8 January - MATH 345

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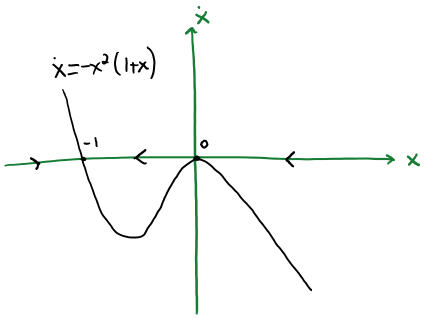
Fixed points and Stability

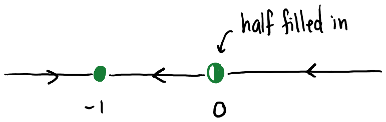
$\dot{x} = f(x)$, $x \in \mathbb{R}$ or an interval

- a fixed point is a solution $x = x^*$ of $f(x) = 0$
- a fixed point x^* is (asymptotically) stable if all sufficiently small perturbations from x^* give solutions $x(t)$ that stay arbitrarily close to x^* for all $t \geq 0$ and approach x^* as $t \to \infty$
- a fixed point x^* is unstable if at least some arbitrarily small perturbations from x^* that do not remain sufficiently close to x^* for all $t \geq 0$

Example 1.2

$\dot{x} = -x^2(1+x)$



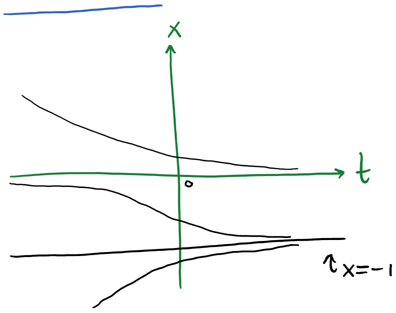


- fixed points:

$$-x^2(1+x) = 0$$

$$x^* = 0, -1$$

- $x^* = -1$ is a stable fixed point
- $x^* = 0$ is an unstable fixed point, but also called half-stable (semistable)



Population Growth

$N(t)$ = population at time t (number of individuals)

- approximate this with a differentiable function
- this is reasonable if N is large (since N is discrete)
- simple model (exponential growth)

$$\dot{N} = rN$$

$$\frac{\dot{N}}{N} = r \quad \text{per-capita growth rate}$$

positive in this model

- the per-capita growth rate is the net instantaneous change in the number of individuals per unit time, per average individual
- units: $\frac{\dot{N}}{N} = \frac{[\text{individuals} \cdot \text{time}^{-1}]}{[\text{individuals}]} = [\text{time}^{-1}]$
- solution:

$$N(t) = N(0)e^{rt} \quad \text{exponential growth}$$

- this is unbounded, unrealistic for some models
- another model: per-capita growth rate is not constant but decreases as N increases (e.g. competition for resources)
- the per-capita growth rate $\frac{\dot{N}}{N} = g(N)$ is some decreasing function of N
- e.g. logistic model

$$g(N) = r\left(1 - \frac{N}{k}\right)$$

r: maximum per-capita growth rate (const.)

k: constant with units of [individuals]

- logistic equation

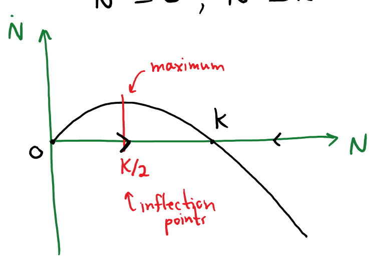
$$\dot{N} = rN\left(1 - \frac{N}{k}\right), \quad N \geq 0$$

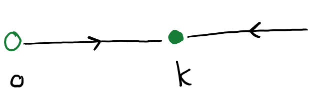
biologically, $N<0$ is irrelevant

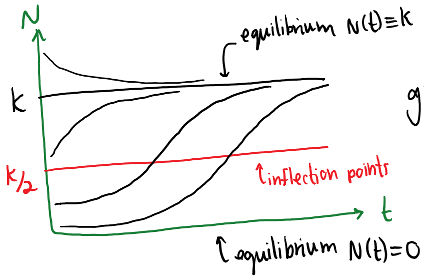
- fixed points:

$$0 = rN\left(1 - \frac{N}{k}\right)$$

$$N^* = 0, \quad N^* = k$$







- K is called the carrying capacity: sustainable population size

Dimensional Analysis and Scaling

- reduce $\dot{N} = rN(1-\frac{N}{k})$ to $\dot{x} = x(1-x)$
- two parameters: r, K (in logistic eq.)
- reduce parameters by nondimensionalising the logistic equation
- dimensionless population

$$x = \frac{N}{A}$$ ← constant with dimensions [individuals]

- dimensionless time

$$\tau = \frac{t}{B}$$ ← constant with dimensions [time]

- how do we choose A, B?

$$N = Ax, \quad \tau = \frac{t}{B}$$

- chain rule

$$\dot{N} = \frac{dN}{dt} = \frac{dN}{d\tau}\frac{d\tau}{dt} = \frac{d(Ax)}{d\tau}\frac{1}{B}$$

$$= \frac{A}{B}\frac{dx}{d\tau}$$

$$\frac{dN}{dt} = rN\left(1-\frac{N}{k}\right)$$

$$\frac{A}{B}\frac{dx}{d\tau} = rAx\left(1-\frac{Ax}{k}\right)$$

$$\frac{dx}{d\tau} = rBx\left(1-\frac{A}{k}x\right)$$

- to simplify the equation, choose

  $A = K$ (both [individuals])

  $B = \frac{1}{r}$ (both [time])

- then $\frac{dN}{dt} = rN(1-\frac{N}{K})$ is equivalent to

  $\frac{dx}{d\tau} = x(1-x)$ where $x = \frac{N}{K}$ and $\tau = rt$

- reduces to Example 1.1
- graph of N vs. t is the same as x vs. τ (with the axes rescaled)

Linear Stability Analysis

- computational way of (sometimes) predicting stability of a fixed point x^*

  $x^* = f(x)$, $x \in \mathbb{R}$ or an interval

- suppose x^* is a fixed point
- stability: what happens to solutions for small perturbations from x^*

  let $x(t) = x^* + \eta(t)$ where η is a small perturbation

  $\eta = x - x^*$

  $\dot{\eta} = \dot{x} - 0 = f(x) = f(x^* + \eta)$

- expand as Taylor series

  $$\dot{\eta} = f(x^*) + \underbrace{f'(x^*)}_{\lambda}\eta + \underbrace{\frac{1}{2!}f''(x^*)\eta^2 + \dots}_{O(\eta^2)}$$

  ↑ we are ignoring these terms since η is small

- by definition

  $f(x^*) = 0$ since x^* is a fixed point

  $\dot{\eta} = \lambda\eta + O(\eta^2)$ (equivalent to $\dot{x} = f(x)$)

- if $\lambda = f'(x^*) \neq 0$, then for small η, the effects of the $O(\eta^2)$ terms are negligible
- approximate the ODE by its <u>linearisation</u> at x^*

  $\dot{\eta} = \lambda\eta$ where $\lambda = f'(x^*)$

- solution:

  $\eta(t) = \eta_0 e^{\lambda t}$

· sign of λ determines stability

if $\lambda > 0$ exponential growth } of perturbations
$\lambda < 0$ exponential decay } $\eta(t)$

$\lambda = f'(x^*) > 0$ perturbations $\eta(t) = x(t) - x^*$ grow $\rightarrow$ x^* is unstable
$\lambda = f'(x^*) < 0$ " " decay $\rightarrow$ x^* is stable

· e.g. $\dot{x} = x(1-x)$