

# REU Practice Problems

## 1 Topology and measurability

We let  $\Sigma$  denote a set  $\llbracket p, q \rrbracket = \{p, p+1, \dots, q-1, q\}$  for  $p \in \mathbb{N}$ ,  $q \in \mathbb{N} \cup \{\infty\}$ , and let  $\Lambda$  denote an interval in  $\mathbb{R}$ . We write  $C(X)$  for the space of continuous real-valued functions on  $X$  with the topology of compact convergence and the Borel  $\sigma$ -algebra  $\mathcal{C}$ . Recall that this topology is generated by the basis of sets

$$B_K(f, \epsilon) := \{g \in C(X) : \sup_{x \in K} |f(x) - g(x)| < \epsilon\},$$

with  $K \subset X$  is compact,  $f \in C(X)$ , and  $\epsilon > 0$ . When  $X = \Sigma \times \Lambda$ , we write  $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ .

### Problem 1

We aim to construct a metric  $d : C(\Sigma \times \Lambda) \times C(\Sigma \times \Lambda) \rightarrow [0, \infty)$  which induces the topology of compact convergence on  $C(\Sigma \times \Lambda)$ . The idea is to write  $\Sigma \times \Lambda$  as a union of compact sets  $K_n$ , such that every compact subset of  $\Sigma \times \Lambda$  is contained in one of these sets  $K_n$ . We then construct  $d$  from the sup-metrics on each of these sets  $K_n$ . We define the sets

$$K_n := \llbracket p, \min(p+n, q) \rrbracket \times \Lambda_n$$

as follows. If  $\Lambda = [a, b]$  is compact, then  $\Lambda_n = \Lambda$  for all  $n$ . If  $\Lambda = (a, b)$ , then

$$\Lambda_n := \left[ a + \frac{1}{n}, b - \frac{1}{n} \right],$$

when this set makes sense, and  $\Lambda_n = \emptyset$  otherwise. If  $\Lambda$  is half-open, we define  $\Lambda_n$  similarly, but only modify the endpoint on the open side. (This might not be necessary, not sure if  $\Lambda$  is assume to be closed.) In any case, we see that the sets  $K_1 \subset K_2 \subset \dots \subset \Sigma \times \Lambda$  are compact, they cover  $\Sigma \times \Lambda$ , and any compact subset  $K$  of  $\Sigma \times \Lambda$  is contained in all  $K_n$  for sufficiently large  $n$ .

We now define, for each  $n$  and  $f, g \in C(\Sigma \times \Lambda)$ ,

$$d_n(f, g) := \sup_{x \in K_n} |f(x) - g(x)|, \quad d'_n(f, g) := \min\{d_n(f, g), 1\}.$$

Clearly each  $d_n$  is nonnegative and satisfies the triangle inequality, and it is then easy to see that the same properties hold for  $d'_n$ . Furthermore,  $d'_n \leq 1$ , so we can define

$$d(f, g) := \sum_{n=1}^{\infty} 2^{-n} d'_n(f, g).$$

We first observe that  $d$  is a metric on  $C(\Sigma \times \Lambda)$ . Indeed, it is nonnegative, and if  $f = g$ , then each  $d'_n(f, g) = 0$ , so the sum is 0. Conversely, if  $f \neq g$ , then since the  $K_n$  cover  $\Sigma \times \Lambda$ , we

can choose  $n$  large enough so that  $K_n$  contains an  $x$  with  $f(x) \neq g(x)$ . Then  $d'_n(f, g) \neq 0$ , and hence  $d(f, g) \neq 0$ . The triangle inequality holds for  $d$  since it holds for each  $d'_n$ .

Now we prove that the topology  $\tau_d$  on  $C(\Sigma \times \Lambda)$  induced by  $d$  is the same as the topology of compact convergence, which we will denote  $\tau_c$ . First, choose  $\epsilon > 0$  and  $f \in C(\Sigma \times \Lambda)$ . Let  $g \in B_\epsilon^d(f)$ , i.e.,  $d(f, g) < \epsilon$ . We will find a set  $A_g \in \tau_c$  such that  $g \in A_g \subset B_\epsilon^d(f)$ . Let  $\delta := d(f, g)$ , and choose  $n$  large enough so that  $\sum_{k>n} 2^{-k} < \frac{\epsilon - \delta}{2}$ . Define  $A_g := B_{K_n}(g, \frac{\epsilon - \delta}{n})$ , and suppose  $h \in A_g$ . Then since  $K_m \subseteq K_n$  for  $m \leq n$ , we have

$$\begin{aligned} d(f, h) &\leq d(f, g) + d(g, h) \\ &\leq \delta + \sum_{k=1}^n 2^{-k} d_n(g, h) + \sum_{k>n} 2^{-k} \\ &\leq \delta + \frac{\epsilon - \delta}{2} + \frac{\epsilon - \delta}{2} = \epsilon. \end{aligned}$$

Therefore  $g \in A_g \subset B_\epsilon^d(f)$ . It follows that  $B_\epsilon^d(f) \in \tau_c$ . Indeed, we can write

$$B_\epsilon^d(f) = \bigcup_{g \in B_\epsilon^d(f)} A_g,$$

a union of elements of  $\tau_c$ . This proves that  $\tau_d \subseteq \tau_c$ .

To prove the converse, let  $K \subset \Sigma \times \Lambda$  be compact,  $f \in C(\Sigma \times \Lambda)$ , and  $\epsilon > 0$ . Choose  $n$  so that  $K \subset K_n$ , and let  $g \in B_K(f, \epsilon)$  and  $\delta := \sup_{x \in K} |f(x) - g(x)|$ . If  $d(g, h) < 2^{-n}(\epsilon - \delta)$ , then  $d'_n(g, h) \leq 2^n d(g, h) < \epsilon - \delta$ , hence  $d_n(g, h) < \epsilon - \delta$ . It follows that

$$\begin{aligned} \sup_{x \in K} |f(x) - h(x)| &\leq \delta + \sup_{x \in K} |g(x) - h(x)| \leq \delta + d_n(g, h) \\ &\leq \delta + \epsilon - \delta = \epsilon. \end{aligned}$$

Thus  $g \in B_{2^{-n}(\epsilon - \delta)}^d(f) \subset B_K(f, \epsilon)$ . It follows that  $\tau_c \subseteq \tau_d$ , and we conclude that  $\tau_d = \tau_c$ .

Next, we show that  $(C(\Sigma \times \Lambda), d)$  is a complete metric space. Let  $(f_n)_{n \geq 1}$  be Cauchy with respect to  $d$ . Then we claim that  $(f_n)$  must be Cauchy with respect to  $d'_n$ , on each  $K_n$ . Indeed,  $d(f_\ell, f_m) \geq 2^{-n} d'_n(f_\ell, f_m)$ , so if  $(f_n)$  were not Cauchy with respect to  $d'_n$ , it would not be Cauchy with respect to  $d$  either. Thus  $(f_n)$  is uniformly Cauchy on each  $K_n$ , and hence converges uniformly to a limit  $f^{K_n}$  on each  $K_n$ . Since the limit must be unique at each point of  $\Sigma \times \Lambda$ , we have  $f^{K_n}(x) = f^{K_m}(x)$  if  $x \in K_n \cap K_m$ . Since  $\bigcup K_n = \Sigma \times \Lambda$ , we obtain a well-defined function  $f$  on all of  $\Sigma \times \Lambda$  given by  $f(x) = f^{K_n}(x)$ , where  $x \in K_n$ . Given any compact  $K \subset \Sigma \times \Lambda$ , if  $n$  is large enough so that  $K \subset K_n$ , then because  $f_n \rightarrow f^{K_n} = f|_{K_n}$  uniformly on  $K_n$ , we have  $f_n \rightarrow f|_K$  uniformly on  $K$ . That is, for any  $K \subset \Sigma \times \Lambda$  compact and  $\epsilon > 0$ , we have  $f_n \in B_K(f, \epsilon)$  for all sufficiently large  $n$ . Therefore  $(f_n)$  converges to  $f$  in the topology of compact convergence, and equivalently in the metric  $d$ .

Lastly, we prove separability. We consider the subspace  $P_{\mathbb{Q}}$  of  $C(\Sigma \times \Lambda)$  consisting of “polynomials” with rational coefficients. That is,  $p \in P_{\mathbb{Q}}$  if  $p(n, \cdot)$  is a polynomial on  $\Lambda$  with rational coefficients for each  $n \in \Sigma$ . If  $f \in C(\Sigma \times \Lambda)$ , then on any compact set  $K \subset \Sigma \times \Lambda$  we can find a sequence of polynomials converging uniformly to  $f$  by the Stone-Weierstrass theorem. These polynomials in turn can be uniformly approximated by polynomials with rational coefficients, so by diagonalization we obtain a sequence in  $P_{\mathbb{Q}}$  converging uniformly

to  $f$  on  $K$ . [Can we patch together the sequences for the sets  $K_n$  to get one sequence  $(p_n)$  in  $P_{\mathbb{Q}}$  that converges uniformly to  $f$  on *all* compact subsets? Maybe use diagonalization again?][The result from McCoy, “Second Countable and Separable Function Spaces,” shows that  $C(\Sigma \times \Lambda)$  is second-countable and hence separable because  $\Sigma \times \Lambda$  is second-countable. But the proof is a bit difficult.]

## Problem 2

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $X, Y$  random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  taking values in  $C(\Sigma \times \Lambda)$ , where  $\Sigma = \llbracket 1, N \rrbracket$  with  $N \in \mathbb{N}$  or  $N = \infty$ . We consider the collection  $\mathcal{S}_X$  of sets of the form

$$\{\omega \in \Omega : X(\omega)(i_1, t_1) \leq x_1, \dots, X(\omega)(i_n, t_n) \leq x_n\} = \bigcap_{k=1}^n X(i_k, t_k)^{-1}(-\infty, x_k],$$

ranging over all  $n \in \mathbb{N}$ ,  $(i_1, t_1), \dots, (i_n, t_n) \in \Sigma \times \Lambda$ , and  $x_1, \dots, x_n \in \mathbb{R}$ . We first prove that  $\mathcal{S}_X \subset \mathcal{F}$ . We can write

$$\{X(i_k, t_k) \leq x_k\} = X^{-1}(\{f \in C(\Sigma \times \Lambda) : f(i_k, t_k) \leq x_k\}).$$

We claim that the set  $\{f \in C(\Sigma \times \Lambda) : f(i_k, t_k) \leq x_k\}$  is closed in the topology of compact convergence. If  $f_n(i_k, t_k) \leq x_k$  for all  $n$  and  $f_n \rightarrow f$  in the (metrizable) topology of compact convergence, then by taking limits on a compact set containing  $(i_k, t_k)$ , we find  $f(i_k, t_k) \leq x_k$  as well. This proves the claim, and it follows from the measurability of  $X$  that  $\{X(i_k, t_k) \leq x_k\} = X^{-1}(\{f(i_k, t_k) \leq x_k\}) \in \mathcal{F}$ . The finite intersection is thus also in  $\mathcal{F}$ , proving that  $\mathcal{S}_X \subset \mathcal{F}$ . On the other hand, it is clear that  $\{\omega \in \Omega : X(\omega) \in A\} = X^{-1}(A) \in \mathcal{F}$  for any  $A \in \mathcal{C}_{\Sigma}$  since  $X$  is measurable.

Now we prove that  $\mathbb{P}|_{\mathcal{S}_X}$  determines the distribution  $\mathbb{P} \circ X^{-1}$ . To do so, note that  $\mathcal{S}_X = \sigma(\{X^{-1}(A) : A \in \mathcal{S}\})$ , where  $\mathcal{S}$  is the collection of cylinder sets

$$\{f \in C(\Sigma \times \Lambda) : f(i_1, t_1) \in A_1, \dots, f(i_n, t_n) \in A_n\}, \quad A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}).$$

This follows from the fact that  $\mathcal{B}(\mathbb{R})$  is generated by intervals of the form  $(-\infty, x]$ . Observe that the intersection of two elements of  $\mathcal{S}$  is clearly another element of  $\mathcal{S}$ , so  $\mathcal{S}$  is a  $\pi$ -system. We now argue that  $\mathcal{S}$  generates the Borel sets, i.e.,  $\sigma(\mathcal{S}) = \mathcal{C}_{\Sigma}$ . By the argument above,  $\mathcal{S} \subset \mathcal{C}_{\Sigma}$ . Conversely, we will show that every basis element of the topology of compact convergence on  $C(\Sigma \times \Lambda)$  is contained in  $\sigma(\mathcal{S})$ , and consequently so is every Borel set. More precisely, let  $K \subset \Sigma \times \Lambda$  be compact,  $f \in C(\Sigma \times \Lambda)$ , and  $\epsilon > 0$ , and let  $H$  be a countable dense subset of  $K$ . (Recall that every compact metric space is separable, and  $K$  is homeomorphic to a product of finitely many compact sets in  $\mathbb{R}$ , which are metrizable. So  $K$  is separable.) We claim that

$$B_K(f, \epsilon) = \bigcup_{n=1}^{\infty} \bigcap_{(i,t) \in H} \{g \in C(\Sigma \times \Lambda) : g(i, t) \in (f(i, t) - (1 - 2^{-n})\epsilon, f(i, t) + (1 - 2^{-n})\epsilon)\}.$$

Indeed, if  $g \in B_K(f, \epsilon)$ , i.e.,  $\sup_{(i,t) \in K} |g(i,t) - f(i,t)| < \epsilon$ . Then since  $1 - 2^{-m} \nearrow 1$ , we can choose  $m$  large enough so that

$$|g(i,t) - f(i,t)| < (1 - 2^{-n})\epsilon$$

for all  $(i,t) \in K$  (in particular with  $(i,t) \in H$ ). Conversely, suppose  $g$  is in the set on the right. Then since  $g$  is continuous and  $H$  is dense in  $K$ , we find that for some  $n \geq 1$ ,

$$|g(i,t) - f(i,t)| \leq (1 - 2^{-n})\epsilon < \epsilon$$

for all  $(i,t) \in K$ . Hence  $g \in B_K(f, \epsilon)$ . This proves the claim. Since  $H$  is countable,  $B_K(f, \epsilon)$  is formed from countably many unions and intersections of sets in  $\mathcal{S}$ , thus  $B_K(f, \epsilon) \in \sigma(\mathcal{S})$ .

In summary, we have shown that the collection  $\mathcal{S}$  is a  $\pi$ -system generating  $\mathcal{C}_\Sigma$ , so the probability measure  $\mathbb{P} \circ X^{-1}$  on  $\mathcal{C}_\Sigma$  is uniquely determined by its restriction to  $\mathcal{S}$ . Suppose

$$\begin{aligned} \mathbb{P}(\{\omega \in \Omega : X(\omega)(i_1, t_1) \leq x_1, \dots, X(\omega)(i_n, t_n) \leq x_n\}) = \\ \mathbb{P}(\{\omega \in \Omega : Y(\omega)(i_1, t_1) \leq x_1, \dots, Y(\omega)(i_n, t_n) \leq x_n\}) \end{aligned}$$

for all  $(i_1, t_1), x_1, \dots, x_n$ . This says that the two probability measures  $\mathbb{P} \circ X^{-1}$  and  $\mathbb{P} \circ Y^{-1}$  agree on  $\mathcal{S}$ . Then they must agree on all of  $\mathcal{C}_\Sigma$ , i.e.,

$$\mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) = \mathbb{P}(\{\omega \in \Omega : Y(\omega) \in A\})$$

for all  $A \in \mathcal{C}_\Sigma$ . In other words, the law of a line ensemble is determined by its finite dimensional distributions.

## 2 Algebra

Problem 3

Problem 4

## 3 Weak convergence

Problem 5

Problem 6

Problem 7

## 4 Tightness

Problem 8

Problem 9

## 5 Lozenge tilings of the hexagon

Problem 10

Problem 11