13 January - MATH 345

January 13, 2015 1:57 PM

- homework 1 due Thursday, 22 January
- midterm Tuesday, 24 February
- get XPPAUT onto computer
- files are on website for homework and installing XPPAUT
- can get account on lab computers in LSK building; can't access remotely
- lab time reserved for this class wednesdays 3-4 PM; class has priority

Linear Stability Analysis

for $\dot{x} = f(x)$, $x \in \mathbb{R}^1$

- calculate derivative $\lambda = f'(x^*) = \frac{df}{dx}(x^*)$ at fixed point x^*

$\lambda > 0$, x^* is unstable

$\lambda < 0$, x^* is stable

e.g. $\dot{x} = x(1-x) = x - x^2 = f(x)$

$\frac{df}{dx}(x) = 1 - 2x$

fixed points: $0 = x(1-x) \rightarrow x^* = 0, 1$

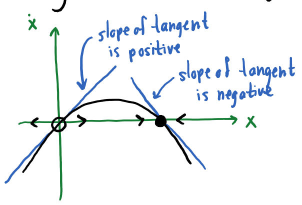
$x^* = 0$: $\lambda = \frac{df}{dx}(0) = 1 > 0$

$x^* = 0$ is unstable

$x^* = 1$: $\lambda = \frac{df}{dx}(1) = -1 < 0$

$x^* = 1$ is stable

- looking at the example geometrically



- if $\lambda = f'(x^*) \neq 0$ then the fixed point x^* is called hyperbolic and the sign of λ determines stability
- if $\lambda = 0$ then x^* is nonhyperbolic and linear stability analysis fails; the linearisation $\dot{\eta} = \lambda \eta$ cannot be used to determine stability of x^*

Exercise

(a) Find fixed points, do linear stability analysis for

i) $\dot{x} = x^2$, ii) $\dot{x} = x^3$, iii) $\dot{x} = -x^3$, iv) $\dot{x} = 0$

(b) sketch phase portraits, determine stability

Existence and Uniqueness

Theorem (p.27 in textbook)

If f and f' are both continuous on an open interval R of x-values and if $x_0 \in R$, then the initial value problem

this R is NOT THE SAME as $\mathbb{R}$

$$\dot{x} = f(x), \quad x(0) = x_0$$

has a (differentiable) solution $x(t)$ defined for all t belonging to some maximal open interval that includes $t=0$ and this solution is unique

- notice that this theorem doesn't guarantee $x(t)$ exists for all $t \in \mathbb{R}$

e.g. $\dot{x} = x(1-x)$, $x(0) = x_0$ where $x_0 < 0$ or $x_0 > 1$

Solving Equations on a Computer

- pre-requisite: simple explicit methods Euler's method, "improved" Euler's method, fourth-order Runge-Kutta method (RK4)
- default method used by XPPAUT is RK4
- default step size is fine for homework 1, but not always appropriate
- adaptive methods: user provides accuracy requirement, program chooses step size
- adaptive methods take longer to run
- XPPAUT can use Gear's method, an adaptive method (to be used in later homework)

Bifurcations

- parameter-dependent vector field

$$\dot{x} = f(x,r), \quad \underset{\substack{\uparrow \\ \text{phase variable}}}{x \in \mathbb{R}}, \quad \underset{\substack{\uparrow \\ \text{control parameter}}}{r \in \mathbb{R}}$$

control parameter } constant

e.g. mass or spring constant as control parameter

- for each r, get phase portrait
- for special, critical parameter values r_c, called bifurcation points, the phase portraits qualitatively change
- these qualitative changes are called bifurcations

Saddle-node Bifurcations

$\boxed{\text{Example 1.3}}$ $\dot{x} = r + x^2$

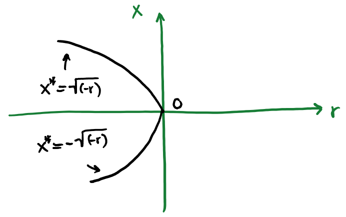
fixed points: $0 = r + x^2$

$x^2 = -r$

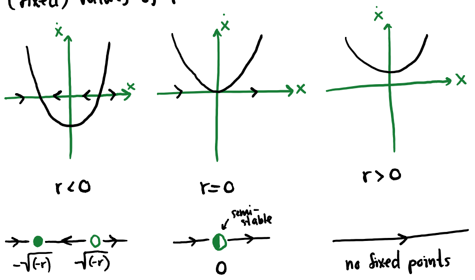
$x^* = \pm\sqrt{(-r)}$, only for $r \le 0$

i.e. if $r > 0$, there are no fixed points

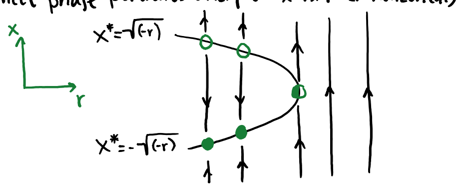
Plot of $Z = \{(r,x) : r + x^2 = 0\}$



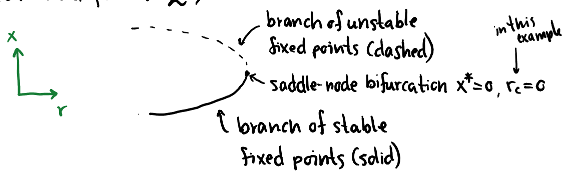
phase portraits (and $\dot{x}$ vs. x) for different (fixed) values of r



- bifurcation point $r_c = 0$ (in this example)
- this type of bifurcation is called a saddle-node bifurcation
- collect phase portraits and plot x vs. r (r horizontal)



- summarise with bifurcation diagram (enhanced plot of $\mathcal{Z}$)



- other names: fold bifurcation, limit point, turning point
- at the bifurcation point $r_c=0$, the fixed point is nonhyperbolic – we can find bifurcations by looking for nonhyperbolic fixed points

Normal Forms

$\dot{x} = f(x, r)$, $x \in \mathbb{R}$, $r \in \mathbb{R}$

- how to verify if a saddle-node bifurcation occurs: first, find a fixed point x^* that is nonhyperbolic at r_c

(SN1) $f(x^*, r_c) = 0$, $x=x^*$ is a fixed point when $r=r_c$

(SN2) $\dfrac{\partial f}{\partial x}(x^*, r_c) = 0$, x^* is nonhyperbolic when $r=r_c$