26 March - MATH 345

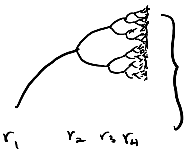
March 26, 2015 2:01 PM

· HW due April 2

Logistic map

$$x_{n+1} = r x_n (1-x_n), \quad x_n \in [0,1], \quad r \in (0,4]$$

· numerics: seems to be infinite sequence $\{r_m\}_{m=1}^{\infty}$ of bifurcation values with a stable 2^{m-1}-cycle for $r_m < r < r_{m+1}$



r_m appear to converge to a limit $r_\infty = 3.569946$ as $m \to \infty$

· seems to converge geometrically

$$\lim_{m\to\infty} \frac{r_m - r_{m-1}}{r_{m+1} - r_m} = \delta \approx 4.699$$

· this implies $|r_m - r_\infty| \sim C\left(\frac{1}{\delta}\right)^m$ for some constant C

· this is called a period doubling cascade; its existence is confirmed analytically

· for some, but not all $r_\infty < r \leq 4$, there appear to be a strange or chaotic attractor: iterates seem "random" even after waiting for transients to decay

· picture in text p. 364

· also XPP files on course website (under "Supplementary Material")

Chaotic attractors

· an attractor for a map $x_{n+1} = f(x_n)$ is a closed and bounded set A (e.g. fixed point, cycles, more complicated set) such that

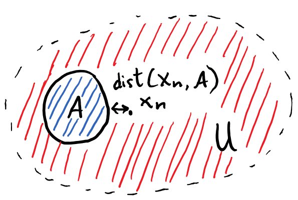
(1) A is invariant: if $x \in A$ then $f(x) \in A$

(2) A attracts an open set of initial values:

there is an open set U containing A such that

$x_0 \in U$ then $\operatorname{dist}(x_n, A) = \min_{x \in A} |x_n - x| \to 0$

e.g.



The largest such U is the basin of attraction of A

(3) A is minimal: no proper closed subset of A satisfies properties (1) and (2)

e.g. A = two stable fixed points

satisfies (1) and (2) but not (3)

- For the logistic map

i) if $1 < r \le 3$ then $A = \{x^*\}$, $x^* = 1 - \frac{1}{r}$ is an attractor,

basin of attraction is $(0,1)$

ii) if $3 < r \le 1+\sqrt{6}$ then $A = \{p, q\}$ is an attractor. What is the basin of attraction?

$\{p, x^*, q\}$ satisfies (1) and (2) but not (3)

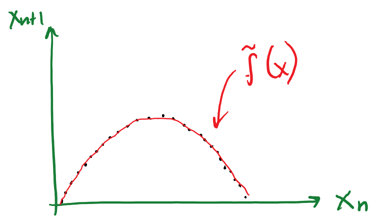
- an attractor is strange or chaotic if it exhibits (or has)

sensitive dependence on initial conditions (SDIC): any two initial values $x_0, \hat{x}_0$ arbitrarily close to each other generate orbits $\{x_n\}$, $\{\hat{x}_n\}$ that eventually diverge

Ruelle plot

- given an orbit $\{x_n\} = \{x_0, x_1, x_2, \dots\}$ on a chaotic attractor, how can we find the map $x_{n+1} = f(x_n)$ that generated it?

- simple trick: plot x_{n+1} vs x_n. Each point $(x,y) = (x_n, x_{n+1})$ lies on the curve $y = f(x)$. If the orbit $\{x_n\}$ samples a wide variety of points in the domain of f, then can fit a curve $y = \tilde{f}(x)$ where $\tilde{f}$ approximates f



Lyapunov exponent

- a way to check for SDIC, works for exponentially diverging orbits

$x_{n+1} = f(x_n)$

- take two nearby initial values x_0, $\hat{x}_0 = x_0 + \delta_0$
  $\uparrow$ $|\delta_0|$ is small
- corresponding orbits $\{x_n\}$, $\{\hat{x}_n\}$. Define $\delta_n = \hat{x}_n - x_n$
- suppose $|\delta_n| = |\delta_0| e^{\lambda n}$ orbits diverge (or converge) at exponential rate
- then $\lambda = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right|$. If $\lambda > 0$ then orbits diverge and this is SDIC
- more generally, do not assume $|\delta_n| = |\delta_0| e^{\lambda n}$
- in general:

$$\delta_n = \hat{x}_n - x_n = f^n(\hat{x}_0) - f^n(x_0)$$

$$= f^n(x_0 + \delta_0) - f^n(x_0)$$

Taylor series expand

$$= (f^n)'(x_0)\,\delta_0 + O(|\delta_0|^2)$$

$$\left| \frac{\delta_n}{\delta_0} \right| = |(f^n)'(x_0)| + O(|\delta_0|)$$

Exercise: show $(f^n)'(x_0) = f'(x_{n-1}) \cdots f'(x_1) f'(x_0)$ (by chain rule)

- then

product

$$\left| \frac{\delta_n}{\delta_0} \right| = \prod_{i=0}^{n-1} |f'(x_i)| + O(|\delta_0|)$$

$$\frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| + \frac{1}{n} O(|\delta_0|)$$

- the Lyapunov exponent of x_0 (or of its orbit $\{x_0, x_1, \cdots\}$) is

$$\lambda = \lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \quad \text{provided this limit exists}$$

- compute approximation to $\lambda(x_0)$ by taking large n
- in many situations it can be proved that $\lambda(x_0)$ is independent of x_0
  e.g. if x_0 is in the basin of attraction or some attractor

$\boxed{\text{Example 3.3}}$

$x_{n+1} = f(x_n)$ x^* is a hyperbolic attracting fixed point $|f'(x^*)| < 1$

$\ln|f'(x^*)| < 0$

· if x_0 is in the basin of attraction of $\{x^*\}$, then $\lim_{n\to\infty} x_n = x^*$

$$\frac{1}{n}\sum_{i=0}^{n-1} \ln|f'(x_i)| = \frac{1}{n}\Big[\underbrace{\ln|f'(x_0)| + \cdots + \ln|f'(x_{N-1})|}_{\text{finite number}} + \underbrace{\ln|f'(x_N)| + \cdots + \ln|f'(x_{n-1})|}_{\text{for large } N,\ f'(x_i) \approx f'(x^*) \text{ for } i \geq N}\Big]$$

$$= \underbrace{\frac{\text{finite number}}{n}}_{\to 0} + \underbrace{\frac{n-N}{n}}_{\to 1} \underbrace{\ln|f'(x^*)|}_{\text{approx.}}$$

as $n \to \infty$

$$\to \underbrace{\ln|f'(x^*)|}_{\text{independent of } x_0} \text{ as } n\to\infty$$