17 March - MATH 345

March 17, 2015 2:00 PM

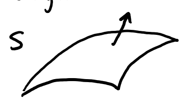
- HW 4 due Thursday, March 19

Poincaré maps, cont.

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \ , \ \vec{x} \in \mathbb{R}^n \ , \ n \geq 2$$

- S: $(n-1)$-dimensional manifold in $\mathbb{R}^n$
  e.g. $S = \{\vec{x} \in \mathbb{R}^n : g(\vec{x}) = 0 \ , \ \vec{x} \in \text{open set}\}$

- S is transverse to the flow i.e. $\vec{f}(\vec{x})$ is never $\vec{0}$ or tangent to S
  e.g. $\nabla g(\vec{x}) \cdot \vec{f}(\vec{x}) \neq 0$ for $\vec{x}$ in S



- S is called a section (or surface of section)

- choose $\vec{x}_0 \in S$, solve the IVP (initial value problem)

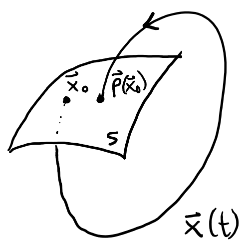
$$\dot{\vec{x}} = \vec{f}(\vec{x}) \ , \ \vec{x}(0) = \vec{x}_0$$

and wait some time $t = T(\vec{x}_0) > 0$, called the time of flight, for $\vec{x}(t)$ to first return to S

- define the Poincaré map as

$$\vec{P}(\vec{x}_0) = \vec{x}(T(\vec{x}_0))$$

e.g. $n = 3$ (3D)



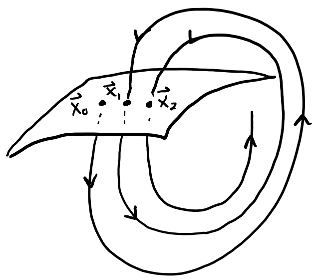
- letting $\vec{x}_0$ vary over S we get a mapping (i.e. function)

$$\vec{P} : S \to S$$

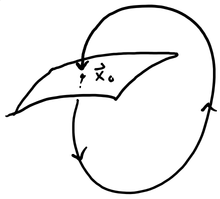
- form a sequence of points in S

$$\vec{x}_1 = \vec{P}(\vec{x}_0) \ , \ \vec{x}_2 = \vec{P}(\vec{x}_1) \ , \ \vec{x}_3 = \vec{P}(\vec{x}_2) \ , \ \ldots$$

$$\vec{x}_{k+1} = \vec{P}(\vec{x}_k) \ , \ k = 0, 1, 2, 3, \ldots$$



- suppose $\vec{P}(\vec{x}^*) = \vec{x}^*$ for some $\vec{x}^* \in S$, called a fixed point of $\vec{P}$



- any fixed point of $\vec{P}$ corresponds to a closed orbit of $\dot{\vec{x}} = \vec{f}(\vec{x})$
- note: $\vec{x}_1 = \vec{P}(\vec{x}_0)$, $\vec{x}_2 = \vec{P}(\vec{x}_1) = \vec{P}(\vec{P}(\vec{x}_0)) = \vec{P} \circ \vec{P}(\vec{x}_0) = \vec{P}^2(\vec{x}_0)$ doesn't mean squared the map is applied twice

  $\vec{x}_3 = \vec{P}(\vec{x}_2) = \vec{P}(\vec{P}(\vec{P}(\vec{x}_0))) = \vec{P} \circ \vec{P} \circ \vec{P}(\vec{x}_0) = \vec{P}^3(\vec{x}_0)$

  and $\vec{x}_i = \vec{P}^i(\vec{x}_0)$

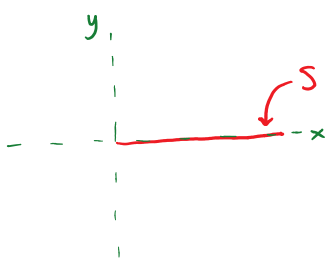
Example 2.14

$$\begin{cases} \dot{x} = x - y - x^3 - xy^2 \\ \dot{y} = x + y - x^2 y - y^3 \end{cases}, \quad (x,y) \in \mathbb{R}^2$$

- in polar coords

$$\begin{cases} \dot{r} = r - r^3 \\ \dot{\theta} = 1 \end{cases}, \quad (r,\theta) \in [0,\infty] \times \mathbb{S}^1$$

- let S be the positive x-axis



$$S = \{(x,y) \in \mathbb{R}^2 : y = 0, 0 < x < \infty\}$$
$$= \{(r,\theta) : \theta = 0 \ (\mathrm{mod}\ 2\pi), 0 < r < \infty\}$$

- let $\vec{x}_0 = (x_0, 0) \in S$ or in polar coords $r_0 = x_0 > 0$, $\theta_0 = 0 \ (\mathrm{mod}\ 2\pi)$
- now solve system

$$\begin{cases} \dot{r} = r - r^3 \\ \dot{\theta} = 1 \end{cases}$$

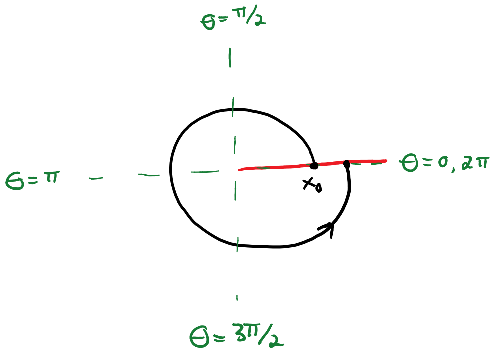
$r(0) = x_0$, $\theta(0) = 0 \ (\mathrm{mod}\ 2\pi)$

- explicit solution (Exercise: $\frac{dr}{dt} = r - r^3$, $\frac{dr}{r-r^3} = dt$, $\int \frac{dr}{r-r^3} = \int dt$, etc)

$$r(t) = r(t, x_0) = \frac{x_0}{\sqrt{(1-x_0^2)e^{-2t} + x_0^2}}$$

$$\theta(t) = t \ (\mathrm{mod}\ 2\pi)$$

- time of flight $T(x_0) = 2\pi$ (independent of x_0)



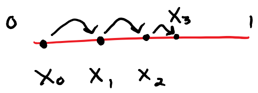
- Poincaré map is

$$P(x_0) = r(2\pi, x_0)$$

$$= \frac{x_0}{\sqrt{(1-x_0^2)e^{-4\pi} + x_0^2}}, \quad x_0 > 0$$

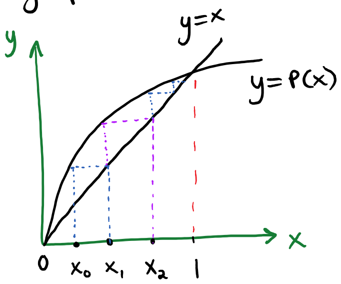
- observe $P(1) = 1$ so $x^* = 1$ is a fixed point

- iterates: $x_k = P^k(x_0)$ think of k as discrete time



- graphical method (staircase / cobweb diagram)

- plot graph



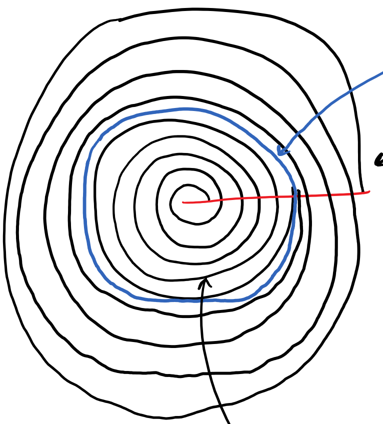
↑ move up to $P(x)$
move horizontal to $y=x$
move down to x-axis
repeat

- fixed points are solutions $x = x^*$ of $\underbrace{P(x) = x}$ — intersection of $y=x$ and $y=P(x)$

- graphically, we see $x_k \to 1$ as $k \to \infty$ for $0 < x_0 < 1$ and for $x_0 > 1$

- so we conclude (non rigourously) that $x^* = 1$ is a stable fixed point

- this stable fixed point corresponds to the stable limit cycle of the flow



III Chaos

One dimensional maps

- smooth function $f:\mathbb{R}\to\mathbb{R}$
- difference equation $x_{n+1}=f(x_n)$ (iterates)
- the term "map" can refer to the function f itself or its difference eqn.
- as seen above,

$$x_n = f^n(x_0) \quad \text{where} \quad f^n=\underbrace{f\circ f\circ f\ldots\circ f}_{n\text{ times}}(x_0),\ n \text{ iterates (compositions) of } f$$

- orbit (or trajectory) starting at x_0 is the sequence $\{x_0, x_1, x_2, x_3, \ldots\}$
- fixed points: solutions $x=x^*$ of $f(x)=x$ *this is important to remember*
  NOT $f(x)=0$

$\boxed{\text{Example 3.1}}$

linear map $f(x)=\lambda x$ i.e. $x_{n+1}=\lambda x_n$

- fixed points:

$$\lambda x = x \rightarrow \lambda x - x = 0 \rightarrow x(\lambda-1)=0$$

- $x^*=0$ is always a fixed point (i.e. for any λ)
- if $\lambda=1$, then any x is a fixed point ($x^*=0$ also applies)
- iterates:

$$x_1=\lambda x_0,\ x_2=\lambda(\lambda x_0)=\lambda^2 x_0,\ x_3=\lambda^3 x_0,\ \ldots$$

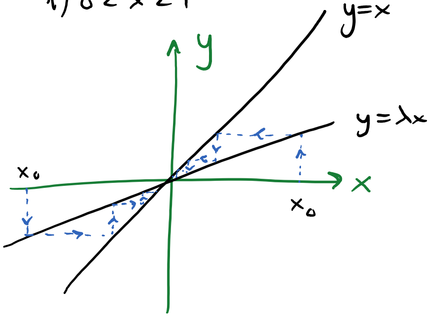
- λ is a multiplier
- analytically, we see that
  (a) If $|\lambda|<1$ then $x_n\to 0$ as $n\to\infty$
  (monotone if $\lambda>0$, alternating if $\lambda<0$)

(b) If $|\lambda| > 1$ then $|x_n| \to \infty$ as $n \to \infty$

- in (a), 0 is a stable fixed point
- in (b), 0 is an unstable fixed point
- graphically,

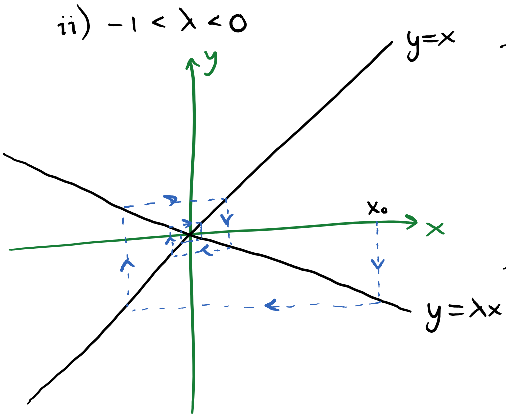
(a) $|\lambda| < 1$

i) $0 < \lambda < 1$



looks like a staircase

ii) $-1 < \lambda < 0$



looks like a cob web

Exercise: plot (b) $|\lambda| > 1$

i) $\lambda > 1$

ii) $\lambda < -1$