3 March - MATH 345

March 3, 2015 2:03 PM

- HW 4 due March 12

Ruling out closed orbits (cont.)

(a) cont. Gradient system $\dot{\vec{x}} = -\nabla V(\vec{x})$

fixed points ($\dot{\vec{x}}=0$) are critical points of V

($\nabla V(\vec{x})=0$)

along any trajectory $\vec{x}(t)$:

$$\dot{V} = \frac{d}{dt} V(\vec{x}(t)) = \nabla V(\vec{x}(t)) \cdot \dot{\vec{x}}(t) = \nabla V(\vec{x}(t)) \cdot [-\nabla V(\vec{x}(t))]$$

$$\dot{V} = -\|\nabla V(\vec{x}(t))\|^2 \leq 0 \text{ for all } \vec{x}(t)$$

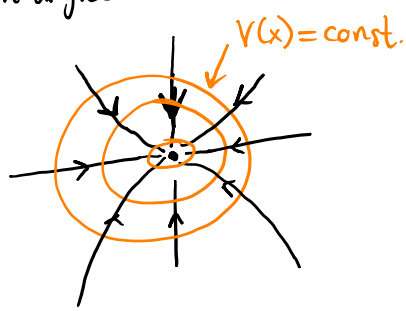
$$< 0 \text{ except at fixed points}$$

V decreases along all trajectories except at fixed points

there are no closed orbits

trajectories of gradient systems cross level sets of V at right angles

e.g.



(b) more general than gradient systems:

sometimes there is a Lyapunov function:

a C^1 scalar function $V(\vec{x})$ such that

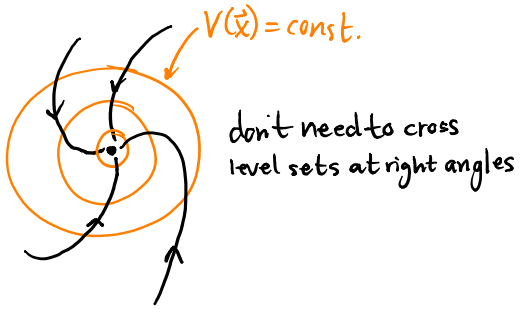
i) V decreases along all trajectories (at least in some region) except at fixed points, and

ii) a fixed point $\vec{x}^*$ is an isolated local minimum of V

then there are no closed orbits entirely in the region where i) holds

also, if i) holds in a region containing $\vec{x}^*$ and ii) holds, then $\vec{x}^*$ is (asymptotically) stable

e.g.



(c) Sometimes, Dulac's or Bendixson's criterion can be used:

Suppose $\dot{\vec{x}} = \vec{f}(\vec{x})$ is C^1 on an open simply-connected region R in $\mathbb{R}^2$
↑ i.e. no holes

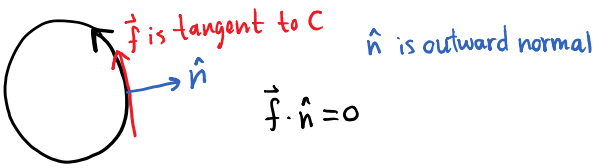
If there is a C^1 scalar function $g(\vec{x})$ such that

$$\nabla \cdot (g\vec{f})$$

is not identically zero and does not change sign throughout R, then there are no closed orbits lying entirely in R.

(the case $g(\vec{x}) \equiv 1$ is called Bendixson's criterion; the case for general g is called Dulac's criterion)

Proof: If there was a closed orbit C lying entirely in R



$$\vec{f} \cdot \hat{n} = 0$$

By Green's theorem

$$\iint_A \underbrace{\nabla \cdot (g\vec{f})}_{\neq 0} \, dx\,dy = \oint_C g \underbrace{\vec{f} \cdot \hat{n}}_{=0} \, dl$$

This is a contradiction, therefore there can not be a closed orbit entirely in R

$\boxed{\text{Example 2.7}}$

$$\begin{cases} \dot{x} = 3x - x^2 - 2xy \\ \dot{y} = 2y - y^2 - xy \end{cases}$$

let $g(\vec{x}) = g(x,y) = \frac{1}{xy}$ then,

$$\nabla \cdot (g\vec{f}) = \frac{\partial}{\partial x}\left[\frac{1}{xy}(3x - x^2 - 2xy)\right] + \frac{\partial}{\partial y}\left[\frac{1}{xy}(2y - y^2 - xy)\right]$$

$$= -\frac{1}{y} - \frac{1}{x}$$ } this does not change sign in $Q = \{(x,y) : 0 < x < \infty, 0 < y < \infty\}$

check this

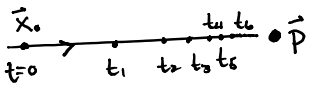
by Dulac's criterion, there are no closed orbits entirely in Q

<u>Remark</u>: (a), (b), (c) don't always work (can't find a suitable V or g)
- don't always get conclusive results

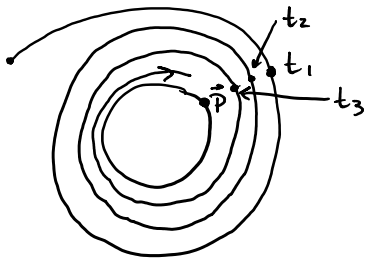
Poincaré-Bendixson Theorem

- a way to prove the existence of closed orbits (only valid for 2D flows)
- recall definition of ω-limit point for an initial value $\vec{x}_0$ (see January 6)

e.g.



e.g.



- the <u>omega-limit set</u> (or ω-limit set) $\omega(\vec{x}_0)$ of $\vec{x}_0$ is the set of all ω-limit points of $\vec{p}$

- <u>Theorem</u> (Poincaré-Bendixson): Suppose

$\bar{R}$ — the line means closed and bounded

(1) $\bar{R}$ is a closed and bounded subset of $\mathbb{R}^2$
(2) $\dot{\vec{x}} = \vec{f}(\vec{x})$ is C^1 on an open subset of $\mathbb{R}^2$ that contains $\bar{R}$
(3) there is a forward trajectory $C = \{\vec{x}(t) : 0 \le t < \infty\}$ contained in $\bar{R}$, with $\vec{x}(0) = \vec{x}_0$

Then, $\omega(\vec{x}_0)$ either contains a fixed point or is a closed orbit

- to apply this theorem (PBT), find a closed and bounded <u>trapping</u> (or <u>positively invariant</u>) region $\bar{R}$ that contains no fixed points

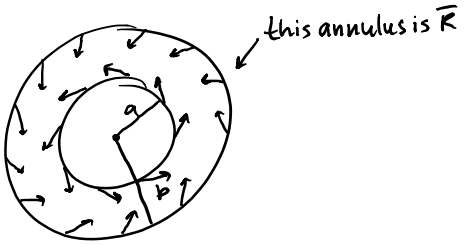
- $\bar{R}$ is trapping (or forward invariant) if every forward trajectory $\vec{x}(t)$, $0 \le t < \infty$, that starts in $\bar{R}$ at $t=0$ remains in $\bar{R}$ for all $0 < t < \infty$ (i.e. if $\vec{x}_0 = \vec{x}(0) \in \bar{R}$ then $\vec{x}(t) \in \bar{R}$ for all $t > 0$)
- we verify $\bar{R}$ is trapping by examining $\vec{f}$ on the boundaries of $\bar{R}$
- if $\bar{R}$ is a closed and bounded trapping region that contains no fixed points, then by PBT there is a (at least one) closed orbit

Example 2.8

In polar coordinates in the plane

$$\begin{cases} \dot{r} = r - r^3 + \varepsilon r^2 \cos\theta, & 0 < \varepsilon < 1 \\ \dot{\theta} = 1 \end{cases}$$

Exercise: What is this system in rectangular x,y coordinates?

When $\varepsilon = 0$, there is a closed orbit $r^* = 1$ (Example 2.6 Fcb.26)

Show that for all $0 < \varepsilon < 1$, a closed orbit exists