Asymptotics of Bernoulli Line Ensembles

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July 27, 2020

Let $\{X_i\}$ be a sequence of independent identically distributed random variables with mean μ and variance σ^2. Let $S_n = X_1 + \cdots + X_n$.

- **Law of Large Numbers:** $\dfrac{S_n}{n} \longrightarrow \mu$ as $n \to \infty$ almost surely.
- **Central Limit Theorem:** $\dfrac{S_n - n\mu}{\sigma\sqrt{n}} \Longrightarrow \mathcal{N}(0,1)$ as $n \to \infty$.
- **Donsker's Theorem:** For $t \in [0,1]$, let $W^{(n)}(t) = \dfrac{S_{nt} - nt\mu}{\sigma\sqrt{n}}$ if $nt \in \mathbb{N}$, and linearly interpolate. Then $W^{(n)} \in C([0,1])$ and $W^{(n)} \Longrightarrow W$ as $n \to \infty$, a standard Brownian motion on $[0,1]$.

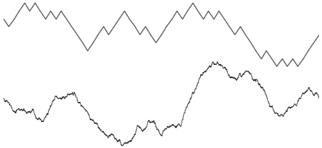
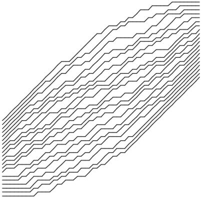


Figure: An example of a random walk and a Brownian motion.

Multiple random walks

If $S_{n+1} - S_n \in \{0,1\}$, then $\{S_n\}_{n=1}^{\infty}$ is a *Bernoulli random walk*. An *avoiding Bernoulli line ensemble* $\mathfrak{L} = (L_1, \dots, L_k)$ consists of k avoiding Bernoulli random walks on an interval $[T_0, T_1]$, such that $L_1(s) \geq L_2(s) \geq \cdots \geq L_k(s)$ for $s \in [T_0, T_1]$.



When dealing with a family of avoiding Brownian Motions, we speak of Dyson Brownian Motion:

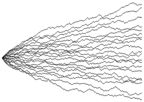


Figure: Dyson Brownian Motion

Airy Line Ensemble

As $k \to \infty$, k avoiding random walks are conjectured to converge to the Airy line ensemble, $\mathcal{A}$, and the top curve to the Airy process, $\mathcal{A}_1$.

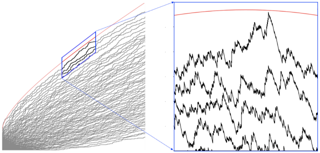


Figure: Multiple Dyson Brownian walks

- Increasing the number of paths pushes us outside of the Gaussian universality class and into Kardar-Parisi-Zhang (KPZ) universality class.
- Open problem: Show that "generic" random walks with "generic" initial conditions converge to the Airy line ensemble.
- We treat this problem for Bernoulli random walks; the proof is only known if all walks start from 0.

Two sufficient conditions for convergence in distribution:

- *Finite dimensional* convergence – difficult, requires exact algebraic formulas
- *Tightness* (existence of weak subsequential limits) – easier, more qualitative/analytic

We focused on tightness, which we prove by controlling the maximum, the minimum, and the modulus of continuity

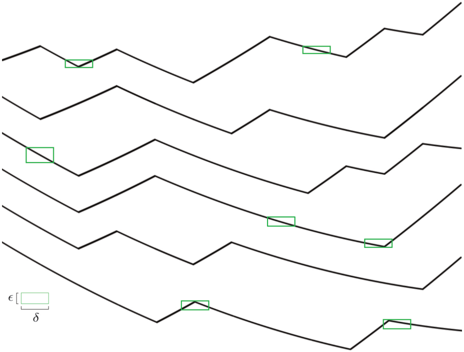


Figure: The Modulus of Continuity

Our Result

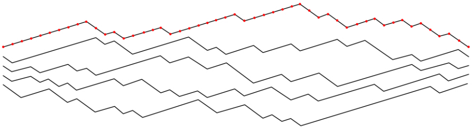
Theorem

Let $\{\mathfrak{L}^N = (L_1^N, \dots, L_k^N)\}_{N=1}^\infty$ be a sequence of avoiding Bernoulli Gibbsian line ensembles. Fix $p \in (0,1)$ and $\lambda > 0$, and suppose that for all $n \in \mathbb{Z}$ we have

$$\lim_{N\to\infty} \mathbb{P}\big(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \le N^{1/3}x\big) = F_{TW}(x).$$

Then $\{\mathfrak{L}^N\}$ is a tight sequence.

- F_{TW} denotes the *Tracy-Widom distribution* – a common limiting distribution in the KPZ universality class.
- [Dauvergne-Nica-Virág '19] showed that finite dimensional convergence of all curves implies tightness, hence convergence to the Airy line ensemble.
- Our result shows that it suffices for the *top curve* to converge in the f.d. sense.



Arguments in this paper are inspired by

1. *Brownian Gibbs property for Airy line ensembles* and *KPZ line ensemble*[Corwin-Hammond '11, '13], which address the issues of continuous line ensembles
2. *Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood line ensembles* [Corwin-Dimitrov '17], which consider similar questions in a discrete setting

Proving tightness

Recall that to show tightness, we want to control:

1. **Minimum** of bottom curve L_k^N,
2. **Maximum** of top curve L_1^N,
3. **Modulus of continuity** of each curve L_i^N.

We will focus on bounding the minimum:

Lemma (——)

Fix $r, \epsilon > 0$. Then there exist constants $R > 0$ and $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,

$$\mathbb{P}\Big(\inf_{x \in [-r,r]} \big(L_k^N(xN^{2/3}) - pxN^{2/3} \big) < -RN^{1/3} \Big) < \epsilon.$$

Monotone coupling

Proof (mention monotone coupling lemmas somewhere) - say MC with picture 2min

Strong coupling with Brownian bridges

Proof (mention strong coupling somewhere) - say SC with picture L = Bernoulli bridge B is a Brownian bridge with variance. There is a probability space such that $P(sup|L - B| \geq k(\log N)^2) < \epsilon$. This is a comparison that allows for example to compare the modulus of continuity of the two. [Dimitrov-Wu '19] 2 min

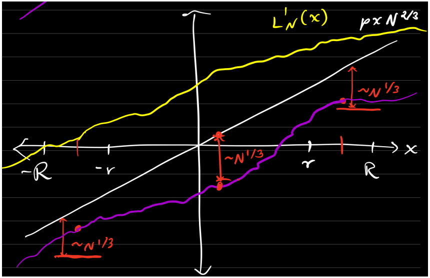
Controlling the minimum: pinning the bottom curve

Lemma (——)

For any $r, \epsilon > 0$, there exists $R > r$ and a constant $M > 0$ so that for large N,

$$\mathbb{P}\Big(\max_{x \in [r,R]} \big(L_k^N(xN^{2/3}) - pxN^{2/3}\big) < -MN^{1/3} \Big) < \epsilon.$$

The same is true of the maximum on $[-R, -r]$.

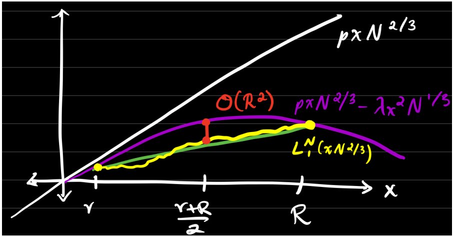


- Couple with a Brownian bridge: if "pinned" at two points $> r$ and $-r$, it cannot be low on scale $N^{1/3}$ on $[-r, r]$.

- Recall our assumption:

$$\mathbb{P}\left(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \le xN^{1/3} \right) \underset{N\to\infty}{\longrightarrow} F_{TW}(x).$$

- The top curve looks like a parabola with an affine shift on large scales.



- Two curves: if L_2^N is low on $[r, R]$, L_1^N looks like a free Brownian bridge.

$$\lambda\left(\frac{R^2+r^2}{2}\right) - \lambda\left(\frac{R+r}{2}\right)^2 = \lambda\frac{R^2+r^2}{4} - \frac{\lambda rR}{2} = O(R^2).$$

- For large R, the top curve would be far from the parabola at the midpoint!