19 March - MATH 345

March 19, 2015 2:01 PM

- HW 5 due April 2 (Thursday 2 weeks from today)

Fixed Points and Linear Stability

$$x_{n+1} = f(x_n), \quad x_n \in \mathbb{R}^1$$

- suppose $x = x^*$ is a fixed point, so $f(x^*) = x^*$
- orbits near x^*: let $\eta = x - x^*$ be the perturbation from x^*

  i.e. $x = x^* + \eta$, $x_n = x^* + \eta_n$
- plug into difference equation

$$x^* + \eta_{n+1} = f(x^* + \eta_n)$$

- expand in Taylor series

$$x^* + \eta_{n+1} = \underbrace{f(x^*)}_{x^*} + \underbrace{f'(x^*)}_{\lambda}\eta_n + \underbrace{O(|\eta_n|^2)}_{\text{higher order terms}}$$

$$\eta_{n+1} = f'(x^*)\eta_n + O(|\eta_n|^2)$$

of the map at the fixed point

- if we neglect the $O(|\eta_n|^2)$ terms, we get the linearisation

$$\eta_{n+1} = \lambda \eta_n \quad \text{where} \quad \lambda = f'(x^*) \text{ is called the multiplier}$$

- x^* is hyperbolic if $|\lambda| \neq 1$

  (a) if $|\lambda| < 1$, x^* is an attractor and is stable

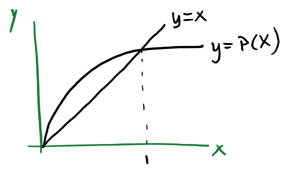
  (b) if $|\lambda| > 1$, x^* is a repeller and is unstable

  (c) if $\lambda = 0$, then x^* is called superstable

note: $\lambda = 0$ is still valid for our Taylor series expansion because f is a map NOT a differential equation

- if $|\lambda| = 1$ (i.e. $\lambda = \pm 1$), then x^* is nonhyperbolic, and for a nonlinear map, stabilisation cannot be determined by linearisation alone

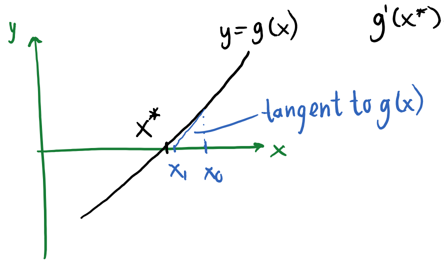
- e.g. Poincaré map of Example 2.14



Exercise: Evaluate $\lambda = P'(1)$ and thus verify $x^* = 1$ is a hyperbolic attractor and stable

Example 3.2

- Newton's method near a $\underbrace{\text{simple root}}_{g'(x^*)\neq 0}$ x^* of $g(x)$



1) go up from x_0
2) draw tangent line of $g(x_0)$ to x-axis → this is x_1
3) continue from x_1

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

i.e. $x_{n+1} = f(x_n)$ where $f(x) = x - \frac{g(x)}{g'(x)}$

- root of $g \iff$ fixed point of f

  $g(x) = 0 \iff f(x) = x$

- x^* is a fixed point of f
- linear stability

$$\lambda = f'(x^*) = 1 - \frac{[g'(x^*)]^2 - \overbrace{g(x^*)}^{=0}g''(x^*)}{[g'(x^*)]^2}$$

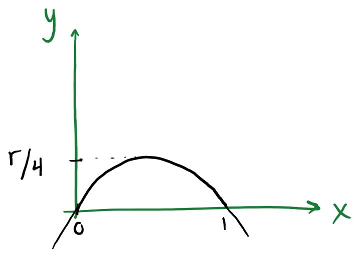
$$= 1 - 1 = 0$$

- the "Newton's method map" $f(x) = x - \frac{g(x)}{g'(x)}$ has a superstable fixed point at any simple root of g
- for sufficiently small η_0, perturbations go to 0 like $\eta_0 \approx C\eta_0^{2^n}$ which is much faster than $\lambda^n \eta_0$ for $0 < |\lambda| < 1$

Logistic map: analytics (1)

- discrete time analogue of logistic ODE model

$$x_{n+1} = \underbrace{r x_n (1 - x_n)}_{f(x_n, r) \text{ or } f(x_n) \text{ for brevity}}, \quad \text{where } r > 0$$



· consider only $0 \le x_n \le 1$, $0 < r \le 4$ so that $0 \le x_{n+1} = f(x_n) \le 1$

· all iterates remain in $[0,1]$ (a positively invariant set; discrete time trapping region)

· find fixed points

$$rx(1-x) = x \qquad \text{since } f(x^*) = x^*$$

$$rx - rx^2 - x = 0$$

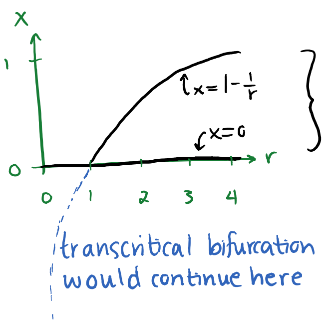
$$x(r - rx - 1) = 0$$

$$x = 0$$

$$r - rx - 1 = 0$$
$$r(1-x) = 1$$
$$x = 1 - \frac{1}{r}$$

· $x^* = 0$, $x^* = 1 - \frac{1}{r}$ (if $r \ge 1$ to keep x^* positive)

· plot of $Z = \{(r,x) \in (0,4] \times [0,1] : f(x,r) = x\} = \underbrace{\{x=0\} \cup \{x = 1 - \frac{1}{r}\}}_{\text{union of sets}}$



plot of Z, looks like transcritical bifurcation

· from picture of Z, we have a transcritical bifurcation at

$$x^*_{c1} = 0, \quad r_{c1} = 1$$

· linear stability

$$\frac{\partial f}{\partial x}(x,r) = f'(x) = r - 2rx$$

· at $x^*=0$, $\lambda=f'(0)=r$

$0<r<1$, $x^*=0$ is a hyperbolic attractor, stable

$r=1$, $x^*=0$ is a nonhyperbolic, with multiplier 1

$1<r\leq 4$, $x^*=0$ is a hyperbolic repeller, unstable

· since there is a stability change at $r=1$, expect a bifurcation

· at $x^*=1-\frac{1}{r}$, $\lambda=f'(1-\frac{1}{r})=r-2r(1-\frac{1}{r})=2-r$ $(1\leq r\leq 4)$

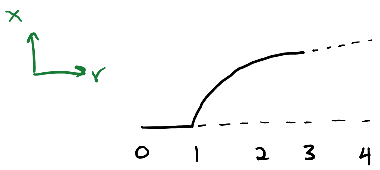
$r=1$, $x^*=1-\frac{1}{r}=0$ is nonhyperbolic, with multiplier 1 (same fixed point)

$1<r<3$, $x^*=1-\frac{1}{r}$ is hyperbolic and stable $(-1<\lambda<1)$, superstable when $r=2$

$r=3$, $x^*=1-\frac{1}{r}=\frac{2}{3}$ is nonhyperbolic, with multiplier -1 $(\lambda=2-3=-1)$

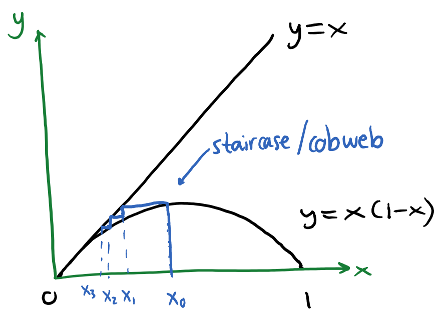
$3<r\leq 4$, $x^*=1-\frac{1}{r}$ is hyperbolic and unstable $(\lambda=2-r\rightarrow|\lambda|>1)$

· stability change at $r_{c_2}=3$, expect bifurcation

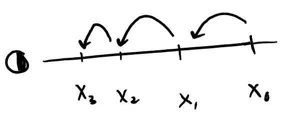


Exercise: Draw staircase/cobweb diagrams and phase portraits for various r

e.g. $r=1$



phase portrait



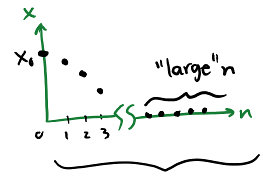
non hyperbolic, stable (from right

Logistic map: numerics (1)

· expect bifurcations at $r_{c_2}=3$

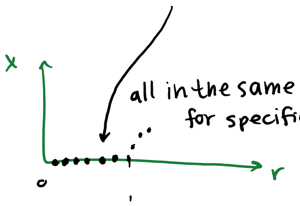
· numerical experiments (XPP) "Numerical bifurcation diagram"

· choose r-value, choose x_0, compute iterates until iterates "settle down", any transients mostly finished choose $0 < r < 1$



plot x_n for large n only on

x vs. r



all in the same spot (very close)

for specific r