Summer 2020

REU Practice Problems

1 Topology and measurability

We let Σ denote a set $[\![p,q]\!] = \{p, p+1, \dots, q-1, q\}$ for $p \in \mathbb{N}$, $q \in \mathbb{N} \cup \{\infty\}$, and let Λ denote an interval in $\mathbb{R}$ with endpoints $a \le b$. We write $C(X)$ for the space of continuous real-valued functions on X with the topology of compact convergence and the Borel σ-algebra $\mathcal{C}$. Recall that this topology is generated by the basis of sets

$$B_K(f,\epsilon) := \Big\{ g \in C(X) : \sup_{x \in K} |f(x) - g(x)| < \epsilon \Big\},$$

with $K \subset X$ is compact, $f \in C(X)$, and $\epsilon > 0$. When $X = \Sigma \times \Lambda$, we write $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$.

Problem 1

We aim to construct a metric $d : C(\Sigma \times \Lambda) \times C(\Sigma \times \Lambda) \to [0,\infty)$ which induces the topology of compact convergence on $C(\Sigma \times \Lambda)$. The idea is to obtain a compact exhaustion of $\Sigma \times \Lambda$, i.e., a countable collection of compact sets $K_n \subset \Sigma \times \Lambda$ such that $\bigcup_n K_n = \Sigma \times \Lambda$, and such that every compact subset of $\Sigma \times \Lambda$ is contained in some K_n. We then construct d from the sup-metrics on each of these sets K_n. We define the sets

$$K_n := \Sigma_n \times \Lambda_n = [\![p, q_n]\!] \times [a_n, b_n]$$

as follows. We let $q_n = \min(p+n, q)$. If $a \in \Lambda$, i.e, Λ is closed at the left, then $a_n = a$ for all n, and likewise $b_n = b$ if $b \in \Lambda$. If $a \notin \Lambda$, we let $a_n \in \mathbb{R}$, $a_n > a$ be a sequence decreasing to a, for instance $a_n = a + \frac{1}{n}$ if $a > -\infty$, or $a_n = -n$ if $a_n = -\infty$. If $b \notin \Lambda$, we let $b_n \nearrow b$. In any case, we see that the sets $K_1 \subset K_2 \subset \cdots \subset \Sigma \times \Lambda$ are compact, they cover $\Sigma \times \Lambda$, and any compact subset K of $\Sigma \times \Lambda$ is contained in all K_n for sufficiently large n.

We now define, for each n and $f, g \in C(\Sigma \times \Lambda)$,

$$d_n(f,g) := \sup_{(i,t) \in K_n} |f(i,t) - g(i,t)|, \quad d_n'(f,g) := \min\{d_n(f,g), 1\}$$

Clearly each d_n is nonnegative and satisfies the triangle inequality, and it is then easy to see that the same properties hold for d_n'. Furthermore, $d_n' \le 1$, so we can define

$$d(f,g) := \sum_{n=1}^{\infty} 2^{-n} d_n'(f,g).$$

We first observe that d is a metric on $C(\Sigma \times \Lambda)$. Indeed, it is nonnegative, and if $f = g$, then each $d_n'(f,g) = 0$, so the sum is 0. Conversely, if $f \neq g$, then since the K_n cover $\Sigma \times \Lambda$, we can choose n large enough so that K_n contains an x with $f(x) \neq g(x)$. Then $d_n'(f,g) \neq 0$, and hence $d(f,g) \neq 0$. The triangle inequality holds for d since it holds for each d_n'.

Now we prove that the topology τ_d on $C(\Sigma\times\Lambda)$ induced by d is the same as the topology of compact convergence, which we will denote τ_c. First, choose $\epsilon > 0$ and $f \in C(\Sigma\times\Lambda)$. Let $g \in B^d_\epsilon(f)$, i.e., $d(f,g) < \epsilon$. We will find a set $A_g \in \tau_c$ such that $g \in A_g \subset B^d_\epsilon(f)$. Let $\delta := d(f,g)$, and choose n large enough so that $\sum_{k>n} 2^{-k} < \frac{\epsilon-\delta}{2}$. Define $A_g := B_{K_n}(g, \frac{\epsilon-\delta}{n})$, and suppose $h \in A_g$. Then since $K_m \subseteq K_n$ for $m \le n$, we have

$$\begin{aligned} d(f,h) &\le d(f,g) + d(g,h) \\ &\le \delta + \sum_{k=1}^{n} 2^{-k} d_n(g,h) + \sum_{k>n} 2^{-k} \\ &\le \delta + \frac{\epsilon-\delta}{2} + \frac{\epsilon-\delta}{2} = \epsilon. \end{aligned}$$

Therefore $g \in A_g \subset B^d_\epsilon(f)$. It follows that $B^d_\epsilon(f) \in \tau_c$. Indeed, we can write

$$B^d_\epsilon(f) = \bigcup_{g\in B^d_\epsilon(f)} A_g,$$

a union of elements of τ_c. This proves that $\tau_d \subseteq \tau_c$.

To prove the converse, let $K \subset \Sigma\times\Lambda$ be compact, $f \in C(\Sigma\times\Lambda)$, and $\epsilon > 0$. Choose n so that $K \subset K_n$, and let $g \in B_K(f,\epsilon)$ and $\delta := \sup_{x\in K} |f(x) - g(x)|$. If $d(g,h) < 2^{-n}(\epsilon-\delta)$, then $d'_n(g,h) \le 2^n d(g,h) < \epsilon - \delta$, hence $d_n(g,h) < \epsilon-\delta$. It follows that

$$\begin{aligned} \sup_{x\in K} |f(x) - h(x)| &\le \delta + \sup_{x\in K} |g(x)-h(x)| \le \delta + d_n(g,h) \\ &\le \delta + \epsilon - \delta = \epsilon. \end{aligned}$$

Thus $g \in B^d_{2^{-n}(\epsilon-\delta)}(f) \subset B_K(f,\epsilon)$. It follows that $\tau_c \subseteq \tau_d$, and we conclude that $\tau_d = \tau_c$.

Next, we show that $(C(\Sigma\times\Lambda), d)$ is a complete metric space. Let $(f_n)_{n\ge1}$ be Cauchy with respect to d. Then we claim that (f_n) must be Cauchy with respect to d'_n, on each K_n. Indeed, $d(f_\ell, f_m) \ge 2^{-n} d'_n(f_\ell, f_m)$, so if (f_n) were not Cauchy with respect to d'_n, it would not be Cauchy with respect to d either. Thus (f_n) is uniformly Cauchy on each K_n, and hence converges uniformly to a limit f^{K_n} on each K_n. Since the limit must be unique at each point of $\Sigma\times\Lambda$, we have $f^{K_n}(x) = f^{K_m}(x)$ if $x \in K_n \cap K_m$. Since $\bigcup K_n = \Sigma\times\Lambda$, we obtain a well-defined function f on all of $\Sigma\times\Lambda$ given by $f(x) = f^{K_n}(x)$, where $x \in K_n$. Given any compact $K \subset \Sigma\times\Lambda$, if n is large enough so that $K \subset K_n$, then because $f_n \to f^{K_n} = f|_{K_n}$ uniformly on K_n, we have $f_n \to f^{K_n}|_K = f|_K$ uniformly on K. That is, for any $K \subset \Sigma\times\Lambda$ compact and $\epsilon > 0$, we have $f_n \in B_K(f,\epsilon)$ for all sufficiently large n. Therefore (f_n) converges to f in the topology of compact convergence, and equivalently in the metric d.

Lastly, we prove separability, c.f. example 1.3 in Billingsley, *Convergence of Probability Measures*. For each pair of positive integers n,k, let $D_{n,k}$ be the subcollection of $C(\Sigma\times\Lambda)$ consisting of polygonal functions that are piecewise linear on $\{j\}\times I_{n,k,i}$ for each $j \in \Sigma_n$ and each subinterval

$$I_{n,k,i} := [a_n + \tfrac{i-1}{k}(b_n - a_n), a_n + \tfrac{i}{k}(b_n - a_n)], \quad 1 \le i \le k,$$

taking rational values at the endpoints of these subintervals, and extended linearly to all of $\Lambda = [a,b]$. Then $D := \bigcup_{n,k} D_{n,k}$ is countable, and we claim that it is dense in the topology of compact convergence. To see this, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and choose n so that $K \subset K_n$. Since f is uniformly continuous on K_n, we can choose k large enough so that for $0 \le i \le k$, if $t \in I_{n,k,i}$, then $|f(j,t) - f(j, a_n + \frac{i}{k}(b_n - a_n))| < \epsilon/2$ for all $j \in \Sigma_n$. We then choose $g \in \bigcup_k D_{n,k}$ with $|g(j, a_n + \frac{i}{k}(b_n - a_n)) - f(j, a_n + \frac{i}{k}(b_n - a_n))| < \epsilon/2$. Then $f(j,t)$ is within ϵ of both $g(j, a_n + \frac{i-1}{k}(b_n - a_n))$ and $g(j, a_n + \frac{i}{k}(b_n - a_n))$. Since $g(j,t)$ lies between these two values, $f(j,t)$ is with ϵ of $g(j,t)$ as well. In summary,

$$\sup_{(j,t)\in K} |f(j,t) - g(j,t)| \le \sup_{(j,t)\in K_n} |f(j,t) - g(j,t)| < \epsilon,$$

so $g \in B_K(f, \epsilon)$. This proves that D is a countable dense subset of $C(\Sigma \times \Lambda)$. We conclude that $(C(\Sigma \times \Lambda), \tau_c)$ is a Polish space.

Problem 2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X, Y random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $C(\Sigma \times \Lambda)$, where $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$. We consider the collection $\mathcal{S}_X$ of sets of the form

$$\{\omega \in \Omega : X(\omega)(i_1, t_1) \le x_1, \dots, X(\omega)(i_n, t_n) \le x_n\} = \bigcap_{k=1}^{n} X(i_k, t_k)^{-1}(-\infty, x_k],$$

ranging over all $n \in \mathbb{N}$, $(i_1, t_1), \dots, (i_n, t_n) \in \Sigma \times \Lambda$, and $x_1, \dots, x_n \in \mathbb{R}$. We first prove that $\mathcal{S}_X \subset \mathcal{F}$. We can write

$$\{X(i_k, t_k) \le x_k\} = X^{-1}(\{f \in C(\Sigma \times \Lambda) : f(i_k, t_k) \le x_k\}).$$

We claim that the set $\{f \in C(\Sigma \times \Lambda) : f(i_k, t_k) \le x_k\}$ is closed in the topology of compact convergence. If $f_n(i_k, t_k) \le x_k$ for all n and $f_n \to f$ in the topology of compact convergence, then by taking limits on a compact set containing (i_k, t_k), we find $f(i_k, t_k) \le x_k$ as well. This proves the claim, and it follows from the measurability of X that $\{X(i_k, t_k) \le x_k\} = X^{-1}(\{f(i_k, t_k) \le x_k\}) \in \mathcal{F}$. The finite intersection is thus also in $\mathcal{F}$, proving that $\mathcal{S}_X \subset \mathcal{F}$. On the other hand, it is clear that $\{\omega \in \Omega : X(\omega) \in A\} = X^{-1}(A) \in \mathcal{F}$ for any $A \in \mathcal{C}_\Sigma$ since X is measurable.

Now we prove that $\mathbb{P}|_{\mathcal{S}_X}$ determines the distribution $\mathbb{P} \circ X^{-1}$. To do so, note that $\mathcal{S}_X = \sigma(\{X^{-1}(A) : A \in \mathcal{S}\})$, where $\mathcal{S}$ is the collection of cylinder sets

$$\{f \in C(\Sigma \times \Lambda) : f(i_1, t_1) \in A_1, \dots, f(i_n, t_n) \in A_n\}, \quad A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}).$$

This follows from the fact that $\mathcal{B}(\mathbb{R})$ is generated by intervals of the form $(-\infty, x]$. Furthermore, this fact, along with the fact proven above that $\{f(i_k, t_k) \in (-\infty, x_k]\}$ is closed, show that $\mathcal{S} \subset \mathcal{C}_\Sigma$. Observe that the intersection of two elements of $\mathcal{S}$ is clearly another element of $\mathcal{S}$, so $\mathcal{S}$ is a π-system. We now argue that $\mathcal{S}$ generates the Borel sets, i.e., $\sigma(\mathcal{S}) = \mathcal{C}_\Sigma$. Since $\mathcal{S} \subset \mathcal{C}_\Sigma$, we have $\sigma(\mathcal{S}) \subseteq \mathcal{C}_\Sigma$. To prove the opposite inclusion, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and let H be a countable dense subset of K. (Recall that every compact metric space is separable, and K is homeomorphic to a product of finitely many compact sets in $\mathbb{R}$, which are metrizable. So K is separable.) We claim that

$$B_K(f,\epsilon) = \bigcup_{n=1}^{\infty} \bigcap_{(i,t)\in H} \{g \in C(\Sigma\times\Lambda) : g(i,t) \in (f(i,t) - (1-2^{-n})\epsilon, f(i,t) + (1-2^{n})\epsilon)\}.$$

Indeed, if $g \in B_K(f,\epsilon)$, i.e., $\sup_{(i,t)\in K} |g(i,t) - f(i,t)| < \epsilon$. Then since $1 - 2^{-m} \nearrow 1$, we can choose m large enough so that

$$|g(i,t) - f(i,t)| < (1-2^{-n})\epsilon$$

for all $(i,t) \in K$ (in particular with $(i,t) \in H$). Conversely, suppose g is in the set on the right. Then since g is continuous and H is dense in K, we find that for some $n \geq 1$,

$$|g(i,t) - f(i,t)| \leq (1-2^{-n})\epsilon < \epsilon$$

for all $(i,t) \in K$. Hence $g \in B_K(f,\epsilon)$. This proves the claim. Since H is countable, $B_K(f,\epsilon)$ is formed from countably many unions and intersections of sets in $\mathcal{S}$, thus $B_K(f,\epsilon) \in \sigma(\mathcal{S})$.

Now by problem 1, the topology generated by the basis $\mathcal{A} = \{B_K(f,\epsilon)\}$ is separable and metrizable. The balls of rational radii centered at points of a countable dense subset then give a (different) countable basis $\mathcal{B}$ for the same topology. We claim that this implies that every open set is a *countable* union of sets $B_K(f,\epsilon)$. To see this, let $B \in \mathcal{B}$, and write $B = \bigcup_{\alpha\in I} A_\alpha$, for sets $A_\alpha \in \mathcal{A}$. Then for each $x \in B$, pick $\alpha_x \in I$ such that $x \in A_{\alpha_x}$. Since $\mathcal{B}$ is a basis, there is a set $B_x \in \mathcal{B}$ with $x \in B_x \subseteq A_{\alpha_x}$. Then $B = \bigcup_{x\in B} A_{\alpha_x}$. Note that if $y \in B_y \subseteq A_{\alpha_y}$ and $B_y = B_x$, then in fact $y \in A_{\alpha_x}$, so we can remove A_{α_y} from the union. In other words, we can choose the A_{α_x} so that each corresponds to exactly one B_x. But there are only countably many distinct sets B_x, so we see that B is a countable union of elements of $\mathcal{A}$. Since every open set can be written as a countable union of elements of B, this proves the claim. Since $\mathcal{A} \subseteq \sigma(\mathcal{S})$ by the above, it follows that every open set is in $\sigma(\mathcal{S})$, and consequently so is every Borel set, i.e., $\mathcal{C}_\Sigma \subseteq \sigma(\mathcal{S})$.

In summary, we have shown that the collection $\mathcal{S}$ is a π-system generating $\mathcal{C}_\Sigma$, so the probability measure $\mathbb{P}\circ X^{-1}$ on $\mathcal{C}_\Sigma$ is uniquely determined by its restriction to $\mathcal{S}$. Suppose

$$\mathbb{P}\left(\{\omega\in\Omega : X(\omega)(i_1,t_1) \leq x_1, \ldots, X(\omega)(i_n,t_n) \leq x_n\}\right) = \\ \mathbb{P}\left(\{\omega\in\Omega : Y(\omega)(i_1,t_1) \leq x_1, \ldots, Y(\omega)(i_n,t_n) \leq x_n\}\right)$$

for all $(i_1,t_1), x_1, \ldots, x_n$. This says that the two probability measures $\mathbb{P}\circ X^{-1}$ and $\mathbb{P}\circ Y^{-1}$ agree on $\mathcal{S}$. Then they must agree on all of $\mathcal{C}_\Sigma$, i.e.,

$$\mathbb{P}\left(\{\omega\in\Omega : X(\omega)\in A\}\right) = \mathbb{P}\left(\{\omega\in\Omega : Y(\omega)\in A\}\right)$$

for all $A \in \mathcal{C}_\Sigma$. In other words, the law of a line ensemble is determined by its finite dimensional distributions.

2 Algebra

Problem 3

Problem 4

3 Weak convergence

Problem 5

(1)$\phi_n(t) = \mathbb{E}[e^{itY_n}] = \sum_{k=0}^{\infty} p_n(1-p_n)^k e^{itp_nk} = \frac{p_n}{1-(1-p_n)e^{itp_n}}$. Then,

$$\lim_{n\to\infty} \phi_n(t) = \lim_{x\to 0} \frac{x}{1-(1-x)e^{itx}} = \lim_{x\to 0} \frac{1}{1-it(1-x)e^{itx}}(\text{L'Hospital}) = \frac{1}{1-it},$$

which is the characteristic function of exponential random variable with parameter 1. Therefore, Y_n weakly converges to $Z \sim Exp(1)$.
(2) Notice that

$$\begin{aligned}
\frac{d}{dq_n}\mathbb{E}[Y_n^{k-1}] &= \frac{d}{dq_n}[\sum_{x=0}^{\infty} p_n^{k-1}x^{k-1}p_nq_n^x] = \sum_{x=0}^{\infty} x^{k-1}[-kp_n^{k-1}q_n^x + p_n^kxq_n^{x-1}] \\
&= -\frac{k}{p_n}\sum_{x=0}^{\infty}(p_nx)^{k-1}p_nq_n^x + \frac{1}{p_nq_n}\sum_{x=0}^{\infty}(p_nx)^kp_nq_n^x \\
&= -\frac{k}{p_n}\mathbb{E}[Y_n^{k-1}] + \frac{1}{p_nq_n}\mathbb{E}[Y_n^k]
\end{aligned}$$

Therefore, we have

$$\mathbb{E}[Y_n^k] = p_nq_n\frac{d}{dq_n}\mathbb{E}[Y_n^{k-1}] + k \cdot q_n\mathbb{E}[Y^{k-1}]$$

Let $p_n \to 0$, we get $\lim_{n\to\infty} \mathbb{E}[Y_n^k] = k \cdot \lim_{n\to\infty} \mathbb{E}[Y_n^{k-1}]$. Since $\lim_{n\to\infty} \mathbb{E}[Y_n] = \lim_{n\to\infty} p_n \cdot \frac{1-p_n}{p_n} = 1$, we obtain:

$$\lim_{n\to\infty} \mathbb{E}[Y_n^k] = k!$$

which is the k-th moment of exponential random variable with parameter 1.

(3) For a bounded continuous function f which is bounded by M,

$$\mathbb{E}[f(Y_n)] = \sum_{k=0}^{\infty} f(kp_n)p_n(1-p_n)^k \leqslant \frac{M(1-p_n)}{p_n}$$

is well-defined. Notice that $(1-p_n)^k = e^{kln(1-p_n)} = e^{-kp_n+o(p_n)} = e^{-kp_n}(1+o(p_n))$, so

$$\mathbb{E}[f(Y_n)] = \sum_{k=0}^{\infty} f(kp_n)p_ne^{-kp_n} + \sum_{k=0}^{\infty} f(kp_n)p_ne^{-kp_n}o(p_n)$$

For the first term,

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} f(kp_n)p_n e^{-kp_n} = \int_0^{\infty} f(x)e^{-x}dx = \mathbb{E}[f(Y)]$$

by definition of integral, and here we use the continuity of function f. For the second term, it converges to 0. Thus, $\mathbb{E}[f(Y_n)] \xrightarrow{n\to\infty} \mathbb{E}[f(Y)]$.
(4) Consider $\frac{1}{p_n}\cdot p_n(1-p_n)^{k_n}$, where $k_n = x\cdot\frac{1}{p_n}$. Notice that $\frac{1}{p_n}\cdot p_n(1-p_n)^{k_n} = e^{\frac{x}{p_n}ln(1-p_n)} = e^{\frac{x}{p_n}}(-p_n+o(p_n)) = e^{-x+o(1)}$. Consider

$$\begin{aligned}
\mathbb{P}(a\leqslant Y_n\leqslant b) &= \mathbb{P}(\frac{a}{p_n}\leqslant X_n\leqslant\frac{b}{p_n}) \\
&= \sum_{k=m_n}^{M_n}\mathbb{P}(X_n=k) \quad (\text{where } m_n=[\frac{a}{p_n}]+1,\ M_n=[\frac{b}{p_n}]) \\
&= \sum_{k=m_n}^{M_n} p_n e^{x_k+o(1)} \quad (\text{where } x_k=p_nk \text{ and } x_k-x_{k-1}=p_n) \\
&\approx \sum_{k=m_n}^{M_n}\int_{x_k-\frac{1}{2}p_n}^{x_k+\frac{1}{2}p_n} e^{-x}dx = \int_{x_{m_n}-\frac{1}{2}p_n}^{x_{M_n}+\frac{1}{2}p_n} e^{-x}dx \\
&\to \int_a^b e^{-x}dx \quad (\text{as } n\to\infty)
\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty}\mathbb{P}(Y_n\leqslant x) = \int_{-\infty}^{x} e^{-u}du$.

Problem 6

(1)

$$\begin{aligned}
\phi_n(t) = \mathbb{E}[e^{itX_n}] &= \sum_{k=0}^{N_n}\binom{N_n}{k}p_n^k(1-p_n)^{N_n-k}e^{itk} \\
&= (p_ne^{it}+(1-p_n))^{N_n} \\
&= e^{N_nln(1+p_n(e^{it}-1))}
\end{aligned}$$

As $p_n\to 0$, $N_n\to\infty$, $p_nN_n\to\lambda$, we have $ln(1+p_n(e^{it}-1))\to p_n(e^{it}-1)$, and $\lim\limits_{n\to\infty}\phi_n(t) = e^{\lim\limits_{n\to\infty}N_np_n(e^{it}-1)} = e^{\lambda(e^{it}-1)}$, which is the characteristic function of Poisson distribution. Thus, X_n weakly converges to Poisson random variable with parameter λ.
(2) Denote

$$P_{k,n} = \frac{N_n!}{k!(N_n-k)!}\cdot p_n^k(1-p_n)^{N_n-k} = \frac{(p_nN_n)^k}{k!}\cdot\frac{N_n!}{N_n^k(N_n-k)!}(1-p_n)^{N_n-k}$$

Notice that $\frac{N_n!}{N_n^k(N_n-k)!} = \frac{N_n}{N_n}\cdot\frac{N_n-1}{N_n}\cdot\dots\cdot\frac{N_n-k+1}{N_n}\to 1$, as $N_n\to\infty$;

$(1-p_n)^{N_n-k} = e^{(N_n-k)ln(1-p_n)} = e^{(N_n-k)(-p_n+o(p_n))} \to e^{-\lambda}$, as $n \to \infty$; and $\frac{(p_nN_n)^k}{k!} \to \frac{\lambda^k}{k!}$. Therefore, $P_{k,n} \to \frac{\lambda^k}{k!}e^{-\lambda}$ as $n \to \infty$. Then, $\mathbb{P}(X_n \leqslant x) = \sum_{k=1}^{[x]} P_{k,n}$. Let $n \to \infty$, $\mathbb{P}(X_n \leqslant x) = \sum_{k=1}^{[x]} P_{k,n} \to \sum_{k=1}^{[x]} \frac{\lambda^k}{k!}e^{-\lambda}$ is the distribution of Poisson random variable.

Problem 7

(1)

$$
\begin{aligned}
\phi_n(t) = \mathbb{E}[e^{itY_n}] &= \sum_{k=0}^{\infty} e^{-n}\frac{n^k}{k!}e^{it\frac{k-n}{\sqrt{n}}} \\
&= \sum_{k=0}^{\infty} \frac{(ne^{it\frac{1}{\sqrt{n}}})^k}{k!}e^{-it\sqrt{n}-n} \\
&= e^{-it\sqrt{n}-n+ne^{it\frac{1}{\sqrt{n}}}}
\end{aligned}
$$

Notice that $n(e^{it\frac{1}{\sqrt{n}}}-1)-it\sqrt{n} = n(it\frac{1}{\sqrt{n}}+\frac{1}{2}(it\frac{1}{\sqrt{n}})^2+o(\frac{1}{n}))-it\sqrt{n} = -\frac{1}{2}t^2+o(1)$. Therefore, $\phi_n(t) \to e^{-\frac{1}{2}t^2}$ as $n \to \infty$, which is the characteristic function of standard normal random variable.

(2) Let us consider $\lim\limits_{n\to\infty} \sqrt{n}\frac{n^{k_n}}{k_n!}e^{-n}$, where $k_n = x\sqrt{n}+n$. By Stirling's formula, $n! \sim \sqrt{2\pi n}n^ne^{-n}$. Then,

$$
\begin{aligned}
\sqrt{n}\frac{n^{k_n}}{k_n!}e^{-n} &\sim \sqrt{n}\frac{n^{k_n}}{\sqrt{2\pi k_n}k_n^{k_n}e^{-k_n}}e^{-n} \\
&= \frac{\sqrt{n}}{\sqrt{2\pi k_n}}(\frac{n}{k_n})^{k_n}e^{k_n-n} \\
&= \frac{\sqrt{n}}{\sqrt{2\pi k_n}}e^{k_nln(\frac{n}{k_n})+k_n-n}
\end{aligned}
$$

Notice that $k_n = x\sqrt{n}+n \sim O(n)$, we know $\lim\limits_{n\to\infty}\frac{\sqrt{n}}{\sqrt{k_n}} = 1$;

$$
\begin{aligned}
k_nln(\frac{n}{k_n}) &= k_nln(1-\frac{k_n-n}{k_n}) \quad (\frac{k_n-n}{k_n} = \frac{x}{x+\sqrt{n}} \sim O(\frac{1}{\sqrt{n}})) \\
&= k_n(-\frac{k_n-n}{k_n}-\frac{1}{2}(\frac{k_n-n}{k_n})^2+o(\frac{1}{n})) \\
&= -k_n+n-\frac{1}{2}\frac{nx^2}{x\sqrt{n}+n}+o(1) \\
&= -k_n+n-\frac{1}{2}x^2+o(1)
\end{aligned}
$$

Therefore, $\sqrt{n}\frac{n^{k_n}}{k_n!}e^{-n} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2+o(1)}$.

Next, consider: $\mathbb{P}(a \leqslant Y_n \leqslant b) = \mathbb{P}(a\sqrt{n}+n \leqslant X_n \leqslant b_n+n)$. Denote $m_n = [a\sqrt{n}+n]+1$,

$M_n = [b\sqrt{n} + n]$, then

$$\begin{aligned}
\mathbb{P}(a \leqslant Y_n \leqslant b) &= \sum_{k=m_n}^{M_n} \mathbb{P}(X_n = k) \\
&= \sum_{k=m_n}^{M_n} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_k^2}{2} + o(1)} \quad (\text{where } x_k = \frac{k-n}{\sqrt{n}},\ x_k - x_{k-1} = \frac{1}{\sqrt{n}}) \\
&\approx \sum_{k=m_n}^{M_n} \int_{x_k - \frac{1}{2\sqrt{n}}}^{x_k + \frac{1}{2\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \\
&= \int_{x_{m_n} - \frac{1}{2\sqrt{n}}}^{x_{M_n} + \frac{1}{2\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \\
&\to \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \mathbb{P}(Y_n \leqslant x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$.
(3) Suppose $Z_1, Z_2, \ldots, Z_n$, *I.I.D*, are Poisson random variables with parameter 1. Then, $X_n = \sum_{k=1}^{n} Z_k \sim Poisson(n)$, and $\mathbb{E}(X_n) = n$, $Var(X_n) = n$. By Central Limit Theorem, $\frac{X_n - n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1)$.

4 Tightness

Problem 8

Let $\Lambda \subset \mathbb{R}$ be an interval and $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N} \cup \{\infty\}$. Consider the maps

$$\pi_i : C(\Sigma \times \Lambda) \to C(\Lambda), \quad \pi_i(F)(x) = F(i, x), \quad i \in \Sigma.$$

Since $C(X)$ with the topology of compact convergence is metrizable by problem 1, to show that the π_i are continuous, it suffices to show that if $f_n \to f$ in $C(\Sigma \times \Lambda)$, then $\pi_i(f_n) \to \pi_i(f)$ in $C(\Lambda)$. But this is immediate, since if $f_n \to f$ uniformly on compact subsets of $\Sigma \times \Lambda$, then in particular $f_n(i, \cdot) \to f(i, \cdot)$ uniformly on compact subsets of Λ.

Let $(\mathcal{L}^n)$ be a sequence of Σ-indexed line ensembles on Λ, i.e., each $\mathcal{L}^n$ is a $C(\Sigma \times \Lambda)$-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X_i^n := \pi_i(\mathcal{L}^n)$. If A is a Borel set in $C(\Lambda)$, then $(X_i^n)^{-1}(A) = (\mathcal{L}^n)^{-1}(\pi_i^{-1}(A))$. Note $\pi_i^{-1}(A) \in \mathcal{C}_\Sigma$ since π_i is continuous, so it follows that $(X_i^n)^{-1}(A) \in \mathcal{F}$. Thus X_i^n is a $C(\Lambda)$-valued random variable.

Suppose the sequence $(\mathcal{L}^n)$ is tight. Then $(\mathcal{L}^n)$ is relatively compact, that is, every subsequence $(\mathcal{L}^{n_k})$ has a further subsequence $(\mathcal{L}^{n_{k_\ell}})$ converging weakly to some $\mathcal{L}$. Then for each $i \in \Sigma$, since π_i is continuous, the subsequence $(\pi_i(\mathcal{L}^{n_{k_\ell}}))$ of $(\pi_i(\mathcal{L}^{n_k}))$ converges weakly to $\pi_i(\mathcal{L})$ by the continuous mapping theorem. Thus every subsequence of $(\pi_i(\mathcal{L}^n))$ has a convergent subsequence. Since $C(\Lambda)$ is a Polish space by the argument in problem 1, Prohorov's theorem implies that each $(\pi_i(\mathcal{L}^n))$ is tight.

Conversely, suppose $(\pi_i(\mathcal{L}^n))$ is tight for all $i \in \Sigma$. Then for each i, every subsequence $(\pi_i(\mathcal{L}^{n_k}))$ has a further subsequence $(\pi_i(\mathcal{L}^{n_{k_\ell}}))$ converging weakly to some $\mathcal{L}_i$. By diagonalizing the subsequences (n_{k_ℓ}), we obtain a sequence that works for all i, so that $\pi_i(\mathcal{L}^{n_{k_\ell}}) \implies \mathcal{L}_i$ for all i simultaneously. Note that $C(\Sigma\times\Lambda)$ is homeomorphic to $\prod_{i\in\Sigma} C(\Lambda)$ with the product topology, with $f \in C(\Sigma\times\Lambda)$ identified with $(\pi_i(f))_{i\in\Sigma}$. It is not hard to see this by observing that the compact subsets K of $\Sigma \times \Lambda$ are of the form $S \times I$, for S finite and I compact. Thus the homeomorphism identifies the basis elements $B_K(f, \epsilon)$ in $C(\Sigma \times \Lambda)$ with products of open sets U_i in $C(\Lambda)$, such that if $i \notin S$ then simply $U_i = C(\Lambda)$; since S is finite, these products $\prod_i U_i$ are basis elements of the product topology.

Consequently, we can identify the sequence of random variables $\mathcal{L} = (\mathcal{L}_i)_{i\in\Sigma}$ with an element of $C(\Sigma \times \Lambda)$. We argue that $\mathcal{L}^{n_{k_\ell}} \implies \mathcal{L}$. Let U be a basis element in the product topology, i.e., $U = \prod_{i\in\Sigma} U_i$, with each U_i open in $C(\Lambda)$ and all but finitely many $U_i = C(\Lambda)$. Without loss of generality, assume these finitely many $U_i \neq C(\Lambda)$ are $U_1, \dots, U_m$. Then

$$\mathbb{P}(X \in U) = \mathbb{P}(\pi_1(X) \in U_1, \dots, \pi_m(X) \in U_m) = \prod_{i=1}^m \mathbb{P}(\pi_i(X) \in U_i).$$

Therefore, since $\pi_i(\mathcal{L}^{n_{k_\ell}}) \implies \mathcal{L}_i$ for each i,

$$\limsup_{\ell\to\infty} \mathbb{P}(\mathcal{L}^{n_{k_\ell}} \in U) \le \prod_{i=1}^m \limsup_{\ell\to\infty} \mathbb{P}(\pi_i(\mathcal{L}^{n_{k_\ell}}) \in U_i) \le \prod_{i=1}^m \mathbb{P}(\mathcal{L}_i \in U_i) = \mathbb{P}(\mathcal{L} \in U).$$

Now by the same argument as in problem 2, since $C(\Sigma \times \Lambda)$ is a second countable metric space, every open set is a union of countably many sets of the form of U. It follows from countable additivity that the condition above holds if U is replaced by an arbitrary open set. This proves that $\mathcal{L}^{n_{k_\ell}} \implies \mathcal{L}$ as desired. Hence $(\mathcal{L}^n)$ is relatively compact, and it follows from Prohorov's theorem once again that $(\mathcal{L}^n)$ is tight. This completes the proof.

Problem 9

Recall that Theorem 7.3 from Billingsley states that a sequence (P_n) of probability measures on $C[0, 1]$ with the uniform topology is tight if and only if the following hold:

$$\lim_{a\to\infty} \limsup_{n\to\infty} P_n(|x(0)| \ge a) = 0 \tag{1}$$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P_n\left(\sup_{|s-t|\le\delta} |x(s) - x(t)| \ge \epsilon\right) = 0, \quad \forall\, \epsilon > 0. \tag{2}$$

We will find analogous necessary and sufficient conditions for the tightness of $(\mathcal{L}^n)$ on $C(\Sigma \times \Lambda)$ in problem 8. It suffices to find conditions for the tightness of the sequences $(\mathcal{L}_i^n) := (\pi_i(\mathcal{L}_i^n))$ on $C(\Lambda)$, with $i \in \Sigma$. Note $C(\Lambda)$ has the topology of uniform convergence on compact sets, so we must work on the level of compact subsets of Λ. Consider the compact exhaustion $\Lambda = \bigcup_k [a_k, b_k]$ as in problem 1. Recall that $[a_1, b_1] \subseteq [a_2, b_2] \subseteq \cdots$, so $a_1 \in [a_k, b_k]$ for all k. We argue that $(\mathcal{L}_i^n)$ is tight if and only if for every $k \ge 1$, we have

(i)

$$\lim_{a\to\infty} \limsup_{n\to\infty} \mathbb{P}(|\mathcal{L}_i^n(a_1)| \ge a) = 0.$$

(ii) For all $\epsilon > 0$,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{\substack{x,y \in [a_k,b_k], \\ |x-y| \le \delta}} |\mathcal{L}_i^n(x) - \mathcal{L}_i^n(y)| \ge \epsilon \right) = 0.$$

By replacing $[0,1]$ with $[a_k, b_k]$ and 0 with a_1, we see by Theorem 7.3 that these conditions imply that the sequences $(\mathcal{L}_i^n|_{[a_k,b_k]})_n$ are tight, hence relatively compact in the uniform topology on $C[a_k, b_k]$, for every $i \in \Sigma$ and $k \ge 1$. Thus every subsequence $(\mathcal{L}_i^{n_m}|_{[a_k,b_k]})_m$ has a further subsequence $(\mathcal{L}_i^{n_{m_\ell}}|_{[a_k,b_k]})_\ell$ converging weakly to some $\mathcal{L}_i|_{[a_k,b_k]}$. We claim that we can patch these $\mathcal{L}_i|_{[a_k,b_k]}$ together to obtain a well-defined random variable $\mathcal{L}_i$ on all of $C(\Lambda)$, such that $\mathcal{L}_i^{n_{m_\ell}}|_{[a_k,b_k]} \implies \mathcal{L}_i|_{[a_k,b_k]}$ on every $C[a_k, b_k]$ with the uniform topology. To see this, note that this $\mathcal{L}_i$ is uniquely determined by its fdd's, according to problem 2. Given any finite collection of points in Λ, if we take k large enough so that all of these points are in $[a_k, b_k]$, then the corresponding fdd is determined by that of $\mathcal{L}_i|_{[a_k,b_k]}$. Moreover, uniqueness of weak limits in distribution implies that this fdd agrees with that of $\mathcal{L}_i|_{[a_\ell,b_\ell]}$ for any $\ell \ge k$. Thus we have specified well-defined fdd's for $\mathcal{L}_i$, which determines $\mathcal{L}_i$ on all of $C(\Lambda)$. By construction, the restriction of $\mathcal{L}_i$ to any $[a_k, b_k]$ is equal to $\mathcal{L}_i|_{[a_k,b_k]}$ in distribution, so the desired property holds.

If $K \subset \Lambda$ is any compact set, then by taking k large enough so that $K \subset [a_k, b_k]$, we also obtain weak convergence of $\mathcal{L}_i|_K$ in the uniform topology on $C(K)$. Let $B_K(f, \epsilon)$ be a basis element in $C(\Lambda)$, and let $B_\epsilon(f)$ denote the corresponding ball in the uniform topology. Then

$$\begin{aligned}\limsup_{\ell \to \infty} \mathbb{P}(\mathcal{L}_i^{n_{m_\ell}} \in B_K(f,\epsilon)) &= \limsup_{\ell \to \infty} \mathbb{P}(\mathcal{L}_i^{n_{m_\ell}}|_K \in B_\epsilon(f)) \\ &\le \mathbb{P}(\mathcal{L}_i|_K \in B_\epsilon(f)) = \mathbb{P}(\mathcal{L}_i \in B_K(f,\epsilon)).\end{aligned}$$

The inequality follows from weak convergence in the uniform topology on $C(K)$. Since every open set in $C(\Lambda)$ can be written as a countable union of sets $B_K(f, \epsilon)$ (see problem 2), it follows from countable additivity that

$$\limsup_{\ell \to \infty} \mathbb{P}(\mathcal{L}_i^{n_{m_\ell}} \in U) \le \mathbb{P}(\mathcal{L}_i \in U)$$

for any U open in $C(\Lambda)$. Therefore $(\mathcal{L}_i^{n_{m_\ell}})_\ell$ converges weakly to $\mathcal{L}_i$, proving that $(\mathcal{L}_i^n)_n$ is relatively compact, hence tight, for every $i \in \Sigma$. Therefore $(\mathcal{L}^n)_n$ is tight by problem 8.

5 Lozenge tilings of the hexagon

Problem 10

Problem 11