15 January - MATH 345

January 15, 2015 2:00 PM

- homework due Thursday, 22 January

Saddle-node Bifurcations (cont.)

$$\dot{x} = f(x,r), \quad x \in \mathbb{R}, \quad r \in \mathbb{R}$$

$$\left.\begin{array}{ll} (\text{SN } 1) & f(x_c^*, r_c) = 0 \\ (\text{SN } 2) & \dfrac{\partial f}{\partial x}(x_c^*, r_c) = 0 \end{array}\right\} \text{saddle-node hypotheses 1, 2}$$

- expand $f(x,r)$ in 2-variable Taylor Series at (x_c^*, r_c)

$$\dot{x} = f(x,r) = \underbrace{f(x_c^*, r_c)}_{=0\,(\text{SN}1)} + \underbrace{\frac{\partial f}{\partial x}(x_c^*, r_c)}_{=0\,(\text{SN}2)}(x - x_c^*) + \underbrace{\frac{\partial f}{\partial r}(x_c^*, r_c)}_{a}(r - r_c) + \underbrace{\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x_c^*, r_c)}_{b}(x - x_c^*)^2 + \ldots$$

$$\dot{x} = a(r - r_c) + b(x - x_c^*)^2 + \ldots$$

- assume

$$\left.\begin{array}{ll} (\text{SN } 3) & a = \dfrac{\partial f}{\partial r}(x_c^*, r_c) \neq 0 \\ (\text{SN } 4) & b = \dfrac{1}{2!}\dfrac{\partial^2 f}{\partial x^2}(x_c^*, r_c) \neq 0 \end{array}\right\} \text{saddle-node hypotheses 3, 4}$$

- a theorem states essentially if (SN1)–(SN4) are true, then we can ignore the "+ . . ." terms. We get qualitatively the correct phase portraits in some open neighbourhood of (x_c^*, r_c) in $\mathbb{R}^2$ (i.e. the correct local dynamics)
- the (truncated) normal form for a saddle-node bifurcation is

$$\dot{x} = a(r - r_c) + b(x - x_c^*)^2$$

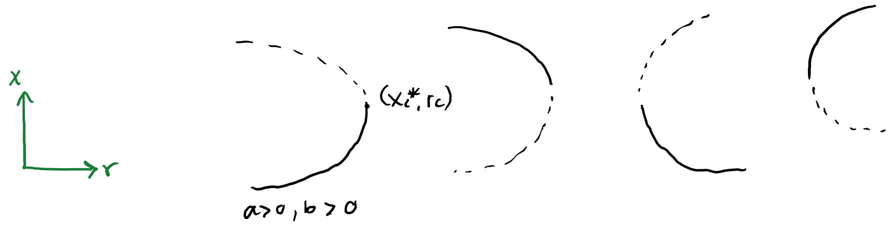
- fixed points:

$$0 = a(r - r_c) + b(x - x_c^*)^2$$

$$x = x^* = x_c^* \pm \sqrt{-\frac{a}{b}(r - r_c)}$$

$-\frac{a}{b}(r-r_c) \geq 0$

- there are four possibilities, depending on the signs of a, b



Exercise: What are the signs of a, b in these cases?

$\boxed{\text{Example 1.4}}$

Find all saddle-node bifurcations in

$$\dot{x} = r - x - e^{-x}$$

$$f(x,r) = r - x - e^{-x}$$

$$\frac{\partial f}{\partial x}(x,r) = -1 + e^{-x}$$

$$\frac{\partial f}{\partial r}(x,r) = 1$$

$$\frac{\partial^2 f}{\partial x^2}(x,r) = -e^{-x}$$

first, find specific solutions of (SN1)-(SN2)

$$\begin{cases} f(x,r) = 0 \\ \frac{\partial f}{\partial x}(x,r) = 0 \end{cases} \qquad \begin{cases} r - x - e^{-x} = 0 & ① \\ -1 + e^{-x} = 0 & ② \end{cases}$$

② $e^{-x} = -1 \rightarrow -x = \ln 1 = 0 \rightarrow x = 0$

① $r - 0 - e^{0} = 0 \rightarrow r = 1$

$x_c^* = 0$, $r_c = 1$ is the only candidate for a saddle-node bifurcation

(SN1), (SN2) are satisfied

verify (SN3), (SN4):

(SN3) $\frac{\partial f}{\partial r}(0,1) = 1 \neq 0$

(SN4) $\quad \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,1) = \frac{1}{2}(-e^{-x})\Big|_{(x,r)=(0,1)} = -\frac{1}{2} \neq 0$

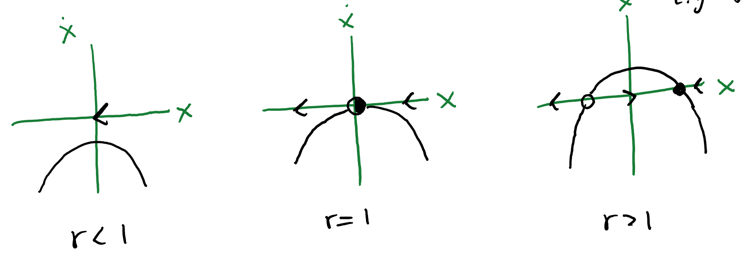
since $a = 1 \neq 0$ and $b = -\frac{1}{2} \neq 0$, there is indeed a saddle-node bifurcation at $x_c^* = 0$, $r_c = 1$

Normal form is

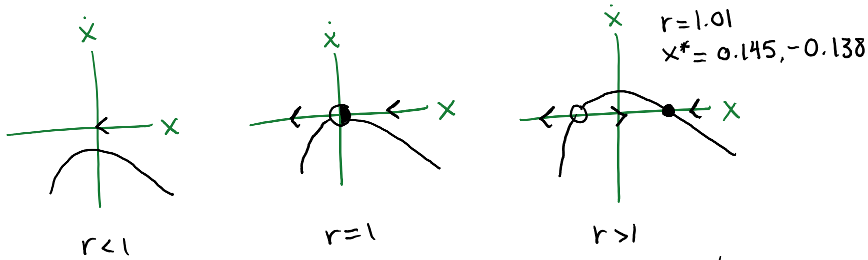
$$\dot{x} = a(r - r_c) + b(x - x_c^*)$$
$$= 1(r-1) + (-\tfrac{1}{2})(x-0)^2$$
$$\dot{x} = r - 1 - \tfrac{1}{2}x^2 \Big\}\text{ this is valid near } (x,r) = (0,1)$$

Phase portraits of normal form

e.g. if $r = 1.01$ then $x^* = 0.141, -0.141$



Phase portraits of actual equation



$\underbrace{\hspace{6cm}}_{\text{these are not symmetric functions}}$

normal form gives qualitatively correct dynamics near $(x,r) = (0,1)$

also a good approximation near $(0,1)$

Transcritical Bifurcation ("exchange of stability")

- in some models there is a fixed point that exists for all parameter values
  e.g. $\dot{N} = rN\left(1 - \frac{N}{K}\right)$

$N=0$ is a fixed point for all r,k

(this makes biological sense)

- in general, consider

$$\dot{x} = f(x,r), \quad x \in \mathbb{R}, \; r \in \mathbb{R}$$

- suppose $x^*=0$ is a fixed point for all r

$$\left.\begin{array}{l}(TC1)\quad f(0,r)=0 \quad \text{for all } r \\ \qquad\qquad g(r)=0\end{array}\right\} \text{transcritical hypothesis \#1}$$

- suppose when $r=r_c$, this fixed point is nonhyperbolic

$$(TC2) \quad \frac{\partial f}{\partial x}(0,r_c)=0$$

no terms constant $\bullet\, (r-r_c)^k$
all have at least one factor of x

- expand into Taylor series

$$\dot{x} = f(x,r) = \underbrace{f(0,r_c)}_{=0\,(TC1)} + \underbrace{\frac{\partial f}{\partial x}(0,r_c)\,x}_{=0\,(TC2)} + \underbrace{\frac{\partial f}{\partial r}(0,r_c)(r-r_c)}_{=0\,(TC1)} + \frac{1}{2!}\underbrace{\frac{\partial^2 f}{\partial x^2}(0,r_c)}_{b}x^2 + \underbrace{\frac{\partial^2 f}{\partial x \partial r}(0,r_c)}_{a}x(r-r_c) + \dots$$

- by (TC1)

$$\frac{\partial^k f}{\partial r^k}(0,1)=0, \quad k=1,2,3,\dots$$

- (TC1) allows us to factor x out of $f(x,r)$

$$\dot{x} = x\left[a(r-r_c)+bx+\dots\right]$$

- assume

$$(TC3) \quad a = \frac{\partial^2 f}{\partial x \partial r}(0,r_c) \neq 0$$

$$(TC4) \quad b = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,r_c) \neq 0$$

- If (TC1) – (TC4) are true, then the correct local dynamics at $(0,r_c)$ is given by the normal form for the transcritical bifurcation

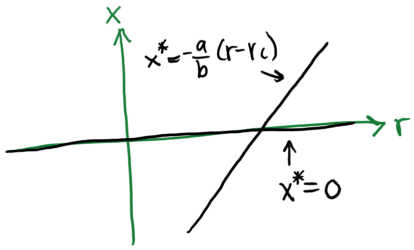
$$\dot{x} = x\left[a(r-r_c)+bx\right] = a(r-r_c)x + bx^2$$

· fixed points

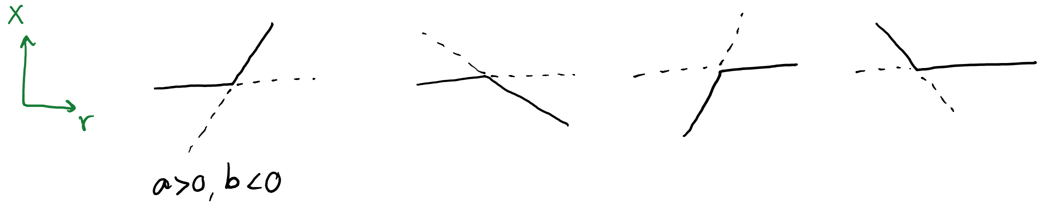
$$0 = x[a(r-r_c)+bx]$$

solutions $x^* = 0$

$$x^* = -\frac{a}{b}(r-r_c)$$



· there are four cases depending on the signs of a, b



Exercise: determine the signs of a, b for each case

$\boxed{\text{Example 1.5}}$

$$\dot{x} = x - 1 + r\ln x, \quad x>0$$

$\ln(1) = 0$ so $x^* = 1$ is a fixed point for all r

$\tilde{f}(1, r) = 0$ for all r

change of variable: $u = x-1$ i.e. $x = 1+u$

u is the deviation from the fixed point $x^* = 1$

$$u = 0 \leftrightarrow x = 1$$

$$\dot{u} = \dot{x} = x - 1 + r\ln x = 1 + u - 1 + r\ln(1+u)$$

$$\dot{u} = \underbrace{u + r\ln(1+u)}_{f(u,r)}, \quad u > -1$$

check (TC1): $f(0, r) = 0 + r\ln 1 = 0$