Asymptotics of Bernoulli Gibbsian Line Ensembles

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Let $\{X_i\}$ be a sequence of independent identically distributed random variables with mean μ and variance σ^2. Let $S_n = X_1 + \cdots + X_n$.

- **Law of Large Numbers:** $\dfrac{S_n}{n} \longrightarrow \mu$ as $n \to \infty$ almost surely.
- **Central Limit Theorem:** $\dfrac{S_n - n\mu}{\sigma\sqrt{n}} \implies \mathcal{N}(0,1)$ as $n \to \infty$.
- **Donsker's Theorem:** For $t \in [0,1]$, let $W^{(n)}(t) = \dfrac{S_{nt} - nt\mu}{\sigma\sqrt{n}}$ if $nt \in \mathbb{N}$, and linearly interpolate. Then $W^{(n)} \in C([0,1])$ and $W^{(n)} \implies W$ as $n \to \infty$, a standard Brownian motion on $[0,1]$.

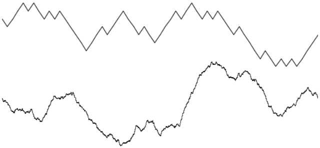
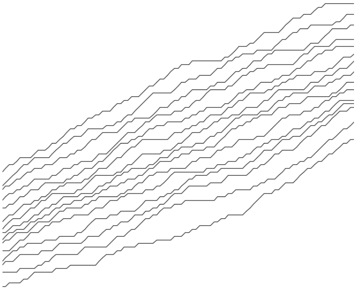


Figure: An example of a random walk and a Brownian motion.

Multiple random walks

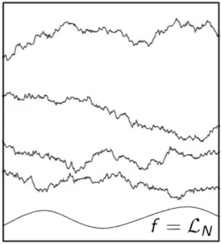
- If $S_{n+1} - S_n \in \{0, 1\}$, then $\{S_n\}_{n=1}^{\infty}$ is a *Bernoulli random walk*
- An *avoiding Bernoulli line ensemble* $\mathfrak{L} = (L_1, \ldots, L_k)$ consists of k avoiding Bernoulli random walks on an interval $[T_0, T_1]$ with random entry and exit data $\mathfrak{L}(T_0), \mathfrak{L}(T_1)$, such that $L_1(s) \geq L_2(s) \geq \cdots \geq L_k(s)$ for $s \in [T_0, T_1]$
- Special case of *Bernoulli Gibbsian line ensembles*



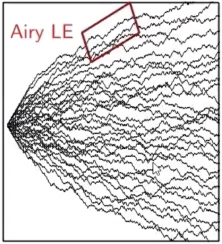
- Question: What does the limit look like as $k \to \infty$?

Airy Line Ensemble

As $k \to \infty$, k avoiding random walks are conjectured to converge to the *Airy line ensemble* $\mathcal{A}$, and the top curve to the *Airy process* $\mathcal{A}_1$



Avoiding Brownian bridges



Dyson Brownian motion

- Increasing the number of paths pushes us outside of the Gaussian universality class and into the *Kardar-Parisi-Zhang (KPZ) universality class*
- Open problem: Show that "generic" random walks with "generic" initial conditions converge to the Airy line ensemble
- We consider this problem for Bernoulli random walks; the proof is only known if all walks start from 0

Convergence to the Airy Line Ensemble

Two sufficient conditions for convergence in distribution:

- *Finite dimensional* convergence – difficult, requires exact algebraic formulas
- *Tightness* (existence of weak subsequential limits) – easier, more qualitative/analytic

We focused on tightness, which we prove by controlling the maximum, the minimum, and the modulus of continuity

$$w(f,\delta) = \sup_{\substack{x,y\in[a,b]\\|x-y|\le\delta}} |f(x)-f(y)|$$

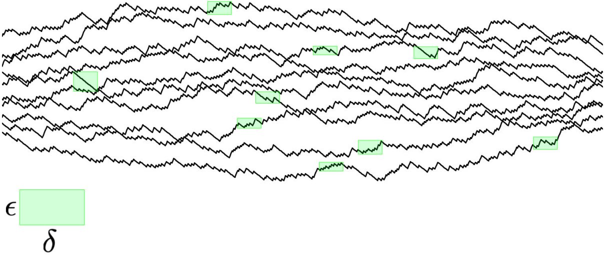


Figure: The modulus of continuity

Our Result

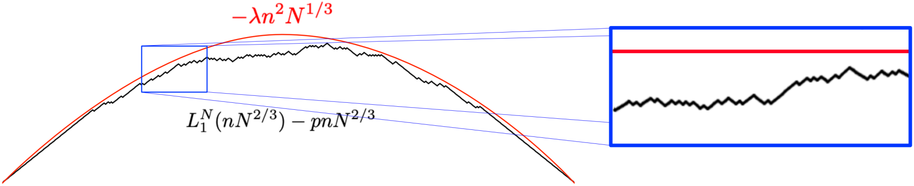
Theorem (DFFSTWZ)

Let $\{\mathfrak{L}^N = (L_1^N, \ldots, L_k^N)\}_{N=1}^\infty$ be a sequence of k avoiding Bernoulli random walks. Fix $p \in (0,1)$ and $\lambda > 0$, and suppose that for all $n \in \mathbb{Z}$ we have

$$\lim_{N\to\infty} \mathbb{P}\big(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \leq N^{1/3}x\big) = F_{TW}(x).$$

Then $\{\mathfrak{L}^N\}$ is a tight sequence.

- F_{TW} is the *Tracy-Widom distribution* – the one-point marginal for the Airy process
- [Dauvergne-Nica-Virág '19] Finite dimensional convergence of all curves implies tightness
- Our result shows that it suffices for the integer time one-point marginals of the top curve to converge to F_{TW}



History of the line ensembles

Arguments in this paper are inspired by

1. *Brownian Gibbs property for Airy line ensembles* [Corwin-Hammond '14] and *KPZ line ensemble* [Corwin-Hammond '16], which address similar issues for continuous line ensembles
2. *Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood line ensembles* [Corwin-Dimitrov '17], which considers similar questions in a discrete setting

Proving tightness

To show tightness, we want to control:

1. **Minimum** of bottom curve L_{k-1}^N
2. **Maximum** of top curve L_1^N
3. **Modulus of continuity** of each curve L_i^N

We will focus on bounding the minimum:

Lemma 1 (DFFSTWZ)

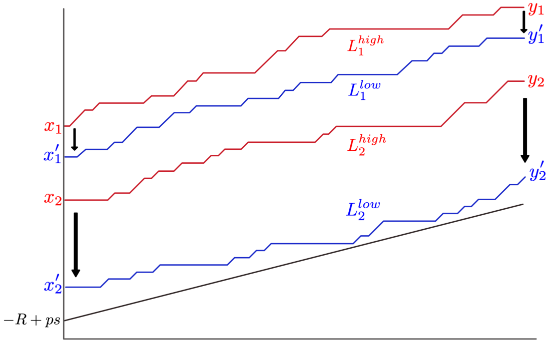
Fix $r, \epsilon > 0$. Then there exist constants $M > 0$ and $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,

$$\mathbb{P}\Big(\inf_{x \in [-r,r]} \big(L_{k-1}^N(xN^{2/3}) - pxN^{2/3} \big) < -MN^{1/3} \Big) < \epsilon.$$

Monotone coupling

Lowering entry and exit data $\vec{x}, \vec{y}$ for the curves $\Longrightarrow$ curves shift down on whole interval

- We proved this by adapting arguments from [Corwin-Hammond '14]

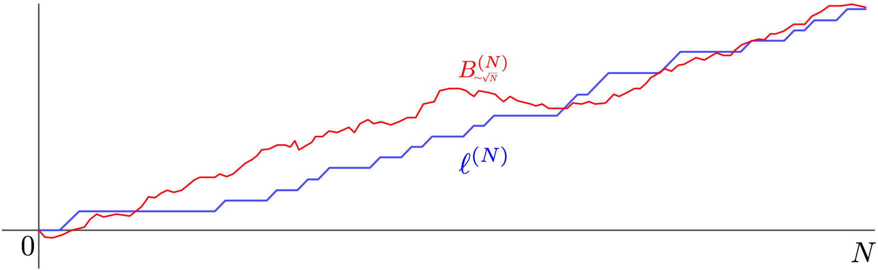


- $\mathfrak{L}^{high}$ and $\mathfrak{L}^{low}$ are coupled, in particular

$$\mathbb{P}(L_2^{high} < -R) \leq \mathbb{P}(L_2^{low} < -R)$$

Strong coupling

A Bernoulli random walk $\ell^{(N)}$ on $[0, N]$ can be coupled with an "exponentially close" Brownian bridge $B^{(N)}$ with variance $O(\sqrt{N})$ [Dimitrov-Wu '19]



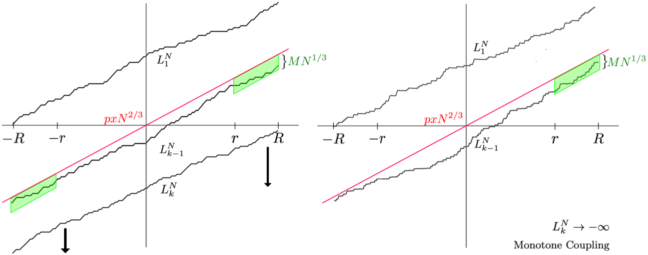
$$\mathbb{P}\left(\sup_{s\in[0,N]} \left| \ell^{(N)}(s) - B^{(N)}(s) \right| \geq M(\log N)^2 + x \right) < Ke^{-Ax}$$

Lemma 2 (DFFSTWZ)

For any $r, \epsilon > 0$, there exists $R > r$ and a constant $M > 0$ so that for large N,

$$\mathbb{P}\left(\max_{x \in [r,R]} \left(L_{k-1}^N(xN^{2/3}) - pxN^{2/3} \right) < -MN^{1/3} \right) < \epsilon.$$

The same is true of the maximum on $[-R, -r]$.

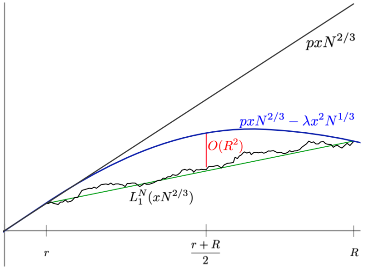


- Use monotone coupling to push L_k^N to $-\infty$
- Strongly couple L_{k-1}^N with a Brownian bridge: if "pinned" at two points in $[r, R]$ and $[-R, -r]$, it cannot be low on $[-r, r]$ on scale $N^{1/3}$

- Recall our assumption:

$$\mathbb{P}\Big(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \le xN^{1/3}\Big) \underset{N\to\infty}{\longrightarrow} F_{TW}(x)$$

- The top curve looks like a parabola with an affine shift on large scales



- Two curves: if L_2^N is low on $[r, R]$, L_1^N looks like a free Brownian bridge

$$\left[-\lambda\left(\frac{R+r}{2}\right)^2\right] - \left[-\lambda\left(\frac{R^2+r^2}{2}\right)\right] = \lambda\,\frac{R^2+r^2-2rR}{4} = O(R^2)$$

- For large R, the top curve would be far from the parabola at the midpoint!