Sorting

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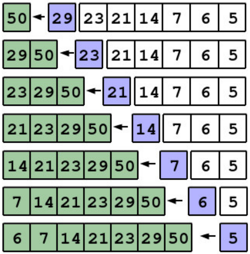


The Sorting Problem

- Sorting data is one of the most thoroughly explored computing problems.

Problem *(Sort)*

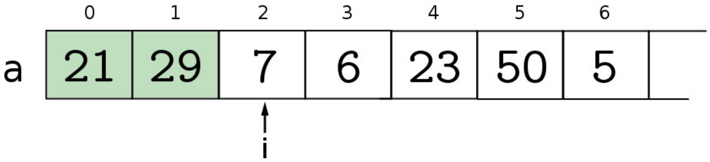
Input: a sequence A of values $\langle a_1, a_2, \ldots, a_N \rangle$

Output: a permutation (reordering) $\langle a'_1, a'_2 \ldots, a'_N \rangle$ of A such that $a'_1 \leq a'_2 \leq \cdots \leq a'_N$

- Sorting is an important problem. It is part of the solution to many other problems.
- Understanding the complexity of sorting algorithms helps design good solutions to these other problems.

Incremental Sorting

The incremental approach of Simple Search can be applied to sorting.

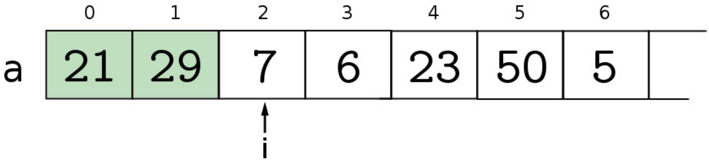


- Proceed from left
- Gradually grow a sorted region (note loop invariant)

EXERCISE

Invent an incremental sorting algorithm.

Incremental Sorting



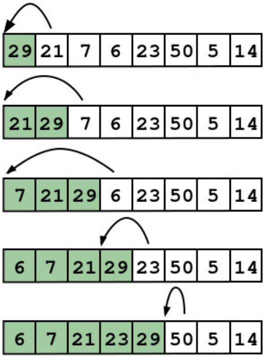
There are two options:

1. Add the next element `a[i]` to the sorted region
2. Add the lowest element outside the sorted region

Option 1 leads to the Insertion Sort algorithm

Insertion Sort

- Insertion Sort divides a into a sorted part, initially just $a[0]$, and the remaining unsorted part
- Elements from the unsorted part are then inserted into the sorted part



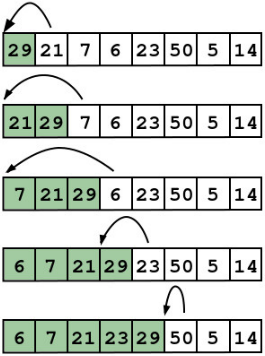
Insertion Sort

Insertion Sort(Input: sequence $a = [a_0, \ldots, a_{N-1}]$)

```
For i = 1 to N              // initialise invariant
  next = a[i]               // save. a[i] overwritten later
  j    = i
  While j > 0 and next < a[j-1]  // use invariant
    a[j] = a[j-1]
    j = j - 1
  EndWhile
  a[j] = next
EndFor
```

- The sorted region can be initialised to contain `a[0]`
- Do not need to compare `next` with all `a[0,..,i-1]` (sorted)

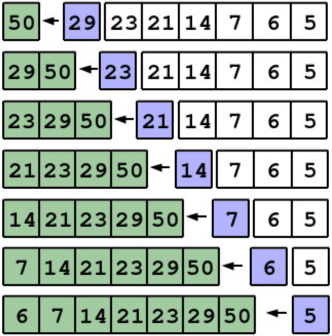
Time Complexity



- What is the worst case input?
- What is the best case input?
- What is the time complexity in the best and worst cases?

Worst Case

- Running time of Insertion Sort has two dimensions:
  - Number of insertions
  - 'Size' of insertion
- Informally: both dimensions are $\Theta(N)$, so $T(N) = \Theta(N^2)$

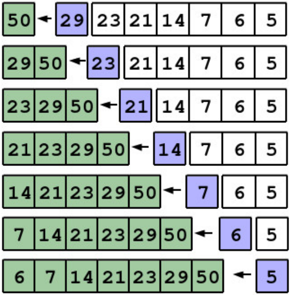


Worst Case

More formally, the total number of iterations of the inner loop is

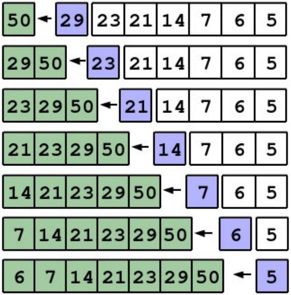
$$t(N) = 1 + 2 + \cdots + (N-1) = \sum_{i=1}^{N-1} i$$

$$= \frac{(N-1+1) \times (N-1)}{2} = \frac{N^2 - N}{2}$$



Worst Case

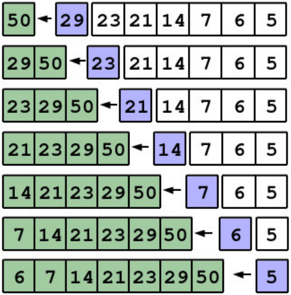
So, the worst case time complexity is $aN^2 + bN + c = \Theta(N^2)$



Best Case

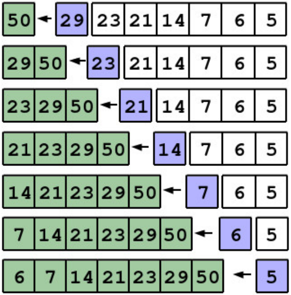
In the best case

- $t(N) = N - 1$
- So, $T(N) = aN + b = \Theta(N)$



Other Properties

- Insertion Sort works in place (space complexity is $\Theta(1)$)
- Assuming randomised input, average case is $T(N) = \Theta(N^2)$
- For any input $T(N) = O(N^2)$



Divide and Conquer

Will a divide and conquer approach work?

29	21	7	6	23	50	5	14

6	7	21	29

5	14	23	50

- Divide into subproblems
- Solve the subproblems
- Combine into overall solution

EXERCISE

Design a combining algorithm.

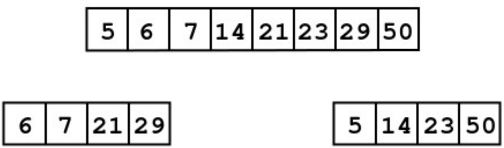
Combining Sorted Sequences

Merge (Input: array a, indices l, m and r, where $r > m \geq l$)

```
left  = a[l,...,m-1]
right = a[m,...,r-1]
i = j = 0, k = l
while k < r
  if (i > (m-l)) or (j < (r-m) and right[j] < left[i])
    a[k] = right[j]
    j = j + 1
  else
    a[k] = left[i]
    i = i + 1
  end
  k = k + 1
end
```

- The procedure takes $\Theta(N)$ time for N total elements

Divide and Conquer



- Time to combine subproblem solutions is $\Theta(N)$

EXERCISE

What is worst case time complexity of divide and conquer algorithm?

- Write recurrence (assume $N = 2^a$ so no floors)
- Solve using master theorem

Time Complexity

The proposed algorithm divides the problem (in constant time) into 2 subproblems of size $N/2$, solves both, and combines the solutions in $\Theta(N)$ time, so the time complexity is:

$$T(N) = \begin{cases} \Theta(1) & \text{, if } N = 1 \\ 2T(N/2) + \Theta(N) & \text{, if } N > 1 \end{cases}$$

So, $N^{\log_b a} = N^{\log_2 2} = N^1 = N$, and therefore

- $f(N) = \Theta(N^{\log_b a}) = \Theta(N^{\log_b a} \times \log_2^0 N)$

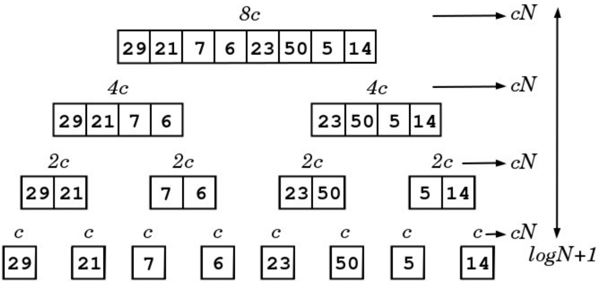
and Case 2, with $k = 0$, applies.

- $T(N) = \Theta(N^{\log_b a} \log_2^1 N) = \Theta(N \log_2 N)$

The divide and conquer algorithm is faster than Insertion Sort. Surprised?

Time Complexity

Alternative informal view of time complexity: recursion tree



- Each level of the tree contributes cN
- There are $\log_2 N + 1$ levels

MergeSort

You have invented Mergesort

MergeSort (Input: array *a*, index *l*, index *r*)

```
if r - l < 2
  return
m = (l + r) / 2
MergeSort(a, l, m)
MergeSort(a, m, r)
Merge(a,l,m,r)
return
```

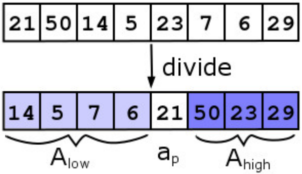
- The sorting appears to be happening in place, but the list is copied during `Merge`
- What is the best case?

Properties of Merge Sort

- Space complexity is $\Theta(N)$
- Time complexity is $\Theta(N \log_2 N)$
- Faster than Insertion Sort for large, unsorted lists
- Slower than Insertion Sort if the list is already sorted
- Slower than Insertion Sort for small N

Alternative Divide and Conquer

- Merge Sort divides the data in half and sorts the halves
- Alternative approach: do a quick (rough) sort into two groups
- a_p is called the pivot and the division ensures that $\forall a \in A_{low}(a < a_p)$ and $\forall a \in A_{high}(a \geq a_p)$



- Left with subproblems of sorting A_{low} and A_{high}
- No combining needed

Quicksort

This procedure sorts the array $a[l, ..., r-1]$

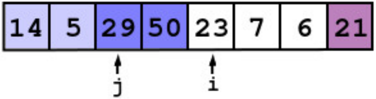
Quicksort (Input: array a, index l, index r)

```
if r < l + 2              // 0 or 1 elements to sort
  return
p = Partition(a, l, r)
Quicksort(a, l, p)
Quicksort(a, p + 1, r)
return
```

- The Quicksort divide step is called partitioning
- The Partition procedure must return the final index of the pivot
- The base case must work for an empty array

Suggested Partition Design

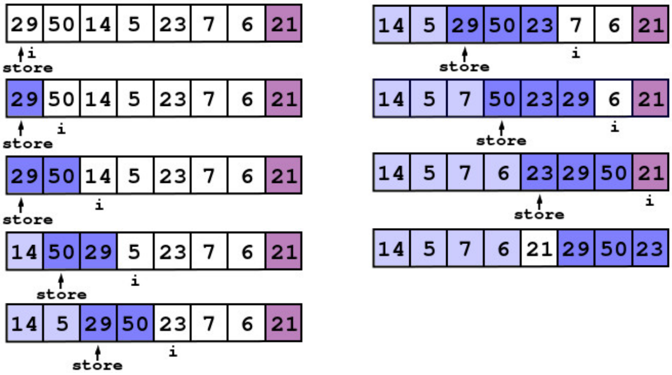
- Use last element as the pivot
- Maintain two subarrays which grow
- Elements before `j` must be less than pivot
- Elements `j,...,i-1` must be equal or greater than the pivot
- Elements `i,...` are unseen so far



Tutorial Exercise

Write the Partition procedure.

Lomuto Partitioning



Partition

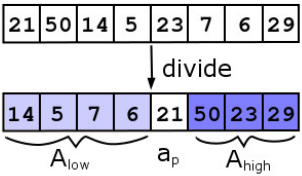
This procedure partitions the array $a[l, ..., r-1]$

Partition (Input: array a, index l, index r)

```
i = j = l              // both partitions are empty
p = r - 1
while i < p
  if a[i] < a[p]
    swap(a, i, j)
    j = j + 1
  i = i + 1
swap(a, p, j)
return j
```

- The time complexity is $\Theta(N)$ (where $N = r - l$)

Quicksort Performance



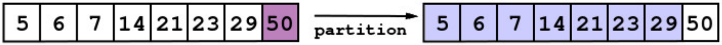
Question

What is the worst case time complexity of Quicksort. And why?

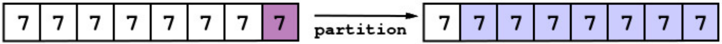
Quicksort Worst Case

The given `partition` procedure:

- will only remove one element from sorted or reverse-sorted data



- will only remove one element from data with many duplicates

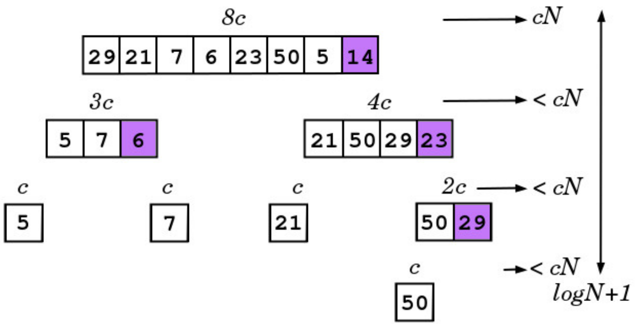


This leads to incremental execution resembling insertion sort

- N levels of recursion
- $N - i$ elements to partition at level i
- So worst case time complexity is $\Theta(N^2)$

Quicksort Best Case

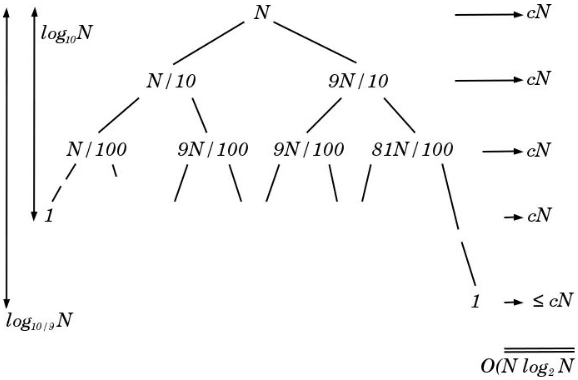
- Fewest levels of recursion when the partitioning is balanced
- Subproblems are no larger than $N/2$
- Same bound as Mergesort: $\Theta(N \log_2 N)$
- As recursion tree suggests, constants smaller than Mergesort



Quicksort Performance

With rather unbalanced partitioning performance is still $O(N \log_2 N)$

- Any dividing factor will introduce a $\log N$ term
- e.g. 9 to 1 (Cormen p. 176)



Randomised Quicksort

If the partition procedure is altered to choose the pivot at random, then the worst case behaviour becomes a matter of chance for all inputs.

Partition (Input: array a, index l, index r)

```
x = random(l, r)    // random integer in [l,r-1]
swap(a, x, r-1)
... as before
```

Assuming N distinct values:

- The probability of choosing the worst pivot in every call is $1/N!$
- This becomes vanishingly small as N increases
- Randomised Quicksort is algorithm of choice if N more than ~ 10

Expected Performance

Possible to give average case time complexity for Randomised Quicksort

- Assume N distinct values again
- Randomisation means all inputs equally likely
- Time depends on number of comparisons performed
- Probability of comparing `a[i]` with `a[j]` determined by their rank by value (see books for full explanation)
- Average number of comparisons is sum of probabilities for all `i,j`

Average case complexity is $\Theta(N \log_2 N)$

- This is called expected running time for randomised algorithm

Partition Variations

There are many ways to implement partitioning. e.g.

- Hoare paritioning
  - Partitions grow inwards from end
  - Handles duplicates better
- Three-way paritioning
  - Includes a region for values equal to pivot
  - Handles duplicates better
- Median of 3
  - Choose pivot as median of three random elements
  - Better balance between subproblems