Propositional Logic cntd.

Fariba Sadri

What we have done so far on Propositional Logic

- Syntax of wffs
- Practice on how to formalise English sentences in propositional logic
- Truth tables for the semantics of the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
- Tautologies, inconsistencies, contingencies
- Equivalences

What we will do now in this set of slides

- Semantic consequence
- Natural deduction proofs
- Soundness and completeness

Recap Exercise

From 2012-13 examination paper:

a. Define a new connective $\otimes$ for *exclusive-or*, using any (combination) of the usual connectives, $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$. Thus $p \otimes q$ is to mean either *p* or *q* but not both.

b. Use the new connective $\otimes$ together with any of the other usual connectives to express the following sentences in propositional logic, where *either ... or* is to be understood as *exclusive-or*. The propositions to be used are given in the text in Italics inside brackets.

Either John will leave the company *(jL)* or Mary will *(mL)*.

If John leaves then either the tax department will close *(closeTax)*, or Peter will be shared between two departments *(pShare)* and an administrator will be recruited *(recruitA)*.

If Mary leaves then either an administrator will be recruited or a secretary will be recruited *(recruitS)*, provided John is shared between two departments *(jShare)*.

Definition: Semantic Consequence

Let

S be a set of wffs, and

W be a wff.

If whenever all the wffs in S are true W is also true, then W is a **semantic consequence** of S.

Semantic Consequence cntd.

Denoted as

$$\mathbf{S} \models \mathbf{W}$$

"|=" is the **semantic turnstile**

(a metasymbol).

We also say W is **semantically entailed** by S.

If W is a tautology then

$$\models \mathbf{W}.$$

Exercise

Show the following:

a. $A \wedge B \models A \vee B$

b. snow, mild $\rightarrow \neg$snow $\models \neg$mild

c. Go back to one of the argument at the beginning of the notes you think is valid and show that the conclusion of the argument is semantically entailed by the premises.

Definitions:Valid, Satisfiable

- *Valid* is just another name for tautology.

So a formula is valid if it is true in every interpretation.

|= A if A is valid.

- A formula is *satisfiable* if it is true in at least one interpretation.

	Validity	**Satisfiability**
A |= B	?? valid	?? unsatisfiable
A |= B and B |= A	?? valid	?? unsatisfiable

Inference

Example: Given

(pass_exams ∧ pass_projects) → pass_MSc

pass_exams

pass_projects

one can infer (conclude)

pass_MSc.

Example: Given

$\textbf{thursday} \rightarrow \textbf{logic\_lecture}$

$\neg\textbf{logic\_lecture}$

Can you infer

$\neg$ **thursday** ?

The “elections” example

Given

- If there are national elections then either the Tory party wins or the Labour party wins.
- If the unions do not support the Labour party then it does not win.
- There are national elections.

Can you infer

If the Tory party does not win then the unions support the Labour party?

The "elections" example: Formalisation in logic

Given

$Elections \rightarrow Labour\_wins \vee Tory\_wins$

$\neg Unions\_support\_Labour \rightarrow \neg Labour\_wins$

$Elections$

can you infer

$\neg Tory\_wins \rightarrow Unions\_support\_Labour$?

The “elections” example: Abbreviation

Premise:

1. $E \to L \vee T$
2. $\neg U \to \neg L$
3. E

Conclusion:

$\neg T \to U$

You can try to use truth tables to see if the conclusion is semantically entailed by the premises.

How many rows?

Too many!

E	L	T	U	$L \vee T$	$E \rightarrow L \vee T$	$\neg U$	$\neg L$	$\neg U \rightarrow \neg L$	$\neg T$	$\neg T \rightarrow U$
T	T	T	T	T	T	F	F	T	F	T

• • • • • • • •

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Can you give an informal proof of the conclusion from the premises without using truth tables?

Performing inferences is very important in many applications of logic

Argument: Premise → Conclusion

Modelling : Theory → Consequences

Programming:

Specification → Program

Specification → Properties

Prolog:

Program → Answers to queries

Rules of Inference
Natural Deduction

(Reasoning purely at the syntactic level)

$\wedge$-elimination ($\wedge$E)

$$\frac{X \wedge Y}{X} \qquad \frac{X \wedge Y}{Y}$$

$\wedge$-introduction ($\wedge$I)

$$\frac{X, Y}{X \wedge Y} \qquad \frac{X, Y}{Y \wedge X}$$

$\vee$-elimination (∨E)

$$\frac{X \vee Y,\ \neg X}{Y} \qquad\qquad \frac{X \vee Y,\ \neg Y}{X}$$

$\vee$-introduction (∨I)

$$\frac{X}{X \vee Y} \qquad\qquad \frac{X}{Y \vee X}$$

$\rightarrow$-elimination (→E) (Modus Ponens)

$$\frac{X,\ X \rightarrow Y}{Y}$$

$\rightarrow$-introduction (→I)

$$\begin{array}{c} X \quad \text{assume} \\ . \\ . \\ \underline{\quad\quad Y \quad\quad} \\ X \rightarrow Y \end{array}$$

$\neg$-elimination and $\neg$-introduction (Reductio Ad Absurdum) (RAA) (Proof by contradiction)

$$
\begin{array}{ll}
\neg X & \text{assume} \\
\cdot & \\
\cdot & \\
\cdot & \\
\hline
Y, \neg Y & \\
X &
\end{array}
\qquad\qquad
\begin{array}{ll}
X & \text{assume} \\
\cdot & \\
\cdot & \\
\cdot & \\
\hline
Y, \neg Y & \\
\neg X &
\end{array}
$$

Note: X and Y may be the same wff.

$$\begin{array}{ll} \neg \mathbf{X} & \textbf{assume} \\ \cdot & \\ \cdot & \\ \cdot & \\ \underline{\neg \mathbf{X}, \mathbf{X}} & \\ \mathbf{X} & \end{array} \qquad\qquad \begin{array}{ll} \mathbf{X} & \textbf{assume} \\ \cdot & \\ \cdot & \\ \cdot & \\ \underline{\mathbf{X}, \neg \mathbf{X}} & \\ \neg \mathbf{X} & \end{array}$$

$\leftrightarrow$-introduction ($\leftrightarrow$I)

$$\frac{X \rightarrow Y,\ Y \rightarrow X}{X \leftrightarrow Y}$$

$\leftrightarrow$-elimination ($\leftrightarrow$E)

$$\frac{X \leftrightarrow Y}{X \rightarrow Y} \qquad \frac{X \leftrightarrow Y}{Y \rightarrow X}$$

Note: In all the inference rules X and Y stand for any wffs. So the following, for example, is an application of the $\rightarrow$elimination rule:

Given $\mathbf{A \wedge (B \vee C)}$ and

$$\mathbf{(A \wedge (B \vee C)) \;\rightarrow\; ((A \rightarrow D) \vee (\neg E \wedge F))}$$

we can infer

$$\mathbf{(A \rightarrow D) \vee (\neg E \wedge F)}$$

Example

You may be wondering why we need the $\vee$I rule. Here is an example that uses it.

Example:

If there is a shortage of petrol or the tax on petrol is high then people are angry. There is a shortage of petrol.

So people are angry.

Premise

1. $(SP \vee HT) \rightarrow Anger$
2. SP

we want to conclude

$Anger$.

1. **$(SP \vee HT) \rightarrow$ Anger** **given**
2. **SP** **given**

?

Anger

1. **$(SP \vee HT) \rightarrow Anger$** **given**
2. **SP** **given**

?

Anger **$\rightarrow E$**

1. **$(SP \lor HT) \rightarrow Anger$** **given**
2. **SP** **given**

?

$SP \lor HT$

Anger **$\rightarrow E$**

Proof

1.	**$(SP \vee HT) \rightarrow Anger$**	**given**
2.	**SP**	**given**
3.	**$SP \vee HT$**	**2, $\vee$ I**
4.	**Anger**	**1,3, $\rightarrow$ E**

Example

Derive **$P \vee Q$** from **$P \wedge Q$**.

1.	**$P \wedge Q$**	**given**
	??????	
	$P \vee Q$	

Derive $\mathbf{P \vee Q}$ from $\mathbf{P \wedge Q}$.

1.	$\mathbf{P \wedge Q}$	**given**
	??????	
	$\mathbf{P \vee Q}$	$\mathbf{\vee I}$

Example

Derive $P \vee Q$ from $P \wedge Q$.

1.	$P \wedge Q$	given
2.	P	1, $\wedge$E
3.	$P \vee Q$	2, $\vee$I

Example

Derive **R** from **P, Q, $(P \land Q) \rightarrow R$**.

1.	**P**	**given**
2.	**Q**	**given**
3.	**$(P \land Q) \rightarrow R$**	**given**
	??????	
	R	

1.	**P**	**given**
2.	**Q**	**given**
3.	**$(P \wedge Q) \rightarrow R$**	**given**
	??????	
	R	**$\rightarrow$E**

1.	P	given
2.	Q	given
3.	$(P \wedge Q) \rightarrow R$	given
4.	$P \wedge Q$	1,2, $\wedge$I
5.	R	3,4, $\rightarrow$E

Example

Derive $\mathbf{Q \rightarrow R}$ from $\mathbf{P, (P \wedge Q) \rightarrow R}$.

1.	$\mathbf{P}$	**given**
2.	$\mathbf{(P \wedge Q) \rightarrow R}$	**given**
	?????	
	?????	
	?????	
	$\mathbf{Q \rightarrow R}$	

1.	P	given
2.	$(P \wedge Q) \rightarrow R$	given
	?????	
	?????	
	?????	
	$Q \rightarrow R$	$\rightarrow$I

1.	P	given
2.	$(P \wedge Q) \rightarrow R$	given
	Q	assume
	????	
	R	
	$Q \rightarrow R$	$\rightarrow$I

1. P given

2. $(P \wedge Q) \rightarrow R$ given

   Q assume

   ????

   R $\rightarrow$E

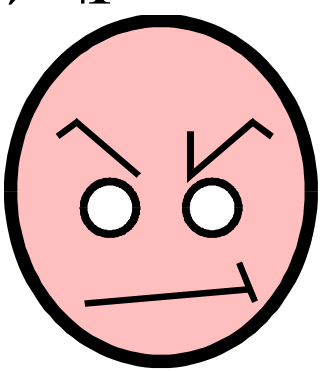
6. $Q \rightarrow R$ 3, 5, $\rightarrow$I

1. P		given	
2. $(P \wedge Q) \rightarrow R$		given	
	3.	Q	assume
	4.	$P \wedge Q$	1, 3, $\wedge$I
	5.	R	2, 4, $\rightarrow$E
6. $Q \rightarrow R$	3, 5, $\rightarrow$I		

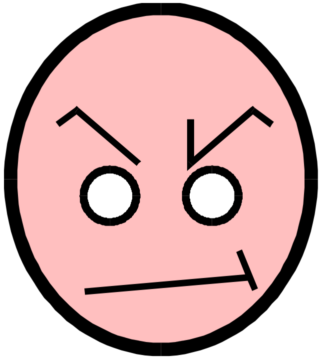
Example

Derive $\neg R$ from $P \wedge Q, R \rightarrow \neg P$.

1.	$P \wedge Q$	given
2.	$R \rightarrow \neg P$	given
????		
????		
????		
	$\neg R$	



1.	$P \wedge Q$	given
2.	$R \rightarrow \neg P$	given
????		
????		
????		
	$\neg R$	RAA



1.	**$P \wedge Q$**	**given**
2.	**$R \rightarrow \neg P$**	**given**
	R	**assume**
	????	
	????	
	$\neg R$	**RAA**

1. $P \wedge Q$		given
2. $R \rightarrow \neg P$		given
3. P		1, $\wedge$E
	<u>4.</u> R	assume
	5. $\neg P$	2, 4, $\rightarrow$E
6. $\neg R$	3, 4, 5, RAA	

$\vdash$

$P \vdash W$

denotes W is (syntactically) derivable from P.

$\vdash$ is called the syntactic turnstile. It is a symbol in the metalanguage.

Example:

In the last example:

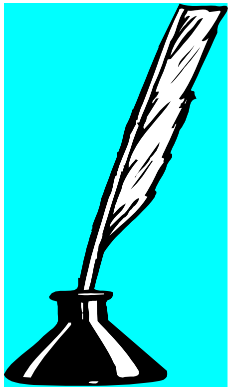
$P \wedge Q,\ R \rightarrow \neg P \vdash \neg R$

Definition

A **derivation** or **proof** of a wff W in propositional logic from a given set P of wffs, called premises, is a finite sequence of wffs such that the last wff is W and each wff in the sequence is one of the following:

- a premise, i.e. a wff in P
- an immediate consequence of one or more wffs preceding it in the sequence, as determined by one of the inference rules of propositional logic.
- An assumption (that is later discharged by an application of $\rightarrow$I or RAA).

Exercise

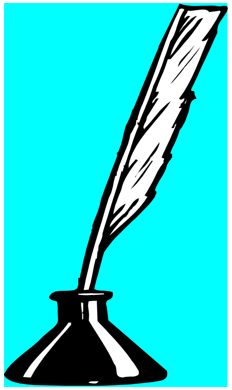


Show

$A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

(Transitivity of the implication.)

Exercise



Show

$Q \rightarrow R \vdash (P \rightarrow Q) \rightarrow (P \rightarrow R)$

Exercise

Consider the following derivation:

1. A	**assume**
2. $A \to A$	**1, 1, $\to$I**
3. A	**1, 2, $\to$E**

Seemingly this proves $\vdash$**A**.

Is there anything wrong with it?

If so, what?

Exercise

Show

$\text{snow}, \text{mild} \rightarrow \neg\text{snow} \vdash \neg\text{mild}$

It is enough to show:

$p, q \rightarrow \neg p \vdash \neg q$

Some useful derived inference rules

Double negation elimination (¬¬E)

$$\frac{\neg\neg X}{X}$$

Double negation introduction (¬¬I)

$$\frac{X}{\neg\neg X}$$

Law of excluded middle

$$\frac{}{X \vee \neg X}$$

Proof by cases

$$\frac{X \vee Y,\ X \rightarrow Z,\ Y \rightarrow Z}{Z}$$

Modus Tollens

$$\frac{X \rightarrow Y,\ \neg Y}{\neg X}$$

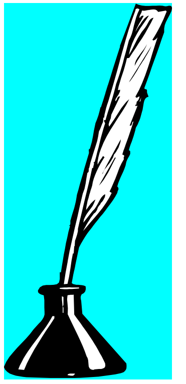
Contraposition

$$\frac{X \rightarrow Y}{\neg Y \rightarrow \neg X}$$

Dilemma

$$\frac{X \rightarrow Y,\ \neg X \rightarrow Y}{Y}$$

Exercises



- Give a formal proof for the "elections" example.

- Using the basic inference rules ($\wedge$I, $\wedge$E, $\vee$I, $\vee$E, $\rightarrow$I, $\rightarrow$E, $\leftrightarrow$I, $\leftrightarrow$E, RAA), show that the derived inference rules hold.

Be careful when you use assumptions in a derivation

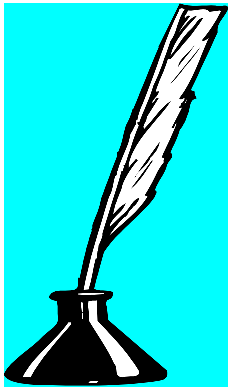
Show $\vdash \neg(\neg A \wedge \neg B) \rightarrow (A \vee B)$

1. $\neg(\neg A \wedge \neg B)$ assume
2. $\neg(A \vee B)$ assume
3. $\neg A$ assume
4. $\neg B$ assume
5. $\neg A \wedge \neg B$ 3,4,$\wedge$I
6. B 4,5,1,RAA
7. $A \vee B$ 6,$\vee$I
8. A 3,2,7,RAA
9. $A \vee B$ 8,$\vee$I
10. $A \vee B$ 2,9,RAA
11. $\neg(\neg A \wedge \neg B) \rightarrow (A \vee B)$ 1,10,$\rightarrow$I

Notes

- The only inference rules that make use of assumptions are RAA and $\rightarrow$I.
- It is very important to be clear about the scope of assumptions.
- Any assumption made during a derivation will remain in force, and ultimately count as one of the premises for the conclusion, unless it gets discharged before the conclusion is reached in the proof.

Exercise



Show

$\vdash ((P \wedge Q) \vee (\neg P \wedge R)) \rightarrow (Q \vee R)$

Exercise

Show

$P, \neg P \vdash Q$

Note:

This exercise shows that anything can be derived from an inconsistent set of premises.

Notes

- $|\!-$**W** denotes W is derivable from an empty set of premises.
- Let A, B be wffs.

If $\mathbf{A} \vdash \mathbf{B}$ then $\vdash \mathbf{A} \rightarrow \mathbf{B}$.

If $\vdash \mathbf{A} \rightarrow \mathbf{B}$ then $\mathbf{A} \vdash \mathbf{B}$.

In general if

P is a set of wffs, and

P' is a conjunction of the wffs in P, and

W is a wff then $\mathbf{P} \vdash \mathbf{W}$ iff $\vdash \mathbf{P'} \rightarrow \mathbf{W}$

Notes cntd.

- Proofs (derivations) are independent of the "meaning" of the propositional symbol.

So a proof is still valid if the symbols are replaced consistently.

Example: If we have a proof for

$$P, Q \rightarrow \neg P \vdash \neg Q$$

Then the following also holds (replacing P with snow and Q with mild)

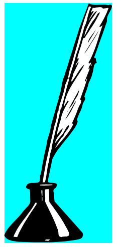
$$\text{snow}, \text{mild} \rightarrow \neg\text{snow} \vdash \neg\text{mild}$$

$$\text{winter} \rightarrow \text{cold}, \ \text{globalWarming} \rightarrow \neg(\text{winter} \rightarrow \text{cold}) \vdash \neg \text{globalWarming}$$

Notes cntd.

- For convenience, in a derivation we can use instances of previous derivations. That is, if we have previously shown

$S \vdash W$,	and we are now attempting a new proof for another wff, but we have so far shown
an instance of S,	then we can write down the same
instance of W	in the derivation without reproducing its entire proof.



Exercise A

Show

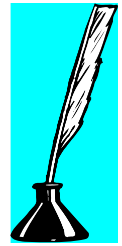
$$P\to Q,\ R\to S \vdash (P\vee R)\to(Q\vee S).$$

Exercise B

Show

$$(P\vee Q)\vee R \vdash P\vee(Q\vee R) \quad \text{and}$$

$$P\vee(Q\vee R) \vdash (P\vee Q)\vee R.$$



Exercise C

Formalise the following argument and show that it is valid. You may use the theorems in A and B, above.

In Britain one of the three parties, Tory, Labour or Liberal Democrat, is in power.

If the Tories are in power the government may support cuts in public spending.

If Labour is in power the government may support tax increases.

If the Liberal Democrats are in power the government may support proportional representation.

So in Britain the government may support cuts in public spending or tax increases or proportional representation.

Soundness and Completeness

Propositional logic is both **sound** and **complete**.

Let S be any set of wffs, and let W be a wff.

Soundness means the following:

If $S \vdash W$ then $S \models W$.

Completeness means the following:

If $S \models W$ then $S \vdash W$.

So in propositional logic we are justified in switching between syntactic proofs and semantic consequences.

Example

$A \equiv B$ iff

$A \models B$ and $B \models A$ iff

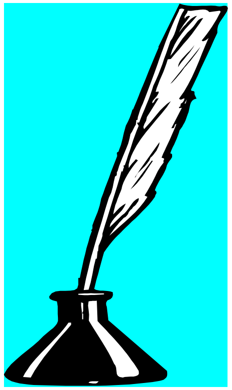
$A \vdash B$ and $B \vdash A$ iff

$A \models B$ and $B \vdash A$

It also means in proofs we can use equivalences.

But note in assessments you need to check the specifications in the questions carefully.

Exercise



We know

$$P \to Q \equiv \neg P \vee Q$$

So

$$P \to Q \vdash \neg P \vee Q$$

$$\neg P \vee Q \vdash P \to Q$$

Also

$$\vdash (P \to Q) \leftrightarrow (\neg P \vee Q)$$

As an exercise show the last using inference rules.

Exercises

Given the equivalence

$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ Show that

$(P \wedge Q) \rightarrow R,\ \neg(R \wedge S) \vdash \neg(P \wedge (Q \wedge S)).$

Using the equivalences

$A \rightarrow B \equiv \neg A \vee B$ and

$\neg(A \vee B) \equiv \neg A \wedge \neg B$

or otherwise show

$\vdash (P \rightarrow Q) \vee (Q \rightarrow P).$