Problem 12

From the previous problem, we have the probability distribution

$$\mathbb{P}(\ell_1, \dots, \ell_N) = \frac{1}{Z} \prod_{1 \le i < j \le N} (\ell_i - \ell_j)^2 \prod_{i=1}^{N} w(\ell_i), \tag{1}$$

where $l_i = x_i d_2 \sqrt{L} + d_1 L$, and

$$w(l_i) = \frac{(A + B - l_i - 1)!(l_i + C - N)!}{(A + N - l_i - 1)! l_i!} \tag{2}$$

By Sterling's Approximation,

$$\begin{aligned}
&(A + B - l_i - 1)! \\
=&\left[(a + b - d_1)L - x_i d_2 \sqrt{L} - 1\right]! \\
\to&\sqrt{2\pi \left[(a + b - d_1)L - x_i d_2 \sqrt{L} - 1\right]} \exp\left[-(a + b - d_1)L + x_i d_2 \sqrt{L} + 1\right] \\
&\exp\left[\left((a + b - d_1)L - x_i d_2 \sqrt{L} - 1\right) \ln\left((a + b - d_1)L - x_i d_2 \sqrt{L} - 1\right)\right]
\end{aligned}$$

Using Taylor expansion,

$$\begin{aligned}
&\ln\left((a + b - d_1)L - x_i d_2 \sqrt{L} - 1\right) \\
=&\ln(a + b - d_1)L - \frac{x_i d_2}{(a + b - d_1)\sqrt{L}} - \frac{x_i^2 d_2^2}{2(a + b - d_1)^2 L} + O(\frac{1}{L})
\end{aligned}$$

The exponent becomes

$$\begin{aligned}
&\left((a + b - d_1)L - x_i d_2 \sqrt{L} - 1\right) \ln\left((a + b - d_1)L - x_i d_2 \sqrt{L} - 1\right) \\
=&\left((a + b - d_1)L - x_i d_2 \sqrt{L} - 1\right)\left(\ln(a + b - d_1)L - \frac{x_i d_2}{(a + b - d_1)\sqrt{L}} - \frac{x_i^2 d_2^2}{2(a + b - d_1)^2 L} + O(\frac{1}{L})\right) \\
=&(a + b - d_1)L \ln(a + b - d_1)L - x_i d_2 \sqrt{L} \ln(a + b - d_1)L - x_i d_2 \sqrt{L} + \frac{x_i^2 d_2^2}{2(a + b - d_1)} + O(\frac{1}{\sqrt{L}})
\end{aligned}$$

Similarly, we can calculate the exponent for other factorials

$$
\begin{aligned}
&\Big((c+d_1)L + x_i d_2\sqrt{L} - N\Big)\ln\Big((c+d_1)L + x_i d_2\sqrt{L} - N\Big)\\
=&(c+d_1)L\ln(c+d_1)L + x_i d_2\sqrt{L}\ln(c+d_1)L - N\ln(c+d_1)L + x_i d_2\sqrt{L} - N - \frac{x_i^2 d_2^2}{2(c+d_1)} + O(\frac{1}{\sqrt{L}})\\
&\Big((a-d_1)L - x_i d_2\sqrt{L} + N - 1\Big)\ln\Big((a-d_1)L - x_i d_2\sqrt{L} + N - 1\Big)\\
=&(a-d_1)L\ln(a-d_1)L - x_i d_2\sqrt{L}\ln(a-d_1)L + (N-1)\ln(a-d_1)L - x_i d_2\sqrt{L}\\
&\qquad + (N-1) - \frac{x_i^2 d_2^2}{2(a-d_1)} + O(\frac{1}{\sqrt{L}})\\
&\Big(d_1 L + x_i d_2\sqrt{L}\Big)\ln\Big(d_1 L + x_i d_2\sqrt{L}\Big)\\
=&d_1 L\ln(d_1 L) + x_i d_2\sqrt{L}\ln(d_1 L) + x_i d_2\sqrt{L} + \frac{x_i^2 d_2^2}{2d_1} + O(\frac{1}{\sqrt{L}})
\end{aligned}
$$

Substitution into yields the limit

$$
\begin{aligned}
&\sqrt{\frac{\Big((a+b-d_1)L - x_i d_2\sqrt{L} - 1\Big)\Big((c+d_1)L + x_i d_2\sqrt{L} - N\Big)}{\Big((a-d_1)L - x_i d_2\sqrt{L} + N - 1\Big)\Big(d_1 L + x_i d_2\sqrt{L}\Big)}}\exp\left(2N - (b+c)L\right)\\
&\exp\Big[(a+b-d_1)L\ln(a+b-d_1)L - x_i d_2\sqrt{L}\ln(a+b-d_1)L - x_i d_2\sqrt{L}\\
&\qquad + \frac{x_i^2 d_2^2}{2(a+b-d_1)} + (c+d_1)L\ln(c+d_1)L + x_i d_2\sqrt{L}\ln(c+d_1)L - N\ln(c+d_1)L\\
&\qquad + x_i d_2\sqrt{L} - N - \frac{x_i^2 d_2^2}{2(c+d_1)} - (a-d_1)L\ln(a-d_1)L + x_i d_2\sqrt{L}\ln(a-d_1)L\\
&\qquad - (N-1)\ln(a-d_1)L + x_i d_2\sqrt{L} - (N-1) + \frac{x_i^2 d_2^2}{2(a-d_1)} - d_1 L\ln(d_1 L) - x_i d_2\sqrt{L}\ln(d_1 L)\\
&\qquad - x_i d_2\sqrt{L} - \frac{x_i^2 d_2^2}{2d_1} + O(\frac{1}{\sqrt{L}})\Big]\\
\to&\exp\left(2N - (b+c)L\right)\exp\left(\frac{x_i^2 d_2^2}{2}(\frac{1}{a+b-d_1} - \frac{1}{c+d_1} + \frac{1}{a-d_1} - \frac{1}{2d_1})\right)\\
&\qquad \cdot\exp\left(x_i d_2\sqrt{L}\ln\frac{(c+d_1)(a-d_1)}{(a+b-d_1)d_1}\right)
\end{aligned}
$$

Solving equations

$$
\begin{aligned}
\frac{(c+d_1)(a-d_1)}{(a+b-d_1)d_1} &= 1\\
\frac{x_i^2 d_2^2}{2}(\frac{1}{a+b-d_1} - \frac{1}{c+d_1} + \frac{1}{a-d_1} - \frac{1}{d_1}) &= -\frac{x_i^2}{2}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
d_1 &= \frac{ac}{b+c} \\
d_2 &= \sqrt{\frac{abc(a+b+c)}{(b^2-c^2)(2a+b+c)}}
\end{aligned}
$$