$\boxed{Var(\hat{\beta}_0)}$

strategy: express $\hat{\beta}_0$ as a sum of constants $\times$ y_i #

$k_i = \frac{x_i - \bar{x}}{SSX}$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$= \frac{1}{n}\sum_i y_i - \underbrace{\left(\sum_{i=1}^{n} k_i y_i\right)}_{*} \bar{x}$$

Claim:

$\hat{\beta}_1 = \sum_i k_i y_i$ where $k_i = \frac{x_i - \bar{x}}{SSX}$

1

$SSX = \sum_i (x_i - \bar{x})^2$

$= \sum_i x_i^2 - n\bar{x}^2$

$\frac{\sum_i x_i y_i - n\bar{x}\bar{y}}{\sum_i x_i^2 - n\bar{x}^2}$

1 PF

$\hat{\beta}_1 = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{SSX}$

$n\bar{x} - n\bar{x}$

$= \sum_i (x_i - \bar{x}) y_i - \sum_i (x_i - \bar{x}) \bar{y}$

$$= \sum_{i=1}^{n} \left(\frac{1}{n} y_i - \bar{x} k_i y_i\right)$$

$$= \sum_{i=1}^{n} \underbrace{\left(\frac{1}{n} - \bar{x} k_i\right)}_{c_i} y_i$$

$$= \sum_{i=1}^{n} c_i y_i$$

$$Var(\hat{\beta}_0) = Var\left(\sum_{i=1}^{n} c_i y_i\right)$$

$$\overset{\text{(1)}}{=} \sum_{i=1}^{n} Var(c_i y_i)$$

$$= \sum_{i=1}^{n} c_i^2 \, Var(y_i)$$

b/c $y_1 \perp\!\!\!\perp y_2 \ldots \perp\!\!\!\perp y_n$

$cor(y_i, y_j) = 0$ if $i \neq j$

$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$

$$= \sum_{i=1}^{n} c_i^2 (\sigma^2)$$

$$= \sigma^2 \sum_{i=1}^{n} c_i^2 \qquad \text{where} \quad c_i = \frac{1}{n} - \bar{x} k_i$$

$$\boxed{= \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{SSX} \right)}$$

$$\sum_{i=1}^{n} c_i^2 = \sum_{i=1}^{n} \left(\frac{1}{n} - \bar{x} k_i \right)^2$$

$$= \sum_{i=1}^{n} \left(\frac{1}{n^2} - \frac{2}{n} \bar{x} k_i + \bar{x}^2 k_i^2 \right)$$

$$= \frac{n}{n^2} - \boxed{\cancel{\frac{2\bar{x}}{n} \sum_i k_i}} + \bar{x}^2 \underline{\sum_i k_i^2} \qquad 0 \qquad \frac{1}{SSX}$$

$$\underline{\sum_{i=1}^{n} k_i} = \sum_{i=1}^{n} \frac{x_i - \bar{x}}{SSX} = \frac{1}{SSX} \left[\sum_i (x_i - \bar{x}) \right] = 0$$

$$\left(\sum_i x_i - \sum_i \bar{x} \right)$$

$$\left(n\bar{x} - n\bar{x} \right)$$

$$\sum_{i=1}^{n} k_i^2 = \sum_{i=1}^{n} \left[\frac{(x_i - \bar{x})}{SSX} \right]^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{(SSX)^2} = \frac{1}{(SSX)^2} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= \frac{SSX}{(SSX)^2} = \frac{1}{SSX}$$

$$SSX = \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{j=1}^{n} (x_j - \bar{x})^2$$

$n = 3$

i	x
1	3
2	4
3	5

$\bar{x} = 4$

$$SSX = (3-4)^2 + (4-4)^2 + (5-4)^2$$
$$= 2$$

$$\sum_{i=1}^{n} (x_i - \bar{x})$$

c. $$\underline{\sum_{i=1}^{n} (e_i)(\hat{y}_i - \bar{y})} = 0$$

ε_i vs. e_i

$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ $\qquad$ $e_i = y_i - \hat{y}_i$

$\varepsilon_i \sim N(0, \sigma^2)$ $\qquad$ $e_i \sim N(\underline{\ 0\ }, \underline{\qquad})$

? ?

$$E(e_i) = E(y_i - \hat{y}_i)$$
$$= E(y_i) - E(\hat{y}_i)$$
$$= \beta_0 + \beta_1 x_i - \beta_0 - \beta_1 x_i$$

$$Var(e_i) = \sigma^2\left(1 - \frac{1}{n} - \frac{(x_i - \bar{x})^2}{n}\right)$$
$$\|$$
$$var(y_i - \hat{y}_i) = Var(y_i) + \underline{Var(\hat{y}_i)} - \underline{2Cov(y_i, \hat{y}_i)}$$
$$= \sigma^2 + \sigma^2\left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{SSx}\right) - 2(\dots)$$

$$\underline{\underline{E(\hat{y}_i)}} = E(\hat{\beta}_0 + \hat{\beta}_1 x_i)$$
$$= E(\hat{\beta}_0) + \boxed{E(\hat{\beta}_1)}x_i$$
$$= \beta_0 + \boxed{\beta_1}x_i$$

SLR Model:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$
$$\varepsilon_i \overset{ind}{\sim} N(0, \sigma^2)$$

$$\underline{\underline{E(y_i)}} = E(\beta_0 + \beta x_i + \varepsilon_i)$$
$$= \beta_0 + \beta_1 x_i + E(\varepsilon_i) = \underline{\beta_0 + \beta_1 x_i}$$

$$Var(A+B) = E\left(\left((A+B) - (E(A)+E(B))\right)^2\right)$$

$$= E\left(\left((A-E(A)) + (B-E(B))\right)^2\right)$$

$$= E\left((A-E(A))^2 + 2(A-E(A))(B-E(B)) + (B-E(B))^2\right)$$

$$= \underbrace{E\left((A-E(A))^2\right)}_{Var(A)} + \underbrace{2E\left((A-E(A))(B-E(B))\right)}_{2Cov(A,B)} + \underbrace{E\left((B-E(B))^2\right)}_{Var(B)}$$

$$Var(A) + 2Cov(A,B) + Var(B)$$

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what about $B = -C$

$Var(A-C)$

$$= Var(A) + 2Cov(A,-C) + Var(-C)$$

$$= Var(A) + (2)(-1)Cov(A,C) + (-1)^2 Var(C)$$

$$= Var(A) - 2Cov(A,C) + Var(C)$$

$$Var(\hat{y}_i) = Var(\hat{\beta}_0 + \hat{\beta}_1 x_i)$$

$$= Var(\hat{\beta}_0) + Var(\hat{\beta}_1)x_i^2 + \underbrace{2Cov(\hat{\beta}_0, \hat{\beta}_1 x_i)}_{-2x_i\bar{x}\,\sigma^2/SSX}$$

$$= \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{SSX}\right) + x_i^2\sigma^2/SSX \quad -2x_i\bar{x}\,\sigma^2/SSX$$

$$= \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2 - 2x_i\bar{x} + \bar{x}^2}{SSX}\right)$$

$$= \sigma^2\left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{SSX}\right)$$

$$Cov(X, Y) = E\left((X - E(X))(Y - E(Y))\right)$$

$$\operatorname{Cov}(\hat\beta_0, \hat\beta_1) = \operatorname{Cov}(\bar y - \hat\beta_1 \bar x, \hat\beta_1)$$

$$= \operatorname{Cov}(\bar y, \hat\beta_1) - \bar x \operatorname{Cov}(\hat\beta_1, \hat\beta_1)$$

$$= 0 \qquad - \bar x \operatorname{Var}(\hat\beta_1)$$

$$\operatorname{Cov}(\bar y, \hat\beta_1) = \operatorname{Cov}\left(\frac{1}{n}\sum_i y_i, \sum_{j=1}^{n} k_j y_j\right)$$

$$= \sum_i \sum_j \frac{1}{n} k_j \operatorname{Cov}(y_i, y_j)$$

$$= \sum_{i=j} \frac{1}{n} k_j \operatorname{Cov}(y_j, y_j)$$

$$= \sum_{i=j} \frac{1}{n} k_j \sigma^2 = \frac{\sigma^2}{n} \sum_{i=j} k_j$$

$$= \frac{\sigma^2}{n} \sum_{j=1}^{n} k_j = \frac{\sigma^2}{n}(0)$$

χ^2 dists.

Given:

$$\frac{\sum_i e_i^2}{\sigma^2} \sim \chi^2_{n-2}$$

$\hat{\sigma}^2$

wanted an est of σ^2

$$SSE \sim \sigma^2 \chi^2_{n-2}$$

$$\frac{SSE}{\sigma^2} \sim \chi^2_{n-2}$$

$$E(\underline{\hat{\sigma}^2}) = \sigma^2$$

$$E\left(\frac{\sum_i e_i^2}{\sigma^2}\right) = n-2$$

$$\Leftrightarrow$$

$$\frac{1}{\sigma^2} E(\sum_i e_i^2) = n-2$$

$$\Leftrightarrow \frac{1}{n-2} E(\sum_i e_i^2) = \sigma^2$$

$$\Leftrightarrow E\left(\frac{\sum_i e_i^2}{n-2}\right) = \sigma^2$$

take this as $\hat{\sigma}^2$

If $X \sim N(\mu_0, \sigma^2)$

$$z = \frac{X-\mu_0}{\sigma} \sim N(0,1)$$

$$t = \frac{z}{\sqrt{\chi^2_{df}/df}}$$

$$t = \frac{X-\mu_0}{\hat{\sigma}}$$

$$= \frac{X-\mu_0}{\sigma} \cdot \frac{\sigma}{\hat{\sigma}}$$

$$= \frac{z}{\hat{\sigma}/\sigma} = \frac{z}{\sqrt{\frac{\sum_i e_i^2}{\sigma^2(n-2)}}}$$

$$= \frac{z}{\sqrt{\frac{\sum_i e_i^2}{\sigma^2}/n-2}} = \frac{z}{\sqrt{\chi^2_{n-2}/n-2}} = t_{n-2}$$

SLR Model

LS Estimators — distribution
— unbiased
— variances?

Fitted values — "
— "
— "

$SE(est) = \sqrt{Var(est)}$

Residuals — "
— "
— "

$est \pm t^* \hat{SE}(est)$

$\hat{\beta}_1 \pm t^* \sqrt{\hat{\sigma}^2 / SSX}$

Estimating σ^2

CI — slope, int, fitted vals

$\hat{y}_i \pm t^* \hat{\sigma} \sqrt{\underline{1} + \frac{1}{n} + \frac{(x_i - \bar{x})^2}{SSX}}$

PI — a new observation

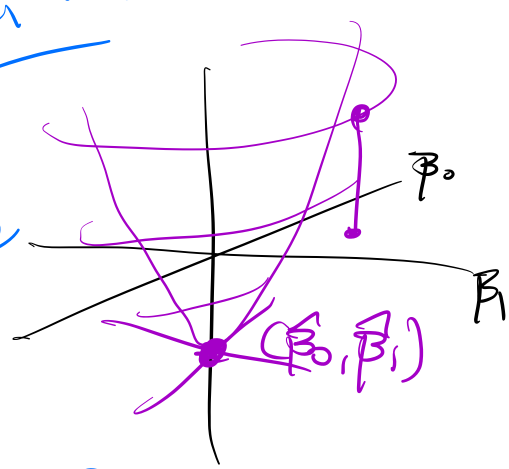
Diagnostics

$$\boxed{\sum_i e_i x_i = 0}$$

② $\sum_i e_i \hat{y}_i$

$$\sum_i e_i (\hat{\beta}_0 + \hat{\beta}_1 x_i) = \sum_{i=1}^{n} e_i \hat{\beta}_0 + \sum_{i=1}^{n} e_i \hat{\beta}_1 x_i$$

$$= \hat{\beta}_0 \sum_{i=1}^{n} e_i + \hat{\beta}_1 \sum_{i=1}^{n} e_i x_i$$



$$Q(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\partial Q}{\partial \beta_1} = \sum_{i=1}^{n} -2x_i (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$= \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) x_i = 0$$

For the specific choices of $\beta_0 = \hat{\beta}_0$, $\beta_1 = \hat{\beta}_1$

$$\Rightarrow \sum_{i=1}^{n} (y_i - \hat{y}_i) x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} e_i x_i = 0$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_1 = \frac{\sum_i (x_i - \bar{x}) y_i}{SSx}$$

$$\sum_{i=1}^{n} (y_i - \hat{y}_i) x_i$$

$$= \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) x_i$$

$$\hat{\beta}_1 = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \sum_{i=1}^{n} k_i y_i$$

$$\boxed{SST = SSE + SSR}$$

F test:

$F_{stat} > F^{\alpha} \Rightarrow$ reject H_0

$F_{stat} < F^{\alpha} \Rightarrow$ fail to reject