Resolution Theorem Proving

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Resolution

- An Alternative to Natural Deduction
- Natural deduction is (we can argue)
  - Good for human use, but maybe too complicated
  - Definitely complicated for automation
  - Not mechanical enough
  - Too many rules of inference

Resolution: Motivation

- Designed to be automated
- Basis of logic programming
- Basis of Prolog

Resolution in Propositional Logic in a Nutshell

- First convert the formulas to a very simple form.
- The form is called Conjunctive Normal Form (CNF).
- CNF has just the connectives $\wedge$, $\vee$, $\neg$.
- Every formula can be put in this form.

Example: The Election

Premise:

1. $E \rightarrow L \vee T$
2. $\neg U \rightarrow \neg L$
3. E

Conclusion:

$\neg T \rightarrow U$

Rewrite in CNF

$\neg E \vee L \vee T$

$U \vee \neg L$

E

- Next Negate the conclusion to be derived and convert that to CNF.

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Negation of conclusion

$\neg(\neg\neg T \vee U) \equiv \neg(T \vee U)$

In CNF $\neg T \wedge \neg U$

The problem now becomes finding an inconsistency amongst

$$\neg E \vee L \vee T$$

$$U \vee \neg L$$

$$E$$

$$\neg T$$

$$\neg U$$

Resolution Proof rule
for Prositional Logic

Unit resolution: $\vee$-elimination ($\vee$E)

$$\frac{X \vee Y,\ \neg X}{Y} \qquad\qquad \frac{X \vee Y,\ \neg Y}{X}$$

$X \vee Y \quad \neg X$

Y

$X \vee Y \quad \neg Y$

X

Take this a little further to **propositional binary resolution**:

$$\frac{X \vee Y,\ \neg X \vee Z}{Y \vee Z} \qquad \frac{X \vee Y,\ Z \vee \neg Y}{X \vee Z}$$

$X \vee Y \quad \neg X \vee Z$

$Y \vee Z$

$X \vee Y \quad Z \vee \neg Y$

$X \vee Z$

Justification of the propositional binary resolution

$X \vee Y \quad \neg X \vee Z$

$Y \vee Z$

1. $X \vee Y$	given
2. $\neg X \vee Z$ given	
3. $X \vee \neg X$ law of excluded middle	
4. X	assume
5. Z	4, 2, $\vee E$
6. $Y \vee Z$	5, $\vee I$
7. $X \rightarrow Y \vee Z$	4, 6, $\rightarrow I$
8. $\neg X$	assume
9. Y	8, 1, $\vee E$
10. $Y \vee Z$	9, $\vee I$
11. $\neg X \rightarrow Y \vee Z$	8, 10, $\rightarrow I$
12. $Y \vee Z$	3, 7, 11, dilemma

Back to the Election Example

We have

$\neg E \vee L \vee T$

$U \vee \neg L$

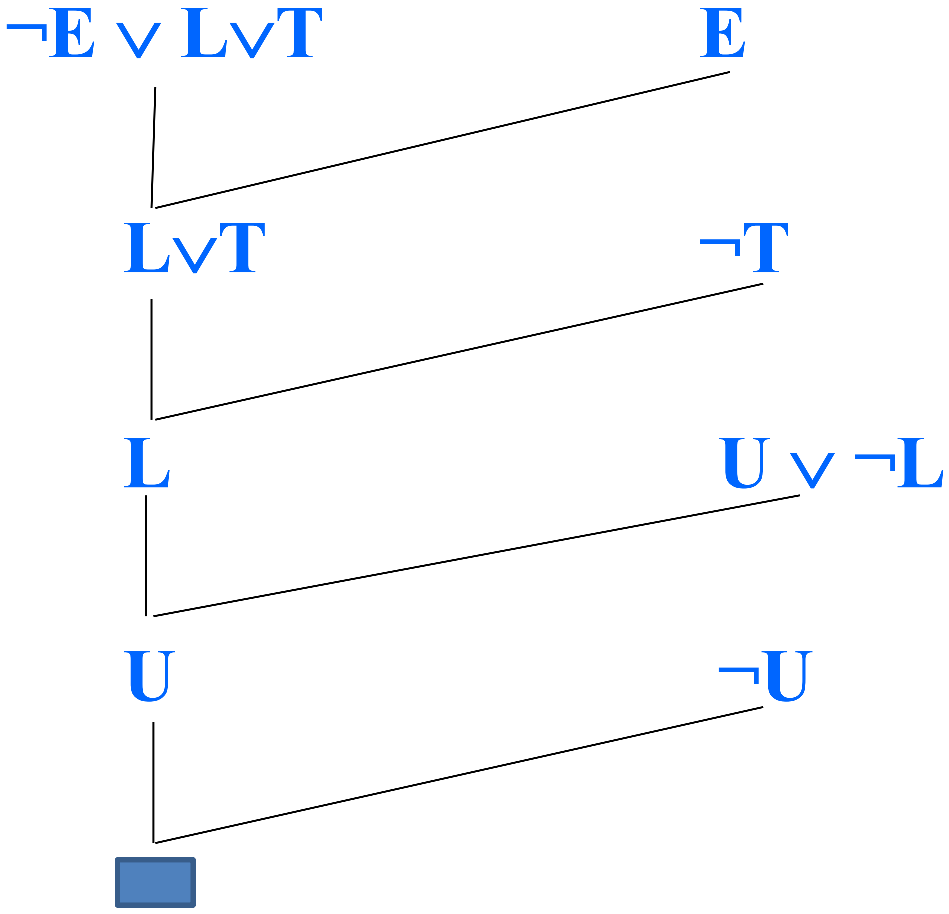
E

$\neg T$

$\neg U$

and just one rule of inference, namely the resolution rule.

$\neg E \vee L \vee T, \quad U \vee \neg L, \quad E, \quad \neg T, \quad \neg U$



This shows

Premise + $\neg$ Conclusion is inconsistent.

So by proof by contradiction

Premise $\vdash$ Conclusion.

Now More Details

- What exactly is CNF
- How do we construct it
- Property of Resolution
- Extension to Predicate Logic
- Relationship to Prolog

Conjunctive Normal Form (CNF)

A wff is in CNF if it is of the form:

$W_1 \wedge W_2 \wedge .... \wedge W_n$, $n \geq 1$

and each W_i is disjunction of literals.

Examples: the following are in CNF

P

$P \vee Q$

$P \vee \neg Q$

$(P \vee \neg Q) \wedge (R \vee \neg S \vee T)$

Converting to CNF

Step 1: Eliminate $\leftrightarrow$ and $\rightarrow$

Using $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$

$P \rightarrow Q \equiv \neg P \vee Q$

Step 2: Push negations in towards atoms

Using $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

$\neg\neg P \equiv P$

Step 3: Use distributatativity and commutativity

$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

$(Q \wedge R) \vee P \equiv P \vee (Q \wedge R)$

Summary: How to do Resolution Proofs in Propositional Logic

Given a set of wffs S and a wff W, to show

$S \vdash W$

by resolution, follow these steps:

1. Convert all sentences in S to conjunctive normal form (CNF).
2. Negate W (to get $\neg$ W).
3. Convert $\neg$ W to CNF.
4. Apply resolution to CNF(S) and CNF($\neg$W) until:

a. Derive a falsity (contradiction). In this case W is proved.
b. Can’t apply any further resolution steps. In this case W is not proved.

Property of Resolution

- Resolution is **refutation-complete** over first order logic (propositional and predicate logic).
- This means that if you write any set of sentences in first order logic which are **contradictory** or **unsatisfiable** (i.e., taken together they have no models), then the resolution method will eventually derive the Falsity, indicating that the sentences are contradictory.

Resolution for Predicate Logic

More general **first-order binary resolution** inference rule:

$$\frac{X \vee Y,\ \neg W \vee Z}{(Y \vee Z)\,\theta}$$

if X and W unify with substitution θ.

Example:

athlete(arnie) $\vee$ actor(arnie) $\neg$athlete(P) $\vee$ healthy(P)

actor(arnie) $\vee$ healthy(arnie)

$\neg$athlete(P) $\vee$ healthy(P) is the CNF of

$\forall$P (athlete(P) $\rightarrow$ healthy(P))

Converting Predicate Logic Sentences to CNF

Step 1: as before

Step 2: as before but add moving negations inwards through quantifies, using

$$\neg \exists X \; p(X) \equiv \forall X \; \neg p(X)$$

$$\neg \forall X \; p(X) \equiv \exists X \; \neg p(X)$$

Step 3: Standardize variables apart by renaming them: each quantifier should use a different variable.

Step 4: Skolemise. (don't worry about it)

Step 5: Drop universal quantifiers.

Step 6: Use distributativity and commutativity

Relationship to Prolog

Prolog is based on unification and resolution.

Example:

```
p :- q.
q :- r.
r.
```

Query: p

p :- q. r.

q :- r. Query: p

$\neg p$ $\quad\quad$ $p \lor \neg q$

$\neg q$ $\quad\quad$ $q \lor \neg r$

$\neg r$ $\quad\quad$ r

$\square$

Example

```
sister_of(X,Y) :-
        siblings(X,Y),
        female(X).
siblings(X,Y) :-
        parent_of(Z,X),
        parent_of(Z,Y).
parent_of(tom, jill).
parent_of(tom, john).
female(jill).
```

$sis(X,Y) \vee \neg sib(X,Y) \vee \neg f(X)$

$sib(X,Y) \vee \neg p(Z,X) \vee \neg p(Z,Y)$

$p(tom, jill)$

$p(tom, john)$

$f(jill)$

Query: sister_of(X, Y)

$\neg sis(X,Y)$

$\neg\, sis(X, Y)$ $\qquad$ $sis(X,Y) \vee \neg sib(X,Y) \vee \neg f(X)$

$\neg sib(X,Y) \vee \neg f\,(X)$ $\qquad$ $sib(X,Y) \vee \neg p(Z,X) \vee \neg p(Z,Y)$

$\neg p(Z,X) \vee \neg p(Z,Y) \vee \neg f\,(X)$ $\qquad$ $p(tom, jill)$

Z=tom, X=jill

$\neg p(tom,Y) \vee \neg f\,(jill)$ $\qquad$ $p(tom, john)$

Y=john

$\neg f\,(jill)$ $\qquad$ $f(jill)$

Answer is X=jill, Y=john