

Asymptotics of Bernoulli Line Ensembles

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The Gaussian universality class

Let $\{X_i\}$ be a sequence of independent identically distributed random variables with mean 0 and variance 1. Let $S_n = X_1 + \cdots + X_n$.

- **Law of Large Numbers:** $\frac{S_n}{n} \rightarrow \mu$ as $n \rightarrow \infty$ almost surely
- **Central Limit Theorem:** $\frac{S_n}{\sqrt{n}} \Rightarrow \mathcal{N}(0, 1)$ as $n \rightarrow \infty$.
- **Donsker's Theorem:** Let $B^{(n)}(t) = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}$ for $t \in [0, 1]$. Then $B^{(n)} \in C([0, 1])$ and $B^{(n)} \Rightarrow B$ as $n \rightarrow \infty$, a standard Brownian motion on $[0, 1]$.

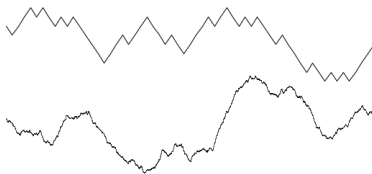
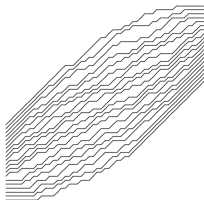


Figure: An example of a random walk and a Brownian motion.

Multiple random walks

If $S_{n+1} - S_n \in \{0, 1\}$, then $\{S_n\}_{n=1}^\infty$ is a *Bernoulli random walk*. A *Bernoulli line ensemble* $\mathfrak{L} = (L_1, \dots, L_k)$ consists of k avoiding Bernoulli random walks on an interval $[T_0, T_1]$, such that $L_1(s) \geq L_2(s) \geq \dots \geq L_k(s)$ for $s \in [T_0, T_1]$.



When dealing with a family of avoiding Brownian Motions, we speak of Dyson Brownian Motion:

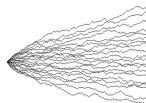


Figure: Dyson Brownian Motion

Airy Line Ensemble

As $k \rightarrow \infty$, k avoiding random walks are conjectured to converge to the **Airy line ensemble**, \mathcal{A} , and the top curve to the **Airy process**, \mathcal{A}_1 .

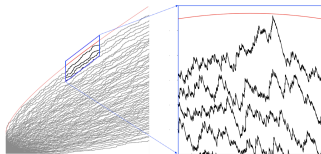


Figure: Multiple Dyson Brownian walks

- Increasing the number of paths pushes us outside of the Gaussian universality class and into Kardar-Parisi-Zhang (KPZ) universality class.
- Open problem: Show that “generic” random walks with “generic” initial conditions converge to the Airy line ensemble.
- We treat this problem for Bernoulli random walks; the proof is only known if all walks start from 0.

Convergence to the Airy Line Ensemble

Two sufficient conditions for convergence in distribution:

- *Finite dimensional* convergence – difficult, requires exact algebraic formulas
- *Tightness* (existence of weak subsequential limits) – easier, more qualitative/analytic

We focused on tightness, which we prove by controlling the maximum, the minimum, and the modulus of continuity

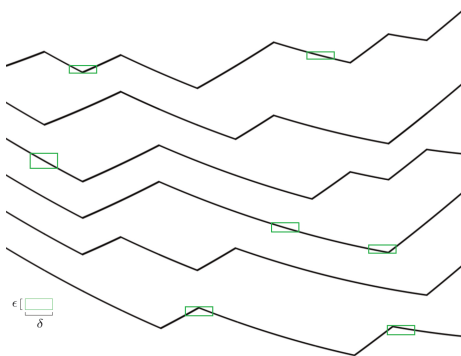


Figure: The Modulus of Continuity

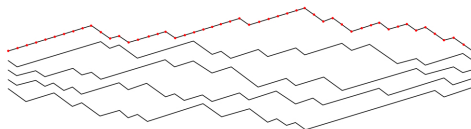
Theorem

Let $\{\mathfrak{L}^N = (L_1^N, \dots, L_k^N)\}_{N=1}^\infty$ be a sequence of avoiding Bernoulli Gibbsian line ensembles. Fix $p \in (0, 1)$ and $\lambda > 0$, and suppose that for all $n \in \mathbb{Z}$ we have

$$\lim_{N \rightarrow \infty} \mathbb{P}(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \leq N^{1/3}x) = F_{TW}(x).$$

Then $\{\mathfrak{L}^N\}$ is a tight sequence.

- F_{TW} denotes the *Tracy-Widom distribution* – a common limiting distribution in the KPZ universality class.
- [Dauvergne-Nica-Virág '19] showed that finite dimensional convergence of all curves implies tightness, hence convergence to the Airy line ensemble.
- Our result shows that it suffices for the *top curve* to converge in the f.d. sense.



Arguments in this paper are inspired by

- 1 *Brownian Gibbs property for Airy line ensembles and KPZ line ensemble* [Corwin-Hammond '11, '13], which address the issues of **continuous** line ensembles
- 2 *Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood line ensembles* [Corwin-Dimitrov '17], which consider similar questions in a **discrete** setting

Recall that to show tightness, we want to control

- ① \min
- ② \max
- ③ modulus of continuity of the line ensembles

We claim that for the **top** curve of our line ensemble to have a **parabolic shift**, the **bottom** curve cannot dip too low, i.e. for any $r, \epsilon > 0$, there exist $R, M > 0$ such that for N large enough,

$$P(\max_{[r,R]} L_k(sN^\alpha) - psN^\alpha \leq -MN^\alpha) < \epsilon$$

(perhaps insert a picture)

Proof (mention monotone coupling lemmas somewhere) - say MC with picture 2min

Proof (mention strong coupling somewhere) - say SC with picture $L = \text{Bernoulli bridge}$ B is a Brownian bridge with variance. There is a probability space such that $P(\sup |L - B| \geq k(\log N)^2) < \epsilon$. This is a comparison that allows for example to compare the modulus of continuity of the two. [Dimitrov-Wu '19] 2 min

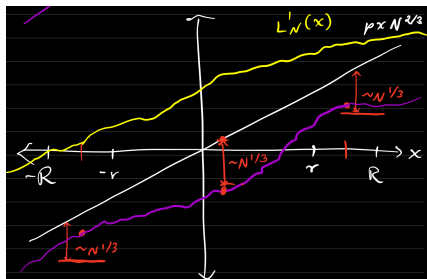
Controlling the minimum: pinning the bottom curve

Lemma (—)

For any $r, \epsilon > 0$, there exists $R > r$ and a constant $M > 0$ so that for large N ,

$$\mathbb{P}\left(\max_{x \in [r, R]} (L_k^N(xN^{2/3}) - pxN^{2/3}) \leq -MN^{1/3}\right) < \epsilon.$$

The same is true of the maximum on $[-R, -r]$.



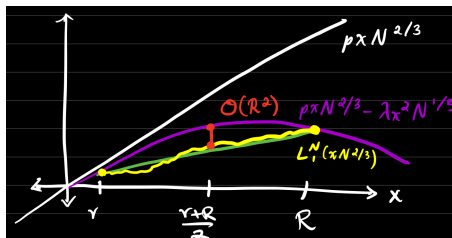
- Couple with a Brownian bridge: if “pinned” at two points $> r$ and $-r$, it cannot be low on scale $N^{1/3}$ on $[-r, r]$.

Proving the pinning lemma

- Recall our assumption:

$$\mathbb{P}\left(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \leq xN^{1/3}\right) \xrightarrow{N \rightarrow \infty} F_{TW}(x).$$

- The top curve looks like a **parabola** with an affine shift on large scales.



- Two curves: if L_2^N is low on $[r, R]$, L_1^N looks like a free Brownian bridge.

$$\lambda\left(\frac{R^2 + r^2}{2}\right) - \lambda\left(\frac{R+r}{2}\right)^2 = \lambda\frac{R^2 + r^2}{4} - \frac{\lambda rR}{2} = O(R^2).$$

- For large R , the top curve would be far from the parabola at the midpoint!