

Asymptotics of Bernoulli Gibbsian Line Ensembles

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The Gaussian universality class

Let $\{X_i\}$ be a sequence of independent identically distributed random variables with mean μ and variance σ^2 . Let $S_n = X_1 + \cdots + X_n$.

- **Law of Large Numbers:** $\frac{S_n}{n} \longrightarrow \mu$ as $n \rightarrow \infty$ almost surely.
- **Central Limit Theorem:** $\frac{S_n - n\mu}{\sigma\sqrt{n}} \implies \mathcal{N}(0, 1)$ as $n \rightarrow \infty$.
- **Donsker's Theorem:** For $t \in [0, 1]$, let $W^{(n)}(t) = \frac{S_{nt} - nt\mu}{\sigma\sqrt{n}}$ if $nt \in \mathbb{N}$, and linearly interpolate. Then $W^{(n)} \in C([0, 1])$ and $W^{(n)} \implies W$ as $n \rightarrow \infty$, a standard Brownian motion on $[0, 1]$.

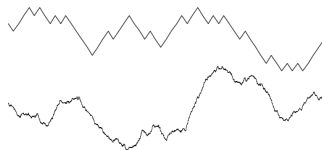
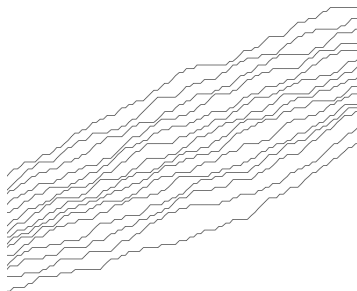


Figure: An example of a random walk and a Brownian motion.

Multiple random walks

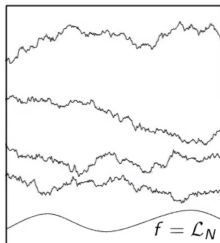
- If $S_{n+1} - S_n \in \{0, 1\}$, then $\{S_n\}_{n=1}^\infty$ is a *Bernoulli random walk*
- An *avoiding Bernoulli line ensemble* $\mathfrak{L} = (L_1, \dots, L_k)$ consists of k avoiding Bernoulli random walks on an interval $[T_0, T_1]$ with random initial and ending points $\mathfrak{L}(T_0), \mathfrak{L}(T_1)$, such that $L_1(s) \geq L_2(s) \geq \dots \geq L_k(s)$ for $s \in [T_0, T_1]$
- Special case of *Bernoulli Gibbsian line ensembles*



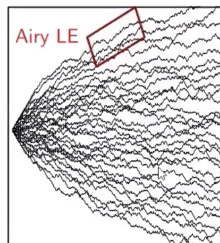
- Question: What does the limit look like as $k \rightarrow \infty$?

Airy Line Ensemble

As $k \rightarrow \infty$, k avoiding random walks are conjectured to converge to the *Airy line ensemble* \mathcal{A} , and the top curve to the *Airy process* \mathcal{A}_1



Avoiding Brownian bridges



Dyson Brownian motion

- Increasing the number of paths pushes us outside of the *Gaussian universality class* and into the *Kardar-Parisi-Zhang (KPZ) universality class*
- Open problem: Show that “generic” random walks with “generic” initial and terminal conditions converge to the *Airy line ensemble*
- We consider this problem for Bernoulli random walks; the proof is only known if all walks start from 0

Convergence to the Airy Line Ensemble

Two sufficient conditions for convergence in distribution:

- *Finite dimensional* convergence – difficult, requires exact algebraic formulas
- *Tightness* (existence of weak subsequential limits) – easier, more qualitative/analytic

We focused on *tightness*, which we prove by controlling the maximum, the minimum, and the modulus of continuity

$$w(f, \delta) = \sup_{\substack{x, y \in [a, b] \\ |x - y| \leq \delta}} |f(x) - f(y)|$$

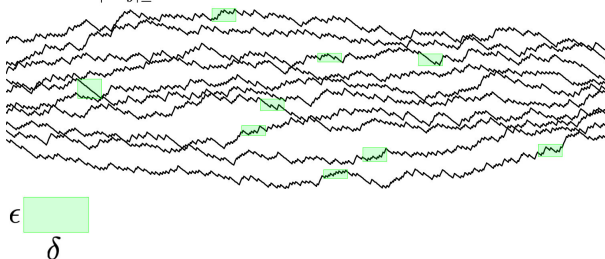


Figure: The modulus of continuity

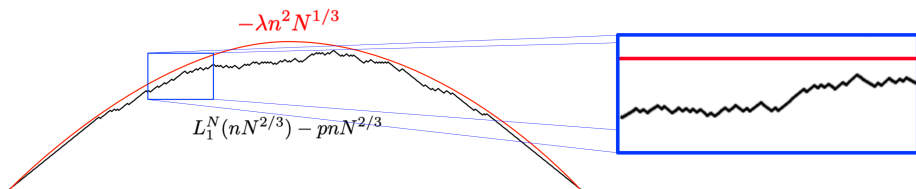
Theorem (DFFSTWZ)

Let $\{\mathfrak{L}^N = (L_1^N, \dots, L_k^N)\}_{N=1}^\infty$ be a sequence of k avoiding Bernoulli random walks. Fix $p \in (0, 1)$ and $\lambda > 0$, and suppose that for all $n \in \mathbb{Z}$ we have

$$\lim_{N \rightarrow \infty} \mathbb{P}(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \leq N^{1/3}x) = F_{TW}(x).$$

Then $\{\mathfrak{L}^N\}$ is a tight sequence.

- F_{TW} is the *Tracy-Widom distribution* – the one-point marginal for the Airy process
- [Dauvergne-Nica-Virág '19] Finite dimensional convergence of all curves implies tightness
- Our result shows that it suffices for the integer time one-point marginals of the **top curve** to converge to F_{TW}



Arguments in this paper are inspired by

- 1 *Brownian Gibbs property for Airy line ensembles* [Corwin-Hammond '14] and *KPZ line ensemble* [Corwin-Hammond '16], which address similar issues for **continuous** line ensembles
- 2 *Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood line ensembles* [Corwin-Dimitrov '17], which considers similar questions in a **discrete** setting

To show tightness, we want to control:

- 1 **Minimum** of bottom curve L_{k-1}^N
- 2 **Maximum** of top curve L_1^N
- 3 **Modulus of continuity** of each curve L_i^N

We will focus on bounding the **minimum**:

Lemma 1 (DFFSTWZ)

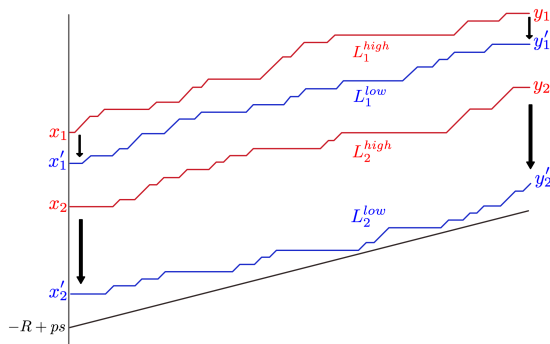
Fix $r, \epsilon > 0$. Then there exist constants $M > 0$ and $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,

$$\mathbb{P}\left(\inf_{x \in [-r, r]} (L_{k-1}^N(xN^{2/3}) - pxN^{2/3}) < -MN^{1/3}\right) < \epsilon.$$

Monotone coupling

Lowering entry and exit data \vec{x}, \vec{y} for the curves \implies curves shift down on whole interval

- We proved this by adapting arguments from [Corwin-Hammond '14]

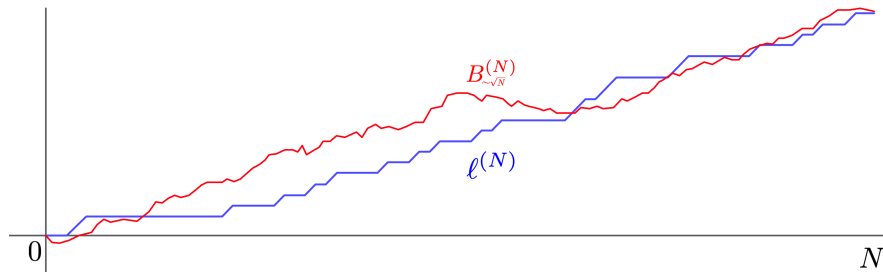


- \mathcal{L}^{high} and \mathcal{L}^{low} are coupled, in particular

$$\mathbb{P}(L_2^{high} < -R) \leq \mathbb{P}(L_2^{low} < -R)$$

Strong coupling

A Bernoulli random walk $\ell^{(N)}$ on $[0, N]$ can be coupled with an “exponentially close” Brownian bridge $B^{(N)}$ with variance $O(\sqrt{N})$ [Dimitrov-Wu '19]



$$\mathbb{P}\left(\sup_{s \in [0, N]} \left| \ell^{(N)}(s) - B^{(N)}(s) \right| \geq M(\log N)^2 + x\right) < K e^{-Ax}$$

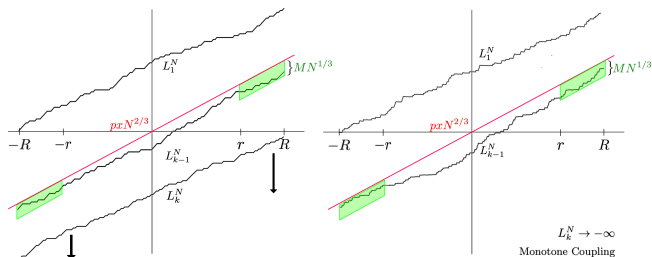
Proving Lemma 1: pinning the bottom curve

Lemma 2 (DFFSTWZ)

For any $r, \epsilon > 0$, there exists $R > r$ and a constant $M > 0$ so that for large N ,

$$\mathbb{P}\left(\max_{x \in [r, R]} (L_{k-1}^N(xN^{2/3}) - \textcolor{red}{px}N^{2/3}) < -\textcolor{green}{MN}^{1/3}\right) < \epsilon.$$

The same is true of the maximum on $[-R, -r]$.



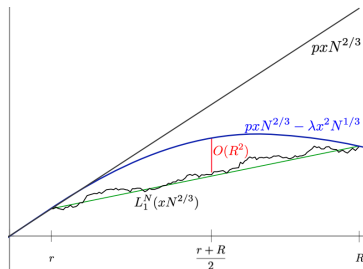
- Use monotone coupling to push L_k^N to $-\infty$
- Strongly couple L_{k-1}^N with a Brownian bridge: if “pinned” at two points in $[r, R]$ and $[-R, -r]$, it cannot be low on $[-r, r]$ on scale $N^{1/3}$

Proving Lemma 2

- Recall our assumption:

$$\mathbb{P}\left(L_1^N(nN^{2/3}) - pnN^{2/3} + \lambda n^2 N^{1/3} \leq xN^{1/3}\right) \xrightarrow{N \rightarrow \infty} F_{TW}(x)$$

- The top curve looks like a **parabola** with an affine shift on large scales



- Two curves: if L_2^N is low on $[r, R]$, L_1^N looks like a free Brownian bridge

$$\left[-\lambda\left(\frac{R+r}{2}\right)^2\right] - \left[-\lambda\left(\frac{R^2+r^2}{2}\right)\right] = \lambda \frac{R^2 + r^2 - 2rR}{4} = O(R^2)$$

- For large R , the top curve would be far from the parabola at the midpoint!