9 April - MATH 345

April 9, 2015 2:00 PM

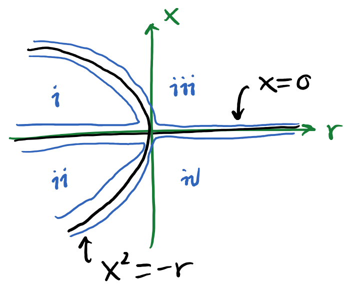
- allowed: one $8\frac{1}{2}$" x 11" sheet of notes/formulae (double sided)
  non-graphing, non-programmable calculator (optional)
- links to past exams on website

Review

I. One dimensional flows

- in an interval or in $\mathbb{S}^1$
- fixed points (have to be real), linear stability (hyperbolic or non hyp.), phase portraits, stability (especially for non-hyperbolic points)
- dimensional analysis & scaling (also in 2D flows)
- models (physics, biology, chemistry - should know typical units)
- bifurcations, normal forms: saddle-node, transcritical, pitchfork
- note: for transcritical & pitchfork, sometimes need preliminary change of variables $x = x^* + u$ if $x^* \neq 0$ (i.e. $u^* = 0$ is fixed point)
  e.g. Example 1.5, midterm #1, practice midterm #1 (b) iii
- "zero-set" $\mathcal{Z} = \{(x,r) \in \mathbb{R}^1 \times \mathbb{R} : f(x,r) = 0\}$ or $\{(\varphi, r) \in \mathbb{S}^1 \times \mathbb{R} : f(\varphi, r) = 0\}$
- like a bifurcation diagram, but without stability info
- factor: e.g. $f(x,r) = xr + x^3 = x(r + x^2) = 0 \Leftrightarrow x = 0$ or $x^2 = -r$

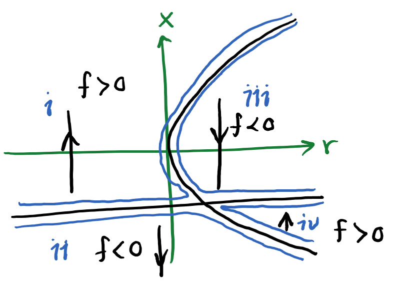
$\mathcal{Z}$:



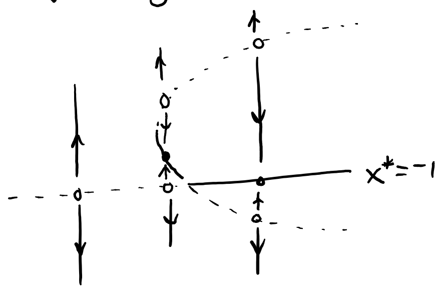
i-iv are the open regions

- if $f(x,r)$ is continuous, then it cannot change sign in the open regions complementary to $\mathcal{Z}$

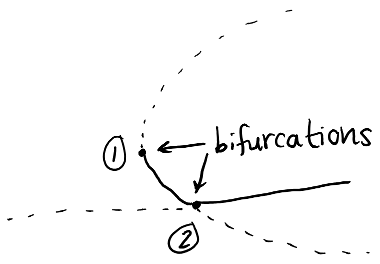
e.g. MT #1



- sign of f is sign of $\dot{x}$



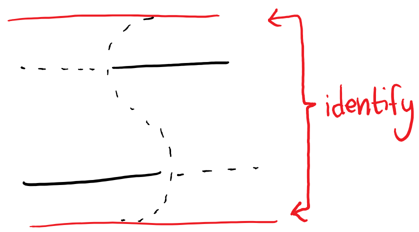
- global bifurcation diagram



- local vs. global bifurcation diagrams
- normal form gives local bifurcation diagram
- plotting Z, finding sign of f gives global bifurcation diagram
- the two should be consistent

- for $\varphi \in S^1$ for global picture, remember to make identification in φ-direction; in local pictures (e.g. normal form) treat φ as in $\mathbb{R}^1$

  e.g. PMT #1



- note: final will probably have 4 Q's: two 2D flows, one 1D flow, one chaos

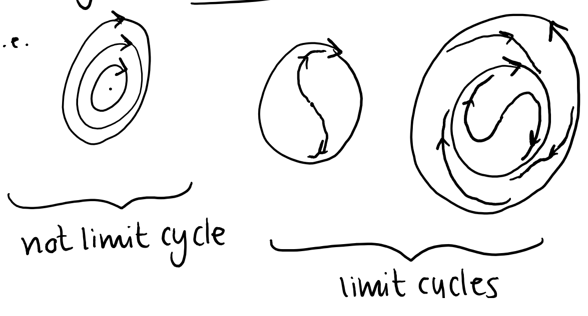
II Two dimensional flows

- in a region in $\mathbb{R}^2$ or in $\mathbb{S}^1 \times \mathbb{R}^1$
- fixed points, linear stability (hyperbolic, non-hyperbolic), implications
- nullclines, direction field, local phase portrait at fixed points
- usually no need to explicitly compute eigenvectors, use nullclines, direction field, classification type
- should be consistent! if not, there is a mistake somewhere

* if you notice a mistake or inconsistency too late in exam, make note that you're aware of it; will get some sympathy

- stable and unstable manifolds (nonlinear) curves tangent to eigenvectors (should be consistent with nullclines & dir. field)
- conservative systems - find conserved quantity by "partial integration" or "guess & check", deduce nonlinear global phase portraits in $\mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{R}^1$ ↑ actually check!
- again, should be consistent with nullclines, dir. field
- rectangular (x,y) or polar coords (r,θ) in $\mathbb{R}^2$, convert ODE by using chain rule (good idea to include conversions on sheet)
- Lyapunov function $V(\vec{x})$: many uses
  - i) to determine (local) stability
  - ii) rule out closed orbits
  - iii) determine global phase portraits
  - iv) help construct trapping regions
- sign of $\dot{V}$ is important parameter - compare to level sets of V
- other ways to rule out closed orbits
- ways to prove closed orbits (Poincaré-Bendixson theorem with suitable trapping region, prove that Poincaré map has fixed point)
- limit cycle = isolated closed orbit

i.e.



- bifurcations in 2D: saddle-node, transcritical, pitchfork, Hopf
- Poincaré map: explicit formula or as conceptual tool

  e.g. fixed point exists for Poincaré map

  ↳ closed orbit exists for flow

  (un)stable fixed point Poincaré map

  ↳ (un)stable closed orbit for flow

III Chaos

- 1D maps: fixed points, linear stability, multiplier

  note: criteria are different than for flows!
- can do $Z = \{(x,r): f(x,r) = x\}$ $f(x,r) = x \Leftrightarrow f(x,r) - x = 0$
- cycles of length N: fixed points of f^N
- linear stability of cycles: chain rule
- Lyapunov exponents: meaning, definition, implications, explicit calculation for "typical" orbits
- doubling / binary shift map, tent map, $\underbrace{\text{coordinate transformations}}_{\text{from one map to another e.g. HW5\#5}}$
- 3D flows (e.g. Lorenz equation, HW5#3)
- conceptual understanding of basic and subtler results

  e.g. what is chaos? how would you prove it?

  e.g. use of approximations of 1D maps
- note: no time for long, hard calculations on exam