Asymptotics of Bernoulli Gibbsian Line Ensembles

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The Gaussian universality class

Let $\{X_i\}$ be a sequence of independent identically distributed random variables with mean μ and variance σ^2. Let $S_n = X_1 + \cdots + X_n$.

- **Law of Large Numbers:** $\dfrac{S_n}{n} \longrightarrow \mu$ as $n \to \infty$ almost surely.
- **Central Limit Theorem:** $\dfrac{S_n - n\mu}{\sigma\sqrt{n}} \implies \mathcal{N}(0,1)$ as $n \to \infty$.
- **Donsker's Theorem:** For $t \in [0,1]$, let $W^{(n)}(t) = \dfrac{S_{nt} - nt\mu}{\sigma\sqrt{n}}$ if $nt \in \mathbb{N}$, and linearly interpolate. Then $W^{(n)} \in C([0,1])$ and $W^{(n)} \implies W$ as $n \to \infty$, a standard Brownian motion on $[0,1]$.

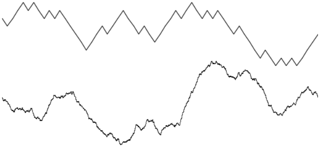
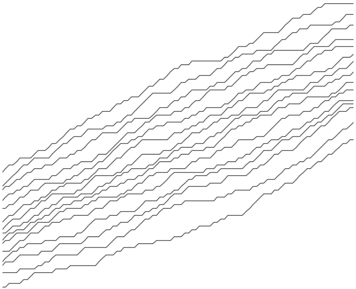


Figure: An example of a random walk and a Brownian motion.

Multiple random walks

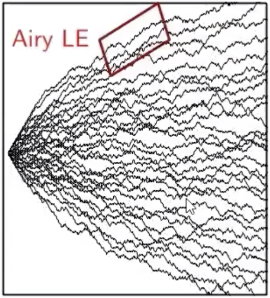
- If $S_{n+1} - S_n \in \{0,1\}$, then $\{S_n\}_{n=1}^{\infty}$ is a *Bernoulli random walk*
- An *avoiding Bernoulli line ensemble* $\mathfrak{L} = (L_1, \dots, L_k)$ consists of k avoiding Bernoulli random walks on an interval $[T_0, T_1]$ with random initial and ending points $\mathfrak{L}(T_0), \mathfrak{L}(T_1)$, such that $L_1(s) \geq L_2(s) \geq \cdots \geq L_k(s)$ for $s \in [T_0, T_1]$
- Special case of *Bernoulli Gibbsian line ensembles*



- Question: What does the limit look like as $k \to \infty$?

Airy Line Ensemble

As $k \to \infty$, k avoiding random walks are conjectured to converge to the *Airy line ensemble* $\mathcal{A}$, and the top curve to the *Airy process* $\mathcal{A}_1$



- Increasing the number of paths pushes us outside of the Gaussian universality class and into the *Kardar-Parisi-Zhang (KPZ) universality class*
- Open problem: Show that "generic" random walks with "generic" initial and terminal conditions converge to the Airy line ensemble
- We consider this problem for Bernoulli random walks; the proof is only known if all walks start from 0

Convergence to the Airy Line Ensemble

Two sufficient conditions for uniform weak convergence:

- *Finite dimensional* convergence – difficult, requires exact algebraic formulas
- *Tightness* (existence of weak subsequential limits) – easier, more qualitative/analytic

Proving tightness requires controlling the maximum, the minimum, and the modulus of continuity

$$w(f,\delta) = \sup_{\substack{x,y\in[a,b]\\ |x-y|\le\delta}} |f(x)-f(y)|$$

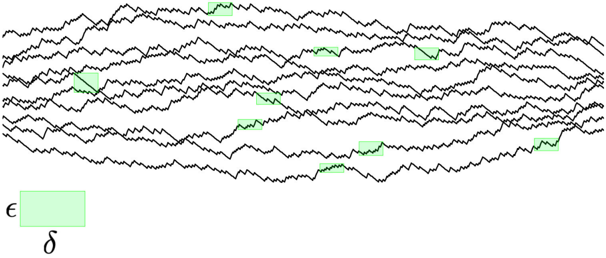


Figure: The modulus of continuity

Our Result

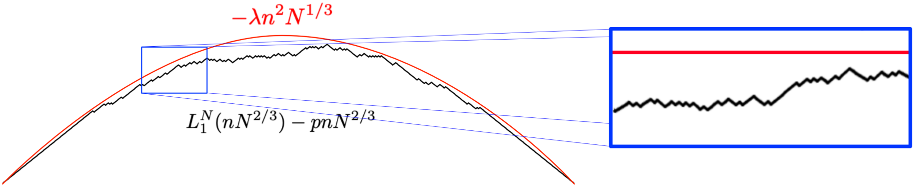
Theorem (DFFSTWZ)

Let $\{\mathfrak{L}^N = (L_1^N, \dots, L_k^N)\}_{N=1}^{\infty}$ *be a sequence of* k *avoiding Bernoulli random walks. Fix* $p \in (0,1)$ *and* $\lambda > 0$*, and suppose that for all* $n \in \mathbb{Z}$ *we have*

$$\lim_{N\to\infty} \mathbb{P}\big(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \le N^{1/3}x\big) = F_{TW}(x).$$

Then the collection formed by the top $k-1$ *curves of* $\{\mathfrak{L}^N\}_{N=1}^{\infty}$ *are a tight sequence.*

- F_{TW} is the *Tracy-Widom distribution* – the one-point marginal for the Airy process
- [Dauvergne-Nica-Virág '19] Finite dimensional convergence of all curves implies tightness
- Our result shows that it suffices for the integer time one-point marginals of the top curve to converge to F_{TW}



History of line ensembles

Arguments in this paper are inspired by

1. *Brownian Gibbs property for Airy line ensembles* [Corwin-Hammond '14] and *KPZ line ensemble* [Corwin-Hammond '16], which address similar issues for continuous line ensembles
2. *Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood line ensembles* [Corwin-Dimitrov '17], which considers similar questions in a discrete setting

Proving tightness

To show tightness, we want to control:

1. **Minimum** of bottom curve L_{k-1}^N
2. **Maximum** of top curve L_1^N
3. **Modulus of continuity** of each curve L_i^N

We will focus on bounding the minimum:

Lemma 1 (DFFSTWZ)

Fix $r, \epsilon > 0$. Then there exist constants $M > 0$ and $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,

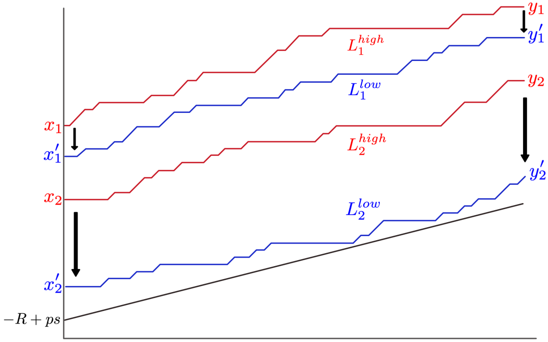
$$\mathbb{P}\Big(\inf_{x \in [-r,r]} \big(L_{k-1}^N(xN^{2/3}) - pxN^{2/3}\big) < -MN^{1/3} \Big) < \epsilon.$$

- For simplicity, we consider the case of two curves $(k = 3)$.

Monotone coupling

Lowering entry and exit data $\vec{x}, \vec{y}$ for the curves $\implies$ curves shift down on whole interval

- We proved this by adapting arguments from [Corwin-Hammond '14]

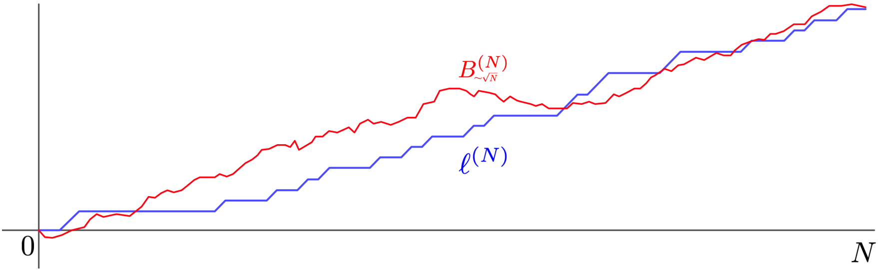


- $\mathfrak{L}^{high}$ and $\mathfrak{L}^{low}$ are coupled, in particular

$$\mathbb{P}(L_2^{high} < -R) \leq \mathbb{P}(L_2^{low} < -R)$$

Strong coupling

A Bernoulli random walk bridge $\ell^{(N)}$ on $[0, N]$ can be coupled with an "exponentially close" Brownian bridge $B^{(N)}$ with standard deviation $O(\sqrt{N})$ [Dimitrov-Wu '19]



$$\mathbb{P}\left(\sup_{s \in [0,N]} \left| \ell^{(N)}(s) - B^{(N)}(s) \right| \geq M(\log N)^2 + x \right) < Ke^{-Ax}$$

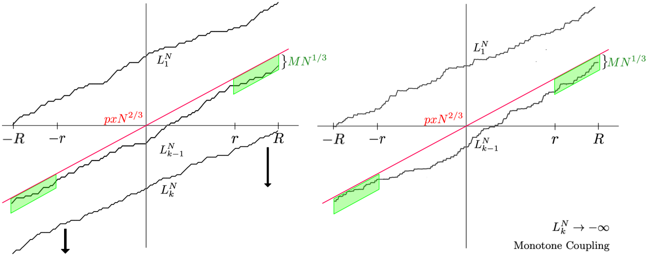
Proving Lemma 1: pinning the bottom curve

Lemma 2 (DFFSTWZ)

For any $r, \epsilon > 0$, there exists $R > r$ and a constant $M > 0$ so that for large N,

$$\mathbb{P}\left(\max_{x \in [r,R]} \left(L_{k-1}^N(xN^{2/3}) - pxN^{2/3} \right) < -MN^{1/3} \right) < \epsilon.$$

The same is true of the maximum on $[-R, -r]$.



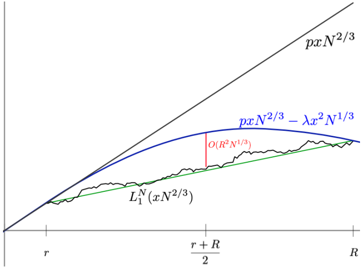
- Use monotone coupling to push L_k^N to $-\infty$
- Strongly couple L_{k-1}^N with a Brownian bridge: if "pinned" at two points in $[r, R]$ and $[-R, -r]$, it cannot be low on $[-r, r]$ on scale $N^{1/3}$

Proving Lemma 2

- Recall our assumption:

$$\mathbb{P}\Big(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \leq xN^{1/3}\Big) \underset{N\to\infty}{\longrightarrow} F_{TW}(x)$$

- The top curve looks like a parabola with an affine shift on large scales



- If L_2^N is low on $[r, R]$, L_1^N looks like a free Brownian bridge

$$\left[-\lambda\left(\frac{R+r}{2}\right)^2 N^{1/3}\right] - \left[-\lambda\left(\frac{R^2+r^2}{2}\right)N^{1/3}\right] = \frac{\lambda}{4}(R-r)^2N^{1/3} = O(R^2N^{1/3})$$

- For large R, the top curve would be far from the parabola at the midpoint!

Thank you!

Thank you for listening. If anyone has any questions, feel free to ask us now!