TIGHTNESS OF BERNOULLI LINE ENSEMBLES

Abstract. Insert abstract here:

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2. Line ensembles

In this section we introduce various definitions and notation that are used throughout the paper.

2.1. **Line ensembles and the Brownian Gibbs property.**

In this section we introduce the notions of a *line ensemble* and the *(partial) Brownian Gibbs property*. Our exposition in this section closely follows that of [6, Section 2] and [4, Section 2].

Given two integers $p \leq q$, we let $[\![p, q]\!]$ denote the set $\{p, p+1, \dots, q\}$. Given an interval $\Lambda \subset \mathbb{R}$ we endow it with the subspace topology of the usual topology on $\mathbb{R}$. We let $(C(\Lambda), \mathcal{C})$ denote the space of continuous functions $f : \Lambda \to \mathbb{R}$ with the topology of uniform convergence over compacts, see [12, Chapter 7, Section 46], and Borel σ-algebra $\mathcal{C}$. Given a set $\Sigma \subset \mathbb{Z}$ we endow it with the discrete topology and denote by $\Sigma \times \Lambda$ the set of all pairs (i, x) with $i \in \Sigma$ and $x \in \Lambda$ with the product topology. We also denote by $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ the space of continuous functions on $\Sigma \times \Lambda$ with the topology of uniform convergence over compact sets and Borel σ-algebra $\mathcal{C}_\Sigma$. Typically, we will take $\Sigma = [\![1, N]\!]$ (we use the convention $\Sigma = \mathbb{N}$ if $N = \infty$) and then we write $\big(C(\Sigma \times \Lambda), \mathcal{C}_{|\Sigma|}\big)$ in place of $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$.

The following defines the notion of a line ensemble.

Definition 2.1. Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. A Σ-*indexed line ensemble* $\mathcal{L}$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that takes values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$. Intuitively, $\mathcal{L}$ is a collection of random continuous curves (sometimes referred to as *lines*), indexed by Σ, each of which maps Λ in $\mathbb{R}$. We will often slightly abuse notation and write $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$, even though it is not $\mathcal{L}$ which is such a function, but $\mathcal{L}(\omega)$ for every $\omega \in \Omega$. For $i \in \Sigma$ we write $\mathcal{L}_i(\omega) = (\mathcal{L}(\omega))(i, \cdot)$ for the curve of index i and note that the latter is a map $\mathcal{L}_i : \Omega \to C(\Lambda)$, which is $(\mathcal{C}, \mathcal{F})-$measurable. If $a, b \in \Lambda$ satisfy $a < b$ we let $\mathcal{L}_i[a, b]$ denote the restriction of $\mathcal{L}_i$ to $[a, b]$.

We will require the following result, whose proof is postponed until Section 8. In simple terms it states that the space $C(\Sigma \times \Lambda)$ where our random variables $\mathcal{L}$ take value has the structure of a complete, separable metric space.

Lemma 2.2. *Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. Suppose that $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ are sequences of real numbers such that $a_n < b_n$, $[a_n, b_n] \subset \Lambda$, $a_{n+1} \leq a_n$, $b_{n+1} \geq b_n$ and $\cup_{n=1}^\infty [a_n, b_n] = \Lambda$. For $n \in \mathbb{N}$ we let $K_n = \Sigma_n \times [a_n, b_n]$ where $\Sigma_n = \Sigma \cap [\![-n, n]\!]$. Define $d : C(\Sigma \times \Lambda) \times C(\Sigma \times \Lambda) \to [0, \infty)$ by*

$$d(f, g) = \sum_{n=1}^\infty 2^{-n} \min \Big\{ \sup_{(i,t) \in K_n} |f(i, t) - g(i, t)|, 1 \Big\}. \tag{2.1}$$

Then d defines a metric on $C(\Sigma \times \Lambda)$ and moreover the metric space topology defined by d is the same as the topology of uniform convergence over compact sets. Furthermore, the metric space $(C(\Sigma \times \Lambda), d)$ is complete and separable.

Definition 2.3. Given a sequence $\{\mathcal{L}^n : n \in \mathbb{N}\}$ of random Σ-indexed line ensembles we say that $\mathcal{L}^n$ *converge weakly* to a line ensemble $\mathcal{L}$, and write $\mathcal{L}^n \implies \mathcal{L}$ if for any bounded continuous function $f : C(\Sigma \times \Lambda) \to \mathbb{R}$ we have that

$$\lim_{n \to \infty} \mathbb{E}\left[f(\mathcal{L}^n)\right] = \mathbb{E}\left[f(\mathcal{L})\right].$$

We also say that $\{\mathcal{L}^n : n \in \mathbb{N}\}$ is *tight* if for any $\epsilon > 0$ there exists a compact set $K \subset C(\Sigma \times \Lambda)$ such that $\mathbb{P}(\mathcal{L}^n \in K) \geq 1 - \epsilon$ for all $n \in \mathbb{N}$.

We call a line ensemble *non-intersecting* if $\mathbb{P}$-almost surely $\mathcal{L}_i(r) > \mathcal{L}_j(r)$ for all $i < j$ and $r \in \Lambda$.

We will require the following sufficient condition for tightness of a sequence of line ensembles, which extends [1, Theorem 7.3]. We give a proof in Section 8.

Lemma 2.4. *Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. Suppose that $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ are sequences of real numbers such that $a_n < b_n$, $[a_n, b_n] \subset \Lambda$, $a_{n+1} \leq a_n$, $b_{n+1} \geq b_n$ and $\cup_{n=1}^\infty [a_n, b_n] = \Lambda$. Then $\{\mathcal{L}^n\}$ is tight if and only if for every $i \in \Sigma$ we have*

(i) $\lim_{a \to \infty} \limsup_{n \to \infty} \mathbb{P}(|\mathcal{L}_i^n(a_0)| \geq a) = 0$;

(ii) For all $\epsilon > 0$ and $k \in \mathbb{N}$, $\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\bigg(\sup_{\substack{x, y \in [a_k, b_k], \\ |x - y| \leq \delta}} |\mathcal{L}_i^n(x) - \mathcal{L}_i^n(y)| \geq \epsilon \bigg) = 0.$

We next turn to formulating the Brownian Gibbs property – we do this in Definition 2.8 after introducing some relevant notation and results. If W_t denotes a standard one-dimensional Brownian motion, then the process

$$\tilde{B}(t) = W_t - tW_1, \quad 0 \leq t \leq 1,$$

is called a *Brownian bridge (from $\tilde{B}(0) = 0$ to $\tilde{B}(1) = 0$) with diffusion parameter* 1. For brevity we call the latter object a *standard Brownian bridge.*

Given $a, b, x, y \in \mathbb{R}$ with $a < b$ we define a random variable on $(C([a, b]), \mathcal{C})$ through

$$B(t) = (b - a)^{1/2} \cdot \tilde{B}\left(\frac{t - a}{b - a}\right) + \left(\frac{b - t}{b - a}\right) \cdot x + \left(\frac{t - a}{b - a}\right) \cdot y, \tag{2.2}$$

and refer to the law of this random variable as a *Brownian bridge (from $B(a) = x$ to $B(b) = y$) with diffusion parameter* 1. Given $k \in \mathbb{N}$ and $\vec{x}, \vec{y} \in \mathbb{R}^k$ we let $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}$ denote the law of k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1.

We next state a couple of results about Brownian bridges from [4] for future use.

Lemma 2.5. *[4, Corollary 2.9]. Fix a continuous function $f : [0,1] \to \mathbb{R}$ such that $f(0) > 0$ and $f(1) > 0$. Let B be a standard Brownian bridge and let $C = \{B(t) > f(t)$ for some $t \in [0,1]\}$ (crossing) and $T = \{B(t) = f(t)$ for some $t \in [0,1]\}$ (touching). Then $\mathbb{P}(T \cap C^c) = 0$.*

Lemma 2.6. *[4, Corollary 2.10]. Let U be an open subset of $C([0,1])$, which contains a function f such that $f(0) = f(1) = 0$. If $B : [0,1] \to \mathbb{R}$ is a standard Brownian bridge then $\mathbb{P}(B[0,1] \subset U) > 0$.*

The following definition introduces the notion of an (f,g)-avoiding Brownian line ensemble, which in simple terms is a collection of k independent Brownian bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two continuous functions f and g.

Definition 2.7. Let $k \in \mathbb{N}$ and W_k° denote the open Weyl chamber in $\mathbb{R}^k$, i.e.

$$W_k^\circ = \{\vec{x} = (x_1, \dots, x_k) \in \mathbb{R}^k : x_1 > x_2 > \cdots > x_k\}$$

(in [4] the notation $\mathbb{R}^k_>$ was used for this set). Let $\vec{x}, \vec{y} \in W_k^\circ$, $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to (-\infty, \infty]$ and $g : [a,b] \to [-\infty, \infty)$ be two continuous functions. The latter condition means that either $f : [a,b] \to \mathbb{R}$ is continuous or $f = \infty$ everywhere, and similarly for g. We also assume that $f(t) > g(t)$ for all $t \in [a,b]$, $f(a) > x_1, f(b) > y_1$ and $g(a) < x_k, g(b) < y_k$.

With the above data we define the *(f,g)-avoiding Brownian line ensemble on the interval $[a,b]$ with entrance data $\vec{x}$ and exit data $\vec{y}$* to be the Σ-indexed line ensemble $\mathcal{Q}$ with $\Sigma = [\![1,k]\!]$ on $\Lambda = [a,b]$ and with the law of $\mathcal{Q}$ equal to $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}$ (the law of k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$) conditioned on the event

$$E = \{f(r) > B_1(r) > B_2(r) > \cdots > B_k(r) > g(r) \text{ for all } r \in [a,b]\}.$$

It is worth pointing out that E is an open set of positive measure and so we can condition on it in the usual way – we explain this briefly in the following paragraph. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that supports k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1. Notice that we can find $\tilde{u}_1, \dots, \tilde{u}_k \in C([0,1])$ and $\epsilon > 0$ (depending on $\vec{x}, \vec{y}, f, g, a, b$) such that $\tilde{u}_i(0) = \tilde{u}_i(1) = 0$ for $i = 1, \dots, k$ and such that if $\tilde{h}_1, \dots, \tilde{h}_k \in C([0,1])$ satisfy $\tilde{h}_i(0) = \tilde{h}_i(1) = 0$ for $i = 1, \dots, k$ and $\sup_{t \in [0,1]} |\tilde{u}_i(t) - \tilde{h}_i(t)| < \epsilon$ then the functions

$$h_i(t) = (b-a)^{1/2} \cdot \tilde{h}_i\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot x_i + \left(\frac{t-a}{b-a}\right) \cdot y_i,$$

satisfy $f(r) > h_1(r) > \cdots > h_k(r) > g(r)$. It follows from Lemma 2.6 that

$$\mathbb{P}(E) \geq \mathbb{P}\left(\max_{1 \leq i \leq k} \sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) = \prod_{i=1}^k \mathbb{P}\left(\sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) > 0,$$

and so we can condition on the event E.

To construct a realization of $\mathcal{Q}$ we proceed as follows. For $\omega \in E$ we define

$$\mathcal{Q}(\omega)(i,r) = B_i(r)(\omega) \text{ for } i = 1, \dots, k \text{ and } r \in [a,b].$$

Observe that for $i \in \{1, \dots, k\}$ and an open set $U \in C([a,b])$ we have that

$$\mathcal{Q}^{-1}(\{i\} \times U) = \{B_i \in U\} \cap E \ \in \ \mathcal{F},$$

and since the sets $\{i\} \times U$ form an open basis of $C([\![1,k]\!] \times [a,b])$ we conclude that $\mathcal{Q}$ is $\mathcal{F}$-measurable. This implies that the law $\mathcal{Q}$ is indeed well-defined and also it is non-intersecting almost surely. Also, given measurable subsets $A_1, \dots, A_k$ of $C([a,b])$ we have that

$$\mathbb{P}(\mathcal{Q}_i \in A_i \text{ for } i = 1, \dots, k) = \frac{\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}\left(\{B_i \in A_i \text{ for } i = 1, \dots, k\} \cap E\right)}{\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}(E)}.$$

We denote the probability distribution of $\mathcal{Q}$ as $\mathbb{P}^{a,b,\vec{x},\vec{y},f,g}_{avoid}$ and write $\mathbb{E}^{a,b,\vec{x},\vec{y},f,g}_{avoid}$ for the expectation with respect to this measure.

The following definition introduces the notion of the Brownian Gibbs property from [4].

Definition 2.8. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and an interval $\Lambda \subset \mathbb{R}$ and let $K = \{k_1, k_1 + 1, \dots, k_2\} \subset \Sigma$ be finite and $a, b \in \Lambda$ with $a < b$. Set $f = \mathcal{L}_{k_1 - 1}$ and $g = \mathcal{L}_{k_2+1}$ with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$ and $g = -\infty$ if $k_2 + 1 \notin \Sigma$. Write $D_{K,a,b} = K \times (a, b)$ and $D^c_{K,a,b} = (\Sigma \times \Lambda) \setminus D_{K,a,b}$. A Σ-indexed line ensemble $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$ is said to have the *Brownian Gibbs property* if it is non-intersecting and

$$\text{Law}\left(\mathcal{L}|_{K \times [a,b]} \text{ conditional on } \mathcal{L}|_{D^c_{K,a,b}}\right) = \text{Law}\left(\mathcal{Q}\right),$$

where $\mathcal{Q}_i = \tilde{\mathcal{Q}}_{i - k_1 + 1}$ and $\tilde{\mathcal{Q}}$ is the (f, g)-avoiding Brownian line ensemble on $[a, b]$ with entrance data $(\mathcal{L}_{k_1}(a), \dots, \mathcal{L}_{k_2}(a))$ and exit data $(\mathcal{L}_{k_1}(b), \dots, \mathcal{L}_{k_2}(b))$ from Definition 2.7. Note that $\tilde{Q}$ is introduced because, by definition, any such (f, g)-avoiding Brownian line ensemble is indexed from 1 to $k_2 - k_1 + 1$ but we want $\mathcal{Q}$ to be indexed from k_1 to k_2.

An equivalent way to express the Brownian Gibbs property is as follows. A Σ-indexed line ensemble $\mathcal{L}$ on Λ satisfies the Brownian Gibbs property if and only if it is non-intersecting and for any finite $K = \{k_1, k_1 + 1, \dots, k_2\} \subset \Sigma$ and $[a, b] \subset \Lambda$ and any bounded Borel-measurable function $F : C(K \times [a, b]) \to \mathbb{R}$ we have $\mathbb{P}$-almost surely

$$\mathbb{E}\left[F\left(\mathcal{L}|_{K \times [a,b]}\right) \middle| \mathcal{F}_{ext}(K \times (a, b))\right] = \mathbb{E}^{a,b,\vec{x},\vec{y},f,g}_{avoid}\left[F(\tilde{\mathcal{Q}})\right], \tag{2.3}$$

where

$$\mathcal{F}_{ext}(K \times (a, b)) = \sigma\left\{\mathcal{L}_i(s) : (i, s) \in D^c_{K,a,b}\right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K\times[a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K \times [a, b]$, $\vec{x} = (\mathcal{L}_{k_1}(a), \dots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \dots, \mathcal{L}_{k_2}(b))$, $f = \mathcal{L}_{k_1-1}[a, b]$ (the restriction of $\mathcal{L}$ to the set $\{k_1 - 1\} \times [a, b]$) with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$, and $g = \mathcal{L}_{k_2+1}[a, b]$ with the convention that $g = -\infty$ if $k_2 + 1 \notin \Sigma$.

Remark 2.9. Let us briefly explain why equation (2.3) makes sense. Firstly, since $\Sigma \times \Lambda$ is locally compact, we know by [12, Lemma 46.4] that $\mathcal{L} \to \mathcal{L}|_{K\times[a,b]}$ is a continuous map from $C(\Sigma \times \Lambda)$ to $C(K \times [a, b])$, so that the left side of (2.3) is the conditional expectation of a bounded measurable function, and is thus well-defined. A more subtle question is why the right side of (2.3) is $\mathcal{F}_{ext}(K \times (a, b))$-measurable. This question was resolved in [6, Lemma 3.4], where it was shown that the right side is measurable with respect to the σ-algebra

$$\sigma\left\{\mathcal{L}_i(s) : \ i \in K \text{ and } s \in \{a, b\}, \text{ or } i \in \{k_1 - 1, k_2 + 1\} \text{ and } s \in [a, b]\right\},$$

which in particular implies the measurability with respect to $\mathcal{F}_{ext}(K \times (a, b))$.

In the present paper it is convenient for us to use the following modified version of the definition above, which we call the partial Brownian Gibbs property – it was first introduced in [6]. We explain the difference between the two definitions, and why we prefer the second one in Remark 2.12.

Definition 2.10. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and an interval $\Lambda \subset \mathbb{R}$. A Σ-indexed line ensemble $\mathcal{L}$ on Λ is said to satisfy the *partial Brownian Gibbs property* if and only if it is non-intersecting and for any finite $K = \{k_1, k_1 + 1, \dots, k_2\} \subset \Sigma$ with $k_2 \le N - 1$ (if $\Sigma \neq \mathbb{N}$), $[a, b] \subset \Lambda$ and any bounded Borel-measurable function $F : C(K \times [a, b]) \to \mathbb{R}$ we have $\mathbb{P}$-almost surely

$$\mathbb{E}\left[F(\mathcal{L}|_{K\times[a,b]})\big|\mathcal{F}_{ext}(K \times (a, b))\right] = \mathbb{E}^{a,b,\vec{x},\vec{y},f,g}_{avoid}\left[F(\tilde{\mathcal{Q}})\right], \tag{2.4}$$

where we recall that $D_{K,a,b} = K \times (a, b)$ and $D^c_{K,a,b} = (\Sigma \times \Lambda) \setminus D_{K,a,b}$, and

$$\mathcal{F}_{ext}(K \times (a, b)) = \sigma\left\{\mathcal{L}_i(s) : (i, s) \in D^c_{K,a,b}\right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K\times[a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K \times [a, b]$, $\vec{x} = (\mathcal{L}_{k_1}(a), \dots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \dots, \mathcal{L}_{k_2}(b))$, $f = \mathcal{L}_{k_1-1}[a, b]$ with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$, and $g = \mathcal{L}_{k_2+1}[a, b]$.

Remark 2.11. Observe that if $N = 1$ then the conditions in Definition 2.10 become void. I.e., any line ensemble with one line satisfies the partial Brownian Gibbs property. Also we mention that (2.4) makes sense by the same reason that (2.3) makes sense, see Remark 2.9.

Remark 2.12. Definition 2.10 is slightly different from the Brownian Gibbs property of Definition 2.8 as we explain here. Assuming that $\Sigma = \mathbb{N}$ the two definitions are equivalent. However, if $\Sigma = \{1, \dots, N\}$ with $1 \le N < \infty$ then a line ensemble that satisfies the Brownian Gibbs property also satisfies the partial Brownian Gibbs property, but the reverse need not be true. Specifically, the Brownian Gibbs property allows for the possibility that $k_2 = N$ in Definition 2.10 and in this case the convention is that $g = -\infty$. As the partial Brownian Gibbs property is more general we prefer to work with it and most of the results later in this paper are formulated in terms of it rather than the usual Brownian Gibbs property.

2.2. **Bernoulli Gibbsian line ensembles.** In this section we introduce the notion of a *Bernoulli line ensemble* and the *Schur Gibbs property*. Our discussion will parallel that of [3, Section 3.1], which in turn goes back to [5, Section 2.1].

Definition 2.13. Let $\Sigma \subset \mathbb{Z}$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. Consider the set Y of functions $f : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(j, i + 1) - f(j, i) \in \{0, 1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$ and let $\mathcal{D}$ denote the discrete topology on Y. We call a function $f : [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(i + 1) - f(i) \in \{0, 1\}$ when $i \in [\![T_0, T_1 - 1]\!]$ an *up-right path* and elements in Y *collections of up-right paths.*

A Σ-*indexed Bernoulli line ensemble* $\mathfrak{L}$ *on* $[\![T_0, T_1]\!]$ is a random variable defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, taking values in Y such that $\mathfrak{L}$ is a $(\mathcal{B}, \mathcal{D})$-measurable function.

Remark 2.14. In [3, Section 3.1] Bernoulli line ensembles $\mathfrak{L}$ were called *discrete line ensembles* in order to distinguish them from the continuous line ensembles from Definition 2.1. In this paper we have opted to use the term Bernoulli line ensembles to emphasize the fact that the functions $f \in Y$ satisfy the property that $f(j, i+1) - f(j, i) \in \{0, 1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$. This condition essentially means that for each $j \in \Sigma$ the function $f(j, \cdot)$ can be thought of as the trajectory of a Bernoulli random walk from time T_0 to time T_1. As other types of discrete line ensembles, see e.g. [16], have appeared in the literature we have decided to modify the notation in [3, Section 3.1] so as to avoid any ambiguity.

The way we think of Bernoulli line ensembles is as random collections of up-right paths on the integer lattice, indexed by Σ (see Figure 1). Observe that one can view an up-right path L on $[\![T_0, T_1]\!]$ as a continuous curve by linearly interpolating the points $(i, L(i))$. This allows us to define $(\mathfrak{L}(\omega))(i, s)$ for non-integer $s \in [T_0, T_1]$ and to view Bernoulli line ensembles as line ensembles in the sense of Definition 2.1. In particular, we can think of $\mathfrak{L}$ as a random variable taking values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ with $\Lambda = [T_0, T_1]$. We will often slightly abuse notation and write $\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$, even

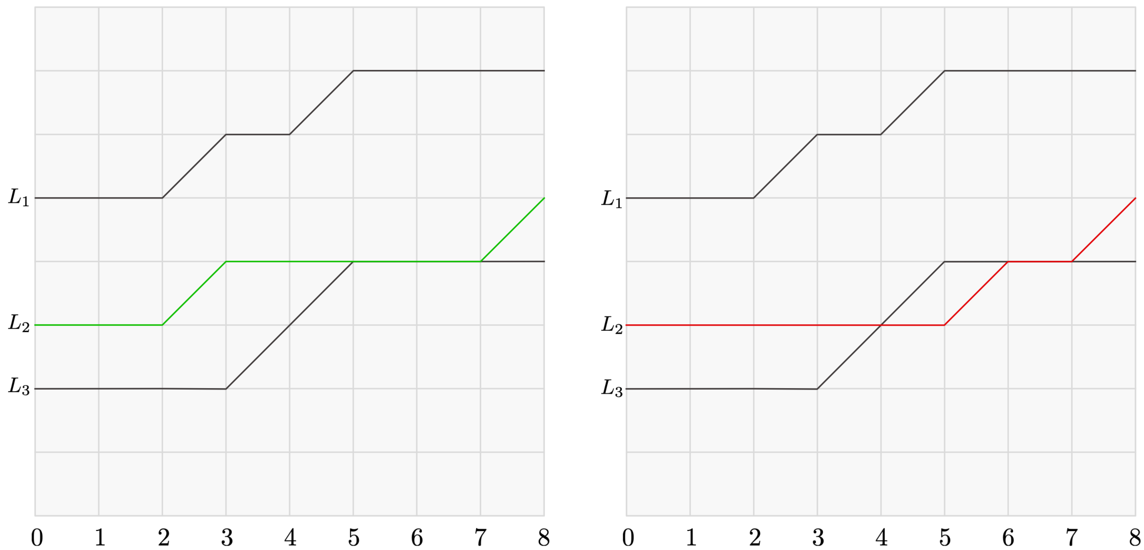

FIGURE 1. Two samples of $[\![1,3]\!]$-indexed Bernoulli line ensembles with $T_0 = 1$ and $T_1 = 8$, with the left ensemble avoiding and the right ensemble nonavoiding.

though it is not $\mathfrak{L}$ which is such a function, but rather $\mathfrak{L}(\omega)$ for each $\omega \in \Omega$. Furthermore we write $L_i = (\mathfrak{L}(\omega))(i, \cdot)$ for the index $i \in \Sigma$ path. If L is an up-right path on $[\![T_0, T_1]\!]$ and $a, b \in [\![T_0, T_1]\!]$ satisfy $a < b$ we let $L[\![a, b]\!]$ denote the resitrction of L to $[\![a, b]\!]$.

Let $t_i, z_i \in \mathbb{Z}$ for $i = 1, 2$ be given such that $t_1 < t_2$ and $0 \le z_2 - z_1 \le t_2 - t_1$. We denote by $\Omega(t_1, t_2, z_1, z_2)$ the collection of up-right paths that start from (t_1, z_1) and end at (t_2, z_2), by $\mathbb{P}_{Ber}^{t_1,t_2,z_1,z_2}$ the uniform distribution on $\Omega(t_1, t_2, z_1, z_2)$ and write $\mathbb{E}_{Ber}^{t_1,t_2,z_1,z_2}$ for the expectation with respect to this measure. One thinks of the distribution $\mathbb{P}_{Ber}^{t_1,t_2,z_1,z_2}$ as the law of a simple random walk with i.i.d. Bernoulli increments with parameter $p \in (0, 1)$ that starts from z_1 at time t_1 and is conditioned to end in z_2 at time t_2 – this interpretation does not depend on the choice of $p \in (0, 1)$. Notice that by our assumptions on the parameters the state space $\Omega(t_1, t_2, z_1, z_2)$ is non-empty.

Given $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$ and $\vec{x}, \vec{y} \in \mathbb{Z}^k$ we let $\mathbb{P}_{Ber}^{T_0,T_1,\vec{x},\vec{y}}$ denote the law of k independent Bernoulli bridges $\{B_i : [\![T_0, T_1]\!] \to \mathbb{Z}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$. Equivalently, this is just k independent random up-right paths $B_i \in \Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ that are uniformly distributed. This measure is well-defined provided that $\Omega(T_0, T_1, x_i, y_i)$ are non-empty for $i = 1, \dots, k$, which holds if $T_1 - T_0 \ge y_i - x_i \ge 0$ for all $i = 1, \dots, k$.

The following definition introduces the notion of an (f, g)-avoiding Bernoulli line ensemble, which in simple terms is a collection of k independent Bernoulli bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two functions f and g.

Definition 2.15. Let $k \in \mathbb{N}$ and $\mathfrak{W}_k$ denote the set of signatures of length k, i.e.

$$\mathfrak{W}_k = \{\vec{x} = (x_1, \dots, x_k) \in \mathbb{Z}^k : x_1 \ge x_2 \ge \cdots \ge x_k\}.$$

Let $\vec{x}, \vec{y} \in \mathfrak{W}_k$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, and $f : [\![T_0, T_1]\!] \to (-\infty, \infty]$ and $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ be two functions.

With the above data we define the (f, g)-*avoiding Bernoulli line ensemble on the interval* $[\![T_0, T_1]\!]$ *with entrance data* $\vec{x}$ *and exit data* $\vec{y}$ to be the Σ-indexed Bernoulli line ensemble $\mathfrak{Q}$ with $\Sigma = [\![1, k]\!]$ on $[\![T_0, T_1]\!]$ and with the law of $\mathfrak{Q}$ equal to $\mathbb{P}_{Ber}^{T_0,T_1,\vec{x},\vec{y}}$ (the law of k independent uniform up-right paths $\{B_i : [\![T_0, T_1]\!] \to \mathbb{R}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$) conditioned on the event

$$E = \{f(r) \ge B_1(r) \ge B_2(r) \ge \cdots \ge B_k(r) \ge g(r) \text{ for all } r \in [\![T_0, T_1]\!]\}.$$

The above definition is well-posed if there exist $B_i \in \Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ that satisfy the conditions in E (i.e. if the set of such up-right paths is not empty). We will denote by $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ the set of collections of k up-right paths that satisfy the conditions in E and then the distribution on $\mathfrak{Q}$ is simply the uniform measure on $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$. We denote the probability distribution of $\mathfrak{Q}$ as $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ and write $\mathbb{E}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ for the expectation with respect to this measure. When $f = +\infty$ and $g = -\infty$, we simply write $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y}}_{avoid,Ber}$ and $\mathbb{E}^{T_0,T_1,\vec{x},\vec{y}}_{avoid,Ber}$.

It will be useful to formulate simple conditions under which $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is non-empty and thus $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ well-defined. We accomplish this in the following lemma, whose proof is postponed until Section 8.

Lemma 2.16. *Suppose that $k \in \mathbb{N}$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. Suppose further that*

(1) $\vec{x}, \vec{y} \in \mathfrak{W}_k$ satisfy $T_1 - T_0 \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$

(2) $f : [\![T_0, T_1]\!] \to (-\infty, \infty]$ and $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ satisfy $f(i+1) = f(i)$ or $f(i+1) = f(i) + 1$, and $g(i+1) = g(i)$ or $g(i+1) = g(i) + 1$ for $i = T_0, \dots, T_1 - 1$

(3) $f(T_0) \geq x_1, f(T_1) \geq y_1$ and $g(T_0) \leq x_k, g(T_1) \leq y_k$

Then the set $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ from Definition 2.15 is non-empty.

The following definition introduces the notion of the Schur Gibbs property, which can be thought of a discrete analogue of the partial Brownian Gibbs property the same way that Bernoulli random walks are discrete analogues of Brownian motion.

Definition 2.17. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. A Σ-indexed Bernoulli line ensemble $\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ is said to satisfy the *Schur Gibbs property* if it is non-crossing, meaning that

$$L_j(i) \geq L_{j+1}(i) \text{ for all } j = 1, \dots, N-1 \text{ and } i \in [\![T_0, T_1]\!],$$

and for any finite $K = \{k_1, k_1 + 1, \dots, k_2\} \subset [\![1, N-1]\!]$ and $a, b \in [\![T_0, T_1]\!]$ with $a < b$ the following holds. Suppose that f, g are two up-right paths drawn in $\{(r, z) \in \mathbb{Z}^2 : a \leq r \leq b\}$ and $\vec{x}, \vec{y} \in \mathfrak{W}_{k_2 - k_1 + 1}$ altogether satisfy that $\mathbb{P}(A) > 0$ where A denotes the event

$$A = \{\vec{x} = (L_{k_1}(a), \dots, L_{k_2}(a)), \vec{y} = (L_{k_1}(b), \dots, L_{k_2}(b)), L_{k_1 - 1}[\![a, b]\!] = f, L_{k_2+1}[\![a, b]\!] = g\},$$

where if $k_1 = 1$ we adopt the convention $f = \infty = L_0$. Then for any $\{B_i \in \Omega(a, b, x_i, y_i)\}_{i=1}^{k_2 - k_1 + 1}$

$$\mathbb{P}\left(L_{i+k_1-1}[\![a, b]\!] = B_i \text{ for } i = 1, \dots, k_2 - k_1 + 1 | A\right) = \mathbb{P}^{a,b,\vec{x},\vec{y},f,g}_{avoid,Ber}\left(\cap_{i=1}^{k}\{\mathfrak{Q}_i = B_i\}\right). \tag{2.5}$$

Remark 2.18. In simple words, a Bernoulli line ensemble is said to satisfy the Schur Gibbs property if the distribution of any finite number of consecutive paths, conditioned on their end-points and the paths above and below them is simply the uniform measure on all collection of up-right paths that have the same end-points and do not cross each other or the paths above and below them.

Remark 2.19. Observe that in Definition 2.17 the index k_2 is assumed to be less than or equal to $N - 1$, so that if $N < \infty$ the N-th path is special and is not conditionally uniform. This is what makes Definition 2.17 a discrete analogue of the partial Brownian Gibbs property rather than the usual Brownian Gibbs property. Similarly to the partial Brownian Gibbs propert, see Remark 2.11, if $N = 1$ then the conditions in Definition 2.17 become void. I.e., any Bernoulli line ensemble with one line satisfies the Schur Gibbs property. Also we mention that the well-posedness of $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ in (2.5) is a consequence of Lemma 2.16 and our assumption that $\mathbb{P}(A) > 0$.

Remark 2.20. In [3] the authors studied a generalization of the Gibbs property in Definition 2.17 depending on a parameter $t \in (0, 1)$, which was called the *Hall-Littlewood Gibbs property* due to its connection to Hall-Littlewood polynomials [11]. The property in Definition 2.17 is the $t \to 0$ limit of the Hall-Littlewood Gibbs property. Since under this $t \to 0$ limit Hall-Littlewood polynomials degenerate to Schur polynomials we have decided to call the Gibbs property in Definition 2.17 the Schur Gibbs property.

Remark 2.21. An immediate consequence of Definition 2.17 is that if $M \leq N$, we have that the induced law on $\{L_i\}_{i=1}^M$ also satisfies the Schur Gibbs property as an $\{1, \dots, M\}$-indexed Bernoulli line ensemble on $[\![T_0, T_1]\!]$.

We end this section with the following definition of the term acceptance probability.

Definition 2.22. Assume the same notation as in Definition 2.15 and suppose that $T_1 - T_0 \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$. We define the *acceptance probability* $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$ to be the ratio

$$Z(T_0, T_1, \vec{x}, \vec{y}, f, g) = \frac{|\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)|}{\prod_{i=1}^k |\Omega(T_0, T_1, x_i, y_i)|}. \tag{2.6}$$

Remark 2.23. The quantity $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is precisely the probability that if B_i are sampled uniformly from $\Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ then the B_i satisfy the condition

$$E = \{f(r) \geq B_1(r) \geq B_2(r) \geq \cdots \geq B_k(r) \geq g(r) \text{ for all } r \in [\![T_0, T_1]\!]\}.$$

Let us explain briefly why we call this quantity an acceptance probability. One way to sample $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ is as follows. Start by sampling a sequence of i.i.d. up-right paths B_i^N uniformly from $\Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ and $N \in \mathbb{N}$. For each n check if $B_1^n, \dots, B_k^n$ satisfy the condition E and let M denote the smallest index that accomplishes this. If $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is non-empty then M is geometrically distributed with parameter $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$, and in particular M is finite almost surely and $\{B_i^M\}_{i=1}^k$ has distribution $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$. In this sampling procedure we construct a sequence of candidates $\{B_i^N\}_{i=1}^k$ for $N \in \mathbb{N}$ and reject those that fail to satisfy condition E, the first candidate that satisfies it is accepted and has law $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ and the probability that a candidate is accepted is precisely $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$, which is why we call it an acceptance probability.

2.3. Main technical result.

In this section we present the main technical result of the paper. We start with the following technical definition.

Definition 2.24. Fix $k \in \mathbb{N}$, $\alpha, \lambda > 0$ and $p \in (0, 1)$. Suppose we are given a sequence $\{T_N\}_{N=1}^\infty$ with $T_N \in \mathbb{N}$ and that $\{\mathfrak{L}^N\}_{N=1}^\infty$, $\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)$ is a sequence of $[\![1, k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N, T_N]\!]$. We call the sequence (α, p, λ)-*good* if

- for each $N \in \mathbb{N}$ we have that $\mathfrak{L}^N$ satisfies the Schur Gibbs property of Definition 2.17;
- there is a function $\psi : \mathbb{N} \to (0, \infty)$ such that $\lim_{N\to\infty} \psi(N) = \infty$ and for each $N \in \mathbb{N}$ we have that $T_N > \psi(N) N^\alpha$;
- there is a function $\phi : (0, \infty) \to (0, \infty)$ such that for any $\epsilon > 0$ we have

$$\sup_{n\in\mathbb{Z}} \limsup_{N\to\infty} \mathbb{P}\left(\left| N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2}) \right| \geq \phi(\epsilon) \right) \leq \epsilon. \tag{2.7}$$

Remark 2.25. Let us elaborate on the meaning of Definition 2.24. In order for a sequence of $\mathfrak{L}^N$ of $[\![1, k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N, T_N]\!]$ to be (α, p, λ)-*good* we want several conditions to be satisfied. Firstly, we want for each N the Bernoulli line ensemble $\mathfrak{L}^N$ to satisfy the Schur Gibbs property. The second condition is that while the interval of definition of $\mathfrak{L}^N$ is finite for each N and given by $[\![-T_N, T_N]\!]$, we want this interval to grow at least with speed N^α. This property is quantified by the function ψ, which can be essentially thought of as an arbitrary unbounded increasing function on $\mathbb{N}$. The third condition is that we want for each $n \in \mathbb{Z}$ the sequence of random variables $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha)$ to be tight but moreover we want globally these random variables to look like the parabola $-\lambda n^2$. This statement is reflected in (2.7), which provides a certain uniform tightness of the random variables $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})$. A particular case when (2.7) is satisfied is for example if we know that for each $n \in \mathbb{Z}$

the random variables $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})$ converge to the same random variable X. In the applications that we have in mind these random variables would converge to the 1-point marginals of the Airy$_2$ process that are all given by the same Tracy-Widom distribution (since the Airy$_2$ process is stationary). Equation (2.7) is a significant relaxation of the requirement that $N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})$ all converge weakly to the Tracy-Widom distribution – the convergence requirement is replaced with a mild but uniform control of all subsequential limits.

The main result of the paper is as follows.

Theorem 2.26. *Fix $k \in \mathbb{N}$ with $k \geq 2$, $\alpha, \lambda > 0$ and $p \in (0,1)$ and let $\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)$ be an (α, p, λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles. Set*

$$f_i^N(s) = N^{-\alpha/2}(L_i^N(sN^\alpha) - psN^\alpha + \lambda s^2 N^{\alpha/2}), \text{ for } s \in [-\psi(N), \psi(N)] \text{ and } i = 1, \dots, k-1,$$

and extend f_i^N to $\mathbb{R}$ by setting for $i = 1, \dots, k-1$

$$f_i^N(s) = f_i^N(-\psi(N)) \text{ for } s \leq -\psi(N) \text{ and } f_i^N(s) = f_N(\psi(N)) \text{ for } s \geq \psi(N).$$

Let $\mathbb{P}_N$ denote the law of $\{f_i^N\}_{i=1}^{k-1}$ as a $[\![1,k-1]\!]$-indexed line ensemble (i.e. as a random variable in $(C([\![1,k-1]\!] \times \mathbb{R}), \mathcal{C})$). Then the sequence $\mathbb{P}_N$ is tight. Moreover, if $\mathcal{L}^\infty = \{f_i^{infty}\}_{i=1}^{k-1}$ denotes any subsequential limit of $\mathbb{P}_N$ then $\mathcal{L}^\infty$ satisfies the Brownian Gibbs property in Definition 2.10.

Roughly, Theorem 2.26 states that if you have a sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles that satisfy the Schur Gibbs property and the top paths of these ensembles under some shift and scaling have tight one-point marginals with a non-trivial parabolic shift, then under the same shift and scaling the top $k-1$ paths of the line ensemble will be tight. The extension of f_i^N to $\mathbb{R}$ is completely arbitrary and irrelevant for the validity of Theorem 2.26 since the topology on $C([\![1,k-1]\!] \times \mathbb{R})$ is that of uniform convergence over compacts. Consequently, only the behavior of these functions on compact intervals matters in Theorem 2.26 and not what these functions do near infinity, which is where the modification happens as $\lim_{N\to\infty} \psi(N) = \infty$ by assumption. The only reason we perform the extension is to embed all Bernoulli line ensembles into the same space $(C([\![1,k-1]\!] \times \mathbb{R}), \mathcal{C})$.

We mention that the k-th up-right path in the sequence of Bernoulli line ensembles is special and Theorem 2.26 provides no tightness result for it. The reason for this stems from the Schur Gibbs property, see Definition 2.17, which assumes less information for the k-th path. In practice, one either has an infinite Bernoulli line ensemble for each N or one has a Bernoulli line ensemble with finite number of paths, which increase with N to infinity. In either of these settings one can use Theorem 2.26 to prove tightness of the full line ensemble - we will have more to say about this in Section 7.

The proof of Theorem 2.26 is presented in Section 4. In the next section we derive various properties for Bernoulli line ensembles.

3. Properties of Bernoulli line ensembles

In this section we derive several results for Bernoulli line ensembles, which will be used in the proof of Theorem 2.26 in Section 4.

3.1. **Monotone coupling lemmas.**

In this section we formulate two lemmas that provide couplings of two Bernoulli line ensembles of non-intersecting Bernoulli bridges on the same interval, which depend monotonically on their boundary data. Schematic depictions of the couplings are provided in Figure 2. We postpone the proof of these lemmas until Section 8.

Lemma 3.1. *Assume the same notation as in Definition 2.15. Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, a function $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ as well as $\vec{x}, \vec{y}, \vec{x}\,', \vec{y}\,' \in \mathfrak{W}_k$. Assume that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}\,', \vec{y}\,', \infty, g)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,*

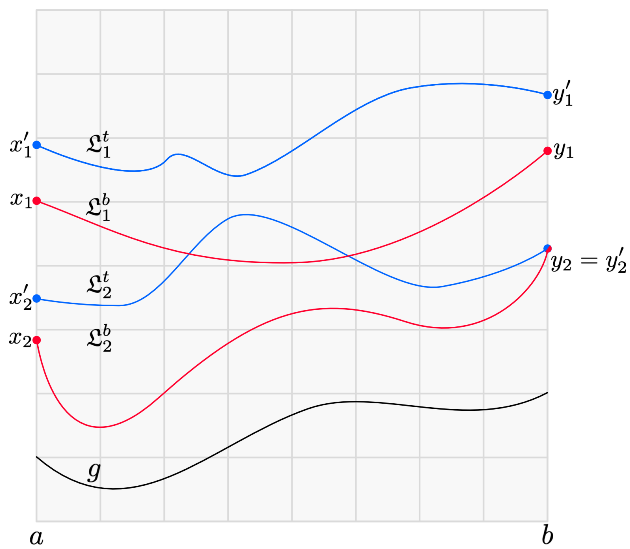

FIGURE 2. Two diagrammatic depictions of the monotone coupling Lemma 3.1 (left part) and Lemma 3.2 (right part). Red depicts the lower line ensemble and accompanying entry data, exit data, and bottom bounding curve, while blue depicts that of the higher ensemble.

which supports two $[\![1,k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0,T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\left(\text{resp. } \mathfrak{L}^b\right)$ under $\mathbb{P}$ is given by $\mathbb{P}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g}_{avoid,Ber}$ $\left(\text{resp. } \mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber}\right)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}^t_i(r) \geq \mathfrak{L}^b_i(r)$ for all $i = 1,\dots,k$ and $r \in [\![T_0,T_1]\!]$.

Lemma 3.2. *Assume the same notation as in Definition 2.15. Fix $k \in \mathbb{N}$, $T_0,T_1 \in \mathbb{Z}$ with $T_0 < T_1$, two functions $g^t, g^b : [\![T_0,T_1]\!] \to [-\infty,\infty)$ and $\vec{x},\vec{y} \in \mathfrak{W}_k$. We assume that $g^t(r) \geq g^b(r)$ for all $r \in [\![T_0,T_1]\!]$ and that $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},\infty,g^t)$ and $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},\infty,g^b)$ are both non-empty. Then there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$, which supports two $[\![1,k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0,T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\left(\text{resp. } \mathfrak{L}^b\right)$ under $\mathbb{P}$ is given by $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^t}_{avoid,Ber}$ $\left(\text{resp. } \mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber}\right)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}^t_i(r) \geq \mathfrak{L}^b_i(r)$ for all $i = 1,\dots,k$ and $r \in [\![T_0,T_1]\!]$.*

In plain words, Lemma 3.1 states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ of non-intersecting Bernoulli bridges, bounded from below by the same function g, in such a way that if all boundary values of $\mathfrak{L}^t$ are above the respective boundary values of $\mathfrak{L}^b$, then all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the left part of Figure 2. Lemma 3.2, states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ that have the same boundary values, but the lower bound g^t of $\mathfrak{L}^t$ is above the lower bound g^b of $\mathfrak{L}^b$, in such a way that all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the right part of Figure 2.

3.2. **Properties of Bernoulli bridges.** In this section we derive several results about Bernoulli bridges, which are random up-right paths that have law $\mathbb{P}^{T_0,T_1,x,y}_{Ber}$ as in Section 2.2. Our results will rely on the two monotonicity Lemmas 3.1 and 3.2 as well as a strong coupling between Bernoulli bridges and Brownian bridges from [3] – recalled here as Theorem 3.3.

If W_t denotes a standard one-dimensional Brownian motion and $\sigma > 0$, then the process

$$B^\sigma_t = \sigma(W_t - tW_1), \quad 0 \leq t \leq 1,$$

is called a *Brownian bridge (conditioned on $B_0 = 0, B_1 = 0$) with variance σ^2*. We note that B^σ is the unique a.s. continuous Gaussian process on $[0,1]$ with $B_0 = B_1 = 0$, $\mathbb{E}[B^\sigma_t] = 0$, and

$$\mathbb{E}[B^\sigma_r B^\sigma_s] = \sigma^2(r \wedge s - rs - sr + sr) = \sigma^2(r \wedge s - rs). \tag{3.1}$$

With the above notation we state the strong coupling result we use.

Theorem 3.3. *Let $p \in (0,1)$. There exist constants $0 < C, a, \alpha < \infty$ (depending on p) such that for every positive integer n, there is a probability space on which are defined a Brownian bridge B^σ with variance $\sigma^2 = p(1-p)$ and a family of random paths $\ell^{(n,z)} \in \Omega(0,n,0,z)$ for $z = 0,\dots,n$ such that $\ell^{(n,z)}$ has law $\mathbb{P}^{0,n,0,z}_{Ber}$ and*

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \le Ce^{\alpha(\log n)^2}e^{|z-pn|^2/n}, \text{ where } \Delta(n,z) := \sup_{0\le t\le n}\left|\sqrt{n}B^\sigma_{t/n} + \tfrac{t}{n}z - \ell^{(n,z)}(t)\right|. \tag{3.2}$$

Remark 3.4. When $p = 1/2$ the above theorem follows (after a trivial affine shift) from [10, Theorem 6.3] and the general $p \in (0,1)$ case was done in [3, Theorem 4.5]. We mention that a significant generalization of Theorem 3.3 for general random walk bridges has recently been proved in [7, Theorem 2.3].

We will use the following simple corollary of Theorem 3.3 in the following to compare Bernoulli bridges with Brownian bridges. We use the same notation as in the theorem.

Corollary 3.5. *Fix $p \in (0,1)$, $\beta > 0$, and $A > 0$. Suppose $|z - pn| \le K\sqrt{n}$ for a constant $K > 0$. Then for any $\epsilon > 0$, there exists N large enough depending on p, ϵ, A, K so that for $n \ge N$,*

$$\mathbb{P}\Big(\Delta(n,z) \ge An^\beta\Big) < \epsilon.$$

Proof. Applying Chebyshev's inequality and (3.2) gives

$$\begin{aligned}\mathbb{P}\Big(\Delta(n,z) \ge An^\beta\Big) &\le e^{-An^\beta}\mathbb{E}\Big[e^{a\Delta(n,z)}\Big] \le C\exp\Big[-An^\beta + \alpha(\log n)^2 + \frac{|z-pn|^2}{n}\Big] \\ &\le C\exp\Big[-An^\beta + \alpha(\log n)^2 + K\Big].\end{aligned}$$

The conclusion is now immediate. □

We also state the following result regarding the distribution of the maximum of a Brownian bridge, which follows from a formula in [9, Chapter 4].

Lemma 3.6. *Fix $p \in (0,1)$, and let B^σ be a Brownian bridge of variance $\sigma^2 = p(1-p)$ on $[0,1]$. Then for any $C, T > 0$ we have*

$$\mathbb{P}\Big(\max_{s\in[0,T]} B^\sigma_{s/T} \ge C\Big) = \exp\left(-\frac{2C^2}{p(1-p)}\right) \quad \text{and} \quad \mathbb{P}\Big(\max_{s\in[0,T]} \big|B^\sigma_{s/T}\big| \ge C\Big) \le 2\exp\left(-\frac{2C^2}{p(1-p)}\right).$$

Proof. Let B^1 be a Brownian bridge with variance 1 on $[0,1]$. Then B^σ_t has the same distribution as σB^1_t. Hence

$$\mathbb{P}\Big(\max_{s\in[0,T]} B^\sigma_{s/T} \ge C\Big) = \mathbb{P}\Big(\max_{t\in[0,1]} B^1_t \ge C/\sigma\Big) = e^{-2(C/\sigma)^2} = e^{-2C^2/p(1-p)}.$$

The second equality follows from [9, Chapter 4, (3.40)]. We now observe that since B^σ_t has mean 0, B^σ_t and $-B^\sigma_t$ have the same distribution. Hence by the equality just proven,

$$\begin{aligned}\mathbb{P}\Big(\max_{s\in[0,T]} \big|B^\sigma_{s/T}\big| \ge C\Big) &\le \mathbb{P}\Big(\max_{s\in[0,T]} B^\sigma_{s/T} \ge C\Big) + \mathbb{P}\Big(\max_{s\in[0,T]} \big(-B^\sigma_{s/T}\big) \ge C\Big) \\ &= 2\,\mathbb{P}\Big(\max_{s\in[0,T]} B^\sigma_{s/T} \ge C\Big) = 2e^{-2C^2/p(1-p)}.\end{aligned}$$

□

We state one more lemma about Brownian bridges, which allows us to decompose a bridge on $[0,1]$ into two independent bridges with Gaussian affine shifts meeting at a point in $(0,1)$.

Lemma 3.7. *Fix $p \in (0,1)$, $T > 0$, $t \in (0,T)$, and let B^σ be a Brownian bridge of variance $\sigma^2 = p(1-p)$ on $[0,1]$. Let ξ be a Gaussian random variable with mean 0 and variance*

$$\mathbb{E}[\xi^2] = \sigma^2 \frac{t}{T}\left(1 - \frac{t}{T}\right).$$

Let B^1, B^2 be two independent Brownian bridges on $[0,1]$ with variances $\sigma^2 t/T$ and $\sigma^2(T-t)/T$ respectively, also independent from B^σ. Define the process

$$\tilde{B}_{s/T} = \begin{cases} \frac{s}{t}\,\xi + B^1\Big(\frac{s}{t}\Big), & s \leq t, \\ \frac{T-s}{T-t}\,\xi + B^2\Big(\frac{s-t}{T-t}\Big), & s \geq t, \end{cases}$$

for $s \in [0,T]$. Then $\tilde{B}$ is a Brownian bridge with variance σ.

Proof. It is clear that the process $\tilde{B}$ is a.s. continuous, and each $\tilde{B}_s$ is Gaussian with mean 0 since it is a linear combination of centered Gaussians. By 3.1, it suffices to verify that if $0 \leq r \leq s \leq T$, then

$$\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \sigma^2 \frac{r}{T}\Big(1 - \frac{s}{T}\Big). \tag{3.3}$$

First assume $s \leq t$ Using the fact that ξ and $B^1_\cdot$ are independent with mean 0, we find

$$\begin{aligned}\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] &= \frac{rs}{t^2}\cdot\sigma^2\frac{t}{T}\Big(1-\frac{t}{T}\Big) + \sigma^2\frac{t}{T}\cdot\frac{r}{t}\Big(1-\frac{s}{t}\Big) \\ &= \sigma^2\frac{r}{T}\Big(\frac{s}{t} - \frac{s}{T} + 1 - \frac{s}{t}\Big) = \sigma^2\frac{r}{T}\Big(1-\frac{s}{T}\Big).\end{aligned}$$

If $r \geq t$, we compute

$$\begin{aligned}\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] &= \frac{(T-r)(T-s)}{(T-t)^2}\cdot\sigma^2\frac{t}{T}\Big(1-\frac{t}{T}\Big) + \sigma^2\frac{T-t}{T}\cdot\frac{r-t}{T-t}\Big(1-\frac{s-t}{T-t}\Big) \\ &= \frac{\sigma^2(T-s)}{T(T-t)}\Big(\frac{t(T-r)}{T} + r - t\Big) = \frac{\sigma^2(T-s)}{T(T-t)}\cdot\frac{r(T-t)}{T} = \sigma^2\frac{r}{T}\Big(1-\frac{s}{T}\Big).\end{aligned}$$

If $r < t < s$, then since ξ, $B^1_\cdot$, and $B^2_\cdot$ are all independent, we have

$$\mathbb{E}[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \frac{r}{t}\cdot\frac{T-s}{T-t}\cdot\sigma^2\frac{t(T-t)}{T^2} = \sigma^2\frac{r(T-s)}{T^2} = \sigma^2\frac{r}{T}\Big(1-\frac{s}{T}\Big).$$

This proves (3.3) in all cases. ☐

Below we list six lemmas about Bernoulli bridges. We provide a brief informal explanation of what each result says after it is stated. All six lemmas are proved in a similar fashion. For the first four lemmas one observes that the event, whose probability is being estimated, is monotone in ℓ. This allows by Lemmas 3.1 and 3.2 to replace x, y in the statements of the lemmas with the extreme values of the ranges specified in each. Once the choice of x and y is fixed one can use our strong coupling results, Theorem 3.3 and Corollary 3.5, to reduce each of the lemmas to an analogous one involving a Brownian bridge with some prescribed variance. The latter statements are then easily confirmed as one has exact formulas for Brownian bridges, such as Lemma 3.6.

Lemma 3.8. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}_{Ber}^{0,T,x,y}$. Let $M_1, M_2 \in \mathbb{R}$ be given. Then we can find $W_0 = W_0(p, M_2 - M_1) \in \mathbb{N}$ such that for $T \geq W_0$, $x \geq M_1 T^{1/2}$, $y \geq pT + M_2 T^{1/2}$ and $s \in [0,T]$ we have*

$$\mathbb{P}_{Ber}^{0,T,x,y}\Big(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\Big) \geq \frac{1}{3}. \tag{3.4}$$

Remark 3.9. If $M_1, M_2 = 0$ then Lemma 3.8 states that if a Bernoulli bridge ℓ is started from $(0,x)$ and terminates at (T,y), which are above the straight line of slope p, then at any given time $s \in [0,T]$ the probability that $\ell(s)$ goes a modest distance below the straight line of slope p is upper bounded by $2/3$.

Proof. Define $A = \lfloor M_1 T^{1/2} \rfloor$ and $B = \lfloor pT + M_2 T^{1/2} \rfloor$. Then since $A \leq x$ and $B \leq y$, it follows from Lemma 3.1 that there is a probability space with measure $\mathbb{P}_0$ supporting random variables L_1 and L_2, whose laws under $\mathbb{P}_0$ are $\mathbb{P}_{Ber}^{0,T,A,B}$ and $\mathbb{P}_{Ber}^{0,T,x,y}$ respectively, and $\mathbb{P}_0$-a.s. we have $L_1 \leq L_2$. Thus

$$\begin{aligned}
&\mathbb{P}_{Ber}^{0,T,x,y}\Big(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\Big) \\
&= \mathbb{P}_0\Big(L_2(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\Big) \\
&\geq \mathbb{P}_0\Big(L_1(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\Big) \\
&= \mathbb{P}_{Ber}^{0,T,A,B}\Big(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\Big).
\end{aligned}$$

Since upright paths on $[\![0,T]\!] \times [\![A,B]\!]$ are equivalent to upright paths on $[\![0,T]\!] \times [\![0,B-A]\!]$ shifted vertically by A, the last line is equal to

$$\mathbb{P}_{Ber}^{0,T,0,B-A}\Big(\ell(s) + A \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\Big).$$

Now we consider the coupling provided by Theorem 3.3. We have another probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random variable $\ell^{(T,B-A)}$ whose law under $\mathbb{P}$ is $\mathbb{P}_{Ber}^{0,T,0,B-A}$, and a Brownian bridge B^σ. Then

$$\begin{aligned}
&\mathbb{P}_{Ber}^{0,T,0,B-A}\Big(\ell(s) + A \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\Big) \\
&= \mathbb{P}\Big(\ell^{(T,B-A)}(s) + A \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\Big) \\
&= \mathbb{P}\Big(\Big[\ell^{(T,B-A)}(s) - \sqrt{T} B^\sigma_{s/T} - \frac{s}{T} \cdot (B-A)\Big] + \sqrt{T} B^\sigma_{s/T} \geq -A - \frac{s}{T} \cdot (B-A) \\
&\qquad + \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2 T^{1/2}\big) - T^{1/4}\Big).
\end{aligned}$$

Recalling the definitions of A and B, we can rewrite the quantity on the right hand side in the last expression and bound it by

$$\begin{aligned}
\frac{T-s}{T} \cdot (M_1 T^{1/2} - A) + \frac{s}{T} \cdot (pT + M_2 T^{1/2} - B) - T^{1/4} &\leq \frac{T-s}{T} + \frac{s}{T} - T^{1/4} \\
&= -T^{1/4} + 1.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{P}^{0,T,0,B-A}_{Ber}\Big(\ell(s) + A \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot (pT + M_2 T^{1/2}) - T^{1/4}\Big) \\
&\geq \mathbb{P}\Big(\Big[\ell^{(T,B-A)}(s) - \sqrt{T} B^\sigma_{s/T} - \frac{s}{T} \cdot (B-A)\Big] + \sqrt{T} B^\sigma_{s/T} \geq -T^{1/4} + 1\Big) \\
&\geq \mathbb{P}\Big(\sqrt{T} B^\sigma_{s/T} \geq 0 \quad \text{and} \quad \Delta(T, B-A) < T^{1/4} - 1\Big) \\
&\geq \mathbb{P}\left(B^\sigma_{s/T} \geq 0\right) - \mathbb{P}\left(\Delta(T, B-A) \geq T^{1/4} - 1\right) \\
&= \frac{1}{2} - \mathbb{P}\left(\Delta(T, B-A) \geq T^{1/4} - 1\right).
\end{aligned}$$

For the second inequality, we used the fact that the quantity in brackets is bounded in absolute value by $\Delta(T, B-A)$. The third inequality follows by dividing the event $\{B^\sigma_{s/T} \geq 0\}$ into cases and applying subadditivity. Since $|B - A - pT| \leq (M_2 - M_1 + 1)\sqrt{T}$, Corollary 3.5 allows us to choose W_0 large enough depending on p and $M_2 - M_1$ so that if $T \geq W_0$, then the last line is bounded above by $1/2 - 1/6 = 1/3$. This proves (3.4). $\square$

Lemma 3.10. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $y, z \in \mathbb{Z}$ such that $T \geq y, z \geq 0$, and suppose that ℓ_y, ℓ_z have distributions $\mathbb{P}^{0,T,0,y}_{Ber}$, $\mathbb{P}^{0,T,0,z}_{Ber}$ respectively. Let $M > 0$ and $\epsilon > 0$ be given. Then we can find $W_1 = W_1(M, p, \epsilon) \in \mathbb{N}$ and $A = A(M, p, \epsilon) > 0$ such that for $T \geq W_1$, $y \geq pT - MT^{1/2}$, $z \leq pT + MT^{1/2}$ we have*

$$\begin{aligned}
&\mathbb{P}^{0,T,0,y}_{Ber}\Big(\inf_{s \in [0,T]} \big(\ell(s) - ps\big) \leq -AT^{1/2}\Big) \leq \epsilon, \\
&\mathbb{P}^{0,T,0,z}_{Ber}\Big(\sup_{s \in [0,T]} \big(\ell(s) - ps\big) \geq AT^{1/2}\Big) \leq \epsilon.
\end{aligned} \tag{3.5}$$

Remark 3.11. Roughly, Lemma 3.10 states that if a Bernoulli bridge ℓ is started from $(0,0)$ and terminates at time T not significantly lower (resp. higher) than the straight line of slope p, then the event that ℓ goes significantly below (resp. above) the straight line of slope p is very unlikely.

Proof. The two inequalities are proven in essentially the same way. We begin with the first inequality. It follows from Lemma 3.1 that

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(\inf_{s \in [0,T]} \big(\ell(s) - ps\big) \leq -AT^{1/2}\Big) \leq \mathbb{P}^{0,T,0,B}_{Ber}\Big(\inf_{s \in [0,T]} \big(\ell(s) - ps\big) \leq -AT^{1/2}\Big),$$

where $B = \lfloor pT - MT^{1/2} \rfloor$. By Theorem 3.3, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random variable $\ell^{(T,B)}$ whose law under $\mathbb{P}$ is that of ℓ, and a Brownian bridge B^σ with variance $\sigma^2 = p(1-p)$. Therefore

$$\begin{aligned}
&\mathbb{P}^{0,T,0,B}_{Ber}\Big(\inf_{s \in [0,T]} \big(\ell(s) - ps\big) \leq -AT^{1/2}\Big) = \mathbb{P}\Big(\inf_{s \in [0,T]} \big(\ell^{(T,B)}(s) - ps\big) \leq -AT^{1/2}\Big) \\
&\leq \mathbb{P}\Big(\inf_{s \in [0,T]} \sqrt{T} B^\sigma_{s/T} \leq -\frac{1}{2} AT^{1/2}\Big) + \mathbb{P}\Big(\sup_{s \in [0,T]} \Big|\sqrt{T} B^\sigma_{s/T} + ps - \ell^{(T,B)}(s)\Big| \geq \frac{1}{2} AT^{1/2}\Big) \\
&\leq \mathbb{P}\Big(\max_{s \in [0,T]} B^\sigma_{s/T} \geq A/2\Big) + \mathbb{P}\Big(\Delta(T, B) \geq \frac{1}{2} AT^{1/2} - MT^{1/2} - 1\Big). \qquad (3.6)
\end{aligned}$$

For the first term in the last line, we used the fact that B^σ and $-B^\sigma$ have the same distribution. For the second term, we used the fact that

$$\sup_{s \in [0,T]} \Big|ps - \frac{s}{T} \cdot B\Big| \leq \sup_{s \in [0,T]} \Big|ps - \frac{pT - MT^{1/2}}{T} \cdot s\Big| + 1 = MT^{1/2} + 1.$$

By Lemma 3.6, the first term in (3.6) is equal to $e^{-A^2/2p(1-p)}$. If we choose $A \geq \sqrt{2p(1-p)\log(2/\epsilon)}$, then this is $\leq \epsilon/2$. If we also take $A > 2M$, then since $|B - pT| \leq (M+1)\sqrt{T}$, Corollary 3.5 gives us a W_1 large enough depending on M, p, ϵ so that the second term in (3.6) is also $< \epsilon/2$ for $T \geq W_1$. Adding the two terms gives the first inequality in (3.5).

If we replace B with $\lceil pT + MT^{1/2} \rceil$ and change signs and inequalities where appropriate, then the same argument proves the second inequality in (3.5). ☐

Lemma 3.12. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,x,y}_{Ber}$. Let $M_1, M_2 > 0$ be given. Then we can find $W_2 = W_2(M_1, M_2, p) \in \mathbb{N}$ such that for $T \geq W_2$, $x \geq -M_1 T^{1/2}$, $y \geq pT - M_1 T^{1/2}$ and $t \in (0,1)$ we have*

$$\mathbb{P}^{0,T,x,y}_{Ber}\Big(\ell(tT) \geq t(M_2 T^{1/2} + pT) - T^{1/4}\Big) \geq (1/2)(1 - \Phi^v(M_1 + M_2)), \tag{3.7}$$

where Φ^v is the cumulative distribution function of a Gaussian random variable with mean 0 and variance $v = pt(1-p)(1-t)$.

Remark 3.13. Lemma 3.12 states that if a Bernoulli bridge ℓ is started from $(0, x)$ and terminates at (T, y) with these points not significantly lower than the straight line of slope p, then its mid-point would lie well above the straight line of slope p at least with some quantifiably tiny probability.

Proof. By Lemma 3.1, we have

$$\begin{aligned}\mathbb{P}^{0,T,x,y}_{Ber}\Big(\ell(tT) \geq t(M_2T^{1/2} + pT) - T^{1/4}\Big) &\geq \mathbb{P}^{0,T,0,B-A}_{Ber}\Big(\ell(tT) + A \geq t(M_2T^{1/2} + pT) - T^{1/4}\Big) \\ &= \mathbb{P}\Big(\ell^{(T,B-A)}(tT) + A \geq t(M_2T^{1/2} + pT) - T^{1/4}\Big),\end{aligned}$$

with $A = \lfloor -M_1T^{1/2} \rfloor$, $B = \lfloor pT - M_1T^{1/2} \rfloor$, and $\mathbb{P}$, and $\ell^{(T,B-A)}$ provided by Theorem 3.3. If B^σ is as in the theorem, we can rewrite the expression on the second line as

$$\mathbb{P}\Big(\big[\ell^{(T,B-A)}(tT) - \sqrt{T}\,B^\sigma_t - t(B-A)\big] + \sqrt{T}\,B^\sigma_t \geq -A - t(B-A) + t(M_2T^{1/2} + pT) - T^{1/4}\Big).$$

We have

$$\begin{aligned}-A - t(B-A) + t(M_2T^{1/2} + pT) - T^{1/4} &\leq M_1T^{1/2} + 1 - t(pT - 1) + t(M_2T^{1/2} + pT) - T^{1/4} \\ &\leq (M_1 + M_2)T^{1/2} - T^{1/4} + 2.\end{aligned}$$

Thus the probability in question is bounded below by

$$\begin{aligned}&\mathbb{P}\Big(\big[\ell^{(T,B-A)}(tT) - \sqrt{T}\,B^\sigma_t - t(B-A)\big] + \sqrt{T}\,B^\sigma_t \geq (M_1+M_2)T^{1/2} - T^{1/4} + 2\Big) \\ &\geq \mathbb{P}\Big(\sqrt{T}\,B^\sigma_t \geq (M_1+M_2)T^{1/2} \quad \text{and} \quad \Delta(T, B-A) < T^{1/4} - 2\Big) \\ &\geq \mathbb{P}\big(B^\sigma_t \geq M_1 + M_2\big) - \mathbb{P}\big(\Delta(T, B-A) \geq T^{1/4} - 2\big).\end{aligned}$$

Note that $B^\sigma_t = \sigma(W_t - tW_1)$ for a standard Brownian motion W on $[0,1]$. Thus B^σ_t is Gaussian with mean 0 and variance $\sigma^2(t - t^2) = p(1-p)t(1-t) = v$. In particular, the first term in the last line is equal to

$$1 - \Phi^v(M_1 + M_2),$$

where Φ^v is the cdf for a Gaussian random variable with mean 0 and variance v. By Corollary 3.5, since $|B - A - pT| \leq 1$, we can choose W_2 depending on M_1, M_2, and p so that the second term is less than $1/2$ the first term for $T \geq W_2$. This proves (3.7). ☐

Lemma 3.14. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,x,y}_{Ber}$. Then we can find $W_3 = W_3(p) \in \mathbb{N}$ such that for $T \geq W_3$, $x \geq T^{1/2}$, $y \geq pT + T^{1/2}$*

$$\mathbb{P}^{0,T,x,y}_{Ber}\Big(\inf_{s \in [0,T]} \big(\ell(s) - ps\big) + T^{1/4} \geq 0\Big) \geq \frac{1}{2}\left(1 - \exp\left(-\frac{2}{p(1-p)}\right)\right). \tag{3.8}$$

Remark 3.15. Lemma 3.14 states that if a Bernoulli bridge ℓ is started from $(0,x)$ and terminates at (T,y) with $(0,x)$ and (T,y) well above the line of slope p then at least with some positive probability ℓ will not fall significantly below the line of slope p.

Proof. By Lemma 3.1,

$$\begin{aligned}
&\mathbb{P}^{0,T,x,y}_{Ber}\Big(\inf_{s \in [0,T]} \big(\ell(s) - ps\big) + T^{1/4} \geq 0\Big) \\
&\geq \mathbb{P}^{0,T,0,B-A}_{Ber}\Big(\inf_{s \in [0,T]} \big(\ell(s) + A - ps\big) + T^{1/4} \geq 0\Big) \\
&= \mathbb{P}\Big(\inf_{s \in [0,T]} \big(\ell^{(T,B-A)}(s) - ps\big) \geq -T^{1/4} - A\Big) \\
&\geq \mathbb{P}\Big(\inf_{s \in [0,T]} \big(\ell^{(T,B-A)}(s) - \frac{s}{T}\cdot(B-A)\big) \geq -T^{1/4} - T^{1/2} + 2\Big),
\end{aligned}$$

with $A = \lfloor T^{1/2} \rfloor$, $B = \lfloor pT + T^{1/2} \rfloor$, and $\mathbb{P}$, and $\ell^{(T,B-A)}$ as in Theorem 3.3. In the last line, we used the facts that $|A - T^{1/2}| \leq 1$ and $|p - (B-A)/T| \leq 1$. With B^σ as in the theorem, the last line is bounded below by

$$\begin{aligned}
&\mathbb{P}\Big(\inf_{s \in [0,T]} \sqrt{T}\, B^\sigma_{s/T} \geq -T^{1/2} \quad \text{and} \quad \Delta(T, B-A) < T^{1/2} - 2\Big) \\
&\geq 1 - \exp\left(-\frac{2}{p(1-p)}\right) - \mathbb{P}\Big(\Delta(T,B-A) \geq T^{1/2} - 2\Big).
\end{aligned}$$

In the second line, we used Lemma 3.6. Since $|B - A - pT| \leq 1$, Corollary 3.5 allows us choose W_3 large enough depending on p so that this term is less than $\frac{1}{2}(1 - e^{-2/p(1-p)})$ for $T \geq W_3$. This implies (3.8). ☐

We need the following definition for our next result. For a function $f \in C([a,b])$ we define its *modulus of continuity* by

$$w(f,\delta) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|. \tag{3.9}$$

Lemma 3.16. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $y \in \mathbb{Z}$ such that $T \geq y \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,0,y}_{Ber}$. For each positive M, ϵ and η, there exist a $\delta(\epsilon,\eta,M) > 0$ and $W_4 = W_4(M,p,\epsilon,\eta) \in \mathbb{N}$ such that for $T \geq W_4$ and $|y - pT| \leq MT^{1/2}$ we have*

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(w\big(f^\ell, \delta\big) \geq \epsilon\Big) \leq \eta, \tag{3.10}$$

where $f^\ell(u) = T^{-1/2}\big(\ell(uT) - puT\big)$ for $u \in [0,1]$.

Remark 3.17. Lemma 3.16 states that if ℓ is a Bernoulli bridge that is started from $(0,0)$ and terminates at (T,y) with y close to pT (i.e. with well-behaved endpoints) then the modulus of continuity of ℓ is also well-behaved with high probability.

Proof. We have

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(w\big(f^\ell, \delta\big) \geq \epsilon\Big) = \mathbb{P}\Big(w\big(f^{\ell^{(T,y)}}, \delta\big) \geq \epsilon\Big),$$

with $\mathbb{P}$, $\ell^{(T,y)}$ as in Theorem 3.3. If B^σ is the Brownian bridge provided by Theorem 3.3, then

$$\begin{aligned}
w\big(f^{\ell^{(T,y)}},\delta\big) &= T^{-1/2}\sup_{\substack{s,t\in[0,1]\\|s-t|\le\delta}}\Big|\ell^{(T,y)}(sT)-psT-\ell^{(T,y)}(tT)+ptT\Big|\\
&\le T^{-1/2}\sup_{\substack{s,t\in[0,1]\\|s-t|\le\delta}}\Big(\big|\sqrt{T}\,B^\sigma_s+sy-psT-\sqrt{T}\,B^\sigma_t-ty+ptT\big|\\
&\qquad+\big|\sqrt{T}\,B^\sigma_s+sy-\ell^{(T,y)}(sT)\big|+\big|\sqrt{T}\,B^\sigma_t+ty-\ell^{(T,y)}(tT)\big|\Big)\\
&\le \sup_{\substack{s,t\in[0,1]\\|s-t|\le\delta}}\Big|B^\sigma_s-B^\sigma_t+T^{-1/2}(y-pT)(s-t)\Big|+2T^{-1/2}\Delta(T,y)\\
&\le w\big(B^\sigma,\delta\big)+M\delta+2T^{-1/2}\Delta(T,y).
\end{aligned}$$

The last line follows from the assumption that $|y-pT|\le MT^{1/2}$. Thus

$$\begin{aligned}
\mathbb{P}\Big(w\big(f^{\ell^{(N,y)}},\delta\big)\ge\epsilon\Big)&\le\mathbb{P}\Big(w\big(B^\sigma,\delta\big)+M\delta+2T^{-1/2}\Delta(T,y)\ge\epsilon\Big)\\
&\le\mathbb{P}\Big(w\big(B^\sigma,\delta\big)+M\delta\ge\epsilon/2\Big)+\mathbb{P}\Big(\Delta(T,y)\ge\epsilon\,T^{1/2}/4\Big).
\end{aligned}\tag{3.11}$$

Corollary 3.5 gives us a W_4 large enough depending on M,p,ϵ,η so that the last expression in equation 3.11 is $\le\eta/2$ for $T\ge W_4$. Since B^σ is a.s. uniformly continuous on the compact interval $[0,1]$, $w(B^\sigma,\delta)\to0$ as $\delta\to0$. Thus we can find $\delta_0>0$ small enough depending on ϵ,η so that $w(B^\sigma,\delta_0)<\epsilon/4$ with probability at least $1-\eta/2$. Then with $\delta=\min(\delta_0,\epsilon/4M)$, the first term is $\le\eta/2$ as well. This implies (3.10). ☐

Lemma 3.18. *Fix $T\in\mathbb{N}$, $p\in(0,1)$, $K>0$, $C\ge\sqrt{8p(1-p)\log3}$, and $a,b\in\mathbb{Z}$ such that $\Omega(0,T,a,b)$ is nonempty. Let $\ell_{bot}\in\Omega(0,T,a,b)$. Suppose $\vec{x},\vec{y}\in\mathfrak{W}_{k-1}$, $k\ge2$, are such that $T\ge y_i-x_i\ge0$ for $1\le i\le k-1$. Write $\vec{z}=\vec{y}-\vec{x}$, and suppose that*

(1) $x_{k-1}+(z_{k-1}/T)s-\ell_{bot}(s)\ge C\sqrt{T}$ for all $s\in[0,T]$
(2) $x_i-x_{i+1}\ge C\sqrt{T}$ and $y_i-y_{i+1}\ge C\sqrt{T}$ for $1\le i\le k-2$,
(3) $|z_i-pT|\le K\sqrt{T}$ for $1\le i\le k-1$, for a constant $K>0$.

Let $\mathfrak{L}=(L_1,\dots,L_{k-1})$ be a line ensemble with law $\mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}$, and let E denote the event

$$E=\{L_1(s)\ge\cdots\ge L_{k-1}(s)\ge\ell_{bot}(s),\text{for } s\in[0,T]\}.$$

Then we can find $W_5=W_5(p,C,K)$ so that for $T\ge W_5$,

$$\mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}(E)\ge\big(1-3e^{-C^2/8p(1-p)}\big)^{k-1}.\tag{3.12}$$

Remark 3.19. This lemma states that if k independent Bernoulli bridges are well-separated from each other and ℓ_{bot}, then there is a positive probability that the curves will intersect neither each other nor ℓ_{bot}. We will use this result to compare curves in an avoiding Bernoulli line ensemble with free Bernoulli bridges.

Proof. Observe that condition (1) simply states that ℓ_{bot} lies a distance of at least $C\sqrt{T}$ uniformly below the line segment connecting x_{k-1} and y_{k-1}. Thus (1) and (2) imply that E occurs if each curve L_i remains within a distance of $C\sqrt{T}/2$ from the line segment connecting x_i and y_i. As in Theorem 3.3, let $\mathbb{P}_i$ be probability measures supporting random variables $\ell^{(T,z_i)}$ with laws $\mathbb{P}^{0,T,0,z_i}_{Ber}$.

Then

$$\mathbb{P}_{Ber}^{0,T,\vec{x},\vec{y}}(E) \geq \mathbb{P}_{Ber}^{0,T,\vec{x},\vec{y}}\Big(\sup_{s\in[0,T]} \big| L_i(s) - x_i - (z_i/T)s \big| \leq C\sqrt{T}/2,\ 1 \leq i \leq k-1 \Big)$$

$$= \prod_{i=1}^{k-1} \Big[\mathbb{P}_{Ber}^{0,T,0,z_i}\Big(\sup_{s\in[0,T]} \big| L_i(s + rN^{\alpha}) - (z_i/T)s \big| \leq C\sqrt{T}/2 \Big) \Big]$$

$$= \prod_{i=1}^{k-1} \Big[1 - \mathbb{P}_i\Big(\sup_{s\in[0,T]} \big| \ell^{(T,z_i)} - (z_i/T)s \big| > C\sqrt{T}/2 \Big) \Big]. \tag{3.13}$$

In the third line, we used the fact that $L_1, \dots, L_{k-1}$ are independent from each other under $\mathbb{P}_{Ber}^{0,T,0,z_i}$. Let $B^{\sigma,i}$ be the Brownian bridge with variance $\sigma^2 = p(1-p)$ coupled with $\ell^{(T,z_i)}$ given by Theorem 3.3. Then we have

$$\mathbb{P}_i\Big(\sup_{s\in[0,T]} \big| \ell^{(T,z_i)}(s) - (z_i/T)s \big| > C\sqrt{T}/2 \Big)$$
$$\leq \mathbb{P}_i\Big(\sup_{s\in[0,T]} |\sqrt{T} B^{\sigma}_{s/T}| > C\sqrt{T}/4 \Big) + \mathbb{P}_i\Big(\Delta(T, z_i) > C\sqrt{T}/4 \Big).$$

By Lemma 3.6, the first term is bounded above by $2e^{-C^2/8p(1-p)}$. For the second term, condition (3) in the hypothesis and Corollary 3.5 give an upper bound of $e^{-C^2/8p(1-p)}$ for sufficiently large N depending on p, C, K, but independent of i. Adding these two terms and referring to (3.13) proves (3.12). □

3.3. **Properties of avoiding Bernoulli line ensembles.** In this section we derive several results about avoiding Bernoulli line ensembles, which are Bernoulli line ensembles with law $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ as in Definition 2.15. The lemmas we prove only involve the case when $f(r) = \infty$ for all $r \in [\![T_0, T_1]\!]$ and we denote the measure in this case by $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g}$. A $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g}$-distributed random variable will be denoted by $\mathfrak{Q} = (Q_1, \dots, Q_k)$ where k is the number of up-right paths in the ensemble. As usual, if $g = -\infty$, we write $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y}}$. Our first result will rely on the two monotonicity Lemmas 3.1 and 3.2 as well as the strong coupling between Bernoulli bridges and Brownian bridges from Theorem 3.3, and the further results make use of the material in Section 9, Appendix B.

Lemma 3.20. *Fix $p \in (0,1)$, $k \in \mathbb{N}$. Let $\vec{x}, \vec{y} \in \mathfrak{W}_k$ be such that $T \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$. Then for any $M, M_1 > 0$ we can find $W_6 \in \mathbb{N}$ depending on p, k, M, M_1 such that if $T \geq W_6$, $x_k \geq -M_1\sqrt{T}$, and $y_k \geq pT - M_1\sqrt{T}$, then*

$$\mathbb{P}_{avoid,Ber}^{0,T,\vec{x},\vec{y}}\Big(Q_k(T/2) - pT/2 \geq M\sqrt{T} \Big) \geq \frac{2^{k/2}\big(1 - 2e^{-4/p(1-p)}\big)^{2k}}{(\pi p(1-p))^{k/2}} \exp\left(-\frac{2k(M + M_1 + 6)^2}{p(1-p)} \right). \tag{3.14}$$

Proof. Define vectors $\vec{x}, \vec{y} \in \mathfrak{W}_k$ by

$$x_i' = \lfloor -M_1\sqrt{T} \rfloor - 10(i-1)\lceil \sqrt{T} \rceil,$$
$$y_i' = \lfloor pT - M_1\sqrt{T} \rfloor - 10(i-1)\lceil \sqrt{T} \rceil.$$

Then $x_i' \leq x_k \leq x_i$ and $y_i' \leq y_k \leq y_i$ for $1 \leq i \leq k-1$. Thus by Lemma 3.1, we have

$$\mathbb{P}_{avoid,Ber}^{0,T,\vec{x},\vec{y}}\Big(Q_k(T/2) - pT/2 \geq M\sqrt{T} \Big) \geq \mathbb{P}_{avoid,Ber}^{0,T,\vec{x}',\vec{y}'}\Big(Q_k(T/2) - pT/2 \geq M\sqrt{T} \Big).$$

Let us write $K_i = pT/2 + M\sqrt{T} + (10(k-i) - 5)\lceil \sqrt{T} \rceil$ for $1 \leq i \leq k$. Note K_i is the midpoint of $pT/2 + M\sqrt{T} + 10(k-i-1)\lceil \sqrt{T} \rceil$ and $pT/2 + M\sqrt{T} + 10(k-i)\lceil \sqrt{T} \rceil$. Let E denote the event that the following conditions hold for $1 \leq i \leq k$:

(1) $\big| Q_i(T/2) - pT/2 - M\sqrt{T} - (10(k-i) - 5)\lceil \sqrt{T} \rceil \big| \leq 2\lceil \sqrt{T} \rceil$,

(2) $\sup_{s\in[0,T/2]} \left|Q_i(s) - x_i' - \frac{K_i - x_i'}{T/2}\, s\right| \leq 3\sqrt{T}$,

(3) $\sup_{s\in[T/2,T]} \left|Q_i(s) - K_i - \frac{y_i' - K_i}{T/2}(s - T/2)\right| \leq 3\sqrt{T}$.

The first condition requires in particular that $Q_k(T/2) - pT/2 \geq M\sqrt{T}$, and also that $Q_i(T/2) - Q_{i+1}(T/2) \geq 6\sqrt{T}$ for each i. The second and third conditions require that each curve Q_i remain within a distance of $3\sqrt{T}$ of the graph of the piecewise linear function on $[0,T]$ passing through the points $(0, x_1')$, $(T/2, K_i)$, and (T, y_i'). We observe that

$$\mathbb{P}^{0,T,\vec{x}',\vec{y}'}_{avoid,Ber}\Big(Q_k(T/2) - ptT \geq M\sqrt{T}\Big) \geq \mathbb{P}^{0,T,\vec{x}',\vec{y}'}_{avoid,Ber}(E) \geq \mathbb{P}^{0,T,\vec{x}',\vec{y}'}_{Ber}(E).$$

The second inequality follows since the event E implies that $Q_1(s) \geq \cdots \geq Q_k(s)$ for all $s \in [0,T]$. Write $z = y_k' - x_k'$. Then we have

$$\begin{aligned}\mathbb{P}^{0,T,\vec{x}',\vec{y}'}_{Ber}(E) = \Big[\mathbb{P}^{0,T,0,z}_{Ber}\Big(&\left|\ell(T/2) - pT/2 - M\sqrt{T} - 5\lceil\sqrt{T}\rceil + x_1'\right| \leq 2\lceil\sqrt{T}\rceil \quad \text{and} \\ &\sup_{s\in[0,T/2]}\left|\ell(s) - \frac{K_1 - x_1'}{T/2}\, s\right| \leq 3\sqrt{T} \quad \text{and} \\ &\sup_{s\in[T/2,T]}\left|\ell(s) - (K_1 - x_1') - \frac{y_1' - K_1}{T/2}(s - T/2)\right| \leq 3\sqrt{T}\Big)\Big]^k.\end{aligned} \tag{3.15}$$

Let $\mathbb{P}$ be a probability space suporting a random variable $\ell^{(T,z)}$ with law $\mathbb{P}^{0,T,0,z}$ coupled with a Brownian bridge B^σ with variance σ^2, as in Theorem 3.3. Then the last probability in brackets is bounded below by

(3.16)

$$\begin{aligned}\mathbb{P}^{0,T,0,z}_{Ber}\Big(&\left|\ell(T/2) - pT/2 - (M + M_1 + 5)\sqrt{T}\right| \leq 2\sqrt{T} - 5 \quad \text{and} \\ &\sup_{s\in[0,T/2]}\left|\ell(s) - ps - \frac{M + M_1 + 5}{\sqrt{T}/2}\, s\right| \leq 3\sqrt{T} - 1 \quad \text{and} \\ &\sup_{s\in[T/2,T]}\left|\ell(s) - ps - (M + M_1 + 5)\sqrt{T} + \frac{M + M_1 + 5}{\sqrt{T}/2}(s - T/2)\right| \leq 3\sqrt{T} - 1\Big) \\ \geq \mathbb{P}\Big(&\left|\sqrt{T}\, B^\sigma_{1/2} - (M + M_1 + 5)\sqrt{T}\right| \leq \sqrt{T} \quad \text{and} \\ &\sup_{s\in[0,T/2]}\left|\sqrt{T}\, B^\sigma_{s/T} - (M + M_1 + 5)\sqrt{T} \cdot \frac{s}{T/2}\right| \leq 2\sqrt{T} \quad \text{and} \\ &\sup_{s\in[T/2,T]}\left|\sqrt{T}\, B^\sigma_{s/T} - (M + M_1 + 5)\sqrt{T} \cdot \frac{T - s}{T/2}\right| \leq 2\sqrt{T}\Big) \\ &- \mathbb{P}\Big(\Delta(T,z) > \sqrt{T}/2\Big).\end{aligned}$$

Note that $B^\sigma_{1/2}$ is a centered Gaussian random variable with variance $p(1-p)/4 = \sigma^2(1/2)(1-1/2)$. Writing $\xi = B^\sigma_{1/2}$, it follows from Lemma 3.7 that there exist independent Brownian bridges B^1, B^2 with variance $\sigma^2/2$ so that B^σ_s has the same law as $\frac{s}{T/2}\xi + B^1_{2s/T}$ for $s \in [0, T/2]$ and $\frac{T-s}{T/2}\xi + B^2_{(2s-T)/T}$

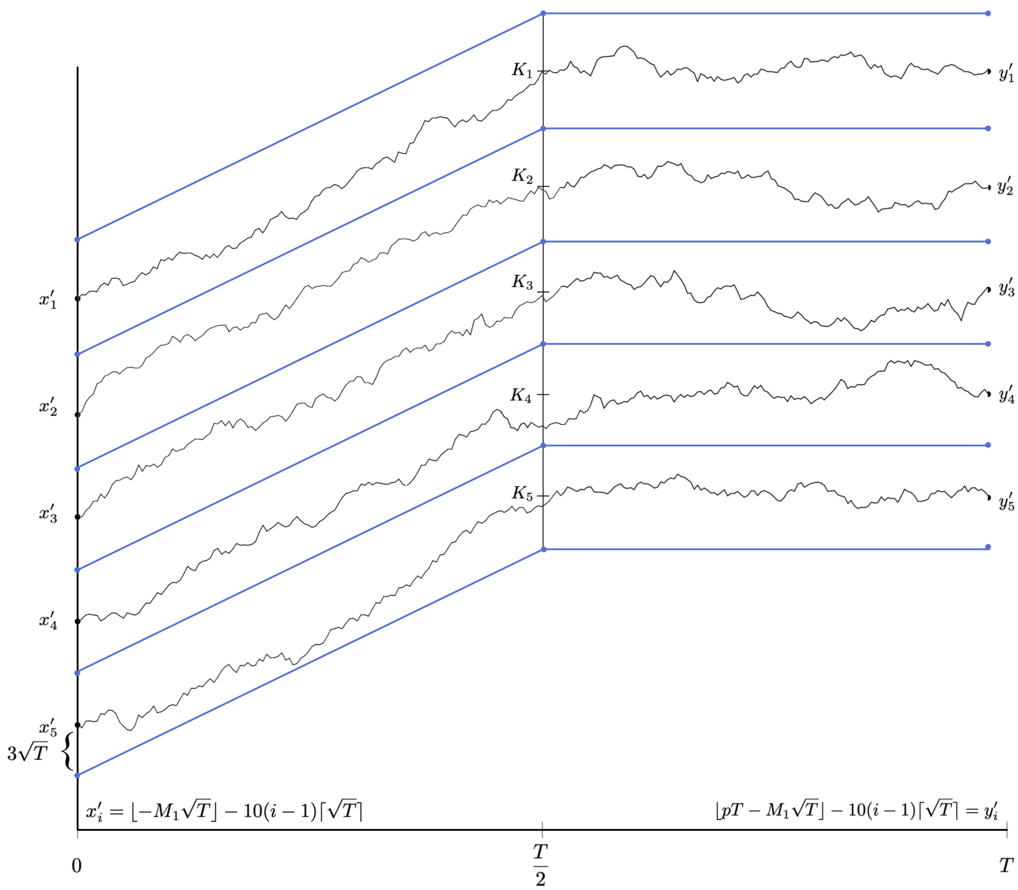

FIGURE 3. Sketch of the argument for Lemma 3.20:
We shift the Bernoulli random walks with entry and exit data $\vec{x}$ and $\vec{y}$ down to Bernoulli random walks with entry data x' and y', applying Monotone Coupling, Lemma 3.1 to find that the probability of the even that midpoint of the random walk from $\vec{x}$ to $\vec{y}$ is greater than $M\sqrt{T}$ is greater than the probability of the same event for random walks from $\vec{x}'$ to $\vec{y}'$. We then define the event E, which is the probability that each random walk lies within the blue bounding lines above after an affine shift. We may then use strong coupling with Brownian motions in accordance with Theorem 3.3 to bound the event E from below, and use previously known results about Brownian motion to bound the original probability, the height of a random walk at its midpoint, from below.

for $s \in [T/2, T]$. The first term in the last expression in (3.16) is thus equal to

$$\begin{aligned}
&\mathbb{P}\Big(|\xi - (M + M_1 + 5)| \le 1 \quad \text{and} \sup_{s\in[0,T/2]} \Big|B^1_{s/T} - (M + M_1 + 5 - \xi)\cdot \frac{s}{T/2}\Big| \le 2 \\
&\qquad\qquad \text{and} \quad \sup_{s\in[T/2,T]} \Big|B^2_{(2s-T)/T} - (M + M_1 + 5 - \xi)\cdot \frac{T-s}{T/2}\Big| \le 2\Big) \\
&\ge \mathbb{P}\Big(|\xi - (M + M_1 + 5)| \le 1 \quad \text{and} \sup_{s\in[0,T/2]} \big|B^1_{2s/T}\big| \le 1 \qquad \text{and} \sup_{s\in[T/2,T]} \big|B^2_{(2s-T)/T}\big| \le 1\Big) \\
&= \mathbb{P}\Big(|\xi - (M + M_1 + 5)| \le 1\Big)\mathbb{P}\Big(\sup_{s\in[0,T/2]} \big|B^1_{2s/T}\big| \le 1\Big)\mathbb{P}\Big(\sup_{s\in[0,T/2]} \big|B^2_{(2s-T)/T}\big| \le 1\Big) \\
&\ge \big(1 - 2e^{-4/p(1-p)}\big)^2 \int_{M+M_1+4}^{M+M_1+6} \frac{e^{-2\xi^2/p(1-p)}}{\sqrt{\pi p(1-p)/2}}\, d\xi \\
&\ge \frac{2\sqrt{2}\, e^{-2(M+M_1+6)^2/p(1-p)}}{\sqrt{\pi p(1-p)}} \big(1 - 2e^{-4/p(1-p)}\big)^2.
\end{aligned}$$

In the fourth line, we used the fact that ξ, $B^1_\cdot$, and $B^2_\cdot$ are independent, and in the second to last line, we used Lemma 3.6. Since $|z - pT| \le (M_1+1)\sqrt{T}$, Lemma 3.5 allows us to choose T large enough so that $\mathbb{P}(\Delta(T,z) > \sqrt{T}/2)$ is less than 1/2 the last expression. Then in view of (3.15) and (3.16), we conclude (3.14).

□

Lemma 3.21. *Fix $p,t \in (0,1)$, $k \in \mathbb{N}$, $\vec{a}$, $\vec{b} \in W_k$. Suppose that $\vec{x}^T = (x_1^T, \cdots, x_k^T)$ and $\vec{y}^T = (y_1^T, \cdots, y_k^T)$ are two sequence of k-dimensional vectors with integer entries such that for $i \in [\![1,k]\!]$,*

$$\lim_{T\to\infty} \frac{x_i^T}{\sqrt{T}} = a_i \text{ and } \lim_{T\to\infty} \frac{y_i^T - pT}{\sqrt{T}} = b_i$$

and define the sequence of random k-dimensional vectors Z^T by

$$Z^T = \Big(\frac{L_1(tT) - ptT}{\sqrt{T}}, \cdots, \frac{L_k(tT) - ptT}{\sqrt{T}}\Big),$$

where $(L_1, \cdots, L_k)$ is $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$-distributed. When $a_i = a_{i+1}$ or $b_j = b_{j+1}$ for any entries of $\vec{a}$ and $\vec{b}$, we write

$$\vec{a} = (a_1, \cdots, a_k) = (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p})$$

$$\vec{b} = (b_1, \cdots, b_k) = (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q})$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_p$, $\beta_1 > \beta_2 > \cdots > \beta_q$ and $\sum_{i=1}^p m_i = \sum_{i=1}^q n_i = k$. Then, the random vector Z^T converges weakly to a random variable $\hat{Z}$ with the density

$$\rho_{\vec{a},\vec{b}}(z_1, \cdots, z_k) = \frac{1}{Z_{\vec{a},\vec{b}}} \cdot \varphi(\vec{a},\vec{z},\vec{m})\psi(\vec{b},\vec{z},\vec{n}) \prod_{i=1}^k e^{-c_3(t,p)z_i^2} \mathbb{1}_{\{z_1 > \cdots > z_k\}}$$

where $\vec{m} = (m_1, \cdots, m_k)$, $\vec{n} = (n_1, \cdots, n_k)$, c_1, c_2, c_3 are constants depending on p, t as given in Proposition 9.1 , $Z_{\vec{a},\vec{b}}$ is a constant depending on $p, t, \vec{a}, \vec{b}$ such that $\rho_{\vec{a},\vec{b}}(z_1, \cdots, z_k)$ integrates to 1 over $\mathbb{R}^k$, and $\varphi(\vec{a},\vec{z},\vec{m})$ and $\psi(\vec{b},\vec{z},\vec{n})$ are determinants:

$$\varphi(\vec{a},\vec{z},\vec{m}) = \det \begin{bmatrix} \big((c_1(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_1 z_j}\big)_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ \big((c_2(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_p z_j}\big)_{\substack{i=1,\cdots,m_p \\ j=1,\cdots,k}} \end{bmatrix}$$

$$\psi(\vec{b},\vec{z},\vec{n}) = \det \begin{bmatrix} \big((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_1 z_j}\big)_{\substack{i=1,\cdots,n_1 \\ j=1,\cdots,k}} \\ \vdots \\ \big((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_q z_j}\big)_{\substack{i=1,\cdots,n_q \\ j=1,\cdots,k}} \end{bmatrix}$$

Proof of this lemma will be postponed until the appendix, in propositions 9.1 and 9.2

Lemma 3.22. *Fix $p,t \in (0,1)$ and $k \in \mathbb{N}$. Suppose that $\vec{x}^T = (x_1^T, \cdots x_k^T)$ and $\vec{y}^T = (y_1^T, \cdots, y_k^T)$ is a sequence of k-dimensional vectors with integer entries such that $T \ge y_i^T - x_i^T \ge 0$ for $i \in [\![1,k]\!]$. Show that for any $M_1, M_2 > 0$ and $\epsilon > 0$ there exists $T_0 \in \mathbb{N}$ and $\delta > 0$ depending on p, k, M_1, M_2 such that if $T \ge T_0$, $\left|x_i^T\right| \le M_1\sqrt{T}$ and $\left|y_i^T - pT\right| \le M_2\sqrt{T}$ then*

$$\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}\left(\min_{1\le i\le k-1} [L_i(tT) - L_{i+1}(tT)] < \delta\sqrt{T}\right) < \epsilon$$

Proof. We prove the claim by contradiction: suppose there exists $M_1, M_2, \epsilon > 0$ such that for any $T_0 \in \mathbb{N}$ and $\delta > 0$ there exists some $T \geq T_0$ such that

$$\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}\left(\min_{1\leq i\leq k-1}[L_i(tT) - L_{i+1}(tT)] < \delta\sqrt{T}\right) \geq \epsilon$$

Therefore, we can obtain sequences T_n, $\delta_n > 0$, $T_n \uparrow \infty$, $\delta_n \downarrow 0$ such that for all n, we have

$$\mathbb{P}^{0,T,\vec{x}^{T_n},\vec{y}^{T_n}}_{avoid,Ber}\left(\min_{1\leq i\leq k-1}\left[\frac{L_i(tT_n) - L_{i+1}(tT_n)}{\sqrt{T_n}}\right] < \delta_n\right) \geq \epsilon$$

As $\left|x_i^{T_n}\right| < M_1\sqrt{T_n}$, $\left|y_i^{T_n} - pT_n\right| \leq M_2\sqrt{T_n}$ we know that there exists some subsequence T_{n_m} such that $\lim_{m\to\infty} \frac{x^{T_{n_m}}}{\sqrt{T_{n_m}}} \to \vec{x}$ and $\lim_{m\to\infty} \frac{y^{T_{n_m}} - pT}{\sqrt{T_{n_m}}} \to \vec{y}$ by the Bolzano-Weierstrass theorem, as these sequences lie in compact intervals. Denote

$$Z_i^m := \frac{L_i(tT_{n_m}) - ptT}{\sqrt{T_{n_m}}}.$$

Then, if we fix $\tilde{\delta} > 0$, because $\delta_n \downarrow 0$, there exists some M such that $m > M$ implies $\delta_m < \tilde{\delta}$. For such $m > M$, we have

$$\epsilon \leq \liminf_{m\to\infty} \mathbb{P}\left(\min_{1\leq i\leq k-1} Z_i^m - Z_{i+1}^m < \delta_{n_m}\right) \leq \liminf_{m\to\infty} \mathbb{P}\left(\min_{1\leq i\leq k-1} Z_i^m - Z_{i+1}^m \leq \tilde{\delta}\right) \tag{3.17}$$

Now, if we apply the results of Lemma 3.21, we that the vector $(Z_1^m, \cdots, Z_k^m)$ converges weakly to some random vector $\hat{Z}$ with probability density function ρ as defined previously in 3.21, the definitions of which we will state in a few lines, when they are needed. Because this sequence has weak convergence we may apply portmanteau's lemma with the closed set $K = [0, \tilde{\delta}]$ to find

$$\limsup_{m\to\infty} \mathbb{P}\left(\min_{1\leq i\leq k-1} Z_i^m - Z_{i+1}^m \in K\right) \leq \mathbb{P}\left(\min_{1\leq i\leq k-1} \hat{Z}_i - \hat{Z}_{i+1} \in K\right) \tag{3.18}$$

Combining (3.17) and (3.18), we arrive at the inequality

$$\epsilon \leq \liminf_{m\to\infty} \mathbb{P}\left(\min_{1\leq i\leq k-1} Z_i^m - Z_{i+1}^m \leq \tilde{\delta}\right) \leq \mathbb{P}\left(\min_{1\leq i\leq k-1} \hat{Z}_i - \hat{Z}_{i+1}\tilde{\delta}\right) \tag{3.19}$$

and so by (3.19) and countable subadditivity,

$$\epsilon \leq \mathbb{P}\left(\min_{1\leq i\leq k-1} \hat{Z}_i - \hat{Z}_{i+1} \leq \tilde{\delta}\right) \leq \sum_{i=1}^{k-1} \mathbb{P}\left(\hat{Z}_i - \hat{Z}_{i+1} \leq \tilde{\delta}\right) \tag{3.20}$$

Since $\tilde{\delta}$ is arbitrary, this inequality holds for any $\tilde{\delta} > 0$. In order to pick a sufficient δ' to find a contradiction, we will now state the definition of ρ, as found in 3.21:

$$\rho(z_1, \cdots, z_k) = \frac{1}{Z_{\vec{a},\vec{b}}} \cdot \varphi(\vec{a}, \vec{z}, \vec{n}) \cdot \psi(\vec{b}, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{\frac{-z_i^2}{t(1-t)}}$$

with $\vec{z}_i \in W_k^\circ$ where we have the functions and constant

$$\varphi(\vec{a},\vec{z},\vec{n}) = \det \begin{bmatrix} \left[(c_1(t,p)z_j)^{i-1}e^{c_1(t,p)\alpha_1 z_j}\right]_{\substack{i=1,\dots,m_1\\ j=1,\dots,k}} \\ \left[(c_1(t,p)z_j)^{i-1}e^{c_1(t,p)\alpha_2 z_j}\right]_{\substack{i=1,\dots,m_2\\ j=1,\dots,k}} \\ \vdots \\ \left[(c_1(t,p)z_j)^{i-1}e^{c_1(t,p)\alpha_r z_j}\right]_{\substack{i=1,\dots,m_r\\ j=1,\dots,k}} \end{bmatrix}$$

$$\psi(\vec{b},\vec{z},\vec{n}) = \det \begin{bmatrix} \left[(c_2(t,p)z_j)^{i-1}e^{c_2(t,p)\beta_1 z_j}\right]_{\substack{i=1,\dots,n_1\\ j=1,\dots,k}} \\ \left[(c_2(t,p)z_j)^{i-1}e^{c_2(t,p)\beta_2 z_j}\right]_{\substack{i=1,\dots,n_2\\ j=1,\dots,k}} \\ \vdots \\ \left[(c_2(t,p)z_j)^{i-1}e^{c_2(t,p)\beta_q z_j}\right]_{\substack{i=1,\dots,n_q\\ j=1,\dots,k}} \end{bmatrix}$$

$$Z_{\vec{a},\vec{b}} = \int_{\mathbb{R}^k} \left(\mathbf{1}_{z_1\geq z_2\geq\cdots\geq z_k}(\vec{z})\right)\varphi(\vec{a},\vec{z},\vec{n})\cdot\psi(\vec{b},\vec{z},\vec{n})\prod_{i=1}^{k} e^{\frac{-z_i^2}{t(1-t)}}\,d\vec{z}$$

we find that ρ is continuous as a sum and product of continuous functions. Additionally, if for any $\chi \in [\![1,k-1]\!]$ such that $z_\chi = z_{\chi+1}$, then we have two equal columns

$$\begin{bmatrix} \left[(c_1(t,p)z_\chi)^{i-1}e^{c_1(t,p)\alpha_1 z_\chi}\right]_{i=1,\dots,m_1} \\ \left[(c_1(t,p)z_\chi)^{i-1}e^{c_1(t,p)\alpha_2 z_\chi}\right]_{i=1,\dots,m_2} \\ \vdots \\ \left[(c_1(t,p)z_\chi)^{i-1}e^{c_1(t,p)\alpha_r z_\chi}\right]_{i=1,\dots,m_r} \end{bmatrix} = \begin{bmatrix} \left[(c_1(t,p)z_{\chi+1})^{i-1}e^{c_1(t,p)\alpha_1 z_{\chi+1}}\right]_{i=1,\dots,m_1} \\ \left[(c_1(t,p)z_{\chi+1})^{i-1}e^{c_1(t,p)\alpha_2 z_{\chi+1}}\right]_{i=1,\dots,m_2} \\ \vdots \\ \left[(c_1(t,p)z_{\chi+1})^{i-1}e^{c_1(t,p)\alpha_r z_{\chi+1}}\right]_{i=1,\dots,m_r} \end{bmatrix}$$

which implies that the determinant in $\varphi(\vec{a},\vec{b},\vec{n}) = 0$, so $\rho(\vec{z}) = 0$. Hence, if we let $E_i^\delta := \{z \mid z_i - z_{i+1} \leq \delta\}$ we know by the continuity of ρ that the accumulation function

$$R_i(\delta) := \int_{E_i^\delta} \rho$$

is well defined and continuous. We can find that $R_i(0) = \int_{E_i^0} 0$ by our previous calculation since $\rho = 0$ on E_i^0 as $z_i = z_{i+1}$. By the continuity of R we know that for any ϵ there exists a $\delta_i(\epsilon)$ such that $0 < \delta < \delta_i(\epsilon)$ implies $|R_i(\delta)| < \epsilon$. Therefore, let $\tilde{\delta} = \min_{1\leq i\leq k-1}\delta_i\left(\frac{\epsilon}{k-1}\right)$ and we find that

$$\mathbb{P}\left(\hat{Z}_i - \hat{Z}_{i+1} \leq \tilde{\delta}\right) = R_i(\tilde{\delta}) < \frac{\epsilon}{k-1} \tag{3.21}$$

for each integer $i \in [\![1,k-1]\!]$. Combining equations (3.20) and (3.21) yields the inequality

$$\epsilon \leq \mathbb{P}\left(\min_{1\leq i\leq k-1}\hat{Z}_i - \hat{Z}_{i+1} \leq \delta'\right) \leq \sum_{i=1}^{k-1}\mathbb{P}\left(\hat{Z}_i - \hat{Z}_{i+1} \leq \delta'\right) < \epsilon$$

which is a contradiction. Thus we have proved the Lemma. □

Remark 3.23. Lemma 3.22 provides us with some amount of separation between the lines in the ensemble at any point along the Beroulli Random Walkers as T tends to infinity. For any ϵ, it allows us to find a distance δ such that the probability that any lines get closer than δ is less than ϵ. The proof essentially passes to subsequential limits in order to use the convergence result of Lemma (3.21), and then the fact that the density ρ is continuous allows us to integrate $\rho(\vec{z})$ over areas where z_i values collide, and find that these integrals are 0. Then we may use continuity around areas where z_i values are equal to bound the probability where z_i are close.

4. Proof of Theorem 2.26

The goal of this section is to prove Theorem 2.26 and for the remainder we assume that $k \in \mathbb{N}$ with $k \geq 2$, $p \in (0,1)$, $\alpha, \lambda > 0$ are all fixed and

$$\{\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)\}_{N=1}^{\infty}, \tag{4.1}$$

is an (α, p, λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles as in Definition 2.24 that are all defined on a probability space with measure $\mathbb{P}$. The main technical result we will require is contained in Proposition 4.1 below and its proof is the content of Section 4.1. The proof of Theorem 2.26 is given in Section 4.2.

4.1. **Bounds on the acceptance probability.** The main result in this section is presented as Proposition 4.1 below. In order to formulate it and some of the lemmas below it will be convenient to adopt the following notation for any $r > 0$:

$$t_1 = \lfloor (r+1)N^{\alpha} \rfloor, \quad t_2 = \lfloor (r+2)N^{\alpha} \rfloor, \quad \text{and } t_3 = \lfloor (r+3)N^{\alpha} \rfloor. \tag{4.2}$$

Proposition 4.1. *For any $\epsilon > 0$, $r > 0$ and any (α, p, λ)-good sequence of Bernoulli line ensembles $\{\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)\}_{N=1}^{\infty}$ there exist $\delta > 0$ and N_1 (both depending on ϵ, r as well as α, p, λ and the functions ϕ, ψ in Definition 2.24) such that for all $N \geq N_1$ we have*

$$\mathbb{P}\Big(Z\big(-t_1, t_1, \vec{x}, \vec{y}, L_k[\![-t_1, t_1]\!]\big) < \delta\Big) < \epsilon,$$

where $\vec{x} = (L_1^N(-t_1), \dots, L_{k-1}^N(-t_1))$, $\vec{y} = (L_1^N(t_1), \dots, L_{k-1}^N(t_1))$, $L_k[\![-t_1, t_1]\!]$ is the restriction of L_k^N to the set $[\![-t_1, t_1]\!]$, and Z is the acceptance probability of Definition 2.22. $\mathbb{P}$ is the measure on a probability space that supports $\{\mathfrak{L}^N\}_{N=1}^{\infty}$.

The general strategy we use to prove Proposition 4.1 is inspired by the proof of Proposition 6.5 in [5]. We begin by stating three key lemmas that will be required. Their proofs are postponed to Section 5. All constants in the statements below will depend implicitly on α, r, p, λ, and the functions ϕ, ψ from Definition 2.24, which are fixed throughout. We will not list this dependence explicitly.

Lemma 4.2 controls the upward deviation of the top curve $L_1^N(s)$ from the line ps in the scale $N^{\alpha/2}$.

Lemma 4.2. *For each $\epsilon > 0$ there exist $R_1 = R_1(\epsilon) > 0$ and $N_2 = N_2(\epsilon)$ such that for $N \geq N_2$*

$$\mathbb{P}\Big(\sup_{s \in [-t_3, t_3]} \big(L_1^N(s) - ps\big) \geq R_1 N^{\alpha/2}\Big) < \epsilon.$$

Lemma 4.3 controls the downward deviation of the bottom curve $L_k^N(s)$ from the line ps in the scale $N^{\alpha/2}$.

Lemma 4.3. *For each $\epsilon > 0$ there exist $R_2 = R_2(\epsilon) > 0$ and $N_3 = N_3(\epsilon)$ such that for $N \geq N_3$*

$$\mathbb{P}\Big(\inf_{s \in [-t_2, t_2]} \big(L_k^N(s) - ps\big) \leq -R_2 N^{\alpha/2}\Big) < \epsilon.$$

Lemma 4.4. *Fix $k \in \mathbb{N}$, $p \in (0,1)$, $M_1, M_2 > 0$. Suppose that $\ell_{bot} : [\![-t_2, t_2]\!] \to \mathbb{R} \cup \{-\infty\}$, and $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$ are such that $|\Omega_{avoid}(-t_2, t_2, \vec{x}, \vec{y}, \infty, \ell_{bot})| \geq 1$. Suppose further that*

(1) $\sup_{s \in [-t_2, t_2]} \big(\ell_{bot}(s) - ps\big) \leq M_2 (2t_2)^{1/2}$,
(2) $-pt_2 + M_1(2t_2)^{1/2} \geq x_1 \geq x_{k-1} \geq \max\big(\ell_{bot}(-t_2), -pt_2 - M_1(2t_2)^{1/2}\big)$,
(3) $pt_2 + M_1(2t_2)^{1/2} \geq y_1 \geq y_{k-1} \geq \max\big(\ell_{bot}(t_2), pt_2 - M_1(2t_2)^{1/2}\big)$.

Then there exist constants g, h depending on M_1, M_2, p, k, r and $N_4 = N_4(M_1, M_2, k) \in \mathbb{N}$ such that for any $\tilde{\epsilon} > 0$ and $N \geq N_4$ we have

$$\mathbb{P}^{-t_2,t_2,\vec{x},\vec{y},\infty,\ell_{bot}}_{avoid,Ber}\Big(Z\big(-t_1,t_1,Q(-t_1),Q(t_1),\ell_{bot}[\![-t_1,t_1]\!]\big) \leq gh\tilde{\epsilon}\Big) \leq \tilde{\epsilon}, \tag{4.3}$$

where $\ell_{bot}[\![-t_1,t_1]\!]$ is the vector, whose coordinates match those of ℓ_{bot} on $[\![-t_1,t_1]\!]$ and $Q(a) = (Q_1(a),\dots,Q_{k-1}(a))$ is the value of the line ensemble Q whose law is $\mathbb{P}^{-t_2,t_2,\vec{x},\vec{y},\infty,\ell_{bot}}_{avoid,Ber}$ at location a.

Proof of Proposition 4.1. Let $\epsilon > 0$ be given. Define the event

$$E_N = \Big\{L^N_{k-1}(\pm t_2) \mp pt_2 \geq -M_1(2t_2)^{1/2}\Big\} \cap \Big\{\sup_{s\in[-t_2,t_2]}\big(L^N_k(s) - ps\big) \leq M_2(2t_2)^{1/2}\Big\},$$

where M_1 and M_2 are sufficiently large so that for $N > N_1 = \max\{N_2, N_3, N_4\}$ we have $\mathbb{P}(E_N^c) < \epsilon/2$. The existence of such M_1 and M_2 is assured from Lemmas 4.2 and 4.3.

Let $\delta = (\epsilon/2)\cdot gh$, where g, h are as in Lemma 4.4 for the values M_1, M_2 as above and r as in the statement of the proposition. We denote

$$V = \Big\{Z\big(-t_1,t_1,\vec{x},\vec{y},L_k[\![-t_1,t_1]\!]\big) < \delta\Big\}$$

and make the following deduction

$$\begin{aligned}
&\mathbb{P}\big(V\cap E_N\big) = \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{E_N}\cdot\mathbf{1}_V\Big|\mathcal{F}_{ext}\big(\{1,\dots,k-1\}\times[\![-t_2+1,t_2-1]\!]\big)\Big]\Big] = \\
&\mathbb{E}\Big[\mathbf{1}_{E_N}\cdot\mathbb{E}\Big[\mathbf{1}\{Z\big(-t_1,t_1,\vec{x},\vec{y},L_k[\![-t_1,t_1]\!]\big)<\delta\}\Big|\mathcal{F}_{ext}\big(\{1,\dots,k-1\}\times[\![-t_2+1,t_2-1]\!]\big)\Big]\Big] = \\
&\mathbb{E}\left[\mathbf{1}_{E_N}\cdot\mathbb{E}^{-t_2,t_2,L^N(-t_2),L^N(t_2),\infty,L^N_k[\![-t_2,t_2]\!]}_{avoid,Ber}\big[\mathbf{1}\{Z\big(-t_1,t_1,\ell(-t_1),\ell(t_1),L^N_k[\![-t_1,t_1]\!]\big)<\delta\}\big]\right] \leq \\
&\mathbb{E}\left[\mathbf{1}_{E_N}\cdot\epsilon/2\right] \leq \epsilon/2.
\end{aligned} \tag{4.4}$$

The first equality follows from the tower property for conditional expectations. The second equality uses the definition of V and the fact that $\mathbf{1}_{E_N}$ is $\mathcal{F}_{ext}\big(\{1\}\times[\![-t_2+1,t_2-1]\!]\big)$-measurable and can thus be taken outside of the conditional expectation. The third equality uses the Schur Gibbs property. The inequality on the third line uses Lemma 4.4 with $\tilde{\epsilon} = \epsilon/2$ as well as the fact that on the event E_N^c the random variables $L^N(-t_2), L^N(t_2)$ and $L^N_k[\![-t_2,t_2]\!]$ (that play the roles of $\vec{x},\vec{y}$ and ℓ_{bot}) satisfy the inequalities

$$\begin{aligned}
L^N_{k-1}(-t_2) &\geq -pt_2 - M_1(2t_2)^{1/2}\\
L^N_{k-1}(t_2) &\geq pt_2 - M_1(2t_2)^{1/2}\\
\sup_{s\in[-t_2,t_2]}\big(L^N_k(s)-ps\big) &\leq M_2(2t_2)^{1/2}
\end{aligned}$$

The last inequality is trivial.

Combining 4.4 with $\mathbb{P}(E_N^c) < \epsilon/2$, we see that for all $N > N_1$ we have

$$\mathbb{P}\left(V\right) = \mathbb{P}(V\cap E_N) + \mathbb{P}(V\cap E_N^c) \leq \epsilon/2 + \mathbb{P}(E_N^c) < \epsilon.$$

□

4.2. **Proof of Theorem 2.26.** By Lemma 2.4, it suffices to verify the following two conditions for all $1 \in [\![1,k-1]\!]$, $R > 0$, and $\epsilon > 0$:

$$\lim_{a\to\infty}\limsup_{N\to\infty}\mathbb{P}(|f^N_i(0)| \geq a) = 0 \tag{4.5}$$

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\leq\delta}}|f^N_i(x)-f^N_i(y)| \geq \epsilon\Bigg) = 0. \tag{4.6}$$

For the sake of clarity, we will prove these conditions in three steps.

Step 1. We first prove condition (4.5), making use of Lemmas 4.2 and 4.3 in order to obtain upper and lower bounds for the top and bottom curves respectively, thus bounding all curves.

Fix $\epsilon > 0$. We show that there exists an $a > 0$ and N_5 depending on ϵ, such that $N > N_5$ implies

$$\mathbb{P}(|f_i^N(0)| \geq a) = \mathbb{P}(|L_i^N(0)| \geq aN^{\alpha/2}) < \epsilon.$$

By Lemmas 4.2 and 4.3, there exist $R_1 := R_1(\epsilon/2) > 0$, $R_2 := R_2(\epsilon/2) > 0$ and $N_2 := N_2(\epsilon/2), N_3 := N_3(\epsilon/2)$ such that

$$N \geq N_2 \text{ implies } \mathbb{P}\Big(\sup_{s\in[-t_3,t_3]} \big(L_1^N(s) - ps\big) \geq R_1 N^{\alpha/2}\Big) < \epsilon/2,$$
$$N \geq N_3 \text{ implies } \mathbb{P}\Big(\inf_{s\in[-t_2,t_2]} \big(L_k^N(s) - ps\big) \leq -R_2 N^{\alpha/2}\Big) < \epsilon/2.$$

In particular, taking $s = 0$, we find that for $N \geq N_5 := \max\{N_2, N_3\}$,

$$\mathbb{P}\big(L_1^N(0) \geq R_1 N^{\alpha/2}\big) \leq \mathbb{P}\Big(\sup_{s\in[-t_3,t_3]} \big(L_1^N(s) - ps\big) \geq R_1 N^{\alpha/2}\Big) < \epsilon/2,$$
$$\mathbb{P}\big(L_k^N(0) \leq -R_2 N^{\alpha/2}\big) \leq \mathbb{P}\Big(\inf_{s\in[-t_2,t_2]} \big(L_k^N(s) - ps\big) \leq -R_2 N^{\alpha/2}\Big) < \epsilon/2.$$

Letting $a = \max\{R_1, R_2\}$ and noting that $L_1^N(0) > L_2^N(0) > ... > L_k^N(0)$, we find that for $i \in [\![1, k]\!]$ and $N \geq N_5$,

$$\mathbb{P}\big(|L_i^N(0)| \geq aN^{\alpha/2}\big) \leq \mathbb{P}\big(L_1^N(0) \geq R_1 N^{\alpha/2}\big) + \mathbb{P}\big(L_k^N(0) \leq -R_2 N^{\alpha/2}\big) < \epsilon.$$

This proves (4.5).

Step 2. This step is set up for proving condition (4.6). We will establish sets to size bias condition (4.6) for a fixed i. First, we will reorganize the statement of condition (4.6) in order to get it in terms of Bernoulli random walks, instead of f_i^N and then we will establish large events E_1 and E_2 in order so that if we prove (4.6) on the intersection of these two sets, it is equivalent to showing it on the entire probability space.

We must show that for all $\epsilon, \eta > 0$ and $R > 0$, there exists a δ and N_0 such that $N > N_0$ implies

$$\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\leq\delta}} |f_i^N(x) - f_i^N(y)| \geq \epsilon\Bigg) < \eta.$$

We rewrite the left hand side as

$$\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\leq\delta}} \Big|N^{-\alpha/2}\left(L_i^N(xN^\alpha) - L_i^N(yN^\alpha)\right) - p(x-y)N^{\alpha/2} + \lambda(x^2-y^2)\Big| \geq \epsilon\Bigg). \tag{4.7}$$

By the bounds of the supremum, we know that $|x+y| \leq 2R$ and $|x-y| < \delta$, hence $|x^2 - y^2| \leq 2R\delta$. Thus if we take $\delta < \frac{\epsilon}{8\lambda R}$, then the last probability is bounded below by

$$\mathbb{P}\Bigg(\sup_{\substack{x,y \in [-R,R], \\ |x-y| \leq \delta}} N^{-\alpha/2} \big| L_i^N(xN^\alpha) - L_i^N(yN^\alpha) - p(x-y)N^\alpha \big| + 2\lambda R\delta \geq \epsilon \Bigg)$$

$$\leq \mathbb{P}\Bigg(\sup_{\substack{x,y \in [-R,R], \\ |x-y| \leq \delta}} \big| L_i^N(xN^\alpha) - L_i^N(yN^\alpha) - p(x-y)N^\alpha \big| \geq \frac{3N^{\alpha/2}\epsilon}{4} \Bigg)$$

$$= \mathbb{P}\Bigg(\sup_{\substack{x,y \in [-R_N,R_N], \\ |x-y| \leq \delta N^\alpha}} \big| L_i^N(x) - L_i^N(y) - p(x-y) \big| \geq \frac{3N^{\alpha/2}\epsilon}{4} \Bigg).$$

Where $R_N := \lfloor RN^\alpha \rfloor$We will take the event in this final equation and define:

$$A_\delta = \Bigg\{ \sup_{\substack{x,y \in [-R_N,R_N], \\ |x-y| \leq \delta N^\alpha}} \big| L_i^N(x) - L_i^N(y) - p(x-y) \big| \geq \frac{3N^{\alpha/2}\epsilon}{4} \Bigg\}. \tag{4.8}$$

Further, define events

$$E_1 = \left\{ \max_{1 \leq j \leq i} |f_j(\pm R)| \leq M_1 \right\},$$
$$E_2 = \left\{ Z(-R_N, R_N, \vec{x}, \vec{y}, \infty, L_{i+1}^N[-R_N, R_N]) > \delta_1 \right\}.$$

Here, $\vec{x} = (L_1^N(-R_N), \ldots, L_i(-R_N))$ and $\vec{y} = (L_1^N(R_N), \ldots, L_i(R_N))$. We argue that E_1, E_2 have high probability for appropriately chosen M_1, δ_1, and it then suffices to bound the probability of A_δ on these events.

Firstly, we observe that $L_j^N(\pm R_N) > L_{j+1}^N(\pm R_N)$, so $f_j^N(\pm R) > f_{j+1}^N(\pm R)$ as well. Thus

$$E_1^c = \{f_1(\pm R) > M_1\} \cup \{f_i(\pm R) < -M_1\}$$
$$= \Big\{ \big(L_1^N(\pm R_N) \mp pR_N\big) > (M_1 - \lambda R^2)N^{\alpha/2} \Big\}$$
$$\cup \Big\{ \big(L_i^N(\pm R_N) \mp pR_N\big) < -(\lambda R^2 + M_1)N^{\alpha/2} \Big\}.$$

Now take $r > R$. Then in particular $R_N \leq t_3$, so we have

$$\mathbb{P}\left(L_1^N(\pm R_N) \mp pR_N > (M_1 - \lambda R^2)N^{\alpha/2} \right) \leq \mathbb{P}\Big(\sup_{s \in [-t_3, t_3]} L_1^N(s) - ps > (M_1 - \lambda R^2)N^{\alpha/2} \Big).$$

By Lemma 4.2, we find that if $M_1 > R_1(\eta/8) + \lambda R^2$ and $N > N_2(\eta/8)$, then this probability is less than $\eta/8$. Next, we have

$$\mathbb{P}\left(L_i^N(\pm R_N) \mp pR_N < -(\lambda R^2 + M_1)N^{\alpha/2} \right) \leq \mathbb{P}\left(L_i^N(\pm R_N) \mp pR_N < -M_1 N^{\alpha/2} \right)$$
$$\leq \mathbb{P}\Big(\inf_{s \in [-t_2, t_2]} \big(L_i^N(s) - ps\big) < -M_1 N^{\alpha/2} \Big),$$

and this last probability is $< \eta/8$ for $M_1 \geq R_2(\eta/8)$ and $N > N_3(\eta/8)$ by Lemma 4.3. Therefore taking $M_1 = \max\{R_1(\eta/8) + \lambda R^2, R_2(\eta/8)\}$, we find

$$\mathbb{P}(E_1^c) < \frac{\eta}{4}.$$

Now by Proposition 4.1 with $r = R - 1$, there exist $\delta_1(\eta/4)$ and $N_1(\eta/4)$ such that $N \geq N_1$ implies

$$\mathbb{P}(E_2^c) < \frac{\eta}{4}.$$

and so by countable subadditivity,

$$\mathbb{P}\left(E_1^c \cup E_2^c\right) < \frac{\eta}{2}$$

Hence, for $N > \hat{N} := \max\{N_1(\eta/4), N_2(\eta/8), N_3(\eta/8)\}$,

$$\mathbb{P}(A_\delta) = \mathbb{P}(A_\delta \cap E_1 \cap E_2) + \mathbb{P}(A_\delta \cap (E_1^c \cup E_2^c)) \leq \mathbb{P}(A_\delta \cap E_1 \cap E_2) + \frac{\eta}{2}. \tag{4.9}$$

Step 3. In this step, we will prove that condition 4.6 holds, using the results of the previous step, namely the inequality 4.9. We will do so by passing from Bernoulli avoiding random walks to random walks using a Radon-Nikodym derivative, and properties of the condtional expectation.

We define a σ-algebra

$$\mathcal{F} = \sigma\left(L_{i+1}^N, L_1^N(\pm R_N), L_2^N(\pm R_N), \dots, L_i^N(\pm R_N)\right).$$

Clearly $E_1, E_2 \in \mathcal{F}$, so the indicator random variables $\mathbf{1}_{E_1}$ and $\mathbf{1}_{E_2}$ are $\mathcal{F}$-measurable. It follows from the tower property of conditional expectation that

$$\mathbb{P}\left(A_\delta \cap E_1 \cap E_2\right) = \mathbb{E}\left[\mathbf{1}_{A_\delta}\, \mathbf{1}_{E_1}\, \mathbf{1}_{E_2}\right] = \mathbb{E}\left[\mathbf{1}_{E_1}\, \mathbf{1}_{E_2}\, \mathbb{E}[\mathbf{1}_{A_\delta} \mid \mathcal{F}]\right]. \tag{4.10}$$

By the Schur-Gibbs property (see Definition 2.17),

$$\mathbb{E}\left[\mathbf{1}_{A_\delta} \mid \mathcal{F}\right] = \mathbb{E}_{avoid,Ber}^{-R_N, R_N, \vec{x}, \vec{y}, \infty, L_{i+1}^N}\left[\mathbf{1}_{A_\delta}\right].$$

We now observe that the Radon-Nikodym derivative of $\mathbb{P}_{avoid,Ber}^{-R_N,R_N,\vec{x},\vec{y},\infty,L_{m+1}^N}$ with respect to $\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}$ [with $\vec{x}, \vec{y}$ defined as in the definition of E_2] is given by

$$\frac{d\mathbb{P}_{avoid,Ber}^{-R_N,R_N,\vec{x},\vec{y},\infty,L_{i+1}^N}}{d\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}} = \frac{\mathbf{1}_{\{L_1 \geq \cdots \geq L_{i+1}\}}}{Z(-R_N, R_N, \vec{x}, \vec{y}, L_{i+1}^N)}. \tag{4.11}$$

To see this, note that for any event A,

$$\mathbb{P}_{avoid,Ber}^{-R_N,R_N,\vec{x},\vec{y},\infty,L_{i+1}^N}(A) = \frac{\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}(A \cap \{L_1 \geq \cdots \geq L_{i+1}\})}{\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}(L_1 \geq \cdots \geq L_{i+1})}$$

$$= \frac{\mathbb{E}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}\left[\mathbf{1}_A\, \mathbf{1}_{\{L_1 \geq \cdots \geq L_{i+1}\}}\right]}{Z(-R_N, R_N, \vec{x}, \vec{y}, L_{i+1}^N)} = \int_A \frac{\mathbf{1}_{\{L_1 \geq \cdots \geq L_{i+1}\}}}{Z(-R_N, R_N, \vec{x}, \vec{y}, L_{i+1}^N)}\, d\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}.$$

It follows from (4.10), (4.11), and the definition of E_2 that

$$\begin{aligned}
\mathbb{P}(A_\delta \cap E_1 \cap E_2) &= \mathbb{E}\left[\mathbf{1}_{E_1}\, \mathbf{1}_{E_2}\, \mathbb{E}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}\left[\frac{\mathbf{1}_{A_\delta} \cdot \mathbf{1}_{\{L_1 \geq \cdots \geq L_{i+1}\}}}{Z(-R_N, R_N, \vec{x}, \vec{y}, L_{i+1}^N)}\right]\right] \\
&\leq \mathbb{E}\left[\mathbf{1}_{E_1}\, \mathbb{E}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}\left[\frac{\mathbf{1}_{A_\delta}}{\delta_1}\right]\right] \\
&\leq \frac{1}{\delta_1}\, \mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}(A_\delta).
\end{aligned}$$

By Lemma 3.16, there exist W_4 and δ such that $N > W_4$ implies

$$\mathbb{P}_{Ber}^{-R_N,R_N,\vec{x},\vec{y}}(A_\delta) < \frac{\eta\, \delta_1}{2},$$

and hence

$$\mathbb{P}\left(A_\delta \cap E_1 \cap E_2\right) \leq \frac{\eta}{2}.$$

We conclude from (4.9) that $\mathbb{P}(A_\delta) < \eta$ for $N \geq N_0 := \max\{\hat{N}, W_4\}$. Given the definition of A_δ (see 4.8), this implies we have shown condition 2, which, in conjuncton with condition 1, implies tightness. This completes the proof.

5. Bounding the max and min

5.1. **Proof of Lemma 4.2.** Our proof of Lemma 4.2 is similar to that of [3, Lemma 5.2]. We exploit the one-point tightness of L_1^N at two appropriately chosen points, and we use Lemma 3.8 to control the upward deviation of L_1^N from the line of slope p away from these points.

Proof. We write $s_4 = \lceil r+3 \rceil N^\alpha$, $s_3 = \lfloor r+3 \rfloor N^\alpha$, so that $s_3 \leq t_3 \leq s_4$, and take N large enough so that L_1^N is defined at s_4. We define events

$$E(M) = \Big\{ \big|L_1^N(-s_4) + ps_4\big| > MN^{\alpha/2} \Big\}, \quad F(M) = \Big\{ L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} \Big\},$$

$$G(M) = \left\{ \sup_{s \in [0,t_3]} \big(L_1^N(s) - ps \big) \geq (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2} \right\}.$$

For $a, b \in \mathbb{Z}$, $s \in [\![0, t_3]\!]$, and $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$ with $z_1 \leq a$, $z_2 \leq b$, we also define $E(a, b, s, \ell_{bot})$ to be the event that $L_1^N(-s_4) = a$, $L_1^N(s) = b$, and L_2^N agrees with ℓ_{bot} on $[-s_4, s]$.

We claim that the set $G(M) \setminus E(M)$ can be written as a *countable disjoint* union of sets $E(a, b, s, \ell_{bot})$. Let $D(M)$ be the set of tuples (a, b, s, ℓ_{bot}) satisfying

(1) $0 \leq s \leq t_3$,
(2) $0 \leq b - a \leq s + s_4$, $|a + ps_4| \leq MN^{\alpha/2}$, and $b - ps \geq (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2}$,
(3) If $0 \leq s' < s$, then $b - ps' < (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2}$,
(4) $z_1 \leq a, z_2 \leq b$, and $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$.

Conditions (1), (2), and (3) show that the union of these sets $E(a, b, s, \ell_{bot})$ for $(a, b, s, \ell_{bot}) \in D(M)$ is $G(M) \setminus E(M)$. In particular, to see that condition (3) is not too restrictive, note that if L^N is contained in $G(M)$, then there must be a first time $s \in [\![0, t_3]\!]$ when $L_1^N(s) - ps \geq (6r+22)(2r+6)^{1/2}(M+1)N^\alpha/2$. Observe that $D(M)$ is countable, since there are finitely many possible choices of s, countably many a, b and z_1, z_2 for each s, and finitely many ℓ_{bot} for each z_1, z_2. Moreover, condition (3) implies that the sets $E(a, b, s, \ell_{bot})$ are pairwise disjoint for distinct tuples in $D(M)$. This proves the claim.

Now by one-point tightness of L_1^N at integer multiples of N^α, we can choose M large enough depending on ϵ so that

$$\mathbb{P}(E(M)) < \epsilon/4, \quad \mathbb{P}(F(M)) < \epsilon/12 \tag{5.1}$$

for all $N \in \mathbb{N}$. If $(a, b, s, \ell_{bot}) \in D(M)$, then

$$\begin{aligned} \mathbb{P}_{Ber}^{-s_4,s,a,b}\Big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\Big) &= \mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\Big(\ell(s_4 - s_3) + a \geq -ps_3 + MN^{\alpha/2}\Big) \\ &\geq \mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\Big(\ell(s_4 - s_3) \geq p(s_4 - s_3) + 2MN^{\alpha/2}\Big). \end{aligned}$$

The inequality follows from the assumption in (2) that $a + ps_4 \geq -MN^{\alpha/2}$. Moreover, since $b - ps > (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2}$ and $a + ps_4 \leq MN^{\alpha/2}$, we have

$$b - a \geq p(s + s_4) + (6r+21)(2r+6)^{1/2}(M+1)N^{\alpha/2} \geq p(s + s_4) + (6r+21)(M+1)(s+s_4)^{1/2}.$$

The second inequality follows since $s + s_4 \leq 2s_4 \leq (2r+6)N^\alpha$. It follows from Lemma 3.8 with $M_1 = 0$, $M_2 = (6r+21)(M+1)$ that for sufficiently large N, we have

$$\mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\Big(\ell(s_4 - s_3) \geq \frac{s_4 - s_3}{s + s_4}[p(s+s_4) + M_2 N^{\alpha/2}] - (s+s_4)^{1/4}\Big) \geq 1/3, \tag{5.2}$$

for all $(a, b, s, \ell_{bot}) \in D(M)$ simultaneously. Note that $\frac{s_4 - s_3}{s + s_4} \geq \frac{N^\alpha - 1}{(2r+6)N^\alpha} \geq \frac{1}{2r+7}$ for large N. Hence $\frac{s_4 - s_3}{s+s_4}[p(s+s_4)+M_2N^{\alpha/2}]-(s+s_4)^{1/4} \geq p(s_4-s_3)+3(M+1)N^{\alpha/2}-(s+s_4)^{1/4} \geq p(s_4-s_3)+2MN^{\alpha/2}$ for all large enough N. We conclude from (5.2) that

$$\mathbb{P}_{Ber}^{-s_4,s,a,b}\Big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\Big) \geq 1/3$$

uniformly in a, b for large N. Now by the Schur Gibbs property for L^N, we have for any $\ell \in \Omega(-s_4, s, a, b)$ that

$$\mathbb{P}(L_1^N|_{[-s_4,s]} = \ell \,|\, E(a, b, s, \ell_{bot})) = \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}(\ell).$$

Also observe that the event that $\ell(-s_3) > -ps_3 + MN^{\alpha/2}$ decreases in probability if ℓ is lowered. It follows from Lemma 3.2 that

$$\mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}\big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\big) \geq \mathbb{P}_{Ber}^{-s_4,s,a,b}\big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\big).$$

Therefore

$$\begin{aligned}
&\mathbb{P}\big(L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} \,\big|\, E(a, b, s, \ell_{bot})\big) \\
&= \sum_{\ell \in \Omega(-s_4,s,a,b)} \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}(\ell) \cdot \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}\big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\big) \\
&\geq \sum_{\ell \in \Omega(-s_4,s,a,b)} \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}(\ell) \cdot \mathbb{P}_{Ber}^{-s_4,s,a,b}\big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\big) \\
&\geq \frac{1}{3} \sum_{\ell \in \Omega(-s_4,s,a,b)} \mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}}(\ell) = \frac{1}{3}.
\end{aligned}$$

Note once again that this bound holds independent of a, b for all sufficiently large N. It follows from (5.1) that

$$\epsilon/12 > \mathbb{P}(F(M)) \geq \sum_{(a,b,s,\ell_{bot}) \in D(M)} \mathbb{P}(F(M) \cap E(a, b, s, \ell_{bot}))$$

$$= \sum_{(a,b,s,\ell_{bot}) \in D(M)} \mathbb{P}(F(M) \,|\, E(a, b, s, \ell_{bot}))\mathbb{P}(E(a, b, s, \ell_{bot})) \geq \frac{1}{3}\mathbb{P}(G(M) \setminus E(M))$$

for large N. Since in addition $\mathbb{P}(E(M)) < \epsilon/4$, we find that

$$\mathbb{P}\Big(\sup_{s \in [0,t_3]} \big(L_1^N(s) - ps\big) \geq (6r + 22)(2r + 6)^{1/2}(M + 1)N^{\alpha/2}\Big) = \mathbb{P}(G(M)) < \epsilon/2$$

for large enough N. A similar argument proves the same inequality with $[-t_3, 0]$ in place of $[0, t_3]$. Thus we can find an $N_2 = N_2(\epsilon)$ so that

$$\mathbb{P}\Big(\sup_{s \in [-t_3,t_3]} \big(L_1^N(s) - ps\big) \geq R_1N^{\alpha/2}\Big) < \epsilon$$

for all $N \geq N_2$, with $R_1 = (6r + 22)(2r + 6)^{1/2}(M + 1)$.

□

5.2. **Proof of Lemma 4.3.** We begin by proving the following importnat lemma, which allows us to prevent the bottom curve of an ensemble from falling too low on a large enough interval.

Lemma 5.1. *Fix $p \in (0, 1)$, $k \in \mathbb{N}$, and $\alpha, \lambda > 0$. Suppose that $\mathfrak{L}^N = (L_1^N, \dots, L_k^N)$ is a (α, p, λ)-good sequence of $[\![1, k]\!]$-indexed Bernoulli line ensembles. Then for any $r, \epsilon > 0$, there exists $R > 0$ depending on $\lambda, k, p, \epsilon, r, \phi$ and $N_0 \in \mathbb{N}$ depending on $\lambda, k, p, \epsilon, r, \phi, \psi, \alpha$ such that for all $N \geq N_0$,*

$$\mathbb{P}\Big(\max_{x \in [r,R]} \big(L_k^N(xN^\alpha) - pxN^\alpha\big) \leq -(\lambda R^2 + \phi(\epsilon/16))N^{\alpha/2}\Big) < \epsilon.$$

The same statement holds if $[r, R]$ is replaced with $[-R, -r]$.

Remark 5.2. The key to this lemma is the parabolic shift implicit in the definition of an (α, p, λ)-good sequence. This requires the deviation of the top curve from the line of slope p to appear roughly parabolic. Using monotone coupling, we separate the curves of the ensemble so that L_1^N is nearly independent of the other curves. Then we would expect the value of L_1^N at the midpoint of r and R to be close to the midpoint of the straight line segment connecting two points of the parabola. But the parabola is convex, so for large enough R this violates the one-point tightness assumpion at $(R + r)/2$.

Proof. Fix $r > 0$. Note that for any $R > r$,

$$\max_{r \le x \le R} \big(L_k^N(xN^\alpha) - pxN^\alpha\big) \ge \max_{\lceil r \rceil \le x \le R} \big(L_k^N(xN^\alpha) - pxN^\alpha\big).$$

Thus by replacing r with $\lceil r \rceil$, we can assume that $r \in \mathbb{Z}$. Before beginning the proof, we introduce notation. Define constants

$$C = \sqrt{8p(1-p) \log \frac{3}{1 - (11/12)^{1/(k-1)}}}, \tag{5.3}$$

$$R_0 = Ck + \sqrt{C^2k^2 + 2\phi(\epsilon/16)} + r. \tag{5.4}$$

Note that $R_0 \ge r$. We define $R = R_0 + \mathbf{1}_{R_0 + r \text{ odd}}$, so that $R \ge R_0$ and the midpoint $(R + r)/2$ is an integer. In the following, we always assume N is large enough depending on ψ, R so that L_1^N is defined at R. We may do so by the second condition in the definition of an (α, p, λ)-good sequence (see Definition 2.24). Define events

$$A = \left\{ L_1^N\left(\frac{R+r}{2} N^\alpha\right) - pN^\alpha \frac{R+r}{2} + \lambda \left(\frac{R+r}{2}\right)^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2} \right\},$$

$$B = \left\{ \max_{x \in [r,R]} \big(L_k^N(xN^\alpha) - pxN^\alpha\big) \le -(\lambda R^2 + \phi(\epsilon/16))N^{\alpha/2} \right\}.$$

Let F denote the subset of B for which the inequalities

$$\begin{aligned} &prN^\alpha - (\lambda r^2 + \phi(\epsilon/16))N^{\alpha/2} < L_1^N(rN^\alpha) < prN^\alpha - (\lambda r^2 - \phi(\epsilon/16))N^{\alpha/2}, \\ &pRN^\alpha - (\lambda R^2 + \phi(\epsilon/16))N^{\alpha/2} < L_1^N(RN^\alpha) < pRN^\alpha - (\lambda R^2 - \phi(\epsilon/16))N^{\alpha/2} \end{aligned} \tag{5.5}$$

hold. Let D denote the set of pairs $(\vec{x}, \vec{y})$, with $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$ satisfying

(1) $0 \le y_i - x_i \le (R - r)N^\alpha$ for $1 \le i \le k$,
(2) $prN^\alpha - (\lambda r^2 + \phi(\epsilon/16))N^{\alpha/2} < x_1 < prN^\alpha - (\lambda r^2 - \phi(\epsilon/16))N^{\alpha/2}$ and $pRN^\alpha - (\lambda R^2 + \phi(\epsilon/16))N^{\alpha/2} < y_1 < pRN^\alpha - (\lambda R^2 - \phi(\epsilon/16))N^{\alpha/2}$.

Let $E(\vec{x}, \vec{y})$ denote the subset of F consisting of L^N for which $L_i^N(rN^\alpha) = x_i$ and $L_i^N(RN^\alpha) = y_i$ for $1 \le i \le k$, and $L_1^N(s) \ge \cdots \ge L_k^N(s)$ for all s. Then D is countable, the $E(\vec{x}, \vec{y})$ are pairwise disjoint, and $F = \bigcup_{(\vec{x},\vec{y}) \in D} E(\vec{x}, \vec{y})$.

To prove the lemma, we argue that $\mathbb{P}(B) < \epsilon$ for large N. We split the proof into several steps.

Step 1. We will argue in the following steps that for large enough N,

$$\mathbb{P}(A \,|\, E(\vec{x}, \vec{y})) > 1/4 \tag{5.6}$$

uniformly in $\vec{x}, \vec{y}$. In this step, we prove the lemma assuming this fact.

It follows from (5.6) that

$$\mathbb{P}(A \,|\, F) = \sum_{(\vec{x},\vec{y}) \in D} \frac{\mathbb{P}(A \,|\, E(\vec{x}, \vec{y}))\mathbb{P}(E(\vec{x}, \vec{y}))}{\mathbb{P}(F)} \ge \frac{1}{4} \cdot \frac{\sum_{(\vec{x},\vec{y}) \in D} \mathbb{P}(E(\vec{x}, \vec{y}))}{\mathbb{P}(F)} = \frac{1}{4}. \tag{5.7}$$

From the third condition in Definition 2.24, we have $\mathbb{P}(A) < \epsilon/8$ for large enough N. Hence

$$\mathbb{P}(F) = \frac{\mathbb{P}(A \cap F)}{\mathbb{P}(A \,|\, F)} \leq 4\mathbb{P}(A) < \epsilon/2.$$

Now with probability $> 1 - \epsilon/2$, the two inequalities in (5.5) hold. We conclude that

$$\mathbb{P}(B) \leq \mathbb{P}(F) + \epsilon/2 \leq \epsilon.$$

Step 2. We will now prove (5.6), assuming results from Steps 3 and 4 below. We first note that by Lemma 3.1, if we raise the endpoints of each curve, then the probability of the event A will decrease. In particular, write $T = (R-r)N^\alpha$, and define $\vec{x}\,', \vec{y}\,'$ by

$$x_i' = \lceil prN^\alpha - (\lambda r^2 - \phi(\epsilon/8))N^{\alpha/2} \rceil + (k-i)\lceil C\sqrt{T}\,\rceil,$$
$$y_i' = \lceil pRN^\alpha - (\lambda R^2 - \phi(\epsilon/8))N^{\alpha/2} \rceil + (k-i)\lceil C\sqrt{T}\,\rceil.$$

Note that $x_i' \geq x_1 \geq x_i$ for each i by condition (2) above. Furthermore, $x_i' - x_{i+1}' \geq C\sqrt{T}$. The same observations hold for y_i'. Using Lemma 3.1, we have

$$\begin{aligned}
\mathbb{P}(A \,|\, E(\vec{x},\vec{y})) &= \mathbb{P}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}, \infty, L_k}_{avoid,Ber}(A \,|\, F) \geq \mathbb{P}^{rN^\alpha, RN^\alpha, \vec{x}\,', \vec{y}\,', \infty, L_k}_{avoid,Ber}(A \,|\, F) \\
&\geq \mathbb{P}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}_{Ber}(A \cap \{L_1 \geq \cdots \geq L_k\} \,|\, F) \\
&\geq \mathbb{P}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}_{Ber}(A \,|\, F) - \big(1 - \mathbb{P}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}_{Ber}(L_1 \geq \cdots \geq L_k \,|\, F)\big) \\
&= \mathbb{P}^{rN^\alpha, RN^\alpha, x_1', y_1'}_{Ber}(A) - \big(1 - \mathbb{P}^{rN^\alpha, RN^\alpha, \vec{x}', \vec{y}'}_{Ber}(L_1 \geq \cdots \geq L_k \,|\, F)\big).
\end{aligned} \tag{5.8}$$

For the first term in the last line, we used the Schur Gibbs property and the fact that A and F are independent under $\mathbb{P}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}}_{Ber}$. In Step 3, we will argue that the two probabilities in (5.8) are bounded below by $1/3$ and $11/12$, respectively. Then $\mathbb{P}(A \,|\, E(\vec{x},\vec{y})) \geq 1/3 - 1/12 = 1/4$ for large N independent of $\vec{x}, \vec{y}$, proving (5.6).

Step 3. We first argue that $\mathbb{P}^{rN^\alpha, RN^\alpha, x_1', y_1'}_{Ber}(A) > 1/3$ for sufficiently large N. Write

$$\overline{x} = x_1' - (k-1)\lceil C\sqrt{T}\rceil, \quad \overline{y} = y_1' - (k-1)\lceil C\sqrt{T}\rceil,$$

and $\overline{z} = \overline{y} - \overline{x}$. We have

$$\begin{aligned}
&\mathbb{P}^{rN^\alpha, RN^\alpha, x_1', y_1'}_{Ber}(A) \\
&= \mathbb{P}^{0,T,x_1',y_1'}_{Ber}\Big(L_1(T/2) - pN^\alpha \frac{R+r}{2} + \lambda\Big(\frac{R+r}{2}\Big)^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\Big) \\
&= \mathbb{P}^{0,T,\overline{x},\overline{y}}_{Ber}\Big(L_1(T/2) - pN^\alpha \frac{R+r}{2} + \lambda\Big(\frac{R+r}{2}\Big)^2 N^{\alpha/2} < -\big(\phi(\epsilon/16) + (k-1)\lceil C\sqrt{R-r}\,\rceil\big)N^{\alpha/2}\Big) \\
&\geq \mathbb{P}^{0,T,\overline{x},\overline{y}}_{Ber}\Big(L_1(T/2) - \frac{\overline{x}+\overline{y}}{2} < \Big(\lambda\Big(\frac{R^2+r^2}{2}\Big) - \lambda\Big(\frac{R+r}{2}\Big)^2 - Ck\sqrt{R-r} - 2\phi(\epsilon/16)\Big)N^{\alpha/2}\Big).
\end{aligned} \tag{5.9}$$

The inequality in the last line follows from the definitions of $\overline{x}, \overline{y}$. Observe that

$$\frac{R^2+r^2}{2} - \left(\frac{R+r}{2}\right)^2 = \frac{R^2+r^2}{4} - \frac{rR}{2} = O(R^2)$$

for fixed r. Our choice of R from (5.4) ensures that the constant factor multiplying $N^{\alpha/2}$ in (5.4) is positive. Denoting this constant by γ_0 and letting $\gamma = \gamma_0/\sqrt{R-r}$, we see that (5.9) is equal to

$$\mathbb{P}^{0,T,0,\overline{z}}_{Ber}\Big(L_1(T/2) - \overline{z}/2 < \gamma\sqrt{T}\Big).$$

Let $\ell^{(T,\overline{z})}$ have the same law as L_1 under a probability measure $\mathbb{P}$ as in Theorem 3.3, and let B^σ, $\sigma^2 = p(1-p)$, be the Brownian bridge provided by the theorem. Then the last probability is

$$\mathbb{P}\Big(\ell^{(T,\overline{z})}(T/2) - \overline{z}/2 < \gamma\sqrt{T}\Big) = \mathbb{P}\Big(\Big[\ell^{(T,\overline{z})}(T/2) - \overline{z}/2 - \sqrt{T}B^\sigma_{1/2}\Big] + \sqrt{T}B^\sigma_{1/2} < \gamma\sqrt{T}\Big)$$
$$\geq \mathbb{P}\Big(\sqrt{T}B^\sigma_{1/2} < 0 \quad \text{and} \quad \Delta(T,\overline{z}) < \gamma\sqrt{T}\Big) \geq \frac{1}{2} - \mathbb{P}\Big(\Delta(T,\overline{z}) \geq \gamma\sqrt{T}\Big).$$

Here, $\Delta(T,\overline{z})$ is as defined in (3.2). Observe that

$$\frac{|\overline{z} - pT|^2}{T} \leq \frac{(\lambda(R^2 - r^2)N^{\alpha/2} + 1)^2}{(R-r)N^\alpha} \leq 4\lambda^2(R+r)^2(R-r). \tag{5.10}$$

Thus Corollary 3.5 shows that $\mathbb{P}(\Delta(T,\overline{z}) \geq \gamma\sqrt{T}) < 1/6$ for large enough N. This gives a lower bound on $\mathbb{P}^{rN^\alpha,RN^\alpha,x_1',y_1'}_{Ber}(A)$ of $1/2 - 1/6 = 1/3$ as desired.

Lastly, we show that $\mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x}',\vec{y}'}_{Ber}(L_1 \geq \cdots \geq L_k \,|\, F) > 11/12$ for large N. Note that on the event F, L_k^N lies uniformly below the line segment connecting $L_1^N(rN^\alpha)$ and $L_1^N(RN^\alpha)$. Thus after raising the endpoints to $\vec{x}\,', \vec{y}\,'$, the bottom curve L_k lies uniformly at a distance of at least $C\sqrt{T}$ below the segment ℓ_{k-1} connecting $L_{k-1}(rN^\alpha)$ and $L_{k-1}(RN^\alpha)$, and moreover the endpoints of all adjacent curves are at least $C\sqrt{T}$ apart. Let ℓ_{bot} denote the segment $\ell_{k-1} - C\sqrt{T}$, so that $L_k \leq \ell_{bot}$. Then since $L_1,\dots,L_{k-1},\ell_{bot}$ are independent of F under $\mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x},\vec{y}}_{Ber}$, we have

$$\mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x},\vec{y}}_{Ber}(L_1 \geq \cdots \geq L_k \,|\, F) \geq \mathbb{P}^{rN^\alpha,RN^\alpha,\vec{x},\vec{y}}_{Ber}(L_1 \geq \cdots \geq L_{k-1} \geq \ell_{bot}).$$

In view of (5.10), we see from Lemma 3.18 that for large enough N depending on λ, r, R, C, the probability on right is bounded below by

$$\big(1 - 3e^{-C^2/8p(1-p)}\big)^{k-1}.$$

Our choice of C in (5.3) implies that this quantity is at least $11/12$.

Essentially the same argument proves the statement if $[r,R]$ is replaced by $[-R,-r]$.

□

We now prove Lemma 4.3. We exploit Lemma 5.1 in order to find two far away points where L_k^N cannot be too low. After separating the curves in order to treat L_k^N as a free curve as in the previous argument, we employ Lemma 3.10 to bound the deviation of L_k^N below the line of slope p.

Proof. We first introduce notation used in the proof. Define events

$$A_N(R_2) = \left\{\inf_{s\in[-t_2,t_2]} \big(L_k^N(s) - ps\big) \leq -R_2N^{\alpha/2}\right\},$$

$$B_N = \left\{\max_{x\in[r+2,R]} \big(L_k^N(xN^\alpha) - pxN^\alpha\big) > -MN^{\alpha/2}\right\}$$
$$\cap \left\{\max_{x\in[-R,-r-2]} \big(L_k^N(xN^\alpha) - pxN^\alpha\big) > -MN^{\alpha/2}\right\}.$$

Here, we define M, R via Lemma 5.1, taking R large enough so that with $M = \lambda R^2 + \phi(\epsilon/64)$, we have

$$\mathbb{P}(B_N^c) < \epsilon/2 \tag{5.11}$$

for sufficiently large N.

For $0 < a,b \in \mathbb{Z}$ and $\vec{x},\vec{y} \in \mathfrak{W}_k$, we define $E(a,b,\vec{x},\vec{y})$ to be the event that $L_i^N(-a) = x_i$ and $L_i^N(b) = y_i$ for $1 \leq i \leq k$, and $L_1^N(s) > \cdots > L_k^N(s)$ for all $s \in [-RN^\alpha, RN^\alpha]$.

We claim that $B_N(M,R)$ can be written as a countable disjoint union of sets $E(a,b,\vec{x},\vec{y})$. Let $D_N(M)$ be the collection of tuples $(a,b,\vec{x},\vec{y})$ satisfying

(1) $a,b \in [(r+2)N^\alpha, RN^\alpha]$.
(2) $0 \le y_i - x_i \le b+a$, $x_k + pa > -MN^{\alpha/2}$, and $y_k - pb > -MN^{\alpha/2}$.
(3) If $c,d \in \mathbb{Z}$, $c > a$, and $d > b$, then $L_k^N(-c) + pc \le -MN^{\alpha/2}$ and $L_k^N(d) - pd \le -MN^{\alpha/2}$.

Since there are finitely many integers a,b satisfying (1), the x_i, y_i are integers, and there are finitely many choices of L_i^N on $[-aN^\alpha, bN^\alpha]$ given a,b,x_i,y_i, we see that $D_N(M)$ is countable. The third condition ensures that the $E(a,b,\vec{x},\vec{y})$ are pairwise disjoint. To see that their union over $D_N(M)$ is all of $B_N(M,R)$, note that $B_N(M,R)$ occurs if and only if there is a first integer time $s=-a$ and a last integer time $s=b$ when $L_k^N(s)-ps$ crosses $-MN^{\alpha/2}$.

Lastly, define the constant

$$C = \sqrt{16p(1-p)\log\frac{3}{1-2^{-1/(k-1)}}}. \tag{5.12}$$

We will prove that $\mathbb{P}(A_N(R_2)) < \epsilon$ for large N, if R_2 is chosen large enough depending on M,C,k,r,ϵ. We specify how we choose R_2 after (5.17) below. We split the proof into steps for clarity.

Step 1. We will prove in the steps below that for large enough N,

$$\mathbb{P}(A_N(R_2)\,|\,E(a,b,\vec{x},\vec{y})) < \epsilon/2 \tag{5.13}$$

uniformly in $a,b,\vec{x},\vec{y}$. In this step, we prove the lemma assuming this fact.

Since the $E(a,b,\vec{x},\vec{y})$ are disjoint, (5.13) implies

$$\begin{aligned}\mathbb{P}(A_N(R_2)\cap B_N(M,R)) &= \sum_{(a,b,\vec{x},\vec{y})\in D_N} \mathbb{P}(A_N(R_2)\,|\,E(a,b,\vec{x},\vec{y}))\mathbb{P}(E(a,b,\vec{x},\vec{y}))\\ &\le \frac{\epsilon}{2}\sum_{(a,b,\vec{x},\vec{y})\in D_N}\mathbb{P}(E(a,b,\vec{x},\vec{y})) \le \frac{\epsilon}{2}.\end{aligned}$$

It follows from (5.11) that

$$\mathbb{P}(A_N(R_2)) \le \mathbb{P}(A_N(R_2)\cap B_N) + \epsilon/2 < \epsilon$$

for large enough N.

Step 2. We next prove (5.13), assuming results from Steps 3 below. Define $\vec{x}\,',\vec{y}\,'$ by

$$\begin{aligned} x_i' &= \lfloor -pa - MN^{\alpha/2}\rfloor - (i-1)\lceil CN^{\alpha/2}\rceil,\\ y_i' &= \lfloor pb - MN^{\alpha/2}\rfloor - (i-1)\lceil CN^{\alpha/2}\rceil.\end{aligned}$$

Observe that by condition (2) above, $x_i' \le -pa - MN^{\alpha/2} \le x_k \le x_i$, and similarly for $\vec{y}$. It follows from Lemma 3.1 that

$$\begin{aligned}\mathbb{P}(A_N(R_2)\,|\,E(a,b,\vec{x},\vec{y})) &\le \mathbb{P}^{-a,b,\vec{x},\vec{y}}_{avoid,Ber}\Big(\inf_{s\in[-a,b]}\big(L_k(s)-ps\big) \le -R_2N^{\alpha/2}\Big) \tag{5.14}\\ &= \mathbb{P}^{0,a+b,\vec{x},\vec{y}}_{avoid,Ber}\Big(\inf_{s\in[0,a+b]}\big(L_k(s-a)-p(s-a)\big) \le -R_2N^{\alpha/2}\Big)\\ &\le \mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{avoid,Ber}\Big(\inf_{s\in[0,a+b]}\big(L_k'(s)-p(s-a)\big) \le -R_2N^{\alpha/2}\Big).\end{aligned}$$

In the last line, we have written $L_k'(s) = L_k(s-a)$. The last probability is bounded above by

$$\frac{\mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{Ber}\Big(\inf_{s\in[0,a+b]}\big(\ell(s)-p(s-a)\big) \le -R_2N^{\alpha/2}\Big)}{\mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{Ber}(F)}, \tag{5.15}$$

where
$$F = \{L_1'(s) > \cdots > L_k'(s),\, s \in [0, a+b]\}.$$
In Step 3 below, we will prove that the numerator and denominator in (5.15) are $< \epsilon/4$ and $> 1/2$ for sufficiently large N. It then follows that (5.15) is bounded above by $\epsilon/2$, proving (5.13).

Step 3. We first argue that
$$\mathbb{P}_{Ber}^{0,a+b,\vec{x}\,',\vec{y}\,'}\Big(\inf_{s\in[0,a+b]}\big(\ell(s) - p(s-a)\big) \leq -R_2 N^{\alpha/2}\Big) < \epsilon/4 \tag{5.16}$$
for sufficiently large N. Writing $\vec{z} = \vec{y}\,' - \vec{x}\,'$, (5.16) is equal to
$$\mathbb{P}_{Ber}^{0,a+b,x_k',y_k'}\Big(\inf_{s\in[0,a+b]}\big(\ell(s) - p(s-a)\big) \leq -R_2 N^{\alpha/2}\Big)$$
$$= \mathbb{P}_{Ber}^{0,a+b,0,z_k}\Big(\inf_{s\in[0,a+b]}\big(\ell(s) - ps + pa - \lceil pa + MN^{\alpha/2}\rceil - (k-1)\lceil CN^{\alpha/2}\rceil\big) \leq -R_2 N^{\alpha/2}\Big)$$
$$\leq \mathbb{P}_{Ber}^{0,a+b,0,z_k}\Big(\inf_{s\in[0,a+b]}\big(\ell(s) - ps\big) \leq -(R_2 - M - Ck)N^{\alpha/2}\Big). \tag{5.17}$$
Since $z_k \geq p(a+b)$, Lemma 3.10 allows us to find $R_2 > 0$ depending on M, C, k, r, ϵ so that this probability is $< \epsilon/4$ for all large N, such that $a+b$ is larger than some constant W_1. But observe that $a+b \geq 2rN^\alpha$, so it suffices to take $N > (W_1/2r)^{1/\alpha}$. Thus we obtain (5.16), *independent* of $a, b, \vec{x}, \vec{y}$.

Lastly, we argue that
$$\mathbb{P}_{Ber}^{0,a+b,\vec{x}\,',\vec{y}\,'}(F) > 1/2 \tag{5.18}$$
for large N. Write $a = a'N^\alpha, b = b'N^\alpha$, $T = a+b = (a'+b')N^\alpha$, and $z = y_k' - x_k'$. Also let $C' = C/\sqrt{a'+b'}$, so that $x_i' - x_{i+1}' \geq CN^{\alpha/2} = C'\sqrt{T}$ and likewise for y_i'. Note that $|z - pT| < 1$. It follows from Lemma 3.18, applied with $k+1$ in place of k, $\ell_{bot} = -\infty$, and C' in place of C, that for T larger than some T_0,
$$\mathbb{P}_{Ber}^{0,a+b,\vec{x}\,',\vec{y}\,'}(F) \geq \big(1 - 3e^{-(C')^2/8p(1-p)}\big)^k \geq \big(1 - 3e^{-C^2/16p(1-p)R}\big)^k. \tag{5.19}$$
Here, we used the fact that $a'+b' \leq 2R$, hence $C' \geq C/\sqrt{2R}$. The constant T_0 depends in particular on C', hence possibly on $a+b$. Referring to the proofs of Lemmas 3.18 and 3.5, we see that the dependency of T_0 on C' amounts to requiring that $e^{-C'\sqrt{T_0}}$ be sufficiently small. But $C' \geq C/\sqrt{2R}$, so for this it suffices to choose T_0 depending on C and R. Moreover, $T \geq 2rN^\alpha$, so as long as $N \geq (T_0/2r)^{1/\alpha}$, we have the bound in (5.19) independent $a, b, \vec{x}, \vec{y}$. Our choice of C in (5.12) ensures that the expression on the right in (5.19) is $> 1/2$, proving (5.18)

□

6. Lower bounds on the acceptance probability

6.1. **Proof of Lemma 4.4.** Throughout this section we assume the same notation as in Lemma 4.4, i.e., we assume that we have fixed $k \in \mathbb{N}$, $p \in (0,1)$, $M_1, M_2 > 0$, $\ell_{bot} : [\![-t_2, t_2]\!] \to \mathbb{R} \cup \{-\infty\}$, and $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$ such that $|\Omega_{avoid}(-t_2, t_2, \vec{x}, \vec{y}, \infty, \ell_{bot})| \geq 1$. We also assume that

(1) $\sup_{s\in[-t_2,t_2]}\big(\ell_{bot}(s) - ps\big) \leq M_2(2t_2)^{1/2}$,
(2) $-pt_2 + M_1(2t_2)^{1/2} \geq x_1 \geq x_{k-1} \geq \max\big(\ell_{bot}(-t_2), -pt_2 - M_1(2t_2)^{1/2}\big)$,
(3) $pt_2 + M_1(2t_2)^{1/2} \geq y_1 \geq y_{k-1} \geq \max\big(\ell_{bot}(t_2), pt_2 - M_1(2t_2)^{1/2}\big)$.

Definition 6.1. Let $\tilde{\ell}_{bot}$ denote the path which is equal to ℓ_{bot} on $[\![-t_2, -t_1]\!] \cup [\![t_1, t_2]\!]$ and equal to $-\infty$ on $[\![-t_1+1, t_1-1]\!]$. We also denote by $\mathfrak{L} = (L_1, \ldots, L_{k-1})$ and $\tilde{\mathfrak{L}} = (\tilde{L}_1, \ldots, \tilde{L}_{k-1})$ the $[\![1, k-1]\!]$-indexed line ensembles which are uniformly distributed on $\Omega_{avoid}(-t_2, t_2, \vec{x}, \vec{y}, \infty, \ell_{bot})$ and $\Omega_{avoid}(-t_2, t_2, \vec{x}, \vec{y}, \infty, \tilde{\ell}_{bot})$ respectively, and we let $\mathbb{P}_{\mathfrak{L}}$ and $\mathbb{P}_{\tilde{\mathfrak{L}}}$ denote these uniform measures.

In other words, $\tilde{\mathfrak{L}}$ has the law of $k-1$ independent Bernoulli bridges that have been conditioned on not-crossing each other on the entire interval $[-t_2, t_2]$ and also staying above the graph of ℓ_{bot} but only on the intervals $[-t_2, -t_1]$ and $[t_1, t_2]$. The latter restriction means that $\tilde{L}_{k-1}$ is allowed to dip below ℓ_{bot} on $(-t_1, t_1)$.

Lemma 6.2. *There exists $N_5 \in \mathbb{N}$ such that for $N \geq N_5$,*

$$\mathbb{P}_{\tilde{\mathfrak{L}}}\Big(Z\big(-t_1, t_1, \tilde{L}(-t_1), \tilde{L}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\big) \geq g\Big) \geq h, \tag{6.1}$$

where the functions g and h are as in Lemma 4.4.

We will prove Lemma 6.2 in Section 6.2. In the remainder of this section, we give the proof of Lemma 4.4.

Lemma 4.4. Let us denote by $\mathbb{P}_{\mathfrak{L}'}$ and $\mathbb{P}_{\tilde{\mathfrak{L}}'}$ the measures on $[\![1, k-1]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}'$, $\tilde{\mathfrak{L}}'$ on the set $S = [\![-t_2, -t_1]\!] \cup [\![t_1, t_2]\!]$, induced by the restrictions of the measures $\mathbb{P}_{\mathfrak{L}}$, $\mathbb{P}_{\tilde{\mathfrak{L}}}$ in Definition 6.1 to S. We claim that the Radon-Nikodym derivative between these two restricted measures is given on Bernoulli line ensembles $\mathfrak{B}$ on $[\![1, k-1]\!] \times S$ by

$$\frac{d\mathbb{P}_{\mathfrak{L}'}}{d\mathbb{P}_{\tilde{\mathfrak{L}}'}}(\mathfrak{B}) = (Z')^{-1} Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right), \tag{6.2}$$

with $Z' = \mathbb{E}_{\tilde{\mathfrak{L}}'}\left[Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right)\right]$. As proof for this claim, consider the following: Here, Ω denotes Ω_{avoid} and $\Omega(S)$ is the set $\Omega_{avoid}(-t_2, t_2, \tilde{\mathfrak{L}}(-t_2), \tilde{\mathfrak{L}}(t_2), \ell_{bot})$ yet each element has its data stripped on $[\![[-t_1+1, t_1-1]$. For each $\omega \in \Omega(S)$ we have

$$\mathbb{P}_{\mathfrak{L}'}(\omega) = \frac{\Omega\left(-t_1, t_1, \mathfrak{B}(-t_1, \omega), \mathfrak{B}(t_1, \omega), \ell_{bot}[\![-t_1, t_1]\!]\right)}{\Omega\left(-t_2, t_2, \mathfrak{B}(-t_2, \omega), \mathfrak{B}(t_2, \omega), \ell_{bot}\right)}$$

$$\mathbb{P}_{\tilde{\mathfrak{B}}'}(\omega) = \frac{\Omega\left(-t_1, t_1, \tilde{\mathfrak{B}}(-t_1, \omega), \tilde{\mathfrak{B}}(t_1, \omega)\right)}{\Omega\left(-t_2, t_2, \tilde{\mathfrak{B}}(-t_2, \omega), \tilde{\mathfrak{B}}(t_2, \omega), \tilde{\ell}_{bot}\right)}$$

$$Z(-t_1, t_1, \tilde{\mathfrak{B}}(-t_1, \omega), \tilde{\mathfrak{B}}(t_1, \omega), \ell_{bot}) = \frac{\Omega(-t_1, t_1, \tilde{\mathfrak{B}}(-t_1, \omega), \tilde{\mathfrak{B}}(t_1, \omega), \ell_{bot})}{\Omega(-t_1, t_1, \tilde{\mathfrak{B}}(-t_1, \omega), \tilde{\mathfrak{B}}(t_1, \omega))}$$

These all follow from definitions [Get Definitions] and the nature of the restrictions. Hence, we may now calculate the value of Z'. We find that

$$\begin{aligned}
\mathbb{E}_{\tilde{\mathfrak{L}}'}\left[Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}\right)\right] &= \sum_{\omega \in \Omega(S)} \frac{\Omega(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1))}{\Omega(-t_2, t_2, \mathfrak{B}(-t_2), \mathfrak{B}(t_2), \tilde{\ell}_{bot})} \cdot \frac{\Omega(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot})}{\Omega(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1))} \\
&= \frac{\sum_{\omega \in \Omega(S)} \Omega(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot})}{\Omega(-t_2, t_2, \mathfrak{B}(-t_2), \mathfrak{B}(t_2), \tilde{\ell}_{bot})} \\
&= \frac{\Omega(-t_2, t_2, \mathfrak{B}(-t_2), \mathfrak{B}(t_2), \ell_{bot}}{\Omega(-t_2, t_2, \mathfrak{B}(-t_2), \mathfrak{B}(t_2), \tilde{\ell}_{bot}}
\end{aligned}$$

Using this equation as well as the earlier stated definitions, we find

$$\begin{aligned}
(Z')^{-1} Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}) &= \frac{\Omega(-t_1, t_1, \tilde{\mathfrak{B}}(-t_1, \omega), \tilde{\mathfrak{B}}(t_1, \omega), \ell_{bot})}{\Omega(-t_1, t_1, \tilde{\mathfrak{B}}(-t_1, \omega), \tilde{\mathfrak{B}}(t_1, \omega))} \cdot \left(\frac{\Omega(-t_2, t_2, \mathfrak{B}(-t_2), \mathfrak{B}(t_2), \ell_{bot}}{\Omega(-t_2, t_2, \mathfrak{B}(-t_2), \mathfrak{B}(t_2), \tilde{\ell}_{bot}}\right)^{-1} \\
&= \frac{\mathbb{P}_{\mathfrak{L}'}(\omega)}{\mathbb{P}_{\tilde{\mathfrak{L}}'}(\omega)}
\end{aligned}$$

Hence, we find that equation 6.2 is correct. Note that $Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right)$ is a deterministic function of $((\mathfrak{B}(-t_1), \mathfrak{B}(t_1))$. In fact, the law of $((\mathfrak{B}(-t_1), \mathfrak{B}(t_1))$ under $\mathbb{P}_{\tilde{\mathfrak{L}}'}$ is the same as that of $\big(\tilde{\mathfrak{L}}(-t_1), \tilde{\mathfrak{L}}(t_1)\big)$ by way of the restriction. It follows from Lemma 6.2 that

$$\begin{aligned} Z' &= \mathbb{E}_{\tilde{\mathfrak{L}}'}\left[Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right)\right] \\ &= \mathbb{E}_{\tilde{\mathfrak{L}}}\left[Z\left(-t_1, t_1, \mathfrak{L}(-t_1), \mathfrak{L}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right)\right] \geq gh, \end{aligned}$$

which gives us

$$(Z')^{-1} \leq \frac{1}{gh}. \tag{6.3}$$

Similarly, the law of $(\mathfrak{B}(-t_1), \mathfrak{B}(t_1))$ under $\mathbb{P}_{\mathfrak{L}'}$ is the same as that of $(\mathfrak{L}(-t_1), \mathfrak{L}(t_1))$ under $\mathbb{P}_{\mathfrak{L}}$. Hence

$$\begin{aligned} &\mathbb{P}_{\mathfrak{L}}\left(Z\left(-t_1, t_1, \mathfrak{L}(-t_1), \mathfrak{L}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right) \leq gh\tilde{\epsilon}\right) = \\ &\qquad \mathbb{P}_{\mathfrak{L}'}\left(Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right) \leq gh\tilde{\epsilon}\right). \end{aligned} \tag{6.4}$$

Now let Ω denote the space of paths of $\mathfrak{B}$, and let us write $E = \{Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right) \leq gh\tilde{\epsilon}\} \subset \Omega$. Then according to (6.2), we have

$$\mathbb{P}_{\mathfrak{L}'}(E) = \int_\Omega \mathbf{1}_E \, d\mathbb{P}_{\mathfrak{L}'} = (Z')^{-1} \int_\Omega \mathbf{1}_E \cdot Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right) \, d\mathbb{P}_{\tilde{\mathfrak{L}}'}(\mathfrak{B}).$$

From the definition of E and the inequality (6.3), it follows that

$$\mathbb{P}_{\mathfrak{L}'}(E) \leq (Z')^{-1} \int_\Omega \mathbf{1}_E \cdot gh\tilde{\epsilon} \, d\mathbb{P}_{\tilde{\mathfrak{L}}'} \leq \frac{1}{gh} \int_\Omega gh\tilde{\epsilon} \, d\mathbb{P}_{\tilde{\mathfrak{L}}'} \leq \tilde{\epsilon}.$$

In combination with (6.4), this proves (4.3).

□

6.2. **Proof of Lemma 6.2.** In this section, we prove Lemma 6.2. We first state and prove two auxiliary lemmas necessary for the proof. The first lemma establishes a set of conditions under which we have the desired lower bound on the acceptance probability.

Lemma 6.3. *Let $\epsilon > 0$ and $V^{top} > 0$ be given such that $V^{top} > M_2 + 6(k-1)\epsilon$. Suppose further that $\vec{a}, \vec{b} \in \mathfrak{W}_{k-1}$ are such that*

(1) $V^{top}(2t_2)^{1/2} \geq a_1 + pt_1 \geq a_{k-1} + pt_1 \geq (M_2 + 2\epsilon)(2t_2)^{1/2}$;
(2) $V^{top}(2t_2)^{1/2} \geq b_1 - pt_1 \geq b_{k-1} - pt_1 \geq (M_2 + 2\epsilon)(2t_2)^{1/2}$;
(3) $a_i - a_{i+1} \geq 3\epsilon(2t_2)^{1/2}$ and $b_i - b_{i+1} \geq 3\epsilon(2t_2)^{1/2}$ for $i = 1, \dots, k-2$.

Then we can find $g = g(\epsilon, V^{top}, M_2) > 0$ and $N_6 \in \mathbb{N}$ such that for all $N \geq N_6$ we have

$$Z\big(-t_1, t_1, \vec{a}, \vec{b}, \ell_{bot}[\![-t_1, t_1]\!]\big) \geq g. \tag{6.5}$$

Proof. Observe by the rightmost inequalities in conditions (1) and (2) in the hypothesis, as well as condition (1) in Lemma 4.4, that ℓ_{bot} lies a distance of at least $2\epsilon(2t_2)^{1/2}$ uniformly below the line segment connecting a_{k-1} and b_{k-1}. Also note that (1) and (2) imply $|b_i - a_i - 2pt_1| \leq (V^{top} - M_2)(2pt_1)^{1/2}$ for each i. Lastly noting (3), we see that the conditions of Lemma 3.18 are satisfied with $C = 2\epsilon$. This implies (6.5), with

$$g = \big(1 - 3e^{-\epsilon^2/2p(1-p)}\big)^{k-1}.$$

□

The next lemma helps us derive the lower bound h in (6.1). Let us put

$$t_{12} = \left\lfloor \frac{t_1 + t_2}{2} \right\rfloor. \tag{6.6}$$

Lemma 6.4. *For any $R > 0$ we can find $V_1^t, V_1^b \geq M_2 + R$, $h_1 > 0$ and $N_7 \in \mathbb{N}$ (depending on R) such that if $N \geq N_7$ we have*

$$\mathbb{P}_{\tilde{\mathfrak{L}}}\Big((2t_2)^{1/2}V_1^t \geq \tilde{L}_1(\pm t_{12}) \mp pt_{12} \geq \tilde{L}_{k-1}(\pm t_{12}) \mp pt_{12} \geq (2t_2)^{1/2}V_1^b\Big) \geq h_1. \tag{6.7}$$

Proof. Let the constant C be as in (5.3), and put

$$V_1^b = \frac{(2r+5)(M_1 + Ck + M_2 + R)}{2}, \tag{6.8}$$

$$h_1 = \frac{2^{k/2-5}\big(1 - 2e^{-4/p(1-p)}\big)^{2k}}{(\pi p(1-p))^{k/2}} \exp\left(-\frac{2k(K_1 + M_1 + 6)^2}{p(1-p)}\right). \tag{6.9}$$

Note in particular that $V_1^b > M_2 + R$. We will choose $V_1^t > V_1^b$ in the below depending on h_1. We claim that for these choices of V_1^b, V_1^t, h_1, and for large enough N, we have

$$\mathbb{P}_{\tilde{\mathfrak{L}}}\Big(\tilde{L}_{k-1}(\pm t_{12}) \mp pt_{12} \geq (2t_2)^{1/2}V_1^b\Big) \geq 2h_1, \tag{6.10}$$

$$\mathbb{P}_{\tilde{\mathfrak{L}}}\Big(\tilde{L}_1(\pm t_{12}) \mp pt_{12} \geq (2t_2)^{1/2}V_1^t\Big) \leq h_1. \tag{6.11}$$

Assuming the validity of the claim, we then observe that the probability in (6.7) is bounded below by $2h_1 - h_1 = h_1$, proving the lemma. We will prove (6.10) and (6.11) in two steps.

Step 1. In this step we prove (6.10). We first condition on the value of $\tilde{\mathfrak{L}}$ at 0 and use the Schur Gibbs property to divide $\tilde{\mathfrak{L}}$ into two independent line ensembles on $[-t_2, 0]$ and $[0, t_2]$, and we then apply Lemma 3.8 to lower bound the probability of these two halves falling low at $\pm t_{12}$. Observe by Lemma 3.2 that

$$\mathbb{P}_{\tilde{\mathfrak{L}}}\Big(\tilde{L}_{k-1}(\pm t_{12}) \mp pt_{12} \geq (2t_2)^{1/2}V_1^b\Big) \geq \mathbb{P}_{avoid,Ber}^{-t_2,t_2,\vec{x},\vec{y}}\Big(\tilde{L}_{k-1}(\pm t_{12}) \mp pt_{12} \geq (2t_2)^{1/2}V_1^b\Big).$$

Let $K_1 = (2r+5)(M_1 + Ck + M_2 + R)$, so that $V_1^b = K_1/2$ according to (6.8). We define events

$$E_{\vec{z}} = \big\{(\tilde{L}_1(0), \dots, \tilde{L}_{k-1}(0)) = \vec{z}\big\}, \quad X = \Big\{\vec{z} \in \mathfrak{W}_{k-1} : z_{k-1} \geq K_1(2t_2)^{1/2} \text{ and } \mathbb{P}_{avoid,Ber}^{-t_2,t_2,\vec{x},\vec{y}}(E_{\vec{z}}) > 0\Big\},$$

and $E = \bigsqcup_{\vec{z}\in X} E_{\vec{z}}$. Note that X is non-empty if N is sufficiently large depending on M_1, C, k, M_2, R. By Lemma 3.20, we have

$$\mathbb{P}_{avoid,Ber}^{-t_2,t_2,\vec{x},\vec{y}}(E) \geq \mathbb{P}_{avoid,Ber}^{-t_2,t_2,\vec{x},\vec{y}}\Big(\tilde{L}_{k-1}(0) \geq K_1(2t_2)^{1/2}\Big) \geq A\exp\left(-\frac{2k(K_1 + M_1 + 6)^2}{p(1-p)}\right) \tag{6.12}$$

for sufficiently large N, where $A = A(p,k)$ is a constant given explicitly in (3.14).

Now let $\tilde{L}_i^1$ and $\tilde{L}_i^2$ denote the restrictions of $\tilde{L}_i$ to $[-t_2, 0]$ and $[0, t_2]$ respectively, for $1 \leq i \leq k-1$. We observe that if $\vec{z} \in X$, then

$$\mathbb{P}_{avoid,Ber}^{-t_2,t_2,\vec{x},\vec{y}}\big(\tilde{L}_{k-1}^1 = \ell_1, \tilde{L}_{k-1}^2 = \ell_2 \,|\, E_{\vec{z}}\big) = \mathbb{P}_{avoid,Ber}^{-t_2,0,\vec{x},\vec{z}}(\ell_1) \cdot \mathbb{P}_{avoid,Ber}^{0,t_2,\vec{z},\vec{y}}(\ell_2). \tag{6.13}$$

Let us define vectors $\vec{x}\,', \vec{z}\,'$ by

$$\begin{aligned} x_i' &= \lfloor -pt_2 - M_1(2t_2)^{1/2} \rfloor - (i-1)\lceil C(2t_2)^{1/2} \rceil, \\ z_i' &= \lfloor K_1(2t_2)^{1/2} \rfloor - (i-1)\lceil C(2t_2)^{1/2} \rceil, \\ y_i' &= \lfloor pt_2 - M_1(2t_2)^{1/2} \rfloor - (i-1)\lceil C(2t_2)^{1/2} \rceil. \end{aligned}$$

Note that $x_i' \leq x_{k-1} \leq x_i$ and $x_i' - x_{i+1}' \geq C(2t_2)^{1/2}$ for $1 \leq i \leq k-1$, and likewise for z_i', y_i'. By Lemma 3.1 and the Schur Gibbs property, we have

(6.14)
$$\mathbb{P}_{avoid,Ber}^{-t_2,0,\vec{x},\vec{z}}\Big(\tilde{L}_{k-1}^1(-t_{12}) + pt_{12} \geq (2t_2)^{1/2}V_1^b\Big) \geq \mathbb{P}_{avoid,Ber}^{-t_2,0,\vec{x}\,',\vec{z}\,'}\Big(\tilde{L}_{k-1}^1(-t_{12}) + pt_{12} \geq (2t_2)^{1/2}V_1^b\Big)$$
$$\geq \mathbb{P}_{Ber}^{-t_2,0,x_{k-1}',z_{k-1}'}\Big(\ell_1(-t_{12}) + pt_{12} \geq (2t_2)^{1/2}V_1^b\Big) - \big(1 - \mathbb{P}_{Ber}^{-t_2,t_2,\vec{x}\,',\vec{z}\,'}\big(\tilde{L}_1^1 \geq \cdots \geq \tilde{L}_{k-1}^1\big)\big).$$

To bound the first term on the second line, we first note that $\frac{t_{12}}{t_2 - t_{12}} \leq 2r+3$ and $\frac{t_2}{t_2-t_{12}} \leq r+5$. It follows from our choice of K_1 and $V_1^b = K_1/2$ that

$$\frac{t_{12}}{t_2}x_{k-1}' + \frac{t_2 - t_{12}}{t_2}z_{k-1}' - (2t_2)^{1/4} > -pt_{12} + (2t_2)^{1/2}V_1^b.$$

Now Lemma 3.8 implies that for all large N,

$$\begin{aligned}&\mathbb{P}_{Ber}^{-t_2,0,x_{k-1}',z_{k-1}'}\Big(\ell_1(-t_{12}) + pt_{12} \geq (2t_2)^{1/2}V_1^b\Big)\\ &\geq \mathbb{P}_{Ber}^{-t_2,0,x_{k-1}',z_{k-1}'}\Big(\ell_1(-t_{12}) \geq \frac{t_{12}}{t_2}x_{k-1}' + \frac{t_2-t_{12}}{t_2}z_{k-1}' - (2t_2)^{1/4}\Big) \geq \frac{1}{3}.\end{aligned} \tag{6.15}$$

Since $|z_i' - x_i' - pt_2| \leq (K_1 + M_1 + 1)(2t_2)^{1/2}$, we have by Lemma 3.18 and our choice of C that the second probability in the second line of (6.14) is bounded below by

$$\big(1 - 3e^{-C^2/8p(1-p)}\big)^{k-1} \geq 11/12.$$

It follows from (6.14) that

$$\mathbb{P}_{avoid,Ber}^{-t_2,0,\vec{x},\vec{z}}\Big(\tilde{L}_{k-1}^1(-t_{12}) + pt_{12} \geq (2t_2)^{1/2}V_1^b\Big) \geq \frac{1}{3} - \frac{1}{12} = \frac{1}{4}. \tag{6.16}$$

Similar arguments show that

$$\mathbb{P}_{Ber}^{0,t_2,\vec{z},\vec{y}}\Big(\ell_1(t_{12}) - pt_{12} \geq (2t_2)^{1/2}V_1^b\Big) \geq \frac{1}{4}. \tag{6.17}$$

Using (6.12) and (6.13), we conclude that

$$\mathbb{P}_{avoid,Ber}^{-t_2,t_2,\vec{x},\vec{y}}\Big(\tilde{L}_{k-1}(\pm t_{12}) \mp pt_{12} \geq (2t_2)^{1/2}V_1^b\Big) \geq \frac{A}{16}\exp\left(-\frac{2k(K_1+M_1+6)^2}{p(1-p)}\right).$$

This proves (6.10) with h_1 as in (6.9).

Step 2. Here we prove (6.11). Let C be as in Step 1, and define vectors $\vec{x}\,'', \vec{y}\,'' \in \mathfrak{W}_{k-1}$ by

$$\begin{aligned}x_i'' &= \lceil -pt_2 + M_1(2t_2)^{1/2}\rceil + (k-i)\lceil C(2t_2)^{1/2}\rceil,\\ y_i'' &= \lceil pt_2 + M_1(2t_2)^{1/2}\rceil + (k-i)\lceil C(2t_2)^{1/2}\rceil.\end{aligned}$$

Note that $x_i'' \geq x_1 \geq x_i$ and $x_i'' - x_{i+1}'' \geq C(2t_2)^{1/2}$, and likewise for y_i''. Moreover, $\tilde{\ell}_{bot}$ lies a distance of at least $C(2t_2)^{1/2}$ uniformly below the line segment connecting x_{k-1}'' and y_{k-1}''. By Lemma 3.1 and the Schur Gibbs property, we have

$$\begin{aligned}\mathbb{P}_{\tilde{\mathfrak{L}}}\Big(\tilde{L}_1(\pm t_{12}) \mp pt_{12} \geq (2t_2)^{1/2}V_1^t\Big) &\leq \mathbb{P}_{avoid,Ber}^{-t_2,t_2,\vec{x}\,'',\vec{y}\,'',\infty,\tilde{\ell}_{bot}}\Big(\sup_{s\in[-t_2,t_2]}\big(\tilde{L}_1(s) - ps\big) \geq (2t_2)^{1/2}V_1^t\Big)\\ &\leq \frac{\mathbb{P}_{Ber}^{-t_2,t_2,x_1'',y_1''}\Big(\sup_{s\in[-t_2,t_2]}\big(\tilde{L}_1(s) - ps\big) \geq (2t_2)^{1/2}V_1^t\Big)}{\mathbb{P}_{Ber}^{-t_2,t_2,\vec{x}\,'',\vec{y}\,''}\big(\tilde{L}_1 \geq \cdots \geq \tilde{L}_{k-1} \geq \tilde{\ell}_{bot}\big)}.\end{aligned}$$

By Lemma 3.10, since $\min(x_1'' + pt_2,\, y_1'' - pt_2) \leq (M_1 + C(k-1))(2t_2)^{1/2}$, we can choose $V_1^t > V_1^b$ large enough so that the numerator is bounded above by $h_1/2$. Since $|y_i'' - x_i'' - 2pt_2| \leq 1$, our choice of C and Lemma 3.18 imply that the denominator is at least $11/12$. This gives an upper bound of $12/11 \cdot h_1/2 < h_1/2$ in the above, proving (6.11).

$\square$

We are now equipped to prove Lemma 6.2.

Proof. We first introduce some notation to be used in the proof. For $\vec{c}, \vec{d} \in \mathfrak{W}_{k-1}$, let us write $\Omega(\vec{c}, \vec{d}) = \Omega_{avoid}(-t_{12}, t_{12}, \vec{c}, \vec{d}, \infty, -\infty)$ and $\tilde{\Omega}(\vec{c}, \vec{d}) = \Omega_{avoid}(-t_{12}, t_{12}, \vec{c}, \vec{d}, \infty, \tilde{\ell}_{bot})$, and define events

$$
\begin{aligned}
&\tilde{A}(\vec{c}, \vec{d}) = \{\mathfrak{L} \in \tilde{\Omega}(\vec{c}, \vec{d}) : L_{k-1}(\pm t_1) \mp pt_1 \geq (M_2+1)(2t_2)^{1/2}\},\\
&\tilde{B}(\vec{c}, \vec{d}, \epsilon) = \left\{ \mathfrak{L} \in \tilde{\Omega}(\vec{c}, \vec{d}) : \min_{\substack{1 \leq i \leq k-2 \\ \varsigma \in \{-1,1\}}} \big(L_i(\varsigma t_1) - L_{i+1}(\varsigma t_1)\big) \geq 3\epsilon (2t_2)^{1/2} \right\},\\
&A(\vec{c}, \vec{d}) = \{\mathfrak{L} \in \Omega(\vec{c}, \vec{d}) : L_{k-1}(\pm t_1) \mp pt_1 \geq (M_2+1)(2t_2)^{1/2}\}, \qquad (6.18)\\
&B(\vec{c}, \vec{d}, \epsilon) = \left\{ \mathfrak{L} \in \Omega(\vec{c}, \vec{d}) : \min_{\substack{1 \leq i \leq k-2 \\ \varsigma \in \{-1,1\}}} \big(L_i(\varsigma t_1) - L_{i+1}(\varsigma t_1)\big) \geq 3\epsilon (2t_2)^{1/2} \right\},\\
&\tilde{C}(\vec{c}, \vec{d}, V^{top}) = \{\mathfrak{L} \in \tilde{\Omega}(\vec{c}, \vec{d}) : L_1(\pm t_1) \mp pt_1 \leq V^{top}(2t_2)^{1/2}\}.
\end{aligned}
$$

Here, ϵ and V^{top} are constants which we will specify later. By Lemma 6.3, for N sufficiently large we have

$$\tilde{A}(\vec{c}, \vec{d}) \cap \tilde{B}(\vec{c}, \vec{d}, \epsilon) \cap \tilde{C}(\vec{c}, \vec{d}, V^{top}) \subset \big\{ Z\big(-t_1, t_1, \vec{a}, \vec{b}, \ell_{bot}[\![-t_1, t_1]\!]\big) > g \big\}$$

for some g depending on ϵ, V^{top}, M_2. Thus we will prove that probability of the event on the left under $\mathbb{P}_{\tilde{\mathfrak{L}}}$ is bounded below by $h = h_1/2$, with h_1 as in (6.9). We split the proof into several steps.

Step 1. In this step, we show that there is an $R > 0$ sufficiently large so that if $c_{k-1} + pt_{12} \geq (2t_2)^{1/2}(M_2 + R)$ and $d_{k-1} - pt_{12} \geq (2t_2)^{1/2}(M_2 + R)$, then we have

$$\frac{|\tilde{A}(\vec{c}, \vec{d})|}{|\tilde{\Omega}(\vec{c}, \vec{d})|} \geq \frac{|A(\vec{c}, \vec{d})|}{|\Omega(\vec{c}, \vec{d})|} \geq \frac{9}{10} \quad \text{and} \quad \frac{|\tilde{\Omega}(\vec{c}, \vec{d})|}{|\Omega(\vec{c}, \vec{d})|} \geq \frac{99}{100}. \qquad (6.19)$$

The first inequality follows immediately from Lemma 3.2.

For the second inequality, define the constant

$$C = \sqrt{8p(1-p) \log \frac{3}{1 - (199/200)^{1/(k-1)}}} \qquad (6.20)$$

and vectors $\vec{c}', \vec{d}' \in \mathfrak{W}_k$ by

$$
\begin{aligned}
c_i' &= \lfloor -pt_{12} + (M_2 + R)(2t_2)^{1/2} \rfloor - (i-1)\lceil C(2t_{12})^{1/2} \rceil,\\
d_i' &= \lfloor pt_{12} + (M_2 + R)(2t_2)^{1/2} \rfloor - (i-1)\lceil C(2t_{12})^{1/2} \rceil.
\end{aligned}
$$

Then by Lemma 3.1 and the Schur Gibbs property,

$$
\begin{aligned}
\mathbb{P}^{-t_{12}, t_{12}, \vec{c}, \vec{d}}_{avoid, Ber}(A(\vec{c}, \vec{d})) &\geq \mathbb{P}^{-t_{12}, t_{12}, \vec{c}', \vec{d}'}_{avoid, Ber}(A(\vec{c}', \vec{d}'))\\
&\geq \mathbb{P}^{-t_{12}, t_{12}, c_{k-1}', d_{k-1}'}_{Ber}\Big(\ell(\pm t_1) \mp pt_1 \geq (M_2+1)(2t_2)^{1/2}\Big) \qquad (6.21)\\
&\quad - \big(1 - \mathbb{P}^{-t_{12}, t_{12}, \vec{c}', \vec{d}'}_{Ber}(L_1 \geq \cdots \geq L_{k-1})\big).
\end{aligned}
$$

By Lemma 3.18 and our choice of C, $\mathbb{P}_{Ber}^{-t_{12},t_{12},\vec{c}',\vec{d}'}(L_1 \geq \cdots \geq L_{k-1}) > 199/200 > 19/20$ for sufficiently large N. Writing $z = d'_{k-1} - c'_{k-1}$, the term in the second line of (6.21) is equal to

$$\begin{aligned}
&\mathbb{P}_{Ber}^{-t_{12},t_{12},0,z}\Big(\ell(\pm t_1) \mp pt_1 + c'_{k-1} \geq (M_2+1)(2t_2)^{1/2}\Big) \\
\geq\ &\mathbb{P}_{Ber}^{0,2t_{12},0,z}\Big(\ell(\pm t_1) \mp pt_1 \geq (-R + Ck + 1)(2t_2)^{1/2}\Big) \\
\geq\ &\mathbb{P}_{Ber}^{0,2t_{12},0,z}\Big(\inf_{s\in[0,2t_{12}]} \big(\ell(s) - ps\big) \geq -(R - Ck - 1)(2t_{12})^{1/2}\Big).
\end{aligned}$$

In the second line, we used the estimate $c'_{k-1} \geq -pt_{12} + (M_2 + R - Ck)(2t_2)^{1/2}$. Now by Lemma 3.10, we can choose R large enough depending on C, k, M_2, p so that this probability is greater than 19/20 for sufficiently large N. This gives a lower bound in (6.21) of $19/20 - 1/20 = 9/10$, proving the second inequality in (6.19).

We prove the third inequality in (6.19)similarly. Note that since $\ell_{bot}(s) \leq ps + M_2(2t_2)^{1/2}$ on $[-t_2, t_2]$ by assumption, we have

$$\begin{aligned}
\frac{|\tilde{\Omega}(\vec{c},\vec{d})|}{|\Omega(\vec{c},\vec{d})|} &\geq \mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d}}\Big(\inf_{s\in[-t_{12},t_{12}]} \big(L_{k-1}(s) - ps\big) \geq M_2(2t_2)^{1/2}\Big) \\
&\geq \mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c}',\vec{d}'}\Big(\inf_{s\in[-t_{12},t_{12}]} \big(L_{k-1}(s) - ps\big) \geq M_2(2t_2)^{1/2}\Big) \\
&\geq \mathbb{P}_{Ber}^{0,2t_{12},0,z}\Big(\inf_{s\in[0,2t_{12}]} \big(\ell(s) - ps\big) \geq -(R - Ck)(2t_2)^{1/2}\Big) \\
&\qquad - \big(1 - \mathbb{P}_{Ber}^{-t_{12},t_{12},\vec{c}',\vec{d}'}(L_1 \geq \cdots \geq L_{k-1})\big).
\end{aligned} \tag{6.22}$$

We enlarge R if necessary so that the probability in the third line of (6.22) is $> 199/200$ by Lemma 3.10, and 3.18 implies as above that the expression in the last line of (6.22) is $> -1/200$. This gives us a lower bound of $199/200 - 1/200 = 99/100$ as desired.

Step 2. With R fixed from Step 1, let V_1^t, V_1^b and h_1 be as in Lemma 6.4 for this choice of R. Define the event

$$E = \{\vec{c},\vec{d} \in \mathfrak{W}_{k-1} : (2t_2)^{1/2}V_1^t \geq \max(c_1 + pt_{12}, d_1 - pt_{12}) \text{ and } \min(c_{k-1} + pt_{12}, d_{k-1} - pt_{12}) \geq (2t_2)^{1/2}V_1^b\}.$$

We show in this step that there exists $V^{top} \geq M_2 + 6(k-1)$ such that for all $(\vec{c},\vec{d}) \in E$, we have

$$\frac{|\tilde{C}(\vec{c},\vec{d},V^{top})|}{|\tilde{\Omega}(\vec{c},\vec{d})|} \geq \frac{9}{10}. \tag{6.23}$$

Let C be as in (6.20), and define $\vec{c}'', \vec{d}'' \in \mathfrak{W}_{k-1}$ by

$$\begin{aligned}
c''_i &= \lceil -pt_{12} + (2t_2)^{1/2}V_1^t \rceil + (k-1-i)\lceil C(2t_{12})^{1/2} \rceil, \\
d''_i &= \lceil pt_{12} + (2t_2)^{1/2}V_1^t \rceil + (k-1-i)\lceil C(2t_{12})^{1/2} \rceil.
\end{aligned}$$

Then $c''_i \geq c_1 \geq c_i$ and $c''_i - c''_{i+1} \geq C(2t_2)^{1/2}$ for each i, and likewise for d''_i. Furthermore, since $V_1^b \geq M_2 + R$, we see that $\tilde{\ell}_{bot}$ lies a distance of at least $R(2t_2)^{1/2}$ uniformly below the line segment connecting c''_{k-1} and d''_{k-1}. By construction, $R > C$. By Lemma 3.1, we have

$$\begin{aligned}
\frac{|\tilde{C}(\vec{c},\vec{d},V^{top})|}{|\tilde{\Omega}(\vec{c},\vec{d})|} &\geq \mathbb{P}^{-t_{12},t_{12},\vec{c}'',\vec{d}'',\infty,\tilde{\ell}_{bot}}_{avoid,Ber}\Big(L_1(\pm t_1) \mp pt_1 \leq V^{top}(2t_2)^{1/2}\Big) \\
&\geq \mathbb{P}^{-t_{12},t_{12},\vec{c}'',\vec{d}'',\infty,\tilde{\ell}_{bot}}_{avoid,Ber}\Big(\sup_{s\in[-t_{12},t_{12}]}\big(L_1(s)-ps\big) \leq V^{top}(2t_2)^{1/2}\Big) \\
&\geq \mathbb{P}^{0,2t_{12},0,z'}_{Ber}\Big(\sup_{s\in[-t_{12},t_{12}]}\big(\ell(s)-ps\big) \leq (V^{top}-V_1^t-Ck)(2t_2)^{1/2}\Big) \\
&\quad - \big(1-\mathbb{P}^{-t_{12},t_{12},\vec{c}'',\vec{d}'',\infty,\tilde{\ell}_{bot}}_{avoid,Ber}\big(L_1\geq\cdots\geq L_{k-1}\geq\tilde{\ell}_{bot}\big)\big).
\end{aligned} \tag{6.24}$$

In the last line, we have used the Schur Gibbs property and written $z' = d''_1 - c''_1$. We also used the fact that $c''_1 \leq -pt_{12} + (V_1^t + Ck)(2t_2)^{1/2}$. By Lemma 3.10, we can find V^{top} large enough depending on V_1^t, C, k, p so that the probability in the third line of (6.24) is at least 19/20 for sufficiently large N. On the other hand, the above observations regarding $\vec{c}''$, $\vec{d}''$, and $\tilde{\ell}_{bot}$, as well as the fact that $|d''_1 - c''_1 - 2pt_{12}| \leq 1$, allow us to conclude from Lemma 3.18 that the probability in the last line of (6.24) is at least 19/20 for sufficiently large N. This gives a lower bound of $19/20 - 1/20 = 9/10$ in (6.24) as desired.

Step 3. In this step, we show that with V_1^t and V_1^b as in Step 2, there is an $\epsilon > 0$ sufficiently small such that for $(\vec{c},\vec{d}) \in E$ we have

$$\frac{|B(\vec{c},\vec{d},\epsilon)|}{|\Omega(\vec{c},\vec{d})|} \geq \frac{9}{10} \quad \text{and} \quad \frac{|\tilde{B}(\vec{c},\vec{d},\epsilon)|}{|\tilde{\Omega}(\vec{c},\vec{d})|} \geq \frac{4}{5}. \tag{6.25}$$

We prove this inequality using Lemma 3.22. In order to apply this result, we approximate t_1 in the form $s \cdot 2t_{12}$, for $s \in (0,1)$.

Observe from (6.6) and the definitions of t_1, t_2 that the ratio $\frac{t_1}{2t_{12}}$ depends on N and satisfies the inequality

$$\frac{(r+1)N^\alpha - 1}{(2r+3)N^\alpha} \leq \frac{t_1}{2t_{12}} \leq \frac{(r+1)N^\alpha}{(2r+3)N^\alpha - 4}.$$

We put $s = \frac{r+1}{2r+3}$, and observe that $s < 1/2$. Also observe that

$$(2r+3)N^\alpha - 4 \leq 2t_{12} \leq (2r+3)N^\alpha,$$

and thus

$$(r+1)N^\alpha \geq s\cdot 2t_{12} \geq (r+1)N^\alpha - 4s \geq t_1 - 2.$$

It follows that $t_1 - 2 \leq \lfloor s\cdot 2t_{12}\rfloor \leq t_1$, from which we conclude that

$$0 \leq L_i(t_1) - L_i(s\cdot 2t_{12}) \leq 2. \tag{6.26}$$

Now applying Lemma 3.22 with $M_1, M_2 = \max(V_1^t, V_1^b)$, we obtain N_0 and $\delta > 0$ such that if $N \geq N_0$, then

$$\mathbb{P}^{-t_{12},t_{12},\vec{c},\vec{d}}_{avoid,Ber}\Big(\min_{1\leq i\leq k-1}\big(L_i(s\cdot 2t_{12}) - L_{i+1}(s\cdot 2t_{12})\big) < \delta(2t_{12})^{1/2}\Big) < \frac{1}{20}.$$

Together with (6.26) and the fact that $t_2/4 < t_1 < t_{12}$, this implies that

$$\mathbb{P}^{-t_{12},t_{12},\vec{c},\vec{d}}_{avoid,Ber}\left(\min_{1\leq i\leq k-1}\big(L_i(t_1) - L_{i+1}(t_1)\big) < (\delta/2)(2t_2)^{1/2} - 2\right) < \frac{1}{20} \tag{6.27}$$

for $N \geq N_0$. Now we observe that if $N \geq \left(\frac{1+1/\delta}{r+2}\right)^{2/\alpha}$, then $(\delta/4)(2t_2)^{1/2} \leq (\delta/2)(2t_2)^{1/2} - 1$. Thus for all sufficiently large N, we have

$$\mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d}}\Big(\min_{1\leq i\leq k-1}\big(L_i(t_1) - L_{i+1}(t_1)\big) < (\delta/4)(2t_2)^{1/2}\Big) < \frac{1}{20}.$$

A similar argument gives us a $\tilde{\delta} > 0$ such that

$$\mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d}}\Big(\min_{1\leq i\leq k-2}\big(L_i(-t_1) - L_{i+1}(-t_1)\big) < (\tilde{\delta}/4)(2t_2)^{1/2}\Big) < \frac{1}{20}$$

for large enough N. Then putting $\epsilon = \min(\delta, \tilde{\delta})/12$, we find

$$\mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d}}\Big(\min_{1\leq i\leq k-2}\big(L_i(\pm t_1) - L_{i+1}(\pm t_1)\big) < 3\epsilon(2t_2)^{1/2}\Big) < \frac{1}{20} + \frac{1}{20} = \frac{1}{10},$$

and the first inequality in (6.25) follows.

For the second inequality in (6.25), we observe via the second inequality in (6.19) that

$$\frac{|\tilde{B}(\vec{c},\vec{d},\epsilon)|}{|\tilde{\Omega}(\vec{c},\vec{d})|} \geq \frac{|B(\vec{c},\vec{d},\epsilon)\cap\tilde{\Omega}(\vec{c},\vec{d})|}{|\Omega(\vec{c},\vec{d})|} \geq \frac{|B(\vec{c},\vec{d},\epsilon)|}{|\Omega(\vec{c},\vec{d})|} - \frac{|\tilde{\Omega}^c(\vec{c},\vec{d})|}{|\Omega(\vec{c},\vec{d})|} \geq \frac{9}{10} - \frac{1}{100} > \frac{4}{5}.$$

Step 4. In this last step, we complete the proof of the lemma. Let $g = g(\epsilon, V^{top}, M_2)$ be as in Lemma 6.3 for the choices of ϵ, V^{top} in Steps 2 and 3, and let $h = h_1/2$, with h_1 as in Step 2. If $(\vec{c},\vec{d}) \in E$, then it follows from (6.19), (6.23), and (6.25) that

$$\begin{aligned}\frac{|\tilde{B}^c(\vec{c},\vec{d},\epsilon)\cup\tilde{A}^c(\vec{c},\vec{d})\cup\tilde{C}^c(\vec{c},\vec{d},V^{top})|}{|\tilde{\Omega}(\vec{c},\vec{d})|} &\leq \frac{|\tilde{B}^c(\vec{c},\vec{d},\epsilon)|}{|\tilde{\Omega}(\vec{c},\vec{d})|} + \frac{|\tilde{A}^c(\vec{c},\vec{d})|}{|\tilde{\Omega}(\vec{c},\vec{d})|} + \frac{|\tilde{C}^c(\vec{c},\vec{d},V^{top})|}{|\tilde{\Omega}(\vec{c},\vec{d})|}\\ &\leq \frac{1}{5} + \frac{1}{10} + \frac{1}{10} < \frac{1}{2}.\end{aligned}$$

In particular,

$$\frac{|\tilde{B}(\vec{c},\vec{d},\epsilon)\cap\tilde{A}(\vec{c},\vec{d})\cap\tilde{C}(\vec{c},\vec{d},V^{top})|}{|\tilde{\Omega}(\vec{c},\vec{d})|} \geq \frac{1}{2}, \tag{6.28}$$

for all large N, independent of $\vec{c},\vec{d}$. That is, we have for all $(\vec{c},\vec{d}) \in E$ that

$$\mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d},\infty,\tilde{\ell}_{bot}}(G) \geq \frac{1}{2},$$

where G is the event that

(1) $V^{top}(2t_2)^{1/2} \geq L_1(-t_1) + pt_1 \geq L_{k-1}(-t_1) + pt_1 \geq (M_2+1)(2t_2)^{1/2}$;
(2) $V^{top}(2t_2)^{1/2} \geq L_1(t_1) - pt_1 \geq L_{k-1}(t_1) - pt_1 \geq (M_2+1)(2t_2)^{1/2}$;
(3) $L_i(-t_1) - L_{i+1}(-t_1) \geq 3\epsilon(2t_2)^{1/2}$ and $L_i(t_1) - L_{i+1}(t_1) \geq 3\epsilon(2t_2)^{1/2}$ for $i = 1, \dots, k-2$.

It follows from Lemma 6.4 that

$$\mathbb{P}_{\tilde{\mathfrak{L}}}(G) = \mathbb{P}_{\tilde{\mathfrak{L}}}(E)\mathbb{P}_{\tilde{\mathfrak{L}}}(G\,|\,E) \geq h_1 \sum_{(\vec{c},\vec{d})\in E} \mathbb{P}_{avoid,Ber}^{-t_{12},t_{12},\vec{c},\vec{d},\infty,\tilde{\ell}_{bot}}(G) \geq \frac{1}{2}h_1$$

for large N. Now Lemma 6.3 implies (6.1), completing the proof. □

7. Applications to uniform lozenge tilings

8. Appendix A

8.1. **Proof of Lemma 2.2.** Observe that the sets $K_1 \subset K_2 \subset \cdots \subset \Sigma \times \Lambda$ are compact, they cover $\Sigma \times \Lambda$, and any compact subset K of $\Sigma \times \Lambda$ is contained in all K_n for sufficiently large n. To see this last fact, let π_1, π_2 denote the canonical projection maps of $\Sigma \times \Lambda$ onto Σ and Λ respectively. Since these maps are continuous, $\pi_1(K)$ and $\pi_2(K)$ are compact in Σ and Λ. This implies that $\pi_1(K)$ is finite, so it is contained in $\Sigma_{n_1} = \Sigma \cap [\![-n_1, n_1]\!]$ for some n_1. On the other hand, $\pi_2(K)$ is closed and bounded in $\mathbb{R}$, thus contained in some closed interval $[\alpha, \beta] \subseteq \Lambda$. Since $a_n \searrow a$ and $b_n \nearrow b$, we can choose n_2 large enough so that $\pi_2(K) \subseteq [\alpha, \beta] \subseteq [a_{n_2}, b_{n_2}]$. Then taking $n = \max(n_1, n_2)$, we have $K \subseteq \pi_1(K) \times \pi_2(K) \subseteq \Sigma_n \times [a_n, b_n] = K_n$.

We now split the proof into several steps.

Step 1. In this step, we show that the function d defined in the statement of the lemma is a metric. For each n and $f, g \in C(\Sigma \times \Lambda)$, we define

$$d_n(f,g) = \sup_{(i,t) \in K_n} |f(i,t) - g(i,t)|, \quad d_n'(f,g) = \min\{d_n(f,g), 1\}$$

Then we have

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} d_n'(f,g).$$

Clearly each d_n is nonnegative and satisfies the triangle inequality, and it is then easy to see that the same properties hold for d_n'. Furthermore, $d_n' \leq 1$, so d is well-defined. Observe that d is nonnegative, and if $f = g$, then each $d_n'(f,g) = 0$, so the sum $d(f,g)$ is 0. Conversely, if $f \neq g$, then since the K_n cover $\Sigma \times \Lambda$, we can choose n large enough so that K_n contains an x with $f(x) \neq g(x)$. Then $d_n'(f,g) \neq 0$, and hence $d(f,g) \neq 0$. Lastly, the triangle inequality holds for d since it holds for each d_n'.

Step 2. Now we prove that the topology τ_d on $C(\Sigma \times \Lambda)$ induced by d is the same as the topology of uniform convergence over compacts, which we denote by τ_c. Recall that τ_c is generated by the basis consisting of sets

$$B_K(f, \epsilon) = \Big\{ g \in C(\Sigma \times \Lambda) : \sup_{(i,t) \in K} |f(i,t) - g(i,t)| < \epsilon \Big\},$$

for $K \subset \Sigma \times \Lambda$ compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and τ_d is generated by sets of the form $B_\epsilon^d(f) = \{g : d(f,g) < \epsilon\}$.

We first show that $\tau_d \subseteq \tau_c$. It suffices to prove that every set $B_\epsilon^d(f)$ is a union of sets $B_K(f, \epsilon)$. First, choose $\epsilon > 0$ and $f \in C(\Sigma \times \Lambda)$. Let $g \in B_\epsilon^d(f)$. We will find a basis element A_g of τ_c such that $g \in A_g \subset B_\epsilon^d(f)$. Let $\delta = d(f,g) < \epsilon$, and choose n large enough so that $\sum_{k>n} 2^{-k} < \frac{\epsilon - \delta}{2}$. Define $A_g = B_{K_n}(g, \frac{\epsilon-\delta}{n})$, and suppose $h \in A_g$. Then since $K_m \subseteq K_n$ for $m \leq n$, we have

$$d(f,h) \leq d(f,g) + d(g,h) \leq \delta + \sum_{k=1}^{n} 2^{-k} d_n(g,h) + \sum_{k>n} 2^{-k} \leq \delta + \frac{\epsilon - \delta}{2} + \frac{\epsilon - \delta}{2} = \epsilon.$$

Therefore $g \in A_g \subset B_\epsilon^d(f)$. Then we can write

$$B_\epsilon^d(f) = \bigcup_{g \in B_\epsilon^d(f)} A_g,$$

a union of basis elements of τ_c.

We now prove conversely that $\tau_c \subseteq \tau_d$. Let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$. Choose n so that $K \subset K_n$, and let $g \in B_K(f, \epsilon)$ and $\delta = \sup_{x \in K} |f(x) - g(x)| < \epsilon$. If

$d(g,h) < 2^{-n}(\epsilon-\delta)$, then $d_n'(g,h) \le 2^n d(g,h) < \epsilon-\delta$, hence $d_n(g,h) < \epsilon-\delta$, assuming without loss of generality that $\epsilon \le 1$. It follows that

$$\sup_{x\in K}|f(x)-h(x)| \le \delta + \sup_{x\in K}|g(x)-h(x)| \le \delta + d_n(g,h) \le \delta+\epsilon-\delta = \epsilon.$$

Thus $g \in B^d_{2^{-n}(\epsilon-\delta)}(g) \subset B_K(f,\epsilon)$, proving that $B_K(f,\epsilon) \in \tau_d$ by the same argument as above. We conclude that $\tau_d = \tau_c$.

Step 3. In this step, we show that $(C(\Sigma\times\Lambda), d)$ is a complete metric space. Let $\{f_n\}_{n\ge1}$ be Cauchy with respect to d. Then we claim that $\{f_n\}$ must be Cauchy with respect to d_n', on each K_n. This follows from the observation that $d_n'(f_\ell, f_m) \le 2^n d(f_\ell, f_m)$. Thus $\{f_n\}$ is Cauchy with respect to the uniform metric on each K_n, and hence converges uniformly to a continuous limit f^{K_n} on each K_n (see [14, Theorem 7.15]). Since the pointwise limit must be unique at each $x \in \Sigma\times\Lambda$, we have $f^{K_n}(x) = f^{K_m}(x)$ if $x \in K_n \cap K_m$. Since $\bigcup K_n = \Sigma\times\Lambda$, we obtain a well-defined function f on all of $\Sigma\times\Lambda$ given by $f(x) = \lim_{n\to\infty} f^{K_n}(x)$. We have $f \in C(\Sigma\times\Lambda)$ since $f|_{K_n} = f^{K_n}$ is continuous on K_n for all n. Moreover, if $K \subset \Sigma\times\Lambda$ is compact and n is large enough so that $K \subset K_n$, then because $f_n \to f^{K_n} = f|_{K_n}$ uniformly on K_n, we have $f_n \to f^{K_n}|_K = f|_K$ uniformly on K. That is, for any $K \subset \Sigma\times\Lambda$ compact and $\epsilon > 0$, we have $f_n \in B_K(f,\epsilon)$ for all sufficiently large n. Therefore $f_n \to f$ in τ_c, and equivalently in the metric d by Step 2.

Step 4. Lastly, we prove separability, c.f. [1, Example 1.3]. For each pair of positive integers n,k, let $D_{n,k}$ be the subcollection of $C(\Sigma\times\Lambda)$ consisting of polygonal functions that are piecewise linear on $\{j\}\times I_{n,k,i}$ for each $j \in \Sigma_n$ and each subinterval

$$I_{n,k,i} = \left[a_n + \tfrac{i-1}{k}(b_n-a_n),\, a_n + \tfrac{i}{k}(b_n-a_n)\right], \quad 1 \le i \le k,$$

taking rational values at the endpoints of these subintervals, and extended linearly to all of $\Lambda = [a,b]$. Then $D = \bigcup_{n,k} D_{n,k}$ is countable, and we claim that it is dense in τ_c. To see this, let $K \subset \Sigma\times\Lambda$ be compact, $f \in C(\Sigma\times\Lambda)$, and $\epsilon > 0$, and choose n so that $K \subset K_n$. Since f is uniformly continuous on K_n, we can choose k large enough so that for $0 \le i \le k$, if $t \in I_{n,k,i}$, then

$$\left|f(j,t) - f(j, a_n + \tfrac{i}{k}(b_n-a_n))\right| < \epsilon/2$$

for all $j \in \Sigma_n$. We then choose $g \in \bigcup_k D_{n,k}$ with $|g(j,a_n+\frac{i}{k}(b_n-a_n)) - f(j,a_n+\frac{i}{k}(b_n-a_n))| < \epsilon/2$. Then we have

$$\left|f(j,t) - g(j, a_n + \tfrac{i-1}{k}(b_n-a_n))\right| < \epsilon \quad \text{and} \quad \left|f(j,t) - g(j, a_n + \tfrac{i}{k}(b_n-a_n))\right| < \epsilon.$$

Since $g(j,a_n+\frac{i-1}{k}(b_n-a_n)) \le g(j,t) \le g(j,a_n+\frac{i}{k}(b_n-a_n))$, it follows that

$$|f(j,t) - g(j,t)| < \epsilon$$

as well. In summary,

$$\sup_{(j,t)\in K}|f(j,t)-g(j,t)| \le \sup_{(j,t)\in K_n}|f(j,t)-g(j,t)| < \epsilon,$$

so $g \in B_K(f,\epsilon)$. This proves that D is a countable dense subset of $C(\Sigma\times\Lambda)$.

8.2. **Proof of Lemma 2.4.** We first prove two lemmas that will be used in the proof of Lemma 2.4. The first result allows us to identify the space $C(\Sigma\times\Lambda)$ with a product of copies of $C(\Lambda)$. In the following, we assume the notation of Lemma 2.4.

Lemma 8.1. *Let $\pi_i : C(\Sigma\times\Lambda) \to C(\Lambda)$, $i \in \Sigma$, be the projection maps given by $\pi_i(F)(x) = F(i,x)$ for $x \in \Lambda$. Then the π_i are continuous. Endow the space $\prod_{i\in\Sigma} C(\Lambda)$ with the product topology* *induced by the topology of uniform convergence over compacts on $C(\Lambda)$. Then the mapping*

$$F : C(\Sigma \times \Lambda) \longrightarrow \prod_{i \in \Sigma} C(\Lambda), \quad f \mapsto (\pi_i(f))_{i \in \Sigma}$$

is a homeomorphism.

Proof. We first prove that the π_i are continuous. Since $C(\Sigma \times \Lambda)$ is metrizable by Lemma 2.2, and by a similar argument so is $C(\Lambda)$, it suffices to assume that $f_n \to f$ in $C(\Sigma \times \Lambda)$ and show that $\pi_i(f_n) \to \pi_i(f)$ in $C(\Lambda)$. Let K be compact in Λ. Then $\{i\} \times K$ is compact in $\Sigma \times \Lambda$, and $f_n \to f$ on $\{i\} \times K$ by assumption, so we have $\pi_i(f_n)|_K = f_n|_{\{i\}\times K} \to f|_{\{i\}\times K} = \pi_i(f)|_K$ uniformly on K. Since K was arbitrary, we conclude that $\pi_i(f_n) \to \pi_i(f)$ in $C(\Lambda)$ as desired.

We now observe that F is invertible. If $(f_i)_{i\in\Sigma} \in \prod_{i\in\Sigma} C(\Lambda)$, then the function f defined by $f(i, \cdot) = f_i(\cdot)$ is in $C(\Sigma \times \Lambda)$, since Σ has the discrete topology. This gives a well-defined inverse for F. It suffices to prove that F and F^{-1} are open maps.

We first show that F sends each basis element $B_K(f, \epsilon)$ of $C(\Sigma \times \Lambda)$ to a basis element in $\prod_{i\in\Sigma} C(\Lambda)$. Note that a basis for the product topology is given by products $\prod_{i\in\Sigma} B_{K_i}(f_i, \epsilon)$, where at most finitely many of the K_i are nonempty. Here, we use the convention that $B_\varnothing(f_i, \epsilon) = C(\Lambda)$. Let π_Σ, π_Λ denote the canonical projections of $\Sigma \times \Lambda$ onto Σ, Λ. The continuity of π_Σ implies that if $K \subset \Sigma \times \Lambda$ is compact, then $\pi_\Sigma(K)$ is compact in Σ, hence finite. Observe that the set $K \cap (\{i\} \times \Lambda)$ is an intersection of two compacts sets, hence compact in $\Sigma \times \Lambda$. Therefore the sets $K_i = \pi_\Lambda(K \cap (\{i\} \times \Lambda))$ are compact in Λ for each $i \in \Sigma$ since π_Λ is continuous. We observe that $F(B_K(f, \epsilon)) = \prod_{i\in\Sigma} U_i$, where

$$U_i = B_{K_i}(\pi_i(f), \epsilon), \quad \text{if} \quad i \in \pi_\Sigma(K),$$

and $U_i = C(\Lambda)$ otherwise. Since $\pi_\Sigma(K)$ is finite and the K_i are compact, we see that $F(B_K(f, \epsilon))$ is a basis element in the product topology as claimed.

Lastly, we show that F^{-1} sends each basis element $U = \prod_{i\in\Sigma} B_{K_i}(f_i, \epsilon)$ for the product topology to a set of the form $B_K(f, \epsilon)$. We have $K_i = \varnothing$ for all but finitely many i. Write $f = F^{-1}((f_i)_{i\in\Sigma})$ and $K = \prod_{i\in\Sigma} K_i$. By Tychonoff's theorem, [12, Theorem 37.3], K is compact in $\Sigma \times \Lambda$, and

$$F^{-1}(U) = B_K(f, \epsilon).$$

□

We next prove a lemma which states that a sequence of line ensembles is tight if and only if all individual curves form tight sequences.

Lemma 8.2. *Suppose that $\{\mathcal{L}^n\}_{n\geq 1}$ is a sequence of Σ-indexed line ensembles on Λ, and let $X_i^n = \pi_i(\mathcal{L}^n)$. Then the X_i^n are $C(\Lambda)$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{L}^n\}$ is tight if and only if for each $i \in \Sigma$ the sequence $\{X_i^n\}_{n\geq 1}$ is tight.*

Proof. The fact that the X_i^n are random variables follows from the continuity of the π_i in Lemma 8.1 and [8, Theorem 1.3.5]. First suppose the sequence $\{\mathcal{L}^n\}$ is tight. By Lemma 2.2, $C(\Sigma \times \Lambda)$ is a Polish space, so it follows from Prohorov's theorem, [1, Theorem 5.1], that $\{\mathcal{L}^n\}$ is relatively compact. That is, every subsequence $\{\mathcal{L}^{n_k}\}$ has a further subsequence $\{\mathcal{L}^{n_{k_\ell}}\}$ converging weakly to some $\mathcal{L}$. Then for each $i \in \Sigma$, since π_i is continuous by the above, the subsequence $\{\pi_i(\mathcal{L}^{n_{k_\ell}})\}$ of $\{\pi_i(\mathcal{L}^{n_k})\}$ converges weakly to $\pi_i(\mathcal{L})$ by the continuous mapping theorem, [8, Theorem 3.2.10]. Thus every subsequence of $\{\pi_i(\mathcal{L}^n)\}$ has a convergent subsequence. Since $C(\Lambda)$ is a Polish space by the same argument as in the proof of Lemma 2.2, Prohorov's theorem implies that each $\{\pi_i(\mathcal{L}^n)\}$ is tight.

Conversely, suppose $\{X_i^n\}$ is tight for all $i \in \Sigma$. Then given $\epsilon > 0$, we can find compact sets $K_i \subset C(\Lambda)$ such that

$$\mathbb{P}(X_i^n \notin K_i) \leq \epsilon/2^i$$

for each $i \in \Sigma$. By Tychonoff's theorem, [12, Theorem 37.3], the product $\tilde{K} = \prod_{i\in\Sigma} K_i$ is compact in $\prod_{i\in\Sigma} C(\Lambda)$. We have

$$\mathbb{P}\big((X_i^n)_{i\in\Sigma} \notin \tilde{K}\big) \leq \sum_{i\in\Sigma} \mathbb{P}(X_i^n \notin K_i) \leq \sum_{i=1}^{\infty} \epsilon/2^i = \epsilon. \tag{8.1}$$

By Lemma 8.1, we have a homeomorphism $G : \prod_{i\in\Sigma} C(\Lambda) \to C(\Sigma \times \Lambda)$. We observe that $G((X_i^n)_{i\in\Sigma}) = \mathcal{L}^n$, and $K = G(\tilde{K})$ is compact in $C(\Sigma\times\Lambda)$. Thus $\mathcal{L}^n \in K$ if and only if $(X_i^n)_{i\in\Sigma} \in \tilde{K}$, and it follows from (8.1) that

$$\mathbb{P}(\mathcal{L}^n \in K) \geq 1 - \epsilon.$$

This proves that $\{\mathcal{L}^n\}$ is tight.

□

We are now ready to prove Lemma 2.4.

Proof. Fix an $i \in \Sigma$. By Lemma 8.2, it suffices to show that the sequence $\{\mathcal{L}_i^n\}_{n\geq 1}$ of $C(\Lambda)$-valued random variables is tight. By [1, Theorem 7.3], a sequence $\{P_n\}$ of probability measures on $C[0,1]$ with the uniform topology is tight if and only if the following conditions hold:

$$\lim_{a\to\infty} \limsup_{n\to\infty} P_n(|x(0)| \geq a) = 0,$$
$$\lim_{\delta\to 0} \limsup_{n\to\infty} P_n\Big(\sup_{|s-t|\leq\delta} |x(s) - x(t)| \geq \epsilon\Big) = 0 \quad \text{for all } \epsilon > 0.$$

By replacing $[0,1]$ with $[a_m, b_m]$ and 0 with a_0, we see that the hypotheses in the lemma imply that the sequence $\{\mathcal{L}_i^n|_{[a_m,b_m]}\}_n$ is tight for every $m \geq 1$. Let $\pi_m : C(\Lambda) \to C([a_m, b_m])$ denote the map $f \mapsto f|_{[a_m,b_m]}$. Then π_m is continuous, since $C(\Lambda)$ and $C([a_m, b_m])$ with the topologies of uniform convergence over compacts are metrizable by Lemma 2.2, and if $f_n \to f$ uniformly on compact subsets of Λ, then $f_n|_{[a_m,b_m]} \to f|_{[a_m,b_m]}$ uniformly on compact subsets of $[a_m, b_m]$. It follows from [8, Theorem 1.3.5] that $\pi_m(\mathcal{L}^n) = \mathcal{L}_i^n|_{[a_m,b_m]}$ is a $C([a_m, b_m])$-valued random variable. Tightness of the sequence implies that for any $\epsilon > 0$, we can find compact sets $K_m \subset C([a_m, b_m])$ so that

$$\mathbb{P}\big(\pi_m(\mathcal{L}_i^n) \notin K_m\big) \leq \epsilon/2^m$$

for each $m \geq 1$. Writing $K = \bigcap_{m=1}^{\infty} \pi_m^{-1}(K_m)$, it follows that

$$\mathbb{P}\big(\mathcal{L}_i^n \in K\big) \geq 1 - \sum_{m=1}^{\infty} \epsilon/2^m = 1 - \epsilon.$$

To conclude tightness of $\{\mathcal{L}_i^n\}$, it suffices to prove that $K = \bigcap_{m=1}^{\infty} \pi_m^{-1}(K_m)$ is sequentially compact in $C(\Lambda)$. We argue by diagonalization. Let $\{f_n\}$ be a sequence in K, so that $f_n|_{[a_m,b_m]} \in K_m$ for every m, n. Since K_1 is compact, there is a sequence $\{n_{1,k}\}$ of natural numbers such that the subsequence $\{f_{n_{1,k}}|_{[a_1,b_1]}\}_k$ converges in $C([a_1, b_1])$. Since K_2 is compact, we can take a further subsequence $\{n_{2,k}\}$ of $\{n_{1,k}\}$ so that $\{f_{n_{2,k}}|_{[a_2,b_2]}\}_k$ converges in $C([a_2, b_2])$. Continuing in this manner, we obtain sequences $\{n_{1,k}\} \supseteq \{n_{2,k}\} \supseteq \cdots$ so that $\{f_{n_{m,k}}|_{[a_m,b_m]}\}_k$ converges in $C([a_m, b_m])$ for all m. Writing $n_k = n_{k,k}$, it follows that the sequence $\{f_{n_k}\}$ converges uniformly on each $[a_m, b_m]$. If K is any compact subset of $C(\Lambda)$, then $K \subset [a_m, b_m]$ for some m, and hence $\{f_{n_k}\}$ converges uniformly on K. Therefore $\{f_{n_k}\}$ is a convergent subsequence of $\{f_n\}$.

□

8.3. **Proof of Lemmas 3.1 and 3.2.** We will prove the following lemma, of which the two lemmas are immediate consequences. In particular, Lemma 3.1 is the special case when $g^b = g^t$, and Lemma 3.2 is the case when $\vec{x} = \vec{x}\,'$ and $\vec{y} = \vec{y}\,'$. We argue in analogy to [6, Lemma 5.6].

Lemma 8.3. *Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, and two functions $g^b, g^t : [\![T_0, T_1]\!] \to [-\infty, \infty)$ with $g^b \leq g^t$. Also fix $\vec{x}, \vec{y}, \vec{x}\,', \vec{y}\,' \in \mathfrak{W}_k$ such that $x_i \leq x_i'$, $y_i \leq y_i'$ for $1 \leq i \leq k$. Assume that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}\,', \vec{y}\,', \infty, g^t)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\big($resp. $\mathfrak{L}^b\big)$ under $\mathbb{P}$ is given by $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}$ $\big($resp. $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}\big)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}_i^t(r) \geq \mathfrak{L}_i^b(r)$ for all $i = 1, \dots, k$ and $r \in [\![T_0, T_1]\!]$.*

Proof. Throughout the proof, we will write Ω_a to mean $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$ and Ω_a' to mean $\Omega_{avoid}(T_0, T_1, \vec{x}\,', \vec{y}\,', \infty, g^t)$. We split the proof into two steps.

Step 1. We first aim to construct a Markov chain $(X^n, Y^n)_{n\geq 0}$, with $X^n \in \Omega_a$, $Y^n \in \Omega_a'$, with initial distribution given by the maximal paths

$$X_1^0(t) = \min(x_1 + t - T_0,\, y_1), \qquad Y_1^0(t) = \min(x_1' + t - T_0,\, y_1'),$$
$$X_i^0(t) = \min(x_i + t - T_0,\, y_i,\, X_{i-1}^0(t)), \qquad Y_i^0(t) = \min(x_i' + t - T_0,\, y_i',\, Y_{i-1}^0(t)).$$

for $t \in [\![T_0, T_1]\!]$ and $2 \leq i \leq k$. First observe that we do in fact have $X^0 \in \Omega_a$, since $X_i^0(T_0) = x_i$, $X_i^0(T_1) = y_i$, $X_i^0(t) \leq X_{i-1}^0(t)$, and $X_k^0(t) \geq g^b(t)$. To see why this last inequality holds, note that $x_i + t - T_0 \geq g^b(T_0) + t - T_0 \geq g^b(t)$, and after observing that $X_1^0(t) \geq g^b(t)$, induction implies that $X_k^0(t) \geq g^b(t)$. Likewise, we have $Y^0 \in \Omega_a'$.

We want the chain (X^n, Y^n) to have the following properties:

(1) $(X^n)_{n\geq 0}$ and $(Y^n)_{n\geq 0}$ are both Markov in their own filtrations,
(2) (X^n) is irreducible and aperiodic, with invariant distribution $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$,
(3) (Y^n) is irreducible and aperiodic, with invariant distribution $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}$,
(4) $X_i^n \leq Y_i^n$ on $[\![T_0, T_1]\!]$ for all $n \geq 0$ and $1 \leq i \leq k$.

This will allow us to conclude convergence of X^n and Y^n to these two uniform measures.

We specify the dynamics of (X^n, Y^n) as follows. At time n, we uniformly sample a triplet $(i, t, z) \in [\![1, k]\!] \times [\![T_0, T_1]\!] \times [\![x_k, y_1' - 1]\!]$. We also flip a fair coin, with $\mathbb{P}(\text{heads}) = \mathbb{P}(\text{tails}) = 1/2$. We update X^n and Y^n using the following procedure. If $j \neq i$, we leave X_j, Y_j unchanged, and for all points $s \neq t$, we set $X_i^{n+1}(s) = X_i^n(s)$. If $T_0 < t < T_1$ and $X_i^n(t-1) = z$ and $X_i^n(t+1) = z + 1$ (note that this implies $X_i^n(t) \in \{z, z+1\}$), then we set

$$X_i^{n+1}(t) = \begin{cases} z + 1, & \text{if heads}, \\ z, & \text{if tails}, \end{cases}$$

assuming that this move does not cause $X_i^{n+1}(t)$ to fall below $X_{i+1}^n(t)$, with the convention that $X_{k+1}^n = g^b$. In all other cases, we leave $X_i^{n+1}(t) = X_i^n(t)$. We update Y^n using the same rule, with g^t in place of g^b. [Maybe add a figure here.]

We first observe that X^n and Y^n are in fact non-intersecting for all n. Note X^0 is non-intersecting, and if X^n is non-intersecting, then the only way X^{n+1} could be non-intersecting is if the update were to push $X_i^{n+1}(t)$ below $X_{i+1}^n(t)$ for some i, t. But any update of this form is suppressed, so it follows by induction that $X^n \in \Omega_a$ for all n. Similarly, we see that $Y^n \in \Omega_a'$.

It is easy to see that (X^n, Y^n) is a Markov chain, since at each time n, the value of (X^{n+1}, Y^{n+1}) depends only on the current state (X^n, Y^n), and not on the time n or any of the states prior to time n. Moreover, the value of X^{n+1} depends only on the state X^n, not on Y^n, so (X^n) is a Markov chain in its own filtration. The same applies to (Y^n). This proves the property (1) above.

We now argue that (X^n) and (Y^n) are irreducible. Observe that the initial distribution X^0 is by construction maximal, in the sense that for any $Z \in \Omega_a$, we have $Z_i \leq X_i^0$ for all i. Thus to reach Z from the initial state X_0, we only need to move the paths downward, and there is no danger of the paths X_i crossing when we do so. We start by ensuring $X_k^n = Z_k$. We successively sample triples (k, t, z) for each $t \in [\![T_0 + 1, T_1 - 1]\!]$ where Z_k differs from X_k, with $z = X_k^n(t-1)$, and flip tails in order to push $X_k^{n+1}(t)$ downwards toward $Z_k(t)$. We continue in this manner until X_k agrees with Z_k on all of $[\![T_0, T_1]\!]$, and we do the same for X_i^n and Z^i, with i descending. Since each of these samples and flips has positive probability, and this process terminates in finitely many steps, the probability of transitioning from X^n to Z after some number of steps is positive. The same reasoning applies to show that (Y^n) is irreducible.

To see that the chains are aperiodic, simply observe that if we sample a triplet (k, T_0, z), then the states of both chains will be unchanged.

To see that the uniform measure $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$ on Ω_a is invariant for (X^n), fix any $\omega \in \Omega_a$. For simplicity, write μ for the uniform measure. Then for all ensembles $\tau \in \Omega_a$, we have $\mu(\tau) = 1/|\Omega_a|$. Hence

$$\sum_{\tau \in \Omega_a} \mu(\tau) \mathbb{P}(X^{n+1} = \omega \,|\, X^n = \tau) = \frac{1}{|\Omega_a|} \sum_{\tau \in \Omega_a} \mathbb{P}(X^{n+1} = \omega \,|\, X^n = \tau)$$
$$= \frac{1}{|\Omega_a|} \sum_{\tau \in \Omega_a} \mathbb{P}(X^{n+1} = \tau \,|\, X^n = \omega) = \frac{1}{|\Omega_a|} \cdot 1 = \mu(\omega).$$

The second equality is clear if $\tau = \omega$. Otherwise, note that $\mathbb{P}(X_{n+1} = \omega \,|\, X_n = \tau) \neq 0$ if and only if τ and ω differ only in one indexed path (say the ith) at one point t, where $|\tau_i(t) - \omega_i(t)| = 1$, and this condition is also equivalent to $\mathbb{P}(X^{n+1} = \tau \,|\, X^n = \omega) \neq 0$. If $X^n = \tau$, there is exactly one choice of triplet (i, t, z) and one coin flip which will ensure $X_i^{n+1}(t) = \omega(t)$, i.e., $X^{n+1} = \omega$. Conversely, if $X^n = \omega$, there is one triplet and one coin flip which will ensure $X^{n+1} = \tau$. Since the triplets are sampled uniformly and the coin flips are fair, these two conditional probabilities are in fact equal. This proves (2), and an analogous argument proves (3).

Lastly, we argue that $X_i^n \leq Y_i^n$ for all $n \geq 0$ and $1 \leq i \leq k$. This is of course true at $n = 0$. Suppose it holds at some $n \geq 0$, and suppose that we sample a triplet (i, t, z). Then the update rule can only change the values of the $X_i^n(t)$ and $Y_i^n(t)$. Notice that the values can change by at most 1, and if $Y_i^n(t) - X_i^n(t) = 1$, then the only way the ordering could be violated is if Y_i were lowered and X_i were raised at the next update. But this is impossible, since a coin flip of heads can only raise or leave fixed both curves, and tails can only lower or leave fixed both curves. Thus it suffices to assume $X_i^n(t) = Y_i^n(t)$.

There are two cases to consider that violate the ordering of $X_i^{n+1}(t)$ and $Y_i^{n+1}(t)$. Either (i) $X_i(t)$ is raised but $Y_i(t)$ is left fixed, or (ii) $Y_i(t)$ is lowered yet $X_i(t)$ is left fixed. These can only occur if the curves exhibit one of two specific shapes on $[\![t-1, t+1]\!]$. For $X_i(t)$ to be raised, we must have $X_i^n(t-1) = X_i^n(t) = X_i^n(t+1) - 1$, and for $Y_i(t)$ to be lowered, we must have $Y_i^n(t-1) - 1 = Y_i^n(t) = Y_i^n(t+1)$. From the assumptions that $X_i^n(t) = Y_i^n(t)$, and $X_i^n \leq Y_i^n$, we observe that both of these requirements force the other curve to exhibit the same shape on $[\![t-1, t+1]\!]$. Then the update rule will be the same for both curves for either coin flip, proving that both (i) and (ii) are impossible.

Step 2. It follows from (2) and (3) and [13, Theorem 1.8.3] that $(X^n)_{n\geq 0}$ and $(Y^n)_{n\geq 0}$ converge weakly to $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$ and $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g^t}$ respectively. In particular, (X^n) and (Y^n) are tight, so $(X^n, Y^n)_{n\geq 0}$ is tight as well. By Prohorov's theorem, it follows that (X^n, Y^n) is relatively compact. Let (n_m) be a sequence such that (X^{n_m}, Y^{n_m}) converges weakly. Then by the Skorohod representation theorem [1, Theorem 6.7], it follows that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting random variables $\mathfrak{X}^n$, $\mathfrak{Y}^n$ and $\mathfrak{X}, \mathfrak{Y}$ taking values in Ω_a, Ω'_a respectively, such that

(1) The law of $(\mathfrak{X}^n, \mathfrak{Y}^n)$ under $\mathbb{P}$ is the same as that of (X^n, Y^n),
(2) $\mathfrak{X}^n(\omega) \longrightarrow \mathfrak{X}(\omega)$ for all $\omega \in \Omega$,
(3) $\mathfrak{Y}^n(\omega) \longrightarrow \mathfrak{Y}(\omega)$ for all $\omega \in \Omega$.

In particular, (1) implies that $\mathfrak{X}^{n_m}$ has the same law as X^{n_m}, which converges weakly to $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber}$. It follows from (2) and the uniqueness of limits that $\mathfrak{X}$ has law $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber}$. Similarly, $\mathfrak{Y}$ has law $\mathbb{P}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}_{avoid,Ber}$. Moreover, condition (4) in Step 1 implies that $\mathfrak{X}^n_i \leq \mathfrak{Y}^n_i$, $\mathbb{P}$-a.s., so $\mathfrak{X}_i \leq \mathfrak{Y}_i$ for $1 \leq i \leq k$, $\mathbb{P}$-a.s. Thus we can take $\mathfrak{L}^b = \mathfrak{X}$ and $\mathfrak{L}^t = \mathfrak{Y}$.

□

9. Appendix B

9.1. **Weak Convergence of Scaled avoiding Bernoulli Line Ensemble.** We consider there $\{1, \dots, k\}$-indexed line ensembles with distribution given by $\mathbb{P}^{0,T,\vec{x},\vec{y},\infty,-\infty}_{avoid,Ber}$ in the sense of Definition 2.15. Recall that this is just the law of k independent Bernoulli random walks that have been conditioned to start from $(x_1, \dots, x_k)$ at time 0 and and at $(y_1, \cdots, y_k)$ at time T and are always ordered. Here $x_1 \geq x_2 \geq \cdots \geq x_k$, $y_1 \geq y_2 \geq \cdots \geq y_k$ and x_i, $y_i \in \mathbb{Z}$ satisfy $T \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$. We will drop the infinities and simply write $\mathbb{P}^{0,T,\vec{x},\vec{y}}_{avoid,Ber}$ for the measure.

Fix $p, t \in (0,1)$, $k \in \mathbb{N}$, a_i, $b_i \in \mathbb{R}$ for $i = 1, \dots, k$ such that $a_1 \geq \cdots \geq a_k$ and $b_1 \geq \cdots \geq b_k$. Suppose that $\vec{x}^T = (x^T_1, \cdots, x^T_k)$ and $\vec{y}^T = (y^T_1, \cdots, y^T_k)$ are two sequence of k-dimensional vectors with integer entries such that

$$\lim_{T\to\infty} \frac{x^T_i}{\sqrt{T}} = a_i \text{ and } \lim_{T\to\infty} \frac{y^T_i - pT}{\sqrt{T}} = b_i$$

for $i = 1, \dots, k$. Define the sequence of random k-dimensional vectors Z^T by

$$Z^T = \Big(\frac{L_1(tT) - ptT}{\sqrt{T}}, \cdots, \frac{L_k(tT) - ptT}{\sqrt{T}}\Big),$$

where $(L_1, \cdots, L_k)$ is $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$-distributed. In this section, we will prove that the random vector Z^T weakly converges to some continuous distribution and give the corresponding density. The followings are two main results of this section, which give the limiting distribution when $(a_1, \cdots, a_k)$ and $(b_1, \cdots, b_k)$ contain distinct values and when they contain collided values.

Proposition 9.1. *When $a_1 > \cdots > a_k$ and $b_1 > \cdots > b_k$ are all distinct, the random vector Z^T converges weakly to a continuous distribution with the density*

$$\rho(z_1, \cdots, z_k) = \frac{1}{Z} \cdot \det\left[e^{c_1(t,p)a_i z_j}\right]^k_{i,j=1} \det\left[e^{c_2(t,p)b_i z_j}\right]^k_{i,j=1} \prod_{i=1}^k e^{-c_3(t,p)z_i^2} \mathbb{1}_{\{z_1 > \cdots > z_k\}}$$

where c_1, c_2, c_3 are constants depending on p, t:

$$c_1(p,t) = \frac{1}{p(p+1)t}, \quad c_2(p,t) = \frac{1}{p(p+1)(1-t)}, \quad c_3(p,t) = \frac{1}{2p(p+1)t(1-t)}$$

and Z is a constant depending on $p, t, \vec{a}, \vec{b}$ such that $\rho(z_1, \cdots, z_k)$ integrates to 1 over $\mathbb{R}^k$:

$$Z = (2\pi)^{\frac{k}{2}} (p(p+1)t(1-t))^{\frac{k}{2}} \cdot e^{c_1(t,p)\sum_{i=1}^k a_i^2} \cdot e^{c_2(t,p)\sum_{i=1}^k b_i^2} \det\left(e^{-\frac{1}{2p(p+1)}(b_i - a_j)^2}\right)^k_{i,j=1}$$

Proposition 9.2. *When $a_1 \geq \cdots \geq a_k$ and $b_1 \geq \cdots \geq b_k$ contain collided values, we suppose*

$$\vec{a}_0 = (a_1, \cdots, a_k) = (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p})$$

$$\vec{b}_0 = (b_1, \cdots, b_k) = (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q})$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_p$, $\beta_1 > \beta_2 > \cdots > \beta_q$ and $\sum_{i=1}^p m_i = \sum_{i=1}^q n_i = k$. Then, the random vector Z^T converges weakly to a continuous distribution with the density

$$\rho_{\vec{a}_0,\vec{b}_0}(z_1, \cdots, z_k) = \frac{1}{Z} \cdot \varphi(\vec{a}_0, \vec{z}, \vec{m})\psi(\vec{b}_0, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{-c_3(t,p)z_i^2} \mathbb{1}_{\{z_1 > \cdots > z_k\}} \tag{9.1}$$

where $\vec{m} = (m_1, \cdots, m_k)$, $\vec{n} = (n_1, \cdots, n_k)$, c_1, c_2, c_3 are constants depending on p, t as given in the proposition, Z is a constant depending on $p, t, \vec{a}, \vec{b}$ such that $\rho_{\vec{a}_0,\vec{b}_0}(z_1, \cdots, z_k)$ integrates to 1 over $\mathbb{R}^k$, and $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ and $\psi(\vec{b}_0, \vec{z}, \vec{n})$ are determinants:

$$\varphi(\vec{a}_0, \vec{z}, \vec{m}) = \det \begin{bmatrix} ((c_1(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_1 z_j})_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ ((c_2(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_p z_j})_{\substack{i=1,\cdots,m_p \\ j=1,\cdots,k}} \end{bmatrix}$$

$$\psi(\vec{b}_0, \vec{z}, \vec{n}) = \det \begin{bmatrix} ((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_1 z_j})_{\substack{i=1,\cdots,n_1 \\ j=1,\cdots,k}} \\ \vdots \\ ((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_q z_j})_{\substack{i=1,\cdots,n_q \\ j=1,\cdots,k}} \end{bmatrix}$$

In order to prove these two propositions, we need to introduce some lemmas.
The following lemma gives the distribution of avoiding Bernoulli line ensembles at time $\lfloor tT \rfloor$.

Lemma 9.3. *The avoiding Bernoulli line ensemble at position $\lfloor tT \rfloor$ has the following distribution:*

$$\mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \cdots, L_k(\lfloor tT \rfloor) = \lambda_k) = \frac{s_{\lambda/\mu}(1^{\lfloor tT \rfloor}) \cdot s_{\kappa/\lambda}(1^{T-\lfloor tT \rfloor})}{s_{\kappa/\mu}(1^T)}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ are positive integers, $s_{\lambda/\mu}$ denote skew Schur polynomials and they are specialized in all parameters equal to 1. The μ partition is just the vector $\vec{x}^T$ and the κ partition should be $\vec{y}^T$.

Proof. Let $\Omega(0, T, \vec{x}^T, \vec{y}^T)$ be the set of all non-intersecting Bernoulli line ensembles from $\vec{x}^T$ to $\vec{y}^T$. For each line ensemble $\mathfrak{B} \in \Omega(0, T, \bar{x}^T, \bar{y}^T)$ with $\mathfrak{B} = (B_1, ..., B_k)$, we may define $\lambda_i(\mathfrak{B}) := (B_1(i), B_2(i), ..., B_k(i))$, where $1 \leqslant i \leqslant T$ is an integer. The λ_i form partitions since by the definition of avoiding Bernoulli line ensembles, we have the inequality $B_\alpha(i) > B_\beta(i)$ if $\alpha < \beta$. Now because $B_\alpha(i+1) - B_\alpha(i) \in \{0, 1\}$ we know that $B_\alpha(i+1) \geq B_\alpha(i)$ but also since $B_\alpha(i+1) \in \mathbb{Z}$ and $B_{\alpha+1}(i+1) < B_\alpha(i+1)$ by the earlier stated inequality, we know that $B_{\alpha+1}(i+1) + 1 \leq B_\alpha(i+1)$ and so we find that

$$B_{\alpha+1}(i+1) \leq B_\alpha(i) \leq B_\alpha(i+1)$$

We therefore find that for all i, $\lambda_i \preceq \lambda_{i+1}$. Note that when $i = 0$, we get $\lambda_0 = \bar{x}^T$ and $\lambda_T = \bar{y}^T$.

Now, let us define the set

$$TB^T_{\kappa/\mu} := \{(\lambda_0, ..., \lambda_T) \mid \lambda_0 = \mu, \lambda_T = \kappa, \lambda_i \preceq \lambda_{i+1}\}$$

Now, if we take $f : \Omega(0, T, \bar{x}^T, \bar{y}^T) \to TB^T_{\kappa/\mu}$ with $f(\mathfrak{B}) = (\lambda_0(\mathfrak{B}), \cdots \lambda_T(\mathfrak{B}))$. We find that this function is in fact a bijection.

First, to show for injectivity, suppose that there are two Bernoulli line ensembles, $\mathfrak{B}, \mathfrak{B}' \in \Omega(0, T, \bar{x}^T, \bar{y}^T)$ such that $\mathfrak{B} \neq \mathfrak{B}'$. Because Bernoulli line ensembles are determined by their values at integer times, we find that this would imply that there exists some (q, r) such that $0 \leq r \leq T$, $0 \leq q \leq k$ and $B_q(r) \neq B'_q(r)$ where B_q and B'_q are components of $\mathfrak{B}$ and $\mathfrak{B}'$ respectively. This implies that $\lambda_r(\mathfrak{B}) \neq \lambda'_r(\mathfrak{B}')$, and we have injectivity.

Now, surjectivity follows since for any $\bar{\lambda} = (\lambda_0, ..., \lambda_T)$ we may define $\mathfrak{B}(\bar{\lambda}) = (B_1(\bar{\lambda}), ..., B_k(\bar{\lambda}))$ where $B_r(\bar{\lambda})(i) = \lambda^r_i$ where λ^r_i is the ith entry of λ_r. The restrictions on $TB^T_{\kappa/\mu}$ ensure that each $\mathfrak{B}(\bar{\lambda}) \in \Omega(0, T, \bar{x}^T, \bar{y}^T)$, and so $f(\mathfrak{B}(\bar{\lambda})) = (\lambda_0, \cdots \lambda_T)$ by the definition $\mathfrak{B}(\bar{\lambda})$.

Applying the result regarding the relationship between number of partitions and skew Schur polynomial [11, Chapter 1, (5.11)], we have

$$s_{\kappa/\mu}(1^T) = \sum_{(\nu)} \prod_{i=1}^{n} s_{\nu^{(i)}/\nu^{i-1}} = \sum_{(\nu)} 1 = |TB^T_{\mu/\kappa}|$$

Therefore, we can find that

$$\begin{aligned}\mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \cdots, L_k(\lfloor tT \rfloor) = \lambda_k) =& \frac{|\Omega(0, \lfloor Tt \rfloor, \vec{x}^T, \lambda)| \cdot |\Omega(\lfloor Tt \rfloor, T, \lambda, \vec{y}^T)|}{|\Omega(0, T, \vec{x}^T, \vec{y}^T)|} \\ =& \frac{s_{\lambda/\vec{x}^T}(1^{\lfloor Tt \rfloor}) \cdot s_{\vec{y}^T/\lambda}(1^{T - \lfloor Tt \rfloor})}{s_{\vec{y}^T/\vec{x}^T}(1^T)}\end{aligned}$$

and proved the result. $\square$

The following lemma helps to prove Proposition 9.1. It shows the asymptotic formula for the distribution of avoiding Bernoulli line ensembles at time $\lfloor tT \rfloor$.

Lemma 9.4. *Let $\mathbb{W}^o_k$ denote the open Weyl chamber in $\mathbb{R}^N$:*

$$\mathbb{W}^o_N = \{(x_1, \cdots, x_k) \in \mathbb{R}^N : x_1 > x_2 > \cdots > x_k\}$$

Fix a real number $A > 0$, $p, t \in (0, 1)$, take $z = (z_1, \cdots, z_k) \in \mathbb{W}^o_k$ such that $A > z_1 > \cdots > z_k > -A$. Choose sufficiently large T_0 such that $ptT_0 - A\sqrt{T_0} \geqslant 1$, then for $T \geqslant T_0$, define $\lambda_i(T) = \lfloor z_i\sqrt{T} + ptT \rfloor \geqslant 1$ for $i = 1, \cdots, k$. Denote $P_T(z) = (\sqrt{T})^k \mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1(T), \cdots, L_k(\lfloor tT \rfloor) = \lambda_k(T))$, then we have for almost every $z \in [-A, A]^k$:

$$\lim_{T \to \infty} P_T(z) = \rho(z_1, \cdots, z_k)$$

and $P_T(z)$ is bounded on $[-A, A]^k$, where $\rho(z)$ is given in Proposition 9.1.

Proof. (i) First, we discuss the pointwise convergence of $P_T(z)$. By Jacobi-Trudi formula([11, Chapter 1, (5.4)]), we conclude:

(9.2)

$$\mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \cdots, L_k(\lfloor tT \rfloor) = \lambda_k) = \frac{\det\left(h_{\lambda_i - x^T_j + j - i}(1^{\lfloor tT \rfloor})\right)^k_{i,j=1} \cdot \det\left(h_{y^T_i - \lambda_j + j - i}(1^{T - \lfloor tT \rfloor})\right)^k_{i,j=1}}{\det\left(h_{y^T_i - x^T_j + j - i}(1^T)\right)^k_{i,j=1}}$$

We first compute the first determinant in the numerator. Using the identity for complete symmetric functions([11, Example 1, Section I.2]) that $h_r(1^n) = \binom{n+r-1}{r}$, we get the resulting equation

$$h_{\lambda_i - x^T_j + j - i}(1^{\lfloor tT \rfloor}) = \frac{(\lambda_i - x^T_j - i + j + \lfloor tT \rfloor - 1)!}{(\lambda_i - x^T_j - i + j)!(\lfloor tT \rfloor - 1)!} \tag{9.3}$$

We have the following Stirling's formula,

$$n! = \sqrt{2\pi n}n^n e^{-n} e^{r_n}, \text{ where } \frac{1}{12n+1} < r_n < \frac{1}{12n}$$

Denote $K = \lambda_i - x_j^T - i + j + \lfloor tT \rfloor - 1$ and apply the above Stirling's formula, we get

$$K! = \sqrt{2\pi}\sqrt{K} \cdot e^{K\log K - K + r_K}$$

Additionally, since $\lambda_i = \lfloor z_i \sqrt{T} + ptT \rfloor$ and $x_i^T = a_i\sqrt{T} + o(\sqrt{T})$, we get

$$K = \lambda_i - x_j^T - i + j + \lfloor tT \rfloor - 1 = (z_i - a_j)\sqrt{T} + (p+1)tT - i + j - 1 + o(\sqrt{T})$$

$$\begin{aligned}\log K &= \log\left(1 + \frac{-i+j-1}{(z_i-a_j)\sqrt{T} + (p+1)tT} + o(\frac{1}{\sqrt{T}})\right) + \log((z_i-a_j)\sqrt{T} + (p+1)tT) \\ &= \frac{-i+j-1}{(z_i-a_j)\sqrt{T} + (p+1)tT} + \log((z_i-a_j)\sqrt{T} + (p+1)tT) + o(\frac{1}{\sqrt{T}})\end{aligned}$$

where the constant in little o notation only depends on A, p and does not depend on T. Next, we compute $K\log K$:

$$\begin{aligned}K\log K = (-i+j-1) + \Big[(z_i-a_j)\sqrt{T} + (p+1)tT\Big] \cdot \log\Big((z_i-a_j)\sqrt{T} + (p+1)tT\Big) \\ + (-i+j-1)\log((p+1)tT) + o(\sqrt{T})\end{aligned}$$

Now we further compute the term $\Big[(z_i-a_j)\sqrt{T} + (p+1)tT\Big] \cdot \log\Big((z_i-a_j)\sqrt{T} + (p+1)tT\Big)$. Notice that

$$\begin{aligned}\log\Big((z_i-a_j)\sqrt{T} + (p+1)tT\Big) &= \log((p+1)tT) + \log\left(1 + \frac{z_i-a_j}{(p+1)t\sqrt{T}}\right) \\ &= \log((p+1)tT) + \frac{z_i-a_j}{(p+1)t\sqrt{T}} - \frac{1}{2}\frac{(z_i-a_j)^2}{(p+1)^2t^2T} + o(\frac{1}{T})\end{aligned}$$

Then,

$$\begin{aligned}&\Big[(z_i-a_j)\sqrt{T} + (p+1)tT\Big] \cdot \log\Big((z_i-a_j)\sqrt{T} + (p+1)tT\Big) \\ &= \Big[(z_i-a_j)\sqrt{T} + (p+1)tT\Big] \cdot \left[\log((p+1)tT) + \frac{z_i-a_j}{(p+1)t\sqrt{T}} - \frac{1}{2}\frac{(z_i-a_j)^2}{(p+1)^2t^2T} + o(\frac{1}{T})\right] \\ &= ((p+1)tT)\log((p+1)tT) + (z_i-a_j)\sqrt{T}\cdot\log((p+1)tT) + (z_i-a_j)\sqrt{T} - \frac{1}{2}\cdot\frac{(z_i-a_j)^2}{(p+1)t} + o(1)\end{aligned}$$

Therefore, we find that $(\lambda_i - x_j^T - i + j + \lfloor tT \rfloor - 1)! =$

$$\begin{aligned}&\sqrt{2\pi}\sqrt{(p+1)tT} \cdot \mathrm{Exp}\{(-i+j-1) + ((p+1)tT)\log((p+1)tT) + (z_i-a_j)\sqrt{T}\cdot((p+1)tT) \\ &\qquad + (z_i-a_j)\sqrt{T} - \frac{1}{2}\cdot\frac{(z_i-a_j)^2}{(p+1)t} + (-i+j-1)\log((p+1)tT) \\ &\qquad - \Big((z_i-a_j)\sqrt{T} + (p+1)tT - i + j - 1\Big) + o(1)\}\end{aligned} \tag{9.4}$$

Similarly, $(\lambda_i - x_j^T - i + j)! =$

$$\begin{aligned}&\sqrt{2\pi}\sqrt{ptT} \cdot \mathrm{Exp}\{(-i+j) + (ptT)\log(ptT) + (z_i-a_j)\sqrt{T}\cdot\log(ptT) + (z_i-a_j)\sqrt{T} \\ &\qquad + \frac{1}{2}\cdot\frac{(z_i-a_j)^2}{pt} + (-i+j)\log(ptT) - ((z_i-a_j)\sqrt{T} + ptT - i + j) + o(1)\}\end{aligned} \tag{9.5}$$

and the final term:

$$(\lfloor tT \rfloor - 1)! = \sqrt{2\pi}\sqrt{tT} \cdot \mathrm{Exp}\{(tT)\log(tT) - 1 - \log(tT) + o(1)\} \tag{9.6}$$

Plugging (9.4), (9.5) and (9.6) into equation (9.3) we get $h_{\lambda_i - x_j^T + j - i}(1^{\lfloor tT \rfloor}) =$

$$\sqrt{2\pi}^{-1}\sqrt{\frac{p+1}{pt}}\sqrt{T}^{-1} \cdot \mathrm{Exp}\Big\{((p+1)tT)\log((p+1)tT) - (ptT)\log(ptT) - (tT)\log(tT)$$
$$+ (-i+j)\log(\frac{p+1}{p}) - \log(p+1) + (z_i - a_j)\sqrt{T} \cdot \log(\frac{p+1}{p}) - \frac{1}{2}\frac{(z_i - a_j)^2}{p(p+1)t} + o(1)\Big\}$$

where the constant in little o notation only depends on A, p and does not depend on T. Denote $S_1(p,t,T) = \big((p+1)tT)\log((p+1)tT\big) - (ptT)\log(ptT) - (tT)\log(tT)$ and we further calculate the determinant

$$\begin{aligned}
\det(h_{\lambda_i - x_j - i + j}(1^{\lfloor tT \rfloor}))_{i,j=1}^k &= \left[(\sqrt{2\pi})^{-1}\sqrt{\frac{p+1}{pt}}\sqrt{T}^{-1} e^{S_1(p,t,T) - \log(p+1)}\right]^k \\
&\quad \cdot \det\left(e^{(-i+j)\log(\frac{p+1}{p}) + (z_i - a_j)\sqrt{T}\cdot\log(\frac{p+1}{p}) - \frac{1}{2}\frac{(z_i - a_j)^2}{p(p+1)t} + o(1)}\right)_{i,j=1}^k \\
&= \left[(\sqrt{2\pi})^{-1}\sqrt{\frac{p+1}{pt}}\sqrt{T}^{-1} e^{S_1(p,t,T) - \log(p+1)}\right]^k \left(\frac{p+1}{p}\right)^{\sum_{i=1}^k (z_i - a_i)\cdot\sqrt{T}} \\
&\quad \cdot e^{-\frac{1}{2p(p+1)t}\sum_{i=1}^k (a_i^2 + z_i^2)} \cdot \det\left(e^{c_1(p,t) z_i a_j + o(1)}\right)_{i,j=1}^k
\end{aligned}$$

where the constant $c_1(p,t) = \frac{1}{p(p+1)t}$.

Analogously, we calculate the other two determinants in equation 9.2:

$$\begin{aligned}
\det\left(h_{y_i - \lambda_j - i + j}(1^{T - \lfloor tT \rfloor})\right) &= \left[(\sqrt{2\pi})^{-1}\sqrt{\frac{p+1}{p(1-t)}}\sqrt{T}^{-1} \cdot e^{S_2(p,t,T) - \log(p+1)}\right]^k \left(\frac{p+1}{p}\right)^{\sum_{i=1}^k (b_i - z_i)\cdot\sqrt{T}} \\
&\quad \cdot e^{-\frac{1}{2p(p+1)(1-t)}\sum_{i=1}^k (b_i^2 + z_i^2)} \cdot \det\left(e^{c_2(p,t) b_i z_j + o(1)}\right)_{i,j=1}^k \\
\det(h_{y_i^T - x_j^T - i + j}(1^T)) &= \left[(\sqrt{2\pi})^{-1}\sqrt{\frac{p+1}{p}}\sqrt{T}^{-1} \cdot e^{S_3(p,t,T) - \log(p+1)}\right]^k \left(\frac{p+1}{p}\right)^{\sum_{i=1}^k (b_i - a_i)\cdot\sqrt{T}} \\
&\quad \cdot \det\left(e^{-\frac{1}{2p(p+1)}(b_i - a_j)^2 + o(1)}\right)_{i,j=1}^k
\end{aligned}$$

where the constants $S_2(p,t,T)$, $S_3(p,t,T)$, $c_2(p,t)$ are:

$$S_2(p,t,T) = ((p+1)(1-t)T)\log((p+1)tT) - (p(1-t)T)\log(ptT) - ((1-t)T)\log((1-t)T)$$
$$S_3(p,t,T) = ((p+1)T)\log((p+1)tT) - (pT)\log(pT) - T\log T, \quad c_2(p,t) = \frac{1}{p(p+1)(1-t)}$$

Notice that $S_1(p,t,T) + S_2(p,t,T) - S_3(p,t,T) = 0$. Plugging the above three determinants into equation 9.2, we get

$$
\begin{aligned}
&\mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \cdots, L_k(\lfloor tT \rfloor) = \lambda_k) \\
=&(2\pi)^{-\frac{k}{2}} \left[\sqrt{\frac{p+1}{pt(1-t)}}\right]^k \cdot T^{-\frac{k}{2}} \cdot e^{-\frac{1}{2p(p+1)t}\sum_{i=1}^k (a_i^2+z_i^2) - \frac{1}{2p(p+1)(1-t)}\sum_{i=1}^k (b_i^2+z_i^2)} \\
&\cdot e^{-k\log(p+1)} \cdot \frac{\det(e^{c_1(p,t)z_i a_j + o(1)})_{i,j=1}^k \cdot \det(e^{c_2(p,t)b_i z_j + o(1)})_{i,j=1}^k}{\det(e^{-\frac{1}{2p(p+1)}(b_i-a_j)^2 + o(1)})_{i,j=1}^k} \\
=&(2\pi)^{-\frac{k}{2}} \left[\sqrt{\frac{1}{p(p+1)t(1-t)}}\right]^k \cdot T^{-\frac{k}{2}} \cdot e^{-\frac{1}{2p(p+1)t}\sum_{i=1}^k a_i^2 - \frac{1}{2p(p+1)(1-t)}\sum_{i=1}^k b_i^2} \\
&\cdot \frac{\det\left(e^{c_1(p,t)z_i a_j}\right)_{i,j=1}^k \cdot \det\left(e^{c_2(p,t)b_i z_j}\right)_{i,j=1}^k}{\det\left(e^{-\frac{1}{2p(p+1)}(b_i-a_j)^2}\right)_{i,j=1}^k} \cdot \exp\{o(1)\} \cdot \prod_{i=1}^k e^{-c_3(t,p)z_i^2}
\end{aligned} \tag{9.7}
$$

where $c_3(t,p) = \frac{1}{2p(p+1)t(1-t)}$, and the constant in little o notation depends on A, p.

In conclusion, $P_T(z) = (\sqrt{T})^k \cdot \mathbb{P}(L_1(\lfloor tT \rfloor) = \lambda_1, \cdots, L_k(\lfloor tT \rfloor) = \lambda_k)$ converges to $\rho(z_1, \ldots, z_k)$ in Proposition 9.1 as $T \to \infty$ and

$$
c_1(p,t) = \frac{1}{p(p+1)t}, \quad c_2(p,t) = \frac{1}{p(p+1)(1-t)}, \quad c_3(p,t) = \frac{1}{2p(p+1)t(1-t)}
$$

$$
Z = (2\pi)^{\frac{k}{2}} (p(p+1)t(1-t))^{\frac{k}{2}} \cdot e^{c_1(t,p)\sum_{i=1}^k a_i^2} \cdot e^{c_2(t,p)\sum_{i=1}^k b_i^2} \det\left(e^{-\frac{1}{2p(p+1)}(b_i-a_j)^2}\right)_{i,j=1}^k
$$

(ii) Second, we discuss the boundedness. By the Equation 9.7 we just derived, $P_T(z) = \rho(z) \cdot \exp\{o(1)\}$ on the compact set $[-A, A]^k$, where the constant in little o notation only depends on A, p. Since continuous function $\rho(z)$ is bounded on $[-A, A]^k$, and $\exp\{o(1)\}$ is uniformly bounded on $[-A, A]^k$, we conclude that $P_T(z)$ is bounded as well. □

Now we are ready to prove Proposition 9.1.

Proof of Proposition 9.1. In the following, we prove the weak convergence of the random vector Z^T, when $\vec{a} = (a_1, \cdots, a_k)$ and $\vec{b} = (b_1, \cdots, b_k)$ consist of distinct entries. In order to show the weak convergence, it is sufficient to show that for every open set $O \in \mathbb{R}^k$, we have:

$$
\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O) \geqslant \int_O \rho(z_1, \cdots, z_k) dz_1 dz_2 \cdots dz_k
$$

according to [8, Theorem 3.2.11]. Actually, it suffices to show that for any open set $U \in \mathbb{W}_k^o$, we have:

$$
\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in U) \geqslant \int_U \rho(z_1, \cdots, z_k) dz_1 dz_2 \cdots dz_k \tag{9.8}
$$

which implies that:

$$
\begin{aligned}
&\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O) \geqslant \liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O \cap \mathbb{W}_k^o) \\
\geqslant &\int_{\mathbb{W}_k^o \cap O} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k = \int_O \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k
\end{aligned}
$$

The second inequality uses the above result (9.8), since $\mathbb{W}_k^o \cap O$ is an open set in $\mathbb{W}_k^o$. The last equality is because $\rho(z)$ is zero outside $\mathbb{W}_k^o$. The rest of the proof will be divided into 4 steps. In Step 1, we prove the weak convergence holds on every closed rectangle. In Step 2, we prove the result 9.8 using Lemma 9.4. In Step 3, we prove that $\rho(z)$ is actually a density and conclude the weak convergence.

Step 1. In this step, we establish the following result:
For any closed rectangle $R = [u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_N, v_N] \in \mathbb{W}_k^o$,

$$\lim_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_R \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \tag{9.9}$$

where $\rho(z)$ is given in Proposition 9.1.

Define $m_i^T = \lceil u_i\sqrt{T} + ptT\rceil$ and $M_i^T = \lfloor v_i\sqrt{T} + ptT\rfloor$, and we have:

$$\begin{aligned}
&\mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \mathbb{P}(u_1 \leqslant Z_1^T \leqslant v_1, \dots, u_k \leqslant Z_k^T \leqslant v_k)\\
&= \mathbb{P}(u_i\sqrt{T} + ptT \leqslant L_i(\lfloor tT\rfloor) \leqslant v_i\sqrt{T} + ptT, i = 1, \dots, k)\\
&= \sum_{\lambda_1(T)=m_1^T}^{M_1^T} \cdots \sum_{\lambda_k(T)=m_k^T}^{M_k^T} \mathbb{P}(L_1(\lfloor tT\rfloor) = \lambda_1(T), \dots, L_k(\lfloor tT\rfloor) = \lambda_k(T))\\
&= \sum_{\lambda_1(T)=m_1^T}^{M_1^T} \cdots \sum_{\lambda_k(T)=m_k^T}^{M_k^T} (\sqrt{T})^{-k} \cdot (\sqrt{T})^k \mathbb{P}(L_1(\lfloor tT\rfloor) = \lambda_1(T), \dots, L_k(\lfloor tT\rfloor) = \lambda_k(T))
\end{aligned}$$

Find sufficiently large A such that $R \subset [-A, A]^k$, for example, $A = 1 + \max_{1\leqslant i\leqslant k} |a_i| + \max_{1\leqslant i\leqslant k} |b_i|$. Define $f_T(z_1, \cdots, z_k)$ as a simple function on $\mathbb{R}^k$: When $(z_1, \cdots, z_k) \in R$, it takes value $(\sqrt{T})^k \cdot \mathbb{P}(L_1(\lfloor tT\rfloor) = \lambda_1(T), \cdots, L_k(\lfloor tT\rfloor) = \lambda_k(T))$ if there exist $\lambda_1(T), \cdots, \lambda_k(T)$ such that $\lambda_i(T) \leqslant z_i\sqrt{T} + ptT < \lambda_i(T) + 1$; It takes value 0 otherwise, when $(z_1, \cdots, z_k) \notin R$. Since the Lebesgue measure of the set $\{z : \lambda_i(T) \leqslant z_i\sqrt{T} + ptT < \lambda_i(T) + 1, i = 1, \cdots, k\}$ is $(\sqrt{T})^{-k}$, the above probability can be further written as an integral of simple function $f_T(z_1, \cdots, z_k)$:

$$\mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_{[-A,A]^k} f_T(z_1, \cdots, z_k) dz_1 \cdots dz_k$$

By Lemma 9.4, the function $f_T(z_1, \cdots, z_k)$ pointwise converges to $\rho(z)$ and is bounded on the compact set $[-A, A]^k$. Since the Lebesgue measure of $[-A, A]^k$ is finite, by bounded convergence theorem we have:

$$\lim_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_R \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \tag{9.10}$$

Step 2. In this step, we prove the statement 9.8. Take any open set $U \in \mathbb{W}_k^o$, it can be written as a countable union of closed rectangles with disjoint interiors: $U = \bigcup_{i=1}^\infty R_i$, where $R_i = [a_1^i, b_1^i] \times \cdots \times [a_k^i, b_k^i]$([15, Theorem 1.4]). Choose sufficiently small $\epsilon > 0$, and denote $R_i^\epsilon = [a_1^i + \epsilon, b_1^i - \epsilon] \times \cdots \times [a_k^i + \epsilon, b_k^i - \epsilon]$, then R_i^ϵ are disjoint. Therefore,

$$\begin{aligned}
&\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in U) \geqslant \liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \bigcup_{i=1}^n R_i^\epsilon)\\
&= \liminf_{T\to\infty} \sum_{i=1}^n \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R_i^\epsilon) = \sum_{i=1}^n \int_{R_i^\epsilon} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k\\
&= \int_{\bigcup_{i=1}^n R_i^\epsilon} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \xrightarrow{\epsilon\downarrow 0} \int_U \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k
\end{aligned}$$

The last line uses monotone convergence theorem. Thus, we proved the inequality 9.8.

Step 3. In this step, we prove that $\rho(z)$ is actually a density. First, it is nonnegative because it's the limit of a sequence of probabilities. Next, we prove it integrates to 1 over $\mathbb{R}^k$. Let the open set U in Step 2 be $\mathbb{W}_k^o$, and we get:

$$1 = \liminf_{T\to\infty} \mathbb{P}(Z^T \in \mathbb{W}_k^o) \geqslant \int_{\mathbb{W}_k^o} \rho(z)dz$$

On the other hand, write the open set $\mathbb{W}_k^o$ as a countable union of almost disjoint closed rectangles: $\mathbb{W}_k^o = \bigcup_{i=1}^{\infty} R_i$. Then we have:

$$1 = \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \mathbb{W}_k^o) = \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \bigcup_{i=1}^{\infty} R_i) \leqslant \sum_{i=1}^{\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R_i)$$

where the inequality uses sub-additivity of probability measure. Take an arbitrary $\epsilon > 0$. For each R_i, we can find a closed rectangle $R_i^{\epsilon-}$ contained in R_i such that $\mathbb{P}(Z^T \in R_i) \leqslant \mathbb{P}(Z^T \in R_i^{\epsilon-}) + \frac{\epsilon}{2^i}$. Then, we have

$$\begin{aligned}1 &= \lim_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \mathbb{W}_k^o) \leqslant \lim_{T\to\infty} \sum_{i=1}^{\infty} \left[\mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R_i^{\epsilon-}) + \frac{\epsilon}{2^i}\right] \\ &= \sum_{i=1}^{\infty} \int_{R_i^{\epsilon-}} \rho(z)dz + \epsilon = \int_{\bigcup_{i=1}^{\infty} R_i^{\epsilon-}} \rho(z)dz + \epsilon\end{aligned}$$

We can interchange the limit and the infinite sum because the sum is bounded by 1. Then let $\epsilon \downarrow 0$, we get $1 \leqslant \int_{\mathbb{W}_k^o} \rho(z)dz$, implying $\int_{\mathbb{W}_k^o} \rho(z)dz = 1$ and we conclude that $\rho(z)$ is actually a density. $\square$

Next we are going to prove Proposition 9.2. Before that, we first introduce some notations and results about multivariate functions.

Suppose $\sigma = (\sigma_1, \cdots, \sigma_n)$ is a multi-index of *length* n. In our context, we require $\sigma_1, \cdots, \sigma_n$ be all non-negative integers(some of them might be equal). We define $|\sigma| = \sum_{i=1}^n \sigma_i$ as the *order* of σ. Suppose $\tau = (\tau_1, \cdots, \tau_n)$ is another multi-index of length n. We say $\tau \leqslant \sigma$ if $\tau_i \leqslant \sigma_i$ for $i = 1, \cdots, n$. We say $\tau < \sigma$ if $\tau \leqslant \sigma$ and there exists at least one index i such that $\tau_i < \sigma_i$. Then, define the partial derivative with respect to the multi-index σ:

$$D^\sigma f(x_1, \cdots, x_n) = \frac{\partial^{|\sigma|} f(x_1, \cdots, x_n)}{\partial x_1^{\sigma_1} \partial x_2^{\sigma_2} \cdots \partial x_n^{\sigma_n}}$$

We have the general Leibniz rule:

$$D^\sigma(fg) = \sum_{\tau \leqslant \sigma} \binom{\sigma}{\tau} D^\tau f \cdot D^{\sigma-\tau} g$$

where $\binom{\sigma}{\tau} = \frac{\sigma_1! \cdots \sigma_n!}{\tau_1! \cdots \tau_n! (\sigma_1 - \tau_1)! \cdots (\sigma_n - \tau_n)!}$.
We also have the Taylor expansion for multi-variable functions:

$$f(x_1, \cdots, x_n) = \sum_{|\sigma| \leqslant r} \frac{1}{\sigma!} D^\sigma f(\vec{x}_0)(\vec{x} - \vec{x}_0)^\sigma + R_{r+1}(\vec{x}, \vec{x}_0)$$

In the equation, $\sigma! = \sigma_1! \sigma_2! \cdots \sigma_n!$ is the factorial with respect to the multi-index σ, $\vec{x}_0 = (x_1^0, \cdots, x_n^0)$ is a constant vector at which we expands the function f, $(\vec{x} - \vec{x}_0)^\sigma$ stands for $(x_1 - x_1^0)^{\sigma_1} \cdots (x_n - x_n^0)^{\sigma_n}$, and

$$R_{r+1}(\vec{x}, \vec{x}_0) = \sum_{\sigma : |\sigma| = r+1} \frac{1}{\sigma!} D^\sigma f(\vec{x}_0 + \theta(\vec{x} - \vec{x}_0))(\vec{x} - \vec{x}_0)^\sigma$$

is the remainder, where $\theta \in (0, 1)$([2, Theorem 3.18 & Corollary 3.19]).

We also need some knowledge about *permutation*. Suppose s_n is a permutation of n non-negative integers, for example $\{1, \cdots, n\}$, and $s_n(i)$ represents the *i-th* element in the permutation s_n. We define *the number of inversions* of s_n by $I(s_n) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathbb{1}_{\{s_n(i)>s_n(j)\}}$. For example, the permutation $s_n = (1, \cdots, n)$ has 0 number of inversions, while the permutation $s_5 = (3, 2, 5, 1, 4)$ has number of inversions 5(2+1+2+0+0). Define the sign of permutation s_n by $sgn(s_n) = (-1)^{I(s_n)}$. For instance, $sgn((1, \cdots, n)) = 1$ and $sgn(s_5) = -1$ in the previous example.

Then, we introduce some notations associated with Proposition 9.2 in order to better discuss the problem. Recall that

$$\vec{a}_0 = (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p})$$
$$\vec{b}_0 = (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q})$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_p$, $\beta_1 > \beta_2 > \cdots > \beta_q$ and $\sum_{i=1}^{p}\alpha_i = \sum_{i=1}^{q}\beta_i = k$. Denote $\vec{a} = (a_1, \cdots, a_k)$, $\vec{b} = (b_1, \cdots, b_k)$. Also denote $\vec{a}^{(1)} = (a_1, \cdots, a_{m_1})$, $\vec{a}^{(2)} = (a_{m_1+1}, \cdots, a_{m_1+m_2})$, $\cdots$, $\vec{a}^{(p)} = (a_{m_1+\cdots+m_{p-1}+1}, \cdots, a_{m_1+\cdots+m_p})$ and $\vec{a} = (\vec{a}^{(1)}, \cdots, \vec{a}^{(p)})$. That is, we divide the vector $\vec{a}$ into p blocks according to the shape of $\vec{a_0}$. Similarly, we write $\vec{b} = (b^{(1)}, \cdots, b^{(q)})$ according to the shape of $\vec{b}_0$. We will keep using similar notations in the following discussion, when we need to divide the vector according to the shape of $\vec{a}_0$ and $\vec{b}_0$. Next, denote

$$f(a_1, \cdots, a_k) \equiv f(\vec{a}) = \det[e^{c_1(t,p)a_i z_j}]_{i,j=1}^{k}, \quad g(b_1, \cdots, b_k) \equiv g(\vec{b}) = \det[e^{c_2(t,p)b_i z_j}]_{i,j=1}^{k}$$

and it's not difficult to see that they are all smooth multi-variable functions with respect to corresponding vectors. In addition, $\lim\limits_{\vec{a}\to\vec{a}_0} f(\vec{a}) = 0$ and $\lim\limits_{\vec{b}\to\vec{b}_0} g(\vec{b}) = 0$.

The last thing before we formally prove Proposition 9.2 is to introduce the following lemmas about the non-vanishing of the determinants.

Lemma 9.5. *Suppose $\sigma_a = (\sigma_1^a, \cdots, \sigma_k^a)$ and $\sigma_b = (\sigma_1^b, \cdots, \sigma_k^b)$ are two multi-indices of length k. We divide them into p and q parts according to the shape of $\vec{a}_0$ and $\vec{b}_0$ as mentioned before: $\sigma_a = (\sigma_a^{(1)}, \sigma_a^{(2)}, \cdots, \sigma_a^{(p)})$, $\sigma_b = (\sigma_b^{(1)}, \sigma_b^{(2)}, \cdots, \sigma_b^{(q)})$. Denote*

$$f(a_1, \cdots, a_k) \equiv f(\vec{a}) = \det[e^{a_i z_j}]_{i,j=1}^{k}, \quad g(b_1, \cdots, b_k) \equiv g(\vec{b}) = \det[e^{b_i z_j}]_{i,j=1}^{k}$$

where we ignore the constants $c_1(t,p)$ and $c_2(t,p)$ temporarily for simplicity. Suppose S_{m_i} is the set of all permutations of $\{0, 1, \cdots, m_i - 1\}$. If $\sigma_a^{(i)} \in S_{m_i}$ for $i = 1, \cdots, p$, then

$$D^{\sigma_a} f(\vec{a}_0) = \det \begin{bmatrix} (z_j^{\sigma_i^a} e^{\alpha_1 z_j})_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ (z_j^{\sigma_i^a} e^{\alpha_p z_j})_{\substack{i=m_1+\cdots+m_{p-1}+1, \\ \cdots, m_1+\cdots+m_p \\ j=1,\cdots,k}} \end{bmatrix}$$

is non-zero for any $(z_1, \cdots, z_k)$ whose elements are distinct. Analogous result also holds for $D^{\sigma_b} g(\vec{b}_0)$.

Proof. Since $f(\vec{a})$ is actually a determinant and its i-th row only depends on the variable a_i, taking derivative of $f(\vec{a})$ with respect to a_i is taking derivative of every entries in the i-th row, and we can get the determinant above. Next, we prove that it is non-zero. WLOG, we can assume $\sigma_a^{(i)} = \{0, 1, \cdots, m_i - 1\}$, because the determinant will only change by -1 when $\sigma_a^{(i)}$ is replaced by other permutations in S_{m_i}. We claim that, the following equation with respect to z:

$$(\xi_1 + \xi_2 z + \cdots + \xi_{m_1} z^{m_i - 1})e^{\alpha_1 z} + \cdots (\xi_{m_1+\cdots+m_{p-1}+1} + \cdots + \xi_k z^{m_p - 1})e^{\alpha_p z} = 0$$

has at most $(k-1)$ distinct roots, where $\sum_{i=1}^{p} m_i = k$ and $(\xi_1, \cdots, \xi_k) \in \mathbb{R}^k$ is non-zero.

Denote the above determinant by $\det\begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$. If this claim holds, we can conclude that we cannot find non-zero $(\xi_1, \cdots, \xi_k) \in \mathbb{R}^k$ such that $\xi_1 v_1 + \cdots + \xi_k v_k = 0$. Thus, the k row vectors of the determinant are linear independent and the determinant is non-zero. Then we prove the claim by induction on k.

1° If $k = 2$, the equation is $(\xi_1 + \xi_2 z)e^{\alpha_1 z} = 0$ or $\xi_1 e^{\alpha_1 z} + \xi_2 e^{\alpha_2 z} = 0$, where $\xi_1, \xi_2 \in \mathbb{R}$ cannot be zero at the same time. Then, it's easy to see that the equation has at most 1 root in two scenarios.

2° Suppose the claim holds for $k \leqslant n$.

3° When $k = n+1$, we have the equation

$$(\xi_1 + \xi_2 z + \cdots + \xi_{m_1} z^{m_i - 1})e^{\alpha_1 z} + \cdots (\xi_{m_1 + \cdots + m_{p-1}+1} + \cdots + \xi_k z^{m_p - 1})e^{\alpha_p z} = 0$$

but now $\sum_{i=1}^{p} m_i = n+1$. WLOG, suppose $(\xi_1, \cdots, \xi_{m_1})$ has a non-zero element and ξ_ℓ is the first non-zero element. Notice that the above equation has the same roots as the following one:

$$F(z) = (\xi_\ell z^{\ell-1} + \cdots + \xi_{m_1} z^{m_1 - 1}) + \cdots + (\xi_{m_1 + \cdots + m_{p-1}+1} + \cdots + \xi_k z^{m_p - 1})e^{(\alpha_p - \alpha_1)z} = 0$$

Assume it has at least $(n+1)$ distinct roots $\eta_1 < \eta_2 < \cdots < \eta_{n+1}$. Then $F'(z) = 0$ has at least n distinct roots $\delta_1 < \cdots < \delta_n$ such that $\eta_1 < \delta_1 < \eta_2 < \cdots < \delta_n < \eta_{n+1}$, by Rolle's Theorem. Actually, $F'(z) = (\xi_\ell(\ell-1))z^{\ell-2} + \cdots + \xi_{m_1}(m_1 - 1)z^{m_1-2}) + \cdots + [\xi'_{m_1+\cdots+m_{p-1}+1} + \cdots + \xi'_k z^{m_p-1})]e^{(\alpha_p - \alpha_1)z} = 0$ where $\xi'_{m_1+\cdots+m_{p-1}+1}$ and ξ'_k are coefficients that can be calculated. This equation has at most $(m_1 - 1) + m_2 + \cdots + m_p - 1 = n - 1$ roots by 2°, which leads to a contradiction. Therefore, our claim holds and we proved Lemma 9.5. □

Remark 9.6. Denote the set $\Lambda_a = \{\sigma_a = (\sigma_a^{(1)}, \cdots, \sigma_a^{(p)}) : \sigma_a^{(i)} \in S_{m_i}, i = 1, \cdots, p\}$, and we have if $\sigma_a \in \Lambda_a$, then $D^{\sigma_a} f(\vec{a}_0)$ is non-zero. Similarly, if $\sigma_b^{(j)} \in S_{n_j}$ for $j = 1, \cdots, q$, then $D^{\sigma_b} g(\vec{b})$ is non-zero, and define $\Lambda_b = \{\sigma_b = (\sigma_b^{(1)}, \cdots, \sigma_b^{(q)}) : \sigma_b^{(j)} \in S_{n_j}, j = 1, \cdots, q\}$.

Lemma 9.7. *The smallest order of σ_a that makes the partial derivative $D^{\sigma_a} f(\vec{a}_0)$ non-zero is $u = \sum_{i=1}^{p} \sum_{j=0}^{m_i - 1} j = \sum_{i=1}^{p} \frac{m_i(m_i - 1)}{2}$. Similarly, $v = \sum_{j=1}^{q} \frac{n_j(n_j-1)}{2}$ is the smallest order of σ_b that makes $D^{\sigma_b} f(\vec{b}_0)$ non-zero.*

Proof. If the order of derivative is less than u, then there exists a $i \in \{1, \cdots, p\}$ such that $\sigma_a^{(i)}$ contains two equal elements, and the determinant $D^{\sigma_a} f(\vec{a}_0)$ would have two equal rows, thus equal to zero. If the order of derivative is u, then when $\sigma_a \in \Lambda_a$, $D^{\sigma_a} f(\vec{a}_0)$ is non-zero by Lemma 9.5. Thus, Lemma 9.7 holds. □

Finally, we give the proof for Proposition 9.2.

Proof of Proposition 9.2. For clarity, the proof will be split into 3 steps. In Step 1, we use multivariate Taylor expansion to find the speed of convergence of $f(\vec{a})$ and $g(\vec{b})$ to zero. In Step 2, we construct a new density function based on Step 1, and we will prove that Z^T weakly converges to the this newly constructed density in Step 3. In Step 3, we use monotone coupling lemma to prove the weak convergence.

Step 1. In this step, we estimate the converging speed of $f(\vec{a})$. Take $\epsilon \in (0, k^{-1} \min_{1 \leqslant i \leqslant p-1} (\alpha_i - \alpha_{i+1}))$ and construct the following vectors:

$$\vec{A}_{\epsilon,+} = (\alpha_1 + m_1\epsilon, \alpha_1 + (m_1 - 1)\epsilon, \cdots, \alpha_1 + \epsilon, \cdots, \alpha_p + m_p\epsilon, \cdots, \alpha_p + \epsilon)$$
$$\vec{A}_{\epsilon,-} = (\alpha_1 - \epsilon, \alpha_1 - 2\epsilon, \cdots, \alpha_1 - m_1\epsilon, \cdots, \alpha_p - \epsilon, \cdots, \alpha_p - m_p\epsilon)$$

That is, the vector $\vec{A}_{\epsilon,+}$(resp. $\vec{A}_{\epsilon,-}$) upwardly(resp. downwardly) spreads out the vector $\vec{a}_0$ such that $\vec{A}_{\epsilon,+}$(resp. $\vec{A}_{\epsilon,-}$) has distinct elements. In addition, when $\epsilon \downarrow 0$, we have $\vec{A}_{\epsilon,\pm}$ converges to $\vec{a}_0$.

The main result of this step is the following:

$$\lim_{\epsilon \downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,\pm}) = \varphi(\vec{a}_0, \vec{z}, \vec{m}) \tag{9.11}$$

where $u = \sum_{i=1}^{p} \frac{m_i(m_i-1)}{2}$ in Lemma 9.7, $\vec{m} = (m_1, \cdots, m_p)$, and $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ is a non-zero function associated with $\vec{a}_0$ and $\vec{m}$.

To prove this result, we first expand the function $f(\vec{a})$ to the order of u at $\vec{a}_0$:

$$\begin{aligned} f(\vec{a}) &= \sum_{|\sigma_a| \leqslant u} \frac{D^{\sigma_a} f(\vec{a}_0)}{\sigma_a!} (\vec{a} - \vec{a}_0)^{\sigma_a} + R_{u+1}(\vec{a}, \vec{a}_0) \\ &= \sum_{\sigma_a \in \Lambda_a} \frac{D^{\sigma_a} f(\vec{a}_0)}{\sigma_a!} (\vec{a} - \vec{a}_0)^{\sigma_a} + R_{u+1}(\vec{a}, \vec{a}_0) \end{aligned}$$

where the $R_{u+1}(\vec{a}, \vec{a}_0) = \sum_{\sigma_a : |\sigma_a| = u+1} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0 + \theta(\vec{a} - \vec{a}_0))(\vec{a} - \vec{a}_0)^{\sigma_a}$, $\theta \in (0, 1)$ is the remainder. The second equality results from Lemma 9.7, since it indicates that all the terms of order less than u are zero, and for the terms of order u, they are non-zero only when $\sigma_a \in \Lambda_a$.

Consider the first term. Denote $sgn(\sigma_a^{(i)})$ as the sign of the permutation $\sigma_a^{(i)} \in S_{m_i}$, and define the sign of σ_a by: $sgn(\sigma_a) = \prod_{i=1}^{p} sgn(\sigma_a^{(i)})$. Denote $\sigma_a^{\star} = (\sigma_a^{(1)\star}, \cdots, \sigma_a^{(p)\star})$, where $\sigma_a^{(i)\star} = (0, 1, \cdots, m_i - 1)$. Thus, $\sigma_a^{\star}$ is a special element in Λ_a and $sgn(\sigma_a^{\star}) = 1$ because $\sigma_a^{(1)\star}, \cdots, \sigma_a^{(p)\star}$ all have 0 number of inversions. Notice that for any $\sigma_a \in \Lambda_a$, we have $D^{\sigma_a} f(\vec{a}_0) = sgn(\sigma_a) \cdot D^{\sigma_a^{\star}} f(\vec{a}_0)$ by the property of determinant. Then we obtain:

$$\sum_{\sigma_a \in \Lambda_a} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0)(\vec{a} - \vec{a}_0)^{\sigma_a} = \frac{D^{\sigma_a^{\star}} f(\vec{a}_0)}{\prod_{i=1}^{p} (m_i - 1)!} \sum_{\sigma_a \in \Lambda_a} (\vec{a} - \vec{a}_0)^{\sigma_a} \cdot sgn(\sigma_a)$$

Notice that

$$\begin{aligned} \sum_{\sigma_a \in \Lambda_a} (\vec{a} - \vec{a}_0)^{\sigma_a} \cdot sgn(\sigma_a) &= \prod_{i=1}^{p} \Big[\sum_{\sigma_a^{(i)} \in S_{m_i}} (\vec{a}^{(i)} - \vec{a}_0^{(i)})^{\sigma_a^{(i)}} \cdot sgn(\sigma_a^{(i)}) \Big] \\ &= \prod_{i=1}^{p} \Delta_{m_i}(a_1^{(i)} - \alpha_i, a_2^{(i)} - \alpha_i, \cdots, a_{m_i}^{(i)} - \alpha_i) \equiv \prod_{i=1}^{p} \Delta_{m_i}^{a} \end{aligned}$$

where $\Delta_n(x_1, x_2, \cdots, x_n)$ is the Vandermonde Determinant, $a_j^{(i)} = a_{m_1 + \cdots + m_{i-1} + j}$ is the j-th element of $\vec{a}^{(i)}$, and the last equality holds by the definition of determinant and Vandermonde Determinant. Now replace $\vec{a}$ with $\vec{A}_{\epsilon,+}$, we get the Vandermonde determinant $\Delta_{m_i}^{a}$ is actually $(m_i - 1)! \cdot \epsilon^{\frac{1}{2} m_i(m_i-1)}$. Therefore, we have:

$$\sum_{\sigma_a \in \Lambda_a} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0)(\vec{a} - \vec{a}_0)^{\sigma_a} = D^{\sigma_a^{\star}} f(\vec{a}_0) \cdot \epsilon^{u}$$

Now we consider the remainder $R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0)$. Since $D^{\sigma_a} f(\vec{a})$ is a continuous function of vector $\vec{a}$, we have that the quantity $D^{\sigma_a^{\star}} f(\vec{a}_0 + \theta(\vec{a} - \vec{a}_0))$ can be bounded by a constant $M(\vec{a}, \vec{a}_0)$. In addition, $\sigma_a!$ only have finitely many possible outcomes when its order is $u + 1$, thus $\frac{1}{\sigma_a!}$ can be bounded by a constant $N(u)$. Also, $|(\vec{A}_{\epsilon,+} - \vec{a}_0)|^{\sigma_a} \leqslant (\max_{1 \leqslant i \leqslant p} m_i \cdot \epsilon)^{u+1}$. Therefore,

$$|R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0)| \leqslant N \cdot M \cdot (\max_{1 \leqslant i \leqslant p} m_i \cdot \epsilon)^{u+1}$$

and this indicates that $R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0)$ is $O(\epsilon^{u+1})$, where the constant in Big O notation only depends on $\vec{a}_0$, $\vec{a}$, $\vec{m}$ and u and does not depend on ϵ. Therefore, we conclude that

$$\lim_{\epsilon \downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,+}) = D^{\sigma_a^{\star}} f(\vec{a}_0)$$

By Lemma 9.5, $D^{\sigma_a^\star} f(\vec{a}_0)$ is non-zero. Thus, we find the limit function $\varphi(\vec{a}_0, \vec{z}, \vec{m}) = D^{\sigma_a^\star} f(\vec{a}_0)$, and its expression can be found in Lemma 9.5. Following similar procedure we can prove $\lim_{\epsilon \downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,-}) = D^{\sigma_a^\star} f(\vec{a}_0)$ also holds, and we established the equation (9.11).

We can construct vectors $\vec{B}_{\epsilon,\pm}$ analogously, which spread out from vector $\vec{b}_0$ upward and downward, and get similar results for $g(\vec{B}_{\epsilon,\pm})$ and then we have:

$$\lim_{\epsilon \downarrow 0} \epsilon^{-v} f(\vec{B}_{\epsilon,\pm}) = D^{\sigma_b^\star} g(\vec{b}_0) \equiv \psi(\vec{b}_0, \vec{z}, \vec{n})$$

where $v = \sum_{i=1}^{q} \frac{n_i(n_i-1)}{2}$ in Lemma 9.7, $\vec{n} = (n_1, \cdots, n_q)$ and the expression of non-zero function $\psi(\vec{b}_0, \vec{z}, \vec{n})$ can be found by Lemma 9.5.

Step 2. In this step, we mainly prove the following result:

The function of $\vec{z}$

$$H(\vec{z}) = \varphi(\vec{a}_0, \vec{z}, \vec{m}) \psi(\vec{b}_0, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{-c_3(t,p) z_i^2}$$

is integrable over $\mathbb{R}^k$.

For simplicity, we ignore the constants c_1, c_2, c_3 temporarily and prove the function $H(\vec{z})$ without those constants is integrable. It's not difficult to see $H(\vec{z})$ is still integrable when adding those constants. Notice that $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ is a determinant whose expression is given in Lemma 9.5. Suppose $z_{j_1}^{i_1-1} e^{a_{i_1} z_{j_1}}$ is the entry that has the largest absolute value. Then

$$|\varphi(\vec{a}_0, \vec{z}, \vec{m})| \leqslant k! |z_{j_1}^{i_1-1} e^{a_{i_1} z_{j_1}}|^k = k! |z_{j_1}^{k(i_1-1)}| e^{k a_{i_1} z_{j_1}}$$

Similarly, we can find index i_2 and j_2 such that $|\varphi(\vec{b}_0, \vec{z}, \vec{n})| \leqslant k! |z_{j_2}^{k(i_2-1)}| e^{k b_{i_2} z_{j_2}}$. Then, we get

$$|H(\vec{z})| \leqslant (k!)^2 \big[\prod_{j \neq j_1, j_2} e^{-z_j^2} |z_{j_1}^{k(i_1-1)} z_{j_2}^{k(i_2-1)}| \big] e^{k a_{i_1} z_{j_1} - z_{j_1}^2} e^{k b_{i_2} z_{j_2} - z_{j_2}^2}$$

The right hand side is integrable over $\mathbb{R}^k$ because the exponential terms have power of some quadratic functions with negative quadratic coefficients. Thus, $H(\vec{z})$ is integrable.

Since $H(\vec{z})$ is integrable, we can define the constant $Z_{\vec{a}_0, \vec{b}_0} = \int_{\mathbb{R}^k} H(\vec{z}) \mathbb{1}_{\{z_1 > z_2 > \cdots > z_k\}} dz < \infty$ and the function

$$\rho_{\vec{a}_0, \vec{b}_0}(z_1, \cdots, z_k) = Z_{\vec{a}_0, \vec{b}_0}^{-1} \varphi(\vec{a}_0, \vec{z}, \vec{m}) \psi(\vec{b}_0, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{-c_3(t,p) z_i^2} \mathbb{1}_{\{z_1 > z_2 > \cdots > z_k\}} \tag{9.12}$$

is a density because it's non-negative and integrates to 1 over $\mathbb{R}^k$.

Step 3. Denote $Z_{\vec{a}_0, \vec{b}_0}^T$ as the random vector Z^T associated with vectors $\vec{a}_0$ and b_0, and in this step we prove it weakly converges to the continuous distribution with the density $\rho_{\vec{a}_0, \vec{b}_0}(z)$ we just constructed in (9.12). Suppose $\mathfrak{L}_+^T$ is an avoiding Bernoulli line ensemble starting with $\vec{x}_+^T$ and ending with $\vec{y}_+^T$ and follows the distribution $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}_+^T,\vec{y}_+^T}$. The vectors $\vec{x}_+^T$ and $\vec{y}_+^T$ are two signatures of length k that satisfies the following:

(i)

$$\lim_{T \to \infty} \frac{\vec{x}_+^T}{\sqrt{T}} = \vec{A}_{\epsilon,+}, \quad \lim_{T \to \infty} \frac{\vec{y}_+^T - pT 1_k}{\sqrt{T}} = \vec{B}_{\epsilon,+}$$

(ii) $\vec{x}_+^T \geqslant \vec{x}^T$, $\vec{y}_+^T \geqslant \vec{y}^T$, which means the endpoints of the newly constructed line ensembles dominate the original ones. This can be achieved due to the limiting behavior of $\vec{x}_+^T$ and $\vec{y}_+^T$ and the construction of $\vec{A}_{\epsilon,+}$ and $\vec{B}_{\epsilon,+}$. Analogously, we construct another avoiding Bernoulli line ensemble $\mathfrak{L}_-^T$ with endpoints $\vec{x}_-^T$ and $\vec{y}_-^T$ and distribution $\mathbb{P}_{Avoid,Ber}^{0,T,\vec{x}_-^T,\vec{y}_-^T}$ such that $\lim_{T \to \infty} \frac{\vec{x}_-^T}{\sqrt{T}} =$

$\vec{A}_{\epsilon,-}$, $\lim_{T\to\infty} \frac{\vec{y}_-^T - pT1_k}{\sqrt{T}} = \vec{B}_{\epsilon,-}$, and $\vec{x}_-^T \leqslant \vec{x}^T$, $\vec{y}_-^T \leqslant \vec{y}^T$.

Since now $\vec{A}_{\epsilon,+}$, $\vec{A}_{\epsilon,-}$, $\vec{B}_{\epsilon,+}$, $\vec{B}_{\epsilon,-}$ have distinct elements, we can apply the results in Proposition 9.1 and conclude the weak convergence:

$$Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} \Rightarrow \rho_{\epsilon,+}(z), \quad Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}} \Rightarrow \rho_{\epsilon,-}(z)$$

where $Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}$ and $Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}}$ are obtained by scaling the line ensembles $\mathfrak{L}^T_+$, and $\mathfrak{L}^T_-$, $\rho_{\epsilon,+}(z)$ and $\rho_{\epsilon,-}(z)$ are densities which are obtained by plugging $\vec{A}_{\epsilon,+}$, $\vec{B}_{\epsilon,+}$ and $\vec{A}_{\epsilon,-}$, $\vec{B}_{\epsilon,-}$ into the formula of $\rho(z)$ in Proposition 9.1.

In order to prove the weak convergence of $Z^T_{\vec{a}_0,\vec{b}_0}$, it is sufficient to prove for any $R = (-\infty, u_1] \times (-\infty, u_2] \times \cdots \times (-\infty, u_k]$, where $u_i \in \mathbb{R}$, we have

$$\lim_{T\to\infty} (Z^T_{\vec{a}_0,\vec{b}_0} \in R) = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$$

Actually, by Lemma 3.1, we can construct a sequence of probability spaces $(\Omega_T, \mathcal{F}_T, \mathbb{P}_T)_{T\geqslant 1}$ such that for each $T \in \mathbb{Z}^+$, we have random variables $\mathfrak{L}^T_+$ and $\mathfrak{L}^T$ have law $\mathbb{P}^{0,T,\vec{x}^T_+,\vec{y}^T_+}_{Avoid,Ber}$, and $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{Avoid,Ber}$ under measure $\mathbb{P}_T$, respectively. Also, we have $\mathfrak{L}^T_+(i,r) \geqslant \mathfrak{L}^T(i,r)$ with probability 1, where $\mathfrak{L}^T_+(i,r)$(resp., $\mathfrak{L}^T(i,r)$) is the value of the *i-th* up-right path of $\mathfrak{L}^T_+$(resp., $\mathfrak{L}^T$) at $r \in [\![0,T]\!]$. Similarly, we can construct another sequence of probability spaces $(\Omega'_T, \mathcal{F}'_T, \mathbb{Q}_T)_{T\geqslant 1}$ such that for each $T \in \mathbb{Z}^+$, we have random variables $\mathfrak{L}^T_-$ and $\mathfrak{L}^T$ have law $\mathbb{P}^{0,T,\vec{x}^T_-,\vec{y}^T_-}_{avoid,Ber}$, and $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$ under measure $\mathbb{Q}_T$, respectively, along with $\mathbb{Q}_T(\mathfrak{L}^T_-(i,r) \leqslant \mathfrak{L}^T(i,r), i = 1,\cdots,k, r \in [\![0,T]\!]) = 1$.

Therefore, we have that under measure $\mathbb{P}_T$ and $\mathbb{Q}_T$:

$$\mathbb{P}_T(Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} \in R) \leqslant \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R), \quad \mathbb{Q}_T(Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}} \in R) \geqslant \mathbb{Q}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R)$$

Take $liminf$ and $limsup$ on both side of the first and second inequality respectively, we get

$$\int_R \rho_{\epsilon,+}(z)dz \leqslant \liminf_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R), \quad \int_R \rho_{\epsilon,-}(z)dz \geqslant \limsup_{T\to\infty} \mathbb{Q}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \tag{9.13}$$

because of the weak convergence of $Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}$ and $Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}}$. Since the distribution of $Z^T_{\vec{a}_0,\vec{b}_0}$ under measure $\mathbb{P}_T$ and $\mathbb{Q}_T$ are the same, we can combine the above two inequalities (9.13) and get

$$\int_R \rho_{\epsilon,+}(z)dz \leqslant \liminf_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \leqslant \limsup_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \leqslant \int_R \rho_{\epsilon,-}(z)dz \tag{9.14}$$

The rest of the proof establishes the following statement:

$$\lim_{\epsilon\downarrow 0} \int_R \rho_{\epsilon,+}(z)dz = \lim_{\epsilon\downarrow 0} \int_R \rho_{\epsilon,-}(z)dz = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz \tag{9.15}$$

and thereby concluding

$$\lim_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$$

by letting $\epsilon \downarrow 0$ in the inequality (9.14), and we prove the weak convergence of $Z^T_{\vec{a}_0,\vec{b}_0}$.

To prove the statement (9.15), first notice that

$$\begin{aligned}
Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} &= \int_{\mathbb{R}^k} f(\vec{a},\vec{z})g(\vec{b},\vec{z})\prod_{i=1}^{k} e^{-c_3(t,p)z_i^2}dz \\
&= \int_{\mathbb{R}^k}\left[\epsilon^{u+v}\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n})+o(\epsilon^{u+v})\right]\prod_{i=1}^{k} e^{-c_3(t,p)z_i^2}dz \\
&= \epsilon^{u+v}\int_{\mathbb{R}^k}\left[\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n})+o(1)\right]\prod_{i=1}^{k} e^{-c_3(t,p)z_i^2}dz
\end{aligned}$$

Then, we get

$$\lim_{\epsilon\downarrow 0}\epsilon^{-(u+v)}Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} = \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^k}\left[\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n})+o(1)\right]\prod_{i=1}^{k} e^{-c_3(t,p)z_i^2}dz = Z_{\vec{a}_0,\vec{b}_0}$$

by definition of the constant $Z_{\vec{a}_0,\vec{b}_0}$. Therefore, we conclude

$$\lim_{\epsilon\downarrow 0}\rho_{\epsilon,+}(z) = \lim_{\epsilon\downarrow 0}(\epsilon^{-(u+v)}Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}})(\epsilon^{u}f(\vec{a},\vec{z}))(\epsilon^{v}g(\vec{b},\vec{z})) = Z_{\vec{a},\vec{b}_0}\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n}) = \rho_{\vec{a}_0,\vec{b}_0}(z)$$

Since $\rho_{\epsilon,+}(z)\mathbb{1}_R dz \leqslant \rho_{\epsilon,+}(z)dz$ is bounded by an integrable function, by Dominated Convergence Theorem we have:

$$\lim_{\epsilon\downarrow 0}\int_R \rho_{\epsilon,+}(z)dz = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$$

Analogously, we can get $\lim_{\epsilon\downarrow 0}\int_R \rho_{\epsilon,-}(z)dz = \int_R \rho_{\vec{a}_0,\vec{b}_0}(z)dz$ and we proved the statement (9.15), which completes the proof. □

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