

# Asymptotics of Bernoulli Gibbsian Line Ensembles

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July 30, 2020

# The Gaussian universality class

Let  $\{X_i\}$  be a sequence of independent identically distributed random variables with mean  $\mu$  and variance  $\sigma^2$ . Let  $S_n = X_1 + \cdots + X_n$ .

- **Law of Large Numbers:**  $\frac{S_n}{n} \longrightarrow \mu$  as  $n \rightarrow \infty$  almost surely.
- **Central Limit Theorem:**  $\frac{S_n - n\mu}{\sigma\sqrt{n}} \implies \mathcal{N}(0, 1)$  as  $n \rightarrow \infty$ .
- **Donsker's Theorem:** For  $t \in [0, 1]$ , let  $W^{(n)}(t) = \frac{S_{nt} - nt\mu}{\sigma\sqrt{n}}$  if  $nt \in \mathbb{N}$ , and linearly interpolate. Then  $W^{(n)} \in C([0, 1])$  and  $W^{(n)} \implies W$  as  $n \rightarrow \infty$ , a standard Brownian motion on  $[0, 1]$ .

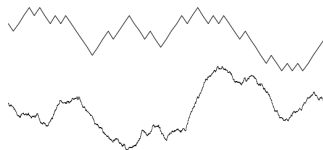
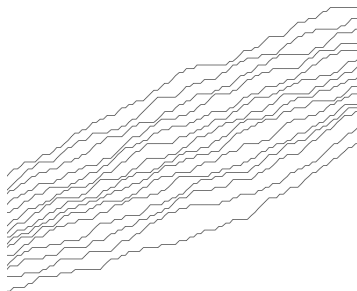


Figure: An example of a random walk and a Brownian motion.

# Multiple random walks

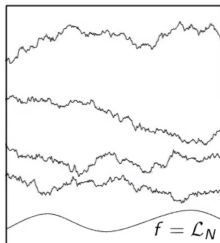
- If  $S_{n+1} - S_n \in \{0, 1\}$ , then  $\{S_n\}_{n=1}^\infty$  is a *Bernoulli random walk*
- An *avoiding Bernoulli line ensemble*  $\mathfrak{L} = (L_1, \dots, L_k)$  consists of  $k$  avoiding Bernoulli random walks on an interval  $[T_0, T_1]$  with random initial and ending points  $\mathfrak{L}(T_0), \mathfrak{L}(T_1)$ , such that  $L_1(s) \geq L_2(s) \geq \dots \geq L_k(s)$  for  $s \in [T_0, T_1]$
- Special case of *Bernoulli Gibbsian line ensembles*



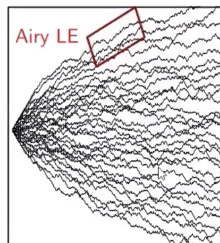
- Question: What does the limit look like as  $k \rightarrow \infty$ ?

# Airy Line Ensemble

As  $k \rightarrow \infty$ ,  $k$  avoiding random walks are conjectured to converge to the *Airy line ensemble*  $\mathcal{A}$ , and the top curve to the *Airy process*  $\mathcal{A}_1$



Avoiding Brownian bridges



Dyson Brownian motion

- Increasing the number of paths pushes us outside of the *Gaussian universality class* and into the *Kardar-Parisi-Zhang (KPZ) universality class*
- Open problem: Show that “generic” random walks with “generic” initial and terminal conditions converge to the *Airy line ensemble*
- We consider this problem for Bernoulli random walks; the proof is only known if all walks start from 0

# Convergence to the Airy Line Ensemble

Two sufficient conditions for convergence in distribution:

- *Finite dimensional* convergence – difficult, requires exact algebraic formulas
- *Tightness* (existence of weak subsequential limits) – easier, more qualitative/analytic

We focused on *tightness*, which we prove by controlling the maximum, the minimum, and the modulus of continuity

$$w(f, \delta) = \sup_{\substack{x, y \in [a, b] \\ |x - y| \leq \delta}} |f(x) - f(y)|$$

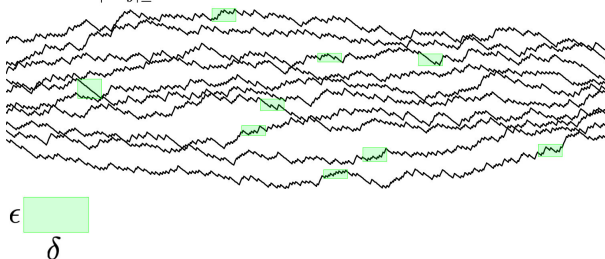


Figure: The modulus of continuity

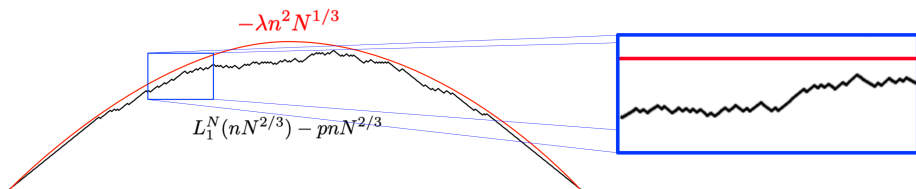
## Theorem (DFFSTWZ)

Let  $\{\mathfrak{L}^N = (L_1^N, \dots, L_k^N)\}_{N=1}^\infty$  be a sequence of  $k$  avoiding Bernoulli random walks. Fix  $p \in (0, 1)$  and  $\lambda > 0$ , and suppose that for all  $n \in \mathbb{Z}$  we have

$$\lim_{N \rightarrow \infty} \mathbb{P}(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \leq N^{1/3}x) = F_{TW}(x).$$

Then  $\{\mathfrak{L}^N\}$  is a tight sequence.

- $F_{TW}$  is the *Tracy-Widom distribution* – the one-point marginal for the Airy process
- [Dauvergne-Nica-Virág '19] Finite dimensional convergence of all curves implies tightness
- Our result shows that it suffices for the integer time one-point marginals of the **top curve** to converge to  $F_{TW}$



Arguments in this paper are inspired by

- 1 *Brownian Gibbs property for Airy line ensembles* [Corwin-Hammond '14] and *KPZ line ensemble* [Corwin-Hammond '16], which address similar issues for **continuous** line ensembles
- 2 *Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood line ensembles* [Corwin-Dimitrov '17], which considers similar questions in a **discrete** setting

To show tightness, we want to control:

- 1 **Minimum** of bottom curve  $L_{k-1}^N$
- 2 **Maximum** of top curve  $L_1^N$
- 3 **Modulus of continuity** of each curve  $L_i^N$

We will focus on bounding the **minimum**:

## Lemma 1 (DFFSTWZ)

Fix  $r, \epsilon > 0$ . Then there exist constants  $M > 0$  and  $N_0 \in \mathbb{N}$  such that for all  $N \geq N_0$ ,

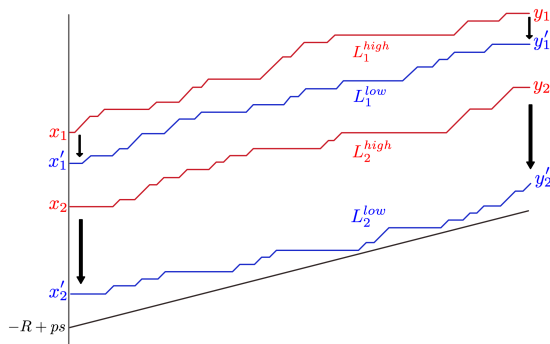
$$\mathbb{P}\left(\inf_{x \in [-r, r]} (L_{k-1}^N(xN^{2/3}) - pxN^{2/3}) < -MN^{1/3}\right) < \epsilon.$$



# Monotone coupling

Lowering entry and exit data  $\vec{x}, \vec{y}$  for the curves  $\implies$  curves shift down on whole interval

- We proved this by adapting arguments from [Corwin-Hammond '14]

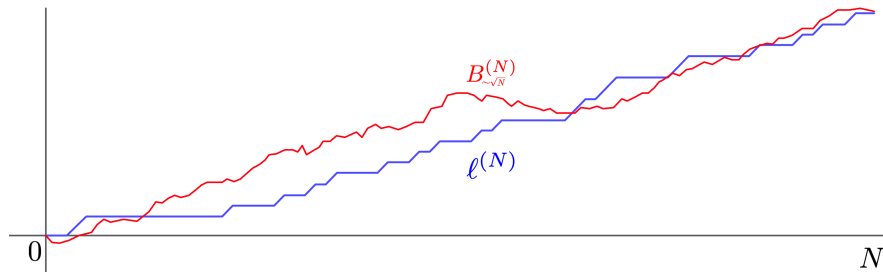


- $\mathcal{L}^{high}$  and  $\mathcal{L}^{low}$  are coupled, in particular

$$\mathbb{P}(L_2^{high} < -R) \leq \mathbb{P}(L_2^{low} < -R)$$

# Strong coupling

A Bernoulli random walk  $\ell^{(N)}$  on  $[0, N]$  can be coupled with an “exponentially close” Brownian bridge  $B^{(N)}$  with variance  $O(\sqrt{N})$  [Dimitrov-Wu '19]



$$\mathbb{P}\left(\sup_{s \in [0, N]} \left| \ell^{(N)}(s) - B^{(N)}(s) \right| \geq M(\log N)^2 + x\right) < K e^{-Ax}$$

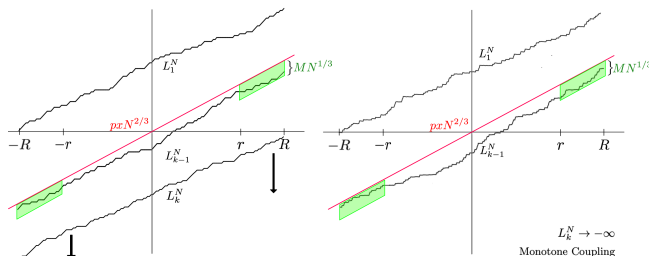
# Proving Lemma 1: pinning the bottom curve

## Lemma 2 (DFFSTWZ)

For any  $r, \epsilon > 0$ , there exists  $R > r$  and a constant  $M > 0$  so that for large  $N$ ,

$$\mathbb{P}\left(\max_{x \in [r, R]} (L_{k-1}^N(xN^{2/3}) - \textcolor{red}{px}N^{2/3}) < -\textcolor{green}{MN}^{1/3}\right) < \epsilon.$$

The same is true of the maximum on  $[-R, -r]$ .



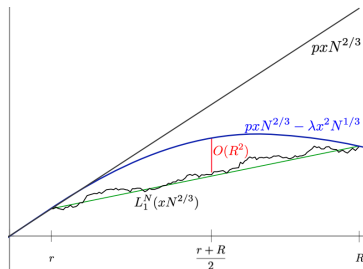
- Use monotone coupling to push  $L_k^N$  to  $-\infty$
- Strongly couple  $L_{k-1}^N$  with a Brownian bridge: if “pinned” at two points in  $[r, R]$  and  $[-R, -r]$ , it cannot be low on  $[-r, r]$  on scale  $N^{1/3}$

# Proving Lemma 2

- Recall our assumption:

$$\mathbb{P}\left(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \leq xN^{1/3}\right) \xrightarrow{N \rightarrow \infty} F_{TW}(x)$$

- The top curve looks like a **parabola** with an affine shift on large scales



- Two curves: if  $L_2^N$  is low on  $[r, R]$ ,  $L_1^N$  looks like a free Brownian bridge

$$\left[-\lambda\left(\frac{R+r}{2}\right)^2\right] - \left[-\lambda\left(\frac{R^2+r^2}{2}\right)\right] = \lambda \frac{R^2 + r^2 - 2rR}{4} = O(R^2)$$

- For large  $R$ , the top curve would be far from the parabola at the midpoint!