5 March - MATH 345

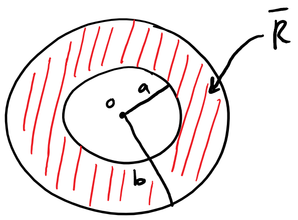
March 5, 2015 7:21 PM

- HW 4 due March 12
- example using Poincaré-Bendixson to show that a closed orbit exists

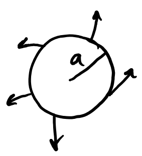
$\boxed{\text{Example 2.8 (cont)}}$

$$\begin{cases} \dot{r} = r - r^3 + \varepsilon r^2 \cos\theta \\ \dot{\theta} = 1 \end{cases} \quad 0 < \varepsilon < 1$$

- construct a region that is closed and bounded, is trapping, and contains no fixed points
- a fixed point exists at the origin, so we exclude it from our candidate region by using an annulus ("washer" shape)



- need to choose a and b such that $\bar{R}$ is trapping
- on the inner boundary $r=a$, we want $\dot{r}>0$ so that the vector field and direction field point into $\bar{R}$



- when $r=a$

$$\dot{r} = a - a^3 + \varepsilon a^2 \cos\theta$$
$$\geq a - a^3 + \varepsilon a^2(-1)$$

lower bound since $-1 \leq \cos\theta \leq 1$

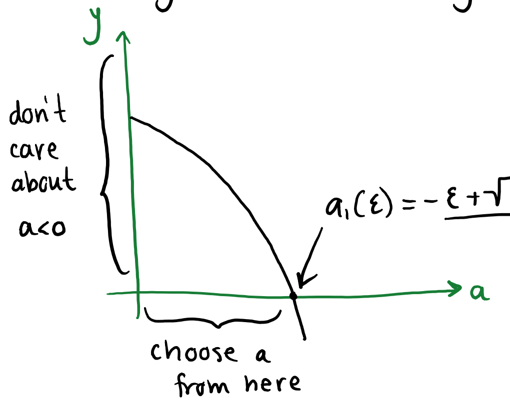
$$= a\underbrace{(-a^2 - \varepsilon a + 1)}$$

a: >0 by definition

$(-a^2 - \varepsilon a + 1)$: we want this to be >0 so $\dot{r}>0$

- let $y = -a^2 - \varepsilon a + 1$

• we want $y>0$ → find roots of y using quadratic formula



• it is possible to choose an a so that $a>0$ and $y>0$
i.e. $0<a<a_1(\varepsilon)$ and $\dot{r}>0$ at $r=a$

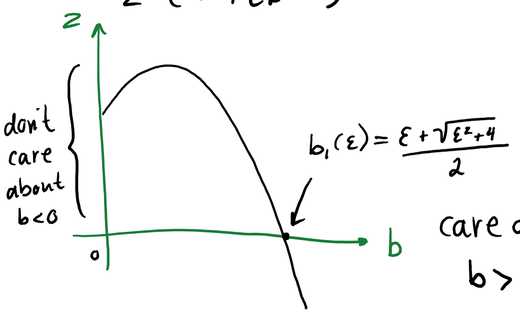
• that means the vector field is pointing into $\bar{R}$
for $0<a<a_1(\varepsilon)$

• for the boundary $r=b$

$$\begin{aligned}\dot{r} &= b - b^3 + \varepsilon b^2 \cos\theta \\ &\le b - b^3 + \varepsilon b^2 (1) \\ &= \underset{\substack{\uparrow \\ b>0 \text{ by} \\ \text{definition}}}{b}\underbrace{(-b^2+\varepsilon b+1)}_{\uparrow \text{ can this be } <0?}\end{aligned} \qquad -1 \le \cos\theta \le 1$$

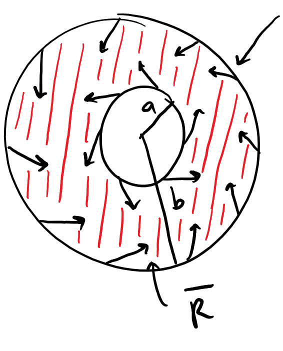
• should be possible for large enough b

$z = (-b^2 + \varepsilon b + 1)$



care about $b > b_1(\varepsilon)$

• choose $b > b_1(\varepsilon)$ so $\dot{r}<0$ on $r=b$
• note: $a_1(\varepsilon) < b_1(\varepsilon)$ therefore $a<b$
• so for choices of a & b



note: the angle depends on θ but the radial (in/out) direction determined by $\dot{r}$

- therefore $\bar{R}$ contains at least one closed orbit by PBT, but there might be more than one; however PBT does not tell us how many closed orbits exist

$\boxed{\text{Example 2.9}}$

- glycolyois from biology (a "more realistic" and thus "messier" example)
- $x(t)$ ("ADP"), $y(t)$ ("fructose b-phosphate") are dimensionless concentrations of interesting chemicals that will appear in the model
- dimensionless in time and concentration
- assume everything is well-mixed so there is no spatial dependence

$$\begin{cases} \dot{x} = -x + ay + x^2 y \\ \dot{y} = b - ay - x^2 y \end{cases} \quad \text{in } \bar{Q} = \{(x,y) : 0 \le x < \infty,\ 0 \le y < \infty\}$$

- $a > 0$, $b > 0$ are concentrations of other chemicals which are large enough to be assumed constant
- fixed points will occur when

$$0 = -x + ay + x^2 y \rightarrow y = \frac{x}{a + x^2}$$

$$0 = b - ay - x^2 y \rightarrow y = \frac{b}{a + x^2}$$

$$y = \frac{x}{a + x^2} = \frac{b}{a + x^2} \rightarrow x = b$$

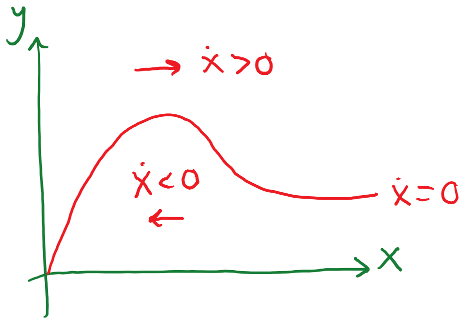
$$(x^*, y^*) = \left(b, \frac{b}{a + b^2}\right)$$

Exercise: determine the linearised stability

hint: look at determinant and trace rather than fully completing calculation

- nullclines: $\dot{x} = 0 \rightarrow y = \frac{x}{a + x^2}$

$\dot{y} = 0 \rightarrow y = \frac{b}{a + x^2}$



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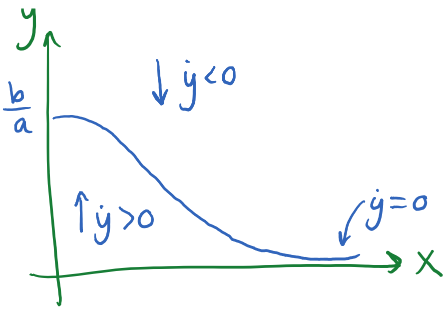
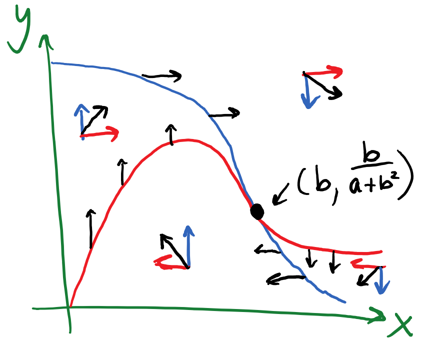


- need to superimpose pictures and that requires some care in properly plotting max/mins
- calculations skipped



- you can see from the direction field arrows that it is likely a closed orbit exists
- there seems to be some circulating
- must construct a suitable $\bar{R}$ (closed, trapping)
- no formula for how to choose a possible $\bar{R}$
- need to look at situations and think where this might be possible

- consider x-axis ($y=0$)

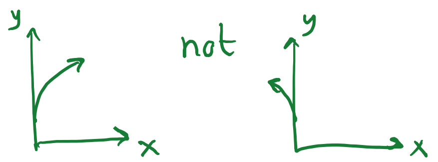
$$\dot{y}=b>0$$

- along that boundary, all vectors are pointing upward
- consider y-axis ($x=0$)

$$\dot{x}=ay>0 \text{ (when } y>0\text{), pointing right}$$

Exercise: if $y=0$ then $x=0$. show that a trajectory

does



- what is happening at large x and large y?

$$\dot{x} \simeq x^2 y, \quad \dot{y} \simeq -x^2 y \rightarrow \dot{x} + \dot{y} \simeq 0$$

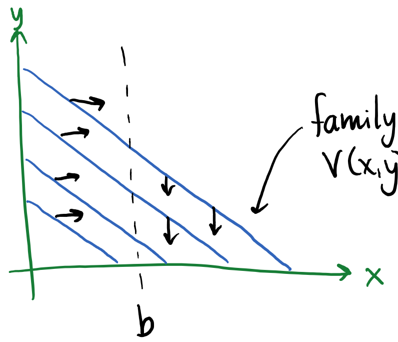
- more precisely, consider a sort of Lyapunov function

$$V(x,y) = x + y$$

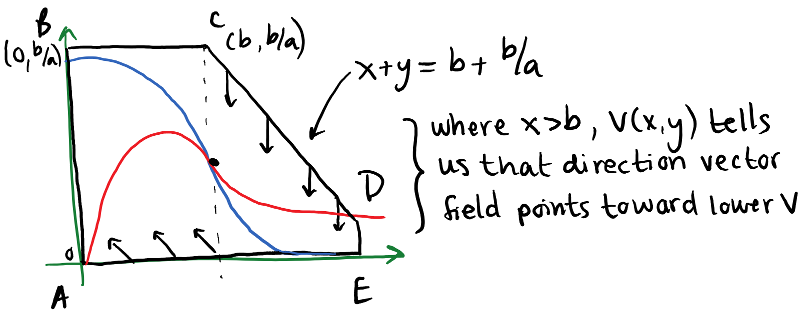
- along trajectories

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$
$$= \dot{x} + \dot{y}$$
$$= -x + ay + x^2 y - b - ay - x^2 y$$
$$= b - x$$

- along contours $x + y = c$, you will either go to higher values of V or lower values of V depending on if $\dot{V}$ is positive or negative

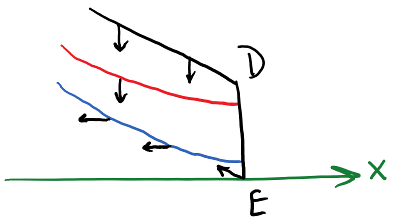


- if $x > b$, then $\dot{V} < 0$ and trajectories will cross the contour lines pointing towards lower values of V
- otherwise $\dot{V} > 0$ and the opposite happens
- constructing $\bar{R}$



$\bar{R}$ is contained in ABCDE region

- zooming in around D, E



- we know these directions from the nullclines and $V(x,y)$
- on all open line segments AB, BC, CD, DE, EA and at corner E, the vector field points in to $\bar{R}$

Exercise: show that trajectories starting at A, B, C, or D all must enter $\bar{R}$

- assuming the results of the exercise, $\bar{R}$ is trapping
- therefore by PBT, $\bar{R}$ contains a fixed point or a closed orbit
- but we already know that there is a fixed point in $\bar{R}$, so this is useless
- we have to rule out/exclude the fixed point, which we will do later

Theory of Bifurcations in 2D flows

- chapter is called 'Bifurcations revisited" in textbook
- bifurcations can exist in any higher dimensional system

$$\dot{\vec{x}} = \vec{f}(\vec{x}, \mu) \quad , \mu \in \mathbb{R}$$

- back in 1D we looked at 3 different types of bifurcations and these happen in n-D systems as well
- in a certain sense, these bifurcations can be reduced to a "1D bifurcation"

Saddlenode, transcritical, & pitchfork bifurcations

- can occur in 1D and higher
- near the critical parameter value, the dynamics can be "reduced" to a 1D curve called the invariant centre manifold

↑ centre between stable and unstable points

Example 2.10

$$\begin{cases} \dot{x} = \mu - x^2 \\ \dot{y} = -y \end{cases}$$

- fixed points $(x^*, y^*) = (\pm\sqrt{\mu}, 0)$ for $\mu \geq 0$
- no fixed point if $\mu < 0$ since we need real points for a real system
- we can do linear stability analysis, phase portraits, etc.