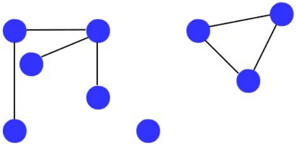
Graphs

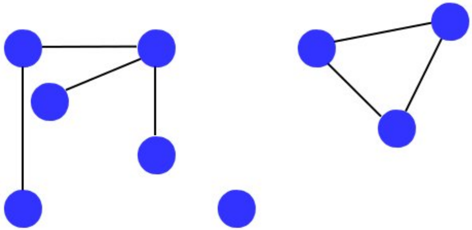
Dr Timothy Kimber

February 2018



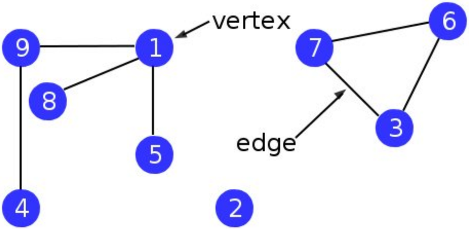
Introduction

Graphs are fundamental to much of computer science



- We have already seen how trees are used as data structures
- All sorts of problems can be modelled using graphs
- Networks, images, programs, anything involving related objects

Graph Terminology

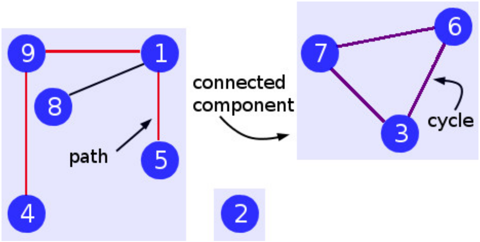


Definition

A graph G is a pair (V, E) where V is a finite set (of objects) and E is a binary relation on V. Elements of V are called vertices and elements of E are called edges.

- E is a set of pairs of vertices: $\{u, v\}$ such that there is an edge between u and v
- Vertices u and v are adjacent if there is an edge $\{u, v\}$

Graph Terminology



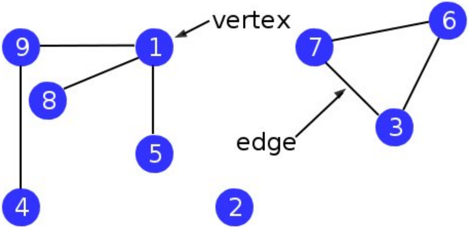
- A path from v_1 to v_n, written $v_1 \rightsquigarrow v_n$, is a sequence $\langle v_1, v_2, \ldots, v_n \rangle$ such that there is an edge $\{v_i, v_{i+1}\}$ for all i, $1 \le i < n$
- A cycle exists if there is a path from v to v, containing at least 4 vertices, for some vertex v
- Vertex v is reachable from vertex u if $u = v$, or if there is a path $u \rightsquigarrow v$
- A connected component (also just called a component) is a set of vertices all reachable from each other

Graph Representation

Question

How should a graph be represented as a data structure?

- A graph vertex is connected to 0-to-many other vertices
- Going to assume that $|V|$ is fixed

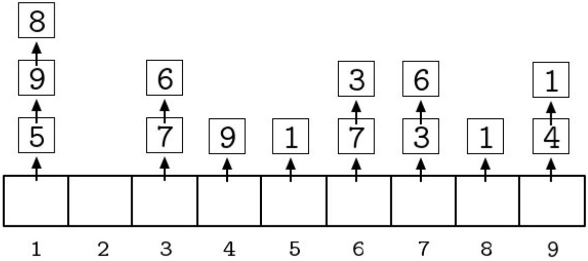


Graph Representation

Two common ways:

- Adjacency List(s): $adj[u]$ contains v if there is an edge $\{u, v\}$
- Adjacency Matrix: $adj_{uv} = 1$ if there is an edge $\{u, v\}$, else 0

Adjacency lists good for sparse graphs

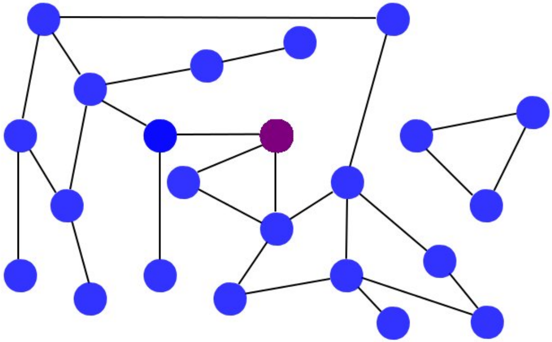


Graph Search

Question

Why search a graph?

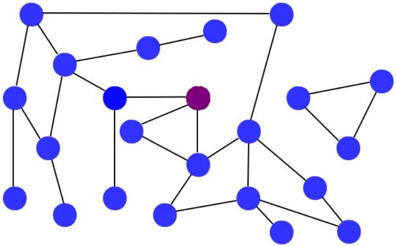
- Searching a graph is like iterating through an ordered structure
- Want to use data in the graph for some computation



Graph Search Actions

Searching a graph has two actions:

- Find adjacent vertices
- Visit vertices not found before

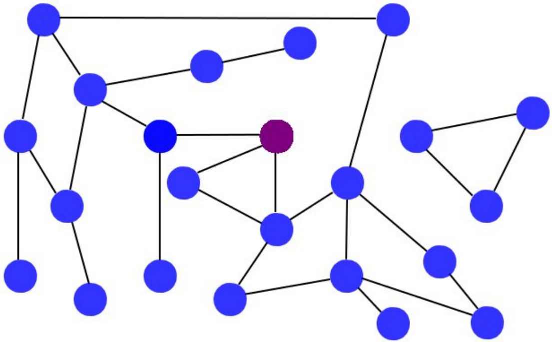


- Visiting means using the vertex: includes finding further vertices
- Vertices are visited in the order they are first found
- Vertices are coloured when they are first found/visited

Breadth-First Search

Question

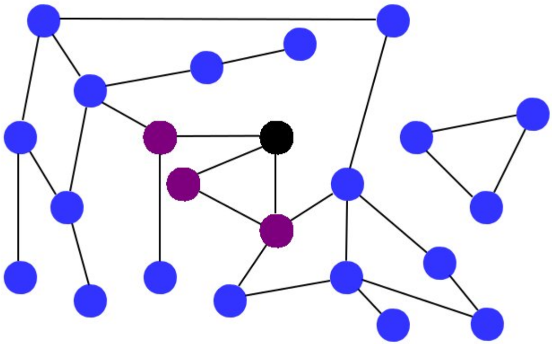
What is a breadth-first search of a graph?



Breadth-First Search

In breadth-first search

- Visit a vertex v (starting with source vertex s)
- Find all vertices adjacent to v before visiting another
- Result: search proceeds gradually down every path at the same rate

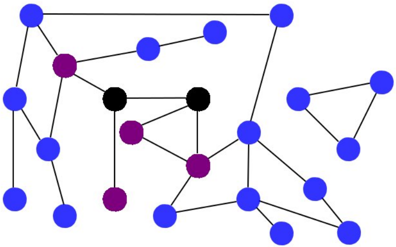


BFS Procedure

Question

How would you implement BFS?

- Inputs are graph g and vertex s
- $g.adj[u]$ returns list of vertices
- $g.vertices$ is number of vertices
- Objective: find all reachable vertices (will add actions later)



Breadth-First Search

BFS (Input: graph g, vertex s)

```
found    = new boolean[g.vertices]
found[s] = true
q        = new Queue(s)        // FIFO queue
while q is not empty
  u = q.remove()
  for v in g.adj[u]
    if not found[v]            // avoid loops
      found[v] = true
      q.add(v)
```

- The use of a (FIFO) queue is characteristic of BFS
- By convention only search from given s

Shortest Paths

BFS searches all paths at the same rate, so ...

Question

How would you modify the BFS procedure to find the length (number of edges) of the shortest path from s to every other vertex?

BFS (Input: graph g, vertex s)

```
found    = new boolean[g.vertices]
found[s] = true
q        = new Queue(s)      // FIFO queue
while q is not empty
  u = q.remove()
  for v in g.adj[u]
    if not found[v]          // avoid loops
      found[v] = true
      q.add(v)
```

Shortest Paths

BFS (Input: graph g, vertex s)

```
q    = new Queue(s)
dist = new int[g.vertices]
dist.fill(-1)
dist[s] = 0
while q is not empty
  u = q.remove()
  for v in g.adj[u]
    if dist[v] == -1          // not found
      dist[v] = dist[u] + 1
      q.add(v)
```

- The distance is recorded when a vertex is (first) found
- Arrays of size $|V|$ like *dist* are also common in graph search
- Unreachable vertices have `dist[v] = -1`

Time

Question

For a connected graph with V vertices and E edges, how long does BFS take?

BFS (Input: graph g, vertex s)

```
found    = new boolean[g.vertices]
found[s] = true
q        = new Queue(s)      // FIFO queue
while q is not empty
  u = q.remove()
  for v in g.adj[u]
    if not found[v]          // avoid loops
      found[v] = true
      q.add(v)
```

BFS Time Complexity

- Each vertex is added and removed from the queue exactly once
- Each adjacency list is used exactly once
- Each edge contributes exactly two vertices to the adjacency lists
- Time depends on both variables: $O(V + E)$

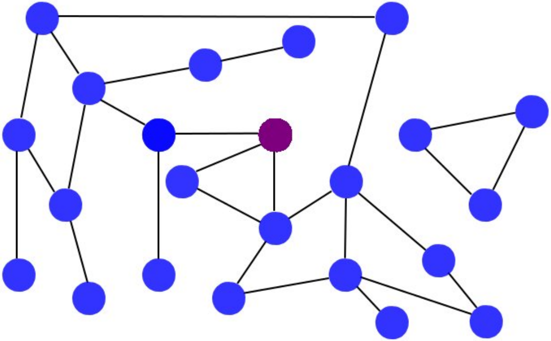
BFS (Input: graph g, vertex s)

```
found    = new boolean[g.vertices]
found[s] = true
q        = new Queue(s)
while q is not empty
  u = q.remove()            runs once per vertex
  for v in g.adj[u]
    if not found[v]          runs twice per edge
      found[v] = true
      q.add(v)
```

Depth-First Search

Question

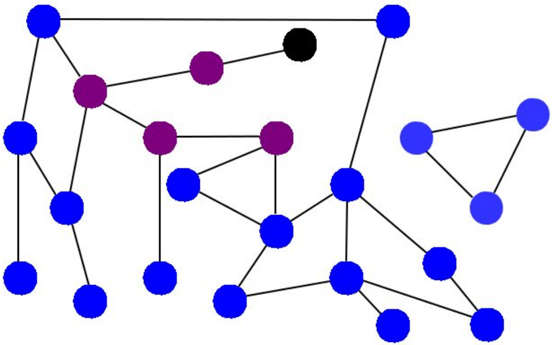
What is a depth-first search of a graph?



Depth-First Search

In depth-first search

- Visit every vertex as soon as it is found
- i.e. start the next visit before completing current visit
- Result: search follows a single path as far as possible and then backtracks to the last alternative path

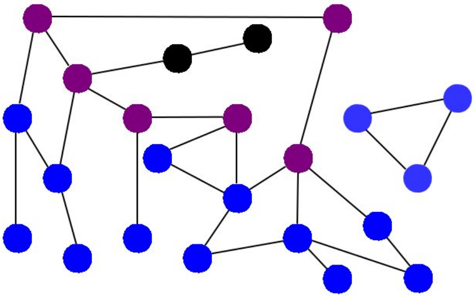


DFS Procedure

Question

How would you implement DFS?

- Input is graph g
- Assume $g.adj[u]$ returns list of vertices
- Objective: find all vertices



Depth-First Search

DepthFirstSearch (Input: graph g)

```
found = new boolean[g.vertices]
for v in g
  if not found[v]
    DFS(g, v, found)
```

DFS (Input: graph g, vertex s, array found)

```
found[s] = true
for v in g.adj[s]
  if not found[v]
    DFS(g, v, found)
```

- DFS can use call stack instead of explicit queue
- Restart until whole graph searched (or not)

An Application

- Program checks if (whole) graph is acyclic
- Returns true or false

DepthFirstAcyclic (Input: graph g)

```
parent = new int[g.vertices]
parent.fill(-1)          // nothing found
for v in g
  if parent[v] == -1   // not found
    parent[v] = -2     // found, no parent
    if not DFSAcyclic(g, v, parent)
      return false
return true
```

Depth-First Search

DFSAcyclic (Input: graph g, vertex u, array parent)

```
for v in g.adj[u]
  if parent[v] == -1            // not found
    parent[v] = u
    if not DFSAcyclic(g, v, parent)
      return false
  else if parent[u] != v        // cycle detected
    return false
return true
```

- A cycle exists if `v` was already found, unless it is `u`'s parent
- Since `u` was just found, and not from `v`, the edge {`u`,`v`} completes an alternative path to `u` from the source

Time

Question

For a connected graph with V vertices and E edges, how long does DFS take?

DFS (Input: graph g, vertex s, array found)

```
found[s] = true
for v in g.adj[s]
  if not found[v]
    DFS(g, v, found)
```

DFS Time Complexity

- DFS is called exactly once per vertex
- Each adjacency list is used exactly once
- Each edge contributes exactly two vertices to the adjacency lists
- Time depends on both variables: $O(V + E)$

DFS (Input: graph g, vertex s, array found)

```
found[s] = true          runs V times
for v in g.adj[s]
  if not found[v]        runs 2E times
    DFS(g, v, found)
```