Imperial College London – Department of Computing

MSc in Computing Science

580: Algorithms
Background: Series

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To determine the time taken by an algorithm we are often faced with evaluating the sum of a sequence of related terms such as this:

$$T(N) = c + 2c + \cdots + (N-1)c + Nc \tag{1}$$

where each term might be the time taken in one iteration of a loop. Such a sum is called a *series*. If each iteration takes the same amount of time the answer is simple, but in the example above, the time increases incrementally, by an amount c. This complicates the calculation. To calculate a bound we can probably do away with the constant c and just count the number of times some representative instruction is executed. In this case the series is simplified to:

$$1 + 2 + \cdots + (N-1) + N \tag{2}$$

but it is essentially the same problem.

A second common form of series is:

$$Nc + (N/2)c + (N/4)c + \cdots + 2c + c \tag{3}$$

in which each term is a multiple (here half) of the previous one. This might represent the time taken at each level of recursion in a divide and conquer algorithm.

Arithmetic Series

The first form, such as (1) or (2), is called an *arithmetic series*. In an arithmetic series there is always a *common difference* between succesive terms. The general form of an arithmetic series with k terms is

$$a + (a+d) + (a+2d) + \cdots + (a + (k-1)d) \tag{4}$$

where a is the first term and d is the common difference. This is also written, more succinctly, as:

$$\sum_{i=0}^{k-1} a + id \tag{5}$$

In (1) both a and d equal c, and in (2) both a and d are 1. However, a does not have to equal d: each can be any real number.

Solving Arithmetic Series

Any arithmetic series can be solved using the following method. Ultimately, this method provides us with a simple formula to solve the series, but understanding where this formula comes from certainly helps me remember what it is.

We start by creating a second series. The new series is simply the reverse of the first. Taking (2) as an example, these two series are:

$$\begin{array}{ccccccccc} 1 & + & 2 & + & \dots & + & (N-1) & + & N \\ N & + & (N-1) & + & \dots & + & 2 & + & 1 \end{array}$$

Adding these together, term-by-term, gives another series

$$(N+1) + (N+1) + \cdots + (N+1) + (N+1) \tag{6}$$

As you can see, every term in (6) is the sum of the first and last terms of (2). Series (6) obviously equates to $N(N+1)$, since it contains N terms that are all $N+1$. So, if the solution (sum) of the original series (2) is s, then $N(N+1) = 2s$, and so

$$s = \frac{N(N+1)}{2} \quad . \tag{7}$$

Since the difference between successive terms is always a constant d, the same method can be applied to any arithmetic series. So, given

$$S_k = a_1 + \cdots + a_k \quad , \tag{8}$$

we have

$$S_k = \frac{k(a_1 + a_k)}{2} \quad . \tag{9}$$

Geometric Series

The second form, such as (3), is called an *geometric series*. In a geometric series there is always a *common ratio* between succesive terms. The general form of a geometric series with k terms is:

$$a + ar + ar^2 + \cdots + ar^{(k-1)} \tag{10}$$

where a is the first term and r is the common ratio. This is also written:

$$\sum_{i=0}^{k-1} ar^i \quad . \tag{11}$$

Solving Geometric Series

The method to solve geometric series is similar, and alomst as simple, as the one for arithmetic series. Rather than collapsing all the terms to be the same value, this method eliminates all but two terms from the calculation.

A second series is again created. This time we multiply the original series by r to get the new one. Taking (3) as an example, $r = 1/2$ and these two series are:

$$\begin{array}{ccccccccc} Nc & + & (N/2)c & +\cdots+ & 2c & + & c & & \\ & & (N/2)c & +\cdots+ & 2c & + & c & + & c/2 \end{array}$$

All but two of the terms in these are identical, and taking the *difference* we are left with $Nc - c/2$. If the solution (sum) of (3) is s, then we have $Nc - c/2 = s - s/2$, so

$$s/2 = Nc - c/2 \quad ,$$

and

$$s = 2Nc - c \quad .$$

Naturally, this works for any geometric series. Given

$$S_k = a + ar + ar^2 + \cdots + ar^{(k-1)}$$

we have

$$S_k - rS_k = a - ar^k \quad , \tag{12}$$

and so

$$S_k = \frac{a(1 - r^k)}{(1 - r)} \quad . \tag{13}$$

If we remember how we got to (13) we can easily adapt it for different situations. Firstly, applying (13) suggests we know how many terms there are. If we simply know the first and last terms of the series (and the common ratio), we can rewrite a and ar^k in (12) as a_1 and ra_k, giving

$$S_k = \frac{a_1 - ra_k}{(1-r)} \quad . \tag{14}$$

The forms of the solution above are most suitable when $r < 1$, because the result of $S_k - rS_k$ and $1 - r$ will be positive. If $r > 1$ then it makes sense to reorganise (12) to be

$$rS_k - S_k = ar^k - a \quad , \tag{15}$$

and so

$$S_k = \frac{a(r^k - 1)}{(r-1)} \quad , \tag{16}$$

or equivalently

$$S_k = \frac{ra_k - a_1}{(r-1)} \quad . \tag{17}$$