Predicate Logic

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Example: MSc regulations

- Passing the exams and the project implies passing the MSc.

- You do not pass the MSc and you do not get a certificate if you do not pass the exams or you do not pass the project.

Let us take the following propositions for formulating the MSc regulations:

pe:	pass exams
pp:	pass project
pm:	pass MSc
gc:	get certificate

In propositional logic:

$\mathbf{pe \wedge pp \rightarrow pm}$

$\mathbf{\neg pe \vee \neg\, pp \rightarrow \neg\, pm \wedge \neg\, gc}$

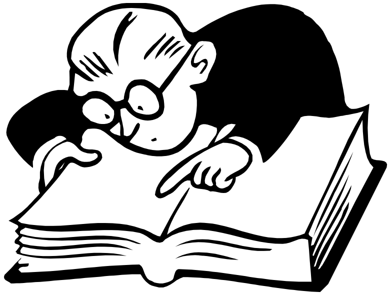
Not expressive enough if we want to consider individual students, to check who has passed the MSc, and who has not, for example.

Example

John:

passes the project

but not the exams



Mary:

passes the exams

passes the project

Who passes the MSc?

Example

For all individuals X:

$$(pe(X) \wedge pp(X) \rightarrow pm(X))$$

For all individuals X:

$$(\neg pe(X) \vee \neg pp(X) \rightarrow \neg pm(X) \wedge \neg gc(X))$$

Increase the expressive power of the Propositional Logic language by adding:

- Predicates: that take arguments (extending propositions)
- Parameters: as arguments of the predicates
- Variables: as arguments of the predicates
- Quantification

More formal expression of the MSc regulations

$$\forall X\ (pe(X) \wedge pp(X) \rightarrow pm(X))$$

$$\forall X(\neg pe(X) \vee \neg pp(X) \rightarrow \neg\ pm(X) \wedge \neg\ gc(X))$$

$\forall$: Universal Quantifier

Now given:

pe(mary)

pp(mary)

Using $\forall X\ (pe(X) \wedge pp(X) \rightarrow pm(X))$

With instance X=mary, i.e.

$$pe(mary) \wedge pp(mary) \rightarrow pm(mary)$$

We can conclude:

pm(mary)

Also given:

pp(john)

¬pe(john)

Using

$$\forall X(\neg pe(X) \vee \neg pp(X) \rightarrow \neg pm(X) \wedge \neg gc(X))$$

With instance X=john, i.e.

$$\neg pe(john) \vee \neg pp(john) \rightarrow$$
$$\neg pm(john) \wedge \neg gc(john)$$

We can conclude:

$$\neg pm(john) \wedge \neg gc(john)$$

Another example

Every student has a tutor.

for all X

(if X is a student then

there is a Y such that Y is tutor of X)

$\forall X\ (student(X) \rightarrow \exists\ Y\ tutor(Y,X))$

$\exists$: **Existential Quantifier**

The Predicate Logic Language Alphabet:

- **Logical connectives (same as propositional logic): $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$**
- **Predicate symbols (as opposed to propositional symbols):a set of symbols each with an associated arity>=0.**
- **A set of constant symbols.**

  E.g. mary, john, 101, 10a, peter_jones
- **Quantifiers $\forall$, $\exists$**
- **A set of variable symbols. E.g. X, Y, X1, YZ.**

Arity

In the previous examples:

Predicate Symbol	Arity
student	1
tutor	2
pm	1
pp	1

A predicate symbol with

arity = 0 is called a **nullary predicate** (it is a proposition),

arity = 1 is called a **unary predicate,**

arity = 2 is called a **binary predicate.**

A predicate symbol with arity=n (usually n>2) is called an **n-ary** predicate.

Definition:

A **Term** is any constant or variable symbol.

Syntax of a grammatically correct sentence (wff) in predicate logic

- **$p(t_1,\ldots, t_n)$ is a wff if p is an n-ary predicate symbol and the t_i are terms.**
- **If W, W1, and W2 are wffs then so are the following:**

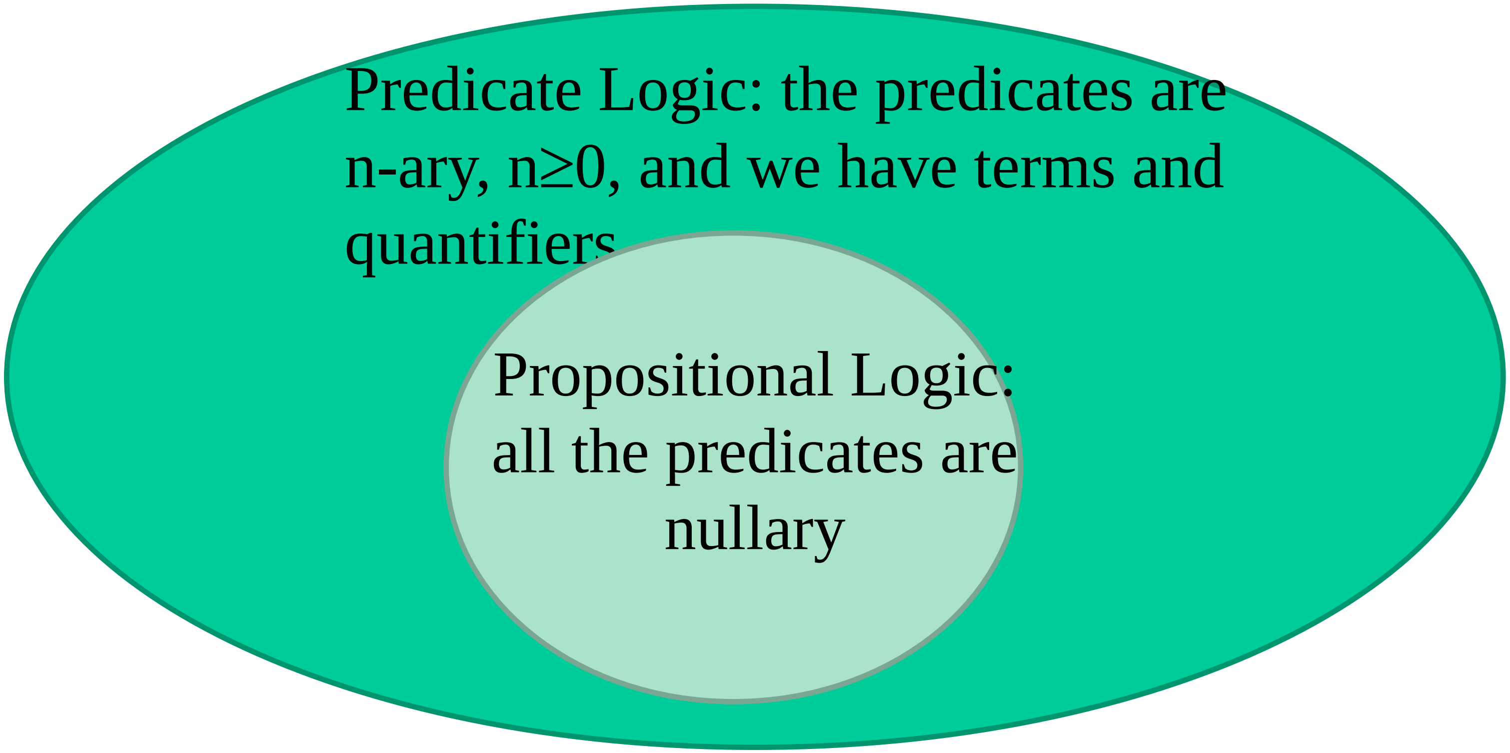
  $\neg W$ $W1 \wedge W2$ $W1 \vee W2$

  $W1 \rightarrow W2$ $W1 \leftrightarrow W2$

  $\forall X(W)$ $\exists X(W)$

  where X is a variable symbol.
- **There are no other wffs.**

From the description above you can see that propositional logic is a special case of predicate logic.



Convention used in most places in these notes:

- Predicate and constant symbols start with lower case letters.
- Variable symbols start with upper case letters.

Examples

The following are wffs:

1. **$\neg$ married(john)**

2. **$\forall$X(alive(X) $\wedge$ adult_human(X) $\wedge$**
   $\neg$ married(X) $\rightarrow$
   single(X) $\vee$ divorced(X) $\vee$ widowed(X))

3. **$\exists$X (bird(X) $\wedge$ $\neg$ fly(X))**

The following are not wffs:

4. $\neg X$

5. $single(X) \rightarrow \forall Y$

6. $\forall \exists X\ (bird(X) \rightarrow feathered(X))$

Exercise
which of the following are wffs?

1. $\forall X\ p(X)$
2. $\forall X\ p(Y)$
3. $\forall X\ \exists Y\ p(Y)$
4. $q(X,Y,Z)$
5. $p(a) \rightarrow \exists q(a,X,b)$
6. $p(a) \vee p(a,b)$

7. $\neg \neg \forall X\ r(X)$
8. $\exists X\ \exists Y\ p(X,Y)$
9. $\exists X,Y\ p(X,Y)$
10. $\forall X\ (\neg\exists Y)$
11. $\forall x\ (\neg\exists Y\ p(x,Y))$

Exercise

Formalise the following in predicate logic using the following predicates (with their more or less obvious meaning):

lecTheatre/1, office/1, contains/2, lecturer/1, has/2, same/2, phd/1, supervises/2, happy/1, completePhd/1.

1. 311 is a lecture theatre and 447 is an office.
2. Every lecture theatre contains a projector.
3. Every office contains a telephone and either a desktop or a laptop computer.
4. Every lecturer has at least one office.
5. No lecturer has more than one office.

6. No lecturers share offices with anyone.
7. Some lecturers supervise PhD students and some do not.
8. Each PhD student has an office, but all PhD students share their office with at least one other PhD student.

9. A lecturer is happy if the PhD students he/she supervises successfully complete their PhD.
10. Not all PhD students complete their PhD.

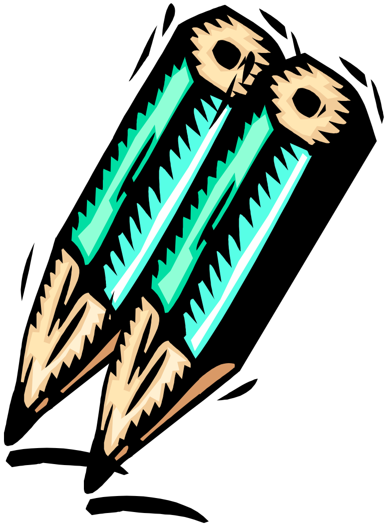
Note:

$\exists$X p(X) states that there is **at least** one X such that p is true of X.

E.g. **$\exists$X father(X, john)**

says John has **at least** one father (assuming *father(X,Y)* is to be read as X is father of Y).

Exercise



Assuming a predicate *same(X,Y)* that expresses that X and Y are the same individual, express the statement that John has exactly one father. You may also assume a binary predicate "father" as above.