

REU Practice Problems

1 Topology and measurability

We let Σ denote a set $\llbracket p, q \rrbracket = \{p, p+1, \dots, q-1, q\}$ for $p \in \mathbb{N}$, $q \in \mathbb{N} \cup \{\infty\}$, and let Λ denote an interval in \mathbb{R} with endpoints $a \leq b$. We write $C(X)$ for the space of continuous real-valued functions on X with the topology of compact convergence and the Borel σ -algebra \mathcal{C} . Recall that this topology is generated by the basis of sets

$$B_K(f, \epsilon) := \{g \in C(X) : \sup_{x \in K} |f(x) - g(x)| < \epsilon\},$$

with $K \subset X$ is compact, $f \in C(X)$, and $\epsilon > 0$. When $X = \Sigma \times \Lambda$, we write $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$.

Problem 1

We aim to construct a metric $d : C(\Sigma \times \Lambda) \times C(\Sigma \times \Lambda) \rightarrow [0, \infty)$ which induces the topology of compact convergence on $C(\Sigma \times \Lambda)$. The idea is to obtain a compact exhaustion of $\Sigma \times \Lambda$, i.e., a countable collection of compact sets $K_n \subset \Sigma \times \Lambda$ such that $\bigcup_n K_n = \Sigma \times \Lambda$, and such that every compact subset of $\Sigma \times \Lambda$ is contained in some K_n . We then construct d from the sup-metrics on each of these sets K_n . We define the sets

$$K_n := \Sigma_n \times \Lambda_n = \llbracket p, q_n \rrbracket \times [a_n, b_n]$$

as follows. We let $q_n = \min(p+n, q)$. If $a \in \Lambda$, i.e., Λ is closed at the left, then $a_n = a$ for all n , and likewise $b_n = b$ if $b \in \Lambda$. If $a \notin \Lambda$, we let $a_n \in \mathbb{R}$, $a_n > a$ be a sequence decreasing to a , for instance $a_n = a + \frac{1}{n}$ if $a > -\infty$, or $a_n = -n$ if $a = -\infty$. If $b \notin \Lambda$, we let $b_n \nearrow b$. In any case, we see that the sets $K_1 \subset K_2 \subset \dots \subset \Sigma \times \Lambda$ are compact, they cover $\Sigma \times \Lambda$, and any compact subset K of $\Sigma \times \Lambda$ is contained in all K_n for sufficiently large n .

We now define, for each n and $f, g \in C(\Sigma \times \Lambda)$,

$$d_n(f, g) := \sup_{(i,t) \in K_n} |f(i, t) - g(i, t)|, \quad d'_n(f, g) := \min\{d_n(f, g), 1\}$$

Clearly each d_n is nonnegative and satisfies the triangle inequality, and it is then easy to see that the same properties hold for d'_n . Furthermore, $d'_n \leq 1$, so we can define

$$d(f, g) := \sum_{n=1}^{\infty} 2^{-n} d'_n(f, g).$$

We first observe that d is a metric on $C(\Sigma \times \Lambda)$. Indeed, it is nonnegative, and if $f = g$, then each $d'_n(f, g) = 0$, so the sum is 0. Conversely, if $f \neq g$, then since the K_n cover $\Sigma \times \Lambda$, we can choose n large enough so that K_n contains an x with $f(x) \neq g(x)$. Then $d'_n(f, g) \neq 0$, and hence $d(f, g) \neq 0$. The triangle inequality holds for d since it holds for each d'_n .

Now we prove that the topology τ_d on $C(\Sigma \times \Lambda)$ induced by d is the same as the topology of compact convergence, which we will denote τ_c . First, choose $\epsilon > 0$ and $f \in C(\Sigma \times \Lambda)$. Let $g \in B_\epsilon^d(f)$, i.e., $d(f, g) < \epsilon$. We will find a set $A_g \in \tau_c$ such that $g \in A_g \subset B_\epsilon^d(f)$. Let $\delta := d(f, g)$, and choose n large enough so that $\sum_{k>n} 2^{-k} < \frac{\epsilon - \delta}{2}$. Define $A_g := B_{K_n}(g, \frac{\epsilon - \delta}{n})$, and suppose $h \in A_g$. Then since $K_m \subseteq K_n$ for $m \leq n$, we have

$$\begin{aligned} d(f, h) &\leq d(f, g) + d(g, h) \\ &\leq \delta + \sum_{k=1}^n 2^{-k} d_n(g, h) + \sum_{k>n} 2^{-k} \\ &\leq \delta + \frac{\epsilon - \delta}{2} + \frac{\epsilon - \delta}{2} = \epsilon. \end{aligned}$$

Therefore $g \in A_g \subset B_\epsilon^d(f)$. It follows that $B_\epsilon^d(f) \in \tau_c$. Indeed, we can write

$$B_\epsilon^d(f) = \bigcup_{g \in B_\epsilon^d(f)} A_g,$$

a union of elements of τ_c . This proves that $\tau_d \subseteq \tau_c$.

To prove the converse, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$. Choose n so that $K \subset K_n$, and let $g \in B_K(f, \epsilon)$ and $\delta := \sup_{x \in K} |f(x) - g(x)|$. If $d(g, h) < 2^{-n}(\epsilon - \delta)$, then $d'_n(g, h) \leq 2^n d(g, h) < \epsilon - \delta$, hence $d_n(g, h) < \epsilon - \delta$. It follows that

$$\begin{aligned} \sup_{x \in K} |f(x) - h(x)| &\leq \delta + \sup_{x \in K} |g(x) - h(x)| \leq \delta + d_n(g, h) \\ &\leq \delta + \epsilon - \delta = \epsilon. \end{aligned}$$

Thus $g \in B_{2^{-n}(\epsilon - \delta)}^d(f) \subset B_K(f, \epsilon)$. It follows that $\tau_c \subseteq \tau_d$, and we conclude that $\tau_d = \tau_c$.

Next, we show that $(C(\Sigma \times \Lambda), d)$ is a complete metric space. Let $(f_n)_{n \geq 1}$ be Cauchy with respect to d . Then we claim that (f_n) must be Cauchy with respect to d'_n , on each K_n . Indeed, $d(f_\ell, f_m) \geq 2^{-n} d'_n(f_\ell, f_m)$, so if (f_n) were not Cauchy with respect to d'_n , it would not be Cauchy with respect to d either. Thus (f_n) is uniformly Cauchy on each K_n , and hence converges uniformly to a limit f^{K_n} on each K_n . Since the limit must be unique at each point of $\Sigma \times \Lambda$, we have $f^{K_n}(x) = f^{K_m}(x)$ if $x \in K_n \cap K_m$. Since $\bigcup K_n = \Sigma \times \Lambda$, we obtain a well-defined function f on all of $\Sigma \times \Lambda$ given by $f(x) = f^{K_n}(x)$, where $x \in K_n$. Given any compact $K \subset \Sigma \times \Lambda$, if n is large enough so that $K \subset K_n$, then because $f_n \rightarrow f^{K_n} = f|_{K_n}$ uniformly on K_n , we have $f_n \rightarrow f^{K_n}|_K = f|_K$ uniformly on K . That is, for any $K \subset \Sigma \times \Lambda$ compact and $\epsilon > 0$, we have $f_n \in B_K(f, \epsilon)$ for all sufficiently large n . Therefore (f_n) converges to f in the topology of compact convergence, and equivalently in the metric d .

Lastly, we prove separability, following example 1.3 in Billingsley, *Convergence of Probability Measures*. For each pair of positive integers n, k , let $D_{n,k}$ be the subcollection of $C(\Sigma \times \Lambda)$ consisting of polygonal functions that are piecewise linear on $\{j\} \times I_{n,k,i}$ for each $j \in \Sigma_n$ and each subinterval

$$I_{n,k,i} := [a_n + \frac{i-1}{k}(b_n - a_n), a_n + \frac{i}{k}(b_n - a_n)], \quad 1 \leq i \leq k,$$

taking rational values at the endpoints of these subintervals, and extended linearly to all of $\Lambda = [a, b]$. Then $D := \bigcup_{n,k} D_{n,k}$ is countable, and we claim that it is dense in the topology

of compact convergence. To see this, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and choose n so that $K \subset K_n$. Since f is uniformly continuous on K_n , we can choose k large enough so that $|f(j, t) - f(j, a_n + \frac{i}{k}(b_n - a_n))| < \epsilon/2$ for $j \in \Sigma_n$ and $1 \leq i \leq k$. We then choose $g \in \bigcup_k D_{n,k}$ with $|g(j, a_n + \frac{i}{k}(b_n - a_n)) - f(j, a_n + \frac{i}{k}(b_n - a_n))| < \epsilon/2$. Then $f(j, t)$ is within ϵ of both $g(j, a_n + \frac{i-1}{k}(b_n - a_n))$ and $g(j, a_n + \frac{i}{k}(b_n - a_n))$. Since $g(j, t)$ lies between these two values, $f(j, t)$ is within ϵ of $g(j, t)$ as well. In summary,

$$\sup_{(j,t) \in K} |f(j, t) - g(j, t)| \leq \sup_{(j,t) \in K_n} |f(j, t) - g(j, t)| < \epsilon,$$

so $g \in B_K(f, \epsilon)$. This proves that D is a countable dense subset of $C(\Sigma \times \Lambda)$. We conclude that $(C(\Sigma \times \Lambda), \tau_c)$ is a Polish space.

Problem 2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X, Y random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $C(\Sigma \times \Lambda)$, where $\Sigma = \llbracket 1, N \rrbracket$ with $N \in \mathbb{N}$ or $N = \infty$. We consider the collection \mathcal{S}_X of sets of the form

$$\{\omega \in \Omega : X(\omega)(i_1, t_1) \leq x_1, \dots, X(\omega)(i_n, t_n) \leq x_n\} = \bigcap_{k=1}^n X(i_k, t_k)^{-1}(-\infty, x_k],$$

ranging over all $n \in \mathbb{N}$, $(i_1, t_1), \dots, (i_n, t_n) \in \Sigma \times \Lambda$, and $x_1, \dots, x_n \in \mathbb{R}$. We first prove that $\mathcal{S}_X \subset \mathcal{F}$. We can write

$$\{X(i_k, t_k) \leq x_k\} = X^{-1}(\{f \in C(\Sigma \times \Lambda) : f(i_k, t_k) \leq x_k\}).$$

We claim that the set $\{f \in C(\Sigma \times \Lambda) : f(i_k, t_k) \leq x_k\}$ is closed in the topology of compact convergence. If $f_n(i_k, t_k) \leq x_k$ for all n and $f_n \rightarrow f$ in the topology of compact convergence, then by taking limits on a compact set containing (i_k, t_k) , we find $f(i_k, t_k) \leq x_k$ as well. This proves the claim, and it follows from the measurability of X that $\{X(i_k, t_k) \leq x_k\} = X^{-1}(\{f(i_k, t_k) \leq x_k\}) \in \mathcal{F}$. The finite intersection is thus also in \mathcal{F} , proving that $\mathcal{S}_X \subset \mathcal{F}$. On the other hand, it is clear that $\{\omega \in \Omega : X(\omega) \in A\} = X^{-1}(A) \in \mathcal{F}$ for any $A \in \mathcal{C}_\Sigma$ since X is measurable.

Now we prove that $\mathbb{P}|_{\mathcal{S}_X}$ determines the distribution $\mathbb{P} \circ X^{-1}$. To do so, note that $\mathcal{S}_X = \sigma(\{X^{-1}(A) : A \in \mathcal{S}\})$, where \mathcal{S} is the collection of cylinder sets

$$\{f \in C(\Sigma \times \Lambda) : f(i_1, t_1) \in A_1, \dots, f(i_n, t_n) \in A_n\}, \quad A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}).$$

This follows from the fact that $\mathcal{B}(\mathbb{R})$ is generated by intervals of the form $(-\infty, x]$. Furthermore, this fact, along with the fact proven above that $\{f(i_k, t_k) \in (-\infty, x_k]\}$ is closed, show that $\mathcal{S} \subset \mathcal{C}_\Sigma$. Observe that the intersection of two elements of \mathcal{S} is clearly another element of \mathcal{S} , so \mathcal{S} is a π -system. We now argue that \mathcal{S} generates the Borel sets, i.e., $\sigma(\mathcal{S}) = \mathcal{C}_\Sigma$. Since $\mathcal{S} \subset \mathcal{C}_\Sigma$, we have $\sigma(\mathcal{S}) \subseteq \mathcal{C}_\Sigma$. To prove the opposite inclusion, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and let H be a countable dense subset of K . (Recall that every

compact metric space is separable, and K is homeomorphic to a product of finitely many compact sets in \mathbb{R} , which are metrizable. So K is separable.) We claim that

$$B_K(f, \epsilon) = \bigcup_{n=1}^{\infty} \bigcap_{(i,t) \in H} \{g \in C(\Sigma \times \Lambda) : g(i, t) \in (f(i, t) - (1 - 2^{-n})\epsilon, f(i, t) + (1 - 2^{-n})\epsilon)\}.$$

Indeed, if $g \in B_K(f, \epsilon)$, i.e., $\sup_{(i,t) \in K} |g(i, t) - f(i, t)| < \epsilon$. Then since $1 - 2^{-m} \nearrow 1$, we can choose m large enough so that

$$|g(i, t) - f(i, t)| < (1 - 2^{-n})\epsilon$$

for all $(i, t) \in K$ (in particular with $(i, t) \in H$). Conversely, suppose g is in the set on the right. Then since g is continuous and H is dense in K , we find that for some $n \geq 1$,

$$|g(i, t) - f(i, t)| \leq (1 - 2^{-n})\epsilon < \epsilon$$

for all $(i, t) \in K$. Hence $g \in B_K(f, \epsilon)$. This proves the claim. Since H is countable, $B_K(f, \epsilon)$ is formed from countably many unions and intersections of sets in \mathcal{S} , thus $B_K(f, \epsilon) \in \sigma(\mathcal{S})$.

Now by problem 1, the topology generated by the basis $\mathcal{A} = \{B_K(f, \epsilon)\}$ is separable and metrizable. The balls of rational radii centered at points of a countable dense subset then give a (different) countable basis \mathcal{B} for the same topology. We claim that this implies that every open set is a *countable* union of sets $B_K(f, \epsilon)$. To see this, let $B \in \mathcal{B}$, and write $B = \bigcup_{\alpha \in I} A_\alpha$, for sets $A_\alpha \in \mathcal{A}$. Then for each $x \in B$, pick $\alpha_x \in I$ such that $x \in A_{\alpha_x}$. Since \mathcal{B} is a basis, there is a set $B_x \in \mathcal{B}$ with $x \in B_x \subseteq A_{\alpha_x}$. Then $B = \bigcup_{x \in B} B_x$. Note that if $y \in B_y \subseteq A_{\alpha_y}$ and $B_y = B_x$, then in fact $y \in A_{\alpha_x}$, so we can remove A_{α_y} from the union. In other words, we can choose the A_{α_x} so that each corresponds to exactly one B_x . But there are only countably many distinct sets B_x , so we see that B is a countable union of elements of \mathcal{A} . Since every open set can be written as a countable union of elements of B , this proves the claim. Since $\mathcal{A} \subseteq \sigma(\mathcal{S})$ by the above, it follows that every open set is in $\sigma(\mathcal{S})$, and consequently so is every Borel set, i.e., $\mathcal{C}_\Sigma \subseteq \sigma(\mathcal{S})$.

In summary, we have shown that the collection \mathcal{S} is a π -system generating \mathcal{C}_Σ , so the probability measure $\mathbb{P} \circ X^{-1}$ on \mathcal{C}_Σ is uniquely determined by its restriction to \mathcal{S} . Suppose

$$\begin{aligned} \mathbb{P}(\{\omega \in \Omega : X(\omega)(i_1, t_1) \leq x_1, \dots, X(\omega)(i_n, t_n) \leq x_n\}) = \\ \mathbb{P}(\{\omega \in \Omega : Y(\omega)(i_1, t_1) \leq x_1, \dots, Y(\omega)(i_n, t_n) \leq x_n\}) \end{aligned}$$

for all $(i_1, t_1), x_1, \dots, x_n$. This says that the two probability measures $\mathbb{P} \circ X^{-1}$ and $\mathbb{P} \circ Y^{-1}$ agree on \mathcal{S} . Then they must agree on all of \mathcal{C}_Σ , i.e.,

$$\mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) = \mathbb{P}(\{\omega \in \Omega : Y(\omega) \in A\})$$

for all $A \in \mathcal{C}_\Sigma$. In other words, the law of a line ensemble is determined by its finite dimensional distributions.

2 Algebra

Problem 3

Problem 4

3 Weak convergence

Problem 5

Problem 6

Problem 7

4 Tightness

Problem 8

Problem 9

5 Lozenge tilings of the hexagon

Problem 10

Problem 11