TIGHTNESS OF BERNOULLI LINE ENSEMBLES

Abstract. Insert abstract here:

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2. Line ensembles

In this section we introduce various definitions and notation that are used throughout the paper.

2.1. Line ensembles and the Brownian Gibbs property.

In this section we introduce the notions of a *line ensemble* and the *(partial) Brownian Gibbs property*. Our exposition in this section closely follows that of [4, Section 2] and [2, Section 2].

Given two integers $p \leq q$, we let $[\![p, q]\!]$ denote the set $\{p, p+1, \dots, q\}$. Given an interval $\Lambda \subset \mathbb{R}$ we endow it with the subspace topology of the usual topology on $\mathbb{R}$. We let $(C(\Lambda), \mathcal{C})$ denote the space of continuous functions $f : \Lambda \to \mathbb{R}$ with the topology of uniform convergence over compacts, see [8, Chapter 7, Section 46], and Borel σ-algebra $\mathcal{C}$. Given a set $\Sigma \subset \mathbb{Z}$ we endow it with the discrete topology and denote by $\Sigma \times \Lambda$ the set of all pairs (i, x) with $i \in \Sigma$ and $x \in \Lambda$ with the product topology. We also denote by $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ the space of continuous functions on $\Sigma \times \Lambda$ with the topology of uniform convergence over compact sets and Borel σ-algebra $\mathcal{C}_\Sigma$. Typically, we will take $\Sigma = [\![1, N]\!]$ (we use the convention $\Sigma = \mathbb{N}$ if $N = \infty$) and then we write $\big(C(\Sigma \times \Lambda), \mathcal{C}_{|\Sigma|}\big)$ in place of $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$.

The following defines the notion of a line ensemble.

Definition 2.1. Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. A Σ-*indexed line ensemble* $\mathcal{L}$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that takes values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$. Intuitively, $\mathcal{L}$ is a collection of random continuous curves (sometimes referred to as *lines*), indexed by Σ, each of which maps Λ in $\mathbb{R}$. We will often slightly abuse notation and write $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$, even though it is not $\mathcal{L}$ which is such a function, but $\mathcal{L}(\omega)$ for every $\omega \in \Omega$. For $i \in \Sigma$ we write $\mathcal{L}_i(\omega) = (\mathcal{L}(\omega))(i, \cdot)$ for the curve of index i and note that the latter is a map $\mathcal{L}_i : \Omega \to C(\Lambda)$, which is $(\mathcal{C}, \mathcal{F})-$measurable.

We will require the following result, whose proof is postponed until Section [Appendix]. In simple terms it states that the space $C(\Sigma \times \Lambda)$ where our random variables $\mathcal{L}$ take value has the structure of a complete, separable metric space. We say that a collection $(K_n)_{n\geq 1}$ of compact subsets $K_n \subset \Sigma \times \Lambda$ is a *compact exhaustion* if $\bigcup_n K_n = \Sigma \times \Lambda$ and every compact subset of $\Sigma \times \Lambda$ is contained in some K_n.

Lemma 2.2. *Let $(K_n)_{n\geq 1}$ be a compact exhaustion of $\Sigma \times \Lambda$. Define $d : C(\Sigma \times \Lambda) \times C(\Sigma \times \Lambda) \to [0, \infty)$ by*

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \min \Big\{ \sup_{(i,t)\in K_n} |f(i, t) - g(i, t)|, 1 \Big\}. \tag{2.1}$$

Then d defines a metric on $C(\Sigma \times \Lambda)$ and moreover the metric space topology defined by d is the same as the topology of uniform convergence over compact sets. Furthermore, the metric space $(C(\Sigma \times \Lambda), d)$ is complete and separable.

Definition 2.3. Given a sequence $\{\mathcal{L}^n : n \in \mathbb{N}\}$ of random Σ-indexed line ensembles we say that $\mathcal{L}^n$ *converge weakly* to a line ensemble $\mathcal{L}$, and write $\mathcal{L}^n \implies \mathcal{L}$ if for any bounded continuous function $f : C(\Sigma \times \Lambda) \to \mathbb{R}$ we have that

$$\lim_{n\to\infty} \mathbb{E}\left[f(\mathcal{L}^n)\right] = \mathbb{E}\left[f(\mathcal{L})\right].$$

We also say that $\{\mathcal{L}^n : n \in \mathbb{N}\}$ is *tight* if for any $\epsilon > 0$ there exists a compact set $K \subset C(\Sigma \times \Lambda)$ such that $\mathbb{P}(\mathcal{L}^n \in K) \geq 1 - \epsilon$ for all $n \in \mathbb{N}$.

We call a line ensemble *non-intersecting* if $\mathbb{P}$-almost surely $\mathcal{L}_i(r) > \mathcal{L}_j(r)$ for all $i < j$ and $r \in \Lambda$.

We next turn to formulating the Brownian Gibbs property – we do this in Definition 2.7 after introducing some relevant notation and results. If W_t denotes a standard one-dimensional Brownian motion, then the process

$$\tilde{B}(t) = W_t - tW_1, \quad 0 \leq t \leq 1,$$

is called a *Brownian bridge (from $\tilde{B}(0) = 0$ to $\tilde{B}(1) = 0$) with diffusion parameter* 1. For brevity we call the latter object a *standard Brownian bridge.*

Given $a, b, x, y \in \mathbb{R}$ with $a < b$ we define a random variable on $(C([a, b]), \mathcal{C})$ through

$$B(t) = (b - a)^{1/2} \cdot \tilde{B}\left(\frac{t - a}{b - a}\right) + \left(\frac{b - t}{b - a}\right) \cdot x + \left(\frac{t - a}{b - a}\right) \cdot y, \tag{2.2}$$

and refer to the law of this random variable as a *Brownian bridge (from $B(a) = x$ to $B(b) = y$) with diffusion parameter* 1. Given $k \in \mathbb{N}$ and $\vec{x}, \vec{y} \in \mathbb{R}^k$ we let $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}$ denote the law of k independent Brownian bridges $\{B_i : [a, b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1.

We next state a couple of results about Brownian bridges from [2] for future use.

Lemma 2.4. *[2, Corollary 2.9]. Fix a continuous function $f : [0, 1] \to \mathbb{R}$ such that $f(0) > 0$ and $f(1) > 0$. Let B be a standard Brownian bridge and let $C = \{B(t) > f(t)$ for some $t \in [0, 1]\}$ (crossing) and $T = \{B(t) = f(t)$ for some $t \in [0, 1]\}$ (touching). Then $\mathbb{P}(T \cap C^c) = 0$.*

Lemma 2.5. *[2, Corollary 2.10]. Let U be an open subset of $C([0, 1])$, which contains a function f such that $f(0) = f(1) = 0$. If $B : [0, 1] \to \mathbb{R}$ is a standard Brownian bridge then $\mathbb{P}(B[0, 1] \subset U) > 0$.*

The following definition introduces the notion of an (f, g)-avoiding Brownian line ensemble, which in simple terms is a collection of k independent Brownian bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two continuous functions f and g.

Definition 2.6. Let $k \in \mathbb{N}$ and W_k° denote the open Weyl chamber in $\mathbb{R}^k$, i.e.

$$W_k^\circ = \{\vec{x} = (x_1, \dots, x_k) \in \mathbb{R}^k : x_1 > x_2 > \cdots > x_k\}$$

(in [2] the notation $\mathbb{R}^k_>$ was used for this set). Let $\vec{x}, \vec{y} \in W_k^\circ$, $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to (-\infty, \infty]$ and $g : [a,b] \to [-\infty, \infty)$ be two continuous functions. The latter condition means that either $f : [a,b] \to \mathbb{R}$ is continuous or $f = \infty$ everywhere, and similarly for g. We also assume that $f(t) > g(t)$ for all $t \in [a,b]$, $f(a) > x_1, f(b) > y_1$ and $g(a) < x_k, g(b) < y_k$.

With the above data we define the *(f,g)-avoiding Brownian line ensemble on the interval* $[a,b]$ *with entrance data $\vec{x}$ and exit data $\vec{y}$* to be the Σ-indexed line ensemble $\mathcal{Q}$ with $\Sigma = [\![1,k]\!]$ on $\Lambda = [a,b]$ and with the law of $\mathcal{Q}$ equal to $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}$ (the law of k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$) conditioned on the event

$$E = \{f(r) > B_1(r) > B_2(r) > \cdots > B_k(r) > g(r) \text{ for all } r \in [a,b]\}.$$

It is worth pointing out that E is an open set of positive measure and so we can condition on it in the usual way – we explain this briefly in the following paragraph. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that supports k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1. Notice that we can find $\tilde{u}_1, \dots, \tilde{u}_k \in C([0,1])$ and $\epsilon > 0$ (depending on $\vec{x}, \vec{y}, f, g, a, b$) such that $\tilde{u}_i(0) = \tilde{u}_i(1) = 0$ for $i = 1, \dots, k$ and such that if $\tilde{h}_1, \dots, \tilde{h}_k \in C([0,1])$ satisfy $\tilde{h}_i(0) = \tilde{h}_i(1) = 0$ for $i = 1, \dots, k$ and $\sup_{t \in [0,1]} |\tilde{u}_i(t) - \tilde{h}_i(t)| < \epsilon$ then the functions

$$h_i(t) = (b-a)^{1/2} \cdot \tilde{h}_i\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot x_i + \left(\frac{t-a}{b-a}\right) \cdot y_i,$$

satisfy $f(r) > h_1(r) > \cdots > h_k(r) > g(r)$. It follows from Lemma 2.5 that

$$\mathbb{P}(E) \geq \mathbb{P}\left(\max_{1 \leq i \leq k} \sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) = \prod_{i=1}^k \mathbb{P}\left(\sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) > 0,$$

and so we can condition on the event E.

To construct a realization of $\mathcal{Q}$ we proceed as follows. For $\omega \in E$ we define

$$\mathcal{Q}(\omega)(i,r) = B_i(r)(\omega) \text{ for } i = 1, \dots, k \text{ and } r \in [a,b].$$

Observe that for $i \in \{1, \dots, k\}$ and an open set $U \in C([a,b])$ we have that

$$\mathcal{Q}^{-1}(\{i\} \times U) = \{B_i \in U\} \cap E \ \in \ \mathcal{F},$$

and since the sets $\{i\} \times U$ form an open basis of $C([\![1,k]\!] \times [a,b])$ we conclude that $\mathcal{Q}$ is $\mathcal{F}$-measurable. This implies that the law $\mathcal{Q}$ is indeed well-defined and also it is non-intersecting almost surely. Also, given measurable subsets $A_1, \dots, A_k$ of $C([a,b])$ we have that

$$\mathbb{P}(\mathcal{Q}_i \in A_i \text{ for } i = 1, \dots, k) = \frac{\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}\left(\{B_i \in A_i \text{ for } i = 1, \dots, k\} \cap E\right)}{\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}(E)}.$$

We denote the probability distribution of $\mathcal{Q}$ as $\mathbb{P}^{a,b,\vec{x},\vec{y},f,g}_{avoid}$ and write $\mathbb{E}^{a,b,\vec{x},\vec{y},f,g}_{avoid}$ for the expectation with respect to this measure.

The following definition introduces the notion of the Brownian Gibbs property from [2].

Definition 2.7. Fix a set $\Sigma = [\![1,N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and an interval $\Lambda \subset \mathbb{R}$ and let $K = \{k_1, k_1+1, \dots, k_2\} \subset \Sigma$ be finite and $a, b \in \Lambda$ with $a < b$. Set $f = \mathcal{L}_{k_1-1}$ and $g = \mathcal{L}_{k_2+1}$ with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$ and $g = -\infty$ if $k_2 + 1 \notin \Sigma$. Write $D_{K,a,b} = K \times (a,b)$ and $D^c_{K,a,b} = (\Sigma \times \Lambda) \setminus D_{K,a,b}$. A Σ-indexed line ensemble $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$ is said to have the *Brownian Gibbs property* if it is non-intersecting and

$$\mathrm{Law}\left(\mathcal{L}|_{K \times [a,b]} \text{ conditional on } \mathcal{L}|_{D^c_{K,a,b}}\right) = \mathrm{Law}(\mathcal{Q}),$$

where $\mathcal{Q}_i = \tilde{\mathcal{Q}}_{i-k_1+1}$ and $\tilde{\mathcal{Q}}$ is the (f,g)-avoiding Brownian line ensemble on $[a,b]$ with entrance data $(\mathcal{L}_{k_1}(a),\dots,\mathcal{L}_{k_2}(a))$ and exit data $(\mathcal{L}_{k_1}(b),\dots,\mathcal{L}_{k_2}(b))$ from Definition 2.6. Note that $\tilde{Q}$ is introduced because, by definition, any such (f,g)-avoiding Brownian line ensemble is indexed from 1 to $k_2 - k_1 + 1$ but we want $\mathcal{Q}$ to be indexed from k_1 to k_2.

A more precise way to express the Brownian Gibbs property is as follows. A Σ-indexed line ensemble $\mathcal{L}$ on Λ satisfies the Brownian Gibbs property if and only if it is non-intersecting and for any finite $K = \{k_1, k_1+1, \dots, k_2\} \subset \Sigma$ and $[a,b] \subset \Lambda$ and any bounded Borel-measurable function $F : C(K \times [a,b]) \to \mathbb{R}$ we have $\mathbb{P}$-almost surely

$$\mathbb{E}\left[F\left(\mathcal{L}|_{K\times[a,b]}\right)\big|\mathcal{F}_{ext}(K \times (a,b))\right] = \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}\big[F(\tilde{\mathcal{Q}})\big], \tag{2.3}$$

where

$$\mathcal{F}_{ext}(K \times (a,b)) = \sigma\left\{\mathcal{L}_i(s) : (i,s) \in D^c_{K,a,b}\right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K\times[a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K \times [a,b]$, $\vec{x} = (\mathcal{L}_{k_1}(a),\dots,\mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b),\dots,\mathcal{L}_{k_2}(b))$, $f = \mathcal{L}_{k_1-1}[a,b]$ (the restriction of $\mathcal{L}$ to the set $\{k_1-1\} \times [a,b]$) with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$, and $g = \mathcal{L}_{k_2+1}[a,b]$ with the convention that $g = -\infty$ if $k_2 + 1 \notin \Sigma$.

Remark 2.8. Let us briefly explain why equation (2.3) makes sense. Firstly, since $\Sigma \times \Lambda$ is locally compact, we know by [8, Lemma 46.4] that $\mathcal{L} \to \mathcal{L}|_{K\times[a,b]}$ is a continuous map from $C(\Sigma\times\Lambda)$ to $C(K \times [a,b])$, so that the left side of (2.3) is the conditional expectation of a bounded measurable function, and is thus well-defined. A more subtle question is why the right side of (2.3) is $\mathcal{F}_{ext}(K \times (a,b))$-measurable. This question was resolved in [4, Lemma 3.4], where it was shown that the right side is measurable with respect to the σ-algebra

$$\sigma\left\{\mathcal{L}_i(s):\ i \in K \text{ and } s \in \{a,b\}, \text{ or } i \in \{k_1-1, k_2+1\} \text{ and } s \in [a,b]\right\},$$

which in particular implies the measurability with respect to $\mathcal{F}_{ext}(K \times (a,b))$.

In the present paper it is convenient for us to use the following modified version of the definition above, which we call the partial Brownian Gibbs property – it was first introduced in [4]. We explain the difference between the two definitions, and why we prefer the second one in Remark 2.11.

Definition 2.9. Fix a set $\Sigma = [\![1,N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and an interval $\Lambda \subset \mathbb{R}$. A Σ-indexed line ensemble $\mathcal{L}$ on Λ is said to satisfy the *partial Brownian Gibbs property* if and only if it is non-intersecting and for any finite $K = \{k_1, k_1+1,\dots,k_2\} \subset \Sigma$ with $k_2 \leq N-1$ (if $\Sigma \neq \mathbb{N}$), $[a,b] \subset \Lambda$ and any bounded Borel-measurable function $F : C(K \times [a,b]) \to \mathbb{R}$ we have $\mathbb{P}$-almost surely

$$\mathbb{E}\left[F(\mathcal{L}|_{K\times[a,b]})\big|\mathcal{F}_{ext}(K \times (a,b))\right] = \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}\big[F(\tilde{\mathcal{Q}})\big], \tag{2.4}$$

where we recall that $D_{K,a,b} = K \times (a,b)$ and $D^c_{K,a,b} = (\Sigma \times \Lambda) \setminus D_{K,a,b}$, and

$$\mathcal{F}_{ext}(K \times (a,b)) = \sigma\left\{\mathcal{L}_i(s) : (i,s) \in D^c_{K,a,b}\right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K\times[a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K \times [a,b]$, $\vec{x} = (\mathcal{L}_{k_1}(a),\dots,\mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b),\dots,\mathcal{L}_{k_2}(b))$, $f = \mathcal{L}_{k_1-1}[a,b]$ with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$, and $g = \mathcal{L}_{k_2+1}[a,b]$.

Remark 2.10. Observe that if $N = 1$ then the conditions in Definition 2.9 become void. I.e., any line ensemble with one line satisfies the partial Brownian Gibbs property. Also we mention that (2.4) makes sense by the same reason that (2.3) makes sense, see Remark 2.8.

Remark 2.11. Definition 2.9 is slightly different from the Brownian Gibbs property of Definition 2.7 as we explain here. Assuming that $\Sigma = \mathbb{N}$ the two definitions are equivalent. However, if $\Sigma = \{1,\dots,N\}$ with $1 \leq N < \infty$ then a line ensemble that satisfies the Brownian Gibbs property also satisfies the partial Brownian Gibbs property, but the reverse need not be true. Specifically, the Brownian Gibbs property allows for the possibility that $k_2 = N$ in Definition 2.9 and in this case the convention is that $g = -\infty$. As the partial Brownian Gibbs property is more general we prefer to work with it and most of the results later in this paper are formulated in terms of it rather than the usual Brownian Gibbs property.

2.2. **Bernoulli Gibbsian line ensembles.** In this section we introduce the notion of a *Bernoulli line ensemble* and the *Schur Gibbs property*. Our discussion will parallel that of [1, Section 3.1], which in turn goes back to [3, Section 2.1].

Definition 2.12. Let $\Sigma \subset \mathbb{Z}$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. Consider the set Y of functions $f : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(j, i+1) - f(j,i) \in \{0,1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$ and let $\mathcal{D}$ denote the discrete topology on Y. We call a function $f : [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(i+1) - f(i) \in \{0,1\}$ when $i \in [\![T_0, T_1 - 1]\!]$ an *up-right path* and elements in Y *collections of up-right paths.*

A Σ-*indexed Bernoulli line ensemble* $\mathfrak{L}$ *on* $[\![T_0, T_1]\!]$ is a random variable defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, taking values in Y such that $\mathfrak{L}$ is a $(\mathcal{B}, \mathcal{D})$-measurable function.

Remark 2.13. In [1, Section 3.1] Bernoulli line ensembles $\mathfrak{L}$ were called *discrete line ensembles* in order to distinguish them from the continuous line ensembles from Definition 2.1. In this paper we have opted to use the term Bernoulli line ensembles to emphasize the fact that the functions $f \in Y$ satisfy the property that $f(j, i+1) - f(j,i) \in \{0,1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$. This condition essentially means that for each $j \in \Sigma$ the function $f(j, \cdot)$ can be thought of as the trajectory of a Bernoulli random walk from time T_0 to time T_1. As other types of discrete line ensembles, see e.g. [9], have appeared in the literature we have decided to modify the notation in [1, Section 3.1] so as to avoid any ambiguity.

The way we think of Bernoulli line ensembles is as random collections of up-right paths on the integer lattice, indexed by Σ (see Figure 1). Observe that one can view an up-right path L on $[\![T_0, T_1]\!]$ as a continuous curve by linearly interpolating the points $(i, L(i))$. This allows us to define $(\mathfrak{L}(\omega))(i, s)$ for non-integer $s \in [T_0, T_1]$ and to view Bernoulli line ensembles as line ensembles in the sense of Definition 2.1. In particular, we can think of $\mathfrak{L}$ as a random variable taking values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ with $\Lambda = [T_0, T_1]$. We will often slightly abuse notation and write $\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$, even

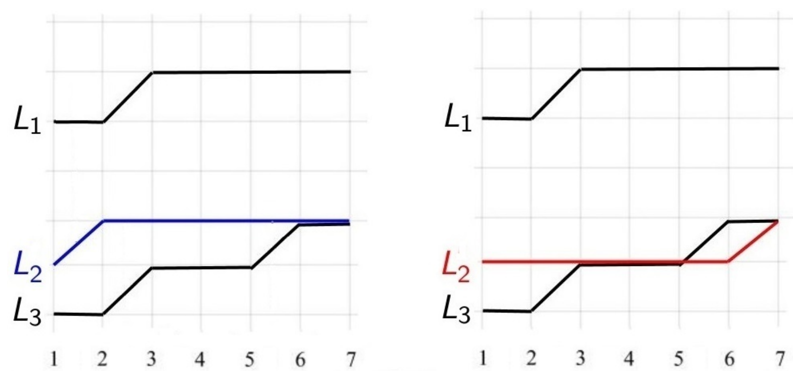

FIGURE 1. Two samples of $\{1,2,3\}$-indexed Bernoulli line ensembles with $T_0 = 1$ and $T_1 = 7$.

though it is not $\mathfrak{L}$ which is such a function, but rather $\mathfrak{L}(\omega)$ for each $\omega \in \Omega$. Furthermore we write $L_i = (\mathfrak{L}(\omega))(i, \cdot)$ for the index $i \in \Sigma$ path. If L is an up-right path on $[\![T_0, T_1]\!]$ and $a, b \in [\![T_0, T_1]\!]$ satisfy $a < b$ we let $L[\![a, b]\!]$ denote the resitrction of L to $[\![a, b]\!]$.

Let $t_i, z_i \in \mathbb{Z}$ for $i = 1, 2$ be given such that $t_1 < t_2$ and $0 \le z_2 - z_1 \le t_2 - t_1$. We denote by $\Omega(t_1, t_2, z_1, z_2)$ the collection of up-right paths that start from (t_1, z_1) and end at (t_2, z_2), by

$\mathbb{P}_{Ber}^{t_1,t_2,z_1,z_2}$ the uniform distribution on $\Omega(t_1,t_2,z_1,z_2)$ and write $\mathbb{E}_{Ber}^{t_1,t_2,z_1,z_2}$ for the expectation with respect to this measure. One thinks of the distribution $\mathbb{P}_{Ber}^{t_1,t_2,z_1,z_2}$ as the law of a simple random walk with i.i.d. Bernoulli increments with parameter $p \in (0,1)$ that starts from z_1 at time t_1 and is conditioned to end in z_2 at time t_2 – this interpretation does not depend on the choice of $p \in (0,1)$. Notice that by our assumptions on the parameters the state space $\Omega(t_1,t_2,z_1,z_2)$ is non-empty.

Given $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$ and $\vec{x}, \vec{y} \in \mathbb{Z}^k$ we let $\mathbb{P}_{Ber}^{T_0,T_1,\vec{x},\vec{y}}$ denote the law of k independent Bernoulli bridges $\{B_i : [\![T_0,T_1]\!] \to \mathbb{Z}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$. Equivalently, this is just k independent random up-right paths $B_i \in \Omega(T_0,T_1,x_i,y_i)$ for $i = 1,\dots,k$ that are uniformly distributed. This measure is well-defined provided that $\Omega(T_0,T_1,x_i,y_i)$ are non-empty for $i = 1,\dots,k$, which holds if $T_1 - T_0 \geq y_i - x_i \geq 0$ for all $i = 1,\dots,k$.

The following definition introduces the notion of an (f,g)-avoiding Bernoulli line ensemble, which in simple terms is a collection of k independent Bernoulli bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two functions f and g.

Definition 2.14. Let $k \in \mathbb{N}$ and $\mathfrak{W}_k$ denote the set of signatures of length k, i.e.

$$\mathfrak{W}_k = \{\vec{x} = (x_1,\dots,x_k) \in \mathbb{Z}^k : x_1 \geq x_2 \geq \cdots \geq x_k\}.$$

Let $\vec{x}, \vec{y} \in \mathfrak{W}_k$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, and $f : [\![T_0,T_1]\!] \to (-\infty,\infty]$ and $g : [\![T_0,T_1]\!] \to [-\infty,\infty)$ be two functions.

With the above data we define the (f,g)-*avoiding Bernoulli line ensemble on the interval* $[\![T_0,T_1]\!]$ *with entrance data* $\vec{x}$ *and exit data* $\vec{y}$ to be the Σ-indexed Bernoulli line ensemble $\mathfrak{Q}$ with $\Sigma = [\![1,k]\!]$ on $[\![T_0,T_1]\!]$ and with the law of $\mathfrak{Q}$ equal to $\mathbb{P}_{Ber}^{T_0,T_1,\vec{x},\vec{y}}$ (the law of k independent uniform up-right paths $\{B_i : [\![T_0,T_1]\!] \to \mathbb{R}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$) conditioned on the event

$$E = \{f(r) \geq B_1(r) \geq B_2(r) \geq \cdots \geq B_k(r) \geq g(r) \text{ for all } r \in [\![T_0,T_1]\!]\}.$$

The above definition is well-posed if there exist $B_i \in \Omega(T_0,T_1,x_i,y_i)$ for $i = 1,\dots,k$ that satisfy the conditions in E (i.e. if the set of such up-right paths is not empty). We will denote by $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},f,g)$ the set of collections of k up-right paths that satisfy the conditions in E and then the distribution on $\mathfrak{Q}$ is simply the uniform measure on $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},f,g)$. We denote the probability distribution of $\mathfrak{Q}$ as $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ and write $\mathbb{E}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ for the expectation with respect to this measure.

It will be useful to formulate simple conditions under which $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},f,g)$ is non-empty and thus $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ well-defined. We accomplish this in the following lemma, whose proof is postponed until Section [Appendix].

Lemma 2.15. *Suppose that* $k \in \mathbb{N}$ *and* $T_0, T_1 \in \mathbb{Z}$ *with* $T_0 < T_1$. *Suppose further that* $\vec{x}, \vec{y} \in \mathfrak{W}_k$ *satisfy* $T_1 - T_0 \geq y_i - x_i \geq 0$ *for* $i = 1,\dots,k$. *Suppose further that* $f : [\![T_0,T_1]\!] \to (-\infty,\infty]$ *and* $g : [\![T_0,T_1]\!] \to [-\infty,\infty)$ *satisfy* $f(i+1) = f(i)$ *or* $f(i+1) = f(i)+1$, *and* $g(i+1) = g(i)$ *or* $g(i+1) = g(i)+1$ *for* $i = T_0,\dots,T_1-1$. *Finally, suppose that* $f(T_0) \geq x_1, f(T_1) \geq y_1$ *and* $g(T_0) \leq x_k, g(T_1) \leq y_k$. *Then the set* $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},f,g)$ *from Definition 2.14 is non-empty.*

The following definition introduces the notion of the Schur Gibbs property, which can be thought of a discrete analogue of the partial Brownian Gibbs property the same way that Bernoulli random walks are discrete analogues of Brownian motion.

Definition 2.16. Fix a set $\Sigma = [\![1,N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. A Σ-indexed Bernoulli line ensemble $\mathfrak{L} : \Sigma \times [\![T_0,T_1]\!] \to \mathbb{Z}$ is said to satisfy the *Schur Gibbs property* if it is non-crossing, meaning that

$$L_j(i) \geq L_{j+1}(i) \text{ for all } j = 1,\dots,N-1 \text{ and } i \in [\![T_0,T_1]\!],$$

and for any finite $K = \{k_1, k_1+1, \dots, k_2\} \subset [\![1, N-1]\!]$ and $a, b \in [\![T_0, T_1]\!]$ with $a < b$ the following holds. Suppose that f, g are two up-right paths drawn in $\{(r,z) \in \mathbb{Z}^2 : a \leq r \leq b\}$ and $\vec{x}, \vec{y} \in \mathfrak{W}_{k_2-k_1+1}$ altogether satisfy that $\mathbb{P}(A) > 0$ where A denotes the event

$$A = \{(L_{k_1}(a), \dots, L_{k_2}(a)), \vec{y} = (L_{k_1}(b), \dots, L_{k_2}(b)), L_{k_1-1}[\![a,b]\!] = f, L_{k_2+1}[\![a,b]\!] = g\},$$

where if $k_1 = 1$ we adopt the convention $f = \infty = L_0$. Then for any $\{B_i \in \Omega(a, b, x_i, y_i)\}_{i=1}^{k_2-k_1+1}$

$$\mathbb{P}\left(L_{i+k_1-1}[\![a,b]\!] = B_i \text{ for } i = 1, \dots, k_2-k_1+1 | A\right) = \mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}\left(\cap_{i=1}^{k}\{\mathfrak{Q}_i = B_i\}\right). \tag{2.5}$$

Remark 2.17. In simple words, a Bernoulli line ensemble is said to satisfy the Schur Gibbs property if the distribution of any finite number of consecutive paths, conditioned on their end-points and the paths above and below them is simply the uniform measure on all collection of up-right paths that have the same end-points and do not cross each other or the paths above and below them.

Remark 2.18. Observe that in Definition 2.16 the index k_2 is assumed to be less than or equal to $N-1$, so that if $N < \infty$ the N-th path is special and is not conditionally uniform. This is what makes Definition 2.16 a discrete analogue of the partial Brownian Gibbs property rather than the usual Brownian Gibbs property. Similarly to the partial Brownian Gibbs propert, see Remark 2.10, if $N = 1$ then the conditions in Definition 2.16 become void. I.e., any Bernoulli line ensemble with one line satisfies the Schur Gibbs property. Also we mention that the well-posedness of $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ in (2.5) is a consequence of Lemma 2.15 and our assumption that $\mathbb{P}(A) > 0$.

Remark 2.19. In [1] the authors studied a generalization of the Gibbs property in Definition 2.16 depending on a parameter $t \in (0,1)$, which was called the *Hall-Littlewood Gibbs property* due to its connection to Hall-Littlewood polynomials [7]. The property in Definition 2.16 is the $t \to 0$ limit of the Hall-Littlewood Gibbs property. Since under this $t \to 0$ limit Hall-Littlewood polynomials degenerate to Schur polynomials we have decided to call the Gibbs property in Definition 2.16 the Schur Gibbs property.

Remark 2.20. An immediate consequence of Definition 2.16 is that if $M \leq N$, we have that the induced law on $\{L_i\}_{i=1}^{M}$ also satisfies the Schur Gibbs property as an $\{1, \dots, M\}$-indexed Bernoulli line ensemble on $[\![T_0, T_1]\!]$.

We end this section with the following definition of the term acceptance probability.

Definition 2.21. Assume the same notation as in Definition 2.14 and suppose that $T_1 - T_0 \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$. We define the *acceptance probability* $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$ to be the ratio

$$Z(T_0, T_1, \vec{x}, \vec{y}, f, g) = \frac{|\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)|}{\prod_{i=1}^{k} |\Omega(T_0, T_1, x_i, y_i)|}. \tag{2.6}$$

Remark 2.22. The quantity $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is precisely the probability that if B_i are sampled uniformly from $\Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ then the B_i satisfy the condition

$$E = \{f(r) \geq B_1(r) \geq B_2(r) \geq \cdots \geq B_k(r) \geq g(r) \text{ for all } r \in [\![T_0, T_1]\!]\}.$$

Let us explain briefly why we call this quantity an acceptance probability. One way to sample $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ is as follows. Start by sampling a sequence of i.i.d. up-right paths B_i^N uniformly from $\Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ and $N \in \mathbb{N}$. For each n check if $B_1^n, \dots, B_k^n$ satisfy the condition E and let M denote the smallest index that accomplishes this. If $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is non-empty then M is geometrically distributed with parameter $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$, and in particular M is finite almost surely and $\{B_i^M\}_{i=1}^{k}$ has distribution $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$. In this sampling procedure we construct a sequence of candidates $\{B_i^N\}_{i=1}^{k}$ for $N \in \mathbb{N}$ and reject those that fail to satisfy condition E, the first candidate that satisfies it is accepted and has law $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ and the probability that a candidate is accepted is precisely $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$, which is why we call it an acceptance probability.

2.3. **Main result.** In this section we present the main result of the paper. We start with the following technical definition.

Definition 2.23. Fix $k \in \mathbb{N}$, $\alpha, \lambda > 0$ and $p \in (0,1)$. Suppose we are given a sequence $\{T_N\}_{N=1}^{\infty}$ with $T_N \in \mathbb{N}$ and that $\{\mathfrak{L}^N\}_{N=1}^{\infty}$, $\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)$ is a sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N, T_N]\!]$. We call the sequence (α, p, λ)-*good* if

- for each $N \in \mathbb{N}$ we have that $\mathfrak{L}^N$ satisfies the Schur Gibbs property of Definition 2.16;
- there is a function $\psi : \mathbb{N} \to (0, \infty)$ such that $\lim_{N\to\infty} \psi(N) = \infty$ and for each $N \in \mathbb{N}$ we have that $T_N > \psi(N) N^{\alpha}$;
- there is a function $\phi : (0,\infty) \to (0,\infty)$ such that for any $\epsilon > 0$ we have

$$\sup_{n\in\mathbb{Z}} \limsup_{N\to\infty} \mathbb{P}\left(\left| N^{-\alpha/2}(L_1^N(nN^{\alpha}) - pnN^{\alpha} + \lambda n^2 N^{\alpha/2}) \right| \geq \phi(\epsilon) \right) \leq \epsilon. \tag{2.7}$$

Remark 2.24. Let us elaborate on the meaning of Definition 2.23. In order for a sequence of $\mathfrak{L}^N$ of $[\![1,k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N, T_N]\!]$ to be (α, p, λ)-*good* we want several conditions to be satisfied. Firstly, we want for each N the Bernoulli line ensemble $\mathfrak{L}^N$ to satisfy the Schur Gibbs property. The second condition is that while the interval of definition of $\mathfrak{L}^N$ is finite for each N and given by $[\![-T_N, T_N]\!]$, we want this interval to grow at least with speed N^{α}. This property is quantified by the function ψ, which can be essentially thought of as an arbitrary unbounded increasing function on $\mathbb{N}$. The third condition is that we want for each $n \in \mathbb{Z}$ the sequence of random variables $N^{-\alpha/2}(L_1^N(nN^{\alpha}) - pnN^{\alpha})$ to be tight but moreover we want globally these random variables to look like the parabola $-\lambda n^2$. This statement is reflected in (2.7), which provides a certain uniform tightness of the random variables $N^{-\alpha/2}(L_1^N(nN^{\alpha}) - pnN^{\alpha} + \lambda n^2 N^{\alpha/2})$. A particular case when (2.7) is satisfied is for example if we know that for each $n \in \mathbb{Z}$ the random variables $N^{-\alpha/2}(L_1^N(nN^{\alpha}) - pnN^{\alpha} + \lambda n^2 N^{\alpha/2})$ converge to the same random variable X. In the applications that we have in mind these random variables would converge to the 1-point marginals of the Airy$_2$ process that are all given by the same Tracy-Widom distribution (since the Airy$_2$ process is stationary). Equation (2.7) is a significant relaxation of the requirement that $N^{-\alpha/2}(L_1^N(nN^{\alpha}) - pnN^{\alpha} + \lambda n^2 N^{\alpha/2})$ all converge weakly to the Tracy-Widom distribution – the convergence requirement is replaced with a mild but uniform control of all subsequential limits.

The main result of the paper is as follows.

Theorem 2.25. *Fix $k \in \mathbb{N}$ with $k \geq 2$, $\alpha, \lambda > 0$ and $p \in (0,1)$ and let $\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)$ be an (α, p, λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles. Set*

$$f_i^N(s) = N^{-\alpha/2}(L_i^N(sN^{\alpha}) - psN^{\alpha} + \lambda s^2 N^{\alpha/2}), \text{ for } s \in [-\psi(N), \psi(N)] \text{ and } i = 1, \dots, k-1,$$

and extend f_i^N to $\mathbb{R}$ by setting for $i = 1, \dots, k-1$

$$f_i^N(s) = f_i^N(-\psi(N)) \text{ for } s \leq -\psi(N) \text{ and } f_i^N(s) = f_N(\psi(N)) \text{ for } s \geq \psi(N).$$

Let $\mathbb{P}_N$ denote the law of $\{f_i^N\}_{i=1}^{k-1}$ as a $[\![1,k-1]\!]$-indexed line ensemble (i.e. as a random variable in $(C([\![1,k-1]\!] \times \mathbb{R}), \mathcal{C})$). Then the sequence $\mathbb{P}_N$ is tight.

Roughly, Theorem 2.25 states that if you have a sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles that satisfy the Schur Gibbs property and the top paths of these ensembles under some shift and scaling have tight one-point marginals with a non-trivial parabolic shift, then under the same shift and scaling the top $k-1$ paths of the line ensemble will be tight. The extension of f_i^N to $\mathbb{R}$ is completely arbitrary and irrelevant for the validity of Theorem 2.25 since the topology on $C([\![1,k-1]\!] \times \mathbb{R})$ is that of uniform convergence over compacts. Consequently, only the behavior of these functions on compact intervals matters in Theorem 2.25 and not what these functions do near infinity, which is where the modification happens as $\lim_{N\to\infty} \psi(N) = \infty$ by assumption. The only reason we perform the extension is to embed all Bernoulli line ensembles into the same space $(C([\![1,k-1]\!] \times \mathbb{R}), \mathcal{C})$.

We mention that the k-th up-right path in the sequence of Bernoulli line ensembles is special and Theorem 2.25 provides no tightness result for it. The reason for this stems from the Schur Gibbs property, see Definition 2.16, which assumes less information for the k-th path. In practice, one either has an infinite Bernoulli line ensemble for each N or one has a Bernoulli line ensemble with finite number of paths, which increase with N to infinity. In either of these settings one can use Theorem 2.25 to prove teightness of the full line ensemble - we will have more to say about this in Section [Applications].

The proof of Theorem 2.25 is presented in Section 4. In the next section we derive various properties for Bernoulli line ensembles.

3. Properties of Bernoulli line ensembles

In this section we derive several results for non-intersecting Bernoulli bridges, which will be used in the proof of Theorem 2.25 in Section 4.

3.1. **Monotone coupling lemmas.** In this section we formulate two lemmas that provide couplings of two Bernoulli line ensembles of non-intersecting Bernoulli bridges on the same interval, which depend monotonically on their boundary data. Schematic depictions of the couplings are provided in Figure 2. We postpone the proof of these lemmas until Section [Appendix].

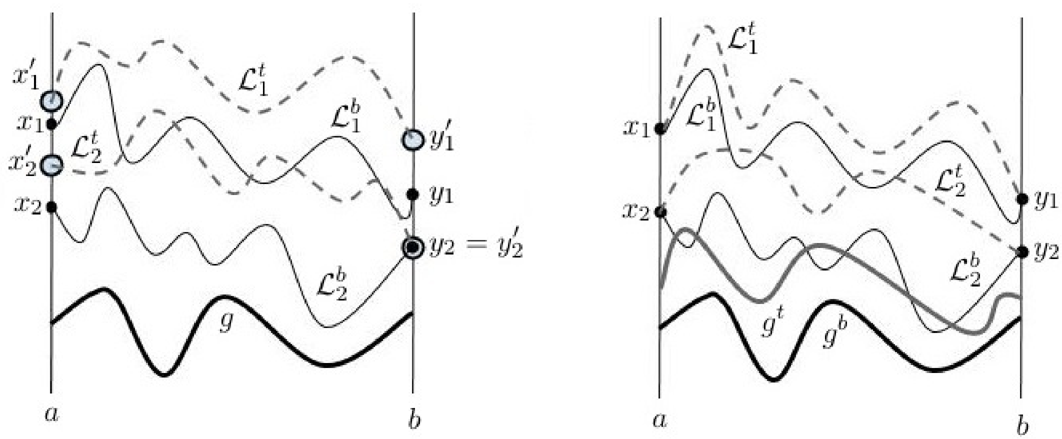

Figure 2. Two diagrammatic depictions of the monotone coupling Lemma 3.1 (left part) and Lemma 3.2 (right part).

Lemma 3.1. *Assume the same notation as in Definition 2.14. Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, a function $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ as well as $\vec{x}, \vec{y}, \vec{x}', \vec{y}' \in \mathfrak{W}_k$. Assume that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}', \vec{y}', \infty, g)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\big($resp. $\mathfrak{L}^b\big)$ under $\mathbb{P}$ is given by $\mathbb{P}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g}_{avoid,Ber}$ $\big($resp. $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber}\big)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}^t_i(r) \geq \mathfrak{L}^b_i(r)$ for all $i = 1, \dots, k$ and $r \in [\![T_0, T_1]\!]$.*

Lemma 3.2. *Assume the same notation as in Definition 2.14. Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, two functions $g^t, g^b : [\![T_0, T_1]\!] \to [-\infty, \infty)$ and $\vec{x}, \vec{y} \in \mathfrak{W}_k$. We assume that $g^t(r) \geq g^b(r)$ for all $r \in [\![T_0, T_1]\!]$ and that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^t)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\big($resp. $\mathfrak{L}^b\big)$ under $\mathbb{P}$ is given by $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^t}_{avoid,Ber}$ $\big($resp. $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber}\big)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}^t_i(r) \geq \mathfrak{L}^b_i(r)$ for all $i = 1, \dots, k$ and $r \in [\![T_0, T_1]\!]$.*

In plain words, Lemma 3.1 states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ of non-intersecting Bernoulli bridges, bounded from below by the same function g, in such a way that if all boundary values of $\mathfrak{L}^t$ are above the respective boundary values of $\mathfrak{L}^b$, then all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the left part of Figure 2. Lemma 3.2, states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ that have the same boundary values, but the lower bound g^t of $\mathfrak{L}^t$ is above the lower bound g^b of $\mathfrak{L}^b$, in such a way that all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the right part of Figure 2.

3.2. **Properties of Bernoulli bridges.** In this section we derive several results about Bernoulli bridges, which are random up-right paths that have law $\mathbb{P}^{T_0,T_1,x,y}_{Ber}$ as in Section 2.2. Our results will rely on the two monotonicity Lemmas 3.1 and 3.2 as well as a strong coupling between Bernoulli bridges and Brownian bridges from [1] – recalled here as Theorem 3.3.

If W_t denotes a standard one-dimensional Brownian motion and $\sigma > 0$, then the process

$$B^\sigma_t = \sigma(W_t - tW_1), \quad 0 \le t \le 1,$$

is called a *Brownian bridge (conditioned on $B_0 = 0, B_1 = 0$) with variance σ^2*. With the above notation we state the strong coupling result we use.

Theorem 3.3. *Let $p \in (0,1)$. There exist constants $0 < C, a, \alpha < \infty$ (depending on p) such that for every positive integer n, there is a probability space on which are defined a Brownian bridge B^σ with variance $\sigma^2 = p(1-p)$ and a family of random paths $\ell^{(n,z)} \in \Omega(0,n,0,z)$ for $z = 0,\dots,n$ such that $\ell^{(n,z)}$ has law $\mathbb{P}^{0,n,0,z}_{Ber}$ and*

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \le Ce^{\alpha(\log n)^2}e^{|z-pn|^2/n}, \text{ where } \Delta(n,z) := \sup_{0\le t\le n}\left|\sqrt{n}B^\sigma_{t/n} + \tfrac{t}{n}z - \ell^{(n,z)}(t)\right|. \tag{3.1}$$

Remark 3.4. When $p = 1/2$ the above theorem follows (after a trivial affine shift) from [6, Theorem 6.3] and the general $p \in (0,1)$ case was done in [1, Theorem 4.5]. We mention that a significant generalization of Theorem 3.3 for general random walk bridges has recently been proved in [5, Theorem 2.3].

Below we list five lemmas about Bernoulli bridges. We provide a brief informal explanation of what each result says after it is stated. All five lemmas are proved in a similar fashion. For the first four lemmas one observes that the event, whose probability is being estimated, is monotone in ℓ. This allows by Lemmas 3.1 and 3.2 to replace x, y in the statements of the lemmas with the extreme values of the ranges specified in each. Once the choice of x and y is fixed one can use our strong coupling result, Theorem 3.3, to reduce each of the lemmas to an analogous one involving a Brownian bridge with some prescribed variance. The latter statements are then easily confirmed as one has exact formulas for Brownian bridges.

Lemma 3.5. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \ge y - x \ge 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,x,y}_{Ber}$. Let $M_1, M_2 \in \mathbb{R}$ be given. Then we can find $W_0 = W_0(p, M_2 - M_1) \in \mathbb{N}$ such that for $T \ge W_0$, $x \ge M_1T^{1/2}$, $y \ge pT + M_2T^{1/2}$ and $s \in [0,T]$ we have*

$$\mathbb{P}^{0,T,x,y}_{Ber}\Big(\ell(s) \ge \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\big(pT + M_2T^{1/2}\big) - T^{1/4}\Big) \ge \frac{1}{3}. \tag{3.2}$$

Remark 3.6. If $M_1, M_2 = 0$ then Lemma 3.5 states that if a Bernoulli bridge ℓ is started from $(0,x)$ and terminates at (T,y), which are above the straight line of slope p, then at any given time $s \in [0,T]$ the probability that $\ell(s)$ goes a modest distance below the straight line of slope p is upper bounded by $2/3$.

Proof. Define $A = \lfloor MT_1^{1/2} \rfloor$ and $B = \lfloor pT + M_2T^{1/2} \rfloor$. Then since $A \leq x$ and $B \leq y$, it follows from Lemma 3.1 that there is a probability space with measure $\mathbb{P}_0$ supporting random variables $\mathfrak{L}_1$ and $\mathfrak{L}_2$, whose laws under $\mathbb{P}_0$ are $\mathbb{P}_{Ber}^{0,T,A,B}$ and $\mathbb{P}_{Ber}^{0,T,x,y}$ respectively, and $\mathbb{P}_0$-a.s. we have $\mathfrak{L}_1 \leq \mathfrak{L}_2$. Thus

$$\begin{aligned}
&\mathbb{P}_{Ber}^{0,T,x,y}\Big(\ell(s) \geq \frac{T-s}{T} \cdot M_1T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2T^{1/2}\big) - T^{1/4}\Big) \\
&= \mathbb{P}_0\Big(\mathfrak{L}_2(s) \geq \frac{T-s}{T} \cdot M_1T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2T^{1/2}\big) - T^{1/4}\Big) \\
&\geq \mathbb{P}_0\Big(\mathfrak{L}_1(s) \geq \frac{T-s}{T} \cdot M_1T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2T^{1/2}\big) - T^{1/4}\Big) \\
&= \mathbb{P}_{Ber}^{0,T,A,B}\Big(\ell(s) \geq \frac{T-s}{T} \cdot M_1T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2T^{1/2}\big) - T^{1/4}\Big).
\end{aligned}$$

Since upright paths on $[\![0, T]\!] \times [\![A, B]\!]$ are equivalent to upright paths on $[\![0, T]\!] \times [\![0, B - A]\!]$ shifted vertically by A, the last line is equal to

$$\mathbb{P}_{Ber}^{0,T,0,B-A}\Big(\ell(s) + A \geq \frac{T-s}{T} \cdot M_1T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2T^{1/2}\big) - T^{1/4}\Big).$$

Now we consider the coupling provided by Theorem 3.3. We have another probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random variable $\ell^{(T,B-A)}$ whose law under $\mathbb{P}$ is that of ℓ, and a Brownian bridge B^σ. Then

$$\begin{aligned}
&\mathbb{P}_{Ber}^{0,T,0,B-A}\Big(\ell(s) + A \geq \frac{T-s}{T} \cdot M_1T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2T^{1/2}\big) - T^{1/4}\Big) \\
&= \mathbb{P}\Big(\ell^{(T,B-A)}(s) + A \geq \frac{T-s}{T} \cdot M_1T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2T^{1/2}\big) - T^{1/4}\Big) \\
&= \mathbb{P}\Big(\Big[\ell^{(T,B-A)}(s) - \sqrt{T}B^\sigma_{s/T} - \frac{s}{T} \cdot (B - A)\Big] + \sqrt{T}B^\sigma_{s/T} \geq -A - \frac{s}{T} \cdot (B - A) \\
&\qquad + \frac{T-s}{T} \cdot M_1T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2T^{1/2}\big) - T^{1/4}\Big).
\end{aligned}$$

Recalling the definitions of A and B, we can rewrite the quantity on the right hand side in the last expression and bound it by

$$\begin{aligned}
\frac{T-s}{T} \cdot (M_1T^{1/2} - A) + \frac{s}{T} \cdot (pT + M_2T^{1/2} - B) - T^{1/4} &\leq \frac{T-s}{T} + \frac{s}{T} - T^{1/4} \\
&= -T^{1/4} + 1.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{P}_{Ber}^{0,T,0,B-A}\Big(\ell(s) + A \geq \frac{T-s}{T} \cdot M_1T^{1/2} + \frac{s}{T} \cdot \big(pT + M_2T^{1/2}\big) - T^{1/4}\Big) \\
&\geq \mathbb{P}\Big(\Big[\ell^{(T,B-A)}(s) - \sqrt{T}B^\sigma_{s/T} - \frac{s}{T} \cdot (B - A)\Big] + \sqrt{T}B^\sigma_{s/T} \geq -T^{1/4} + 1\Big) \\
&\geq \mathbb{P}\Big(\sqrt{T}B^\sigma_{s/T} \geq 0 \quad \text{and} \quad \Delta(T, B - A) < T^{1/4} - 1\Big) \\
&\geq \mathbb{P}\left(B^\sigma_{s/T} \geq 0\right) - \mathbb{P}\left(\Delta(T, B - A) \geq T^{1/4} - 1\right) \\
&= \frac{1}{2} - \mathbb{P}\left(\Delta(T, B - A) \geq T^{1/4} - 1\right).
\end{aligned}$$

For the second inequality, we used the fact that the quantity in brackets is bounded in absolute value by $\Delta(T, B - A)$. The third inequality follows by splitting the event $\{B^\sigma_{s/T} \geq 0\}$ into cases and applying subadditivity. It remains to bound the second term on the last line. Applying Chebyshev's inequality and Theorem 3.3, we obtain constants C, a, α depending only on p such that

$$\begin{aligned}\mathbb{P}\left(\Delta(T, B-A) \geq T^{1/4}-1\right) &\leq e^{-a(T^{1/4}-1)}\mathbb{E}\left[e^{a\Delta(T,B-A)}\right] \\ &\leq C\exp\left[-a(T^{1/4}-1)+\alpha(\log T)^2+\frac{|B-A-pT|^2}{T}\right] \\ &\leq C\exp\left[-a(T^{1/4}-1)+\alpha(\log T)^2+(M_2-M_1)^2+\frac{1}{T}\right] \\ &= O(e^{-T^{1/4}}).\end{aligned}$$

Thus we can choose W_0 large enough, depending on p and $M_2 - M_1$, so that if $T \geq W_0$, then this probability does not exceed $1/6$. Combining this with the above inequalities completes the proof. ☐

Lemma 3.7. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $y \in \mathbb{Z}$ such that $T \geq y \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,0,y}_{Ber}$. Let $M > 0$ and $\epsilon > 0$ be given. Then we can find $W_1 = W_1(M,p,\epsilon) \in \mathbb{N}$ and $A = A(M,p,\epsilon) > 0$ such that for $T \geq W_1$, $y \geq pT - MT^{1/2}$ we have*

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(\inf_{s\in[0,T]}\big(\ell(s)-ps\big) \leq -AT^{1/2}\Big) \leq \epsilon. \tag{3.3}$$

Remark 3.8. Roughly, Lemma 3.7 states that if a Bernoulli bridge ℓ is started from $(0,0)$ and terminates at (T,y) with (T,y) not significantly lower than the straight line of slope p, then the event that ℓ goes significantly below the straight line of slope p is very unlikely.

Proof. As in the previous proof, it follows from Lemma 3.1 that

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(\inf_{s\in[0,T]}\big(\ell(s)-ps\big) \leq -AT^{1/2}\Big) \leq \mathbb{P}^{0,T,0,B}_{Ber}\Big(\inf_{s\in[0,T]}\big(\ell(s)-ps\big) \leq -AT^{1/2}\Big),$$

where $B = \lfloor pT - MT^{1/2}\rfloor$. By Theorem 3.3, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random variable $\ell^{(T,B)}$ whose law under $\mathbb{P}$ is that of ℓ, and a Brownian bridge B^σ with variance $\sigma^2 = p(1-p)$. Therefore

$$\begin{aligned}&\mathbb{P}^{0,T,0,B}_{Ber}\Big(\inf_{s\in[0,T]}\big(\ell(s)-ps\big) \leq -AT^{1/2}\Big) = \mathbb{P}\Big(\inf_{s\in[0,T]}\big(\ell^{(T,B)}(s)-ps\big) \leq -AT^{1/2}\Big) \\ &\leq \mathbb{P}\Big(\inf_{s\in[0,T]}\sqrt{T}B^\sigma_{s/T} \leq -\frac{1}{2}AT^{1/2}\Big) + \mathbb{P}\Big(\sup_{s\in[0,T]}\Big|\sqrt{T}B^\sigma_{s/T}+ps-\ell^{(T,B)}(s)\Big| \geq \frac{1}{2}AT^{1/2}\Big) \\ &\leq \mathbb{P}\Big(\max_{s\in[0,T]}\sqrt{T}B^\sigma_{s/T} \geq \frac{1}{2}AT^{1/2}\Big) + \mathbb{P}\Big(\Delta(T,B) \geq \frac{1}{2}AT^{1/2}-MT^{1/2}-1\Big).\end{aligned}$$

For the first term in the last line, we used the fact that B^σ and $-B^\sigma$ have the same distribution. For the second term, we used the fact that

$$\sup_{s\in[0,T]}\Big|ps-\frac{s}{T}\cdot B\Big| \leq \sup_{s\in[0,T]}\Big|ps-\frac{pT-MT^{1/2}}{T}\cdot s\Big|+1 = MT^{1/2}+1.$$

To estimate the first term, note that $\sqrt{T}B^\sigma_{s/T} = \sigma\sqrt{T}(W_{s/T}-W_1)$, where W is a standard Brownian motion on $[0,1]$. Hence

$$\begin{aligned}\mathbb{P}\Big(\max_{s\in[0,T]}\sqrt{T}B^\sigma_{s/T} \geq \frac{1}{2}AT^{1/2}\Big) &\leq \mathbb{P}\Big(\sigma\max_{s\in[0,T]}\sqrt{T}\,W_{s/T} \geq \frac{1}{4}AT^{1/2}\Big) + \mathbb{P}\Big(\sigma\sqrt{T}\,W_1 \leq -\frac{1}{4}AT^{1/2}\Big) \\ &= \mathbb{P}\Big(\sigma\big|\sqrt{T}\,W_{T/T}\big| \geq \frac{1}{4}AT^{1/2}\Big) + \mathbb{P}\Big(\sigma W_1 \leq -\frac{1}{4}A\Big) \\ &= 3\,\mathbb{P}\Big(W_1 \geq \frac{A}{4\sqrt{p(p-1)}}\Big).\end{aligned}$$

The equality in the second line follows from the reflection principle, since $\sqrt{T}\,W_{s/T}$ is a standard Brownian motion on $[0,T]$, and the third line follows by symmetry. Since $W_1 \sim \mathcal{N}(0,1)$, we can choose A large enough depending on p and ϵ so that this probability is bounded above by $\epsilon/2$.

For the second term, it follows from Theorem 3.3 and Chebyshev's inequality that

$$\mathbb{P}\Big(\Delta(T,B) \geq \Big(\frac{A}{2} - M\Big)T^{1/2} - 1\Big) \leq C \exp\Big[-a\Big(\frac{A}{2} - M\Big)T^{1/2} + a + \alpha(\log T)^2 + M^2 + \frac{1}{T}\Big].$$

If we take $A > 2M$, then this is $O(e^{-T^{1/2}})$, and then we can find W_1 large enough depending on M, p, ϵ so that this term is also $\leq \epsilon/2$ for $T \geq W_1$. Adding the two terms gives (3.3). ☐

Lemma 3.9. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,x,y}_{Ber}$. Let $M_1, M_2 > 0$ be given. Then we can find $W_2 = W_2(M_1,M_2,p) \in \mathbb{N}$ such that for $T \geq W_2$, $x \geq -M_1T^{1/2}$, $y \geq pT - M_1T^{1/2}$ we have*

$$\mathbb{P}^{0,T,x,y}_{Ber}\left(\ell(T/2) \geq \frac{M_2T^{1/2} + pT}{2} - T^{1/4}\right) \geq (1/2)(1 - \Phi^v(M_1 + M_2)), \tag{3.4}$$

where Φ^v is the cumulative distribution function of a Gaussian random variable with mean 0 and variance $v = p(1-p)/4$.

Remark 3.10. Lemma 3.9 states that if a Bernoulli bridge ℓ is started from $(0,x)$ and terminates at (T,y) with these points not significantly lower than the straight line of slope p, then its mid-point would lie well above the straight line of slope p at least with some quantifiably tiny probability.

Proof. We have

$$\begin{aligned}\mathbb{P}^{0,T,x,y}_{Ber}\left(\ell(T/2) \geq \frac{M_2T^{1/2} + pT}{2} - T^{1/4}\right) &\geq \mathbb{P}^{0,T,0,B-A}_{Ber}\left(\ell(T/2) + A \geq \frac{M_2T^{1/2} + pT}{2} - T^{1/4}\right)\\ &= \mathbb{P}\left(\ell^{(T,B-A)}(T/2) + A \geq \frac{M_2T^{1/2} + pT}{2} - T^{1/4}\right),\end{aligned}$$

with $A = \lfloor -M_1T^{1/2}\rfloor$, $B = \lfloor pT - M_1T^{1/2}\rfloor$, and $\mathbb{P}$, and $\ell^{(T,B-A)}$ provided by Theorem 3.3. If B^σ is as in the theorem, we can rewrite the expression on the second line as

$$\mathbb{P}\left(\left[\ell^{(T,B-A)}(T/2) - \sqrt{T}\,B^\sigma_{1/2} - \frac{B-A}{2}\right] + \sqrt{T}\,B^\sigma_{1/2} \geq -A - \frac{B-A}{2} + \frac{M_2T^{1/2} + pT}{2} - T^{1/4}\right).$$

We have

$$\begin{aligned}-A - \frac{B-A}{2} + \frac{M_2T^{1/2} + pT}{2} - T^{1/4} &\leq M_1T^{1/2} + 1 - \frac{pT-1}{2} + \frac{M_2T^{1/2} + pT}{2} - T^{1/4}\\ &\leq (M_1 + M_2)T^{1/2} - T^{1/4} + 2.\end{aligned}$$

Thus the probability in question is bounded below by

$$\begin{aligned}&\mathbb{P}\left(\left[\ell^{(T,B-A)}(T/2) - \sqrt{T}\,B^\sigma_{1/2} - \frac{B-A}{2}\right] + \sqrt{T}\,B^\sigma_{1/2} \geq (M_1 + M_2)T^{1/2} - T^{1/4} + 2\right)\\ &\geq \mathbb{P}\left(\sqrt{T}\,B^\sigma_{1/2} \geq (M_1 + M_2)T^{1/2} \quad \text{and} \quad \Delta(T, B-A) < T^{1/4} - 2\right)\\ &\geq \mathbb{P}\big(B^\sigma_{1/2} \geq M_1 + M_2\big) - \mathbb{P}\big(\Delta(T,B-A) \geq T^{1/4} - 2\big).\end{aligned}$$

Note that $B^\sigma_{1/2} = \sigma(W_{1/2} - \frac{1}{2}W_1)$ for a standard Brownian motion W on $[0,1]$. Thus $B^\sigma_{1/2}$ is Gaussian with mean 0 and variance $\sigma^2(1/2 - (1/2)^2) = \sigma^2/4$. In particular, the first term in the last line is equal to

$$1 - \Phi^v(M_1 + M_2),$$

where Φ^v is the cdf for a Gaussian random variable with mean 0 and variance $v = \sigma^2/4 = p(1-p)/4$. For the second term, the same argument as in the proof of Lemma 3.5 shows that it is $O(e^{-T^{1/4}})$. In particular, we can choose W_2 depending on M_1, M_2, and p so that the second term is less than $1/2$ the first term for $T \geq W_2$. This proves (3.4). □

Lemma 3.11. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,x,y}_{Ber}$. Then we can find $W_3 = W_3(p) \in \mathbb{N}$ such that for $T \geq W_3$, $x \geq T^{1/2}$, $y \geq pT + T^{1/2}$*

$$\mathbb{P}^{0,T,x,y}_{Ber}\Big(\inf_{s\in[0,T]}\big(\ell(s) - ps\big) + T^{1/4} \geq 0\Big) \geq \frac{1}{2}\left(1 - \exp\left(-\frac{2}{p(1-p)}\right)\right). \tag{3.5}$$

Remark 3.12. Lemma 3.11 states that if a Bernoulli bridge ℓ is started from $(0,x)$ and terminates at (T,y) with $(0,x)$ and (T,y) well above the line of slope p then at least with some positive probability ℓ will not fall significantly below the line of slope p.

Proof. We have

$$\begin{aligned}
&\mathbb{P}^{0,T,x,y}_{Ber}\Big(\inf_{s\in[0,T]}\big(\ell(s) - ps\big) + T^{1/4} \geq 0\Big)\\
&\geq \mathbb{P}^{0,T,0,B-A}_{Ber}\Big(\inf_{s\in[0,T]}\big(\ell(s) + A - ps\big) + T^{1/4} \geq 0\Big)\\
&= \mathbb{P}\Big(\inf_{s\in[0,T]}\big(\ell^{(T,B-A)}(s) - ps\big) \geq -T^{1/4} - A\Big)\\
&\geq \mathbb{P}\Big(\inf_{s\in[0,T]}\big(\ell^{(T,B-A)}(s) - \frac{s}{T}\cdot(B-A)\big) \geq -T^{1/4} - T^{1/2} + 2\Big),
\end{aligned}$$

with $A = \lfloor T^{1/2} \rfloor$, $B = \lfloor pT + T^{1/2} \rfloor$, and $\mathbb{P}$, and $\ell^{(T,B-A)}$ provided by Theorem 3.3. In the last line, we used the facts that $|A - T^{1/2}| \leq 1$ and $|p - (B-A)/T| \leq 1$. With B^σ as in the theorem, the last line is bounded below by

$$\begin{aligned}
&\mathbb{P}\Big(\inf_{s\in[0,T]} \sqrt{T}\, B^\sigma_{s/T} \geq -T^{1/2} \quad \text{and} \quad \Delta(T, B-A) < T^{1/2} - 2\Big)\\
&\geq \mathbb{P}\Big(\max_{s\in[0,T]} B^\sigma_{s/T} \leq 1\Big) - \mathbb{P}\Big(\Delta(T, B-A) \geq T^{1/2} - 2\Big).
\end{aligned}$$

To compute the first term, note that if B^1 is a Brownian bridge with variance 1 on $[0,1]$, then $\sigma\sqrt{T}\, B^1_{s/T}$ on $[0,T]$ has the same distribution as $B^\sigma_{s/T}$. Hence

$$\begin{aligned}
\mathbb{P}\Big(\max_{s\in[0,T]} \sqrt{T}\, B^\sigma_{s/T} \leq T^{1/2}\Big) &= 1 - \mathbb{P}\Big(\max_{s\in[0,T]} \sqrt{T}\, B^1_{s/T} \geq T^{1/2}/\sigma\Big) = 1 - e^{-2(T^{1/2}/\sigma)^2/T}\\
&= 1 - \exp\left(-\frac{2}{p(1-p)}\right).
\end{aligned}$$

For the second equality, see (3.40) in Chapter 4 of Karatzas & Shreve, *Brownian Motion and Stochastic Calculus*.

The second term is $O(e^{-T^{1/2}})$ by the same argument as in the proof of Lemma 3.5, so we can choose W_3 large enough depending on p so that this term is less than $1/2$ the first term for $T \geq W_3$. This implies (3.5). □

We need the following definition for our next result. For a function $f \in C[a,b]$ we define its *modulus of continuity* by

$$w(f,\delta) = \sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}} |f(x) - f(y)|. \tag{3.6}$$

Lemma 3.13. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $y \in \mathbb{Z}$ such that $T \geq y \geq 0$, and suppose that ℓ has distribution $\mathbb{P}^{0,T,0,y}_{Ber}$. For each positive M, ϵ and η, there exist a $\delta(\epsilon,\eta,M) > 0$ and $W_4 = W_4(M,p,\epsilon,\eta) \in \mathbb{N}$ such that for $T \geq W_4$ and $|y - pT| \leq MT^{1/2}$ we have*

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(w\big(f^{\ell},\delta\big) \geq \epsilon\Big) \leq \eta, \tag{3.7}$$

where $f^{\ell}(u) = T^{-1/2}\big(\ell(uT) - puT\big)$ for $u \in [0,1]$.

Remark 3.14. Lemma 3.13 states that if ℓ is a Bernoulli bridge that is started from $(0,0)$ and terminates at (T,y) with y close to pT (i.e. with well-behaved endpoints) then the modulus of continuity of ℓ is also well-behaved with high probability.

Proof. We have

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(w\big(f^{\ell},\delta\big) \geq \epsilon\Big) = \mathbb{P}\Big(w\big(f^{\ell^{(N,y)}},\delta\big) \geq \epsilon\Big),$$

with $\mathbb{P}$, $f^{\ell^{(N,y)}}$. If B^{σ} is the Brownian bridge provided by Theorem 3.3, then

$$\begin{aligned} w\big(f^{\ell^{(N,y)}},\delta\big) &= T^{-1/2} \sup_{\substack{x,y\in[0,1] \\ |x-y|\leq\delta}} \Big|\ell^{(N,y)}(xT) - pxT - \ell^{(N,y)}(yT) + pyT\Big| \\ &\leq T^{-1/2} \sup_{\substack{x,y\in[0,1] \\ |x-y|\leq\delta}} \Big|\sqrt{T}\,B^{\sigma}_x + xy - pxT - \sqrt{T}\,B^{\sigma}_y - y^2 + pyT\Big| \\ &\qquad + T^{-1/2}\cdot\Big|\sqrt{T}\,B^{\sigma}_x + xy - \ell^{(N,y)}(xT)\Big| + T^{-1/2}\cdot\Big|\sqrt{T}\,B^{\sigma}_y + y^2 - \ell^{(N,y)}(yT)\Big| \\ &\leq \sup_{\substack{x,y\in[0,1] \\ |x-y|\leq\delta}} \Big|B^{\sigma}_x - B^{\sigma}_y + T^{-1/2}(y-pT)(x-y)\Big| + 2T^{-1/2}\Delta(T,y) \\ &\leq w\big(B^{\sigma},\delta\big) + M\delta + 2T^{-1/2}\Delta(T,y). \end{aligned}$$

The last line follows from the assumption that $|y - pT| \leq MT^{1/2}$. Thus

$$\begin{aligned} \mathbb{P}\Big(w\big(f^{\ell^{(N,y)}},\delta\big) \geq \epsilon\Big) &\leq \mathbb{P}\Big(w\big(B^{\sigma},\delta\big) + M\delta + 2T^{-1/2}\Delta(T,y) \geq \epsilon\Big) \\ &\leq \mathbb{P}\Big(w\big(B^{\sigma},\delta\big) + M\delta \geq \epsilon/2\Big) + \mathbb{P}\Big(\Delta(T,y) \geq \epsilon\,T^{1/2}/2\Big). \end{aligned}$$

The last term is $O(e^{-T^{1/2}})$ by the argument in the proof of Lemma 3.5, so we can choose W_4 large enough depending on M,p,ϵ,η so that this term is $\leq \eta/2$ for $T \geq W_4$. Since B^{σ} is a.s. uniformly continuous on the compact interval $[0,1]$, $w(B^{\sigma},\delta) \to 0$ as $\delta \to 0$. Thus we can find $\delta_0 > 0$ small enough depending on ϵ,η so that $w(B^{\sigma},\delta_0) < \epsilon/2$ with probability at least $1-\eta/2$. Then with $\delta = \min(\delta_0, \epsilon/4M)$, the first term is $\leq \eta/2$ as well. This implies (3.7). $\square$

3.3. **Properties of avoiding Bernoulli line ensembles.** In this section we derive several results about avoiding Bernoulli line ensembles, which are Bernoulli line ensembles with law $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ as in Definition 2.14. The lemmas we prove only involve the case when $f(r) = \infty$ for all $r \in [\![T_0,T_1]\!]$ and we denote the measure in this case by $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber}$. A $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber}$-distributed random variable will be denoted by $\mathfrak{Q} = (Q_1,\dots,Q_k)$ where k is the number of up-right paths in the ensemble. Our results will rely on the two monotonicity Lemmas 3.1 and 3.2 as well as the strong coupling between Bernoulli bridges and Brownian bridges from Theorem 3.3.

Lemma 3.15. *Fix $k,T \in \mathbb{N}$, $p \in (0,1)$, and $\vec{x},\vec{y} \in \mathfrak{W}_k$ such that $T \geq y_i - x_i \geq 0$ for $i = 1,\dots,k$. Suppose that $g : [\![0,T]\!] \to [-\infty,\infty)$ is such that $g(0) \leq x_k$, $g(T) \leq y_k$ and $g(i+1) = g(i)$ or $g(i+1) = g(i)+1$ for $i = 0,\dots,T-1$. Suppose that ℓ has distribution $\mathbb{P}^{0,T,\vec{x},\vec{y},\infty,g}_{avoid,Ber}$ as in Definition 2.14 (notice this distribtuion is well-defined by Lemma 2.15). Let $M_1,M_2,M_3 > 0$ be given. Then* *we can find* $W_5 = W_5(M_1, M_2, M_3, p) \in \mathbb{N}$ *such that for* $T \geq W_5$, $x_1 \leq M_1 T^{1/2}$, $y_1 \leq pT - M_2 T^{1/2}$ *and* g *satisfying* $g(r) \leq p \cdot r - M_3 T^{1/2}$ *for* $r \in [\![0, T]\!]$ *we have*

$$\mathbb{P}^{0,T,\vec{x},\vec{y},\infty,g}_{avoid,Ber}\left(Q_1(T/2) \leq k(M_1+1)T^{1/2} + \frac{(M_1 - M_2)T^{1/2} + pT}{2}\right) \geq [Something]. \tag{3.8}$$

Proof. □

4. Proof of Theorem 2.25

The goal of this section is to prove Theorem 2.25 and for the remainder we assume that $k \in \mathbb{N}$ with $k \geq 2$, $p \in (0,1)$, $\alpha, \lambda > 0$ are all fixed and

$$\left\{\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)\right\}_{N=1}^{\infty}, \tag{4.1}$$

is an (α, p, λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles as in Definition 2.23 that are all defined on a probability space with measure $\mathbb{P}$. The main technical result we will require is contained in Proposition 4.1 below and its proof is the content of Section 4.1. The proof of Theorem 2.25 is given in Section 4.2.

4.1. Bounds on the acceptance probability.

The main result in this section is presented as Proposition 4.1 below. In order to formulate it and some of the lemmas below it will be convenient to adopt the following notation for any $r > 0$:

$$t_1 = \lfloor (r+1)N^{\alpha} \rfloor, \quad t_2 = \lfloor (r+2)N^{\alpha} \rfloor, \quad \text{and } t_3 = \lfloor (r+3)N^{\alpha} \rfloor. \tag{4.2}$$

Proposition 4.1. *For any* $\epsilon > 0$, $r > 0$ *and any* (α, p, λ)*-good sequence of Bernoulli line ensembles* $\left\{\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)\right\}_{N=1}^{\infty}$ *there exist* $\delta > 0$ *and* N_1 *(both depending on* ϵ, r *as well as* α, p, λ *and the functions* ϕ, ψ *in Definition 2.23) such that for all* $N \geq N_1$ *we have*

$$\mathbb{P}\Big(Z\big(t_1^-, t_1^+, \vec{x}, \vec{y}, L_k[\![t_1^-, t_1^+]\!]\big) < \delta\Big) < \epsilon,$$

where $\vec{x} = (L_1^N(t_1^-), \dots, L_{k-1}^N(t_1^-))$, $\vec{y} = (L_1^N(t_1^+), \dots, L_{k-1}^N(t_1^+))$, $L_k[\![t_1^-, t_1^+]\!]$ *is the restriction of* L_k^N *to the set* $[\![t_1^-, t_1^+]\!]$*, and* Z *is the acceptance probability of Definition 2.21.* $\mathbb{P}$ *is the measure on a probability space that supports* $\left\{\mathfrak{L}^N\right\}_{N=1}^{\infty}$.

The general strategy we use to prove Proposition 4.1 is inspired by the proof of Proposition 6.5 in [3]. We begin by stating three key lemmas that will be required. Their proofs are postponed to Section 5. All constants in the statements below will depend implicitly on α, r, p, λ, and the functions ϕ, ψ from Definition 2.23, which are fixed throughout. We will not list this dependence explicitly.

Lemma 4.2 controls the deviation of the curve $L_1^N(s)$ from the line ps in the scale $N^{\alpha/2}$.

Lemma 4.2. *For each* $\epsilon > 0$ *there exist* $R_1 = R_1(\epsilon) > 0$ *and* $N_2 = N_2(\epsilon)$ *such that for* $N \geq N_2$

$$\mathbb{P}\Big(\sup_{s \in [-t_3, t_3]} \big(L_1^N(s) - ps\big) \geq R_1 N^{\alpha/2}\Big) < \epsilon.$$

Lemma 4.3 controls the upper deviation of the curve $L_2^N(s)$ from the line ps in the scale $N^{\alpha/2}$.

Lemma 4.3. *For each* $\epsilon > 0$ *there exist* $R_2 = R_2(\epsilon) > 0$ *and* $N_3 = N_3(\epsilon)$ *such that for* $N \geq N_3$

$$\mathbb{P}\Big(\inf_{s \in [-t_2, t_2]} \big(L_k^N(s) - ps\big) \leq -R_2 N^{\alpha/2}\Big) < \epsilon.$$

4.2. **Proof of Theorem 2.25.** As shown in Exercise 9, the sequence $\mathbb{P}_N$ is tight if the following two conditions are met:

$$\lim_{a\to\infty} \limsup_{N\to\infty} \mathbb{P}(|f_i^N(0)| \geq a) = 0 \tag{4.3}$$

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\leq\delta}} |f_i^N(x) - f_i^N(y)| \geq \epsilon \Bigg) = 0. \tag{4.4}$$

For the sake of clarity, we will divide this proof into sections in which the first section will prove the former condition, then the second section will prove the latter.

Step 1: In this step, we will prove condition (1).

In order to prove the first condition, we will make great use of Lemmata 4.2 and 4.3, with Lemma 4.2 giving an upper bound for the top line in the line ensemble and Lemma 4.3 giving a lower bound for the bottom line, and thus providing upper and lower bounds for each intermediate line.

First, we will reformulate (1) slightly to find that $\lim_{a\to\infty} \limsup_{N\to\infty} \mathbb{P}(|f_i^N(0)| \geq a) = 0$ is the same statement as for all $\epsilon > 0$, there exists an $a > 0$ and N' such that $N > N'$ implies $\mathbb{P}(|f_i^N(0)| \geq a) < \epsilon$.

Now, using the definition of $f_i^N(s)$, we remember that $f_i^N(0) = N^{-\alpha/2} L_i^N(0)$, which tells us that we need to prove that for all $\epsilon > 0$, there exists an $a > 0$ and N' such that $N > N'$ implies

$$\mathbb{P}(|L_i^N(0)| \geq aN^{-\alpha/2}) < \epsilon$$

Lemmata 4.2 and 4.3 give us that there exist $R_1(\epsilon) > 0$, $R_2(\epsilon) > 0$ and $N_2(\epsilon), N_3(\epsilon)$ such that

$$N > N_2(\epsilon) \text{ implies } \mathbb{P}\left(\sup_{s\in[t_3^-,t_3^+]} \left(L_1^N(s) - ps\right) \geq R_1(\epsilon) N^{\alpha/2} \right) < \epsilon$$

$$N > N_3(\epsilon) \text{ implies } \mathbb{P}\left(\inf_{s\in[t_2^-,t_2^+]} \left(L_k^N(s) - ps\right) \leq -R_2(\epsilon) N^{\alpha/2} \right) < \epsilon$$

where $t_i^\pm = \lfloor \pm(r+i)N^\alpha \rfloor$ for any $r > 0$. In particular, since $0 \in [t_i^-, t_i^+]$ for any t_i^-, t_i^+, we know that

$$N > N_2 \text{ implies } \mathbb{P}\left(L_1^N(0) \geq R_1\left(\frac{\epsilon}{2}\right) N^{\alpha/2} \right) \leq \mathbb{P}\left(\sup_{s\in[t_3^-,t_3^+]} \left(L_1^N(s) - ps\right) \geq R_1\left(\frac{\epsilon}{2}\right) N^{\alpha/2} \right) < \frac{\epsilon}{2}$$

$$N > N_3 \text{ implies } \mathbb{P}\left(L_k^N(0) \leq -R_2\left(\frac{\epsilon}{2}\right) N^{\alpha/2} \right) \leq \mathbb{P}\left(\inf_{s\in[t_2^-,t_2^+]} \left(L_k^N(s) - ps\right) \leq -R_2\left(\frac{\epsilon}{2}\right) N^{\alpha/2} \right) < \frac{\epsilon}{2}$$

Therefore, allow $N' = \max\left\{N_2\left(\frac{\epsilon}{2}\right), N_3\left(\frac{\epsilon}{2}\right)\right\}$ and $a = \max\left\{R_1\left(\frac{\epsilon}{2}\right), R_2\left(\frac{\epsilon}{2}\right)\right\}$, and so because $L_1^N(0) > L_2^N(0) > \dots > L_k^N(0)$, for each $i \in \{1, 2, ..., k\}$, given the above definitions of N' and a we find that

$$\mathbb{P}(|L_i^N(0)| \geq aN^{-\alpha/2}) \leq \mathbb{P}\left(L_1^N(0) \geq R_1\left(\frac{\epsilon}{2}\right) N^{\alpha/2} \right) + \mathbb{P}\left(L_k^N(0) \leq -R_2\left(\frac{\epsilon}{2}\right) N^{\alpha/2} \right) < \epsilon$$

This is the desired result, giving us tightness of $\{f_i^N(0)\}$ and so completing Step 1.

Step 2: Here, we will prove condition (2) in three parts.

Part 1: We will begin here with a reformulation of the conditions on f_i^N to conditions on L_i^N so that we may use existing knowledge about that structure. To remind ourselves of the statement of condition (2), let us restate it here,expanding the definitions of limits fully. We seek that $\forall \epsilon, \eta > 0$

and $R > 0$, there exists a δ and N_0 such that $N > N_0$ implies

$$\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\leq\delta}} |f_i^N(x) - f_i^N(y)| \geq \epsilon \Bigg) < \eta$$

We may then reformulate this expression slightly by expanding the f_i^N to their definition and then contracting them by a factor of N^α.

$$\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\leq\delta}} \Big| N^{-\alpha/2}\left(L_i^N(xN^\alpha) - L_i^N(yN^\alpha)\right) - p(x-y)N^{\alpha/2} + \lambda(x^2-y^2) \Big| \geq \epsilon \Bigg) \tag{4.5}$$

Now given that $|x-y| < \delta$ and $x, y \in [-R, R]$, we know that $|x+y| \leq 2R$ and $|x-y| < \delta$, and so $|x^2 - y^2| \leq 2R\delta$, and so by the triangle inequalty, we know that the probability above is upper bounded by

$$\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\leq\delta}} N^{-\alpha/2}\big|L_i^N(xN^\alpha) - L_i^N(yN^\alpha) - p(x-y)N^\alpha\big| + 2\lambda R\delta \geq \epsilon \Bigg)$$

and so ensure that $\delta \leq \frac{\epsilon}{8\lambda R}$ to find that expression (3) is upper bounded by

$$\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\leq\delta}} \big|L_i^N(xN^\alpha) - L_i^N(yN^\alpha) - p(x-y)N^\alpha\big| \geq \frac{3N^{\alpha/2}\epsilon}{4} \Bigg)$$

which may be scaled by N^α to have the equal expression

$$\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-RN^\alpha,RN^\alpha],\\ |x-y|\leq\delta N^\alpha}} \big|L_i^N(x) - L_i^N(y) - p(x-y)\big| \geq \frac{3N^{\alpha/2}\epsilon}{4} \Bigg) \tag{4.6}$$

This string of arguments has given us the inequality

$$\mathbb{P}\Bigg(\sup_{\substack{x,y\in[-R,R],\\ |x-y|\leq\delta}} |f_i^N(x) - f_i^N(y)| \geq \epsilon \Bigg) < \mathbb{P}\Bigg(\sup_{\substack{x,y\in[-RN^\alpha,RN^\alpha],\\ |x-y|\leq\delta N^\alpha}} \big|L_i^N(x) - L_i^N(y) - p(x-y)\big| \geq \frac{3N^{\alpha/2}\epsilon}{4} \Bigg)$$

which implies that if (4) is less than η, condition (2) has been met.

Step 2, Part 2: In this step, we will establish events in order to size bias our event on arbitrarily large sets that we approximate the entire probability space with certain extra conditions.

Let us denote in the limit above as follows, as well as two other events with high probability:

$$A_\delta = \Bigg\{ \sup_{\substack{x,y\in[-RN^\alpha,RN^\alpha],\\ |x-y|\leq\delta N^\alpha}} \big|L_i^N(x) - L_i^N(y) - p(x-y)\big| \geq 3\epsilon N^{\alpha/2}/4 \Bigg\}$$

$$E_1 = \Big\{ \max_{1\leq i\leq m} |f_i(\pm R)| \leq M_1 \Big\}$$

$$E_2 = \big\{ Z(-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}, \infty, L_{m+1}^N[-RN^\alpha, RN^\alpha]) > \delta_1 \big\}$$

We can understand these events as follows: A_δ is the event whose probability must be bounded by η for tightness to occur, E_1 is an event with a bounding condition on the entrance and exit data of the line ensemble with M_1 being some large constant, and E_2 is an event conditioned on a high enough acceptance probability, as defined in Definition 2.21.

We want to show that these events occur with very high probability for some choice of δ_1 and M_1. In order to do this, we may use Proposition 4.1 and Lemmata 4.2 and 4.3.

For E_1, we first know that $L_i^N(\pm RN^\alpha) > L_{i+1}^N(\pm RN^\alpha)$, since the index i is the ordering of line ensembles by the values of their entry and exit data, which implies that $f_i^N(\pm R) > f_{i+1}^N(\pm R)$ as well. Therefore, we find that

$$\begin{aligned} E_1^c =& \{f_1(\pm R) > M_1\} \cup \{f_m(\pm R) < -M_1\} \\ =& \left\{ \left(L_1^N(\pm RN^\alpha) \mp pRN^\alpha\right) > (M_1 - \lambda R^2)N^{\frac{\alpha}{2}} \right\} \cup \left\{ \left(L_m^N(\pm RN^\alpha) \mp pRN^\alpha\right) < -(\lambda R^2 + M_1)N^{\frac{\alpha}{2}} \right\} \end{aligned}$$

Therefore we will now calculate the probability of both of these events to get an upper bound of E_1^c. For the first event, we find that

$$\mathbb{P}\left(L_1^N(\pm RN^\alpha) \mp prN^\alpha > (M_1 - \lambda R^2)N^{\frac{\alpha}{2}}\right) \leq \mathbb{P}\left(\sup_{s\in[t_3^-,t_3^+]} L_1^N(s) - ps > (M_1 - \lambda R^2)N^{\frac{\alpha}{2}}\right)$$

because $t_3^- < -RN^\alpha < RN^\alpha < t_3^+$ by definition. By Lemma 4.2, we find that if $M_1 > R_1(\frac{\eta}{8}) + \lambda R^2$ and $N > N_2(\frac{\eta}{8})$, then this probability is less than $\frac{\eta}{8}$. Now for the second event,

$$\begin{aligned} \mathbb{P}\left(L_m^N(\pm RN^\alpha) \mp pRN^\alpha < -(\lambda R^2 + M_1)N^{\frac{\alpha}{2}}\right) &\leq \mathbb{P}\left(L_m^N(\pm RN^\alpha) \mp pRN^\alpha < -M_1 N^{\frac{\alpha}{2}}\right) \\ &\leq \mathbb{P}\left(\inf_{s\in[t_2^-,t_2^+]} L_m^N(s) - ps < -M_1 N^{\frac{\alpha}{2}}\right) \end{aligned}$$

with the last step justified by the inequality $t_2^- < -RN^{\frac{\alpha}{2}} < RN^{\frac{\alpha}{2}} < t_2^+$. By Lemma 4.3, we know that if $M_1 \geq R_2(\frac{\eta}{8})$ and $N > N_2(\frac{\eta}{8})$ then this probability is less than $\frac{\eta}{8}$

Therefore, we find that with $M_1 = \max\{R_1(\frac{\eta}{8}) + \lambda R^2, R_2(\frac{\eta}{8})\}$ the probability of each event is bounded by $\frac{\eta}{8}$, so

$$\mathbb{P}(E_1^c) < \frac{\eta}{4}$$

by subadditivity.

Now, for E_2, Proposition 4.1 gives us that for $\frac{\eta}{4}$ and $r = R - 1$, there exists some $\delta_1(\frac{\eta}{4})$ and $N_1(\frac{\eta}{4})$ such that $N \geq N_1$ implies we have $\mathbb{P}\left(Z(-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}, L_m[-RN^\alpha, RN^\alpha]) < \delta_1\right) < \frac{\eta}{4}$. and therefore

$$\mathbb{P}\left(E_2^c\right) < \frac{\eta}{4}$$

Therefore, we have found that given $N > \max\{N_1(\frac{\eta}{4}), N_2(\frac{\eta}{8}), N_3(\frac{\eta}{8})\}$, $\mathbb{P}(E_1^c \cup E_2^c) < \frac{\eta}{2}$ and therefore

$$\mathbb{P}(A_\delta) = \mathbb{P}(A_\delta \cap E_1 \cap E_2) + \mathbb{P}(A_\delta \cap (E_1^c \cup E_2^c)) \leq \mathbb{P}(A_\delta \cap E_1 \cap E_2) + \frac{\eta}{2}$$

and hence $\mathbb{P}(A_\delta \cap E_1 \cap E_2) \leq \frac{\eta}{2} \implies \mathbb{P}(A_\delta) < \eta$.

Step 2, Part 3: In this step we will bound $\mathbb{P}(A_\delta \cap E_1 \cap E_2)$ to prove Condition (2), following from the results of the previous step.
First, let us begin by defining a σ-algebra, the usefulness of which will be shown shortly.

$$\mathcal{F} = \sigma\left(L_{m+1}^N, L_1^N(\pm N^\alpha R), L_2^N(\pm N^\alpha R), \ldots, L_m^N(\pm N^\alpha R)\right).$$

We claim that $E_1, E_2 \in \mathcal{F}$. This is trivial for E_1, and for E_2 we need only apply the definition of the acceptance probability, Definition 2.21, since the values of $\vec{x}$ and $\vec{y}$ are determined by L_i^N for $i \in [\![1, m]\!]$ and the bottom bounding curve is L_{m+1}^N all of which are generators for $\mathcal{F}$ where m is any integer between 1 and k..

The $\mathcal{F}$-measurability of $\mathbf{1}_{E_1}$ and $\mathbf{1}_{E_2}$ as well as the tower property of conditional expectation give us the following equations

$$\begin{aligned} \mathbb{P}(A_\delta \cap E_1 \cap E_2) &= \mathbb{E}(\mathbf{1}_{A_\delta} \cdot \mathbf{1}_{E_1} \cdot \mathbf{1}_{E_2}) \\ &= \mathbb{E}(\mathbf{1}_{E_1} \cdot \mathbf{1}_{E_2} \cdot \mathbb{E}(\mathbf{1}_{A_\delta} \mid \mathcal{F})) \end{aligned}$$

Directly from the Schur-Gibbs property of the line ensemble, as defined in 2.16 we know that

$$\mathbb{E}\left(\mathbf{1}_{A_\delta} \mid \mathcal{F}\right) = \mathbb{E}_{avoid,Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}, \infty, L_{m+1}^N}\left(\mathbf{1}_{A_\delta}\right)$$

We now observe that the Radon-Nikodym derivative of $\mathbb{P}_{avoid,Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}, \infty, L_{m+1}^N}$ with respect to $\mathbb{P}_{Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}}$ is

$$\frac{\mathbf{1}_{\{L_i \le L_{i+1}, \forall i \in [1,m]\}}}{Z(-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}, L_{m+1}^N)}.$$

To see this, note that for any event A,

$$\mathbb{P}_{avoid,Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}, \infty, L_{m+1}^N}(A) = \frac{\mathbb{P}_{Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}}(A \cap \{L_i \le L_{i+1}, \forall i \in [1,m]\})}{\mathbb{P}_{Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}}(L_i \le L_{i+1}, \forall i \in [1,m])}$$

$$= \frac{\mathbb{E}_{Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}}\left[\mathbf{1}_A \, \mathbf{1}_{\{L_i \le L_{i+1}, \forall i \in [1,m]\}}\right]}{Z(-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}, L_{m+1}^N)} = \int_A \frac{\mathbf{1}_{\{L_i \le L_{i+1}, \forall i \in [1,m]\}}}{Z(-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}, L_{m+1}^N)} \, d\mathbb{P}_{Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}}.$$

Therefore, we find that

$$\mathbb{E}\left(\mathbf{1}_{A_\delta} \mid \mathcal{F}\right) = \mathbb{E}_{Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}}\left(\frac{\mathbf{1}_{A_\delta} \cdot \mathbf{1}_{\{L_i \le L_{i+1}, \forall i \in [1,m]\}}}{Z(-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}, L_{m+1}^N)}\right)$$

$$\mathbb{P}(A_\delta \cap E_1 \cap E_2) = \mathbb{E}\left(\mathbf{1}_{E_1} \cdot \mathbf{1}_{E_2} \cdot \mathbb{E}_{Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}}\left(\frac{\mathbf{1}_{A_\delta} \cdot \mathbf{1}_{\{L_i \le L_{i+1}, \forall i \in [1,m]\}}}{Z(-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}, L_{m+1}^N)}\right)\right)$$

Now given the factor $\mathbf{1}_{E_2}$, we know that either $Z(-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}, L_{m+1}^N) > \delta_1$, or the entire expression is 0. Hence, we know that

$$\mathbb{P}(A_\delta \cap E_1 \cap E_2) \le \mathbb{E}\left(\mathbf{1}_{E_1} \cdot \mathbf{1}_{E_2} \cdot \mathbb{E}_{Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}}\left(\frac{\mathbf{1}_{A_\delta}}{\delta_1}\right)\right)$$

$$= \mathbb{E}\left(\frac{\mathbf{1}_{E_1} \cdot \mathbf{1}_{E_2}}{\delta_1} \mathbb{P}_{Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}}(A_\delta)\right).$$

By Lemma 3.13, we know that there exists a N_4 and δ such that $N > N^4$ implies that

$$\mathbb{P}_{Ber}^{-RN^\alpha, RN^\alpha, \vec{x}, \vec{y}}(A_\delta) < \frac{\eta \cdot \delta_1}{2}$$

and therefore we find that

$$\mathbb{P}\left(A_\delta \cap E_1 \cap E_2\right) \le \frac{\eta}{2}$$

which is precisely the bound we found to be required at the end of Step 2, part 2.

Therefore, we have found that the two conditions we set out to prove are correct and this implies the tightness of $\mathbb{P}_N$ as defined in Theorem 2.25.

5. Proof of three key lemmas

Here we prove the three key lemmas from Section 4.1.

5.1. **Proof of Lemma 4.2.** We first establish some notation. Let $a, b, t_1, t_2, z_1, z_2 \in \mathbb{Z}$ be given such that $t_1 + 1 < t_2$, $0 \le z_2 - z_1 \le t_2 - t_1$, $0 \le b - a \le t_2 - t_1$, $z_1 \le a$, and $z_2 \le b$. We write $\ell \in \Omega(t_1, t_2, a, b)$ and $\ell_{bot} \in \Omega(t_1, t_2, z_1, z_2)$ for generic paths in these two spaces, and we consider the event $\{\ell \ge \ell_{bot}\} = \{\ell(s) \ge \ell_{bot}(s), s \in [t_1, t_2]\}$. Note that $\mathbb{P}_{avoid,Ber}^{t_1,t_2,a,b,\infty,\ell_{bot}}(\ell) = \mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell \,|\, \ell \ge \ell_{bot}))$. We now establish some auxiliary results which will be used in the proof of Lemma 4.2.

Lemma 5.1. *If $a \le k_1 \le k_2 \le a + T - t_1$, then with notation as above,*

$$\mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \ge \ell_{bot} \,\big|\, \ell(T) = k_1\big) \le \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \ge \ell_{bot} \,\big|\, \ell(T) = k_2\big).$$

Remark 5.2. This lemma essentially states that a path ℓ is more likely to lie above ℓ_{bot} if its value at a point T is increased. A more general result is proven in [1, Lemma 4.1]

Proof. Let ℓ_1 be a random path distributed according to $\mathbb{P}_{Ber}^{t_1,t_2,a,b}$ conditioned on $\ell_1(T) = k_1$. We can identify ℓ_1 with a sequence of $+$'s and $-$'s of length $t_2 - t_1$, where a $+$ in the ith position means that $\ell_1(t_1 + i + 1) - \ell_1(t_1 + i) = 1$, and a $-$ means that $\ell_1(t_1 + i + 1) - \ell_1(t_1 + i) = 0$. [Maybe include Figure 9 from Corwin-Dimitrov here.] In this representation, the value of $\ell_1(T)$ is a plus the number of $+$'s in the first $T - t_1$ slots, and the value of $\ell_1(t_2)$ is a plus the total number of $+$'s. Note that we must have exactly $(k_1 - a)$ $+$'s in the first $T - t_1$ slots, and $(b - k_1)$ $+$'s in the last $t_2 - T$ slots. We pick uniformly at random $(k_2 - k_1)$ $-$'s in the first $T - t_1$ slots and change them to $+$'s, then pick randomly $(k_2 - k_1)$ $+$'s in the last $t_2 - T$ slots and change them to $-$'s. This defines a new path ℓ_2. Since there are now $k_2 - a$ $+$'s in the first $T - t_1$ slots, we have $\ell_2(T) = k_2$, and we still have $\ell_2(t_2) = b$ since the number of $+$'s is unchanged. Thus we see that ℓ_2 is distributed according to $\mathbb{P}_{Ber}^{t_1,t_2,a,b}$ conditioned on $\ell_2(T) = k_2$.

Now suppose $\ell_1 \geq \ell_{bot}$. We claim that $\ell_2 \geq \ell_1$ on all of $[t_1, t_2]$. To see this, note that for any $s \in [\![t_1, t_2]\!]$, $\ell_2(s) - \ell_1(s)$ is equal to the number of $+$'s in the first $s - t_1$ slots of the sequence representing ℓ_2, minus the corresponding number for ℓ_1. If $s \leq T$, this difference is clearly positive by construction. The difference is equal to $k_2 - k_1 \geq 0$ at $s = T$, and the difference then decreases monotonically as s increases to t_2, since we have removed exactly $k_2 - k_1$ $+$'s from the last $t_2 - T$ slots. The difference is of course 0 at $s = t_2$, so this proves the claim. It follows that

$$\mathbf{1}_{\ell_1 \geq \ell_{bot}} \leq \mathbf{1}_{\ell_2 \geq \ell_{bot}}.$$

Now taking expectations of both sides and recalling the distributions of ℓ_1, ℓ_2 proves the lemma. $\square$

Corollary 5.3. *Let $T \in [\![t_1, t_2]\!]$, and let A, B be nonempty sets of integers such that $a \leq \alpha \leq \beta \leq a + T - t_1$ for all $\alpha \in A, \beta \in B$. Then*

$$\mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) \in A\big) \leq \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) \in B\big).$$

Proof. We have

$$\begin{aligned}
&\mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) \in A\big) = \sum_{\alpha \in A} \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) = \alpha\big) \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \alpha)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in A)} \\
&= \sum_{\alpha \in A} \sum_{\beta \in B} \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) = \alpha\big) \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \alpha)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in A)} \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \beta)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in B)} \\
&\leq \sum_{\alpha \in A} \sum_{\beta \in B} \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) = \beta\big) \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \alpha)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in A)} \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \beta)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in B)} \\
&= \sum_{\beta \in B} \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) = \beta\big) \cdot \frac{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) = \beta)}{\mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \in B)} = \mathbb{P}_{Ber}^{t_1,t_2,a,b}\big(\ell \geq \ell_{bot} \,\big|\, \ell(T) \in B\big).
\end{aligned}$$

The inequality in the third line follows from Lemma 5.1. $\square$

Corollary 5.4. *Let $\alpha \leq a + T - t_1$. Then*

$$\mathbb{P}_{avoid,Ber}^{t_1,t_2,a,b,\infty,\ell_{bot}}(\ell(T) \geq \alpha) \geq \mathbb{P}_{Ber}^{t_1,t_2,a,b}(\ell(T) \geq \alpha).$$

Proof. We write $\mathbb{P} := \mathbb{P}_{Ber}^{t_1,t_2,a,b}$ for brevity. Using Bayes' theorem repeatedly, we find

$$\begin{aligned}\mathbb{P}(\ell(T) \geq \alpha \,|\, \ell \geq \ell_{bot}) &= \frac{\mathbb{P}(\ell \geq \ell_{bot} \,|\, \ell(T) \geq \alpha)\mathbb{P}(\ell(T) \geq \alpha)}{\mathbb{P}(\ell \geq \ell_{bot})} \\ &\geq \frac{\mathbb{P}(\ell \geq \ell_{bot} \,|\, \ell(T) < \alpha)\mathbb{P}(\ell(T) \geq \alpha)}{\mathbb{P}(\ell \geq \ell_{bot})} \\ &= \big(1 - \mathbb{P}(\ell(T) \geq \alpha \,|\, \ell \geq \ell_{bot})\big) \cdot \frac{\mathbb{P}(\ell(T) \geq \alpha)}{\mathbb{P}(\ell(T) < \alpha)}.\end{aligned}$$

The inequality in the second line follows from Corollary 5.3. It follows that

$$\mathbb{P}(\ell(T) \geq \alpha \,|\, \ell \geq \ell_{bot}) \geq \frac{\mathbb{P}(\ell(T) \geq \alpha)}{\mathbb{P}(\ell(T) \geq \alpha) + \mathbb{P}(\ell(T) < \alpha)} = \mathbb{P}(\ell(T) \geq \alpha).$$

□

We are now ready to prove Lemma 4.2. The proof is similar to that of [1, Lemma 5.2]. We exploit the one-point tightness of L_1^N at two appropriately chosen points, and we use Lemma 3.5 to control the deviation of L_1^N from the line of slope p away from these points.

Proof. We write $s_4 = \lceil r+3 \rceil N^\alpha$, $s_3 = \lfloor r+3 \rfloor N^\alpha$, so that $s_3 \leq t_3 \leq s_4$, and take N large enough so that L_1^N is defined at s_4. We define events

$$E(M) = \Big\{ \big|L_1^N(-s_4) + ps_4\big| > MN^{\alpha/2} \Big\}, \quad F(M) = \Big\{ L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} \Big\},$$

$$G(M) = \left\{ \sup_{s \in [0,t_3]} \big(L_1^N(s) - ps\big) \geq (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2} \right\}.$$

For $a, b \in \mathbb{Z}$, $s \in [\![0, t_3]\!]$, and $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$ with $z_1 \leq a$, $z_2 \leq b$, we also define $E(a,b,s,\ell_{bot})$ to be the event that $L_1^N(-s_4) = a$, $L_1^N(s) = b$, and L_2^N agrees with ℓ_{bot} on $[-s_4, s]$.

We claim that the set $G(M) \setminus E(M)$ can be written as a *countable disjoint* union of sets $E(a,b,s,\ell_{bot})$. Let $D(M)$ be the set of tuples (a,b,s,ℓ_{bot}) satisfying

(1) $0 \leq s \leq t_3$,
(2) $0 \leq b - a \leq s + s_4$, $|a + ps_4| \leq MN^{\alpha/2}$, and $b - ps > (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2}$,
(3) $z_1 \leq a, z_2 \leq b$, and $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$.

Conditions (1) and (2) show that the union of these sets $E(a,b,s,\ell_{bot})$ for $(a,b,s,\ell_{bot}) \in D(M)$ is $G(M) \setminus E(M)$. Observe that $D(M)$ is countable, since there are finitely many possible choices of s, countably many a, b and z_1, z_2 for each s, and finitely many ℓ_{bot} for each z_1, z_2. Moreover, the sets $E(a,b,s,\ell_{bot})$ are clearly pairwise disjoint for distinct tuples in $D(M)$. This proves the claim.

Now by one-point tightness of L_1^N at integer multiples of N^α, we can choose M large enough depending on ϵ so that

$$\mathbb{P}(E(M)) < \epsilon/4, \quad \mathbb{P}(F(M)) < \epsilon/12 \tag{5.1}$$

for all $N \in \mathbb{N}$. If $(a,b,s,\ell_{bot}) \in D(M)$, then

$$\begin{aligned}\mathbb{P}_{Ber}^{-s_4,s,a,b}\Big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\Big) &= \mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\Big(\ell(s_4 - s_3) + a \geq -ps_3 + MN^{\alpha/2}\Big) \\ &\geq \mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\Big(\ell(s_4 - s_3) \geq p(s_4 - s_3) + 2MN^{\alpha/2}\Big).\end{aligned}$$

The inequality follows from the assumption in (2) that $a + ps_4 \geq -MN^{\alpha/2}$. Moreover, since $b - ps > (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2}$ and $a + ps_4 \leq MN^{\alpha/2}$, we have

$$b - a \geq p(s + s_4) + (6r+21)(2r+6)^{1/2}(M+1)N^{\alpha/2} \geq p(s + t_3) + (6r+21)(M+1)(s+s_4)^{1/2}.$$

The second inequality follows since $s + s_4 \le 2s_4 \le (2r+6)N^\alpha$. It follows from Lemma 3.5 with $M_1 = 0$, $M_2 = (6r+21)(M+1)$ that for sufficiently large N, we have

$$\mathbb{P}^{0,s+s_4,0,b-a}_{Ber}\Big(\ell(s_4 - s_3) \ge \frac{s_4 - s_3}{s + s_4}[p(s+s_4) + M_2 N^{\alpha/2}] - (s+s_4)^{1/4}\Big) \ge 1/3, \tag{5.2}$$

for all $(a, b, s, \ell_{bot}) \in D(M)$ simultaneously. Note that $\frac{s_4 - s_3}{s+s_4} \ge \frac{N^\alpha - 1}{(2r+6)N^\alpha} \ge \frac{1}{2r+7}$ for large N. Hence $\frac{s_4-s_3}{s+s_4}[p(s+t_3)+M_2N^{\alpha/2}] - (s+s_4)^{1/4} \ge p(s+s_4)+3(M+1)N^{\alpha/2} - (s+s_4)^{1/4} \ge p(s+s_4)+2MN^{\alpha/2}$ for all large enough N. We conclude from (5.2) that

$$\mathbb{P}^{-s_4,s,a,b}_{Ber}\Big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\Big) \ge 1/3$$

uniformly in a, b for large N. Now by the Gibbs property for L^N, we have for any $\ell \in \Omega(-s_4, s, a, b)$ that

$$\mathbb{P}(L^N_1|_{[-s_4,s]} = \ell \,|\, E(a, b, s, \ell_{bot})) = \mathbb{P}^{-s_4,s,a,b,\infty,\ell_{bot}}_{avoid,Ber}(\ell).$$

Hence by Corollary 5.4,

$$\begin{aligned}
&\mathbb{P}\big(L^N_1(-s_3) > -ps_3 + MN^{\alpha/2} \,\big|\, E(a, b, s, \ell_{bot})\big) \\
&= \sum_{\ell \in \Omega(-s_4,s,a,b)} \mathbb{P}^{-s_4,s,a,b,\infty,\ell_{bot}}_{avoid,Ber}(\ell) \cdot \mathbb{P}^{-s_4,s,a,b,\infty,\ell_{bot}}_{avoid,Ber}\big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\big) \\
&\ge \sum_{\ell \in \Omega(-s_4,s,a,b)} \mathbb{P}^{-s_4,s,a,b,\infty,\ell_{bot}}_{avoid,Ber}(\ell) \cdot \mathbb{P}^{-s_4,s,a,b}_{Ber}\big(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\big) \\
&\ge \frac{1}{3} \sum_{\ell \in \Omega(-s_4,s,a,b)} \mathbb{P}^{-s_4,s,a,b,\infty,\ell_{bot}}_{avoid,Ber}(\ell) = \frac{1}{3}.
\end{aligned}$$

Note once again that this bound holds independent of a, b for all sufficiently large N. It follows from (5.1) that

$$\epsilon/12 > \mathbb{P}(F(M)) \ge \sum_{(a,b,s,\ell_{bot}) \in D(M)} \mathbb{P}(F(M) \cap E(a, b, s, \ell_{bot}))$$

$$= \sum_{(a,b,s,\ell_{bot}) \in D(M)} \mathbb{P}(F(M) \,|\, E(a, b, s, \ell_{bot}))\mathbb{P}(E(a, b, s, \ell_{bot})) \ge \frac{1}{3}\mathbb{P}(G(M) \setminus E(M))$$

for large N. Since in addition $\mathbb{P}(E(M)) < \epsilon/4$, we find that

$$\mathbb{P}\Big(\sup_{s \in [0,t_3]} \big(L^N_1(s) - ps\big) \ge (6r+22)(2r+6)^{1/2}(M+1)N^{\alpha/2}\Big) = \mathbb{P}(G(M)) < \epsilon/2$$

for large enough N. A similar argument proves the same inequality with $[-t_3, 0]$ in place of $[0, t_3]$. Thus we can find an $N_2 = N_2(\epsilon)$ so that

$$\mathbb{P}\Big(\sup_{s \in [-t_3,t_3]} \big(L^N_1(s) - ps\big) \ge R_1 N^{\alpha/2}\Big) < \epsilon$$

for all $N \ge N_2$, with $R_1 = (6r+22)(2r+6)^{1/2}(M+1)$.

□

5.2. **Proof of Lemma 4.3.** We begin by proving the following lemma, which allows us to prevent the bottom curve of an ensemble from falling too low on some interval.

Lemma 5.5. *Fix $p \in (0, 1)$, $k \in \mathbb{N}$, and $\alpha, \lambda > 0$. Suppose that $\mathfrak{L}^N = (L^N_1, \dots, L^N_k)$ is a (α, p, λ)-good sequence of $[\![1, k]\!]$-indexed Bernoulli line ensembles. Then for any $r, \epsilon > 0$, there exists $R > 0$ depending on $\lambda, k, p, \epsilon, r, \phi$ and $N_0 \in \mathbb{N}$ depending on $\lambda, k, p, \epsilon, r, \phi, \psi, \alpha$ such that for all $N \ge N_0$,*

$$\mathbb{P}\Big(\max_{x \in [r,R]} \big(L^N_k(xN^\alpha) - pxN^\alpha\big) \le -(\lambda R^2 + \phi(\epsilon/8))N^{\alpha/2}\Big) < \epsilon.$$

The same statement holds if $[r,R]$ is replaced with $[-R,-r]$.

Remark 5.6. The key to this lemma is the parabolic shift implicit in the definition of an (α,p,λ)-good sequence. This requires the deviation of the top curve from the line of slope p to appear roughly parabolic. Using monotone coupling, we separate the curves of the ensemble so that L_1^N is nearly independent of the other curves. Then we would expect the value of L_1^N at the midpoint of r and R to be close to the midpoint of the straight line segment connecting two points of the parabola. But the parabola is convex, so for large enough R this violates the one-point tightness assumpion at $(R+r)/2$.

Proof. Fix $r>0$ and $R>r$. Note that

$$\max_{r\le x\le R}\big(L_k^N(xN^\alpha)-pxN^\alpha\big)\ge\max_{\lceil r\rceil\le x\le\lfloor R\rfloor}\big(L_k^N(xN^\alpha)-pxN^\alpha\big).$$

Thus by replacing r and R with $\lceil r\rceil$ and $\lfloor R\rfloor$ respectively, we can assume that $r,R\in\mathbb{Z}$. Moreover, we will assume that the midpoint $\frac{R+r}{2}$ is an integer. If not, we are free to enlarge R by 1 so that $R+r$ is even. We will always assume N is large enough depending on ψ so that L_1^N is defined at R. Define events

$$A=\left\{L_1^N\left(\frac{R+r}{2}N^\alpha\right)-pN^\alpha\,\frac{R+r}{2}+\lambda\left(\frac{R+r}{2}\right)^2N^{\alpha/2}<-\phi(\epsilon/8)N^{\alpha/2}\right\},$$

$$B=\left\{\max_{x\in[r,R]}\big(L_k^N(xN^\alpha)-pxN^\alpha\big)\le-(\lambda R^2+\phi(\epsilon/8))N^{\alpha/2}\right\}.$$

We aim to bound $\mathbb{P}(B)$, using the fact that $\mathbb{P}(A)\le\epsilon/4$ for large enough N and M by one-point tightness. Recall that with probability $>1-\epsilon/4$, we have

$$prN^\alpha-(\lambda r^2+\phi(\epsilon/8))N^{\alpha/2}<L_1^N(rN^\alpha)<prN^\alpha-(\lambda r^2-\phi(\epsilon/8))N^{\alpha/2},$$
$$pRN^\alpha-(\lambda R^2+\phi(\epsilon/8))N^{\alpha/2}<L_1^N(RN^\alpha)<pRN^\alpha-(\lambda R^2-\phi(\epsilon/8))N^{\alpha/2}$$

Let F denote the subset of B for which these two inequalities hold. Then

$$\mathbb{P}(B)\le\mathbb{P}(F)+\epsilon/4,$$

so it suffices to bound $\mathbb{P}(F)$. To do so, we argue that

$$\mathbb{P}(A\,|\,F)>1/4.$$

for large enough R,N. Let D denote the set of pairs $(\vec{x},\vec{y})$, with $\vec{x},\vec{y}\in\mathfrak{W}_{k-1}$ satisfying

(1) $0\le y_i-x_i\le(R-r)N^\alpha$ for $1\le i\le k$,
(2) $prN^\alpha-(\lambda r^2+\phi(\epsilon/8))N^{\alpha/2}<x_1<prN^\alpha-(\lambda r^2-\phi(\epsilon/8))N^{\alpha/2}$ and $pRN^\alpha-(\lambda R^2+\phi(\epsilon/8))N^{\alpha/2}<y_1<pRN^\alpha-(\lambda R^2-\phi(\epsilon/8))N^{\alpha/2}$.

Let $E(\vec{x},\vec{y})$ denote the subset of F consisting of L^N for which $L_i^N(rN^\alpha)=x_i$ and $L_i^N(RN^\alpha)=y_i$ for $1\le i\le k$, and $L_1^N(s)>\cdots>L_k^N(s)$ for all s. Then D is countable, the $E(\vec{x},\vec{y})$ are pairwise disjoint, and $F=\bigcup_{(\vec{x},\vec{y})\in D}E(\vec{x},\vec{y})$.

We first try to find a lower bound for $\mathbb{P}(A\,|\,E(\vec{x},\vec{y}))$. We first note that by Lemma 3.1, if we raise the endpoints of each curve, then the probability of the event A will decrease. In particular, write $T=(R-r)N^\alpha$, and define $\vec{x}\,',\vec{y}\,'$ by

$$x_i'=\lfloor prN^\alpha-(\lambda r^2-\phi(\epsilon/8))N^{\alpha/2}\rfloor+(k-i)\lceil C\sqrt{T}\,\rceil,$$
$$y_i'=\lfloor pRN^\alpha-(\lambda R^2-\phi(\epsilon/8))N^{\alpha/2}\rfloor+(k-i)\lceil C\sqrt{T}\,\rceil.$$

Here, C is a constant depending only on k,p which we specify in (5.5) below. Note that $x_i'\ge x_1\ge x_i$ for each i by condition (2) above. Furthermore, $x_i'-x_{i+1}'\ge C\sqrt{T}$. The same observations hold for

y_i'. Using Lemma 3.1, we have

$$\begin{aligned}
\mathbb{P}(A \,|\, E(\vec{x},\vec{y})) &= \mathbb{P}_{avoid,Ber}^{rN^\alpha,RN^\alpha,\vec{x},\vec{y},\infty,L_k}(A\,|\,F) \geq \mathbb{P}_{avoid,Ber}^{rN^\alpha,RN^\alpha,\vec{x}\,',\vec{y}\,',\infty,L_k}(A\,|\,F)\\
&\geq \mathbb{P}_{Ber}^{rN^\alpha,RN^\alpha,\vec{x}',\vec{y}'}(A\cap\{L_1>\cdots>L_k\}\,|\,F)\\
&\geq \mathbb{P}_{Ber}^{rN^\alpha,RN^\alpha,\vec{x}',\vec{y}'}(A\,|\,F) - \big(1-\mathbb{P}_{Ber}^{rN^\alpha,RN^\alpha,\vec{x}',\vec{y}'}(L_1>\cdots>L_k\,|\,F)\big)\\
&= \mathbb{P}_{Ber}^{rN^\alpha,RN^\alpha,x_1',y_1'}(A) - \big(1-\mathbb{P}_{Ber}^{rN^\alpha,RN^\alpha,\vec{x}',\vec{y}'}(L_1>\cdots>L_k\,|\,F)\big). \qquad (5.3)
\end{aligned}$$

For the first term in the last line, we used the Gibbs property and the fact that A and F are independent under $\mathbb{P}_{Ber}^{rN^\alpha,RN^\alpha,\vec{x},\vec{y}}$. We now estimate the two terms in (5.3), splitting the remainder of the proof into two steps for clarity.

Step 1. Write $\overline{x}$ and $\overline{y}$ for the floors of the upper bounds on x_1 and y_1 in (2) above, and write $\overline{z} = \overline{y} - \overline{x}$. We can rewrite the first term in (5.3) as

$$\begin{aligned}
&\mathbb{P}_{Ber}^{0,T,x_1',y_1'}\Big(L_1(T/2) - pN^\alpha\frac{R+r}{2} + \lambda\Big(\frac{R+r}{2}\Big)^2 N^{\alpha/2} < -\phi(\epsilon/8)N^{\alpha/2}\Big)\\
&= \mathbb{P}_{Ber}^{0,T,\overline{x},\overline{y}}\Big(L_1(T/2) - pN^\alpha\frac{R+r}{2} + \lambda\Big(\frac{R+r}{2}\Big)^2 N^{\alpha/2} < -\big(\phi(\epsilon/8) + C(k-1)\sqrt{R-r}\big)N^{\alpha/2}\Big)\\
&\geq \mathbb{P}_{Ber}^{0,T,\overline{x},\overline{y}}\Big(L_1(T/2) - \frac{\overline{x}+\overline{y}}{2} < \Big(\lambda\Big(\frac{R^2+r^2}{2}\Big) - \lambda\Big(\frac{R+r}{2}\Big)^2 - C(k-1)\sqrt{R-r} - 2\phi(\epsilon/8)\Big)N^{\alpha/2} - 1\Big).
\end{aligned}$$

The inequality in the last line follows from the definitions of $\overline{x},\overline{y}$. Observe that

$$\frac{R^2+r^2}{2} - \Big(\frac{R+r}{2}\Big)^2 = \frac{R^2+r^2}{4} - \frac{rR}{4} = O(R^2)$$

for fixed r. Thus we can take R large enough depending on $\lambda,k,p,\epsilon,r,\phi$ (recalling that C depends only on k,p) so that the factor multiplying $N^{\alpha/2}$ on the right hand side in the last line is positive. We fix R here for the remainder of the proof. Then we can find a constant $\gamma>0$ so that the last probability is bounded below by

$$\mathbb{P}_{Ber}^{0,T,0,\overline{z}}\Big(L_1(T/2) - \overline{z}/2 < \gamma\sqrt{T}\Big).$$

To estimate this probability, let $\ell^{(T,\overline{z})}$ have the same law as L_1 under a probability measure $\mathbb{P}$ as in Theorem 3.3. Also let B^σ, $\sigma^2 = p(1-p)$, be the Brownian bridge provided by Theorem 3.3. Then the last probability is

$$\begin{aligned}
&\mathbb{P}\Big(\ell^{(T,\overline{z})}(T/2) - \overline{z}/2 < \gamma\sqrt{T}\Big) = \mathbb{P}\Big(\Big[\ell^{(T,\overline{z})}(T/2) - \overline{z}/2 - \sqrt{T}B^\sigma_{1/2}\Big] + \sqrt{T}B^\sigma_{1/2} < \gamma\sqrt{T}\Big)\\
&\geq \mathbb{P}\Big(\sqrt{T}B^\sigma_{1/2} < 0 \quad\text{and}\quad \Delta(T,\overline{z}) < \gamma\sqrt{T}\Big) \geq \frac{1}{2} - \mathbb{P}\Big(\Delta(T,\overline{z}) \geq \gamma\sqrt{T}\Big).
\end{aligned}$$

Here, $\Delta(T,\overline{z})$ is as defined in Theorem 3.3. By Chebyshev's inequality and Theorem 3.3, there are constants K,a,β depending on q, hence on $p,\lambda,k,\epsilon,r,\alpha,\psi,\phi$, such that

$$\mathbb{P}\Big(\Delta(T,\overline{z}) \geq \gamma\sqrt{T}\Big) \leq e^{-a\gamma\sqrt{T}}\mathbb{E}[e^{a\Delta(T,\overline{z})}] \leq K\exp\Big[-a\gamma\sqrt{T} + \beta(\log T)^2 + \frac{|\overline{z}-pT|^2}{T}\Big].$$

Observe that

$$\frac{|\overline{z}-pT|^2}{T} \leq \frac{(\lambda(R^2-r^2)N^{\alpha/2}+1)^2}{(R-r)N^\alpha} \leq 4\lambda^2(R+r)^2(R-r).$$

In particular, we can find $N_{00}\in\mathbb{N}$ large enough depending on $p,\lambda,k,\epsilon,r,\alpha,\psi,\phi$ so that $\mathbb{P}(\Delta(T,\overline{z}) \geq \gamma\sqrt{T}) < 1/6$ for $N \geq N_{00}$. This gives a lower bound of $1/2 - 1/6 = 1/3$ for the first term in (5.3) for $N \geq N_{00}$.

Step 2. It remains to bound $\mathbb{P}_{Ber}^{rN^\alpha,RN^\alpha,\vec{x},\vec{y}}(L_1 > \cdots > L_k \,|\, F)$. Note that on the event F, L_k^N lies uniformly below the line segment connecting $L_1^N(rN^\alpha)$ and $L_1^N(RN^\alpha)$. Thus after raising the endpoints of $\vec{x}\,'$, $\vec{y}\,'$, the bottom curve L_k lies uniformly at a distance of at least $C(k-1)\sqrt{T}$ below the segment. Then in order to have $L_1 > \cdots > L_k$ given F, it suffices to require each L_i to lie within a distance of $C\sqrt{T}/2$ from the line segment connecting its endpoints. That is,

$$
\begin{aligned}
&\mathbb{P}_{Ber}^{rN^\alpha,RN^\alpha,\vec{x}\,',\vec{y}\,'}(L_1 > \cdots > L_k \,|\, F)\\
&\geq \mathbb{P}_{Ber}^{rN^\alpha,RN^\alpha,\vec{x}\,',\vec{y}\,'}\Big(\sup_{x\in[r,R]} \big|L_i(xN^\alpha) - x_i' - (\overline{z}/T)(x-r)N^\alpha\big| \leq C\sqrt{T}/2,\ 1 \leq i \leq k-1 \,\Big|\, F\Big)\\
&= \Big[\mathbb{P}_{Ber}^{0,T,0,\overline{z}}\Big(\sup_{s\in[0,T]} \big|L_1(s + rN^\alpha) - (\overline{z}/T)s\big| > C\sqrt{T}/2\Big)\Big]^{k-1}\\
&= \Big[1 - \mathbb{P}\Big(\sup_{s\in[0,T]} \big|\ell^{(T,\overline{z})} - (\overline{z}/T)s\big| \leq C\sqrt{T}/2\Big)\Big]^{k-1}, \qquad (5.4)
\end{aligned}
$$

with $\mathbb{P}$ and $\ell^{(T,\overline{z})}$ as in Step 1. In the third line, we used the fact that $L_1,\dots,L_{k-1}$ are independent from each other and from L_k under $\mathbb{P}_{Ber}^{rN^\alpha,RN^\alpha,\vec{x},\vec{y}}$. Let B^σ be as in Step 1. Then we have

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{s\in[0,T]} \big|\ell^{(T,\overline{z})}(s) - (\overline{z}/T)s\big| > C\sqrt{T}/2\Big)\\
&\leq \mathbb{P}\Big(\sup_{s\in[0,T]} |\sqrt{T}B^\sigma_{s/T}| > C\sqrt{T}/4\Big) + \mathbb{P}\Big(\Delta(T,\overline{z}) > C\sqrt{T}/4\Big).
\end{aligned}
$$

The first term is equal to

$$2\exp\left(-\frac{2}{\sigma^2}\left(\frac{C}{4}\right)^2\right) = 2e^{-C^2/8p(1-p)}.$$

This follows from (3.40) in Chapter 4 of Karatzas & Shreve. For the second term, Chebyshev's inequality and Theorem 3.3 give an upper bound of

$$K\exp\Big[-aC\sqrt{T}/4 + \beta(\log T)^2 + 4\lambda^2(R+r)^2(R-r)\Big].$$

As in Step 1, we can find N_{01} large enough so that this is $< e^{-C^2/8p(1-p)}$ for $N \geq N_{01}$, after choosing C depending on k,p so that

$$\big(1 - 3e^{-C^2/8p(1-p)}\big)^{k-1} > 11/12. \qquad (5.5)$$

Then for $N \geq N_0 := N_{00} \vee N_{01}$, we get an upper bound of 11/12 in (5.4), independent of $\vec{x},\vec{y}$.

Combining these two estimates, we finally obtain an upper bound of $1/3 - 1/12 = 1/4$ in (5.3), for all $\vec{x},\vec{y}$. It follows that

$$\mathbb{P}(A\,|\,F) = \sum_{(\vec{x},\vec{y})\in D} \frac{\mathbb{P}(A\,|\,E(\vec{x},\vec{y}))\mathbb{P}(E(\vec{x},\vec{y}))}{\mathbb{P}(F)} \geq \frac{1}{4}\cdot\frac{\sum_{(\vec{x},\vec{y})\in D}\mathbb{P}(E(\vec{x},\vec{y}))}{\mathbb{P}(F)} = \frac{1}{4}.$$

Therefore

$$\mathbb{P}(F) \leq 4\mathbb{P}(A) \leq \epsilon$$

for $N \geq N_0$ and R as chosen in Step 1. Essentially the same argument proves the statement if $[r,R]$ is replaced by $[-R,-r]$.

□

We now prove Lemma 4.3. We exploit Lemma 5.5 in order to find two far away points where L_k^N cannot be too low. After separating the curves in order to treat L_k^N as a free curve as in the previous argument, we employ Lemma 3.7 to bound the deviation of L_k^N below the line of slope p.

Proof. We define events

$$A_N(R_2) = \left\{ \inf_{s\in[-t_2,t_2]} \big(L_k^N(s) - ps\big) \le -R_2 N^{\alpha/2} \right\},$$

$$B_N(M,R) = \left\{ \max_{x\in[r+2,R]} \big(L_k^N(xN^\alpha) - pxN^\alpha\big) > -MN^{\alpha/2} \right\} \\ \cap \left\{ \max_{x\in[-R,-r-2]} \big(L_k^N(xN^\alpha) - pxN^\alpha\big) > -MN^{\alpha/2} \right\}.$$

We aim to bound $\mathbb{P}(A_N(R_2))$. By Lemma 5.5, given $\epsilon > 0$, we can find $M, R > 0$ and $N_{30} \in \mathbb{N}$ large enough depending on $\epsilon, \lambda, k, r, p, \alpha, \phi, \psi$ so that $\mathbb{P}(B_N^c(M,R)) < \epsilon/2$ for all $N \ge N_{30}$. Then

$$\mathbb{P}(A_N(R_2)) \le \mathbb{P}(A_N(R_2) \cap B_N(M,R)) + \epsilon/2 \tag{5.6}$$

Thus it suffices to bound $\mathbb{P}(A_N(R_2) \cap B_N(M,R))$. For $0 < a, b \in \mathbb{Z}$ and $\vec{x}, \vec{y} \in \mathfrak{W}_k$, we define $E(a,b,\vec{x},\vec{y})$ to be the event that $L_i^N(-a) = x_i$ and $L_i^N(b) = y_i$ for $1 \le i \le k$, and $L_1^N(s) > \cdots > L_k^N(s)$ for all $s \in [-RN^\alpha, RN^\alpha]$.

We claim that $B_N(M,R)$ can be written as a countable disjoint union of sets $E(a,b,\vec{x},\vec{y})$. Let $D_N(M)$ be the collection of tuples $(a,b,\vec{x},\vec{y})$ satisfying

(1) $a, b \in [rN^\alpha, RN^\alpha]$.
(2) $0 \le y_i - x_i \le b + a$, $x_k + pa > -MN^{\alpha/2}$, and $y_k - pb > -MN^{\alpha/2}$.
(3) If $c, d \in \mathbb{Z}$, $c > a$, and $d > b$, then $L_k^N(-c) + pc \le -MN^{\alpha/2}$ and $L_k^N(d) - pd \le -MN^{\alpha/2}$.

Since there are finitely many integers a, b satisfying (1), the x_i, y_i are integers, and there are finitely many choices of L_i^N on $[-aN^\alpha, bN^\alpha]$ given a, b, x_i, y_i, we see that $D_N(M)$ is countable. The third condition ensures that the $E(a,b,\vec{x},\vec{y})$ are pairwise disjoint. To see that their union over $D_N(M)$ is all of $B_N(M,R)$, note that $B_N(M,R)$ occurs if and only if there is a first integer time $s = -a$ and a last integer time $s = b$ when $L_k^N(s) - ps$ crosses $-MN^{\alpha/2}$. We have

$$\mathbb{P}(A_N(R_2) \cap B_N(M,R)) = \sum_{(a,b,\vec{x},\vec{y})\in D_N(M)} \mathbb{P}(A_N(R_2) \,|\, E(a,b,\vec{x},\vec{y}))\mathbb{P}(E(a,b,\vec{x},\vec{y})).$$

Now

$$\begin{aligned} \mathbb{P}(A_N(R_2) \,|\, E(a,b,\vec{x},\vec{y})) &\le \mathbb{P}^{-a,b,\vec{x},\vec{y}}_{avoid,Ber}\Big(\inf_{s\in[-a,b]} \big(L_k(s) - ps\big) \le -R_2 N^{\alpha/2}\Big) \\ &= \mathbb{P}^{0,a+b,\vec{x},\vec{y}}_{avoid,Ber}\Big(\inf_{s\in[0,a+b]} \big(L_k(s-a) - p(s-a)\big) \le -R_2 N^{\alpha/2}\Big) \\ &\le \mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{avoid,Ber}\Big(\inf_{s\in[0,a+b]} \big(L_k'(s) - p(s-a)\big) \le -R_2 N^{\alpha/2}\Big). \end{aligned} \tag{5.7}$$

Here, we have defined $\vec{x}\,', \vec{y}\,'$ by

$$x_i' = \lfloor -pa - MN^{\alpha/2} \rfloor - (i-1)\lceil CN^{\alpha/2} \rceil,$$
$$y_i' = \lfloor pb - MN^{\alpha/2} \rfloor - (i-1)\lceil CN^{\alpha/2} \rceil.$$

We will specify the constant C below. The last inequality follows from Lemma 3.1 since $x_i' \le -pa - MN^{\alpha/2} \le x_i$ and $y_i' \le pb - MN^{\alpha/2} \le y_i$ by condition (2) above. We also wrote $L_k'(s) = L_k(s-a)$. The last probability is

$$\le \frac{\mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{Ber}\Big(\inf_{s\in[0,a+b]} \big(\ell(s) - p(s-a)\big) \le -R_2 N^{\alpha/2}\Big)}{\mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{Ber}(F)},$$

where

$$F = \{L_1'(s) > \cdots > L_k'(s),\ s \in [0, a+b]\}.$$

Writing $\vec{z} = \vec{y}\,' - \vec{x}\,'$, the numerator is equal to

$$\begin{aligned}
&\mathbb{P}_{Ber}^{0,a+b,x_k',y_k'}\Big(\inf_{s\in[0,a+b]}\big(\ell(s)-p(s-a)\big)\leq -R_2N^{\alpha/2}\Big)\\
&=\mathbb{P}_{Ber}^{0,a+b,0,z_k}\Big(\inf_{s\in[0,a+b]}\big(\ell(s)-ps+pa-\lceil pa+MN^{\alpha/2}\rceil-(k-1)\lceil CN^{\alpha/2}\rceil\big)\leq -R_2N^{\alpha/2}\Big)\\
&\leq \mathbb{P}_{Ber}^{0,a+b,0,z_k}\Big(\inf_{s\in[0,a+b]}\big(\ell(s)-ps\big)\leq -(R_2-M-C(k-1))N^{\alpha/2}+k\Big).
\end{aligned}$$

Since $z_k \geq p(a+b)$ and $a+b \geq 2rN^\alpha$, Lemma 3.7 allows us to find $R_2>0$ depending on $\epsilon, p, \lambda, k, r, \phi$ (see Lemma 5.5 for these dependencies) so that this probability is $<\epsilon/4$ for all large N depending on p,ϵ,α,r, but not on a,b,z_k.

We now bound from below the probability of the event F. The argument is very similar to that in the proof of Lemma 5.5. Write $a=a'N^\alpha, b=b'N^\alpha$, $T=a+b=(a'+b')N^\alpha$, and $z=y_k'-x_k'$. Let $\ell^{(T,z)}$ be a random variable with the same law as the L_i' shifted down by $x_i\,'$ under a measure $\mathbb{P}$, as provided by Theorem 3.3. Let B^σ be a Brownian bridge with variance $\sigma^2=p(1-p)$ coupled with $\ell^{(T,z)}$. Then

$$\begin{aligned}
\mathbb{P}_{Ber}^{0,T,\vec{x}\,',\vec{y}\,'}(F) &\geq \mathbb{P}_{Ber}^{0,T,\vec{x}\,',\vec{y}\,'}\Big(\sup_{s\in[0,T]}\Big|L_i'(s)-x_i'-(z/T)s\Big|<\frac{CN^{\alpha/2}}{2},\ 1\leq i\leq k\Big)\\
&=\Big[1-\mathbb{P}\Big(\sup_{s\in[0,T]}\Big|\ell^{(T,z)}(s)-(z/T)s\Big|\geq C'\sqrt{T}\Big)\Big]^k,
\end{aligned}$$

where in the last line we have written $C'=C/2\sqrt{a'+b'}$. Now

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{s\in[0,T]}\Big|\ell^{(T,z)}(s)-(z/T)s\Big|\geq C'\sqrt{T}\Big)\\
&\leq \mathbb{P}\Big(\sup_{s\in[0,T]}|\sqrt{T}\,B^\sigma_{s/T}|\geq C'\sqrt{T}/2\Big)+\mathbb{P}\Big(\Delta(T,z)\geq C'\sqrt{T}/2\Big),
\end{aligned}$$

where $\Delta(T,z)$ is as defined in Theorem 3.3. The first term is equal to

$$2\exp\left(-\frac{2}{\sigma^2}\Big(\frac{C'}{2}\Big)^2\right)\leq 2e^{-C^2/8\sigma^2(a'+b')}\leq 2e^{-C^2/16p(1-p)R}.$$

This follows from (3.40) in Chapter 4 of Karatzas & Shreve and the fact that $a'+b'\leq 2R$. To estimate the second term, we use Chebyshev's inequality and Theorem 3.3 to find constants K,A,β depending only on p giving an upper bound of

$$\begin{aligned}
e^{-AC'\sqrt{T}/2}\mathbb{E}[e^{A\Delta(T,z)}] &\leq K\exp\Big[-AC'\sqrt{T}/2+\beta(\log T)^2+\frac{|z-pT|^2}{T}\Big]\\
&\leq K\exp\Big[-AC'\sqrt{r/2}\,N^{\alpha/2}+\beta(\log(2RN^\alpha))^2+\frac{1}{2rN^\alpha}\Big].
\end{aligned}$$

For the last line, we used $|z-pT|<1$ and $2rN^\alpha\leq T\leq 2RN^\alpha$. This probability is $<e^{-C^2/16p(1-p)R}$ for large enough N depending on p,k,r,α,R, for all a,b simultaneously. Then

$$\mathbb{P}_{Ber}^{0,T,\vec{x}\,'',\vec{y}\,''}(F)\geq\big(1-3e^{-C^2/16p(1-p)R}\big)^k\geq 1/2,$$

if C is chosen large enough depending on k,p. It follows that the probability in (5.7) is $<2\cdot\epsilon/4=\epsilon/2$ for sufficiently large N depending on p,k,ϵ,r,α, independent of $a,b,\vec{x},\vec{y}$. Thus we can find N_{31} depending on $p,k,\epsilon,\alpha,r,\lambda,\phi$ so that for all $N\geq N_{31}$,

$$\mathbb{P}(A_N(R_2)\cap B_N(M,R))\leq\frac{\epsilon}{2}\sum_{(a,b,\vec{x},\vec{y})\in D_N(M)}\mathbb{P}(E(a,b,\vec{x},\vec{y}))\leq\frac{\epsilon}{2}.$$

Combining with (5.6) proves the result for $N \geq N_3(\epsilon, \lambda, k, r, p, \alpha, \phi, \psi) := N_{30} \vee N_{31}$ and $R_2 := R_2(\epsilon, p, \lambda, k, r, \phi)$.

□

6. Appendix

6.1. **Proof of Lemma 2.2.** We first construct a compact exhaustion of $\Sigma \times \Lambda$. Define the sets

$$K_n := \Sigma_n \times \Lambda_n := \Sigma_n \times [a_n, b_n]$$

as follows. We take Σ_n to be the set of the n smallest elements of Σ, or all of Σ if $n \geq \#(\Sigma)$. If $a \in \Lambda$, i.e, Λ is closed at the left, then $a_n = a$ for all n, and likewise $b_n = b$ if $b \in \Lambda$. If $a \notin \Lambda$, we let $a_n \in \Lambda$, $a_n > a$ be a sequence decreasing to a, for instance $a_n = a + \frac{1}{n}$ if $a > -\infty$, or $a_n = -n$ if $a_n = -\infty$. If $b \notin \Lambda$, we let $b_n \in \Lambda, b_n \nearrow b$. In any case, we see that the sets $K_1 \subset K_2 \subset \cdots \subset \Sigma \times \Lambda$ are compact, they cover $\Sigma \times \Lambda$, and any compact subset K of $\Sigma \times \Lambda$ is contained in all K_n for sufficiently large n.

We now define, for each n and $f, g \in C(\Sigma \times \Lambda)$,

$$d_n(f,g) := \sup_{(i,t)\in K_n} |f(i,t) - g(i,t)|, \quad d_n'(f,g) := \min\{d_n(f,g), 1\}$$

Clearly each d_n is nonnegative and satisfies the triangle inequality, and it is then easy to see that the same properties hold for d_n'. Furthermore, $d_n' \leq 1$, so the function

$$d(f,g) := \sum_{n=1}^{\infty} 2^{-n} d_n'(f,g)$$

in the statement of the lemma is well-defined. We first observe that d is a metric on $C(\Sigma \times \Lambda)$. Indeed, it is nonnegative, and if $f = g$, then each $d_n'(f,g) = 0$, so the sum is 0. Conversely, if $f \neq g$, then since the K_n cover $\Sigma \times \Lambda$, we can choose n large enough so that K_n contains an x with $f(x) \neq g(x)$. Then $d_n'(f,g) \neq 0$, and hence $d(f,g) \neq 0$. The triangle inequality holds for d since it holds for each d_n'.

Now we prove that the topology τ_d on $C(\Sigma \times \Lambda)$ induced by d is the same as the topology of compact convergence, which we will denote τ_c. First, choose $\epsilon > 0$ and $f \in C(\Sigma \times \Lambda)$. Let $g \in B_\epsilon^d(f)$, i.e., $d(f,g) < \epsilon$. We will find a set $A_g \in \tau_c$ such that $g \in A_g \subset B_\epsilon^d(f)$. Let $\delta := d(f,g)$, and choose n large enough so that $\sum_{k>n} 2^{-k} < \frac{\epsilon-\delta}{2}$. Define $A_g := B_{K_n}(g, \frac{\epsilon-\delta}{n})$, and suppose $h \in A_g$. Then since $K_m \subseteq K_n$ for $m \leq n$, we have

$$d(f,h) \leq d(f,g) + d(g,h) \leq \delta + \sum_{k=1}^{n} 2^{-k} d_n(g,h) + \sum_{k>n} 2^{-k} \leq \delta + \frac{\epsilon - \delta}{2} + \frac{\epsilon - \delta}{2} = \epsilon.$$

Therefore $g \in A_g \subset B_\epsilon^d(f)$. It follows that $B_\epsilon^d(f) \in \tau_c$. Indeed, we can write

$$B_\epsilon^d(f) = \bigcup_{g \in B_\epsilon^d(f)} A_g,$$

a union of elements of τ_c. This proves that $\tau_d \subseteq \tau_c$.

To prove the converse, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$. Choose n so that $K \subset K_n$, and let $g \in B_K(f, \epsilon)$ and $\delta := \sup_{x\in K} |f(x) - g(x)|$. If $d(g,h) < 2^{-n}(\epsilon - \delta)$, then $d_n'(g,h) \leq 2^n d(g,h) < \epsilon - \delta$, hence $d_n(g,h) < \epsilon - \delta$. It follows that

$$\sup_{x\in K} |f(x) - h(x)| \leq \delta + \sup_{x\in K} |g(x) - h(x)| \leq \delta + d_n(g,h) \leq \delta + \epsilon - \delta = \epsilon.$$

Thus $g \in B^d_{2^{-n}(\epsilon-\delta)}(f) \subset B_K(f, \epsilon)$. Therefore $\tau_c \subseteq \tau_d$, and we conclude that $\tau_d = \tau_c$.

Next, we show that $(C(\Sigma \times \Lambda), d)$ is a complete metric space. Let $(f_n)_{n\geq 1}$ be Cauchy with respect to d. Then we claim that (f_n) must be Cauchy with respect to d_n', on each K_n. Indeed,

$d(f_\ell, f_m) \geq 2^{-n} d_n'(f_\ell, f_m)$, so if (f_n) were not Cauchy with respect to d_n', it would not be Cauchy with respect to d either. Thus (f_n) is uniformly Cauchy on each K_n, and hence converges uniformly to a limit f^{K_n} on each K_n. Since the limit must be unique at each point of $\Sigma \times \Lambda$, we have $f^{K_n}(x) = f^{K_m}(x)$ if $x \in K_n \cap K_m$. Since $\bigcup K_n = \Sigma \times \Lambda$, we obtain a well-defined function f on all of $\Sigma \times \Lambda$ given by $f(x) = \lim_{n\to\infty} f^{K_n}(x)$. Given any compact $K \subset \Sigma \times \Lambda$, if n is large enough so that $K \subset K_n$, then because $f_n \to f^{K_n} = f|_{K_n}$ uniformly on K_n, we have $f_n \to f^{K_n}|_K = f|_K$ uniformly on K. That is, for any $K \subset \Sigma \times \Lambda$ compact and $\epsilon > 0$, we have $f_n \in B_K(f, \epsilon)$ for all sufficiently large n. Therefore (f_n) converges to f in the topology of compact convergence, and equivalently in the metric d.

Lastly, we prove separability, c.f. [?, Example 1.3]. For each pair of positive integers n, k, let $D_{n,k}$ be the subcollection of $C(\Sigma \times \Lambda)$ consisting of polygonal functions that are piecewise linear on $\{j\} \times I_{n,k,i}$ for each $j \in \Sigma_n$ and each subinterval

$$I_{n,k,i} := \left[a_n + \tfrac{i-1}{k}(b_n - a_n), a_n + \tfrac{i}{k}(b_n - a_n)\right], \quad 1 \leq i \leq k,$$

taking rational values at the endpoints of these subintervals, and extended linearly to all of $\Lambda = [a, b]$. Then $D := \bigcup_{n,k} D_{n,k}$ is countable, and we claim that it is dense in the topology of compact convergence. To see this, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and choose n so that $K \subset K_n$. Since f is uniformly continuous on K_n, we can choose k large enough so that for $0 \leq i \leq k$, if $t \in I_{n,k,i}$, then $|f(j,t) - f(j, a_n + \frac{i}{k}(b_n - a_n))| < \epsilon/2$ for all $j \in \Sigma_n$. We then choose $g \in \bigcup_k D_{n,k}$ with $|g(j, a_n + \frac{i}{k}(b_n - a_n)) - f(j, a_n + \frac{i}{k}(b_n - a_n))| < \epsilon/2$. Then $f(j,t)$ is within ϵ of both $g(j, a_n + \frac{i-1}{k}(b_n - a_n))$ and $g(j, a_n + \frac{i}{k}(b_n - a_n))$. Since $g(j,t)$ lies between these two values, $f(j,t)$ is with ϵ of $g(j,t)$ as well. In summary,

$$\sup_{(j,t)\in K} |f(j,t) - g(j,t)| \leq \sup_{(j,t)\in K_n} |f(j,t) - g(j,t)| < \epsilon,$$

so $g \in B_K(f, \epsilon)$. This proves that D is a countable dense subset of $C(\Sigma \times \Lambda)$.

6.2. **Proof of Lemmas 3.1 and 3.2.** We will prove the following lemma, of which the two lemmas are immediate consequences. In particular, Lemma 3.1 is the special case when $g^b = g^t$, and Lemma 3.2 is the case when $\vec{x} = \vec{x}\,'$ and $\vec{y} = \vec{y}\,'$. We argue in analogy to Lemma 5.6 in Dimitrov-Matestki.

Lemma 6.1. *Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, and two functions $g^b, g^t : [\![T_0, T_1]\!] \to [-\infty, \infty)$ with $g^b \leq g^t$. Also fix $\vec{x}, \vec{y}, \vec{x}\,', \vec{y}\,' \in \mathfrak{W}_k$, such that $g^b(T_0) \leq x_i$, $g^b(T_1) \leq y_i$, $g^t(T_0) \leq x_i'$, $g^t(T_1) \leq y_i'$, and $x_i \leq x_i'$, $y_i \leq y_i'$ for $1 \leq i \leq k$. Assume that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}\,', \vec{y}\,', \infty, g^t)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\big($resp. $\mathfrak{L}^b\big)$ under $\mathbb{P}$ is given by $\mathbb{P}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}_{avoid,Ber}$ $\big($resp. $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber}\big)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}^t_i(r) \geq \mathfrak{L}^b_i(r)$ for all $i = 1, \dots, k$ and $r \in [\![T_0, T_1]\!]$.*

Proof. We split the proof into two steps.

Step 1. We first aim to construct a Markov chain $(X^n, Y^n)_{n\geq 0}$, with
$X^n \in \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$, $Y^n \in \Omega_{avoid}(T_0, T_1, \vec{x}\,', \vec{y}\,', \infty, g^t)$, with initial distribution given by the maximal paths

$$X_1^0(t) = (x_1 + t - T_0) \wedge y_1, \qquad Y_1^0(t) = (x_1' + t - T_0) \wedge y_1'$$
$$X_k^0(t) = (x_k + t - T_0) \wedge y_k \wedge X_{k-1}^0(t), \qquad Y_k^0(t) = (x_k' + t - T_0) \wedge y_k' \wedge Y_{k-1}^0(t).$$

for $t \in [\![T_0, T_1]\!]$. We want this chain to have the following properties:

(1) $(X^n)_{n\geq 0}$ and $(Y^n)_{n\geq 0}$ are both Markov in their own filtrations,
(2) (X^n) is irreducible and has as an invariant distribution the uniform measure $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber}$,

(3) (Y^n) is irreducible and has invariant distribution $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g^t}$,

(4) $X_i^n \leq Y_i^n$ on $[\![T_0,T_1]\!]$ for all $n \geq 0$ and $1 \leq i \leq k$.

This will allow us to conclude convergence of X^n and Y^n to these two uniform measures.

We specify the dynamics of (X^n, Y^n) as follows. At time n, we uniformly sample a segment $\{t\} \times [z, z+1]$, with $t \in [\![T_0,T_1]\!]$ and $z \in [\![x_k, y_1' - 1]\!]$. We also flip a fair coin, with $\mathbb{P}(\text{heads}) = \mathbb{P}(\text{tails}) = 1/2$. We update X^n and Y^n using the following procedure. For all points $s \neq t$, we set $X^{n+1}(s) = X^n(s)$. If $T_0 < t < T_1$ and $X_i^n(t-1) = z$ and $X_i^n(t+1) = z+1$ (note that this implies $X_i^n(t) \in \{z, z+1\}$), then we set

$$X_i^{n+1}(t) = \begin{cases} z+1, & \text{if heads,} \\ z, & \text{if tails,} \end{cases}$$

assuming that this move does not cause $X_i^{n+1}(t)$ to fall below $g^b(t)$. In all other cases, we leave $X_i^{n+1}(t) = X_i^n(t)$. We update Y^n using the same rule, with g^t in place of g^b. [Maybe add a figure here.] We will verify below in the proof of (4) that X^n and Y^n are in fact non-intersecting for all n, but we assume this for now.

It is easy to see that (X^n, Y^n) is a Markov chain, since at each time n, the value of (X^{n+1}, Y^{n+1}) depends only on the current state (X^n, Y^n), and not on the time n or any of the states prior to time n. Moreover, the value of X^{n+1} depends only on the state X^n, not on Y^n, so (X^n) is a Markov chain in its own filtration. The same applies to (Y^n). This proves the property (1) above.

We now argue that (X^n) is each irreducible. Observe that the initial distribution X^0 is by construction maximal, in the sense that for any $Z \in \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$, we have $Z_i \leq X_i^0$ for all i. Thus to reach Z from the initial state X_0, we only need to move the paths downward, and there is no danger of the paths X_i crossing when we do so. We start by ensuring $X_k^n = Z_k$. We successively sample segments which touch Z_k at each point in $[\![T_0,T_1]\!]$ where Z_k differs from X_k, and choose the appropriate coin flips until the two agree on all of $[\![a,b]\!]$. We repeat this procedure for X_i^n and Z^i, with i descending. Since each of these samples and flips has positive probability, and this process terminates in finitely many steps, the probability of transitioning from X^n to Z after some number of steps is positive. The same reasoning applies to show that (Y^n) is irreducible.

To see that the uniform measure $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$ on $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$ is invariant for (X^n), fix any line ensemble $\omega \in \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$. For simplicity, write μ for the uniform measure and $N = |\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)|$ for the (finite) number of allowable ensembles. Then for all ensembles $\tau \in \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b)$, $\mu(\tau) = 1/N$. Hence

$$\sum_\tau \mu(\tau)\mathbb{P}(X^{n+1} = \omega \mid X^n = \tau) = \frac{1}{N} \sum_\tau \mathbb{P}(X^{n+1} = \omega \mid X^n = \tau)$$
$$= \frac{1}{N} \sum_\tau \mathbb{P}(X^{n+1} = \tau \mid X^n = \omega) = \frac{1}{N} \cdot 1 = \mu(\omega).$$

The second equality is clear if $\tau = \omega$. Otherwise, note that $\mathbb{P}(X_{n+1} = \omega \mid X_n = \tau) \neq 0$ if and only if τ and ω differ only in one indexed path (say the ith) at one point t, where $|\tau_i(t) - \omega_i(t)| = 1$, and this condition is also equivalent to $\mathbb{P}(X^{n+1} = \tau \mid X^n = \omega) \neq 0$. If $X^n = \tau$, there is exactly one choice of segment $\{t\} \times [z, z+1]$ and one coin flip which will ensure $X_i^{n+1}(t) = \omega(t)$, i.e., $X^{n+1} = \omega$. Conversely, if $X^n = \omega$, there is one segment and one coin flip which will ensure $X^{n+1} = \tau$. Since the segments are sampled uniformly and the coin flips are fair, these two conditional probabilities are in fact equal. This proves (2), and an analogous argument proves (3).

Lastly, we argue that $X_i^n \leq Y_i^n$ for all $n \geq 0$ and $1 \leq i \leq k$. The same argument will prove that $X_{i+1}^n \leq X_i^n$ for all n, i, so that X^n is in fact non-intersecting for all n, and likewise for Y^n. This is of course true at $n = 0$. Suppose it holds at some $n \geq 0$. Then since the update rule can only change the values of X_i and Y_i at a single point t, it suffices to look at the possible updates to the

ith curve at a single point $t \in [\![T_0, T_1]\!]$. Notice that the update can only change values by at most 1, and if $Y_i^n(t) - X_i^n(t) = 1$, then the only way the ordering could be violated is if Y_i were lowered and X_i were raised at the next update. But this is impossible, since a coin flip of heads can only raise or leave fixed both curves, and tails can only lower or leave fixed both curves. Thus it suffices to assume $X_i^n(t) = Y_i^n(t)$.

There are two cases to consider that violate the ordering of $X_i^{n+1}(t)$ and $Y_i^{n+1}(t)$. Either (i) $X_i(t)$ is raised but $Y_i(t)$ is left fixed, or (ii) $Y_i(t)$ is lowered yet $X_i(t)$ is left fixed. These can only occur if the curves exhibit one of two specific shapes on $[\![t-1, t+1]\!]$. For $X_i(t)$ to be raised, we must have $X_i^n(t-1) = X_i^n(t) = X_i^n(t+1) - 1$, and for $Y_i(t)$ to be lowered, we must have $Y_i^n(t-1) - 1 = Y_i^n(t) = Y_i^n(t+1)$. From the assumptions that $X_i^n(t) = Y_i^n(t)$, and $X_i^n \leq Y_i^n$, we observe that both of these requirements force the other curve to exhibit the same shape on $[\![t-1, t+1]\!]$. Then the update rule will be the same for both curves, proving that both (i) and (ii) are impossible.

Step 2. It follows from (2) and (3) that $(X^n)_{n\geq 0}$ and $(Y^n)_{n\geq 0}$ converge weakly to $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$ and $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g^t}$ respectively, c.f. Norris, Theorem 1.8.3. In particular, (X^n) and (Y^n) are tight, so $(X^n, Y^n)_{n\geq 0}$ is tight as well. By Prohorov's theorem, it follows that (X^n, Y^n) is relatively compact. Let (n_m) be a sequence such that (X^{n_m}, Y^{n_m}) converges weakly. Then by the Skorohod representation theorem, [?, Theorem 6.7], it follows that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting $C([\![1,k]\!] \times [\![T_0, T_1]\!])$-valued random variables $\mathfrak{X}^n$, $\mathfrak{Y}^n$ and $\mathfrak{X}, \mathfrak{Y}$ such that

(1) The law of $(\mathfrak{X}^n, \mathfrak{Y}^n)$ under $\mathbb{P}$ is the same as that of (X^n, Y^n),
(2) $\mathfrak{X}^n(\omega) \longrightarrow \mathfrak{X}(\omega)$ for all $\omega \in \Omega$,
(3) $\mathfrak{Y}^n(\omega) \longrightarrow \mathfrak{Y}(\omega)$ for all $\omega \in \Omega$.

In particular, (1) implies that $\mathfrak{X}^{n_m}$ has the same law as X^{n_m}, which converges weakly to $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$. It follows from (2) and the uniqueness of limits that $\mathfrak{X}$ has law $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$. Similarly, $\mathfrak{Y}$ has law $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x}',\vec{y}',\infty,g^t}$. Moreover, condition (4) in Step 1 implies that $\mathfrak{X}_i^n \leq \mathfrak{Y}_i^n$, $\mathbb{P}$-a.s., so $\mathfrak{X}_i \leq \mathfrak{Y}_i$ for $1 \leq i \leq k$, $\mathbb{P}$-a.s. Thus we can take $\mathfrak{L}^b := \mathfrak{X}$ and $\mathfrak{L}^t := \mathfrak{Y}$.

□

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