Problem 21

This is a rough argument for Problem 21 in the special case when $k = 2$. Fix $r > 0$ and $R > r$; assume that $r, R \in \mathbb{Z}N^\alpha$ for simplicity. Define events

$$A = \left\{ L_1^N \left(\frac{R+r}{2} N^\alpha \right) - pN^\alpha \frac{R+r}{2} + \lambda \left(\frac{R+r}{2} \right)^2 N^{\alpha/2} < -\phi(\epsilon) N^{\alpha/2} \right\},$$
$$B = \left\{ \max_{x \in [r,R]} \left(L_2^N(xN^\alpha) - pxN^\alpha \right) < -R_2 N^{\alpha/2} \right\}.$$

We aim to bound $\mathbb{P}(B)$, using the fact that $\mathbb{P}(A) \leq 2\epsilon$ for large enough N by one-point tightness. Recall that with probability $> 1 - 2\epsilon$, we have

$$prN^\alpha - (\lambda r^2 + \phi(\epsilon))N^{\alpha/2} < L_1^N(rN^\alpha) < prN^\alpha - (\lambda r^2 - \phi(\epsilon))N^{\alpha/2},$$
$$pRN^\alpha - (\lambda R^2 + \phi(\epsilon))N^{\alpha/2} < L_2^N(RN^\alpha) < pRN^\alpha - (\lambda R^2 - \phi(\epsilon))N^{\alpha/2}$$

Let F denote the subset of B for which these two inequalities hold. Then

$$\mathbb{P}(B) \leq \mathbb{P}(F) + 2\epsilon,$$

so it suffices to bound $\mathbb{P}(F)$. To do so, we argue that there is a constant $c > 0$ independent of ϵ (maybe $c = 1/4$) such that

$$\mathbb{P}(A \,|\, F) > c$$

for large enough R and R_2. Let D denote the set of pairs $(\vec{x}, \vec{y})$, with $\vec{x}, \vec{y} \in \mathfrak{W}_2$, satisfying

(1) $0 \leq y_i - x_i \leq (R - r)N^\alpha$,

(2) $prN^\alpha - (\lambda r^2 + \phi(\epsilon))N^{\alpha/2} < x_1 < prN^\alpha - (\lambda r^2 - \phi(\epsilon))N^{\alpha/2}$ and $pRN^\alpha - (\lambda R^2 + \phi(\epsilon))N^{\alpha/2} < x_2 < pRN^\alpha - (\lambda R^2 - \phi(\epsilon))N^{\alpha/2}$,

(3) $x_2 < prN^\alpha - R_2 N^{\alpha/2}$ and $y_2 < pRN^\alpha - R_2 N^{\alpha/2}$.

Let $E(\vec{x}, \vec{y})$ denote the subset of F consisting of L^N for which $L_i^N(rN^\alpha) = x_i$ and $L_i^N(RN^\alpha) = y_i$ for $i = 1, 2$, and $L_1^N(s) > L_2^N(s)$ for all s. Then D is countable, the $E(\vec{x}, \vec{y})$ are pairwise disjoint, and $F = \bigcup_{(\vec{x},\vec{y}) \in D} E(\vec{x}, \vec{y})$. Suppose we can show that $\mathbb{P}(A \,|\, E(\vec{x}, \vec{y})) > c$ for all $(\vec{x}, \vec{y}) \in D$. Then

$$\mathbb{P}(A \,|\, F) = \sum_{(\vec{x},\vec{y}) \in D} \frac{\mathbb{P}(A \,|\, E(\vec{x}, \vec{y}))\mathbb{P}(E(\vec{x}, \vec{y}))}{\mathbb{P}(F)} \geq c \cdot \frac{\sum_{(\vec{x},\vec{y}) \in D} \mathbb{P}(E(\vec{x}, \vec{y}))}{\mathbb{P}(F)} = c.$$

We now try to find a lower bound for $\mathbb{P}(A \,|\, E(\vec{x}, \vec{y}))$. We have

$$\begin{aligned} \mathbb{P}(A \,|\, E(\vec{x}, \vec{y})) &= \mathbb{P}_{avoid,Ber}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}}(A \,|\, F) \geq \mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}}(A \cap \{L_1 > L_2\} \,|\, F) \\ &\geq \mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}}(A \,|\, F) - \left(1 - \mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}}(L_1 > L_2 \,|\, F)\right) \\ &= \mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, x_1, y_1}(A) - \left(1 - \mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}}(L_1 > L_2 \,|\, F)\right) \end{aligned}$$

In the last line, we used the fact that A and F are independent under $\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}}$. (Do we also need the Gibbs property here to replace $\vec{x}, \vec{y}$ with x_1, y_1?) We can bound the first term using a lemma similar to those proven in Section 3. We would need a statement to the effect that with some positive probability, say at least 1/3, $L_1(\frac{R+r}{2}N^\alpha)$ does not lie far above the midpoint of the line segment connecting $L_1(rN^\alpha)$ and $L_1(RN^\alpha)$. Note that this midpoint is close to $\lambda(\frac{R^2+r^2}{2})N^{\alpha/2}$, and

$$\frac{R^2+r^2}{2} - \left(\frac{R+r}{2}\right)^2 = \frac{R^2+r^2-2rR}{4} = O(R^2)$$

for fixed r. Thus for large enough R, A will hold as long as $L_1(\frac{R+r}{2}N^\alpha)$ is not far above the midpoint of the segment connecting $L_1(rN^\alpha)$ and $L_1(RN^\alpha)$, giving a lower bound of 1/3 on $\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, x_1, y_1}(A)$.

It remains to bound $\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}}(L_1 > L_2 \,|\, F)$. We have

$$\begin{aligned}
\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}}(L_1 > L_2 \,|\, F) &\geq \mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}}\left(\inf_{x\in[r,R]} L_1(xN^\alpha) > pxN^\alpha - R_2 N^{\alpha/2} \,\middle|\, F\right) \\
&= \mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, x_1, y_1}\left(\inf_{x\in[r,R]} L_1(xN^\alpha) > pxN^\alpha - R_2 N^{\alpha/2}\right).
\end{aligned}$$

Again, we used the fact that L_1 and L_2 are independent under $\mathbb{P}_{Ber}^{rN^\alpha, RN^\alpha, \vec{x}, \vec{y}}$. We can bound the quantity in the second line using monotone coupling to fix x_1, y_1, and then using strong coupling with a Brownian bridge. For large R_2, we can make this probability $> 11/12$. However, the argument seems to break down at the first inequality if $k > 2$, because then the event F doesn't tell us anything about how low L_2 is.

Combining our estimates, we get

$$\mathbb{P}(A \,|\, E(\vec{x}, \vec{y})) \geq \frac{1}{3} - \frac{1}{12} = \frac{1}{4}.$$

Hence $\mathbb{P}(A \,|\, F) \geq 1/4$. It follows that

$$\mathbb{P}(F) \leq 4\mathbb{P}(A) \leq 8\epsilon$$

for large enough N, R, R_2.