27 January - MATH 345

January 27, 2015 2:00 PM

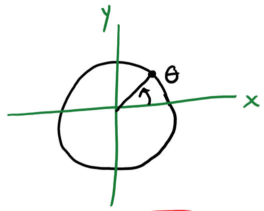
- HW2 due February 5

Flows on the Circle

$\dot{\theta} = f(\theta)$, $\theta \in S^1$ where $f(\theta + 2\pi) = f(\theta)$ for all θ

note: $\dot{\theta}, f(\theta) \in \mathbb{R}^1$ even if $\theta \in S^1$

convention:

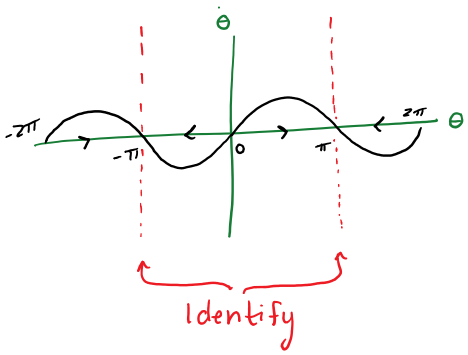


θ in radians is usually measured counter clockwise from positive x-axis (unless application specifies otherwise)

$\boxed{\text{Example 1.8}}$

$\dot{\theta} = \sin\theta$, $\theta \in S^1$

$\dot{\theta}$ vs. θ representation in $\mathbb{R}^2$



- thinking of θ in $\mathbb{R}^1$, there are infinitely many fixed points

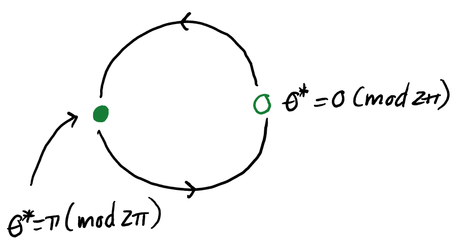
  $\theta^* = 0, \pm 2\pi, \pm 4\pi, \ldots$

  $\pm\pi, \pm 3\pi, \ldots$

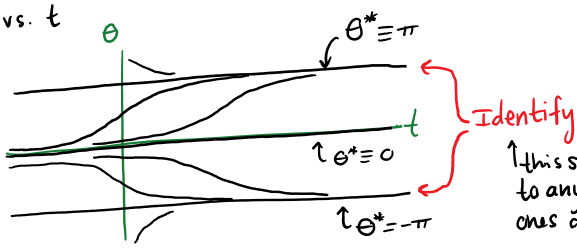
- for $\theta \in S^1$, there are only two fixed points

  $\theta^* = \underbrace{0 \pmod{2\pi}}_{\text{up to an integer multiple of } 2\pi}$, $\theta^* = \pi \pmod{2\pi}$

(Global) phase portrait in S^1



θ vs. t



↑this should be added to any plots including ones done in XPPAUT

- any θ-interval of length 2π could be used
  e.g. $(-\pi,\pi]$, $[-\pi,\pi)$, $[0,2\pi)$, ...

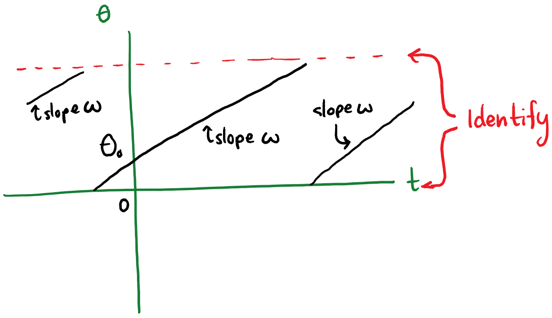
$\boxed{\text{Example 1.9}}$

$\dot{\theta}=\omega$, $\theta\in S^1$ where $\omega>0$ is given

explicit solution

$\theta(t)=\omega t+\theta_0$

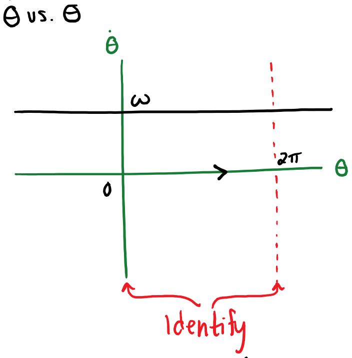
interpreted as follows



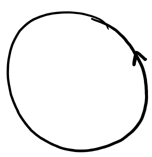
we should actually write

$\theta(t)=\omega t+\theta_0 \pmod{2\pi}$

$\dot{\theta}$ vs. θ



phase portrait in S^1

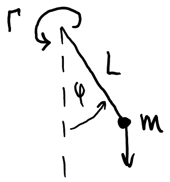


Exercise Plot $\frac{d\varphi}{d\tau}$ vs φ, sketch global phase portraits in S^1, sketch φ vs τ for

$$\frac{d\varphi}{d\tau}=\gamma\sin\varphi\cos\varphi-\sin\varphi$$

if i) $0<\gamma<1$ ii) $\gamma=1$ iii) $\gamma>1$

Over damped pendulum (with steady applied torque)



no forcing, no damping

$$mL\frac{d^2\varphi}{dt^2} = -mg\sin\varphi \quad \text{(units of force)}$$

force x distance

$$mL^2\frac{d^2\varphi}{dt^2} = -mgL\sin\varphi \quad \text{(units of torque)}$$

with constant (any value) applied torque $\Gamma \geq 0$ in direction of increasing φ and viscous damping

$$mL^2\frac{d^2\varphi}{dt^2} = -mgL\sin\varphi + \Gamma - b\frac{d\varphi}{dt}, \quad \varphi \in S^1 \quad \text{(units of torque)}$$

Exercise: use dimensional analysis and scaling to get

$$\varepsilon\frac{d^2\varphi}{d\tau^2} = -\sin\varphi + \gamma - \frac{d\varphi}{d\tau}, \quad \varphi \in S^1 \quad \text{(no units)}$$

what are ε and γ in terms of original parameters?

practice for midterm!!!

$0<\varepsilon<<1$ if damping is "large"

over damped: take $\varepsilon=0$

$$\frac{d\varphi}{d\tau} = \underbrace{\gamma - \sin\varphi}_{f(\varphi,\gamma)}, \quad \varphi \in S^1 \quad (\gamma \geq 0)$$

note: $\frac{\partial f}{\partial\varphi}(\varphi,\gamma) = -\cos\varphi$

- there is a bifurcation, what kind?
- should have some intuition
- pitchfork: odd symmetry? no
- transcritical: always a fixed point for any γ? no
- saddle node is left

fixed points: $0=\gamma-\sin\varphi$

$$\sin\varphi = \gamma$$

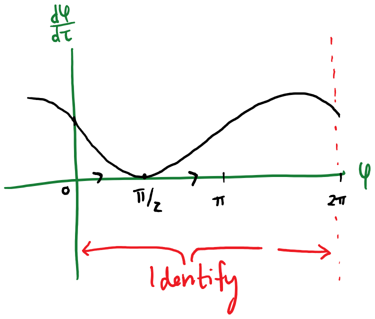
no solutions if $\gamma>1$

let $\gamma_c=1$. when $\gamma=\gamma_c=1$

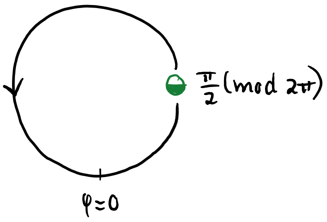
$$\sin\varphi = 1 \Rightarrow \varphi^* = \pi/2 \pmod{2\pi}$$

only one fixed point for $\gamma=\gamma_c=1$

$\frac{d\varphi}{d\tau}$ vs φ when $\gamma=1$



phase portrait in S^1 when $\gamma=1$



$f(\pi/2, 1) = 1 - \sin(\pi/2) = 0$ ✓

$\frac{\partial f}{\partial \varphi}(\frac{\pi}{2}, 1) = -\cos(\pi/2) = 0$ ✓

$\frac{\partial f}{\partial \gamma}(\frac{\pi}{2}, 1) = 1 \neq 0$ ✓

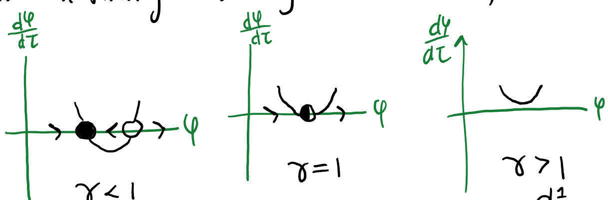
$\frac{1}{2}\frac{\partial^2 f}{\partial \varphi^2}(\pi/2, 1) = \frac{\sin(\pi/2)}{2} = \frac{1}{2} \neq 0$ ✓

checking these 4 conditions verifies there exists a saddle node bifurcation at $\gamma_c = 1$, $\varphi_c^* = \pi/2$

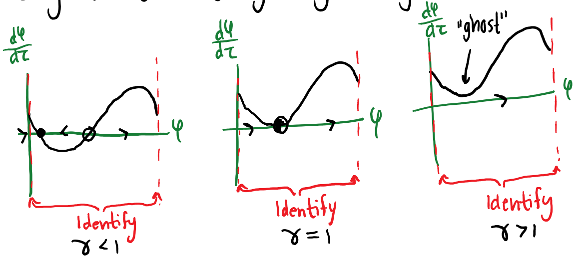
normal form:

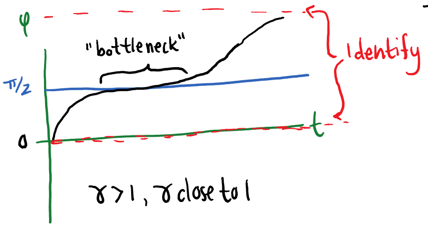
$$\frac{d\varphi}{d\tau} = \underbrace{(\gamma - 1) + \frac{1}{2}(\varphi - \pi/2)^2}_{\text{Taylor polynomial approximation valid for } \varphi \text{ near } \pi/2,\ \gamma \text{ near } 1}$$

· normal form gives local dynamics near $\varphi = \frac{\pi}{2}, \gamma = 1$



· original vector field gives global dynamics, $\varphi \in S^1$





slows down close to $\varphi = \pi/2$
close to being a fixed point
("ghost" of fixed point)

2. Two-dimensional Flows

Linear Systems

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned} \quad \text{OR} \quad \begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} \quad \text{OR} \quad \dot{\vec{x}} = A\vec{x}$$

$\uparrow \begin{pmatrix} a & b \\ c & d\end{pmatrix}$