

# REU Practice Problems

## 1 Topology and measurability

We let  $\Sigma$  denote a set  $\llbracket p, q \rrbracket = \{p, p+1, \dots, q-1, q\}$  for  $p \in \mathbb{N}$ ,  $q \in \mathbb{N} \cup \{\infty\}$ , and let  $\Lambda$  denote an interval in  $\mathbb{R}$  with endpoints  $a \leq b$ . We write  $C(X)$  for the space of continuous real-valued functions on  $X$  with the topology of compact convergence and the Borel  $\sigma$ -algebra  $\mathcal{C}$ . Recall that this topology is generated by the basis of sets

$$B_K(f, \epsilon) := \{g \in C(X) : \sup_{x \in K} |f(x) - g(x)| < \epsilon\},$$

with  $K \subset X$  is compact,  $f \in C(X)$ , and  $\epsilon > 0$ . When  $X = \Sigma \times \Lambda$ , we write  $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ .

### Problem 1

We aim to construct a metric  $d : C(\Sigma \times \Lambda) \times C(\Sigma \times \Lambda) \rightarrow [0, \infty)$  which induces the topology of compact convergence on  $C(\Sigma \times \Lambda)$ . The idea is to obtain a compact exhaustion of  $\Sigma \times \Lambda$ , i.e., a countable collection of compact sets  $K_n \subset \Sigma \times \Lambda$  such that  $\bigcup_n K_n = \Sigma \times \Lambda$ , and such that every compact subset of  $\Sigma \times \Lambda$  is contained in some  $K_n$ . We then construct  $d$  from the sup-metrics on each of these sets  $K_n$ . We define the sets

$$K_n := \Sigma_n \times \Lambda_n = \llbracket p, q_n \rrbracket \times [a_n, b_n]$$

as follows. We let  $q_n = \min(p+n, q)$ . If  $a \in \Lambda$ , i.e.,  $\Lambda$  is closed at the left, then  $a_n = a$  for all  $n$ , and likewise  $b_n = b$  if  $b \in \Lambda$ . If  $a \notin \Lambda$ , we let  $a_n \in \mathbb{R}$ ,  $a_n > a$  be a sequence decreasing to  $a$ , for instance  $a_n = a + \frac{1}{n}$  if  $a > -\infty$ , or  $a_n = -n$  if  $a = -\infty$ . If  $b \notin \Lambda$ , we let  $b_n \nearrow b$ . In any case, we see that the sets  $K_1 \subset K_2 \subset \dots \subset \Sigma \times \Lambda$  are compact, they cover  $\Sigma \times \Lambda$ , and any compact subset  $K$  of  $\Sigma \times \Lambda$  is contained in all  $K_n$  for sufficiently large  $n$ .

We now define, for each  $n$  and  $f, g \in C(\Sigma \times \Lambda)$ ,

$$d_n(f, g) := \sup_{(i,t) \in K_n} |f(i, t) - g(i, t)|, \quad d'_n(f, g) := \min\{d_n(f, g), 1\}$$

Clearly each  $d_n$  is nonnegative and satisfies the triangle inequality, and it is then easy to see that the same properties hold for  $d'_n$ . Furthermore,  $d'_n \leq 1$ , so we can define

$$d(f, g) := \sum_{n=1}^{\infty} 2^{-n} d'_n(f, g).$$

We first observe that  $d$  is a metric on  $C(\Sigma \times \Lambda)$ . Indeed, it is nonnegative, and if  $f = g$ , then each  $d'_n(f, g) = 0$ , so the sum is 0. Conversely, if  $f \neq g$ , then since the  $K_n$  cover  $\Sigma \times \Lambda$ , we can choose  $n$  large enough so that  $K_n$  contains an  $x$  with  $f(x) \neq g(x)$ . Then  $d'_n(f, g) \neq 0$ , and hence  $d(f, g) \neq 0$ . The triangle inequality holds for  $d$  since it holds for each  $d'_n$ .

Now we prove that the topology  $\tau_d$  on  $C(\Sigma \times \Lambda)$  induced by  $d$  is the same as the topology of compact convergence, which we will denote  $\tau_c$ . First, choose  $\epsilon > 0$  and  $f \in C(\Sigma \times \Lambda)$ . Let  $g \in B_\epsilon^d(f)$ , i.e.,  $d(f, g) < \epsilon$ . We will find a set  $A_g \in \tau_c$  such that  $g \in A_g \subset B_\epsilon^d(f)$ . Let  $\delta := d(f, g)$ , and choose  $n$  large enough so that  $\sum_{k>n} 2^{-k} < \frac{\epsilon - \delta}{2}$ . Define  $A_g := B_{K_n}(g, \frac{\epsilon - \delta}{n})$ , and suppose  $h \in A_g$ . Then since  $K_m \subseteq K_n$  for  $m \leq n$ , we have

$$\begin{aligned} d(f, h) &\leq d(f, g) + d(g, h) \\ &\leq \delta + \sum_{k=1}^n 2^{-k} d_n(g, h) + \sum_{k>n} 2^{-k} \\ &\leq \delta + \frac{\epsilon - \delta}{2} + \frac{\epsilon - \delta}{2} = \epsilon. \end{aligned}$$

Therefore  $g \in A_g \subset B_\epsilon^d(f)$ . It follows that  $B_\epsilon^d(f) \in \tau_c$ . Indeed, we can write

$$B_\epsilon^d(f) = \bigcup_{g \in B_\epsilon^d(f)} A_g,$$

a union of elements of  $\tau_c$ . This proves that  $\tau_d \subseteq \tau_c$ .

To prove the converse, let  $K \subset \Sigma \times \Lambda$  be compact,  $f \in C(\Sigma \times \Lambda)$ , and  $\epsilon > 0$ . Choose  $n$  so that  $K \subset K_n$ , and let  $g \in B_K(f, \epsilon)$  and  $\delta := \sup_{x \in K} |f(x) - g(x)|$ . If  $d(g, h) < 2^{-n}(\epsilon - \delta)$ , then  $d'_n(g, h) \leq 2^n d(g, h) < \epsilon - \delta$ , hence  $d_n(g, h) < \epsilon - \delta$ . It follows that

$$\begin{aligned} \sup_{x \in K} |f(x) - h(x)| &\leq \delta + \sup_{x \in K} |g(x) - h(x)| \leq \delta + d_n(g, h) \\ &\leq \delta + \epsilon - \delta = \epsilon. \end{aligned}$$

Thus  $g \in B_{2^{-n}(\epsilon - \delta)}^d(f) \subset B_K(f, \epsilon)$ . It follows that  $\tau_c \subseteq \tau_d$ , and we conclude that  $\tau_d = \tau_c$ .

Next, we show that  $(C(\Sigma \times \Lambda), d)$  is a complete metric space. Let  $(f_n)_{n \geq 1}$  be Cauchy with respect to  $d$ . Then we claim that  $(f_n)$  must be Cauchy with respect to  $d'_n$ , on each  $K_n$ . Indeed,  $d(f_\ell, f_m) \geq 2^{-n} d'_n(f_\ell, f_m)$ , so if  $(f_n)$  were not Cauchy with respect to  $d'_n$ , it would not be Cauchy with respect to  $d$  either. Thus  $(f_n)$  is uniformly Cauchy on each  $K_n$ , and hence converges uniformly to a limit  $f^{K_n}$  on each  $K_n$ . Since the limit must be unique at each point of  $\Sigma \times \Lambda$ , we have  $f^{K_n}(x) = f^{K_m}(x)$  if  $x \in K_n \cap K_m$ . Since  $\bigcup K_n = \Sigma \times \Lambda$ , we obtain a well-defined function  $f$  on all of  $\Sigma \times \Lambda$  given by  $f(x) = f^{K_n}(x)$ , where  $x \in K_n$ . Given any compact  $K \subset \Sigma \times \Lambda$ , if  $n$  is large enough so that  $K \subset K_n$ , then because  $f_n \rightarrow f^{K_n} = f|_{K_n}$  uniformly on  $K_n$ , we have  $f_n \rightarrow f^{K_n}|_K = f|_K$  uniformly on  $K$ . That is, for any  $K \subset \Sigma \times \Lambda$  compact and  $\epsilon > 0$ , we have  $f_n \in B_K(f, \epsilon)$  for all sufficiently large  $n$ . Therefore  $(f_n)$  converges to  $f$  in the topology of compact convergence, and equivalently in the metric  $d$ .

Lastly, we prove separability, following example 1.3 in Billingsley, *Convergence of Probability Measures*. For each pair of positive integers  $n, k$ , let  $D_{n,k}$  be the subcollection of  $C(\Sigma \times \Lambda)$  consisting of polygonal functions that are piecewise linear on  $\{j\} \times I_{n,k,i}$  for each  $j \in \Sigma_n$  and each subinterval

$$I_{n,k,i} := [a_n + \frac{i-1}{k}(b_n - a_n), a_n + \frac{i}{k}(b_n - a_n)], \quad 1 \leq i \leq k,$$

taking rational values at the endpoints of these subintervals, and extended linearly to all of  $\Lambda = [a, b]$ . Then  $D := \bigcup_{n,k} D_{n,k}$  is countable, and we claim that it is dense in the topology

of compact convergence. To see this, let  $K \subset \Sigma \times \Lambda$  be compact,  $f \in C(\Sigma \times \Lambda)$ , and  $\epsilon > 0$ , and choose  $n$  so that  $K \subset K_n$ . Since  $f$  is uniformly continuous on  $K_n$ , we can choose  $k$  large enough so that  $|f(j, t) - f(j, a_n + \frac{i}{k}(b_n - a_n))| < \epsilon/2$  for  $j \in \Sigma_n$  and  $1 \leq i \leq k$ . We then choose  $g \in \bigcup_k D_{n,k}$  with  $|g(j, a_n + \frac{i}{k}(b_n - a_n)) - f(j, a_n + \frac{i}{k}(b_n - a_n))| < \epsilon/2$ . Then  $f(j, t)$  is within  $\epsilon$  of both  $g(j, a_n + \frac{i-1}{k}(b_n - a_n))$  and  $g(j, a_n + \frac{i}{k}(b_n - a_n))$ . Since  $g(j, t)$  lies between these two values,  $f(j, t)$  is within  $\epsilon$  of  $g(j, t)$  as well. In summary,

$$\sup_{(j,t) \in K} |f(j, t) - g(j, t)| \leq \sup_{(j,t) \in K_n} |f(j, t) - g(j, t)| < \epsilon,$$

so  $g \in B_K(f, \epsilon)$ . This proves that  $D$  is a countable dense subset of  $C(\Sigma \times \Lambda)$ . We conclude that  $(C(\Sigma \times \Lambda), \tau_c)$  is a Polish space.

## Problem 2

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $X, Y$  random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  taking values in  $C(\Sigma \times \Lambda)$ , where  $\Sigma = \llbracket 1, N \rrbracket$  with  $N \in \mathbb{N}$  or  $N = \infty$ . We consider the collection  $\mathcal{S}_X$  of sets of the form

$$\{\omega \in \Omega : X(\omega)(i_1, t_1) \leq x_1, \dots, X(\omega)(i_n, t_n) \leq x_n\} = \bigcap_{k=1}^n X(i_k, t_k)^{-1}(-\infty, x_k],$$

ranging over all  $n \in \mathbb{N}$ ,  $(i_1, t_1), \dots, (i_n, t_n) \in \Sigma \times \Lambda$ , and  $x_1, \dots, x_n \in \mathbb{R}$ . We first prove that  $\mathcal{S}_X \subset \mathcal{F}$ . We can write

$$\{X(i_k, t_k) \leq x_k\} = X^{-1}(\{f \in C(\Sigma \times \Lambda) : f(i_k, t_k) \leq x_k\}).$$

We claim that the set  $\{f \in C(\Sigma \times \Lambda) : f(i_k, t_k) \leq x_k\}$  is closed in the topology of compact convergence. If  $f_n(i_k, t_k) \leq x_k$  for all  $n$  and  $f_n \rightarrow f$  in the topology of compact convergence, then by taking limits on a compact set containing  $(i_k, t_k)$ , we find  $f(i_k, t_k) \leq x_k$  as well. This proves the claim, and it follows from the measurability of  $X$  that  $\{X(i_k, t_k) \leq x_k\} = X^{-1}(\{f(i_k, t_k) \leq x_k\}) \in \mathcal{F}$ . The finite intersection is thus also in  $\mathcal{F}$ , proving that  $\mathcal{S}_X \subset \mathcal{F}$ . On the other hand, it is clear that  $\{\omega \in \Omega : X(\omega) \in A\} = X^{-1}(A) \in \mathcal{F}$  for any  $A \in \mathcal{C}_\Sigma$  since  $X$  is measurable.

Now we prove that  $\mathbb{P}|_{\mathcal{S}_X}$  determines the distribution  $\mathbb{P} \circ X^{-1}$ . To do so, note that  $\mathcal{S}_X = \sigma(\{X^{-1}(A) : A \in \mathcal{S}\})$ , where  $\mathcal{S}$  is the collection of cylinder sets

$$\{f \in C(\Sigma \times \Lambda) : f(i_1, t_1) \in A_1, \dots, f(i_n, t_n) \in A_n\}, \quad A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}).$$

This follows from the fact that  $\mathcal{B}(\mathbb{R})$  is generated by intervals of the form  $(-\infty, x]$ . Furthermore, this fact, along with the fact proven above that  $\{f(i_k, t_k) \in (-\infty, x_k]\}$  is closed, show that  $\mathcal{S} \subset \mathcal{C}_\Sigma$ . Observe that the intersection of two elements of  $\mathcal{S}$  is clearly another element of  $\mathcal{S}$ , so  $\mathcal{S}$  is a  $\pi$ -system. We now argue that  $\mathcal{S}$  generates the Borel sets, i.e.,  $\sigma(\mathcal{S}) = \mathcal{C}_\Sigma$ . Since  $\mathcal{S} \subset \mathcal{C}_\Sigma$ , we have  $\sigma(\mathcal{S}) \subseteq \mathcal{C}_\Sigma$ . To prove the opposite inclusion, let  $K \subset \Sigma \times \Lambda$  be compact,  $f \in C(\Sigma \times \Lambda)$ , and  $\epsilon > 0$ , and let  $H$  be a countable dense subset of  $K$ . (Recall that every

compact metric space is separable, and  $K$  is homeomorphic to a product of finitely many compact sets in  $\mathbb{R}$ , which are metrizable. So  $K$  is separable.) We claim that

$$B_K(f, \epsilon) = \bigcup_{n=1}^{\infty} \bigcap_{(i,t) \in H} \{g \in C(\Sigma \times \Lambda) : g(i, t) \in (f(i, t) - (1 - 2^{-n})\epsilon, f(i, t) + (1 - 2^{-n})\epsilon)\}.$$

Indeed, if  $g \in B_K(f, \epsilon)$ , i.e.,  $\sup_{(i,t) \in K} |g(i, t) - f(i, t)| < \epsilon$ . Then since  $1 - 2^{-m} \nearrow 1$ , we can choose  $m$  large enough so that

$$|g(i, t) - f(i, t)| < (1 - 2^{-n})\epsilon$$

for all  $(i, t) \in K$  (in particular with  $(i, t) \in H$ ). Conversely, suppose  $g$  is in the set on the right. Then since  $g$  is continuous and  $H$  is dense in  $K$ , we find that for some  $n \geq 1$ ,

$$|g(i, t) - f(i, t)| \leq (1 - 2^{-n})\epsilon < \epsilon$$

for all  $(i, t) \in K$ . Hence  $g \in B_K(f, \epsilon)$ . This proves the claim. Since  $H$  is countable,  $B_K(f, \epsilon)$  is formed from countably many unions and intersections of sets in  $\mathcal{S}$ , thus  $B_K(f, \epsilon) \in \sigma(\mathcal{S})$ .

Now by problem 1, the topology generated by the basis  $\mathcal{A} = \{B_K(f, \epsilon)\}$  is separable and metrizable. The balls of rational radii centered at points of a countable dense subset then give a (different) countable basis  $\mathcal{B}$  for the same topology. We claim that this implies that every open set is a *countable* union of sets  $B_K(f, \epsilon)$ . To see this, let  $B \in \mathcal{B}$ , and write  $B = \bigcup_{\alpha \in I} A_\alpha$ , for sets  $A_\alpha \in \mathcal{A}$ . Then for each  $x \in B$ , pick  $\alpha_x \in I$  such that  $x \in A_{\alpha_x}$ . Since  $\mathcal{B}$  is a basis, there is a set  $B_x \in \mathcal{B}$  with  $x \in B_x \subseteq A_{\alpha_x}$ . Then  $B = \bigcup_{x \in B} B_x$ . Note that if  $y \in B_y \subseteq A_{\alpha_y}$  and  $B_y = B_x$ , then in fact  $y \in A_{\alpha_x}$ , so we can remove  $A_{\alpha_y}$  from the union. In other words, we can choose the  $A_{\alpha_x}$  so that each corresponds to exactly one  $B_x$ . But there are only countably many distinct sets  $B_x$ , so we see that  $B$  is a countable union of elements of  $\mathcal{A}$ . Since every open set can be written as a countable union of elements of  $B$ , this proves the claim. Since  $\mathcal{A} \subseteq \sigma(\mathcal{S})$  by the above, it follows that every open set is in  $\sigma(\mathcal{S})$ , and consequently so is every Borel set, i.e.,  $\mathcal{C}_\Sigma \subseteq \sigma(\mathcal{S})$ .

In summary, we have shown that the collection  $\mathcal{S}$  is a  $\pi$ -system generating  $\mathcal{C}_\Sigma$ , so the probability measure  $\mathbb{P} \circ X^{-1}$  on  $\mathcal{C}_\Sigma$  is uniquely determined by its restriction to  $\mathcal{S}$ . Suppose

$$\begin{aligned} \mathbb{P}(\{\omega \in \Omega : X(\omega)(i_1, t_1) \leq x_1, \dots, X(\omega)(i_n, t_n) \leq x_n\}) = \\ \mathbb{P}(\{\omega \in \Omega : Y(\omega)(i_1, t_1) \leq x_1, \dots, Y(\omega)(i_n, t_n) \leq x_n\}) \end{aligned}$$

for all  $(i_1, t_1), x_1, \dots, x_n$ . This says that the two probability measures  $\mathbb{P} \circ X^{-1}$  and  $\mathbb{P} \circ Y^{-1}$  agree on  $\mathcal{S}$ . Then they must agree on all of  $\mathcal{C}_\Sigma$ , i.e.,

$$\mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) = \mathbb{P}(\{\omega \in \Omega : Y(\omega) \in A\})$$

for all  $A \in \mathcal{C}_\Sigma$ . In other words, the law of a line ensemble is determined by its finite dimensional distributions.

## 2 Algebra

### Problem 3

### Problem 4

## 3 Weak convergence

### Problem 5

(1)  $\phi_n(t) = \mathbb{E}[e^{itY_n}] = \sum_{k=0}^{\infty} p_n(1-p_n)^k e^{itp_n k} = \frac{p_n}{1-(1-p_n)e^{itp_n}}$ . Then,

$$\lim_{n \rightarrow \infty} \phi_n(t) = \lim_{x \rightarrow 0} \frac{x}{1 - (1-x)e^{itx}} = \lim_{x \rightarrow 0} \frac{1}{1 - it(1-x)e^{itx}} \text{ (L'Hospital) } = \frac{1}{1 - it},$$

which is the characteristic function of exponential random variable with parameter 1. Therefore,  $Y_n$  weakly converges to  $Z \sim \text{Exp}(1)$ .

(2) Notice that

$$\begin{aligned} \frac{d}{dq_n} \mathbb{E}[Y_n^{k-1}] &= \frac{d}{dq_n} \left[ \sum_{x=0}^{\infty} p_n^{k-1} x^{k-1} p_n q_n^x \right] = \sum_{x=0}^{\infty} x^{k-1} [-k p_n^{k-1} q_n^x + p_n^k x q_n^{x-1}] \\ &= -\frac{k}{p_n} \sum_{x=0}^{\infty} (p_n x)^{k-1} p_n q_n^x + \frac{1}{p_n q_n} \sum_{x=0}^{\infty} (p_n x)^k p_n q_n^x \\ &= -\frac{k}{p_n} \mathbb{E}[Y_n^{k-1}] + \frac{1}{p_n q_n} \mathbb{E}[Y_n^k] \end{aligned}$$

Therefore, we have

$$\mathbb{E}[Y_n^k] = p_n q_n \frac{d}{dq_n} \mathbb{E}[Y_n^{k-1}] + k \cdot q_n \mathbb{E}[Y_n^{k-1}]$$

Let  $p_n \rightarrow 0$ , we get  $\lim_{n \rightarrow \infty} \mathbb{E}[Y_n^k] = k \cdot \lim_{n \rightarrow \infty} \mathbb{E}[Y_n^{k-1}]$ . Since  $\lim_{n \rightarrow \infty} \mathbb{E}[Y_n] = \lim_{n \rightarrow \infty} p_n \cdot \frac{1-p_n}{p_n} = 1$ , we obtain:

$$\lim_{n \rightarrow \infty} \mathbb{E}[Y_n^k] = k!$$

which is the  $k$ -th moment of exponential random variable with parameter 1.

(3) For a bounded continuous function  $f$  which is bounded by  $M$ ,

$$\mathbb{E}[f(Y_n)] = \sum_{k=0}^{\infty} f(kp_n) p_n (1-p_n)^k \leq \frac{M(1-p_n)}{p_n}$$

is well-defined. Notice that  $(1-p_n)^k = e^{k \ln(1-p_n)} = e^{-kp_n + o(p_n)} = e^{-kp_n} (1 + o(p_n))$ , so

$$\mathbb{E}[f(Y_n)] = \sum_{k=0}^{\infty} f(kp_n) p_n e^{-kp_n} + \sum_{k=0}^{\infty} f(kp_n) p_n e^{-kp_n} o(p_n)$$

For the first term,

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} f(kp_n) p_n e^{-kp_n} = \int_0^{\infty} f(x) e^{-x} dx = \mathbb{E}[f(Y)]$$

by definition of integral, and here we use the continuity of function  $f$ . For the second term, it converges to 0. Thus,  $\mathbb{E}[f(Y_n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}[f(Y)]$ .

(4) Consider  $\frac{1}{p_n} \cdot p_n(1-p_n)^{k_n}$ , where  $k_n = x \cdot \frac{1}{p_n}$ . Notice that  $\frac{1}{p_n} \cdot p_n(1-p_n)^{k_n} = e^{\frac{x}{p_n} \ln(1-p_n)} = e^{\frac{x}{p_n}(-p_n + o(p_n))} = e^{-x+o(1)}$ . Consider

$$\begin{aligned} \mathbb{P}(a \leq Y_n \leq b) &= \mathbb{P}\left(\frac{a}{p_n} \leq X_n \leq \frac{b}{p_n}\right) \\ &= \sum_{k=m_n}^{M_n} \mathbb{P}(X_n = k) \quad (\text{where } m_n = \lfloor \frac{a}{p_n} \rfloor + 1, M_n = \lfloor \frac{b}{p_n} \rfloor) \\ &= \sum_{k=m_n}^{M_n} p_n e^{x_k + o(1)} \quad (\text{where } x_k = p_n k \text{ and } x_k - x_{k-1} = p_n) \\ &\approx \sum_{k=m_n}^{M_n} \int_{x_k - \frac{1}{2}p_n}^{x_k + \frac{1}{2}p_n} e^{-x} dx = \int_{x_{m_n} - \frac{1}{2}p_n}^{x_{M_n} + \frac{1}{2}p_n} e^{-x} dx \\ &\rightarrow \int_a^b e^{-x} dx \quad (\text{as } n \rightarrow \infty) \end{aligned}$$

Therefore,  $\lim_{n \rightarrow \infty} \mathbb{P}(Y_n \leq x) = \int_{-\infty}^x e^{-u} du$ .

## Problem 6

(1)

$$\begin{aligned} \phi_n(t) &= \mathbb{E}[e^{itX_n}] = \sum_{k=0}^{N_n} \binom{N_n}{k} p_n^k (1-p_n)^{N_n-k} e^{itk} \\ &= (p_n e^{it} + (1-p_n))^{N_n} \\ &= e^{N_n \ln(1+p_n(e^{it}-1))} \end{aligned}$$

As  $p_n \rightarrow 0$ ,  $N_n \rightarrow \infty$ ,  $p_n N_n \rightarrow \lambda$ , we have  $\ln(1+p_n(e^{it}-1)) \rightarrow p_n(e^{it}-1)$ , and  $\lim_{n \rightarrow \infty} \phi_n(t) = \lim_{n \rightarrow \infty} e^{N_n p_n(e^{it}-1)} = e^{\lambda(e^{it}-1)}$ , which is the characteristic function of Poisson distribution. Thus,  $X_n$  weakly converges to Poisson random variable with parameter  $\lambda$ .

(2) Denote

$$P_{k,n} = \frac{N_n!}{k!(N_n-k)!} \cdot p_n^k (1-p_n)^{N_n-k} = \frac{(p_n N_n)^k}{k!} \cdot \frac{N_n!}{N_n^k (N_n-k)!} (1-p_n)^{N_n-k}$$

Notice that  $\frac{N_n!}{N_n^k (N_n-k)!} = \frac{N_n}{N_n} \cdot \frac{N_n-1}{N_n} \cdot \dots \cdot \frac{N_n-k+1}{N_n} \rightarrow 1$ , as  $N_n \rightarrow \infty$ ;

$(1 - p_n)^{N_n - k} = e^{(N_n - k) \ln(1 - p_n)} = e^{(N_n - k)(-p_n + o(p_n))} \rightarrow e^{-\lambda}$ , as  $n \rightarrow \infty$ ;  
and  $\frac{(p_n N_n)^k}{k!} \rightarrow \frac{\lambda^k}{k!}$ . Therefore,  $P_{k,n} \rightarrow \frac{\lambda^k}{k!} e^{-\lambda}$  as  $n \rightarrow \infty$ . Then,  $\mathbb{P}(X_n \leq x) = \sum_{k=1}^{[x]} P_{k,n}$ .  
Let  $n \rightarrow \infty$ ,  $\mathbb{P}(X_n \leq x) = \sum_{k=1}^{[x]} P_{k,n} \rightarrow \sum_{k=1}^{[x]} \frac{\lambda^k}{k!} e^{-\lambda}$  is the distribution of Poisson random variable.

## Problem 7

(1)

$$\begin{aligned} \phi_n(t) &= \mathbb{E}[e^{itY_n}] = \sum_{k=0}^{\infty} e^{-n} \frac{n^k}{k!} e^{it \frac{k-n}{\sqrt{n}}} \\ &= \sum_{k=0}^{\infty} \frac{(n e^{it \frac{1}{\sqrt{n}}})^k}{k!} e^{-it\sqrt{n}-n} \\ &= e^{-it\sqrt{n}-n+ne^{it \frac{1}{\sqrt{n}}}} \end{aligned}$$

Notice that  $n(e^{it \frac{1}{\sqrt{n}}} - 1) - it\sqrt{n} = n(it \frac{1}{\sqrt{n}} + \frac{1}{2}(it \frac{1}{\sqrt{n}})^2 + o(\frac{1}{n})) - it\sqrt{n} = -\frac{1}{2}t^2 + o(1)$ . Therefore,  $\phi_n(t) \rightarrow e^{-\frac{1}{2}t^2}$  as  $n \rightarrow \infty$ , which is the characteristic function of standard normal random variable.

(2) Let us consider  $\lim_{n \rightarrow \infty} \sqrt{n} \frac{n^{k_n}}{k_n!} e^{-n}$ , where  $k_n = x\sqrt{n} + n$ . By Stirling's formula,  $n! \sim \sqrt{2\pi n} n^n e^{-n}$ . Then,

$$\begin{aligned} \sqrt{n} \frac{n^{k_n}}{k_n!} e^{-n} &\sim \sqrt{n} \frac{n^{k_n}}{\sqrt{2\pi k_n} k_n^{k_n} e^{-k_n}} e^{-n} \\ &= \frac{\sqrt{n}}{\sqrt{2\pi k_n}} \left(\frac{n}{k_n}\right)^{k_n} e^{k_n - n} \\ &= \frac{\sqrt{n}}{\sqrt{2\pi k_n}} e^{k_n \ln(\frac{n}{k_n}) + k_n - n} \end{aligned}$$

Notice that  $k_n = x\sqrt{n} + n \sim O(n)$ , we know  $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{k_n}} = 1$ ;

$$\begin{aligned} k_n \ln\left(\frac{n}{k_n}\right) &= k_n \ln\left(1 - \frac{k_n - n}{k_n}\right) \quad \left(\frac{k_n - n}{k_n} = \frac{x}{x + \sqrt{n}} \sim O\left(\frac{1}{\sqrt{n}}\right)\right) \\ &= k_n \left(-\frac{k_n - n}{k_n} - \frac{1}{2} \left(\frac{k_n - n}{k_n}\right)^2 + o\left(\frac{1}{n}\right)\right) \\ &= -k_n + n - \frac{1}{2} \frac{nx^2}{x\sqrt{n} + n} + o(1) \\ &= -k_n + n - \frac{1}{2} x^2 + o(1) \end{aligned}$$

Therefore,  $\sqrt{n} \frac{n^{k_n}}{k_n!} e^{-n} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 + o(1)}$ .

Next, consider:  $\mathbb{P}(a \leq Y_n \leq b) = \mathbb{P}(a\sqrt{n} + n \leq X_n \leq b_n + n)$ . Denote  $m_n = [a\sqrt{n} + n] + 1$ ,

$M_n = \lfloor b\sqrt{n} + n \rfloor$ , then

$$\begin{aligned}
\mathbb{P}(a \leq Y_n \leq b) &= \sum_{k=m_n}^{M_n} \mathbb{P}(X_n = k) \\
&= \sum_{k=m_n}^{M_n} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_k^2}{2} + o(1)} \quad (\text{where } x_k = \frac{k-n}{\sqrt{n}}, x_k - x_{k-1} = \frac{1}{\sqrt{n}}) \\
&\approx \sum_{k=m_n}^{M_n} \int_{x_k - \frac{1}{2\sqrt{n}}}^{x_k + \frac{1}{2\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{x_{m_n} - \frac{1}{2\sqrt{n}}}^{x_{M_n} + \frac{1}{2\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&\rightarrow \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx
\end{aligned}$$

Therefore,  $\lim_{n \rightarrow \infty} \mathbb{P}(Y_n \leq x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$ .

(3) Suppose  $Z_1, Z_2, \dots, Z_n$ , *I.I.D.*, are Poisson random variables with parameter 1. Then,  $X_n = \sum_{k=1}^n Z_k \sim \text{Poisson}(n)$ , and  $\mathbb{E}(X_n) = n$ ,  $\text{Var}(X_n) = n$ . By Central Limit Theorem,  $\frac{X_n - n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1)$ .

## 4 Tightness

### Problem 8

Let  $\Lambda \subset \mathbb{R}$  be an interval and  $\Sigma = \llbracket 1, N \rrbracket$  with  $N \in \mathbb{N} \cup \{\infty\}$ . Consider the maps

$$\pi_i : C(\Sigma \times \Lambda) \rightarrow C(\Lambda), \quad \pi_i(F)(x) = F(i, x), \quad i \in \Sigma.$$

Since  $C(X)$  with the topology of compact convergence is metrizable by problem 1, to show that the  $\pi_i$  are continuous, it suffices to show that if  $f_n \rightarrow f$  in  $C(\Sigma \times \Lambda)$ , then  $\pi_i(f_n) \rightarrow \pi_i(f)$  in  $C(\Lambda)$ . But this is immediate, since if  $f_n \rightarrow f$  uniformly on compact subsets of  $\Sigma \times \Lambda$ , then in particular  $f_n(i, \cdot) \rightarrow f(i, \cdot)$  uniformly on compact subsets of  $\Lambda$ .

Let  $\mathcal{L}^n$  be a sequence of  $\Sigma$ -indexed line ensembles on  $\Lambda$ , i.e., each  $\mathcal{L}^n$  is a  $C(\Sigma \times \Lambda)$ -valued random variable on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let  $X_i^n := \pi_i(\mathcal{L}^n)$ . If  $A$  is a Borel set in  $C(\Lambda)$ , then  $(X_i^n)^{-1}(A) = (\mathcal{L}^n)^{-1}(\pi_i^{-1}(A))$ . Note  $\pi_i^{-1}(A) \in \mathcal{C}_\Sigma$  since  $\pi_i$  is continuous, so it follows that  $(X_i^n)^{-1}(A) \in \mathcal{F}$ . Thus  $X_i^n$  is a  $C(\Lambda)$ -valued random variable.

Suppose the sequence  $(\mathcal{L}^n)$  is tight. Then  $(\mathcal{L}^n)$  is relatively compact, that is, every subsequence  $(\mathcal{L}^{n_k})$  has a further subsequence  $(\mathcal{L}^{n_{k_\ell}})$  converging weakly to some  $\mathcal{L}$ . Then for each  $i \in \Sigma$ , since  $\pi_i$  is continuous, the subsequence  $(\pi_i(\mathcal{L}^{n_{k_\ell}}))$  of  $(\pi_i(\mathcal{L}^{n_k}))$  converges weakly to  $\pi_i(\mathcal{L})$  by the continuous mapping theorem. Thus every subsequence of  $(\pi_i(\mathcal{L}^n))$  has a convergent subsequence. Since  $C(\Lambda)$  is a Polish space by the argument in problem 1, Prohorov's theorem implies that each  $(\pi_i(\mathcal{L}^n))$  is tight.



Conversely, suppose  $(\pi_i(\mathcal{L}^n))$  is tight for all  $i \in \Sigma$ . Then for each  $i$ , every subsequence  $(\pi_i(\mathcal{L}^{n_k}))$  has a further subsequence  $(\pi_i(\mathcal{L}^{n_{k_\ell}}))$  converging weakly to some  $\mathcal{L}_i$ . By diagonalizing the subsequences  $(n_{k_\ell})$ , we obtain a sequence that works for all  $i$ , so that  $\pi_i(\mathcal{L}^{n_{k_\ell}}) \Rightarrow \mathcal{L}_i$  for all  $i$  simultaneously. Since  $C(\Sigma \times \Lambda)$  is homeomorphic to  $\prod_{i \in \Sigma} C(\Lambda)$ , we can identify the sequence of random variables  $(\mathcal{L}_i)_{i \in \Sigma}$  with an element of  $C(\Sigma \times \Lambda)$ . Then  $\mathcal{L}^{n_{k_\ell}} \Rightarrow (\mathcal{L}_i)_{i \in \Sigma}$ . [prove this]

[Or maybe argue this way instead:] If  $Q^n$  is the law of  $\mathcal{L}^n$  as above and  $Q_i$  is the law of  $\mathcal{L}_i$ , we claim that  $Q^{n_{k_\ell}}$  converges weakly to the product measure  $Q_1 \otimes Q_2 \otimes \cdots$ .

## Problem 9

## 5 Lozenge tilings of the hexagon

### Problem 10

### Problem 11