6.825 Exercise Solutions: Week 3

Solutions

September 27, 2004

Converting to CNF

Convert the following sentences to conjunctive normal form.

1. $(A \to B) \to C$
   Answer:
   $\neg(\neg A \vee B) \vee C$
   $(A \wedge \neg B) \vee C$
   $(A \vee C) \wedge (\neg B \vee C)$

2. $A \to (B \to C)$
   Answer:
   $\neg A \vee \neg B \vee C$

3. $(A \to B) \vee (B \to A)$
   Answer:
   $(\neg A \vee B) \vee (\neg B \vee A)$
   $True$

4. $(\neg P \to (P \to Q))$
   Answer:
   $\neg\neg P \vee (\neg P \vee Q)$
   $P \vee \neg P \vee Q$
   $True$

5. $(P \to (Q \to R)) \to (P \to (R \to Q))$
   Answer:
   $\neg(\neg P \vee \neg Q \vee R) \vee (\neg P \vee \neg R \vee Q)$
   $(P \wedge Q \wedge \neg R) \vee (\neg P \vee \neg R \vee Q)$
   $(P \vee \neg P \vee \neg R \vee Q) \wedge (Q \vee \neg P \vee \neg R \vee Q) \wedge (\neg R \vee \neg P \vee \neg R \vee Q)$
   $\neg P \vee Q \vee \neg R$

6. $(P \to Q) \to ((Q \to R) \to (P \to R))$
   Answer:
   $\neg(\neg P \vee Q) \vee (\neg(\neg Q \vee R) \vee (\neg P \vee R))$
   $(P \wedge \neg Q) \vee ((Q \wedge \neg R) \vee (\neg P \vee R))$
   $(P \wedge \neg Q) \vee ((Q \vee \neg P \vee R) \wedge (\neg R \vee \neg P \vee R))$
   $(P \wedge \neg Q) \vee Q \vee \neg P \vee R$
   $(P \vee \neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg Q \vee R)$
   $True$

First Order Logic Sentences

For each of the following English sentences, write a corresponding sentence in FOL.

1. The only good extraterrestrial is a drunk extraterrestrial.
   $\forall x.ET(x) \land Good(x) \rightarrow Drunk(x)$
2. The Barber of Seville shaves all men who do not shave themselves.
   $\forall x.\neg Shaves(x, x) \rightarrow Shaves(BarberOfSeville, x)$
3. There are at least two mountains in England.
   $\exists x, y.Mountain(x) \land Mountain(y) \land InEngland(x) \land InEngland(y) \land x \neq y$
4. There is exactly one coin in the box.
   $\exists x.Coin(x) \land InBox(x) \land \forall y.(Coin(y) \land InBox(y) \rightarrow x = y)$
5. There are exactly two coins in the box.
   $\exists x, y.Coin(x) \land InBox(x) \land Coin(y) \land InBox(y) \land x \neq y \land \forall z.(Coin(z) \land InBox(z) \rightarrow (x = z \lor y = z))$
6. The largest coin in the box is a quarter.
   $\exists x.Coin(x) \land InBox(x) \land Quarter(x) \land \forall y.(Coin(y) \land InBox(y) \land \neg Quarter(y) \rightarrow Smaller(y, x))$
7. No mountain is higher than itself.
   $\forall x.Mountain(x) \rightarrow \neg Higher(x, x)$
8. All students get good grades if they study.
   $\forall x.Student(x) \land Study(x) \rightarrow GetGoodGrade(x)$

FOL Interpretations, Part 1

For each group of sentences, write an interpretation under which the last sentence is false and all the rest are true.

1. $\forall x.h(x) \rightarrow g(x)$

   $\forall x.f(x) \rightarrow g(x)$

   $\exists x.f(x) \land h(x)$

   An interpretation that makes the first two sentences true and the third false:
   $U = \{A, B\}$
   $I(f) = \{A\}$
   $I(g) = \{A, B\}$
   $I(h) = \{B\}$

2. $\forall x.\exists y.f(x, y)$

   $\exists y.\forall x.f(x, y)$

   An interpretation that makes the first sentence true and the second sentence false:
   $U = \{A, B, C\}$
   $I(f) = \{< A, B >, < B, C >, < C, A >\}$

3. $\forall x.(f(x) \rightarrow g(A))$

   $(\forall x.f(x)) \rightarrow g(A)$

   There is no interpretation that makes the first sentence true and the second sentence false.

   Reason: For the second sentence to be false, $\forall x.f(x)$ has to be *true*, **and** $g(A)$ has to be *false*. With these two requirements, we can see that the first sentence cannot be true because $f(x)$ is true for $\forall x$, and $g(A)$ is false.

However, if we replace $\forall x$ with $\exists x$,

$\exists x.(f(x) \rightarrow g(A))$

$(\exists x.f(x)) \rightarrow g(A)$

Then the following interpretation makes the first sentence true and the second sentence false.

$U = \{A, B\}$

$f = \{B\}$

$g = \{B\}$

FOL Interpretations, Part 2

For each group of sentences, give an interpretation in which all sentences are true.

1. $(\forall x.p(x) \vee q(x)) \rightarrow \exists x.r(x)$

   $\forall x.r(x) \rightarrow q(x)$

   $\exists x.p(x) \wedge \neg q(x)$

   Interpretation:

   $U = \{A, B\}$

   $I(p) = \{A\}$

   $I(q) = \{B\}$

   $I(r) = \{B\}$

2. $\forall x.\neg f(x, x)$

   $\forall x, y, z.f(x, y) \wedge f(y, z) \rightarrow f(x, z)$

   $\forall x.\exists y.f(x, y)$

   There is no interpretation in a finite universe that makes all of these sentences true. However, if you consider an infinite universe, (e.g., real numbers) and a *greater than* function ($>$), these sentences are all true.

   Interpretation:

   $U = \mathbb{R}$

   $I(f) = >$

3. $\forall x.\exists y.f(x, y)$

   $\forall x.(g(x) \rightarrow \exists y.f(y, x))$

   $\exists x.g(x)$

   $\forall x.\neg f(x, x)$

   Interpretation:

   $U = \{A, B\}$

   $I(f) = \{< A, B >, < B, A >\}$

   $I(g) = \{A\}$

FOL Semantics

(**6**) Consider a world with objects **A**, **B**, and **C**. We'll look at a logical languge with constant symbols X, Y, and Z, function symbols f and g, and predicate symbols p, q, and r. Consider the following interpretation:

- $I(X) = \mathbf{A}$, $I(Y) = \mathbf{A}$, $I(Z) = \mathbf{B}$
- $I(f) = \{\langle \mathbf{A}, \mathbf{B} \rangle, \langle \mathbf{B}, \mathbf{C} \rangle, \langle \mathbf{C}, \mathbf{C} \rangle\}$
- $I(p) = \{\mathbf{A}, \mathbf{B}\}$
- $I(q) = \{\mathbf{C}\}$
- $I(r) = \{\langle \mathbf{B}, \mathbf{A} \rangle, \langle \mathbf{C}, \mathbf{B} \rangle, \langle \mathbf{C}, \mathbf{C} \rangle\}$

For each of the following sentences, say whether it is true or false in the given interpretation I:

1. $q(f(Z))$
   Answer: T
2. $r(X, Y)$
   Answer: F
3. $\exists w. f(w) = Y$
   Answer: F
4. $\forall w. r(f(w), w)$
   Answer: T
5. $\forall u, v. r(u, v) \rightarrow (\forall w. r(u, w) \rightarrow v = w)$
   Answer: F
6. $\forall u, v. r(u, v) \rightarrow (\forall w. r(w, v) \rightarrow u = w)$
   Answer: T

Clausal form

(**6**) Convert each sentence below to clausal form.

1. $\forall y. \exists x. r(x, y) \vee s(x, y)$
   Answer: $r(f(y), y) \vee s(f(y), y)$
2. $\forall y. (\exists x. r(x, y)) \rightarrow p(y)$
   Answer: $\neg r(x, y) \vee p(y)$
3. $\forall y. \exists x. (r(x, y) \rightarrow p(x))$
   AnsweR: $\neg r(f(y), y) \vee p(f(y))$

Implication vs. Entailment

Show that $P \models Q \leftrightarrow (True \models P \rightarrow Q)$.

Let $M(P)$ and $M(Q)$ be the sets of interpretations (models) under which P and Q are true, respectively.

1. Assume $P \models Q$. By the definition of entailment, we have $M(P) \subseteq M(Q)$. Because $M(Q)$ and $M(\neg Q)$ are disjoint (there are no interpretations under which both Q and $\neg Q$ are true), it follows that $M(P) \cap M(\neg Q) = \emptyset$. Therefore there are no interpretations under which P is true and Q is false, and so $P \rightarrow Q$ is true under all interpretations: $M(P \rightarrow Q) = M(True)$ and consequently $M(True) \subseteq M(P \rightarrow Q)$. By the definition of entailment, this means that $True \models P \rightarrow Q$, and so we have shown that $P \models Q \rightarrow (True \models P \rightarrow Q)$.
2. Assume $True \models P \rightarrow Q$. By the definition of entailment, this means that $P \rightarrow Q$ is true under all models, and so there can be no model such that P is true and Q is false: $M(P) \cap M(\neg Q) = \emptyset$. Therefore $M(P) \subseteq M(Q)$ and we can conclude that $P \models Q$. Thus we have shown that $(True \models P \rightarrow Q) \rightarrow P \models Q$.

(1) and (2) together prove the statement $P \models Q \leftrightarrow (True \models P \rightarrow Q)$.