

580: Algorithms

Tutorial 2

1. (Part of a 2015 exam question.)

- (a) Using either Java or pseudocode, write a recursive procedure $\text{POW}(x, N)$ to compute x^N , where N is a positive integer. Use a divide and conquer strategy. *Hint:*

$$\begin{aligned} x^N &= x^{N/2} \times x^{N/2} && \text{for even } N \\ x^N &= x^{(N-1)/2} \times x^{(N-1)/2} \times x && \text{for odd } N. \end{aligned}$$

- (b) Write recurrence expressions for the time complexity $T(N)$ of your POW procedure in the following cases:

$$T(N) = \begin{cases} , & \text{if } 0 < N \leq c \\ , & \text{if } N > c \end{cases}$$

What is c ?

- (c) Solve your expressions for $T(N)$ using the master method. Show each step.

Answer:

(a)

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1: procedure POW( $x, n$ )
2:   if  $n == 1$  then
3:     return  $x$ 
4:   end if
5:    $tmp = \text{POW}(x, n/2)$  ▷ integer division
6:   if  $n \bmod 2 == 0$  then
7:     return  $tmp \times tmp$ 
8:   end if
9:   return  $tmp \times tmp \times x$ 
10: end procedure
```

Key feature: POW should be called recursively ONCE, in line 5. If it is called more than once then the complexity will be $\Theta(N)$.

(b)

$$T(N) = \begin{cases} \Theta(1) & , \text{ if } N = 1 \\ T(\lfloor N/2 \rfloor) + \Theta(1) & , \text{ if } N > 1 \end{cases}$$

So, $c = 1$. Symbols representing constants would be acceptable in place of the $\Theta(1)$ terms. The second formula must include $\lfloor N/2 \rfloor$, not $N/2$. The expression is not correct for all N if it uses $T(N/2)$.

- (c) To apply the master method you need to first substitute the recurrence into the expression $T(N) = aT(N/b) + f(N)$, while ignoring any floors or ceilings. This gives us $a = 1$, $b = 2$ and $f(N) = \Theta(1)$. Then compute $N^{\log_b a}$, which in this case is $N^0 = 1$. So, $N^{\log_b a} = \Theta(1)$ and $f(N) = \Theta(1) = \Theta(N^{\log_b a} \log_2^0 N)$. Therefore, Case 2 of the master method applies and $T(N) = \Theta(N^{\log_b a} \log_2 N) = \Theta(\log_2 N)$.