31 March - MATH 345

March 31, 2015 2:00 PM

- HW 5 due April 2

- Lyapunov exponent $\lambda(x_0)$ measures eventual, average exponential convergence or divergence of orbits starting infinitesimaly close to x_0
- if $\lambda(x_0) > 0$, then there is SDIC (sensitive dependence on initial condition)
- if $\lambda(x_0) = 0$, can't say from only the Lyapunov exponent

- remark: in N dimensions and also for ODEs, there are N Lyapunov exponents
- if the largest, or maximal, Lyapunov is positive, then there is SDIC

i.e. try out other ICs or parameters on the HW
XPPAUT only gives an approximation

$\boxed{\text{Example 3.4}}$ Doubling map, or binary shift map

$$f : [0,1] \to [0,1], \quad x_{n+1} = f(x_n)$$

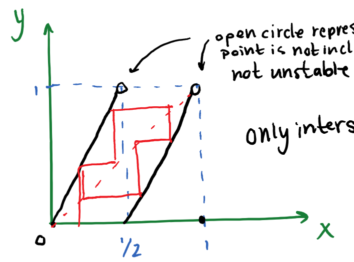
$$f(x) = 2x \pmod 1$$

e.g. $2(\frac{1}{4}) \pmod 1 = \frac{1}{2}$

$2(\frac{1}{2}) \pmod 1 = 0$

$2(\frac{3}{4}) \pmod 1 = \frac{1}{2}$

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2x-1 & \text{if } 1/2 \le x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$



open circle represents point is not included

not unstable fixed points

only intersects $y=x$ at $x=0$

- fixed points: $x^* = 0$
- Lyapunov exponent: $\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)|$
- for a "typical" orbit $\{x_0, x_1, x_2, \dots\}$ (for which $x_n \ne 0, \frac{1}{2}, 1$ for all n) then $\lambda(x_0)$ can be computed explicitly:

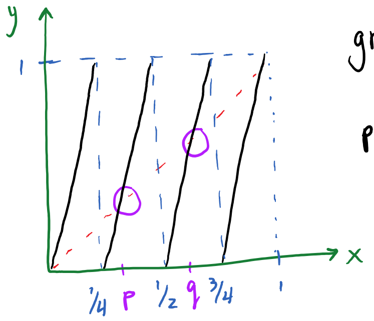
$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)|$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln 2$$

$$= \lim_{n\to\infty} \frac{1}{n} n \ln 2$$

$$= \ln 2 > 0 \quad , \text{SDIC}$$

- cycles of period 2: $f^2(x) = f(f(x)) = x$
- if $0 \le x < \frac{1}{4}$, then $f(x) = 2x$, $0 < 2x < \frac{1}{2}$, $f(f(x)) = 2(2x) = 4x$
- if $\frac{1}{4} \le x < \frac{1}{2}$, then $f(x) = 2x$, $\frac{1}{2} \le 2x < 1$, $f(f(x)) = 2(2x) - 1 = 4x - 1$
- if $\frac{1}{2} \le x < \frac{3}{4}$, then $f(x) = 2x - 1$, $0 \le 2x - 1 < \frac{1}{2}$, $f(f(x)) = 2(2x-1) = 4x - 2$
- if $\frac{3}{4} \le x < 1$, then $f(x) = 2x - 1$, $\frac{1}{2} \le 2x - 1 < 1$, $f(f(x)) = 2(2x-1) - 1 = 4x - 3$
- if $x = 1$, $f(f(x)) = 0$



graph suggests two "new" fixed points of f^2

$p \in [\frac{1}{4}, \frac{1}{2})$, $q \in [\frac{1}{2}, \frac{3}{4})$

$$f^2(x) = x \quad , \quad \frac{1}{4} \le x < \frac{1}{2}$$

$$4x - 1 = x \rightarrow x = \frac{1}{3} \quad (\in [\tfrac{1}{4}, \tfrac{1}{2}))$$

$$p = \frac{1}{3}$$

$$f^2(x) = x \quad , \quad \frac{1}{2} \le x < \frac{3}{4}$$

$$4x - 2 = x \rightarrow x = \frac{2}{3} \ (\in [\tfrac{1}{2}, \tfrac{3}{4}))$$

$$q = \frac{2}{3}$$

- $p = \frac{1}{3}$, $q = \frac{2}{3}$ are "new" fixed points of f^2
- $\{p, q\} = \{\frac{1}{3}, \frac{2}{3}\}$ is a 2-cycle for f
- check: $f(\frac{1}{3}) = 2(\frac{1}{3}) = \frac{2}{3}$

  $f(\frac{2}{3}) = 2(\frac{2}{3}) - 1 = \frac{1}{3}$

· linear stability:

$$(f^2)'(p) = f'(p)\, f'(q)$$
$$= f'(\tfrac{1}{3})\, f'(\tfrac{2}{3})$$
$$= 2 \cdot 2 = 4 > 1$$

the 2-cycle is unstable

Exercise: What is f^3? Does it have fixed points? What are they? What do they represent for $x_{n+1} = f(x_n)$?

· to study cycles more easily, consider binary expansion of $x \in [0,1)$

$$\left.\begin{aligned} x &= 0.b_1 b_2 b_3 \cdots \\ &= \frac{b_1}{2^1} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \cdots \end{aligned}\right\} \quad b_i = 0 \text{ or } 1 \quad i = 1, 2, 3, \ldots$$

binary expansion because this is 2

· what is binary expansion of $f(x) = 2x \pmod 1$?

i) If $0 \le x < \frac{1}{2}$ then $b_1 = 0$ and $x = 0.0 b_2 b_3 b_4 \ldots$

$$x = \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + \cdots$$

$$f(x) = 2x = 2\left(\frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + \cdots\right)$$
$$= \frac{b_2}{2^1} + \frac{b_3}{2^2} + \frac{b_4}{2^3} + \cdots$$
$$= 0.b_2 b_3 b_4 \ldots$$

ii) If $\frac{1}{2} \le x < 1$ then $b_1 = 1$ and $x = 0.1 b_2 b_3 b_4 \cdots$

$$= \frac{1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + \cdots$$

$$f(x) = 2x - 1 = 2\left(\frac{1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + \cdots\right) - 1$$
$$= \frac{b_2}{2^1} + \frac{b_3}{2^2} + \frac{b_4}{2^3} + \cdots$$
$$= 0.b_2 b_3 b_4 \ldots$$

· in either case ($b_1 = 0$ or 1)

$$f(0.b_1b_2b_3{}_4\dots) = 0.b_2b_3b_4\dots$$

- the "binary point" is shifted one position to the right
- easy way to identify cycles

fixed point: $x^* = 0.0000\dots$

2-cycles: $p = 0.010101\dots = 0.\overline{01}$

$q = 0.101010\dots = 0.\overline{10}$

Exercise: show $\frac{1}{3} = 0.\overline{01}$, $\frac{2}{3} = 0.\overline{10}$ in binary

3-cycles: e.g. $r = 0.\overline{101}$ belongs to a 3 cycle

- f has cycles of any length N Exercise: prove that they are all hyperbolic and unstable $(f^N)'(p_0)$ & use chain rule

- periodic points (points that belong to some cycle) $\longleftrightarrow$ rational #'s in $(0,1)$ that do not have denominator 2^N } countably infinite

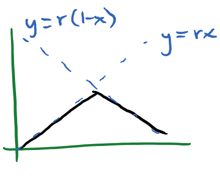
- these can be shown to be dense in $[0,1]$

(for any $x \in [0,1]$ there is an arbitrarily close periodic point)

- non periodic points $\longleftrightarrow$ irrational #'s in $[0,1]$ } uncountability infinite

Tent map

$f: [0,1] \longrightarrow [0,1]$, $r > 0$

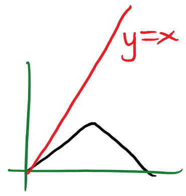


$f(x) = r \min\{x, 1-x\}$

$$f(x) = \begin{cases} rx & \text{if } 0 \le x \le \frac{1}{2} \\ r(1-x) & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

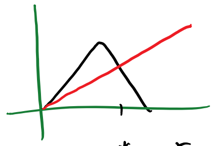
· called "tent map of slope r"

· if $0 < r < 1$ there is one fixed point



$x^* = 0$

· if $r > 1$ there are two fixed points



$x^* = \frac{r}{1+r}$