Predicate Logic Cntd.

Fariba Sadri

Rules of Inference
Natural Deduction

All inference rules for propositional logic + 4 new rules to deal with the quantifiers.

1. ∀-elimination (∀E)

<u>**∀X p(X)**</u>

p(a)

where a is any constant.

The constant a must replace every free occurrence of X in P(X).

E.g.

From $\forall X$ beautiful(X) we can conclude

beautiful(quasimodo).

From

$\forall X$ (lion(X)$\rightarrow$ $\exists Y$(lioness(Y)$\wedge$provides_food(Y,X)))

We can infer

lion(shere_khan)$\rightarrow$

$\exists Y$(lioness(Y) $\wedge$ provides_food(Y,shere_khan))

Exercise

Formalise the argument below and show that it is valid.

MSc Generalist and MSc Software Engineering students do group projects.

Martin is either an MSc Generalist or an MSc Software Engineering student.

So Martin does a group project.

2. **$\forall$-Introduction ($\forall$I)**

(Universal generalisation)

If we know all the ground terms and there are a small number of them, e.g. $a_1, ..., a_n$,

then to show

$$\forall X\ p(X)$$

we show $p(a_1), \ldots, p(a_n)$.

Suppose we wanted to show

$$\forall X\ (\text{student}(X) \rightarrow \text{undergrad}(X) \lor \text{postgrad}(X))$$

then this approach would mean checking every student.

This approach is not practical in general.

But maybe we know some properties of being a student, e.g.

All students are enrolled on a degree programme.

Our only degree programmes are UG and PG.

Then we can show that if you pick any student they will be UG or PG.

In general to show

$$\forall X\ p(X)$$

we show $p(a)$

for an arbitrary constant a on which there are no constraints.

∀-Introduction (∀I)

$$\frac{p(a)}{\forall X\ p(X)}$$

provided the following conditions are met:

1. a is an arbitrary constant.
2. There are no assumptions involving a, left undischarged, used to obtain p(a).
3. Substitution of X for a in p(X) is uniform, i.e. X is substituted for every occurrence of a.

E.g.

From

$\forall Y\ (q(a,Y) \rightarrow \exists Z\ (r(Z) \wedge t(Z,Y,a)))$

we can infer

$\forall X \forall Y\ (q(X,Y) \rightarrow \exists Z\ (r(Z) \wedge t(Z,Y,X)))$

provided a is an arbitrary constant.

Note:

To be on the safe side:

Make sure there is no variable clash when applying the rule.

The safest is to introduce a new variable, i.e. one that does not occur in the original wff.

Exercise

All messages are encrypted.

Anything that is encrypted is secure.

So all messages are secure.

Exercise

Given

1. $\forall X\ (p(X) \rightarrow \exists Y\ q(X,Y))$
2. $\forall Z\ (\exists X\ q(Z,X) \wedge r(a) \rightarrow s(Z,a))$
3. $r(a)$

show

$\forall X\ (\neg p(X) \vee s(X,a))$

3. ∃ -Introduction (∃I)

$$\frac{p(t)}{\exists X\ p(X)}$$

where t is any term, and X does not clash with any occurrence of X in p(t).

X is substituted for one or more occurrences of t in p(t).

Example : Given

logician(charles_dodgson) $\wedge$

writer(charles_dodgson)

Abbreviated as **$l(cd) \wedge w(cd)$**

we can derive each of the following by an application of the $\exists$I rule.

$\exists X\ (l(X) \wedge w(X))$ **$\exists X\ (l(X) \wedge w(cd))$**

$\exists X\ (l(cd) \wedge w(X))$

Beware clash of variables:

Example:

There is a course that Mary likes.

$\exists X\ (course(X) \wedge likes(mary,X))$

We can derive:

$\exists Y\ \exists X(course(X) \wedge likes(Y,X))$

but not

$\exists X\ \exists X(course(X) \wedge likes(X,X))$

(There is a course that likes itself!)

4. ∃-Elimination (∃E)

$$\frac{\exists X p(X), \quad \begin{array}{l} p(a) \quad \text{assume} \\ \vdots \\ W \end{array}}{W}$$

where W is any wff, provided the following conditions are met:

i. a is an arbitrary constant.

ii. In proving W from p(a) the only assumption left undischarged in which a occurs is p(a).

iii. a does not occur in W or in $\exists X\ p(X)$.

Note:

p(a) is an assumption, which is discharged by the application of $\exists E$ rule, above.

Example:

Aliens is an exciting film.

All exciting films make a lot of money.

So there is a film that makes a lot of money.

1.	$film(als) \wedge exciting(als)$	given
2.	$\forall X(film(X) \wedge exciting(X) \rightarrow makes\_money(X))$	given
3.	$film(als) \wedge exciting(als) \rightarrow makes\_money(als)$	$2, \forall E$
4.	$makes\_money(als)$	$3, 1, \rightarrow E$
5.	$film(als)$	$1, \wedge E$
6.	$film(als) \wedge makes\_money(als)$	$5, 4, \wedge I$
7.	$\exists X\ (film(X) \wedge makes\_money(X))$	$6, \exists I$

Compare with:

There is an exciting film.
All exciting films make a lot of money.
So there is a film that makes a lot of money.

1. $\exists X\ (film(X) \land exciting(X))$ given
2. $\forall X(film(X) \land exciting(X) \rightarrow makes\_money(X))$ given
    3. $film(a) \land exciting(a)$ assume
    4. $film(a) \land exciting(a) \rightarrow makes\_money(a)$ 2, $\forall$E
    5. $makes\_money(a)$ 3,4, $\rightarrow$E
    6. $film(a)$ 3, $\land$E
    7. $film(a) \land makes\_money(a)$ 5,6, $\land$I
    8. $\exists X\ (film(X) \land makes\_money(X))$ 7, $\exists$I
9. $\exists X\ (film(X) \land makes\_money(X))$ 1,3,8, $\exists$E

Example:

$\exists X\ (\text{manager}(X) \wedge \text{promoted}(X)) \vdash$

$\exists X\ \text{promoted}(X)$

More generally, the following are useful derivations:

$\exists X\ (p(X) \wedge q(X)) \vdash \exists X\ p(X)$

$\exists X\ (p(X) \wedge q(X)) \vdash \exists X\ q(X)$

1. $\exists X\ (p(X) \wedge q(X))$	given
2. $p(a) \wedge q(a)$	assume
3. $q(a)$	2, $\wedge$E
4. $\exists X\ q(X)$	3, $\exists$I
5. $\exists X\ q(X)$	1, 2, 4, $\exists$E

Exercise

Formalise the following argument and show that it is valid.

Someone hacked into secure file f ('finance').

Anyone who hacks into a secure file either has stolen its password or has had insider help.

So there is someone who has stolen f's password or has had insider help.

Be careful!

- When applying the inference rules identify the dominant connective/quantifier correctly.
- Apply the inference rule applicable to that connective.

Example:

From $\forall X\ (p(X) \rightarrow q(X))$

we can derive $p(a) \rightarrow q(a)$ by $\forall$E.

But

From $\neg\forall X\ (p(X) \rightarrow q(X))$

we cannot derive $\neg(p(a) \rightarrow q(a))$ by $\forall$E.

The following derivation is wrong:

$\neg\forall X\ (\text{business}(X) \rightarrow \text{avoidsTax}(X))$

$\neg\ (\text{business}(\text{amazon}) \rightarrow \text{avoidsTax}(\text{amazon}))$

Be careful!

From $\neg p(a)$

we can derive $\exists X\ \neg p(X)$ by $\exists$I.

But from $\neg p(a)$

we cannot derive $\neg\exists X\ p(X)$ by $\exists$I.

From $\neg$ **happy(tom)** we can derive

$\exists X\ \neg happy(X)$ but not $\neg\exists X\ happy(X)$.

Soundness and Completeness

Predicate logic is sound and complete.

Decidability

Definition:

A logical system is **decidable** iff it is possible to have an effective method (an algorithm) that is guaranteed to recognise correctly whether a wff is a theorem of the system or not. In other words, a logical system is decidable if it satisfies conditions 1 and 2 below.

1) If $|=$ W then there is an algorithm that recognises that W is a theorem.

2) If it is not the case that $|=$W then there is an algorithm that recognises that W is not a theorem.

Propositional logic is decidable.

Predicate logic is not - it is semi-decidable, that is, it satisfies condition 1, above, but not condition 2.