Direct Access Sets?

Have been assuming need to search for a key

- In an array sorted by key
- Better: in a tree sorted by key

Can data just be indexed by key?

Questions

How could such indexing work?

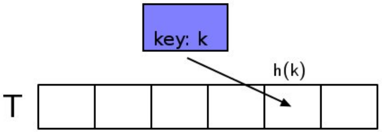
- Want to use any type as a key

Assuming such indexing, how long would put and get take in a set containing N objects?

Indexed Sets

Hash Tables index data (indirectly) by key

- A hash table T is (like) an array with m slots
- The key is converted to an integer index by a hash function h
- So, an object with key k is stored at $T[h(k)]$



The time taken by h depends only on k

- New object added into N object set in $\Theta(1)$ time (theoretical only!)

Numerical Encoding

Map any key object to a natural number

- Requirement: equal keys have same result
- Requirement: unequal keys have different result

Exercise

Design a function to map every ASCII string to a different natural number

Encoding Function

One Possibility

The formula

$$k = s[0] + s[1] * 128 + s[2] * 128^2 + \ldots$$

converts every ASCII string s to a different natural number.

- Treat each character as a digit
- Same principle can be applied (recursively) to any type

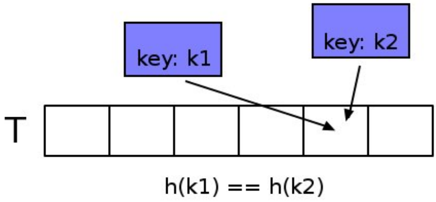
Question

What is the problem with this as a practical solution?

Collisions

Impractical to store every key at a different index

- Very space inefficient, even if it's possible
- Result: collisions



Will need a way to resolve collisions (store both objects)

A Hash Function Part 2

Map the numerical code k from Step 1 to a position in the table

Step 2

If the table has size m:

$$h(k) = k \bmod m$$

New requirement: minimise collisions

- spread the keys as evenly as possible

Question

What happens to the ASCII string keys if $m = 128$?

- All keys starting a.. hash to same slot
- If all keys start a... only one slot used
- Using a prime radix for k limits the problem

Uniform Hashing

- Lots of ways to hash: universal, fingerprint, cryptographic, ...
- Best result is data dependent
- More uniform, often slower

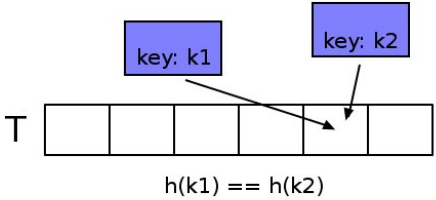
Definition (Simple Uniform Hashing Assumption)

Given a hash table T with m slots, using hash faunction h, the *simple uniform hashing assumption* (SUHA) states that each new key k is equally likely to hash into any of the m slots. So, the probability that $h(k) = i$, for every slot $1 \leq i \leq m$ is $1/m$.

- SUHA is an assumption about both h and input data
- Allows analysis to ignore details of both

Hash Table Memory

Recall: need a way to resolve collisions (store both objects)



Exercise

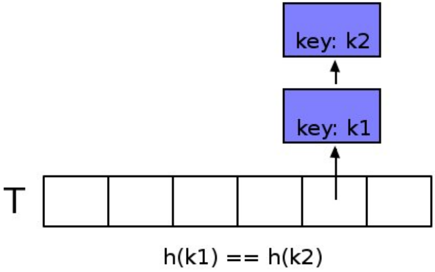
Design a way to resolve collisions

- Table has to store both objects somewhere
- What is the worst case time to add a new object?

Chaining

With collision resolution by Chaining

- All objects whose key hashes to $h(k)$ are placed into a linked list
- The table contains a pointer to the list
- So, $T[i]$ contains a list of objects x where $h(x.key) = i$



Performance of Chaining

Add object x to table T:

Insert as head of list at $T[h(x.key)]$

- takes $\Theta(1)$ time

Search for an object with key k

Search list at $T[h(k)]$ for an object where $x.key == k$

In a table containing N values

- Worst case is N elements in one chain: $O(N)$ search
- Under SUHA, expected time is $O(N/m)$
- N/m is called the load factor

Expected Time To Search

The expected time for an unsuccessful search for key k, in a hash table with m slots, containing N objects, assuming simple uniform hashing:

- By SUHA, expected length of each chain is N/m
- k equally likely to hash to all m positions
- Probability of searching chain at $T[i]$ is $1/m$

Expected number of comparisons is

$$\sum_{i=1}^{m} \frac{N}{m} \times \frac{1}{m} = \frac{N}{m}$$

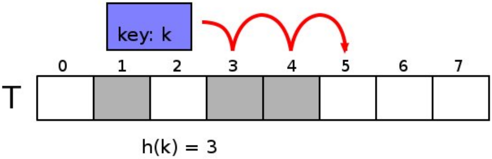
If N is proportional to m, expected running time for Search is $\Theta(1)$

- The design of the table needs to ensure N/m is $\Theta(1)$
- Successful search reasoning is similar: $O(N/m)$

Probing

In an open address hash table objects are stored directly in the table

- We use probing to resolve collisions
- To insert an object we probe the table until we find a space
- The hash function generates a sequence $\langle h(k,0), \ldots, h(k,m-1)\rangle$



The simplest form (above) is linear probing

- Consecutive slots are probed, beginning with $h(k)$, up to $h(k) - 1$

Performance of Probing

Definition (Uniform Hashing)

Given a hash table with m slots, a hash function produces uniform hashing if, for an unknown key k, the probability that the probe sequence of k is p, where p is a permutation of $\langle 0, \ldots, m-1 \rangle$ is the same for all such p.

- Uniform hashing first implies that every permutation is possible
- Linear probing does not produce uniform hashing

Assuming uniform hashing, the expected number of keys compared when inserting an object depends on the load factor N/m

- Each probe is to a random slot, with probability N/m it is occupied

If N is proportional to m, expected time for insert (and search) is $\Theta(1)$

Limitations

Hash tables do not support operations such as:

- In order iteration
- Next key / object
- Minimum key
- Maximum key

since objects are stored, by design, in random order.