Asymptotics of Bernoulli Line Ensembles

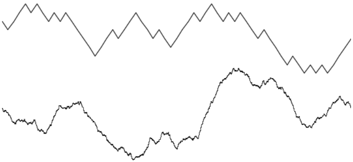
Xiang Fang, Lukas Fesser, Christian Serio, Carson Teitler, and Angela Wang
Advisor: Evgeni Dimitrov
Graduate Student Assistant: Weitao Zhu

Columbia University REU

July 24, 2020

Let $\{X_i\}$ be a sequence of i.i.d. random variables, s.t. $\mathbb{E}[X_1] = \mu$, $Var(X_1^2) = \sigma^2$. Let $S_n = \sum_{i=1}^n X_i$:

- **Law of Large Numbers:** $\frac{S_n}{n} \to \mu$ as $n \to \infty$ almost surely
- **Central Limit Theorem:** $\frac{S_n - n\mu}{\sqrt{n}} \to N(0, \sigma^2)$ as $n \to \infty$
- **Donsker's Theorem:** Let $S(x) = S_k$ if $x = k$ and linearly interpolate for $x \in [0, n]$ Let $\mu = 0$ and $\sigma = 1$. Then $\frac{S(n\cdot)}{\sqrt{n}} \in C([0,1])$ and $\frac{S(n\cdot)}{\sqrt{n}} \to B(\cdot)$, where B denotes a standard Brownian Motion.



Multiple Random Walks

Increase the number of walkers (avoiding Bernoulli random walks and Dyson BM)

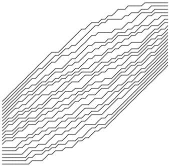
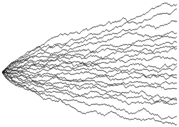


Figure: Multiple Bernoulli Random Walks



What happens as N (number of walkers) goes to infinity? new type of limit occurs Airy line ensemble, top curve is the Airy process. Increasing the number of paths pushes us outside of the Gaussian universality class and into what is called the "KPZ universality class"

Open Question

Big open problem: Show that for "generic random walks" with "generic" initial conditions we have convergence to Airy LE. This problem is open even for Bernoulli random walks (only known if all are started from 0)

Two sufficient conditions:

1. Finite dimensional distribution convergence
2. Tightness, or the existence of weak subsequential limits.

We focused on tightness, which requires a maximum, minimum, and conditions on the Modulus of Continuity

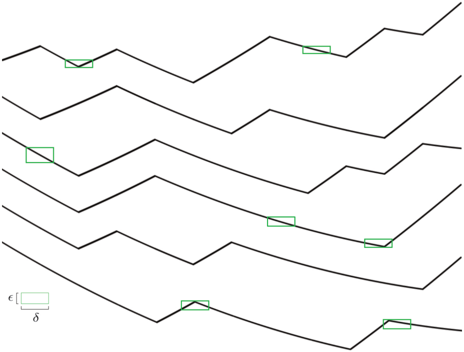


Figure: The Modulus of Continuity

Our Result

Theorem

For $n \in \mathbb{Z}$, there exists a subsequence N_T such that

$$\lim_{T\to\infty} P(L_1^{N_T}(nN^{\alpha}) - nN_T^{\alpha}p + \lambda n^2 N_T^{\alpha/2} \leq N_T^{\alpha/2}x) \to F_{TW}(x)$$

If the one-point marginal probabilities at integer times weakly converge to the Tracy Widom distribution then the Line Ensemble is tight.

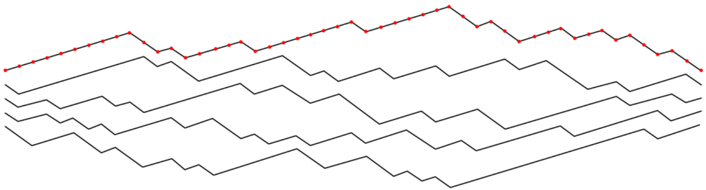


Figure: Integer time points of top line

Improvements

[Duavergne, Nica, & Virag, 2019] - tightness assuming finite dimensional convergence to the Airy Line Ensemble.
We achieve the same result with much less restrictive assumptions

[Unsure of Image Choice]

History of the line ensembles

Arguments in this paper are inspired by

1. *Brownian Gibbs property for Airy line ensembles* and *KPZ line ensemble*[Corwin-Hammond '11, '13], which address the issues of continuous line ensembles
2. *Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood line ensembles* [Corwin-Dimitrov '17], which consider similar questions in a discrete setting

Problem Description

Recall that to show tightness, we want to control

1. min
2. max
3. modulus of continuity of the line ensembles

We claim that for the **top** curve of our line ensemble to have a **parabolic shift**, the **bottom** curve cannot dip too low, i.e. for any $r, \epsilon > 0$, there exist $R, M > 0$ such that for N large enough,

$$P(\max_{[r,R]} L_k(sN^\alpha) - psN^\alpha \leq -MN^\alpha) < \epsilon$$

(perhaps insert a picture)

Proof (mention monotone coupling lemmas somewhere) - say MC with picture 2min

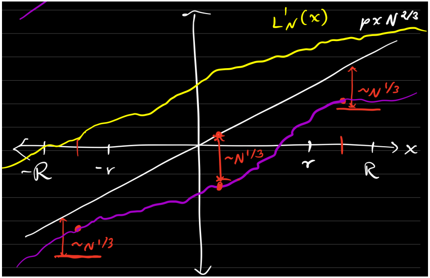
Proof (mention strong coupling somewhere) - say SC with picture L = Bernoulli bridge B is a Brownian bridge with variance. There is a probability space such that $P(sup|L - B| \geq k(\log N)^2) < \epsilon$. This is a comparison that allows for example to compare the modulus of continuity of the two. [Dimitrov-Wu '19] 2 min

Lemma (——)

For any $r, \epsilon > 0$, there exists $R > r$ and a constant $M > 0$ so that for large N,

$$\mathbb{P}\left(\max_{x \in [r,R]} \left(L_k^N(xN^{2/3}) - pxN^{2/3} \right) \leq -MN^{1/3} \right) < \epsilon.$$

The same is true of the maximum on $[-R, -r]$.



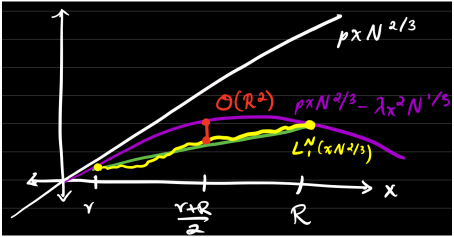
- Couple with a Brownian bridge: if "pinned" at two points $> r$ and $-r$, it cannot be low on scale $N^{1/3}$ on $[-r, r]$.

Proving the pinning lemma

- Recall our assumption:

$$\mathbb{P}\left(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \le xN^{1/3}\right) \xrightarrow[N\to\infty]{} F_{TW}(x).$$

- The top curve looks like a parabola with an affine shift on large scales.



- Two curves: if L_2^N is low on $[r, R]$, L_1^N looks like a free Brownian bridge.

$$\lambda\left(\frac{R^2 + r^2}{2}\right) - \lambda\left(\frac{R+r}{2}\right)^2 = \lambda\frac{R^2 + r^2}{4} - \frac{\lambda rR}{2} = O(R^2).$$

- For large R, the top curve would be far from the parabola at the midpoint!