Dynamic Data Structures

Dr Timothy Kimber

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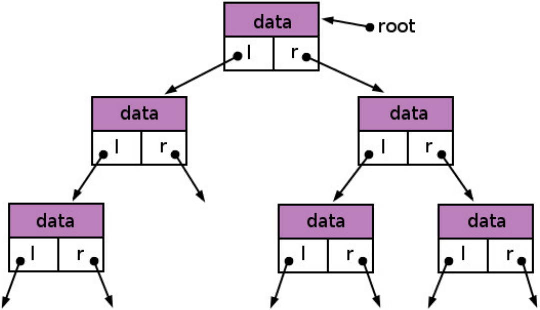


Dynamic Data Structures

Having efficient data structures is crucial for successful algorithms.

- The problems seen so far involved fixed length lists
- In most languages we have a simple way to implement this efficiently — arrays
- Our algorithms assumed some sort of array type was available

Other problems require dynamic data structures such as

- Lists, Stacks and Queues
- Sets and Dictionaries

These are designed to hold variable, essentially unlimited amounts of data.

Ordered Data Structures

A *list* is an ordered collection of {nodes, items, elements}.

- The key property of a list is the ordering of the nodes
- A list might support operations such as

push	adds an element to the end of the list
pop	removes the last element of the list
shift	removes the first element of the list
unshift	adds an element to the front of the list
insert	adds an element at a given position
remove	removes the element at a given position
iterate	returns the items in order

- Plus sorting, searching, copying, joining, splitting ...
- The most appropriate implementation depends on which operations are needed.

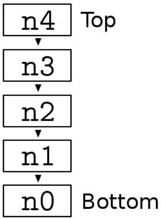
Stacks

A *stack* is a last-in first-out (LIFO) list.

- Stacks support only

  push for adding elements
  pop for removing elements

- Stacks are usually pictured as a vertical (stacked!) structure



- Stacks support recursive algorithms including fundamental operations such as calling subprocedures and evaluating arithmetic expressions

Stacks

Question

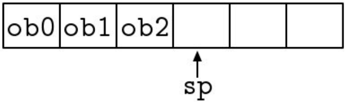
How would you implement a stack?

- Must be able to add “unlimited” objects
- Push and Pop must implement LIFO behaviour

Stack Implementation

Could use array or linked-list as storage model

- Both very simple implementation
- Array has fixed capacity
- Linked list has higher overheads



- Can make array dynamic (variable size)
- Integer sp points to the top of the stack
- Update sp within Push and Pop

Dynamic Array-Based Stack

Push(Input: stack $s[0, \ldots, k-1]$, object x)

```
if s.sp == k                         // array is full
  s' = new stack of size 2k
  for i = 0 to k - 1
    s'[i] = s[i]
    s = s'
s[s.sp] = x
s.sp = s.sp + 1
```

- push increases the capacity of the stack if it is full

Dynamic Array-Based Stack

Pop(Input: stack $s[0, \ldots, k-1]$)

```
if s.sp < k/2            // array is less than half full
  s' = new stack of size k/2
  for i = 0 to k/2 - 1
    s'[i] = s[i]
    s = s'
s.sp = s.sp - 1
return s[s.sp]
```

- pop decreases the capacity if it is too big
- a full implementation should have a minimum size

Performance of Push

Question

Given a stack containing N objects, what is the worst case time complexity of push?

- Assume: time to insert (copy, add) one object to array is c
- Assume: initial capacity is 4

The time taken for pushing objects is:

- c, c, c, c, $c + 4c$, c, ...

So:

- Worst time to push a single object is Nc
- So $T(N) = O(N)$
- Want to reflect fact that most pushes are not $O(N)$

Performance of Push

Revised Question

Given an empty stack, what is the worst case time to push N objects?

- Assume: initial capacity is 4
- Assume: time to insert (copy, add) one object to array is c

The time taken for the each push is still:

- c, c, c, c, $c + 4c$, c, ...
- For N pushes the worst single push is Nc
- $T(N) = O(N^2)$

However, this is a big overestimate

Performance of Push

The time for N pushes is at most

$$T(N) = Nc + (4c + 8c + \cdots + (N/2)c + Nc)$$

where

- the first Nc is the cost of writing N elements
- the rest is for copying to new arrays

The time for copying is $2Nc - 4c$ (see notes on solving series), so

- $T(N) \leq 3Nc - 4c$
- $T(N) = O(N)$

Amortisation

The time for N pushes is $O(N)$, so:

- A single push is *effectively* a constant time operation
- More correctly: `push` is amortised $\Theta(1)$
- NOT the same as $\Theta(1)$

Amortisation

- Related to accountancy method used to defer large costs
- Amortised analysis considers a sequence of operations
- Cost of individual ops is “amortised” across the sequence
- Unlike accountancy, must never be in debt

Amortised Analysis

Rather than calculating cost of full sequence of N steps we can

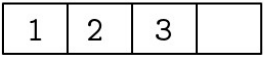
- Pick a representative subsequence
- Subsequence is some “cycle” that repeats
- Pick an amortised cost for operations
- Show that paying amortised cost covers all costs (never in debt)

Exercise

Find a representative cycle (subsequence) of pushes into the stack and show that the amortised cost of $3c$ covers all costs.

Amortised Analysis

- Start after any expensive push
- Up to and including next expensive one

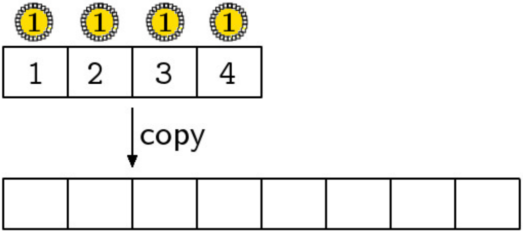


- Cheap pushes each put $2c$ in the bank
- Have enough to cover extra costs when next expensive push occurs
- (If you started with a copy you are immediately over budget)

Amortised Analysis

Argument only works because array is initially empty and size is doubled

- Say we have N objects on stack after a copy
- Before next copy we always push N more
- This is how cost is covered



- Multiplying by any factor will do - will affect amortisation constant

Queues

A *queue* is also a list, but the next object removed is either:

- The earliest one added (FIFO Queue)
- The one with highest priority (Priority Queue)

Questions

- How could you implement a priority queue (PQ)?
- Given a PQ following your design that contains N objects, what would be the worst case time to add a new object? (Each object has a key attribute that determines its priority.)

Priority Queue Design

If we maintain a total ordering of the queue:

- It will take $O(N)$ time to add new elements
- Can search a sorted array quickly but have to shift existing objects
- Finding position in a linked list is $O(N)$

9, 8, 7, 6, 3 ← Max Priority Queue ← 6, 9, 7, 3, 8

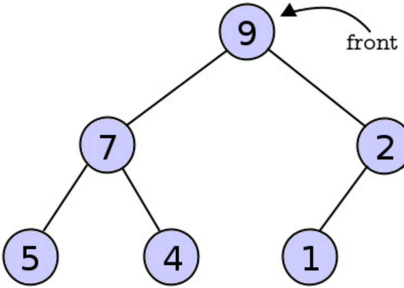
`remove items` `add items`

Do not actually need total ordering.

- Queue does not support indexed access
- Just want to find object with highest priority

Priority Queue Design

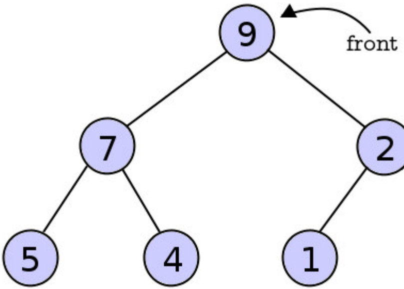
Solution is to divide and conquer the data



- Key Property: Maintain order within each branch
- Highest (or lowest) key will be at the root
- Behaves like lots of mini queues

Priority Queue Design

Solution is to divide and conquer the data



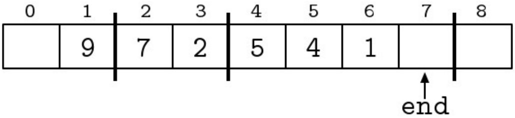
Question

A new object could go in any branch. (Do you agree?) So, where should it go? Why?

Heap: a Tree in an Array

We want to know where the “end” of the tree is:

- Build a tree within an array



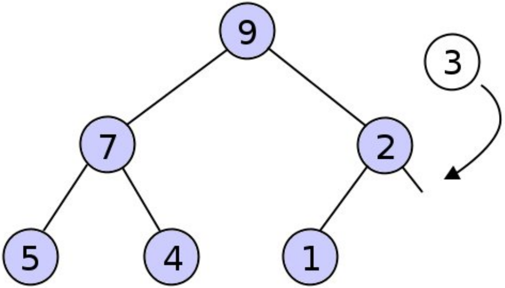
- Track end using “stack pointer”
- Navigate by indices
- Leaving `a[0]` blank means:
  - parent of `a[n]` is `a[n/2]`
  - children of `a[n]` are `a[2*n]` and `a[2*n+1]`

Exercise

How should a new object be added to a max binary heap? (i.e. the greatest key should be at the root).

Binary Heap Operations

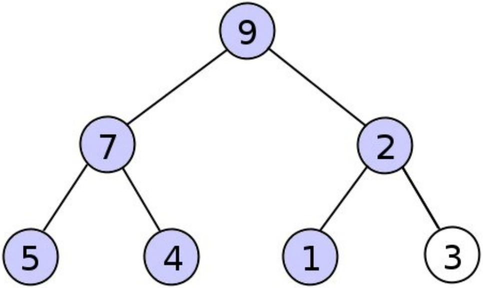
Adding an object to the heap



- Restore the "shape"
- Then restore the order

Binary Heap Operations

Adding an object to the heap



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Binary Heap Operations

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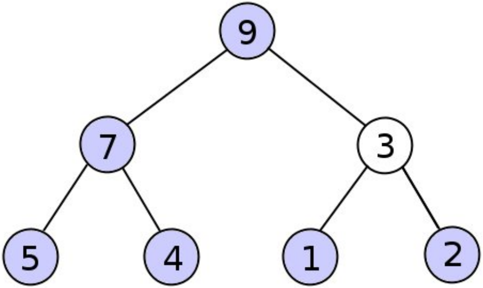

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Binary Heap Operations

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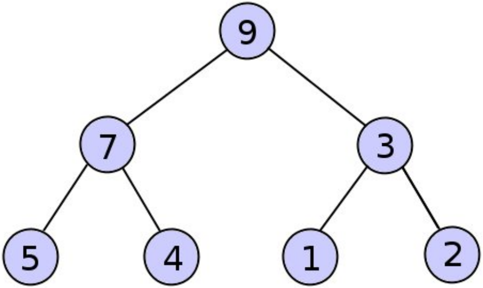

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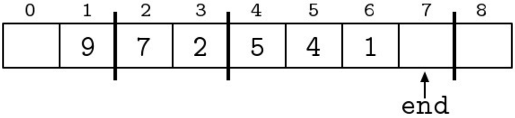
Heap: a Tree in an Array



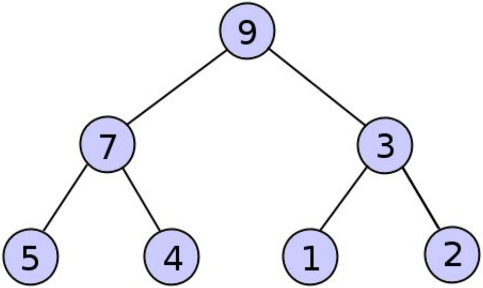
- Track end using "stack pointer"
- Leaving a[0] blank means:
  - parent of a[n] is a[n/2]
  - children of a[n] are a[2*n] and a[2*n+1]

Exercise

How should the object with the greatest key be removed from a max binary heap?

Binary Heap Operations

To remove the object at the root



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Binary Heap Operations

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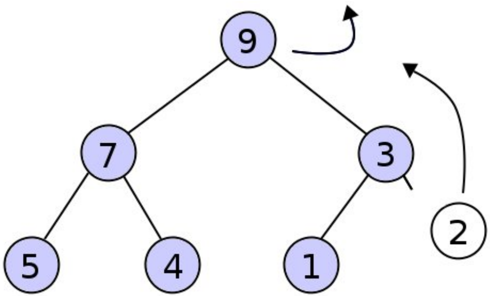

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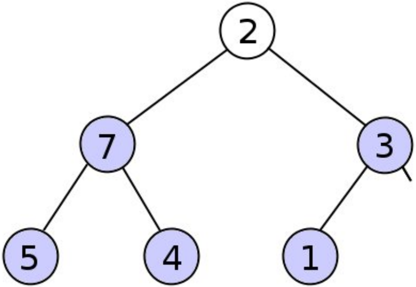
Binary Heap Operations

To remove the object at the root



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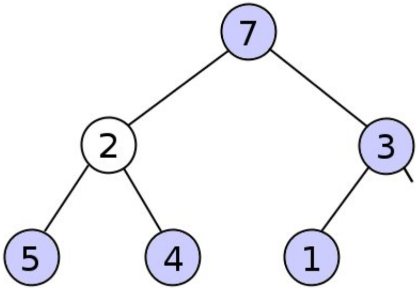
Binary Heap Operations

To remove the object at the root



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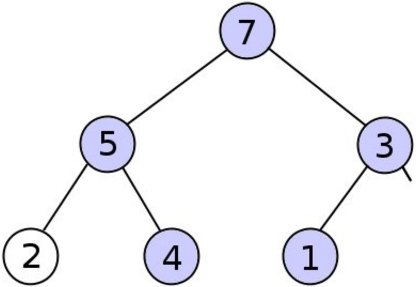
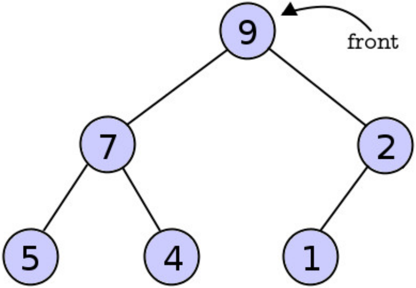
Binary Heap Operations

To remove the object at the root



- Restore the "shape"
- Then restore the order

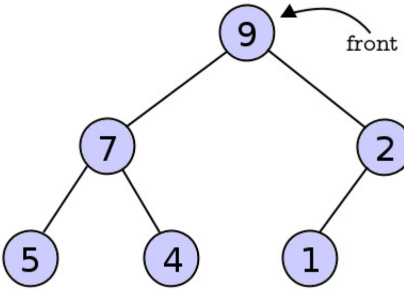
Binary Heap Performance



Question

Given a heap containing N objects, what is the time complexity for adding or removing one object?

Binary Heap Performance



Both operations are $O(\log_2 N)$

- Height of the heap is $\Theta(\log_2 N)$
- Each operation confined to one branch

Heapsort

Heaps also provide us with the Heapsort algorithm (JWJ Williams, 1964)

Heapsort (given a list L)

- Create an empty heap H
- Remove each element of L and add it to H
- Remove each element of H and add it to L
- HALT

- What could be simpler?!
- Performance is again $\Theta(N log_2 N)$
- Can also be implemented in place by setting up list and heap partitions within a single array