Summer 2020

REU Practice Problems

1 Topology and measurability

We let Σ denote a set $[\![p, q]\!] = \{p, p+1, \dots, q-1, q\}$ for $p \in \mathbb{N}$, $q \in \mathbb{N} \cup \{\infty\}$, and let Λ denote an interval in $\mathbb{R}$. We write $C(X)$ for the space of continuous real-valued functions on X with the topology of compact convergence and the Borel σ-algebra $\mathcal{C}$. Recall that this is generated by the basis of sets

$$B_K(f, \epsilon) := \Big\{g \in C(X) : \sup_{x \in K} |f(x) - g(x)| < \epsilon\Big\},$$

with $K \subset X$ is compact, $f \in C(X)$, and $\epsilon > 0$. When $X = \Sigma \times \Lambda$, we write $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$.

Problem 1

We aim to construct a metric $d : C(\Sigma \times \Lambda) \times C(\Sigma \times \Lambda) \to [0, \infty)$ which induces the topology of compact convergence on $C(\Sigma \times \Lambda)$. The idea is to write $\Sigma \times \Lambda$ as a union of compact sets K_n, such that every compact subset of $\Sigma \times \Lambda$ is contained in one of these sets K_n. We then construct d from the sup-metrics on each of these sets K_n. We define the sets

$$K_n := [\![p, \min(p+n, q)]\!] \times \Lambda_n$$

as follows. If $\Lambda = [a, b]$ is compact, then $\Lambda_n = \Lambda$ for all n. If $\Lambda = (a, b)$, then

$$\Lambda_n := \left[a + \frac{1}{n}, b - \frac{1}{n}\right],$$

when this set makes sense, and $\Lambda_n = \varnothing$ otherwise. If Λ is half-open, we define Λ_n similarly, but only modify the endpoint on the open side. (This might not be necessary, not sure if Λ is assume to be closed.) In any case, we see that the sets $K_1 \subset K_2 \subset \cdots \subset \Sigma \times \Lambda$ are compact, they cover $\Sigma \times \Lambda$, and any compact subset K of $\Sigma \times \Lambda$ is contained in all K_n for sufficiently large n.

We now define, for each n and $f, g \in C(\Sigma \times \Lambda)$,

$$d_n(f, g) := \sup_{x \in K_n} |f(x) - g(x)|, \quad d_n'(f, g) := \min\{d_n(f, g), 1\}.$$

Clearly each d_n is nonnegative and satisfies the triangle inequality, and it is then easy to see that the same properties hold for d_n'. Furthermore, $d_n' \le 1$, so we can define

$$d(f, g) := \sum_{n=1}^{\infty} 2^{-n} d_n'(f, g).$$

We first observe that d is a metric on $C(\Sigma \times \Lambda)$. Indeed, it is nonnegative, and if $f = g$, then each $d_n'(f, g) = 0$, so the sum is 0. Conversely, if $f \neq g$, then since the K_n cover $\Sigma \times \Lambda$, we can choose n large enough so that K_n contains an x with $f(x) \neq g(x)$. Then $d'_n(f,g) \neq 0$, and hence $d(f,g) \neq 0$. The triangle inequality holds for d since it holds for each d'_n.

Now we prove that the topology τ_d on $C(\Sigma \times \Lambda)$ induced by d is the same as the topology of compact convergence, which we will denote τ_c. First, choose $\epsilon > 0$ and $f \in C(\Sigma \times \Lambda)$. Let $g \in B^d_\epsilon(f)$, i.e., $d(f,g) < \epsilon$. We will find a set $A_g \in \tau_c$ such that $g \in A_g \subset B^d_\epsilon(f)$. Let $\delta := d(f,g)$, and choose n large enough so that $\sum_{k>n} 2^{-k} < \frac{\epsilon-\delta}{2}$. Define $A_g := B_{K_n}(g, \frac{\epsilon-\delta}{n})$, and suppose $h \in A_g$. Then since $K_m \subseteq K_n$ for $m \leq n$, we have

$$\begin{aligned} d(f,h) &\leq d(f,g) + d(g,h) \\ &\leq \delta + \sum_{k=1}^{n} 2^{-k} d_n(g,h) + \sum_{k>n} 2^{-k} \\ &\leq \delta + \frac{\epsilon-\delta}{2} + \frac{\epsilon-\delta}{2} = \epsilon. \end{aligned}$$

Therefore $g \in A_g \subset B^d_\epsilon(f)$. It follows that $B^d_\epsilon(f) \in \tau_c$. Indeed, we can write

$$B^d_\epsilon(f) = \bigcup_{g \in B^d_\epsilon(f)} A_g,$$

a union of elements of τ_c. This proves that $\tau_d \subseteq \tau_c$.

To prove the converse, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$. Choose n so that $K \subset K_n$, and let $g \in B_K(f,\epsilon)$ and $\delta := \sup_{x\in K} |f(x) - g(x)|$. If $d(g,h) < 2^{-n}(\epsilon - \delta)$, then $d'_n(g,h) \leq 2^n d(g,h) < \epsilon - \delta$, hence $d_n(g,h) < \epsilon - \delta$. It follows that

$$\begin{aligned} \sup_{x\in K} |f(x) - h(x)| &\leq \delta + \sup_{x\in K} |g(x) - h(x)| \leq \delta + d_n(g,h) \\ &\leq \delta + \epsilon - \delta = \epsilon. \end{aligned}$$

Thus $g \in B^d_{2^{-n}(\epsilon-\delta)}(f) \subset B_K(f,\epsilon)$. It follows that $\tau_c \subseteq \tau_d$, and we conclude that $\tau_d = \tau_c$.

Next, we show that $(C(\Sigma \times \Lambda), d)$ is a complete metric space. Let $(f_n)_{n\geq 1}$ be Cauchy with respect to d. Then we claim that (f_n) must be Cauchy with respect to d'_n, on each K_n. Indeed, $d(f_\ell, f_m) \geq 2^{-n} d'_n(f_\ell, f_m)$, so if (f_n) were not Cauchy with respect to d'_n, it would not be Cauchy with respect to d either. Thus (f_n) is uniformly Cauchy on each K_n, and hence converges uniformly to a limit f^{K_n} on each K_n. Since the limit must be unique at each point of $\Sigma \times \Lambda$, we have $f^{K_n}(x) = f^{K_m}(x)$ if $x \in K_n \cap K_m$. Since $\bigcup K_n = \Sigma \times \Lambda$, we obtain a well-defined function f on all of $\Sigma \times \Lambda$ given by $f(x) = f^{K_n}(x)$, where $x \in K_n$. Given any compact $K \subset \Sigma \times \Lambda$, if n is large enough so that $K \subset K_n$, then because $f_n \to f^{K_n} = f|_{K_n}$ uniformly on K_n, we have $f_n \to f^{K_n}|_K = f|_K$ uniformly on K. That is, for any $K \subset \Sigma \times \Lambda$ compact and $\epsilon > 0$, we have $f_n \in B_K(f,\epsilon)$ for all sufficiently large n. Therefore (f_n) converges to f in the topology of compact convergence, and equivalently in the metric d.

Lastly, we prove separability. We consider the subspace $P_{\mathbb{Q}}$ of $C(\Sigma \times \Lambda)$ consisting of "polynomials" with rational coefficients. That is, $p \in P_{\mathbb{Q}}$ if $p(n,\cdot)$ is a polynomial on Λ with rational coefficients for each $n \in \Sigma$. If $f \in C(\Sigma \times \Lambda)$, then on any compact set $K \subset \Sigma \times \Lambda$ we can find a sequence of polynomials converging uniformly to f by the Stone-Weierstrass theorem. These polynomials in turn can be uniformly approximated by polynomials with rational coefficients, so by diagonalization we obtain a sequence in $P_{\mathbb{Q}}$ converging uniformly to f on K. [Can we patch together the sequences for the sets K_n to get one sequence (p_n) in $P_{\mathbb{Q}}$ that converges uniformly to f on *all* compact subsets? Maybe use diagonalization again?][The result from McCoy, "Second Countable and Separable Function Spaces," shows that $C(\Sigma \times \Lambda)$ is second-countable and hence separable because $\Sigma \times \Lambda$ is second-countable. But the proof is a bit difficult.]

Problem 2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X, Y random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $C(\Sigma \times \Lambda)$, where $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$. We consider the collection $\mathcal{S}_X$ of sets of the form

$$\{\omega \in \Omega : X(\omega)(i_1, t_1) \leq x_1, \ldots, X(\omega)(i_n, t_n) \leq x_n\} = \bigcap_{k=1}^{n} X(i_k, t_k)^{-1}(-\infty, x_k],$$

ranging over all $n \in \mathbb{N}$, $(i_1, t_1), \ldots, (i_n, t_n) \in \Sigma \times \Lambda$, and $x_1, \ldots, x_n \in \mathbb{R}$. We first prove that $\mathcal{S}_X \subset \mathcal{F}$. We can write

$$\{X(i_k, t_k) \leq x_k\} = X^{-1}(\{f \in C(\Sigma \times \Lambda) : f(i_k, t_k) \leq x_k\}).$$

We claim that the set $\{f \in C(\Sigma \times \Lambda) : f(i_k, t_k) \leq x_k\}$ is closed in the topology of compact convergence. If $f_n(i_k, t_k) \leq x_k$ for all n and $f_n \to f$ in the (metrizable) topology of compact convergence, then by taking limits on a compact set containing (i_k, t_k), we find $f(i_k, t_k) \leq x_k$ as well. This proves the claim, and it follows from the measurability of X that $\{X(i_k, t_k) \leq x_k\} = X^{-1}(\{f(i_k, t_k) \leq x_k\}) \in \mathcal{F}$. The finite intersection is thus also in $\mathcal{F}$, proving that $\mathcal{S}_X \subset \mathcal{F}$. On the other hand, it is clear that $\{\omega \in \Omega : X(\omega) \in A\} = X^{-1}(A) \in \mathcal{F}$ for any $A \in \mathcal{C}_\Sigma$ since X is measurable.

Now we prove that $\mathbb{P}|_{\mathcal{S}_X}$ determines the distribution $\mathbb{P} \circ X^{-1}$. To do so, note that $\mathcal{S}_X = \sigma(\{X^{-1}(A) : A \in \mathcal{S}\})$, where $\mathcal{S}$ is the collection of cylinder sets

$$\{f \in C(\Sigma \times \Lambda) : f(i_1, t_1) \in A_1, \ldots, f(i_n, t_n) \in A_n\}, \quad A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}).$$

This follows from the fact that $\mathcal{B}(\mathbb{R})$ is generated by intervals of the form $(-\infty, x]$. Observe that the intersection of two elements of $\mathcal{S}$ is clearly another element of $\mathcal{S}$, so $\mathcal{S}$ is a π-system. We now argue that $\mathcal{S}$ generates the Borel sets, i.e., $\sigma(\mathcal{S}) = \mathcal{C}_\Sigma$. By the argument above, $\mathcal{S} \subset \mathcal{C}_\Sigma$. Conversely, we will show that every basis element of the topology of compact convergence on $C(\Sigma \times \Lambda)$ is contained in $\sigma(\mathcal{S})$, and consequently so is every Borel set. More precisely, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and let H be a countable dense subset of K. (Recall that every compact metric space is separable, and K is homeomorphic to a product of finitely many compact sets in $\mathbb{R}$, which are metrizable. So K is separable.) We claim that

$$B_K(f, \epsilon) = \bigcup_{n=1}^{\infty} \bigcap_{(i,t) \in H} \{g \in C(\Sigma \times \Lambda) : g(i, t) \in (f(i, t) - (1 - 2^{-n})\epsilon, f(i, t) + (1 - 2^{n})\epsilon)\}.$$

Indeed, if $g \in B_K(f, \epsilon)$, i.e., $\sup_{(i,t)\in K} |g(i,t) - f(i,t)| < \epsilon$. Then since $1 - 2^{-m} \nearrow 1$, we can choose m large enough so that

$$|g(i,t) - f(i,t)| < (1 - 2^{-n})\epsilon$$

for all $(i,t) \in K$ (in particular with $(i,t) \in H$). Conversely, suppose g is in the set on the right. Then since g is continuous and H is dense in K, we find that for some $n \geq 1$,

$$|g(i,t) - f(i,t)| \leq (1 - 2^{-n})\epsilon < \epsilon$$

for all $(i,t) \in K$. Hence $g \in B_K(f, \epsilon)$. This proves the claim. Since H is countable, $B_K(f, \epsilon)$ is formed from countably many unions and intersections of sets in $\mathcal{S}$, thus $B_K(f, \epsilon) \in \sigma(\mathcal{S})$.

In summary, we have shown that the collection $\mathcal{S}$ is a π-system generating $\mathcal{C}_\Sigma$, so the probability measure $\mathbb{P} \circ X^{-1}$ on $\mathcal{C}_\Sigma$ is uniquely determined by its restriction to $\mathcal{S}$. Suppose

$$\begin{aligned} &\mathbb{P}\left(\{\omega \in \Omega : X(\omega)(i_1, t_1) \leq x_1, \dots, X(\omega)(i_n, t_n) \leq x_n\}\right) = \\ &\qquad \mathbb{P}\left(\{\omega \in \Omega : Y(\omega)(i_1, t_1) \leq x_1, \dots, Y(\omega)(i_n, t_n) \leq x_n\}\right) \end{aligned}$$

for all $(i_1, t_1), x_1, \dots, x_n$. This says that the two probability measures $\mathbb{P} \circ X^{-1}$ and $\mathbb{P} \circ Y^{-1}$ agree on $\mathcal{S}$. Then they must agree on all of $\mathcal{C}_\Sigma$, i.e.,

$$\mathbb{P}\left(\{\omega \in \Omega : X(\omega) \in A\}\right) = \mathbb{P}\left(\{\omega \in \Omega : Y(\omega) \in A\}\right)$$

for all $A \in \mathcal{C}_\Sigma$. In other words, the law of a line ensemble is determined by its finite dimensional distributions.

2 Algebra

Problem 3

Problem 4

3 Weak convergence

Problem 5

Problem 6

Problem 7

4 Tightness

Problem 8

Problem 9

5 Lozenge tilings of the hexagon

Problem 10

Problem 11