H is a projection matrix &

so is $(I-H)$ ....

Are all linear combinations of

H & $I-H$ also projections?

If this is true, we must have:

$aH + b(I-H)$ ① symmetric ② idempotent

① is true since scalars don't affect symmetry.

② ...

$$(aH + b(I-H))^2 = a^2H^2 + ab\,\cancel{H(I-H)}^{\,0} + ba\,\cancel{(I-H)H}^{\,0} + b^2(I-H)^2$$

$$= a^2H^2 + b^2(I-H)^2$$

$$= a^2H + b^2(I-H) \ldots$$

This only is equal to original: $aH + b(I-H)$

If $a^2 = a$ & $b^2 = b$ . . .

i.e. $a = 0$ or 1,

$b = 0$ or 1

So the linear combs of H & $I-H$ which are projections are

- $0H + 0(I-H) = 0$
- $0H + 1(I-H) = I-H$
- $1H + 0(I-H) = H$
- $1 \cdot H + 1 \cdot (I-H) = H + I - H = I$

kind of a boring result sadly lol

but still a very interesting question!!