec{y},L_{i+1}^N)}\, d\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}.$$

It follows from (4.9), (4.10), and the definition of E_2 that

$$\mathbb{P}(A_\delta \cap E_1 \cap E_2) = \mathbb{E}\left[\mathbf{1}_{E_1}\,\mathbf{1}_{E_2}\,\mathbb{E}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}\left[\frac{\mathbf{1}_{A_\delta}\cdot \mathbf{1}_{\{L_1 \geq \cdots \geq L_{i+1}\}}}{Z(-R_N,R_N,\vec{x},\vec{y},L_{i+1}^N)}\right]\right]$$
$$\leq \mathbb{E}\left[\mathbf{1}_{E_1}\,\mathbb{E}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}\left[\frac{\mathbf{1}_{A_\delta}}{\delta_1}\right]\right]$$
$$\leq \frac{1}{\delta_1}\,\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}(A_\delta).$$

By Lemma 3.15, there exist W_4 and δ such that $N > W_4$ implies

$$\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}(A_\delta) < \frac{\eta\,\delta_1}{2},$$

and hence

$$\mathbb{P}\left(A_\delta \cap E_1 \cap E_2\right) \leq \frac{\eta}{2}.$$

We conclude from (4.8) that $\mathbb{P}(A_\delta) < \eta$ for $N \geq N_0 := \hat{N} \vee W_4$. Given the manner in which we defined A_δ, this implies we have shown condition 2, which, in conjuncton with condition 1, implies tightness. This completes the proof.

5. Proof of three key lemmas

Here we prove the three key lemmas from Section 4.1.

5.1. **Proof of Lemma 4.2.**

We first establish some notation. Let $a, b, t_1, t_2, z_1, z_2 \in \mathbb{Z}$ be given such that $t_1 + 1 < t_2$, $0 \leq z_2 - z_1 \leq t_2 - t_1$, $0 \leq b - a \leq t_2 - t_1$, $z_1 \leq a$, and $z_2 \leq b$. We write $\ell \in \Omega(t_1, t_2, a, b)$ and $\ell_{bot} \in \Omega(t_1, t_2, z_1, z_2)$ for generic paths in these two spaces, and we consider the event $\{\ell \geq \ell_{bot}\} = \{\ell(s) \geq \ell_{bot}(s), s \in [t_1, t_2]\}$. Note that $\mathbb{P}_{avoid,Ber}^{t_1,t_2,a,b,\infty,\ell_{bot}}(\ell) = \mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell \,|\, \ell \geq \ell_{bot}))$. We now establish some auxiliary results which will be used in the proof of Lemma 4.2.

Lemma 5.1. *If $a \leq k_1 \leq k_2 \leq a + T - t_1$, then with notation as above,*

$$\mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) = k_1\big) \leq \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) = k_2\big).$$

Remark 5.2. This lemma essentially states that a path ℓ is more likely to lie above ℓ_{bot} if its value at a point T is increased. A more general result is proven in [3, Lemma 4.1]

Proof. Let ℓ_1 be a random path distributed according to $\mathbb{P}_{Ber}^{t_1,t_2,a,b}$ conditioned on $\ell_1(T) = k_1$. We can identify ℓ_1 with a sequence of $+$'s and $-$'s of length $t_2 - t_1$, where a $+$ in the ith position means that $\ell_1(t_1 + i + 1) - \ell_1(t_1 + i) = 1$, and a $-$ means that $\ell_1(t_1 + i + 1) - \ell_1(t_1 + i) = 0$. [Maybe include Figure 9 from Corwin-Dimitrov here.] In this representation, the value of $\ell_1(T)$ is a plus the number of $+$'s in the first $T - t_1$ slots, and the value of $\ell_1(t_2)$ is a plus the total number of $+$'s. Note that we must have exactly $(k_1 - a)$ $+$'s in the first $T - t_1$ slots, and $(b - k_1)$ $+$'s in the last $t_2 - T$ slots. We pick uniformly at random $(k_2 - k_1)$ $-$'s in the first $T - t_1$ slots and change them to $+$'s, then pick randomly $(k_2 - k_1)$ $+$'s in the last $t_2 - T$ slots and change them to $-$'s. This defines a new path ℓ_2. Since there are now $k_2 - a$ $+$'s in the first $T - t_1$ slots, we have $\ell_2(T) = k_2$, and we still have $\ell_2(t_2) = b$ since the number of $+$'s is unchanged. Thus we see that ℓ_2 is distributed according to $\mathbb{P}_{Ber}^{t_1,t_2,a,b}$ conditioned on $\ell_2(T) = k_2$.

Now suppose $\ell_1 \geq \ell_{bot}$. We claim that $\ell_2 \geq \ell_1$ on all of $[t_1, t_2]$. To see this, note that for any $s \in [\![t_1, t_2]\!]$, $\ell_2(s) - \ell_1(s)$ is equal to the number of $+$'s in the first $s - t_1$ slots of the sequence representing ℓ_2, minus the corresponding number for ℓ_1. If $s \leq T$, this difference is clearly positive by construction. The difference is equal to $k_2 - k_1 \geq 0$ at $s = T$, and the difference then decreases monotonically as s increases to t_2, since we have removed exactly $k_2 - k_1$ $+$'s from the last $t_2 - T$ slots. The difference is of course 0 at $s = t_2$, so this proves the claim. It follows that

$$\mathbf{1}_{\ell_1 \geq \ell_{bot}} \leq \mathbf{1}_{\ell_2 \geq \ell_{bot}}.$$

Now taking expectations of both sides and recalling the distributions of ℓ_1, ℓ_2 proves the lemma. $\square$

Corollary 5.3. *Let $T \in [\![t_1, t_2]\!]$, and let A, B be nonempty sets of integers such that $a \leq \alpha \leq \beta \leq a + T - t_1$ for all $\alpha \in A, \beta \in B$. Then*

$$\mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) \in A\big) \leq \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) \in B\big).$$

Proof. We have

$$\begin{aligned}
&\mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) \in A\big) = \sum_{\alpha \in A} \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) = \alpha\big) \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \alpha)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in A)} \\
&= \sum_{\alpha \in A} \sum_{\beta \in B} \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) = \alpha\big) \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \alpha)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in A)} \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \beta)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in B)} \\
&\leq \sum_{\alpha \in A} \sum_{\beta \in B} \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) = \beta\big) \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \alpha)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in A)} \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \beta)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in B)} \\
&= \sum_{\beta \in B} \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) = \beta\big) \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \beta)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in B)} = \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) \in B\big).
\end{aligned}$$

The inequality in the third line follows from Lemma 5.1. $\square$

Corollary 5.4. *Let $\alpha \leq a + T - t_1$. Then*

$$\mathbb{P}_{avoid,Ber}^{t_1,t_2,a,b,\infty,\ell_{bot}}(\ell(T) \geq \alpha) \geq \mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \geq \alpha).$$

Proof. We write $\mathbb{P} := \mathbb{P}_{Ber}^{t_1,t_2,a,b}$ for brevity. Using Bayes' theorem repeatedly, we find

$$\begin{aligned}
\mathbb{P}(\ell(T) \geq \alpha \,|\, \ell \geq \ell_{bot}) &= \frac{\mathbb{P}(\ell \geq \ell_{bot} \,|\, \ell(T) \geq \alpha)\mathbb{P}(\ell(T) \geq \alpha)}{\mathbb{P}(\ell \geq \ell_{bot})} \\
&\geq \frac{\mathbb{P}(\ell \geq \ell_{bot} \,|\, \ell(T) < \alpha)\mathbb{P}(\ell(T) \geq \alpha)}{\mathbb{P}(\ell \geq \ell_{bot})} \\
&= \big(1 - \mathbb{P}(\ell(T) \geq \alpha \,|\, \ell \geq \ell_{bot})\big) \cdot \frac{\mathbb{P}(\ell(T) \geq \alpha)}{\mathbb{P}(\ell(T) < \alpha)}.
\end{aligned}$$

The inequality in the second line follows from Corollary 5.3. It follows that

$$\mathbb{P}(\ell(T) \geq \alpha \,|\, \ell \geq \ell_{bot}) \geq \frac{\mathbb{P}(\ell(T) \geq \alpha)}{\mathbb{P}(\ell(T) \geq \alpha) + \mathbb{P}(\ell(T) < \alpha)} = \mathbb{P}(\ell(T) \geq \alpha).$$

$\square$

We are now ready to prove Lemma 4.2. The proof is similar to that of [3, Lemma 5.2]. We exploit the one-point tightness of L_1^N at two appropriately chosen points, and we use Lemma 3.7 to control the deviation of L_1^N from the line of slope p away from these points.

Proof. We write $s_4 = \lceil r+3 \rceil N^\alpha$, $s_3 = \lfloor r+3 \rfloor N^\alpha$, so that $s_3 \le t_3 \le s_4$, and take N large enough so that L_1^N is defined at s_4. We define events

$$E(M) = \Big\{ \big|L_1^N(-s_4) + ps_4\big| > MN^{\alpha/2} \Big\}, \quad F(M) = \Big\{ L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} \Big\},$$

$$G(M) = \left\{ \sup_{s \in [0,t_3]} \big(L_1^N(s) - ps\big) \ge (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2} \right\}.$$

For $a, b \in \mathbb{Z}$, $s \in [\![0, t_3]\!]$, and $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$ with $z_1 \le a$, $z_2 \le b$, we also define $E(a, b, s, \ell_{bot})$ to be the event that $L_1^N(-s_4) = a$, $L_1^N(s) = b$, and L_2^N agrees with ℓ_{bot} on $[-s_4, s]$.

We claim that the set $G(M) \setminus E(M)$ can be written as a *countable disjoint* union of sets $E(a, b, s, \ell_{bot})$. Let $D(M)$ be the set of tuples (a, b, s, ℓ_{bot}) satisfying

(1) $0 \le s \le t_3$,
(2) $0 \le b - a \le s + s_4$, $|a + ps_4| \le MN^{\alpha/2}$, and $b - ps > (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2}$,
(3) $z_1 \le a, z_2 \le b$, and $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$.

Conditions (1) and (2) show that the union of these sets $E(a, b, s, \ell_{bot})$ for $(a, b, s, \ell_{bot}) \in D(M)$ is $G(M) \setminus E(M)$. Observe that $D(M)$ is countable, since there are finitely many possible choices of s, countably many a, b and z_1, z_2 for each s, and finitely many ℓ_{bot} for each z_1, z_2. Moreover, the sets $E(a, b, s, \ell_{bot})$ are clearly pairwise disjoint for distinct tuples in $D(M)$. This proves the claim.

Now by one-point tightness of L_1^N at integer multiples of N^α, we can choose M large enough depending on ϵ so that

$$\mathbb{P}(E(M)) < \epsilon/4, \quad \mathbb{P}(F(M)) < \epsilon/12 \tag{5.1}$$

for all $N \in \mathbb{N}$. If $(a, b, s, \ell_{bot}) \in D(M)$, then

$$\begin{aligned}\mathbb{P}_{Ber}^{-s_4,s,a,b}\Big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\Big) &= \mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\Big(\ell(s_4 - s_3) + a \ge -ps_3 + MN^{\alpha/2}\Big) \\ &\ge \mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\Big(\ell(s_4 - s_3) \ge p(s_4 - s_3) + 2MN^{\alpha/2}\Big).\end{aligned}$$

The inequality follows from the assumption in (2) that $a + ps_4 \ge -MN^{\alpha/2}$. Moreover, since $b - ps > (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2}$ and $a + ps_4 \le MN^{\alpha/2}$, we have

$$b - a \ge p(s+s_4) + (6r+21)(2r+6)^{1/2}(M+1)N^{\alpha/2} \ge p(s+t_3) + (6r+21)(M+1)(s+s_4)^{1/2}.$$

The second inequality follows since $s + s_4 \le 2s_4 \le (2r+6)N^\alpha$. It follows from Lemma 3.7 with $M_1 = 0$, $M_2 = (6r+21)(M+1)$ that for sufficiently large N, we have

$$\mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\Big(\ell(s_4 - s_3) \ge \frac{s_4 - s_3}{s + s_4}[p(s+s_4) + M_2 N^{\alpha/2}] - (s+s_4)^{1/4}\Big) \ge 1/3, \tag{5.2}$$

for all $(a, b, s, \ell_{bot}) \in D(M)$ simultaneously. Note that $\frac{s_4 - s_3}{s+s_4} \ge \frac{N^\alpha - 1}{(2r+6)N^\alpha} \ge \frac{1}{2r+7}$ for large N. Hence $\frac{s_4-s_3}{s+s_4}[p(s+t_3) + M_2N^{\alpha/2}] - (s+s_4)^{1/4} \ge p(s+s_4) + 3(M+1)N^{\alpha/2} - (s+s_4)^{1/4} \ge p(s+s_4) + 2MN^{\alpha/2}$ for all large enough N. We conclude from (5.2) that

$$\mathbb{P}_{Ber}^{-s_4,s,a,b}\Big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\Big) \ge 1/3$$

uniformly in a, b for large N. Now by the Gibbs property for L^N, we have for any $\ell \in \Omega(-s_4, s, a, b)$ that

$$\mathbb{P}(L_1^N|_{[-s_4,s]} = \ell \,|\, E(a, b, s, \ell_{bot})) = \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}(\ell).$$

Hence by Corollary 5.4,

$$\begin{aligned}
&\mathbb{P}\big(L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} \,\big|\, E(a,b,s,\ell_{bot})\big) \\
&= \sum_{\ell \in \Omega(-s_4,s,a,b)} \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}(\ell) \cdot \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}\big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\big) \\
&\geq \sum_{\ell \in \Omega(-s_4,s,a,b)} \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}(\ell) \cdot \mathbb{P}_{Ber}^{-s_4,s,a,b}\big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\big) \\
&\geq \frac{1}{3} \sum_{\ell \in \Omega(-s_4,s,a,b)} \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}(\ell) = \frac{1}{3}.
\end{aligned}$$

Note once again that this bound holds independent of a, b for all sufficiently large N. It follows from (5.1) that

$$\epsilon/12 > \mathbb{P}(F(M)) \geq \sum_{(a,b,s,\ell_{bot}) \in D(M)} \mathbb{P}(F(M) \cap E(a,b,s,\ell_{bot}))$$

$$= \sum_{(a,b,s,\ell_{bot}) \in D(M)} \mathbb{P}(F(M) \,|\, E(a,b,s,\ell_{bot}))\mathbb{P}(E(a,b,s,\ell_{bot})) \geq \frac{1}{3}\mathbb{P}(G(M) \setminus E(M))$$

for large N. Since in addition $\mathbb{P}(E(M)) < \epsilon/4$, we find that

$$\mathbb{P}\Big(\sup_{s \in [0,t_3]} \big(L_1^N(s) - ps\big) \geq (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2}\Big) = \mathbb{P}(G(M)) < \epsilon/2$$

for large enough N. A similar argument proves the same inequality with $[-t_3, 0]$ in place of $[0, t_3]$. Thus we can find an $N_2 = N_2(\epsilon)$ so that

$$\mathbb{P}\Big(\sup_{s \in [-t_3,t_3]} \big(L_1^N(s) - ps\big) \geq R_1 N^{\alpha/2}\Big) < \epsilon$$

for all $N \geq N_2$, with $R_1 = (6r+22)(2r+6)^{1/2}(M+1)$.

∎

5.2. **Proof of Lemma 4.3.** We begin by proving the following lemma, which allows us to prevent the bottom curve of an ensemble from falling too low on some interval.

Lemma 5.5. *Fix $p \in (0,1)$, $k \in \mathbb{N}$, and $\alpha, \lambda > 0$. Suppose that $\mathfrak{L}^N = (L_1^N, \dots, L_k^N)$ is a (α, p, λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles. Then for any $r, \epsilon > 0$, there exists $R > 0$ depending on $\lambda, k, p, \epsilon, r, \phi$ and $N_0 \in \mathbb{N}$ depending on $\lambda, k, p, \epsilon, r, \phi, \psi, \alpha$ such that for all $N \geq N_0$,*

$$\mathbb{P}\Big(\max_{x \in [r,R]} \big(L_k^N(xN^\alpha) - pxN^\alpha\big) \leq -(\lambda R^2 + \phi(\epsilon/16))N^{\alpha/2}\Big) < \epsilon.$$

The same statement holds if $[r, R]$ is replaced with $[-R, -r]$.

Remark 5.6. The key to this lemma is the parabolic shift implicit in the definition of an (α, p, λ)-good sequence. This requires the deviation of the top curve from the line of slope p to appear roughly parabolic. Using monotone coupling, we separate the curves of the ensemble so that L_1^N is nearly independent of the other curves. Then we would expect the value of L_1^N at the midpoint of r and R to be close to the midpoint of the straight line segment connecting two points of the parabola. But the parabola is convex, so for large enough R this violates the one-point tightness assumpion at $(R+r)/2$.

Proof. Fix $r > 0$. Note that for any $R > r$,

$$\max_{r \leq x \leq R} \big(L_k^N(xN^\alpha) - pxN^\alpha\big) \geq \max_{\lceil r \rceil \leq x \leq R} \big(L_k^N(xN^\alpha) - pxN^\alpha\big).$$

Thus by replacing r with $\lceil r \rceil$, we can assume that $r \in \mathbb{Z}$. Before beginning the proof, we introduce notation. Define constants

$$C = \sqrt{8p(1-p)\log\frac{3}{1-(11/12)^{1/(k-1)}}}, \tag{5.3}$$

$$R_0 = 8\left(r + \frac{Ck}{\lambda}\right) \vee \frac{2\phi(\epsilon/16)}{Ck+\lambda r}. \tag{5.4}$$

Note that $R_0 \geq r$. We define $R = R_0 + \mathbf{1}_{R_0+r \text{ odd}}$, so that $R \geq R_0$ and the midpoint $(R+r)/2$ is an integer. In the following, we always assume N is large enough depending on ψ, R so that L_1^N is defined at R. We may do so by the second condition in the definition of an (α, p, λ)-good sequence (see Definition 2.24). Define events

$$A = \left\{ L_1^N\left(\frac{R+r}{2}N^\alpha\right) - pN^\alpha\frac{R+r}{2} + \lambda\left(\frac{R+r}{2}\right)^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2} \right\},$$
$$B = \left\{ \max_{x\in[r,R]} \left(L_k^N(xN^\alpha) - pxN^\alpha\right) \leq -(\lambda R^2 + \phi(\epsilon/16))N^{\alpha/2} \right\}.$$

Let F denote the subset of B for which the inequalities

$$\begin{aligned} &prN^\alpha - (\lambda r^2 + \phi(\epsilon/16))N^{\alpha/2} < L_1^N(rN^\alpha) < prN^\alpha - (\lambda r^2 - \phi(\epsilon/16))N^{\alpha/2}, \\ &pRN^\alpha - (\lambda R^2 + \phi(\epsilon/16))N^{\alpha/2} < L_1^N(RN^\alpha) < pRN^\alpha - (\lambda R^2 - \phi(\epsilon/16))N^{\alpha/2} \end{aligned} \tag{5.5}$$

hold. Let D denote the set of pairs $(\vec{x}, \vec{y})$, with $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$ satisfying

(1) $0 \leq y_i - x_i \leq (R-r)N^\alpha$ for $1 \leq i \leq k$,
(2) $prN^\alpha - (\lambda r^2 + \phi(\epsilon/8))N^{\alpha/2} < x_1 < prN^\alpha - (\lambda r^2 - \phi(\epsilon/8))N^{\alpha/2}$ and $pRN^\alpha - (\lambda R^2 + \phi(\epsilon/8))N^{\alpha/2} < y_1 < pRN^\alpha - (\lambda R^2 - \phi(\epsilon/8))N^{\alpha/2}$.

Let $E(\vec{x}, \vec{y})$ denote the subset of F consisting of L^N for which $L_i^N(rN^\alpha) = x_i$ and $L_i^N(RN^\alpha) = y_i$ for $1 \leq i \leq k$, and $L_1^N(s) \geq \cdots \geq L_k^N(s)$ for all s. Then D is countable, the $E(\vec{x}, \vec{y})$ are pairwise disjoint, and $F = \bigcup_{(\vec{x},\vec{y})\in D} E(\vec{x}, \vec{y})$.

To prove the lemma, we argue that $\mathbb{P}(B) < \epsilon$ for large N. We split the proof into several steps.

Step 1. We will argue in the following steps that for large enough N,

$$\mathbb{P}(A \,|\, E(\vec{x}, \vec{y})) > 1/4 \tag{5.6}$$

uniformly in $\vec{x}, \vec{y}$. In this step, we prove the lemma assuming this fact.

It follows from (5.6) that

$$\mathbb{P}(A \,|\, F) = \sum_{(\vec{x},\vec{y})\in D} \frac{\mathbb{P}(A \,|\, E(\vec{x}, \vec{y}))\mathbb{P}(E(\vec{x}, \vec{y}))}{\mathbb{P}(F)} \geq \frac{1}{4} \cdot \frac{\sum_{(\vec{x},\vec{y})\in D} \mathbb{P}(E(\vec{x}, \vec{y}))}{\mathbb{P}(F)} = \frac{1}{4}. \tag{5.7}$$

From the third condition in Definition 2.24, we have $\mathbb{P}(A) < \epsilon/8$ for large enough N. Hence

$$\mathbb{P}(F) = \frac{\mathbb{P}(A \cap F)}{\mathbb{P}(A \,|\, F)} \leq 4\mathbb{P}(A) < \epsilon/2.$$

Now with probability $> 1 - \epsilon/2$, the two inequalities in (5.5) hold. We conclude that

$$\mathbb{P}(B) \leq \mathbb{P}(F) + \epsilon/2 \leq \epsilon.$$

Step 2. We will now prove (5.6), assuming results from Steps 3 and 4 below. We first note that by Lemma 3.1, if we raise the endpoints of each curve, then the probability of the event A will decrease. In particular, write $T = (R-r)N^\alpha$, and define $\vec{x}\,', \vec{y}\,'$ by

$$x_i' = \lceil prN^\alpha - (\lambda r^2 - \phi(\epsilon/8))N^{\alpha/2} \rceil + (k-i)\lceil C\sqrt{T}\,\rceil,$$
$$y_i' = \lceil pRN^\alpha - (\lambda R^2 - \phi(\epsilon/8))N^{\alpha/2} \rceil + (k-i)\lceil C\sqrt{T}\,\rceil.$$

Note that $x_i' \geq x_1 \geq x_i$ for each i by condition (2) above. Furthermore, $x_i' - x_{i+1}' \geq C\sqrt{T}$. The same observations hold for y_i'. Using Lemma 3.1, we have

$$\begin{aligned}
\mathbb{P}(A \,|\, E(\vec{x},\vec{y})) &= \mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x},\vec{y},\infty,L_k}_{avoid,Ber}(A\,|\,F) \geq \mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x}\,',\vec{y}\,',\infty,L_k}_{avoid,Ber}(A\,|\,F) \\
&\geq \mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x}',\vec{y}'}_{Ber}(A \cap \{L_1 \geq \cdots \geq L_k\}\,|\,F) \\
&\geq \mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x}',\vec{y}'}_{Ber}(A\,|\,F) - \big(1 - \mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x}',\vec{y}'}_{Ber}(L_1 \geq \cdots \geq L_k\,|\,F)\big) \\
&= \mathbb{P}^{rN^\alpha,RN^\alpha,x_1',y_1'}_{Ber}(A) - \big(1 - \mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x}',\vec{y}'}_{Ber}(L_1 \geq \cdots \geq L_k\,|\,F)\big).
\end{aligned} \tag{5.8}$$

For the first term in the last line, we used the Gibbs property and the fact that A and F are independent under $\mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x},\vec{y}}_{Ber}$. In Steps 3 and 4, we will argue that the two probabilities in (5.8) are bounded below by $1/3$ and $11/12$, respectively. Then $\mathbb{P}(A\,|\,E(\vec{x},\vec{y})) \geq 1/3 - 1/12 = 1/4$ for large N independent of $\vec{x},\vec{y}$, proving (5.6).

Step 3. In this step, we argue that $\mathbb{P}^{rN^\alpha,RN^\alpha,x_1',y_1'}_{Ber}(A) > 1/3$ for sufficiently large N. Write $\overline{x} = x_1' - (k-1)\lceil C\sqrt{T}\rceil$, $\overline{y} = y_1' - (k-1)\lceil C\sqrt{T}\rceil$, and $\overline{z} = \overline{y} - \overline{x}$. We have

$$\begin{aligned}
&\mathbb{P}^{rN^\alpha,RN^\alpha,x_1',y_1'}_{Ber}(A) \\
&= \mathbb{P}^{0,T,x_1',y_1'}_{Ber}\Big(L_1(T/2) - pN^\alpha\frac{R+r}{2} + \lambda\Big(\frac{R+r}{2}\Big)^2 N^{\alpha/2} < -\phi(\epsilon/8)N^{\alpha/2}\Big) \\
&= \mathbb{P}^{0,T,\overline{x},\overline{y}}_{Ber}\Big(L_1(T/2) - pN^\alpha\frac{R+r}{2} + \lambda\Big(\frac{R+r}{2}\Big)^2 N^{\alpha/2} < -\big(\phi(\epsilon/8) + (k-1)\lceil C\sqrt{R-r}\,\rceil\big)N^{\alpha/2}\Big) \\
&\geq \mathbb{P}^{0,T,\overline{x},\overline{y}}_{Ber}\Big(L_1(T/2) - \frac{\overline{x}+\overline{y}}{2} < \Big(\lambda\Big(\frac{R^2+r^2}{2}\Big) - \lambda\Big(\frac{R+r}{2}\Big)^2 - Ck\sqrt{R-r} - 2\phi(\epsilon/8)\Big)N^{\alpha/2}\Big).
\end{aligned} \tag{5.9}$$

The inequality in the last line follows from the definitions of $\overline{x},\overline{y}$. Observe that

$$\frac{R^2+r^2}{2} - \Big(\frac{R+r}{2}\Big)^2 = \frac{R^2+r^2}{4} - \frac{rR}{4} = O(R^2)$$

for fixed r. Our choice of R from (5.4) ensures that the constant factor multiplying $N^{\alpha/2}$ in (5.4) is positive. Denoting this constant by γ, we see that (5.9) is bounded below by

$$\mathbb{P}^{0,T,0,\overline{z}}_{Ber}\Big(L_1(T/2) - \overline{z}/2 < \gamma\sqrt{T}\Big).$$

Let $\ell^{(T,\overline{z})}$ have the same law as L_1 under a probability measure $\mathbb{P}$ as in Theorem 3.3, and let B^σ, $\sigma^2 = p(1-p)$, be the Brownian bridge provided by the theorem. Then the last probability is

$$\begin{aligned}
&\mathbb{P}\Big(\ell^{(T,\overline{z})}(T/2) - \overline{z}/2 < \gamma\sqrt{T}\Big) = \mathbb{P}\Big(\Big[\ell^{(T,\overline{z})}(T/2) - \overline{z}/2 - \sqrt{T}B^\sigma_{1/2}\Big] + \sqrt{T}B^\sigma_{1/2} < \gamma\sqrt{T}\Big) \\
&\geq \mathbb{P}\Big(\sqrt{T}B^\sigma_{1/2} < 0 \quad \text{and} \quad \Delta(T,\overline{z}) < \gamma\sqrt{T}\Big) \geq \frac{1}{2} - \mathbb{P}\Big(\Delta(T,\overline{z}) \geq \gamma\sqrt{T}\Big).
\end{aligned}$$

Here, $\Delta(T,\overline{z})$ is as defined in (3.1). Observe that

$$\frac{|\overline{z} - pT|^2}{T} \leq \frac{(\lambda(R^2-r^2)N^{\alpha/2}+1)^2}{(R-r)N^\alpha} \leq 4\lambda^2(R+r)^2(R-r). \tag{5.10}$$

Thus Corollary 3.5 shows that $\mathbb{P}(\Delta(T,\overline{z}) \geq \gamma\sqrt{T}) < 1/6$ for large enough N. This gives a lower bound on $\mathbb{P}^{rN^\alpha,RN^\alpha,x_1',y_1'}_{Ber}(A)$ of $1/2 - 1/6 = 1/3$ as desired.

Step 4. In this last step, we show that $\mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x},\vec{y}}_{Ber}(L_1 > \cdots > L_k \,|\, F) > 11/12$ for large N. Note that on the event F, L_k^N lies uniformly below the line segment connecting $L_1^N(rN^\alpha)$ and $L_1^N(RN^\alpha)$. Thus after raising the endpoints to $\vec{x}\,'$, $\vec{y}\,'$, the bottom curve L_k lies uniformly at a distance of at least $C\sqrt{T}$ below the segment connecting $L_{k-1}(rN^\alpha)$ and $L_{k-1}(RN^\alpha)$, and moreover the endpoints of adjacent curves are at least $C\sqrt{T}$ apart. Then in order to have $L_1 \geq \cdots \geq L_k$ given F, it suffices to require each L_i to lie within a distance of $C\sqrt{T}/2$ from the line segment connecting its endpoints. That is,

$$
\begin{aligned}
&\mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x}\,',\vec{y}\,'}_{Ber}(L_1 \geq \cdots \geq L_k \,|\, F) \\
&\geq \mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x}\,',\vec{y}\,'}_{Ber}\Big(\sup_{x\in[r,R]} \big| L_i(xN^\alpha) - x_i' - (\overline{z}/T)(x-r)N^\alpha \big| \leq C\sqrt{T}/2,\ 1 \leq i \leq k-1 \,\Big|\, F \Big) \\
&= \Big[\mathbb{P}^{0,T,0,\overline{z}}_{Ber}\Big(\sup_{s\in[0,T]} \big| L_1(s+rN^\alpha) - (\overline{z}/T)s \big| \leq C\sqrt{T}/2 \Big) \Big]^{k-1} \\
&= \Big[1 - \mathbb{P}\Big(\sup_{s\in[0,T]} \big| \ell^{(T,\overline{z})} - (\overline{z}/T)s \big| > C\sqrt{T}/2 \Big) \Big]^{k-1},
\end{aligned} \tag{5.11}
$$

with $\mathbb{P}$ and $\ell^{(T,\overline{z})}$ as in Step 1. In the third line, we used the fact that $L_1, \dots, L_{k-1}$ are independent from each other and from L_k under $\mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x},\vec{y}}_{Ber}$. Let B^σ be as in Step 1. Then we have

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{s\in[0,T]} \big| \ell^{(T,\overline{z})}(s) - (\overline{z}/T)s \big| > C\sqrt{T}/2 \Big) \\
&\leq \mathbb{P}\Big(\sup_{s\in[0,T]} |\sqrt{T} B^\sigma_{s/T}| > C\sqrt{T}/4 \Big) + \mathbb{P}\Big(\Delta(T,\overline{z}) > C\sqrt{T}/4 \Big).
\end{aligned}
$$

By Lemma 3.6, the first term is equal to

$$2\exp\left(-\frac{2}{\sigma^2}\left(\frac{C}{4}\right)^2 \right) = 2e^{-C^2/8p(1-p)}.$$

For the second term, (5.10) and Corollary 3.5 give an upper bound of $< e^{-C^2/8p(1-p)}$ for sufficiently large N. Our choice of C in (5.3) implies a lower bound in (5.11) of

$$(1 - 3e^{-C^2/8p(1-p)})^{k-1} > 11/12.$$

Essentially the same argument proves the statement if $[r,R]$ is replaced by $[-R,-r]$.

□

We now prove Lemma 4.3. We exploit Lemma 5.5 in order to find two far away points where L_k^N cannot be too low. After separating the curves in order to treat L_k^N as a free curve as in the previous argument, we employ Lemma 3.9 to bound the deviation of L_k^N below the line of slope p.

Proof. We first introduce notation used in the proof. Define events

$$A_N(R_2) = \left\{ \inf_{s\in[-t_2,t_2]} \big(L_k^N(s) - ps \big) \leq -R_2 N^{\alpha/2} \right\},$$

$$
\begin{aligned}
B_N = &\left\{ \max_{x\in[r+2,R]} \big(L_k^N(xN^\alpha) - pxN^\alpha \big) > -MN^{\alpha/2} \right\} \\
&\cap \left\{ \max_{x\in[-R,-r-2]} \big(L_k^N(xN^\alpha) - pxN^\alpha \big) > -MN^{\alpha/2} \right\}.
\end{aligned}
$$

Here, we define M, R via Lemma 5.5, taking R large enough so that with $M = \lambda R^2 + \phi(\epsilon/64)$, we have

$$\mathbb{P}(B_N^c) < \epsilon/2 \tag{5.12}$$

for sufficiently large N.

For $0 < a, b \in \mathbb{Z}$ and $\vec{x}, \vec{y} \in \mathfrak{W}_k$, we define $E(a, b, \vec{x}, \vec{y})$ to be the event that $L_i^N(-a) = x_i$ and $L_i^N(b) = y_i$ for $1 \le i \le k$, and $L_1^N(s) > \cdots > L_k^N(s)$ for all $s \in [-RN^\alpha, RN^\alpha]$.

We claim that $B_N(M, R)$ can be written as a countable disjoint union of sets $E(a, b, \vec{x}, \vec{y})$. Let $D_N(M)$ be the collection of tuples $(a, b, \vec{x}, \vec{y})$ satisfying

(1) $a, b \in [rN^\alpha, RN^\alpha]$.
(2) $0 \le y_i - x_i \le b + a$, $x_k + pa > -MN^{\alpha/2}$, and $y_k - pb > -MN^{\alpha/2}$.
(3) If $c, d \in \mathbb{Z}$, $c > a$, and $d > b$, then $L_k^N(-c) + pc \le -MN^{\alpha/2}$ and $L_k^N(d) - pd \le -MN^{\alpha/2}$.

Since there are finitely many integers a, b satisfying (1), the x_i, y_i are integers, and there are finitely many choices of L_i^N on $[-aN^\alpha, bN^\alpha]$ given a, b, x_i, y_i, we see that $D_N(M)$ is countable. The third condition ensures that the $E(a, b, \vec{x}, \vec{y})$ are pairwise disjoint. To see that their union over $D_N(M)$ is all of $B_N(M, R)$, note that $B_N(M, R)$ occurs if and only if there is a first integer time $s = -a$ and a last integer time $s = b$ when $L_k^N(s) - ps$ crosses $-MN^{\alpha/2}$.

Lastly, define the constant

$$C = \sqrt{16p(1-p)\log\frac{3}{1 - 2^{-1/(k-1)}}}. \tag{5.13}$$

We will prove that $\mathbb{P}(A_N(R_2)) < \epsilon$ for large N, if R_2 is chosen large enough depending on M, C, k, r, ϵ. We specify how we choose R_2 after (5.18) below. We split the proof into steps for clarity.

Step 1. We will prove in the steps below that for large enough N,

$$\mathbb{P}(A_N(R_2) \,|\, E(a, b, \vec{x}, \vec{y}) < \epsilon/2 \tag{5.14}$$

uniformly in $a, b, \vec{x}, \vec{y}$. In this step, we prove the lemma assuming this fact.

Since the $E(a, b, \vec{x}, \vec{y})$ are disjoint, (5.14) implies

$$\begin{aligned}\mathbb{P}(A_N(R_2) \cap B_N(M, R)) &= \sum_{(a,b,\vec{x},\vec{y}) \in D_N} \mathbb{P}(A_N(R_2) \,|\, E(a, b, \vec{x}, \vec{y}))\mathbb{P}(E(a, b, \vec{x}, \vec{y})) \\ &\le \frac{\epsilon}{2} \sum_{(a,b,\vec{x},\vec{y}) \in D_N} \mathbb{P}(E(a, b, \vec{x}, \vec{y})) \le \frac{\epsilon}{2}.\end{aligned}$$

It follows from (5.12) that

$$\mathbb{P}(A_N(R_2)) \le \mathbb{P}(A_N(R_2) \cap B_N) + \epsilon/2 < \epsilon$$

for large enough N.

Step 2. We next prove (5.14), assuming results from Steps 3 and 4 below. Define $\vec{x}\,', \vec{y}\,'$ by

$$\begin{aligned} x_i' &= \lfloor -pa - MN^{\alpha/2} \rfloor - (i-1)\lceil CN^{\alpha/2} \rceil, \\ y_i' &= \lfloor pb - MN^{\alpha/2} \rfloor - (i-1)\lceil CN^{\alpha/2} \rceil. \end{aligned}$$

Observe that by condition (2) above, $x_i' \leq -pa - MN^{\alpha/2} \leq x_k \leq x_i$, and similarly for $\vec{y}$. It follows from Lemma 3.1 that

$$\mathbb{P}(A_N(R_2) \,|\, E(a,b,\vec{x},\vec{y})) \leq \mathbb{P}^{-a,b,\vec{x},\vec{y}}_{avoid,Ber}\Big(\inf_{s\in[-a,b]}\big(L_k(s)-ps\big) \leq -R_2N^{\alpha/2}\Big) \tag{5.15}$$

$$= \mathbb{P}^{0,a+b,\vec{x},\vec{y}}_{avoid,Ber}\Big(\inf_{s\in[0,a+b]}\big(L_k(s-a)-p(s-a)\big) \leq -R_2N^{\alpha/2}\Big)$$

$$\leq \mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{avoid,Ber}\Big(\inf_{s\in[0,a+b]}\big(L_k'(s)-p(s-a)\big) \leq -R_2N^{\alpha/2}\Big).$$

In the last line, we have written $L_k'(s) = L_k(s-a)$. The last probability is bounded above by

$$\frac{\mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{Ber}\Big(\inf_{s\in[0,a+b]}\big(\ell(s)-p(s-a)\big) \leq -R_2N^{\alpha/2}\Big)}{\mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{Ber}(F)}, \tag{5.16}$$

where

$$F = \{L_1'(s) > \cdots > L_k'(s),\ s\in[0,a+b]\}.$$

In Steps 3 and 4 below, we will prove that the numerator and denominator in (5.16) are $< \epsilon/4$ and $> 1/2$ for sufficiently large N. It then follows that (5.16) is bounded above by $\epsilon/2$, proving (5.14).

Step 3. We now argue that

$$\mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{Ber}\Big(\inf_{s\in[0,a+b]}\big(\ell(s)-p(s-a)\big) \leq -R_2N^{\alpha/2}\Big) < \epsilon/4 \tag{5.17}$$

for sufficiently large N. Writing $\vec{z} = \vec{y}\,' - \vec{x}\,'$, (5.17) is equal to

$$\mathbb{P}^{0,a+b,x_k',y_k'}_{Ber}\Big(\inf_{s\in[0,a+b]}\big(\ell(s)-p(s-a)\big) \leq -R_2N^{\alpha/2}\Big)$$

$$= \mathbb{P}^{0,a+b,0,z_k}_{Ber}\Big(\inf_{s\in[0,a+b]}\big(\ell(s)-ps+pa-\lceil pa+MN^{\alpha/2}\rceil-(k-1)\lceil CN^{\alpha/2}\rceil\big) \leq -R_2N^{\alpha/2}\Big)$$

$$\leq \mathbb{P}^{0,a+b,0,z_k}_{Ber}\Big(\inf_{s\in[0,a+b]}\big(\ell(s)-ps\big) \leq -(R_2-M-Ck)N^{\alpha/2}\Big). \tag{5.18}$$

Since $z_k \geq p(a+b)$, Lemma 3.9 allows us to find $R_2 > 0$ depending on M, C, k, r, ϵ so that this probability is $< \epsilon/4$ for all large N, such that $a+b$ is larger than some constant W_1. But observe that $a+b \geq 2rN^{\alpha}$, so it suffices to take $N > (W_1/2r)^{1/\alpha}$. Thus we obtain (5.17), *independent* of $a, b, \vec{x}, \vec{y}$.

Step 4. Lastly, we argue that

$$\mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{Ber}(F) > 1/2 \tag{5.19}$$

for large N. The argument is very similar to that in the proof of Lemma 5.5. Write $a = a'N^{\alpha}, b = b'N^{\alpha}$, $T = a+b = (a'+b')N^{\alpha}$, and $z = y_k' - x_k'$. Let $\ell^{(T,z)}$ be a random variable with the same law as the L_i' shifted down by $x_i{}'$ under a measure $\mathbb{P}$, as provided by Theorem 3.3. Let B^{σ} be a Brownian bridge with variance $\sigma^2 = p(1-p)$ coupled with $\ell^{(T,z)}$. Then

$$\mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{Ber}(F) \geq \mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{Ber}\Big(\sup_{s\in[0,T]}\Big|L_i'(s)-x_i'-(z/T)s\Big| < \frac{CN^{\alpha/2}}{2},\ 1\leq i\leq k\Big)$$

$$= \Big[1-\mathbb{P}\Big(\sup_{s\in[0,T]}\Big|\ell^{(T,z)}(s)-(z/T)s\Big| \geq C'\sqrt{T}\Big)\Big]^k, \tag{5.20}$$

where in the last line we have written $C' = C/2\sqrt{a'+b'}$. Now

$$\mathbb{P}\Big(\sup_{s\in[0,T]}\Big|\ell^{(T,z)}(s)-(z/T)s\Big|\geq C'\sqrt{T}\Big)$$
$$\leq \mathbb{P}\Big(\sup_{s\in[0,T]}|\sqrt{T}\,B^{\sigma}_{s/T}|\geq C'\sqrt{T}/2\Big)+\mathbb{P}\Big(\Delta(T,z)\geq C'\sqrt{T}/2\Big), \tag{5.21}$$

where $\Delta(T,z)$ is as defined in Theorem 3.3. Since $a'+b'\leq 2R$, Lemma 3.6 implies that the first term in (5.21) is equal to

$$2e^{-C^2/8\sigma^2(a'+b')}\leq 2e^{-C^2/16p(1-p)R}.$$

Since $|z-pT|<1$, Lemma 3.5 gives a lower bound for the second term in (5.21) of $e^{-C^2/16p(1-p)R}$ for $T\geq T_0$, where T_0 is some constant depending in particular on C', hence possibly on $a+b$. But note that $C'\sqrt{T}=CN^{\alpha/2}$ is independent of a,b, so the C' dependency in Lemma 3.5 reduces to a dependence on C. Moreover, $T\geq 2rN^{\alpha}$, so as long as $N\geq (T_0/2r)^{1/\alpha}$, we have the bound independent $a,b,\vec{x},\vec{y}$. Our choice of C in (5.13) then gives a lower bound in (5.20) of

$$\big(1-3e^{-C^2/16p(1-p)R}\big)^k\geq 1/2.$$

This proves (5.19).

□

5.3. **Proof of Lemma 4.4.** Throughout this section we assume the same notation as in Lemma 4.4. I.e. we assume that we have fixed $k\in\mathbb{N}$, $p\in(0,1)$, $M_1,M_2>0$, $\ell_{bot}:[\![-t_2,t_2]\!]\to\mathbb{R}\cup\{-\infty\}$, and $\vec{x},\vec{y}\in\mathfrak{W}_{k-1}$ such that $|\Omega_{avoid}(-t_2,t_2,\vec{x},\vec{y},\infty,\ell_{bot})|\geq 1$. We also assume that

(1) $\sup_{s\in[-t_2,t_2]}\big(\ell_{bot}(s)-ps\big)\leq M_2(2t_2)^{1/2}$,
(2) $-pt_2+M_1(2t_2)^{1/2}\geq x_1\geq x_{k-1}\geq \max\big(\ell_{bot}(-t_2),-pt_2-M_1(2t_2)^{1/2}\big)$,
(3) $pt_2+M_1(2t_2)^{1/2}\geq y_1\geq y_{k-1}\geq \max\big(\ell_{bot}(t_2),pt_2-M_1(2t_2)^{1/2}\big)$.

Definition 5.7. Let $\tilde{\Omega}_{avoid}(-t_2,t_2,\vec{x},\vec{y},\infty,\ell_{bot})$ denote the set of elements $(\ell_1,\dots,\ell_{k-1})$ in $\Omega_{avoid}(-t_2,t_2,\vec{x},\vec{y},\infty,-\infty)$ such that $\ell_{k-1}(s)\geq\ell_{bot}(s)$ for all $s\in[\![-t_2,t_2]\!]\setminus[\![-t_1+1,t_1-1]\!]$. We also denote by $\tilde{\mathfrak{L}}=(\tilde{L}_1,\dots,\tilde{L}_{k-1})$ the $[\![1,k-1]\!]$-indexed line ensemble that is uniformly distributed on $\tilde{\Omega}_{avoid}(-t_2,t_2,\vec{x},\vec{y},\infty,\ell_{bot})$.

In simple words, $\tilde{L}$ has the law of $k-1$ independent Bernoulli bridges that have been conditioned on not-crossing each other and also staying above the graph of ℓ_{bot} but only on the intervals $[-t_2,-t_1]$ and $[t_1,t_2]$, meaning that $\tilde{L}_{k-1}$ is allowed to be below ℓ_{bot} on $(-t_1,t_1)$.

Lemma 5.8. *We claim that there exists $N_5\in\mathbb{N}$ such that for $N\geq N_5$*

$$\mathbb{P}_{\tilde{\mathfrak{L}}}\big(Z\big(-t_1,t_1,Q(-t_1),Q(t_1),\ell_{bot}[\![-t_1,t_1]\!]\big)\geq g\big)\geq h, \tag{5.22}$$

where the functions g and h are as in Lemma 4.4.

We will prove Lemma 5.8 in the next section. In the remainder of this section we give the proof of Lemma 4.4.

Lemma 4.4. First, define $S:=[t_1^-,t_1^+]$ and $T:=[t_2^-,t_1^-]\cup[t_1^+,t_2^+]$. Define $\mathbb{P}_{\mathfrak{L}'}$ and $\mathbb{P}_{\tilde{\mathfrak{L}}'}$ as the measures on Bernoulli line ensembles $\mathfrak{L}',\tilde{\mathfrak{L}}':T\to\mathbb{R}$ with $\mathfrak{L}'(t_2^-)=\tilde{\mathfrak{L}}':(t_2^-)=\vec{x}$ and $\mathfrak{L}'(t_2^+)=\tilde{\mathfrak{L}}'(t_2^+)=\vec{y}$, which are the restrictions of the previous measures $\mathbb{P}_{\mathfrak{L}}$ and $\mathbb{P}_{\tilde{\mathfrak{L}}}$ to T. The Radon-Nikodym derivative on line ensembles $\mathfrak{B}:T\to\mathbb{R}^k$ is

$$\frac{d\mathbb{P}_{\mathfrak{L}'}}{d\mathbb{P}_{\tilde{\mathfrak{L}}'}}(\mathfrak{B})=\big(Z'\big)^{-1}Z\left(t_1^-,t_1^+,\mathfrak{B}\left(t_1^-\right),\mathfrak{B}\left(t_1^+\right),\ell_{bot};T\right) \tag{5.23}$$

with $Z' := \mathbb{E}_{\tilde{\mathfrak{L}}'}\left(Z\left(t_1^-, t_1^+, \mathfrak{B}(t_1^-), \mathfrak{B}(t_1^+), \ell_{bot}; T\right)\right)$ Now, note that $Z\left(t_1^-, t_1^+, \mathfrak{B}(t_1^-), \mathfrak{B}(t_1^+), \ell_{bot}; T\right)$ is a function of $(\mathfrak{B}(t_1^{-}), \mathfrak{B}(t_1^+))$, and in fact because of the manner of our restriction implies the equality of the laws of $\left(\mathfrak{B}(t_1^{-}), \mathfrak{B}(t_1^+)\right)$ and $(\tilde{\mathfrak{L}}(t_1^{-}), \tilde{\mathfrak{L}}(t_1^+))$ alongside Lemma 5.8 we have

$$Z' = \mathbb{E}_{\tilde{\mathfrak{L}}'}\left[Z(t_1^-, t_1^+, \mathfrak{B}(t_1^-), \mathfrak{B}(t_1^+), \ell_{bot}; S)\right] = \mathbb{E}_{\tilde{\mathfrak{L}}}\left[Z(t_1^-, t_1^+, \tilde{\mathfrak{L}}(t_1^-), \tilde{\mathfrak{L}}(t_1^+), \ell_{bot}; S)\right] \geq gh$$

which gives us

$$(Z')^{-1} \leq \frac{1}{gh} \tag{5.24}$$

For the same reasons as above, the law of $\mathfrak{B}(t_1^-), \mathfrak{B}(t_1^+)$ under $\mathbb{P}_{\mathfrak{L}'}$ is the same as $\mathfrak{L}(t_1^{-}), \mathfrak{L}(t_1^+)$ under $\mathbb{P}_{\mathfrak{L}}$ Hence,

$$\mathbb{P}_{\mathfrak{L}}\left(Z\left(t_1^-, t_1^+, \mathfrak{L}(t_1^-), \mathfrak{L}(t_1^+), \ell_{bot}; S\right) \leq gh\tilde{\epsilon}\right) = \mathbb{P}_{\mathfrak{L}'}\left(Z\left(t_1^-, t_1^+, \mathfrak{B}(t_1^-), \mathfrak{B}(t_1^+), \ell_{bot}; S\right) \leq gh\tilde{\epsilon}\right) \tag{5.25}$$

Now if we let $E = \{Z(t_1^-, t_1^+, \mathfrak{B}(t_1^-), \mathfrak{B}(t_1^+), \ell_{bot}; S) \leq gh\tilde{\epsilon}\} \subset \Omega$ with Ω defined as the probability space of paths of $\mathfrak{B}$. As a result, we have

$$\mathbb{P}_{\mathfrak{L}'}(E) = \int_\Omega \mathbf{1}_E \cdot d\mathbb{P}_{\mathfrak{L}'}(B) = (Z')^{-1} \int_\Omega \mathbf{1}_E \cdot Z(t_1^-, t_1^+, B(t_1^-), B(t_1^+), \ell_{bot}; S) \cdot d\mathbb{P}_{\tilde{\mathfrak{L}}'}$$

using the Radon-Nikodym derivative from 5.23. By the definition of E, we have that the acceptance probability is bounded by $gh\tilde{\epsilon}$, which gives the inequality

$$\mathbb{P}_{\mathfrak{L}'}(E) \leq (Z')^{-1} \int_\Omega \mathbf{1}_E \cdot gh\tilde{\epsilon} \cdot d\mathbb{P}_{\tilde{\mathfrak{L}}'}(B) \leq \frac{1}{gh} \int_\Omega \mathbf{1}_E \cdot gh\tilde{\epsilon} \cdot d\mathbb{P}_{\tilde{\mathfrak{L}}'}(B) \leq \tilde{\epsilon}$$

with 5.24 providing the middle inequality and the fact that $\mathbf{1}_E \leq 1$ providing the final inequality. This inequality is the desired result.

□

5.4. **Proof of Lemma 5.8.**

Lemma 5.9. *Let $\epsilon > 0$ and $V^{top} > 0$ be given such that $V^{top} > M_2 + 6(k-1)\epsilon$. Suppose further that $\vec{a}, \vec{b} \in \mathfrak{W}_{k-1}$ are such that*

(1) $V^{top}(2t_2)^{1/2} \geq a_1 + pt_1 \geq a_{k-1} + pt_1 \geq (M_2 + 2\epsilon)(2t_2)^{1/2}$;
(2) $V^{top}(2t_2)^{1/2} \geq b_1 - pt_1 \geq b_{k-1} - pt_1 \geq (M_2 + 2\epsilon)(2t_2)^{1/2}$;
(3) $a_i - a_{i+1} \geq 3\epsilon(2t_2)^{1/2}$ and $b_i - b_{i+1} \geq 3\epsilon(2t_2)^{1/2}$ for $i = 1, \ldots, k-2$.

Then we can find $g(\epsilon, V^{top}, M_2) > 0$ and $N_6 \in \mathbb{N}$ such that for all $N \geq N_6$ we have

$$Z\big(-t_1, t_1, \vec{a}, \vec{b}, \ell_{bot}[\![-t_1, t_1]\!]\big) \geq g. \tag{5.26}$$

Proof. Observe by conditions (1) and (2) in the hypothesis, as well as condition (1) in Lemma 4.4, that ℓ_{bot} lies a distance of at least $2\epsilon(2t_2)^{1/2}$ uniformly below the line segment connecting a_{k-1} and b_{k-1}. In view of condition (3), we see that the acceptance event $\{Q_1 \geq \cdots \geq Q_{k-1} \geq \ell_{bot}[\![-t_1, t_1]\!]\}$ occurs as long as each curve Q_i remains within a distance $\epsilon(2t)^{1/2}/2$ of the line segment connecting x_i' and y_i'. Write $z_i = b_i - a_i$ and $T = 2t_1$. As in Theorem 3.3, let $\mathbb{P}_i$ be probability measures supporting random variables $\ell^{(T,z_i)}$ with laws $\mathbb{P}_{Ber}^{0,T,0,z_i}$. Then

$$Z(-t_1, t_1, \vec{a}, \vec{b}, \ell_{bot}[\![-t_1, t_1]\!]) \geq \prod_{i=1}^{k-1}\Big[1 - \mathbb{P}_i\Big(\sup_{s\in[0,T]}\big|\ell^{(T,z_i)}(s) - (z_i/T)s\big| \geq \epsilon\sqrt{T}/2\Big)\Big]. \tag{5.27}$$

Let $B^{\sigma,i}$ be the Brownian bridge with variance $\sigma^2 = p(1-p)$ coupled with $\ell^{(T,z_i)}$ given by Theorem 3.3. Then

$$\mathbb{P}_i\Big(\sup_{s\in[0,T]}\big|\ell^{(T,z_i)}(s) - (z_i/T)s\big| \geq \epsilon\sqrt{T}/2\Big) \leq \mathbb{P}_i\Big(\max_{s\in[0,T]}\big|B^{\sigma,i}_{s/T}\big| \geq \epsilon/4\Big) + \mathbb{P}_i\Big(\Delta(T, z_i) \geq \epsilon\sqrt{T}/4\Big).$$

The first term is equal to $2e^{-\epsilon^2/8p(1-p)}$ by Lemma 3.6, and since $|z_i - pT| \le (V^{top} - M_2)\sqrt{T}$ by conditions (1) and (2), it follows from Lemma 3.5 that the second term is $< e^{-\epsilon^2/8p(1-p)}$ for large enough N depending on $p, \epsilon, V^{top}, M_2$, but independent of i. In view of (5.27), we obtain

$$Z(-t_1, t_1, \vec{a}, \vec{b}, \ell_{bot}[\![-t_1, t_1]\!]) \ge \left(1 - 3e^{-\epsilon^2/8p(1-p)}\right)^{k-1}.$$

□

Let us put $t_{12} = \lfloor \frac{t_1+t_2}{2} \rfloor$. We also let $A = \cdots$ and $B = \cdots$.

Lemma 5.10. *We can find $V_1^t = [M_2 + 6kA + B](2t_2)^{1/2}$ and $[M_2 + 6kA](2t_2)^{1/2}$ such that*

$$\mathbb{P}_{\tilde{\mathfrak{L}}}\left(V_1^t \ge \tilde{L}_1(\pm t_{12}) \mp pt_{12} \ge \tilde{L}_{k-1}(\pm t_{12}) \mp pt_{12} \ge V_1^b\right) \ge h_1. \tag{5.28}$$

Lemma 5.11. *We can find $V^t = \cdots$ and ϵ such that*

$$\mathbb{P}_{\tilde{\mathfrak{L}}}(E) \ge h_2, \tag{5.29}$$

where E is the event that $\vec{a} = \tilde{\mathfrak{L}}(-t_1)$ and $\vec{b} = \tilde{\mathfrak{L}}(t_1)$ satisfy the conditions of Lemma 5.9.

6. Appendix

6.1. **Proof of Lemma 2.2.** Without loss of generality, we use the following compact exhaustion of $\Sigma \times \Lambda$. Define the sets

$$K_n := \Sigma_n \times \Lambda_n := \Sigma_n \times [a_n, b_n]$$

as follows. We take Σ_n to be the set of the n smallest elements of Σ, or all of Σ if $n \ge \#(\Sigma)$. If $a \in \Lambda$, i.e, Λ is closed at the left, then $a_n = a$ for all n, and likewise $b_n = b$ if $b \in \Lambda$. If $a \notin \Lambda$, we let $a_n \in \Lambda$, $a_n > a$ be a sequence decreasing to a, for instance $a_n = a + \frac{1}{n}$ if $a > -\infty$, or $a_n = -n$ if $a_n = -\infty$. If $b \notin \Lambda$, we let $b_n \in \Lambda, b_n \nearrow b$. In any case, we see that the sets $K_1 \subset K_2 \subset \cdots \subset \Sigma \times \Lambda$ are compact, they cover $\Sigma \times \Lambda$, and any compact subset K of $\Sigma \times \Lambda$ is contained in all K_n for sufficiently large n.

We now define, for each n and $f, g \in C(\Sigma \times \Lambda)$,

$$d_n(f,g) := \sup_{(i,t) \in K_n} |f(i,t) - g(i,t)|, \quad d_n'(f,g) := \min\{d_n(f,g), 1\}$$

Clearly each d_n is nonnegative and satisfies the triangle inequality, and it is then easy to see that the same properties hold for d_n'. Furthermore, $d_n' \le 1$, so the function

$$d(f,g) := \sum_{n=1}^{\infty} 2^{-n} d_n'(f,g)$$

in the statement of the lemma is well-defined. We first observe that d is a metric on $C(\Sigma \times \Lambda)$. Indeed, it is nonnegative, and if $f = g$, then each $d_n'(f,g) = 0$, so the sum is 0. Conversely, if $f \ne g$, then since the K_n cover $\Sigma \times \Lambda$, we can choose n large enough so that K_n contains an x with $f(x) \ne g(x)$. Then $d_n'(f,g) \ne 0$, and hence $d(f,g) \ne 0$. The triangle inequality holds for d since it holds for each d_n'.

Now we prove that the topology τ_d on $C(\Sigma \times \Lambda)$ induced by d is the same as the topology of uniform convergence over compacts, which we will denote τ_c. Recall that τ_c is generated by the basis consisting of sets

$$B_K(f, \epsilon) = \left\{ g \in C(\Sigma \times \Lambda) : \sup_{(i,t) \in K} |f(i,t) - g(i,t)| < \epsilon \right\},$$

for $K \subset \Sigma \times \Lambda$ compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$. First, choose $\epsilon > 0$ and $f \in C(\Sigma \times \Lambda)$. Let $g \in B_\epsilon^d(f)$, i.e., $d(f,g) < \epsilon$. We will find a set $A_g \in \tau_c$ such that $g \in A_g \subset B_\epsilon^d(f)$. Let $\delta := d(f,g)$, and choose n large enough so that $\sum_{k>n} 2^{-k} < \frac{\epsilon-\delta}{2}$. Define $A_g := B_{K_n}(g, \frac{\epsilon-\delta}{n})$, and suppose $h \in A_g$. Then since $K_m \subseteq K_n$ for $m \leq n$, we have

$$d(f,h) \leq d(f,g) + d(g,h) \leq \delta + \sum_{k=1}^{n} 2^{-k} d_n(g,h) + \sum_{k>n} 2^{-k} \leq \delta + \frac{\epsilon-\delta}{2} + \frac{\epsilon-\delta}{2} = \epsilon.$$

Therefore $g \in A_g \subset B_\epsilon^d(f)$. It follows that $B_\epsilon^d(f) \in \tau_c$. Indeed, we can write

$$B_\epsilon^d(f) = \bigcup_{g \in B_\epsilon^d(f)} A_g,$$

a union of elements of τ_c. This proves that $\tau_d \subseteq \tau_c$.

To prove the converse, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$. Choose n so that $K \subset K_n$, and let $g \in B_K(f, \epsilon)$ and $\delta := \sup_{x \in K} |f(x) - g(x)|$. If $d(g,h) < 2^{-n}(\epsilon - \delta)$, then $d_n'(g,h) \leq 2^n d(g,h) < \epsilon - \delta$, hence $d_n(g,h) < \epsilon - \delta$. It follows that

$$\sup_{x \in K} |f(x) - h(x)| \leq \delta + \sup_{x \in K} |g(x) - h(x)| \leq \delta + d_n(g,h) \leq \delta + \epsilon - \delta = \epsilon.$$

Thus $g \in B^d_{2^{-n}(\epsilon-\delta)}(f) \subset B_K(f, \epsilon)$. Therefore $\tau_c \subseteq \tau_d$, and we conclude that $\tau_d = \tau_c$.

Next, we show that $(C(\Sigma \times \Lambda), d)$ is a complete metric space. Let $(f_n)_{n \geq 1}$ be Cauchy with respect to d. Then we claim that (f_n) must be Cauchy with respect to d_n', on each K_n. Indeed, $d(f_\ell, f_m) \geq 2^{-n} d_n'(f_\ell, f_m)$, so if (f_n) were not Cauchy with respect to d_n', it would not be Cauchy with respect to d either. Thus (f_n) is uniformly Cauchy on each K_n, and hence converges uniformly to a limit f^{K_n} on each K_n. Since the limit must be unique at each point of $\Sigma \times \Lambda$, we have $f^{K_n}(x) = f^{K_m}(x)$ if $x \in K_n \cap K_m$. Since $\bigcup K_n = \Sigma \times \Lambda$, we obtain a well-defined function f on all of $\Sigma \times \Lambda$ given by $f(x) = \lim_{n \to \infty} f^{K_n}(x)$. Given any compact $K \subset \Sigma \times \Lambda$, if n is large enough so that $K \subset K_n$, then because $f_n \to f^{K_n} = f|_{K_n}$ uniformly on K_n, we have $f_n \to f^{K_n}|_K = f|_K$ uniformly on K. That is, for any $K \subset \Sigma \times \Lambda$ compact and $\epsilon > 0$, we have $f_n \in B_K(f, \epsilon)$ for all sufficiently large n. Therefore (f_n) converges to f in τ_c, and equivalently in the metric d.

Lastly, we prove separability, c.f. [1, Example 1.3]. For each pair of positive integers n, k, let $D_{n,k}$ be the subcollection of $C(\Sigma \times \Lambda)$ consisting of polygonal functions that are piecewise linear on $\{j\} \times I_{n,k,i}$ for each $j \in \Sigma_n$ and each subinterval

$$I_{n,k,i} := \left[a_n + \tfrac{i-1}{k}(b_n - a_n), a_n + \tfrac{i}{k}(b_n - a_n)\right], \quad 1 \leq i \leq k,$$

taking rational values at the endpoints of these subintervals, and extended linearly to all of $\Lambda = [a, b]$. Then $D := \bigcup_{n,k} D_{n,k}$ is countable, and we claim that it is dense in τ_c. To see this, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and choose n so that $K \subset K_n$. Since f is uniformly continuous on K_n, we can choose k large enough so that for $0 \leq i \leq k$, if $t \in I_{n,k,i}$, then $|f(j,t) - f(j, a_n + \frac{i}{k}(b_n - a_n))| < \epsilon/2$ for all $j \in \Sigma_n$. We then choose $g \in \bigcup_k D_{n,k}$ with $|g(j, a_n + \frac{i}{k}(b_n - a_n)) - f(j, a_n + \frac{i}{k}(b_n - a_n))| < \epsilon/2$. Then $f(j,t)$ is within ϵ of both $g(j, a_n + \frac{i-1}{k}(b_n - a_n))$ and $g(j, a_n + \frac{i}{k}(b_n - a_n))$. Since $g(j,t)$ lies between these two values, $f(j,t)$ is with ϵ of $g(j,t)$ as well. In summary,

$$\sup_{(j,t) \in K} |f(j,t) - g(j,t)| \leq \sup_{(j,t) \in K_n} |f(j,t) - g(j,t)| < \epsilon,$$

so $g \in B_K(f, \epsilon)$. This proves that D is a countable dense subset of $C(\Sigma \times \Lambda)$.

6.2. **Proof of Lemma 2.4.** We first state and prove two auxiliary results regarding the topology of uniform convergence over compacts. The first lemma states that a $C(\Sigma \times \Lambda)$-valued random variable is determined by its finite-dimensional distributions.

Lemma 6.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X, Y random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $C(\Sigma \times \Lambda)$. Suppose that for all $n \in \mathbb{N}$, $(i_1, t_1), \dots, (i_n, t_n) \in \Sigma \times \Lambda$, and $x_1, \dots, x_n \in \mathbb{R}$, we have that*

$$\mathbb{P}\left(X(i_1, t_1) \le x_1, \dots, X(i_n, t_n) \le x_n\right) = \mathbb{P}\left(Y(i_1, t_1) \le x_1, \dots, Y(i_n, t_n) \le x_n\right).$$

Then X and Y are equal in distribution, i.e., $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$ for all $A \in \mathcal{C}_\Sigma$.

Proof. Let $\mathcal{S}$ denote the collection of cylinder sets

$$\{f \in C(\Sigma \times \Lambda) : f(i_1, t_1) \in A_1, \dots, f(i_n, t_n) \in A_n\}, \quad A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}).$$

Since the Borel sets in $\mathbb{R}$ are generated by intervals of the form $(-\infty, x]$, the hypothesis is equivalent to requiring that the probability measures $\mathbb{P} \circ X^{-1}$ and $\mathbb{P} \circ Y^{-1}$ agree on $\mathcal{S}$. We will show that $\mathcal{S}$ is a π-system generating $\mathcal{C}_\Sigma$, which will imply by [] that $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ Y^{-1}$ on all of $\mathcal{C}_\Sigma$. Observe that the intersection of two elements of $\mathcal{S}$ is clearly another element of $\mathcal{S}$, so $\mathcal{S}$ is a π-system.

We claim that the set $\{f \in C(\Sigma \times \Lambda) : f(i_k, t_k) \le x_k\}$ is closed in the topology of compact convergence. If $f_n(i_k, t_k) \le x_k$ for all n and $f_n \to f$ in the topology of compact convergence, then by taking limits on a compact set containing (i_k, t_k), we find $f(i_k, t_k) \le x_k$ as well. This proves that $\sigma(\mathcal{S}) \subseteq \mathcal{C}_\Sigma$.

To prove the opposite inclusion, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and let H be a countable dense subset of K. (Recall that every compact metric space is separable, and K is homeomorphic to a product of finitely many compact sets in $\mathbb{R}$, which are metrizable. So K is separable.) We claim that

$$B_K(f, \epsilon) = \bigcup_{n=1}^{\infty} \bigcap_{(i,t) \in H} \{g \in C(\Sigma \times \Lambda) : g(i,t) \in (f(i,t) - (1 - 2^{-n})\epsilon, f(i,t) + (1 - 2^{n})\epsilon)\}.$$

Indeed, if $g \in B_K(f, \epsilon)$, i.e., $\sup_{(i,t) \in K} |g(i,t) - f(i,t)| < \epsilon$. Then since $1 - 2^{-n} \nearrow 1$, we can choose n large enough so that

$$|g(i,t) - f(i,t)| < (1 - 2^{-n})\epsilon$$

for all $(i,t) \in K$ (in particular with $(i,t) \in H$). Conversely, suppose g is in the set on the right. Then since g is continuous and H is dense in K, we find that for some $n \ge 1$,

$$|g(i,t) - f(i,t)| \le (1 - 2^{-n})\epsilon < \epsilon$$

for all $(i,t) \in K$. Hence $g \in B_K(f, \epsilon)$. This proves the claim. Since H is countable, $B_K(f, \epsilon)$ is formed from countably many unions and intersections of sets in $\mathcal{S}$, thus $B_K(f, \epsilon) \in \sigma(\mathcal{S})$.

Now by Lemma 2.2, the topology generated by the basis $\mathcal{A} = \{B_K(f, \epsilon)\}$ is separable and metrizable. The balls of rational radii centered at points of a countable dense subset then give a countable basis $\mathcal{B}$ for the same topology. We claim that this implies that every open set is a *countable* union of sets $B_K(f, \epsilon)$. To see this, let $B \in \mathcal{B}$, and write $B = \bigcup_{\alpha \in I} A_\alpha$, for sets $A_\alpha \in \mathcal{A}$. Then for each $x \in B$, pick $\alpha_x \in I$ such that $x \in A_{\alpha_x}$. Since $\mathcal{B}$ is a basis, there is a set $B_x \in \mathcal{B}$ with $x \in B_x \subseteq A_{\alpha_x}$. Then $B = \bigcup_{x \in B} A_{\alpha_x}$. Note that if $y \in B_y \subseteq A_{\alpha_y}$ and $B_y = B_x$, then in fact $y \in A_{\alpha_x}$, so we can remove A_{α_y} from the union. In other words, we can choose the A_{α_x} so that each corresponds to exactly one B_x. But there are only countably many distinct sets B_x, so we see that B is a countable union of elements of $\mathcal{A}$. Since every open set can be written as a countable union of elements of B, this proves the claim. Since $\mathcal{A} \subseteq \sigma(\mathcal{S})$ by the above, it follows that every open set is in $\sigma(\mathcal{S})$, and consequently so is every Borel set, i.e., $\mathcal{C}_\Sigma \subseteq \sigma(\mathcal{S})$. This completes the proof ∎

The next result shows that a sequence of line ensembles is tight if and only if all individual curves form tight sequences.

Lemma 6.2. *Consider the projection maps $\pi_i : C(\Sigma \times \Lambda) \to C(\Lambda)$, $i \in \Sigma$, given by $\pi_i(F)(x) = F(i,x)$ for $x \in \Lambda$. Then the π_i are continuous. Suppose that $(\mathcal{L}^n)_{n\geq 1}$ is a sequence of Σ-indexed line ensembles on Λ. Then $(\mathcal{L}^n)$ is tight if and only if for each $i \in \Sigma$ the sequence $(X_i^n)_{n\geq 1}$ is tight.*

Proof. Since $C(X)$ with the topology of uniform convergence on compacts is metrizable by Lemma 2.2, to show that the π_i are continuous, it suffices to show that if $f_n \to f$ in $C(\Sigma \times \Lambda)$, then $\pi_i(f_n) \to \pi_i(f)$ in $C(\Lambda)$. But this is immediate, since if $f_n \to f$ uniformly on compact subsets of $\Sigma \times \Lambda$, then in particular $f_n(i, \cdot) \to f(i, \cdot)$ uniformly on compact subsets of Λ. Now write $X_i^n := \pi_i(\mathcal{L}^n)$. If A is a Borel set in $C(\Lambda)$, then $(X_i^n)^{-1}(A) = (\mathcal{L}^n)^{-1}(\pi_i^{-1}(A))$. Note $\pi_i^{-1}(A) \in \mathcal{C}_\Sigma$ since π_i is continuous, so it follows that $(X_i^n)^{-1}(A) \in \mathcal{F}$. Thus X_i^n is a $C(\Lambda)$-valued random variable.

Suppose the sequence $(\mathcal{L}^n)$ is tight. By Lemma 2.2, $C(\Sigma\times\Lambda)$ is a Polish space, so it follows from Prohorov's theorem [1, Theorem 5.1], that $(\mathcal{L}^n)$ is relatively compact. That is, every subsequence $(\mathcal{L}^{n_k})$ has a further subsequence $(\mathcal{L}^{n_{k_\ell}})$ converging weakly to some $\mathcal{L}$. Then for each $i \in \Sigma$, since π_i is continuous, the subsequence $(\pi_i(\mathcal{L}^{n_{k_\ell}}))$ of $(\pi_i(\mathcal{L}^{n_k}))$ converges weakly to $\pi_i(\mathcal{L})$ by the continuous mapping theorem. Thus every subsequence of $(\pi_i(\mathcal{L}^n))$ has a convergent subsequence. Since $C(\Lambda)$ is a Polish space by the same argument as in the proof of Lemma 2.2, Prohorov's theorem implies that each $(\pi_i(\mathcal{L}^n))$ is tight.

Conversely, suppose $(\pi_i(\mathcal{L}^n))$ is tight for all $i \in \Sigma$. Then for each i, every subsequence $(\pi_i(\mathcal{L}^{n_k}))$ has a further subsequence $(\pi_i(\mathcal{L}^{n_{k_\ell}}))$ converging weakly to some $\mathcal{L}_i$. By diagonalizing the subsequences (n_{k_ℓ}), we obtain a sequence that works for all i, so that $\pi_i(\mathcal{L}^{n_{k_\ell}}) \implies \mathcal{L}_i$ for all i simultaneously. Note that $C(\Sigma \times \Lambda)$ is homeomorphic to $\prod_{i\in\Sigma} C(\Lambda)$ with the product topology, with $f \in C(\Sigma\times\Lambda)$ identified with $(\pi_i(f))_{i\in\Sigma}$. It is not hard to see this by observing that the compact subsets K of $\Sigma\times\Lambda$ are of the form $S \times I$, for S finite and I compact. Thus the homeomorphism identifies the basis elements $B_K(f, \epsilon)$ in $C(\Sigma \times \Lambda)$ with products of open sets U_i in $C(\Lambda)$, such that if $i \notin S$ then simply $U_i = C(\Lambda)$; since S is finite, these products $\prod_i U_i$ are basis elements of the product topology.

Consequently, we can identify the sequence of random variables $\mathcal{L} = (\mathcal{L}_i)_{i\in\Sigma}$ with an element of $C(\Sigma \times \Lambda)$. We argue that $\mathcal{L}^{n_{k_\ell}} \implies \mathcal{L}$. Let U be a basis element in the product topology, i.e., $U = \prod_{i\in\Sigma} U_i$, with each U_i open in $C(\Lambda)$ and all but finitely many $U_i = C(\Lambda)$. Without loss of generality, assume these finitely many $U_i \neq C(\Lambda)$ are $U_1, \dots, U_m$. Then

$$\mathbb{P}(X \in U) = \mathbb{P}(\pi_1(X) \in U_1, \dots, \pi_m(X) \in U_m) = \prod_{i=1}^m \mathbb{P}(\pi_i(X) \in U_i).$$

Therefore, since $\pi_i(\mathcal{L}^{n_{k_\ell}}) \implies \mathcal{L}_i$ for each i, we have by the portmanteau theorem, [1, Theorem 2.1], that

$$\liminf_{\ell\to\infty} \mathbb{P}(\mathcal{L}^{n_{k_\ell}} \in U) \geq \prod_{i=1}^m \liminf_{\ell\to\infty} \mathbb{P}(\pi_i(\mathcal{L}^{n_{k_\ell}}) \in U_i) \geq \prod_{i=1}^m \mathbb{P}(\mathcal{L}_i \in U_i) = \mathbb{P}(\mathcal{L} \in U).$$

Now by the same argument as in the proof of Lemma 6.1, every open set in $C(\Sigma\times\Lambda)$ is a countable union of sets of the form of U. Therefore by countable additivity, the inequalities above hold if U is replaced by an arbitrary open set. Thus again by the portmanteau theorem, $\mathcal{L}^{n_{k_\ell}} \implies \mathcal{L}$ as desired. Hence $(\mathcal{L}^n)$ is relatively compact, and it follows from Prohorov's theorem once again that $(\mathcal{L}^n)$ is tight.

□

We are now ready to prove Lemma 2.4.

Proof. By [1, Theorem 7.3], a sequence (P_n) of probability measures on $C[0,1]$ with the uniform topology is tight if and only if the following conditions hold:

$$\lim_{a\to\infty}\limsup_{n\to\infty} P_n(|x(0)|\geq a)=0,$$

$$\lim_{\delta\to 0}\limsup_{n\to\infty} P_n\Big(\sup_{|s-t|\leq\delta}|x(s)-x(t)|\geq\epsilon\Big)=0 \quad \text{for all } \epsilon>0.$$

By replacing $[0,1]$ with $[a_k,b_k]$ and 0 with a_0, we see that the hypotheses in the lemma imply that the restricted sequences $(\mathcal{L}_i^n|_{[a_k,b_k]})_n$ are tight, hence relatively compact in the uniform topology on $C[a_k,b_k]$ by Prohorov's theorem, for every $i\in\Sigma$ and $k\geq 1$. Thus every subsequence $(\mathcal{L}_i^{n_m}|_{[a_k,b_k]})_m$ has a further subsequence $(\mathcal{L}_i^{n_{m_\ell}}|_{[a_k,b_k]})_\ell$ converging weakly to some $\mathcal{L}_i^{[a_k,b_k]}$. We claim that we can patch these $\mathcal{L}_i^{[a_k,b_k]}$ together to obtain a well-defined random variable $\mathcal{L}_i$ on all of $C(\Lambda)$, such that $\mathcal{L}_i|_{[a_k,b_k]}=\mathcal{L}_i^{[a_k,b_k]}$ for every k. By Lemma 6.1, it suffices to construct the finite-dimensional distributions of this $\mathcal{L}_i$. Given any finite collection $A=\{x_1,\dots,x_j\}$ of points in Λ, if we take k large enough so that $A\subset[a_k,b_k]$, then the corresponding finite-dimensional distribution $\{\mathcal{L}_i(x_1)\in B_1,\dots,\mathcal{L}_i(x_j)\in B_j\}$ is determined by that of $\mathcal{L}_i^{[a_k,b_k]}$. Moreover, uniqueness of weak limits in distribution implies that this finite-dimensional distribution agrees with that of $\mathcal{L}_i^{[a_\ell,b_\ell]}$ whenever $A\subset[a_\ell,b_\ell]$. Thus we have specified well-defined finite-dimensional distributions for $\mathcal{L}_i$, which determines $\mathcal{L}_i$ on all of $C(\Lambda)$ by Lemma 6.1. By construction, the restriction of $\mathcal{L}_i$ to any $[a_k,b_k]$ is equal to $\mathcal{L}_i^{[a_k,b_k]}$ in distribution.

In particular, we see that $\mathcal{L}_i^{n_{m_\ell}}|_{[a_k,b_k]} \implies \mathcal{L}_i|_{[a_k,b_k]}$ in the uniform topology on $C[a_k,b_k]$, for every k. If $K\subset\Lambda$ is any compact set, then by taking k large enough so that $K\subset[a_k,b_k]$, we also find $\mathcal{L}_i^{n_{m_\ell}}|_K \implies \mathcal{L}_i|_K$ in the uniform topology on $C(K)$. Let $B_K(f,\epsilon)$ be a basis element in $C(\Lambda)$, and let $B_\epsilon(f|_K)$ denote the corresponding ball in the uniform topology on $C(K)$. Then

$$\begin{aligned}\liminf_{\ell\to\infty}\mathbb{P}(\mathcal{L}_i^{n_{m_\ell}}\in B_K(f,\epsilon)) &= \liminf_{\ell\to\infty}\mathbb{P}(\mathcal{L}_i^{n_{m_\ell}}|_K\in B_\epsilon(f|_K))\\ &\geq \mathbb{P}(\mathcal{L}_i|_K\in B_\epsilon(f|_K))=\mathbb{P}(\mathcal{L}_i\in B_K(f,\epsilon)).\end{aligned}$$

The inequality follows from weak convergence in the uniform topology on $C(K)$ and the portmanteau theorem. Since every open set in $C(\Lambda)$ can be written as a countable union of sets $B_K(f,\epsilon)$ (by the same argument as in the proof of Lemma 6.1), it follows from countable additivity that

$$\liminf_{\ell\to\infty}\mathbb{P}(\mathcal{L}_i^{n_{m_\ell}}\in U)\geq\mathbb{P}(\mathcal{L}_i\in U)$$

for any U open in $C(\Lambda)$. Therefore $(\mathcal{L}_i^{n_{m_\ell}})_\ell$ converges weakly to $\mathcal{L}_i$, proving that $(\mathcal{L}_i^n)_n$ is relatively compact, hence tight by Prohorov's theorem, for every $i\in\Sigma$. We conclude that $(\mathcal{L}^n)$ is tight by Lemma 6.2.

□

6.3. **Proof of Lemmas 3.1 and 3.2.** We will prove the following lemma, of which the two lemmas are immediate consequences. In particular, Lemma 3.1 is the special case when $g^b=g^t$, and Lemma 3.2 is the case when $\vec{x}=\vec{x}\,'$ and $\vec{y}=\vec{y}\,'$. We argue in analogy to Lemma 5.6 in Dimitrov-Matestki.

Lemma 6.3. *Fix $k\in\mathbb{N}$, $T_0,T_1\in\mathbb{Z}$ with $T_0<T_1$, and two functions $g^b,g^t:[\![T_0,T_1]\!]\to[-\infty,\infty)$ with $g^b\leq g^t$. Also fix $\vec{x},\vec{y},\vec{x}\,',\vec{y}\,'\in\mathfrak{W}_k$, such that $g^b(T_0)\leq x_i$, $g^b(T_1)\leq y_i$, $g^t(T_0)\leq x_i'$, $g^t(T_1)\leq y_i'$, and $x_i\leq x_i'$, $y_i\leq y_i'$ for $1\leq i\leq k$. Assume that $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},\infty,g^b)$ and $\Omega_{avoid}(T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t)$ are both non-empty. Then there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$, which supports two $[\![1,k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0,T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\big($resp. $\mathfrak{L}^b\big)$ under $\mathbb{P}$ is given by $\mathbb{P}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}_{avoid,Ber}$ $\big($resp. $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber}\big)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}_i^t(r)\geq\mathfrak{L}_i^b(r)$ for all $i=1,\dots,k$ and $r\in[\![T_0,T_1]\!]$.*

Proof. We split the proof into two steps.

Step 1. We first aim to construct a Markov chain $(X^n, Y^n)_{n\geq 0}$, with $X^n \in \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$, $Y^n \in \Omega_{avoid}(T_0, T_1, \vec{x}\,', \vec{y}\,', \infty, g^t)$, with initial distribution given by the maximal paths

$$X_1^0(t) = (x_1 + t - T_0) \wedge y_1, \qquad Y_1^0(t) = (x_1' + t - T_0) \wedge y_1'$$
$$X_k^0(t) = (x_k + t - T_0) \wedge y_k \wedge X_{k-1}^0(t), \qquad Y_k^0(t) = (x_k' + t - T_0) \wedge y_k' \wedge Y_{k-1}^0(t).$$

for $t \in [\![T_0, T_1]\!]$. We want this chain to have the following properties:

(1) $(X^n)_{n\geq 0}$ and $(Y^n)_{n\geq 0}$ are both Markov in their own filtrations,
(2) (X^n) is irreducible and has as an invariant distribution the uniform measure $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$,
(3) (Y^n) is irreducible and has invariant distribution $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g^t}$,
(4) $X_i^n \leq Y_i^n$ on $[\![T_0, T_1]\!]$ for all $n \geq 0$ and $1 \leq i \leq k$.

This will allow us to conclude convergence of X^n and Y^n to these two uniform measures.

We specify the dynamics of (X^n, Y^n) as follows. At time n, we uniformly sample a segment $\{t\} \times [z, z+1]$, with $t \in [\![T_0, T_1]\!]$ and $z \in [\![x_k, y_1' - 1]\!]$. We also flip a fair coin, with $\mathbb{P}(\text{heads}) = \mathbb{P}(\text{tails}) = 1/2$. We update X^n and Y^n using the following procedure. For all points $s \neq t$, we set $X^{n+1}(s) = X^n(s)$. If $T_0 < t < T_1$ and $X_i^n(t-1) = z$ and $X_i^n(t+1) = z+1$ (note that this implies $X_i^n(t) \in \{z, z+1\}$), then we set

$$X_i^{n+1}(t) = \begin{cases} z+1, & \text{if heads,} \\ z, & \text{if tails,} \end{cases}$$

assuming that this move does not cause $X_i^{n+1}(t)$ to fall below $g^b(t)$. In all other cases, we leave $X_i^{n+1}(t) = X_i^n(t)$. We update Y^n using the same rule, with g^t in place of g^b. [Maybe add a figure here.] We will verify below in the proof of (4) that X^n and Y^n are in fact non-intersecting for all n, but we assume this for now.

It is easy to see that (X^n, Y^n) is a Markov chain, since at each time n, the value of (X^{n+1}, Y^{n+1}) depends only on the current state (X^n, Y^n), and not on the time n or any of the states prior to time n. Moreover, the value of X^{n+1} depends only on the state X^n, not on Y^n, so (X^n) is a Markov chain in its own filtration. The same applies to (Y^n). This proves the property (1) above.

We now argue that (X^n) is each irreducible. Observe that the initial distribution X^0 is by construction maximal, in the sense that for any $Z \in \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$, we have $Z_i \leq X_i^0$ for all i. Thus to reach Z from the initial state X_0, we only need to move the paths downward, and there is no danger of the paths X_i crossing when we do so. We start by ensuring $X_k^n = Z_k$. We successively sample segments which touch Z_k at each point in $[\![T_0, T_1]\!]$ where Z_k differs from X_k, and choose the appropriate coin flips until the two agree on all of $[\![a, b]\!]$. We repeat this procedure for X_i^n and Z^i, with i descending. Since each of these samples and flips has positive probability, and this process terminates in finitely many steps, the probability of transitioning from X^n to Z after some number of steps is positive. The same reasoning applies to show that (Y^n) is irreducible.

To see that the uniform measure $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$ on $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$ is invariant for (X^n), fix any line ensemble $\omega \in \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$. For simplicity, write μ for the uniform measure and $N = |\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)|$ for the (finite) number of allowable ensembles. Then for all ensembles $\tau \in \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$, $\mu(\tau) = 1/N$. Hence

$$\sum_{\tau} \mu(\tau)\mathbb{P}(X^{n+1} = \omega \mid X^n = \tau) = \frac{1}{N}\sum_{\tau} \mathbb{P}(X^{n+1} = \omega \mid X^n = \tau)$$

$$= \frac{1}{N}\sum_{\tau} \mathbb{P}(X^{n+1} = \tau \mid X^n = \omega) = \frac{1}{N} \cdot 1 = \mu(\omega).$$

The second equality is clear if $\tau = \omega$. Otherwise, note that $\mathbb{P}(X_{n+1} = \omega \mid X_n = \tau) \neq 0$ if and only if τ and ω differ only in one indexed path (say the ith) at one point t, where $|\tau_i(t) - \omega_i(t)| = 1$, and this condition is also equivalent to $\mathbb{P}(X^{n+1} = \tau \mid X^n = \omega) \neq 0$. If $X^n = \tau$, there is exactly one choice of segment $\{t\} \times [z, z+1]$ and one coin flip which will ensure $X_i^{n+1}(t) = \omega(t)$, i.e., $X^{n+1} = \omega$. Conversely, if $X^n = \omega$, there is one segment and one coin flip which will ensure $X^{n+1} = \tau$. Since the segments are sampled uniformly and the coin flips are fair, these two conditional probabilities are in fact equal. This proves (2), and an analogous argument proves (3).

Lastly, we argue that $X_i^n \leq Y_i^n$ for all $n \geq 0$ and $1 \leq i \leq k$. The same argument will prove that $X_{i+1}^n \leq X_i^n$ for all n, i, so that X^n is in fact non-intersecting for all n, and likewise for Y^n. This is of course true at $n = 0$. Suppose it holds at some $n \geq 0$. Then since the update rule can only change the values of X_i and Y_i at a single point t, it suffices to look at the possible updates to the ith curve at a single point $t \in [\![T_0, T_1]\!]$. Notice that the update can only change values by at most 1, and if $Y_i^n(t) - X_i^n(t) = 1$, then the only way the ordering could be violated is if Y_i were lowered and X_i were raised at the next update. But this is impossible, since a coin flip of heads can only raise or leave fixed both curves, and tails can only lower or leave fixed both curves. Thus it suffices to assume $X_i^n(t) = Y_i^n(t)$.

There are two cases to consider that violate the ordering of $X_i^{n+1}(t)$ and $Y_i^{n+1}(t)$. Either (i) $X_i(t)$ is raised but $Y_i(t)$ is left fixed, or (ii) $Y_i(t)$ is lowered yet $X_i(t)$ is left fixed. These can only occur if the curves exhibit one of two specific shapes on $[\![t-1, t+1]\!]$. For $X_i(t)$ to be raised, we must have $X_i^n(t-1) = X_i^n(t) = X_i^n(t+1) - 1$, and for $Y_i(t)$ to be lowered, we must have $Y_i^n(t-1) - 1 = Y_i^n(t) = Y_i^n(t+1)$. From the assumptions that $X_i^n(t) = Y_i^n(t)$, and $X_i^n \leq Y_i^n$, we observe that both of these requirements force the other curve to exhibit the same shape on $[\![t-1, t+1]\!]$. Then the update rule will be the same for both curves, proving that both (i) and (ii) are impossible.

Step 2. It follows from (2) and (3) and [13, Theorem 1.8.3] that $(X^n)_{n\geq 0}$ and $(Y^n)_{n\geq 0}$ converge weakly to $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber}$ and $\mathbb{P}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g^t}_{avoid,Ber}$ respectively. In particular, (X^n) and (Y^n) are tight, so $(X^n, Y^n)_{n\geq 0}$ is tight as well. By Prohorov's theorem, it follows that (X^n, Y^n) is relatively compact. Let (n_m) be a sequence such that (X^{n_m}, Y^{n_m}) converges weakly. Then by the Skorohod representation theorem [1, Theorem 6.7], it follows that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting $C([\![1,k]\!] \times [\![T_0, T_1]\!])$-valued random variables $\mathfrak{X}^n$, $\mathfrak{Y}^n$ and $\mathfrak{X}, \mathfrak{Y}$ such that

(1) The law of $(\mathfrak{X}^n, \mathfrak{Y}^n)$ under $\mathbb{P}$ is the same as that of (X^n, Y^n),
(2) $\mathfrak{X}^n(\omega) \longrightarrow \mathfrak{X}(\omega)$ for all $\omega \in \Omega$,
(3) $\mathfrak{Y}^n(\omega) \longrightarrow \mathfrak{Y}(\omega)$ for all $\omega \in \Omega$.

In particular, (1) implies that $\mathfrak{X}^{n_m}$ has the same law as X^{n_m}, which converges weakly to $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber}$. It follows from (2) and the uniqueness of limits that $\mathfrak{X}$ has law $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber}$. Similarly, $\mathfrak{Y}$ has law $\mathbb{P}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g^t}_{avoid,Ber}$. Moreover, condition (4) in Step 1 implies that $\mathfrak{X}_i^n \leq \mathfrak{Y}_i^n$, $\mathbb{P}$-a.s., so $\mathfrak{X}_i \leq \mathfrak{Y}_i$ for $1 \leq i \leq k$, $\mathbb{P}$-a.s. Thus we can take $\mathfrak{L}^b := \mathfrak{X}$ and $\mathfrak{L}^t := \mathfrak{Y}$.

□

6.4. **Weak Convergence of Scaled avoiding Bernoulli Line Ensemble.** We consider there $\{1, \dots, k\}$-indexed line ensembles with distribution given by $\mathbb{P}^{0,T,\vec{x},\vec{y},\infty,-\infty}_{avoid,Ber}$ in the sense of Definition

2.15. Recall that this is just the law of k independent Bernoulli random walks that have been conditioned to start from $(x_1, \dots, x_k)$ at time 0 and and at $(y_1, \cdots, y_k)$ at time T and are always ordered. Here $x_1 \geq x_2 \geq \cdots \geq x_k$, $y_1 \geq y_2 \geq \cdots \geq y_k$ and $x_i, y_i \in \mathbb{Z}$ satisfy $T \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$. We will drop the infinities and simply write $\mathbb{P}^{0,T,\vec{x},\vec{y}}_{avoid,Ber}$ for the measure.

Fix $p, t \in (0,1)$, $k \in \mathbb{N}$, a_i, $b_i \in \mathbb{R}$ for $i = 1, \dots, k$ such that $a_1 \geq \cdots \geq a_k$ and $b_1 \geq \cdots \geq b_k$. Suppose that $\vec{x}^T = (x^T_1, \cdots, x^T_k)$ and $\vec{y}^T = (y^T_1, \cdots, y^T_k)$ are two sequence of k-dimensional vectors with integer entries such that

$$\lim_{T\to\infty} \frac{x^T_i}{\sqrt{T}} = a_i \text{ and } \lim_{T\to\infty} \frac{y^T_i - pT}{\sqrt{T}} = b_i$$

for $i = 1, \dots, k$. Define the sequence of random k-dimensional vectors Z^T by

$$Z^T = \Big(\frac{L_1(tT) - ptT}{\sqrt{T}}, \cdots, \frac{L_k(tT) - ptT}{\sqrt{T}}\Big),$$

where $(L_1, \cdots, L_k)$ is $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$-distributed. In this section, we will prove that the random vector Z^T weakly converges to some continuous distribution and give the corresponding density. The followings are two main results of this section, which give the limiting distribution when $(a_1, \cdots, a_k)$ and $(b_1, \cdots, b_k)$ contain distinct values and when they contain collided values.

Proposition 6.4. *When $a_1 > \cdots > a_k$ and $b_1 > \cdots > b_k$ are all distinct, the random vector Z^T converges weakly to a continuous distribution with the density*

$$\rho(z_1, \cdots, z_k) = \frac{1}{Z} \cdot \det\left[e^{c_1(t,p)a_i z_j}\right]^k_{i,j=1} \det\left[e^{c_2(t,p)b_i z_j}\right]^k_{i,j=1} \prod_{i=1}^k e^{-c_3(t,p)z_i^2} \mathbb{1}_{\{z_1 > \cdots > z_k\}}$$

where c_1, c_2, c_3 are constants depending on p, t:

$$c_1(p,t) = \frac{1}{p(p+1)t}, \quad c_2(p,t) = \frac{1}{p(p+1)(1-t)}, \quad c_3(p,t) = \frac{1}{2p(p+1)t(1-t)}$$

and Z is a constant depending on $p, t, \vec{a}, \vec{b}$ such that $\rho(z_1, \cdots, z_k)$ integrates to 1 over $\mathbb{R}^k$:

$$Z = (2\pi)^{\frac{k}{2}} (p(p+1)t(1-t))^{\frac{k}{2}} \cdot e^{c_1(t,p)\sum_{i=1}^k a_i^2} \cdot e^{c_2(t,p)\sum_{i=1}^k b_i^2} \det\left(e^{-\frac{1}{2p(p+1)}(b_i - a_j)^2}\right)^k_{i,j=1}$$

Proposition 6.5. *When $a_1 \geq \cdots \geq a_k$ and $b_1 \geq \cdots \geq b_k$ contain collided values, we suppose*

$$\vec{a}_0 = (a_1, \cdots, a_k) = (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p})$$
$$\vec{b}_0 = (b_1, \cdots, b_k) = (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q})$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_p$, $\beta_1 > \beta_2 > \cdots > \beta_q$ and $\sum_{i=1}^p m_i = \sum_{i=1}^q n_i = k$. Then, the random vector Z^T converges weakly to a continuous distribution with the density

$$\rho_{\vec{a}_0,\vec{b}_0}(z_1, \cdots, z_k) = \frac{1}{Z} \cdot \varphi(\vec{a}_0, \vec{z}, \vec{m})\psi(\vec{b}_0, \vec{z}, \vec{n}) \prod_{i=1}^k e^{-c_3(t,p)z_i^2} \mathbb{1}_{\{z_1 > \cdots > z_k\}}$$

where $\vec{m} = (m_1, \cdots, m_k)$, $\vec{n} = (n_1, \cdots, n_k)$, c_1, c_2, c_3 are constants depending on p, t as given in Proposition 6.4 , Z is a constant depending on $p, t, \vec{a}, \vec{b}$ such that $\rho_{\vec{a}_0,\vec{b}_0}(z_1, \cdots, z_k)$ integrates to 1 *over* $\mathbb{R}^k$*, and* $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ *and* $\psi(\vec{b}_0, \vec{z}, \vec{n})$ *are determinants:*

$$\varphi(\vec{a}_0, \vec{z}, \vec{m}) = \det \begin{bmatrix} ((c_1(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_1 z_j})_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ ((c_2(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_p z_j})_{\substack{i=1,\cdots,m_p \\ j=1,\cdots,k}} \end{bmatrix}$$

$$\psi(\vec{b}_0, \vec{z}, \vec{n}) = \det \begin{bmatrix} ((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_1 z_j})_{\substack{i=1,\cdots,n_1 \\ j=1,\cdots,k}} \\ \vdots \\ ((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_q z_j})_{\substack{i=1,\cdots,n_q \\ j=1,\cdots,k}} \end{bmatrix}$$

In order to prove these two propositions, we need to introduce some lemmas.
The following lemma gives the distribution of avoiding Bernoulli line ensembles at time $\lfloor tT \rfloor$.

Lemma 6.6. *The avoiding Bernoulli line ensemble at position* $\lfloor tT \rfloor$ *has the following distribution:*

$$\mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \cdots, L_k(\lfloor tT \rfloor) = \lambda_k) = \frac{s_{\lambda/\mu}(1^{\lfloor tT \rfloor}) \cdot s_{\kappa/\lambda}(1^{T - \lfloor tT \rfloor})}{s_{\kappa/\mu}(1^T)}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ *are positive integers,* $s_{\lambda/\mu}$ *denote skew Schur polynomials and they are specialized in all parameters equal to* 1*. The* μ *partition is just the vector* $\vec{x}^T$ *and the* κ *partition should be* $\vec{y}^T$*.*

Proof. Let $\Omega(0, T, \vec{x}^T, \vec{y}^T)$ be the set of all non-intersecting Bernoulli line ensembles from $\vec{x}^T$ to $\vec{y}^T$. For each line ensemble $\mathfrak{B} \in \Omega(0, T, \bar{x}^T, \bar{y}^T)$ with $\mathfrak{B} = (B_1, ..., B_k)$, we may define $\lambda_i(\mathfrak{B}) := (B_1(i), B_2(i), ..., B_k(i))$, where $1 \leqslant i \leqslant T$ is an integer. The λ_i form partitions since by the definition of avoiding Bernoulli line ensembles, we have the inequality $B_\alpha(i) > B_\beta(i)$ if $\alpha < \beta$. Now because $B_\alpha(i+1) - B_\alpha(i) \in \{0, 1\}$ we know that $B_\alpha(i+1) \geq B_\alpha(i)$ but also since $B_\alpha(i+1) \in \mathbb{Z}$ and $B_{\alpha+1}(i+1) < B_\alpha(i+1)$ by the earlier stated inequality, we know that $B_{\alpha+1}(i+1) + 1 \leq B_\alpha(i+1)$ and so we find that

$$B_{\alpha+1}(i+1) \leq B_\alpha(i) \leq B_\alpha(i+1)$$

We therefore find that for all i, $\lambda_i \preceq \lambda_{i+1}$. Note that when $i = 0$, we get $\lambda_0 = \bar{x}^T$ and $\lambda_T = \bar{y}^T$.

Now, let us define the set

$$TB^T_{\kappa/\mu} := \{(\lambda_0, ..., \lambda_T) \mid \lambda_0 = \mu, \lambda_T = \kappa, \lambda_i \preceq \lambda_{i+1}\}$$

Now, if we take $f : \Omega(0, T, \bar{x}^T, \bar{y}^T) \to TB^T_{\kappa/\mu}$ with $f(\mathfrak{B}) = (\lambda_0(\mathfrak{B}), \cdots \lambda_T(\mathfrak{B}))$. We find that this function is in fact a bijection.

First, to show for injectivity, suppose that there are two Bernoulli line ensembles, $\mathfrak{B}, \mathfrak{B}' \in \Omega(0, T, \bar{x}^T, \bar{y}^T)$ such that $\mathfrak{B} \neq \mathfrak{B}'$. Because Bernoulli line ensembles are determined by their values at integer times, we find that this would imply that there exists some (q, r) such that $0 \leq r \leq T$, $0 \leq q \leq k$ and $B_q(r) \neq B'_q(r)$ where B_q and B'_q are components of $\mathfrak{B}$ and $\mathfrak{B}'$ respectively. This implies that $\lambda_r(\mathfrak{B}) \neq \lambda'_r(\mathfrak{B}')$, and we have injectivity.

Now, surjectivity follows since for any $\bar{\lambda} = (\lambda_0, ..., \lambda_T)$ we may define $\mathfrak{B}(\bar{\lambda}) = (B_1(\bar{\lambda}), ..., B_k(\bar{\lambda}))$ where $B_r(\bar{\lambda})(i) = \lambda^r_i$ where λ^r_i is the *ith* entry of λ_r. The restrictions on $TB^T_{\kappa/\mu}$ ensure that each $\mathfrak{B}(\bar{\lambda}) \in \Omega(0, T, \bar{x}^T, \bar{y}^T)$, and so $f(\mathfrak{B}(\bar{\lambda})) = (\lambda_0, \cdots \lambda_T)$ by the definition $\mathfrak{B}(\bar{\lambda})$.

Applying the result regarding the relationship between number of partitions and skew Schur polynomial [11, Chapter 1, (5.11)], we have

$$s_{\kappa/\mu}(1^T) = \sum_{(\nu)} \prod_{i=1}^{n} s_{\nu^{(i)}/\nu^{i-1}} = \sum_{(\nu)} 1 = |TB^T_{\mu/\kappa}|$$

Therefore, we can find that

$$\mathbb{P}(L_1(\lfloor tT\rfloor)=\lambda_1,\cdots,L_k(\lfloor tT\rfloor)=\lambda_k)=\frac{|\Omega(0,\lfloor Tt\rfloor,\vec{x}^T,\lambda)|\cdot|\Omega(\lfloor Tt\rfloor,T,\lambda,\vec{y}^T)|}{|\Omega(0,T,\vec{x}^T,\vec{y}^T)|}$$
$$=\frac{s_{\lambda/\vec{x}^T}(1^{\lfloor Tt\rfloor})\cdot s_{\vec{y}^T/\lambda}(1^{T-\lfloor Tt\rfloor})}{s_{\vec{y}^T/\vec{x}^T}(1^T)}$$

and proved the result. □

The following lemma helps to prove Proposition 6.4. It shows the asymptotic formula for the distribution of avoiding Bernoulli line ensembles at time $\lfloor tT\rfloor$.

Lemma 6.7. *Let $\mathbb{W}_k^o$ denote the open Weyl chamber in $\mathbb{R}^N$:*

$$\mathbb{W}_N^o=\{(x_1,\cdots,x_k)\in\mathbb{R}^N:x_1>x_2>\cdots>x_k\}$$

Fix a real number $A>0$, $p,t\in(0,1)$, take $z=(z_1,\cdots,z_k)\in\mathbb{W}_k^o$ such that $A>z_1>\cdots>z_k>-A$. Choose sufficiently large T_0 such that $ptT_0-A\sqrt{T_0}\geqslant 1$, then for $T\geqslant T_0$, define $\lambda_i(T)=\lfloor z_i\sqrt{T}+ptT\rfloor\geqslant 1$ for $i=1,\cdots,k$. Denote $P_T(z)=(\sqrt{T})^k\mathbb{P}(L_1(\lfloor tT\rfloor)=\lambda_1(T),\cdots,L_k(\lfloor tT\rfloor)=\lambda_k(T))$, then we have for almost every $z\in[-A,A]^k$:

$$\lim_{T\to\infty}P_T(z)=\rho(z_1,\cdots,z_k)$$

and $P_T(z)$ is bounded on $[-A,A]^k$, where $\rho(z)$ is given in Proposition 6.4.

Proof. (i) First, we discuss the pointwise convergence of $P_T(z)$. By Jacobi-Trudi formula([11, Chapter 1, (5.4)]), we conclude:

(6.1)
$$\mathbb{P}(L_1(\lfloor tT\rfloor)=\lambda_1,\cdots,L_k(\lfloor tT\rfloor)=\lambda_k)=\frac{\det\left(h_{\lambda_i-x_j^T+j-i}(1^{\lfloor tT\rfloor})\right)_{i,j=1}^k\cdot\det\left(h_{y_i^T-\lambda_j+j-i}(1^{T-\lfloor tT\rfloor})\right)_{i,j=1}^k}{\det\left(h_{y_i^T-x_j^T+j-i}(1^T)\right)_{i,j=1}^k}$$

We first compute the first determinant in the numerator. Using the identity for complete symmetric functions([11, Example 1, Section I.2]) that $h_r(1^n)=\binom{n+r-1}{r}$, we get the resulting equation

$$h_{\lambda_i-x_j^T+j-i}(1^{\lfloor tT\rfloor})=\frac{(\lambda_i-x_j^T-i+j+\lfloor tT\rfloor-1)!}{(\lambda_i-x_j^T-i+j)!(\lfloor tT\rfloor-1)!}\tag{6.2}$$

We have the following Stirling's formula,

$$n!=\sqrt{2\pi n}n^ne^{-n}e^{r_n},\text{ where }\frac{1}{12n+1}<r_n<\frac{1}{12n}$$

Denote $K=\lambda_i-x_j^T-i+j+\lfloor tT\rfloor-1$ and apply the above Stirling's formula, we get

$$K!=\sqrt{2\pi}\sqrt{K}\cdot e^{K\log K-K+r_K}$$

Additionally, since $\lambda_i=\lfloor z_i\sqrt{T}+ptT\rfloor$ and $x_i^T=a_i\sqrt{T}+o(\sqrt{T})$, we get

$$K=\lambda_i-x_j^T-i+j+\lfloor tT\rfloor-1=(z_i-a_j)\sqrt{T}+(p+1)tT-i+j-1+o(\sqrt{T})$$

$$\log K=\log\left(1+\frac{-i+j-1}{(z_i-a_j)\sqrt{T}+(p+1)tT}+o(\frac{1}{\sqrt{T}})\right)+\log((z_i-a_j)\sqrt{T}+(p+1)tT)$$
$$=\frac{-i+j-1}{(z_i-a_j)\sqrt{T}+(p+1)tT}+\log((z_i-a_j)\sqrt{T}+(p+1)tT)+o(\frac{1}{\sqrt{T}})$$

where the constant in little o notation only depends on A, p and does not depend on T. Next, we compute $K \log K$:

$$K \log K = (-i+j-1) + \Big[(z_i - a_j)\sqrt{T} + (p+1)tT\Big] \cdot \log\Big((z_i - a_j)\sqrt{T} + (p+1)tT\Big)$$
$$+ (-i+j-1)\log((p+1)tT) + o(\sqrt{T})$$

Now we further compute the term $\Big[(z_i - a_j)\sqrt{T} + (p+1)tT\Big] \cdot \log\Big((z_i - a_j)\sqrt{T} + (p+1)tT\Big)$. Notice that

$$\log\Big((z_i - a_j)\sqrt{T} + (p+1)tT\Big) = \log((p+1)tT) + \log\left(1 + \frac{z_i - a_j}{(p+1)t\sqrt{T}}\right)$$
$$= \log((p+1)tT) + \frac{z_i - a_j}{(p+1)t\sqrt{T}} - \frac{1}{2}\frac{(z_i - a_j)^2}{(p+1)^2t^2T} + o(\frac{1}{T})$$

Then,

$$\Big[(z_i - a_j)\sqrt{T} + (p+1)tT\Big] \cdot \log\Big((z_i - a_j)\sqrt{T} + (p+1)tT\Big)$$
$$= \Big[(z_i - a_j)\sqrt{T} + (p+1)tT\Big] \cdot \left[\log((p+1)tT) + \frac{z_i - a_j}{(p+1)t\sqrt{T}} - \frac{1}{2}\frac{(z_i - a_j)^2}{(p+1)^2t^2T} + o(\frac{1}{T})\right]$$
$$= ((p+1)tT)\log((p+1)tT) + (z_i - a_j)\sqrt{T} \cdot \log((p+1)tT) + (z_i - a_j)\sqrt{T} - \frac{1}{2} \cdot \frac{(z_i - a_j)^2}{(p+1)t} + o(1)$$

Therefore, we find that $(\lambda_i - x_j^T - i + j + \lfloor tT \rfloor - 1)! =$

$$\begin{aligned}\sqrt{2\pi}\sqrt{(p+1)tT} \cdot \mathrm{Exp}\{&(-i+j-1) + ((p+1)tT)\log((p+1)tT) + (z_i - a_j)\sqrt{T} \cdot ((p+1)tT) \\ &+ (z_i - a_j)\sqrt{T} - \frac{1}{2} \cdot \frac{(z_i - a_j)^2}{(p+1)t} + (-i+j-1)\log((p+1)tT) \\ &- \Big((z_i - a_j)\sqrt{T} + (p+1)tT - i + j - 1\Big) + o(1)\}\end{aligned} \tag{6.3}$$

Similarly, $(\lambda_i - x_j^T - i + j)! =$

$$\begin{aligned}\sqrt{2\pi}\sqrt{ptT} \cdot \mathrm{Exp}\{&(-i+j) + (ptT)\log(ptT) + (z_i - a_j)\sqrt{T} \cdot \log(ptT) + (z_i - a_j)\sqrt{T} \\ &+ \frac{1}{2} \cdot \frac{(z_i - a_j)^2}{pt} + (-i+j)\log(ptT) - ((z_i - a_j)\sqrt{T} + ptT - i + j) + o(1)\}\end{aligned} \tag{6.4}$$

and the final term:

$$(\lfloor tT \rfloor - 1)! = \sqrt{2\pi}\sqrt{tT} \cdot \mathrm{Exp}\{(tT)\log(tT) - 1 - \log(tT) + o(1)\} \tag{6.5}$$

Plugging (6.3), (6.4) and (6.5) into equation (6.2) we get $h_{\lambda_i - x_j^T + j - i}(1^{\lfloor tT \rfloor}) =$

$$\begin{aligned}\sqrt{2\pi}^{-1}\sqrt{\frac{p+1}{pt}}\sqrt{T}^{-1} \cdot \mathrm{Exp}\Big\{&((p+1)tT)\log((p+1)tT) - (ptT)\log(ptT) - (tT)\log(tT) \\ &+ (-i+j)\log(\frac{p+1}{p}) - \log(p+1) + (z_i - a_j)\sqrt{T} \cdot \log(\frac{p+1}{p}) - \frac{1}{2}\frac{(z_i - a_j)^2}{p(p+1)t} + o(1)\Big\}\end{aligned}$$

where the constant in little o notation only depends on A, p and does not depend on T. Denote $S_1(p,t,T) = \big((p+1)tT\big)\log((p+1)tT) - (ptT)\log(ptT) - (tT)\log(tT)$ and we further calculate the determinant

$$\begin{aligned}\det(h_{\lambda_i - x_j - i + j}(1^{\lfloor tT \rfloor}))_{i,j=1}^k &= \left[(\sqrt{2\pi})^{-1} \sqrt{\frac{p+1}{pt}} \sqrt{T}^{-1} e^{S_1(p,t,T) - \log(p+1)} \right]^k \\ &\quad \cdot \det \left(e^{(-i+j)\log(\frac{p+1}{p}) + (z_i - a_j)\sqrt{T} \cdot \log(\frac{p+1}{p}) - \frac{1}{2}\frac{(z_i - a_j)^2}{p(p+1)t} + o(1)} \right)_{i,j=1}^k \\ &= \left[(\sqrt{2\pi})^{-1} \sqrt{\frac{p+1}{pt}} \sqrt{T}^{-1} e^{S_1(p,t,T) - \log(p+1)} \right]^k \left(\frac{p+1}{p} \right)^{\sum_{i=1}^k (z_i - a_i) \cdot \sqrt{T}} \\ &\quad \cdot e^{-\frac{1}{2p(p+1)t} \sum_{i=1}^k (a_i^2 + z_i^2)} \cdot \det \left(e^{c_1(p,t) z_i a_j + o(1)} \right)_{i,j=1}^k\end{aligned}$$

where the constant $c_1(p,t) = \frac{1}{p(p+1)t}$.

Analogously, we calculate the other 2 determinants in equation 6.1:

$$\begin{aligned}\det\left(h_{y_i - \lambda_j - i + j}(1^{T - \lfloor tT \rfloor}) \right) &= \left[(\sqrt{2\pi})^{-1} \sqrt{\frac{p+1}{p(1-t)}} \sqrt{T}^{-1} \cdot e^{S_2(p,t,T) - \log(p+1)} \right]^k \left(\frac{p+1}{p} \right)^{\sum_{i=1}^k (b_i - z_i) \cdot \sqrt{T}} \\ &\quad \cdot e^{-\frac{1}{2p(p+1)(1-t)} \sum_{i=1}^k (b_i^2 + z_i^2)} \cdot \det \left(e^{c_2(p,t) b_i z_j + o(1)} \right)_{i,j=1}^k \\ \det(h_{y_i^T - x_j^T - i + j}(1^T)) &= \left[(\sqrt{2\pi})^{-1} \sqrt{\frac{p+1}{p}} \sqrt{T}^{-1} \cdot e^{S_3(p,t,T) - \log(p+1)} \right]^k \left(\frac{p+1}{p} \right)^{\sum_{i=1}^k (b_i - a_i) \cdot \sqrt{T}} \\ &\quad \cdot \det \left(e^{-\frac{1}{2p(p+1)}(b_i - a_j)^2 + o(1)} \right)_{i,j=1}^k\end{aligned}$$

where the constants $S_2(p,t,T)$, $S_3(p,t,T)$, $c_2(p,t)$ are:

$$\begin{aligned}S_2(p,t,T) &= ((p+1)(1-t)T)\log((p+1)tT) - (p(1-t)T)\log(ptT) - ((1-t)T)\log((1-t)T) \\ S_3(p,t,T) &= ((p+1)T)\log((p+1)tT) - (pT)\log(pT) - T\log T, \quad c_2(p,t) = \frac{1}{p(p+1)(1-t)}\end{aligned}$$

Notice that $S_1(p,t,T) + S_2(p,t,T) - S_3(p,t,T) = 0$. Plugging the above three determinants into equation 6.1, we get

$$\begin{aligned}&\mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \cdots, L_k(\lfloor tT \rfloor) = \lambda_k) \\ =&(2\pi)^{-\frac{k}{2}} \left[\sqrt{\frac{p+1}{pt(1-t)}} \right]^k \cdot T^{-\frac{k}{2}} \cdot e^{-\frac{1}{2p(p+1)t} \sum_{i=1}^k (a_i^2 + z_i^2) - \frac{1}{2p(p+1)(1-t)} \sum_{i=1}^k (b_i^2 + z_i^2)} \\ &\cdot e^{-k\log(p+1)} \cdot \frac{\det(e^{c_1(p,t) z_i a_j + o(1)})_{i,j=1}^k \cdot \det(e^{c_2(p,t) b_i z_j + o(1)})_{i,j=1}^k}{\det(e^{-\frac{1}{2p(p+1)}(b_i - a_j)^2 + o(1)})_{i,j=1}^k} \\ =&(2\pi)^{-\frac{k}{2}} \left[\sqrt{\frac{1}{p(p+1)t(1-t)}} \right]^k \cdot T^{-\frac{k}{2}} \cdot e^{-\frac{1}{2p(p+1)t} \sum_{i=1}^k a_i^2 - \frac{1}{2p(p+1)(1-t)} \sum_{i=1}^k b_i^2} \\ &\cdot \frac{\det\left(e^{c_1(p,t) z_i a_j}\right)_{i,j=1}^k \cdot \det\left(e^{c_2(p,t) b_i z_j}\right)_{i,j=1}^k}{\det\left(e^{-\frac{1}{2p(p+1)}(b_i - a_j)^2}\right)_{i,j=1}^k} \cdot \exp\{o(1)\} \cdot \prod_{i=1}^k e^{-c_3(t,p) z_i^2}\end{aligned} \tag{6.6}$$

where $c_3(t,p) = \frac{1}{2p(p+1)t(1-t)}$, and the constant in little o notation depends on A, p.

In conclusion, $P_T(z) = (\sqrt{T})^k \cdot \mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \cdots, L_k(\lfloor tT \rfloor) = \lambda_k)$ converges to $\rho(z_1, \ldots, z_k)$ in Proposition 6.4 as $T \to \infty$ and

$$c_1(p,t) = \frac{1}{p(p+1)t}, \quad c_2(p,t) = \frac{1}{p(p+1)(1-t)}, \quad c_3(p,t) = \frac{1}{2p(p+1)t(1-t)}$$
$$Z = (2\pi)^{\frac{k}{2}} (p(p+1)t(1-t))^{\frac{k}{2}} \cdot e^{c_1(t,p)\sum_{i=1}^k a_i^2} \cdot e^{c_2(t,p)\sum_{i=1}^k b_i^2} \det\left(e^{-\frac{1}{2p(p+1)}(b_i-a_j)^2}\right)_{i,j=1}^k$$

(ii) Second, we discuss the boundedness. By the Equation 6.6 we just derived, $P_T(z) = \rho(z) \cdot \exp\{o(1)\}$ on the compact set $[-A,A]^k$, where the constant in little o notation only depends on A, p. Since continuous function $\rho(z)$ is bounded on $[-A,A]^k$, and $\exp\{o(1)\}$ is uniformly bounded on $[-A,A]^k$, we conclude that $P_T(z)$ is bounded as well. $\square$

Now we are ready to prove Proposition 6.4.

Proof of Proposition 6.4. In the following, we prove the weak convergence of the random vector Z^T, when $\vec{a} = (a_1, \cdots, a_k)$ and $\vec{b} = (b_1, \cdots, b_k)$ consist of distinct entries. In order to show the weak convergence, it is sufficient to show that for every open set $O \in \mathbb{R}^k$, we have:

$$\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O) \geqslant \int_O \rho(z_1, \cdots, z_k) dz_1 dz_2 \cdots dz_k$$

according to [8, Theorem 3.2.11]. Actually, it suffices to show that for any open set $U \in \mathbb{W}_k^o$, we have:

$$\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in U) \geqslant \int_U \rho(z_1, \cdots, z_k) dz_1 dz_2 \cdots dz_k \tag{6.7}$$

which implies that:

$$\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O) \geqslant \liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O \cap \mathbb{W}_k^o)$$
$$\geqslant \int_{\mathbb{W}_k^o \cap O} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k = \int_O \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k$$

The second inequality uses the above result (6.7), since $\mathbb{W}_k^o \cap O$ is an open set in $\mathbb{W}_k^o$. The last equality is because $\rho(z)$ is zero outside $\mathbb{W}_k^o$. The rest of the proof will be divided into 4 steps. In Step 1, we prove the weak convergence holds on every closed rectangle. In Step 2, we prove the result 6.7 using Lemma 6.7. In Step 3, we prove that $\rho(z)$ is actually a density and conclude the weak convergence.

Step 1. In this step, we establish the following result:
For any closed rectangle $R = [u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_N, v_N] \in \mathbb{W}_k^o$,

$$\lim_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_R \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \tag{6.8}$$

where $\rho(z)$ is given in Proposition 6.4.

Define $m_i^T = \lceil u_i\sqrt{T} + ptT\rceil$ and $M_i^T = \lfloor v_i\sqrt{T} + ptT\rfloor$, and we have:

$$\begin{aligned}
&\mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \mathbb{P}(u_1 \leqslant Z_1^T \leqslant v_1, \ldots, u_k \leqslant Z_k^T \leqslant v_k) \\
&= \mathbb{P}(u_i\sqrt{T} + ptT \leqslant L_i(\lfloor tT\rfloor) \leqslant v_i\sqrt{T} + ptT, i = 1, \ldots, k) \\
&= \sum_{\lambda_1(T)=m_1^T}^{M_1^T} \cdots \sum_{\lambda_k(T)=m_k^T}^{M_k^T} \mathbb{P}(L_1(\lfloor tT\rfloor) = \lambda_1(T), \ldots, L_k(\lfloor tT\rfloor) = \lambda_k(T)) \\
&= \sum_{\lambda_1(T)=m_1^T}^{M_1^T} \cdots \sum_{\lambda_k(T)=m_k^T}^{M_k^T} (\sqrt{T})^{-k} \cdot (\sqrt{T})^k \mathbb{P}(L_1(\lfloor tT\rfloor) = \lambda_1(T), \ldots, L_k(\lfloor tT\rfloor) = \lambda_k(T))
\end{aligned}$$

Find sufficiently large A such that $R \subset [-A, A]^k$, for example, $A = 1+\max_{1\leqslant i\leqslant k}|a_i|+\max_{1\leqslant i\leqslant k}|b_i|$. Define $f_T(z_1, \cdots, z_k)$ as a simple function on $\mathbb{R}^k$: When $(z_1, \cdots, z_k) \in R$, it takes value $(\sqrt{T})^k \cdot \mathbb{P}(L_1(\lfloor tT\rfloor) = \lambda_1(T), \cdots, L_k(\lfloor tT\rfloor) = \lambda_k(T))$ if there exist $\lambda_1(T), \cdots, \lambda_k(T)$ such that $\lambda_i(T) \leqslant z_i\sqrt{T} + ptT < \lambda_i(T) + 1$; It takes value 0 otherwise, when $(z_1, \cdots, z_k) \notin R$. Since the Lebesgue measure of the set $\{z : \lambda_i(T) \leqslant z_i\sqrt{T} + ptT < \lambda_i(T) + 1, i = 1, \cdots, k\}$ is $(\sqrt{T})^{-k}$, the above probability can be further written as an integral of simple function $f_T(z_1, \cdots, z_k)$:

$$\mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_{[-A,A]^k} f_T(z_1, \cdots, z_k)dz_1 \cdots dz_k$$

By Lemma 6.7, the function $f_T(z_1, \cdots, z_k)$ pointwise converges to $\rho(z)$ and is bounded on the compact set $[-A, A]^k$. Since the Lebesgue measure of $[-A, A]^k$ is finite, by bounded convergence theorem we have:

$$\lim_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_R \rho(z_1, \cdots, z_k)dz_1 \cdots dz_k \tag{6.9}$$

Step 2. In this step, we prove the statement 6.7. Take any open set $U \in \mathbb{W}_k^o$, it can be written as a countable union of closed rectangles with disjoint interiors: $U = \bigcup_{i=1}^\infty R_i$, where $R_i = [a_1^i, b_1^i] \times \cdots \times [a_k^i, b_k^i]$([14, Theorem 1.4]). Choose sufficiently small $\epsilon > 0$, and denote $R_i^\epsilon = [a_1^i + \epsilon, b_1^i - \epsilon] \times \cdots \times [a_k^i + \epsilon, b_k^i - \epsilon]$, then R_i^ϵ are disjoint. Therefore,

$$\begin{aligned}
&\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in U) \geqslant \liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \bigcup_{i=1}^n R_i^\epsilon) \\
&= \liminf_{T\to\infty} \sum_{i=1}^n \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R_i^\epsilon) = \sum_{i=1}^n \int_{R_i^\epsilon} \rho(z_1, \cdots, z_k)dz_1 \cdots dz_k \\
&= \int_{\bigcup_{i=1}^n R_i^\epsilon} \rho(z_1, \cdots, z_k)dz_1 \cdots dz_k \xrightarrow{\epsilon\downarrow 0} \int_U \rho(z_1, \cdots, z_k)dz_1 \cdots dz_k
\end{aligned}$$

The last line uses monotone convergence theorem. Thus, we proved the inequality 6.7.

Step 3. In this step, we prove that $\rho(z)$ is actually a density. First, it is nonnegative because it's the limit of a sequence of probabilities. Next, we prove it integrates to 1 over $\mathbb{R}^k$. Let the open set U in Step 2 be $\mathbb{W}_k^o$, and we get:

$$1 = \liminf_{T\to\infty} \mathbb{P}(Z^T \in \mathbb{W}_k^o) \geqslant \int_{\mathbb{W}_k^o} \rho(z)dz$$

On the other hand, write the open set $\mathbb{W}_k^o$ as a countable union of almost disjoint closed rectangles: $\mathbb{W}_k^o = \bigcup_{i=1}^{\infty} R_i$. Then we have:

$$1 = \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \mathbb{W}_k^o) = \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \bigcup_{i=1}^{\infty} R_i) \leqslant \sum_{i=1}^{\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R_i)$$

where the inequality uses sub-additivity of probability measure. Take an arbitrary $\epsilon > 0$. For each R_i, we can find a closed rectangle $R_i^{\epsilon-}$ contained in R_i such that $\mathbb{P}(Z^T \in R_i) \leqslant \mathbb{P}(Z^T \in R_i^{\epsilon-}) + \frac{\epsilon}{2^i}$. Then, we have

$$\begin{aligned} 1 &= \lim_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \mathbb{W}_k^o) \leqslant \lim_{T\to\infty} \sum_{i=1}^{\infty} \left[\mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R_i^{\epsilon-}) + \frac{\epsilon}{2^i}\right] \\ &= \sum_{i=1}^{\infty} \int_{R_i^{\epsilon-}} \rho(z)dz + \epsilon = \int_{\bigcup_{i=1}^{\infty} R_i^{\epsilon-}} \rho(z)dz + \epsilon \end{aligned}$$

We can interchange the limit and the infinite sum because the sum is bounded by 1. Then let $\epsilon \downarrow 0$, we get $1 \leqslant \int_{\mathbb{W}_k^o} \rho(z)dz$, implying $\int_{\mathbb{W}_k^o} \rho(z)dz = 1$ and we conclude that $\rho(z)$ is actually a density. $\square$

Next we are going to prove Proposition 6.5. Before that, we first introduce some notations and results about multivariate functions.

Suppose $\sigma = (\sigma_1, \cdots, \sigma_n)$ is a multi-index of *length* n. In our context, we require $\sigma_1, \cdots, \sigma_n$ be all non-negative integers(some of them might be equal). We define $|\sigma| = \sum_{i=1}^n \sigma_i$ as the *order* of σ. Suppose $\tau = (\tau_1, \cdots, \tau_n)$ is another multi-index of length n. We say $\tau \leqslant \sigma$ if $\tau_i \leqslant \sigma_i$ for $i = 1, \cdots, n$. We say $\tau < \sigma$ if $\tau \leqslant \sigma$ and there exists at least one index i such that $\tau_i < \sigma_i$. Then, define the partial derivative with respect to the multi-index σ:

$$D^\sigma f(x_1, \cdots, x_n) = \frac{\partial^{|\sigma|} f(x_1, \cdots, x_n)}{\partial x_1^{\sigma_1} \partial x_2^{\sigma_2} \cdots \partial x_n^{\sigma_n}}$$

We have the general Leibniz rule:

$$D^\sigma (fg) = \sum_{\tau \leqslant \sigma} \binom{\sigma}{\tau} D^\tau f \cdot D^{\sigma-\tau} g$$

where $\binom{\sigma}{\tau} = \frac{\sigma_1! \cdots \sigma_n!}{\tau_1! \cdots \tau_n! (\sigma_1 - \tau_1)! \cdots (\sigma_n - \tau_n)!}$.

We also have the Taylor expansion for multi-variable functions:

$$f(x_1, \cdots, x_n) = \sum_{|\sigma| \leqslant r} \frac{1}{\sigma!} D^\sigma f(\vec{x}_0)(\vec{x} - \vec{x}_0)^\sigma + R_{r+1}(\vec{x}, \vec{x}_0)$$

In the equation, $\sigma! = \sigma_1! \sigma_2! \cdots \sigma_n!$ is the factorial with respect to the multi-index σ, $\vec{x}_0 = (x_1^0, \cdots, x_n^0)$ is a constant vector at which we expands the function f, $(\vec{x} - \vec{x}_0)^\sigma$ stands for $(x_1 - x_1^0)^{\sigma_1} \cdots (x_n - x_n^0)^{\sigma_n}$, and

$$R_{r+1}(\vec{x}, \vec{x}_0) = \sum_{\sigma : |\sigma| = r+1} \frac{1}{\sigma!} D^\sigma f(\vec{x}_0 + \theta(\vec{x} - \vec{x}_0))(\vec{x} - \vec{x}_0)^\sigma$$

is the remainder, where $\theta \in (0, 1)$([2, Theorem 3.18 & Corollary 3.19]).

We also need some knowledge about *permutation*. Suppose s_n is a permutation of n non-negative integers, for example $\{1, \cdots, n\}$, and $s_n(i)$ represents the *i-th* element in the permutation s_n. We define *the number of inversions* of s_n by $I(s_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbb{1}_{\{s_n(i) > s_n(j)\}}$. For example, the permutation $s_n = (1, \cdots, n)$ has 0 number of inversions, while the permutation $s_5 = (3, 2, 5, 1, 4)$ has number of inversions 5(2+1+2+0+0). Define the sign of permutation s_n by $sgn(s_n) = (-1)^{I(s_n)}$. For instance, $sgn((1, \cdots, n)) = 1$ and $sgn(s_5) = -1$ in the previous example.

Then, we introduce some notations associated with Proposition 6.5 in order to better discuss the problem. Denote

$$\vec{a}_0 = (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p})$$

$$\vec{b}_0 = (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q})$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_p$, $\beta_1 > \beta_2 > \cdots > \beta_q$ and $\sum_{i=1}^p \alpha_i = \sum_{i=1}^q \beta_i = k$. Denote $\vec{a} = (a_1, \cdots, a_k)$, $\vec{b} = (b_1, \cdots, b_k)$. Also denote $\vec{a}^{(1)} = (a_1, \cdots, a_{m_1})$, $\vec{a}^{(2)} = (a_{m_1+1}, \cdots, a_{m_1+m_2})$, $\cdots$, $\vec{a}^{(p)} = (a_{m_1+\cdots+m_{p-1}+1}, \cdots, a_{m_1+\cdots+m_p})$ and $\vec{a} = (\vec{a}^{(1)}, \cdots, \vec{a}^{(p)})$. That is, we divide the vector $\vec{a}$ into p blocks according to the shape of $\vec{a_0}$. Similarly, we write $\vec{b} = (b^{(1)}, \cdots, b^{(q)})$ according to the shape of $\vec{b}_0$. We will keep using similar notations in the following discussion, when we need to divide the vector according to the shape of $\vec{a}_0$ and $\vec{b}_0$. Next, denote

$$f(a_1, \cdots, a_k) \equiv f(\vec{a}) = \det[e^{c_1(t,p)a_i z_j}]_{i,j=1}^k, \quad g(b_1, \cdots, b_k) \equiv g(\vec{b}) = \det[e^{c_2(t,p)b_i z_j}]_{i,j=1}^k$$

and it's not difficult to see that they are all smooth multi-variable functions with respect to corresponding vectors. In addition, $\lim\limits_{\vec{a}\to\vec{a}_0} f(\vec{a}) = 0$ and $\lim\limits_{\vec{b}\to\vec{b}_0} g(\vec{b}) = 0$.

The last thing before we formally prove Proposition 6.5 is to introduce the following lemmas about the non-vanishing of the determinants.

Lemma 6.8. *Suppose $\sigma_a = (\sigma_1^a, \cdots, \sigma_k^a)$ and $\sigma_b = (\sigma_1^b, \cdots, \sigma_k^b)$ are two multi-indices of length k. We divide them into p and q parts according to the shape of $\vec{a}_0$ and $\vec{b}_0$ as mentioned before: $\sigma_a = (\sigma_a^{(1)}, \sigma_a^{(2)}, \cdots, \sigma_a^{(p)})$, $\sigma_b = (\sigma_b^{(1)}, \sigma_b^{(2)}, \cdots, \sigma_b^{(q)})$. Denote*

$$f(a_1, \cdots, a_k) \equiv f(\vec{a}) = \det[e^{a_i z_j}]_{i,j=1}^k, \quad g(b_1, \cdots, b_k) \equiv g(\vec{b}) = \det[e^{b_i z_j}]_{i,j=1}^k$$

where we ignore the constants $c_1(t,p)$ and $c_2(t,p)$ temporarily for simplicity. Suppose S_{m_i} is the set of all permutations of $\{0, 1, \cdots, m_i - 1\}$. If $\sigma_a^{(i)} \in S_{m_i}$ for $i = 1, \cdots, p$, then

$$D^{\sigma_a} f(\vec{a}_0) = \det \begin{bmatrix} (z_j^{\sigma_i^a} e^{\alpha_1 z_j})_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ (z_j^{\sigma_i^a} e^{\alpha_p z_j})_{\substack{i=m_1+\cdots+m_{p-1}+1, \\ \cdots, m_1+\cdots+m_p \\ j=1,\cdots,k}} \end{bmatrix}$$

is non-zero for any $(z_1, \cdots, z_k)$ whose elements are distinct. Analogous result also holds for $D^{\sigma_b} g(\vec{b}_0)$.

Proof. Since $f(\vec{a})$ is actually a determinant and its i-th row only depends on the variable a_i, taking derivative of $f(\vec{a})$ with respect to a_i is taking derivative of every entries in the i-th row, and we can get the determinant above. Next, we prove that it is non-zero. WLOG, we can assume $\sigma_a^{(i)} = \{0, 1, \cdots, m_i - 1\}$, because the determinant will only change by -1 when $\sigma_a^{(i)}$ is replaced by other permutations in S_{m_i}. We claim that, the following equation with respect to z:

$$(\xi_1 + \xi_2 z + \cdots + \xi_{m_1} z^{m_i - 1}) e^{\alpha_1 z} + \cdots (\xi_{m_1+\cdots+m_{p-1}+1} + \cdots + \xi_k z^{m_p - 1}) e^{\alpha_p z} = 0$$

has at most $(k-1)$ distinct roots, where $\sum_{i=1}^p m_i = k$ and $(\xi_1, \cdots, \xi_k) \in \mathbb{R}^k$ is non-zero.

Denote the above determinant by $\det \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$. If this claim holds, we can conclude that we cannot find non-zero $(\xi_1, \cdots, \xi_k) \in \mathbb{R}^k$ such that $\xi_1 v_1 + \cdots + \xi_k v_k = 0$. Thus, the k row vectors of the determinant are linear independent and the determinant is non-zero. Then we prove the claim by induction on k.

1° If $k=2$, the equation is $(\xi_1+\xi_2 z)e^{\alpha_1 z}=0$ or $\xi_1 e^{\alpha_1 z}+\xi_2 e^{\alpha_2 z}=0$, where $\xi_1,\xi_2\in\mathbb{R}$ cannot be zero at the same time. Then, it's easy to see that the equation has at most 1 root in two scenarios.
2° Suppose the claim holds for $k\leqslant n$.
3° When $k=n+1$, we have the equation

$$(\xi_1+\xi_2 z+\cdots+\xi_{m_1}z^{m_i-1})e^{\alpha_1 z}+\cdots(\xi_{m_1+\cdots+m_{p-1}+1}+\cdots+\xi_k z^{m_p-1})e^{\alpha_p z}=0$$

but now $\sum_{i=1}^p m_i=n+1$. WLOG, suppose $(\xi_1,\cdots,\xi_{m_1})$ has a non-zero element and ξ_ℓ is the first non-zero element. Notice that the above equation has the same roots as the following one:

$$F(z)=(\xi_\ell z^{\ell-1}+\cdots+\xi_{m_1}z^{m_1-1})+\cdots+(\xi_{m_1+\cdots+m_{p-1}+1}+\cdots+\xi_k z^{m_p-1})e^{(\alpha_p-\alpha_1)z}=0$$

Assume it has at least $(n+1)$ distinct roots $\eta_1<\eta_2<\cdots<\eta_{n+1}$. Then $F'(z)=0$ has at least n distinct roots $\delta_1<\cdots<\delta_n$ such that $\eta_1<\delta_1<\eta_2<\cdots<\delta_n<\eta_{n+1}$, by Rolle's Theorem. Actually, $F'(z)=(\xi_\ell(\ell-1))z^{\ell-2}+\cdots+\xi_{m_1}(m_1-1)z^{m_1-2})+\cdots+[\xi'_{m_1+\cdots+m_{p-1}+1}+\cdots+\xi'_k z^{m_p-1})]e^{(\alpha_p-\alpha_1)z}=0$ where $\xi'_{m_1+\cdots+m_{p-1}+1}$ and ξ'_k are coefficients that can be calculated. This equation has at most $(m_1-1)+m_2+\cdots+m_p-1=n-1$ roots by 2°, which leads to a contradiction. Therefore, our claim holds and we proved Lemma 6.8. $\square$

Remark 6.9. Denote the set $\Lambda_a=\{\sigma_a=(\sigma_a^{(1)},\cdots,\sigma_a^{(p)}):\sigma_a^{(i)}\in S_{m_i}, i=1,\cdots,p\}$, and we have if $\sigma_a\in\Lambda_a$, then $D^{\sigma_a}f(\vec{a}_0)$ is non-zero. Similarly, if $\sigma_b^{(j)}\in S_{n_j}$ for $j=1,\cdots,q$, then $D^{\sigma_b}g(\vec{b})$ is non-zero, and define $\Lambda_b=\{\sigma_b=(\sigma_b^{(1)},\cdots,\sigma_b^{(q)}):\sigma_b^{(j)}\in S_{n_j}, j=1,\cdots,q\}$.

Lemma 6.10. *The smallest order of σ_a that makes the partial derivative $D^{\sigma_a}f(\vec{a}_0)$ non-zero is $u=\sum_{i=1}^p\sum_{j=0}^{m_i-1}j=\sum_{i=1}^p\frac{m_i(m_i-1)}{2}$. Similarly, $v=\sum_{j=1}^q\frac{n_j(n_j-1)}{2}$ is the smallest order of σ_b that makes $D^{\sigma_b}f(\vec{b}_0)$ non-zero.*

Proof. If the order of derivative is less than u, then there exists a $i\in\{1,\cdots,p\}$ such that $\sigma_a^{(i)}$ contains two equal elements, and the determinant $D^{\sigma_a}f(\vec{a}_0)$ would have two equal rows, thus equal to zero. If the order of derivative is u, then when $\sigma_a\in\Lambda_a$, $D^{\sigma_a}f(\vec{a}_0)$ is non-zero by Lemma 6.8. Thus, Lemma 6.10 holds. $\square$

Finally, we give the proof for Proposition 6.5.

Proof of Proposition 6.5. For clarity, the proof will be split into 3 steps. In Step 1, we use multivariate Taylor expansion to find the speed of convergence of $f(\vec{a})$ and $g(\vec{b})$ to zero. In Step 2, we construct a new density function based on Step 1, and we will prove that Z^T weakly converges to the this newly constructed density in Step 3. In Step 3, we use monotone coupling lemma to prove the weak convergence.
Step 1. In this step, we estimate the converging speed of $f(\vec{a})$. Take $\epsilon\in(0,k^{-1}\min\limits_{1\leqslant i\leqslant p-1}(\alpha_i-\alpha_{i+1}))$ and construct the following vectors:

$$\vec{A}_{\epsilon,+}=(\alpha_1+m_1\epsilon,\alpha_1+(m_1-1)\epsilon,\cdots,\alpha_1+\epsilon,\cdots,\alpha_p+m_p\epsilon,\cdots,\alpha_p+\epsilon)$$
$$\vec{A}_{\epsilon,-}=(\alpha_1-\epsilon,\alpha_1-2\epsilon,\cdots,\alpha_1-m_1\epsilon,\cdots,\alpha_p-\epsilon,\cdots,\alpha_p-m_p\epsilon)$$

That is, the vector $\vec{A}_{\epsilon,+}$(resp. $\vec{A}_{\epsilon,-}$) upwardly(resp. downwardly) spreads out the vector $\vec{a}_0$ such that $\vec{A}_{\epsilon,+}$(resp. $\vec{A}_{\epsilon,-}$) has distinct elements. In addition, when $\epsilon\downarrow 0$, we have $\vec{A}_{\epsilon,\pm}$ converges to $\vec{a}_0$. The main result of this step is the following:

$$\lim_{\epsilon\downarrow 0}\epsilon^{-u}f(\vec{A}_{\epsilon,\pm})=\varphi(\vec{a}_0,\vec{z},\vec{m}) \tag{6.10}$$

where $u=\sum_{i=1}^p\frac{m_i(m_i-1)}{2}$ in Lemma 6.10, $\vec{m}=(m_1,\cdots,m_p)$, and $\varphi(\vec{a}_0,\vec{z},\vec{m})$ is a non-zero function associated with $\vec{a}_0$ and $\vec{m}$.

To prove this result, we first expand the function $f(\vec{a})$ to the order of u at $\vec{a}_0$:

$$\begin{aligned} f(\vec{a}) &= \sum_{|\sigma_a|\leqslant u} \frac{D^{\sigma_a} f(\vec{a}_0)}{\sigma_a!}(\vec{a}-\vec{a}_0)^{\sigma_a} + R_{u+1}(\vec{a},\vec{a}_0) \\ &= \sum_{\sigma_a\in\Lambda_a} \frac{D^{\sigma_a} f(\vec{a}_0)}{\sigma_a!}(\vec{a}-\vec{a}_0)^{\sigma_a} + R_{u+1}(\vec{a},\vec{a}_0) \end{aligned}$$

where the $R_{u+1}(\vec{a},\vec{a}_0) = \sum_{\sigma_a:|\sigma_a|=u+1} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0 + \theta(\vec{a}-\vec{a}_0))(\vec{a}-\vec{a}_0)^{\sigma_a}$, $\theta \in (0,1)$ is the remainder. The second equality results from Lemma 6.10, since it indicates that all the terms of order less than u are zero, and for the terms of order u, they are non-zero only when $\sigma_a \in \Lambda_a$.

Consider the first term. Denote $sgn(\sigma_a^{(i)})$ as the sign of the permutation $\sigma_a^{(i)} \in S_{m_i}$, and define the sign of σ_a by: $sgn(\sigma_a) = \prod_{i=1}^p sgn(\sigma_a^{(i)})$. Denote $\sigma_a^\star = (\sigma_a^{(1)\star}, \cdots, \sigma_a^{(p)\star})$, where $\sigma_a^{(i)\star} = (0, 1, \cdots, m_i - 1)$. Thus, $\sigma_a^\star$ is a special element in Λ_a and $sgn(\sigma_a^\star) = 1$ because $\sigma_a^{(1)\star}, \cdots, \sigma_a^{(p)\star}$ all have 0 number of inversions. Notice that for any $\sigma_a \in \Lambda_a$, we have $D^{\sigma_a} f(\vec{a}_0) = sgn(\sigma_a) \cdot D^{\sigma_a^\star} f(\vec{a}_0)$ by the property of determinant. Then we obtain:

$$\sum_{\sigma_a\in\Lambda_a} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0)(\vec{a}-\vec{a}_0)^{\sigma_a} = \frac{D^{\sigma_a^\star} f(\vec{a}_0)}{\prod_{i=1}^p (m_i-1)!} \sum_{\sigma_a\in\Lambda_a} (\vec{a}-\vec{a}_0)^{\sigma_a} \cdot sgn(\sigma_a)$$

Notice that

$$\begin{aligned} \sum_{\sigma_a\in\Lambda_a} (\vec{a}-\vec{a}_0)^{\sigma_a} \cdot sgn(\sigma_a) &= \prod_{i=1}^p \Big[\sum_{\sigma_a^{(i)}\in S_{m_i}} (\vec{a}^{(i)} - \vec{a}_0^{(i)})^{\sigma_a^{(i)}} \cdot sgn(\sigma_a^{(i)}) \Big] \\ &= \prod_{i=1}^p \Delta_{m_i}(a_1^{(i)} - \alpha_i, a_2^{(i)} - \alpha_i, \cdots, a_{m_i}^{(i)} - \alpha_i) \equiv \prod_{i=1}^p \Delta_{m_i}^a \end{aligned}$$

where $\Delta_n(x_1, x_2, \cdots, x_n)$ is the Vandermonde Determinant, $a_j^{(i)} = a_{m_1+\cdots+m_{i-1}+j}$ is the j-th element of $\vec{a}^{(i)}$, and the last equality holds by the definition of determinant and Vandermonde Determinant. Now replace $\vec{a}$ with $\vec{A}_{\epsilon,+}$, we get the Vandermonde determinant $\Delta_{m_i}^a$ is actually $(m_i-1)! \cdot \epsilon^{\frac{1}{2}m_i(m_i-1)}$. Therefore, we have:

$$\sum_{\sigma_a\in\Lambda_a} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0)(\vec{a}-\vec{a}_0)^{\sigma_a} = D^{\sigma_a^\star} f(\vec{a}_0) \cdot \epsilon^u$$

Now we consider the remainder $R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0)$. Since $D^{\sigma_a} f(\vec{a})$ is a continuous function of vector $\vec{a}$, we have that the quantity $D^{\sigma_a^\star} f(\vec{a}_0 + \theta(\vec{a}-\vec{a}_0))$ can be bounded by a constant $M(\vec{a},\vec{a}_0)$. In addition, $\sigma_a!$ only have finitely many possible outcomes when its order is $u+1$, thus $\frac{1}{\sigma_a!}$ can be bounded by a constant $N(u)$. Also, $|(\vec{A}_{\epsilon,+} - \vec{a}_0)|^{\sigma_a} \leqslant (\max_{1\leqslant i\leqslant p} m_i \cdot \epsilon)^{u+1}$. Therefore,

$$|R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0)| \leqslant N \cdot M \cdot (\max_{1\leqslant i\leqslant p} m_i \cdot \epsilon)^{u+1}$$

and this indicates that $R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0)$ is $O(\epsilon^{u+1})$, where the constant in Big O notation only depends on $\vec{a}_0$, $\vec{a}$, $\vec{m}$ and u and does not depend on ϵ. Therefore, we conclude that

$$\lim_{\epsilon\downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,+}) = D^{\sigma_a^\star} f(\vec{a}_0)$$

By Lemma 6.8, $D^{\sigma_a^\star} f(\vec{a}_0)$ is non-zero. Thus, we find the limit function $\varphi(\vec{a}_0, \vec{z}, \vec{m}) = D^{\sigma_a^\star} f(\vec{a}_0)$, and its expression can be found in Lemma 6.8. Following similar procedure we can prove $\lim_{\epsilon\downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,-}) = D^{\sigma_a^\star} f(\vec{a}_0)$ also holds, and we established the equation (6.10).

We can construct vectors $\vec{B}_{\epsilon,\pm}$ analogously, which spread out from vector $\vec{b}_0$ upward and downward, and get similar results for $g(\vec{B}_{\epsilon,\pm})$ and then we have:

$$\lim_{\epsilon\downarrow 0}\epsilon^{-v}f(\vec{B}_{\epsilon,\pm}) = D^{\sigma_{\vec{b}}^{\star}}g(\vec{b}_0) \equiv \psi(\vec{b}_0,\vec{z},\vec{n})$$

where $v = \sum_{i=1}^{q}\frac{n_i(n_i-1)}{2}$ in Lemma 6.10, $\vec{n} = (n_1,\cdots,n_q)$ and the expression of non-zero function $\psi(\vec{b}_0,\vec{z},\vec{n})$ can be found by Lemma 6.8.

Step 2. In this step, we mainly prove the following result:

The function of $\vec{z}$

$$H(\vec{z}) = \varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n})\prod_{i=1}^{k}e^{-c_3(t,p)z_i^2}$$

is integrable over $\mathbb{R}^k$.

For simplicity, we ignore the constants c_1, c_2, c_3 temporarily and prove the function $H(\vec{z})$ without those constants is integrable. It's not difficult to see $H(\vec{z})$ is still integrable when adding those constants. Notice that $\varphi(\vec{a}_0,\vec{z},\vec{m})$ is a determinant whose expression is given in Lemma 6.8. Suppose $z_{j_1}^{i_1-1}e^{a_{i_1}z_{j_1}}$ is the entry that has the largest absolute value. Then

$$|\varphi(\vec{a}_0,\vec{z},\vec{m})| \leqslant k!|z_{j_1}^{i_1-1}e^{a_{i_1}z_{j_1}}|^k = k!|z_{j_1}^{k(i_1-1)}|e^{ka_{i_1}z_{j_1}}$$

Similarly, we can find index i_2 and j_2 such that $|\varphi(\vec{b}_0,\vec{z},\vec{n})| \leqslant k!|z_{j_2}^{k(i_2-1)}|e^{kb_{i_2}z_{j_2}}$. Then, we get

$$|H(\vec{z})| \leqslant (k!)^2\big[\prod_{j\neq j_1,j_2}e^{-z_j^2}|z_{j_1}^{k(i_1-1)}z_{j_2}^{k(i_2-1)}|\big]e^{ka_{i_1}z_{j_1}-z_{j_1}^2}e^{kb_{i_2}z_{j_2}-z_{j_2}^2}$$

The right hand side is integrable over $\mathbb{R}^k$ because the exponential terms have power of some quadratic functions with negative quadratic coefficients. Thus, $H(\vec{z})$ is integrable.

Since $H(\vec{z})$ is integrable, we can define the constant $Z_{\vec{a}_0,\vec{b}_0} = \int_{\mathbb{R}^k}H(\vec{z})\mathbb{1}_{\{z_1>z_2>\cdots>z_k\}}dz < \infty$ and the function

$$\rho_{\vec{a}_0,\vec{b}_0}(z_1,\cdots,z_k) = Z_{\vec{a}_0,\vec{b}_0}^{-1}\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n})\prod_{i=1}^{k}e^{-c_3(t,p)z_i^2}\mathbb{1}_{\{z_1>z_2>\cdots>z_k\}} \tag{6.11}$$

is a density because it's non-negative and integrates to 1 over $\mathbb{R}^k$.

Step 3. Denote $Z_{\vec{a}_0,\vec{b}_0}^{T}$ as the random vector Z^T associated with vectors $\vec{a}_0$ and b_0, and in this step we prove it weakly converges to the continuous distribution with the density $\rho_{\vec{a}_0,\vec{b}_0}(z)$ we just constructed in (6.11). Suppose $\mathfrak{L}_+^T$ is an avoiding Bernoulli line ensemble starting with $\vec{x}_+^T$ and ending with $\vec{y}_+^T$ and follows the distribution $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}_+^T,\vec{y}_+^T}$. The vectors $\vec{x}_+^T$ and $\vec{y}_+^T$ are two signatures of length k that satisfies the following:

(i)

$$\lim_{T\to\infty}\frac{\vec{x}_+^T}{\sqrt{T}} = \vec{A}_{\epsilon,+}, \quad \lim_{T\to\infty}\frac{\vec{y}_+^T - pT1_k}{\sqrt{T}} = \vec{B}_{\epsilon,+}$$

(ii) $\vec{x}_+^T \geqslant \vec{x}^T$, $\vec{y}_+^T \geqslant \vec{y}^T$, which means the endpoints of the newly constructed line ensembles dominate the original ones. This can be achieved due to the limiting behavior of $\vec{x}_+^T$ and $\vec{y}_+^T$ and the construction of $\vec{A}_{\epsilon,+}$ and $\vec{B}_{\epsilon,+}$. Analogously, we construct another avoiding Bernoulli line ensemble $\mathfrak{L}_-^T$ with endpoints $\vec{x}_-^T$ and $\vec{y}_-^T$ and distribution $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}_-^T,\vec{y}_-^T}$ such that $\lim_{T\to\infty}\frac{\vec{x}_-^T}{\sqrt{T}} = \vec{A}_{\epsilon,-}$, $\lim_{T\to\infty}\frac{\vec{y}_-^T-pT1_k}{\sqrt{T}} = \vec{B}_{\epsilon,-}$, and $\vec{x}_-^T \leqslant \vec{x}^T$, $\vec{y}_-^T \leqslant \vec{y}^T$.

Since now $\vec{A}_{\epsilon,+}$, $\vec{A}_{\epsilon,-}$, $\vec{B}_{\epsilon,+}$, $\vec{B}_{\epsilon,-}$ have distinct elements, we can apply the results in Proposition 6.4 and conclude the weak convergence:

$$Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} \Rightarrow \rho_{\epsilon,+}(z), \quad Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}} \Rightarrow \rho_{\epsilon,-}(z)$$

where $Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}$ and $Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}}$ are obtained by scaling the line ensembles $\mathfrak{L}^T_+$, and $\mathfrak{L}^T_-$, $\rho_{\epsilon,+}(z)$ and $\rho_{\epsilon,-}(z)$ are densities which are obtained by plugging $\vec{A}_{\epsilon,+}$, $\vec{B}_{\epsilon,+}$ and $\vec{A}_{\epsilon,-}$, $\vec{B}_{\epsilon,-}$ into the formula of $\rho(z)$ in Proposition 6.4.

In order to prove the weak convergence of $Z^T_{\vec{a}_0,\vec{b}_0}$, it is sufficient to prove for any $R = (-\infty, u_1] \times (-\infty, u_2] \times \cdots \times (-\infty, u_k]$, where $u_i \in \mathbb{R}$, we have

$$\lim_{T\to\infty} (Z^T_{\vec{a}_0,\vec{b}_0} \in R) = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$$

Actually, by Lemma 3.1, we can construct a sequence of probability spaces $(\Omega_T, \mathcal{F}_T, \mathbb{P}_T)_{T\geqslant 1}$ such that for each $T \in \mathbb{Z}^+$, we have random variables $\mathfrak{L}^T_+$ and $\mathfrak{L}^T$ have law $\mathbb{P}^{0,T,\vec{x}^T_+,\vec{y}^T_+}_{Avoid,Ber}$, and $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{Avoid,Ber}$ under measure $\mathbb{P}_T$, respectively. Also, we have $\mathfrak{L}^T_+(i,r) \geqslant \mathfrak{L}^T(i,r)$ with probability 1, where $\mathfrak{L}^T_+(i,r)$(resp., $\mathfrak{L}^T(i,r)$) is the value of the *i-th* up-right path of $\mathfrak{L}^T_+$(resp., $\mathfrak{L}^T$) at $r \in [\![0,T]\!]$. Similarly, we can construct another sequence of probability spaces $(\Omega'_T, \mathcal{F}'_T, \mathbb{Q}_T)_{T\geqslant 1}$ such that for each $T \in \mathbb{Z}^+$, we have random variables $\mathfrak{L}^T_-$ and $\mathfrak{L}^T$ have law $\mathbb{P}^{0,T,\vec{x}^T_-,\vec{y}^T_-}_{avoid,Ber}$, and $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$ under measure $\mathbb{Q}_T$, respectively, along with $\mathbb{Q}_T(\mathfrak{L}^T_-(i,r) \leqslant \mathfrak{L}^T(i,r), i = 1,\cdots,k, r \in [\![0,T]\!]) = 1$.

Therefore, we have that under measure $\mathbb{P}_T$ and $\mathbb{Q}_T$:

$$\mathbb{P}_T(Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} \in R) \leqslant \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R), \quad \mathbb{Q}_T(Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}} \in R) \geqslant \mathbb{Q}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R)$$

Take *liminf* and *limsup* on both side of the first and second inequality respectively, we get

$$\int_R \rho_{\epsilon,+}(z)dz \leqslant \liminf_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R), \quad \int_R \rho_{\epsilon,-}(z)dz \geqslant \limsup_{T\to\infty} \mathbb{Q}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \tag{6.12}$$

because of the weak convergence of $Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}$ and $Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}}$. Since the distribution of $Z^T_{\vec{a}_0,\vec{b}_0}$ under measure $\mathbb{P}_T$ and $\mathbb{Q}_T$ are the same, we can combine the above two inequalities (6.12) and get

$$\int_R \rho_{\epsilon,+}(z)dz \leqslant \liminf_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \leqslant \limsup_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \leqslant \int_R \rho_{\epsilon,-}(z)dz \tag{6.13}$$

The rest of the proof establishes the following statement:

$$\lim_{\epsilon\downarrow 0} \int_R \rho_{\epsilon,+}(z)dz = \lim_{\epsilon\downarrow 0} \int_R \rho_{\epsilon,-}(z)dz = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz \tag{6.14}$$

and thereby concluding

$$\lim_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$$

by letting $\epsilon \downarrow 0$ in the inequality (6.13), and we prove the weak convergence of $Z^T_{\vec{a}_0,\vec{b}_0}$.

To prove the statement (6.14), first notice that

$$\begin{aligned}
Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} &= \int_{\mathbb{R}^k} f(\vec{a},\vec{z})g(\vec{b},\vec{z})\prod_{i=1}^k e^{-c_3(t,p)z_i^2}dz \\
&= \int_{\mathbb{R}^k}\left[\epsilon^{u+v}\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n}) + o(\epsilon^{u+v})\right]\prod_{i=1}^k e^{-c_3(t,p)z_i^2}dz \\
&= \epsilon^{u+v}\int_{\mathbb{R}^k}\left[\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n}) + o(1)\right]\prod_{i=1}^k e^{-c_3(t,p)z_i^2}dz
\end{aligned}$$

Then, we get

$$\lim_{\epsilon\downarrow 0}\epsilon^{-(u+v)}Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} = \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^k}\left[\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n}) + o(1)\right]\prod_{i=1}^k e^{-c_3(t,p)z_i^2}dz = Z_{\vec{a}_0,\vec{b}_0}$$

by definition of the constant $Z_{\vec{a}_0,\vec{b}_0}$. Therefore, we conclude

$$\lim_{\epsilon\downarrow 0}\rho_{\epsilon,+}(z) = \lim_{\epsilon\downarrow 0}(\epsilon^{-(u+v)}Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}})(\epsilon^u f(\vec{a},\vec{z}))(\epsilon^v g(\vec{b},\vec{z})) = Z_{\vec{a},\vec{b}_0}\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n}) = \rho_{\vec{a}_0,\vec{b}_0}(z)$$

Since $\rho_{\epsilon,+}(z)\mathbb{1}_R dz \leqslant \rho_{\epsilon,+}(z)dz$ is bounded by an integrable function, by Dominated Convergence Theorem we have:

$$\lim_{\epsilon\downarrow 0}\int_R \rho_{\epsilon,+}(z)dz = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$$

Analogously, we can get $\lim_{\epsilon\downarrow 0}\int_R \rho_{\epsilon,-}(z)dz = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$ and we proved the statement (6.14), which completes the proof. □

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