5 February - MATH 345

February 5, 2015 1:55 PM

- HW 3 due February 26
- midterm February 24

Rabbits vs. sheep

$$\frac{dx}{d\tau} = x(3-x-2y)$$

$$\frac{dy}{d\tau} = y(2-y-x)$$

fixed points

$$\left.\begin{array}{l} x=0 \text{ OR } 3-x-2y=0 \\ y=0 \text{ OR } 2-y-x=0 \end{array}\right\} \text{4 fixed points}$$

$(0,0)$, $(3,0)$, $(0,2)$, $(1,1)$

$$\vec{f}(\vec{x}) = \begin{pmatrix} 3x-x^2-2xy \\ 2y-y^2-xy \end{pmatrix}$$

$$D\vec{f}(\vec{x}) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

- linearised stability:

$$D\vec{f}(\vec{x}) = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-2y-x \end{pmatrix}$$

evaluate

i) at $\vec{x}^* = (0,0)$

$$A = D\vec{f}(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

already diagonal → eigenvalues 3, 2

both positive: hyperbolic repeller

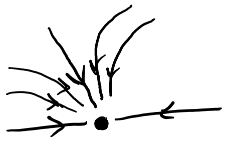
local phase portrait (linear)

from a linear system
look at eigen vectors → x-dir grows faster $\lambda=3$

ii) at $\vec{x}^*=(3,0)$

$$A = D\vec{f}(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

(upper) triangular matrix
eigenvalues $-3, -1$
both negative: hyperbolic attractor
$\Delta = 3$, $\tau = -4$
local phase portrait (linear)
eigenvectors: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for -3
$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for -1



y-dir decays slower ($\lambda=-1$)

iii) at $\vec{x}^*=(0,2)$

$$A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

(lower) triangle → eigenvalues $-1, -2$
hyperbolic attractor
local phase portrait (linear)

eigen vectors:
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for -2
$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for -1

x-dir decays slower

iv) at $\vec{x}^*=(1,1)$

$$A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

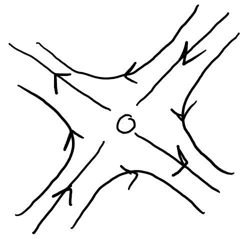
$\Delta = -1 \to$ hyperbolic saddle point

eigenvectors:

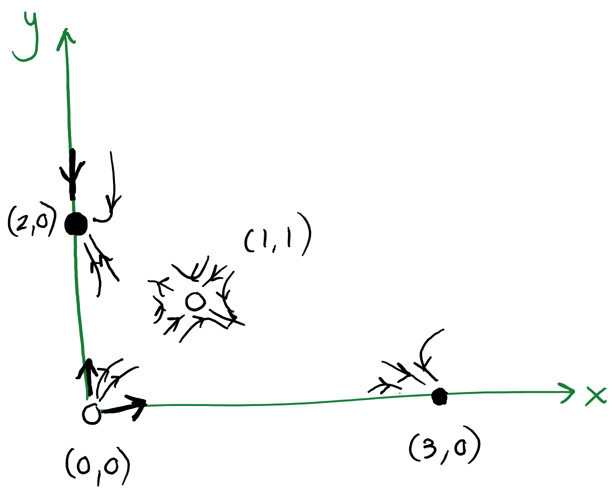
$\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$ for eigenvalue $-1-\sqrt{2} < 0$ decay

$\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$ for eigenvalue $-1+\sqrt{2} > 0$ growth

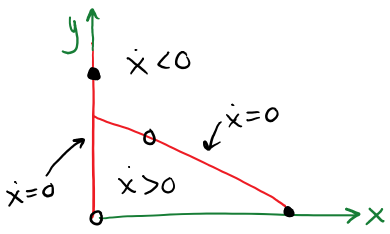
local phase portrait (linear)

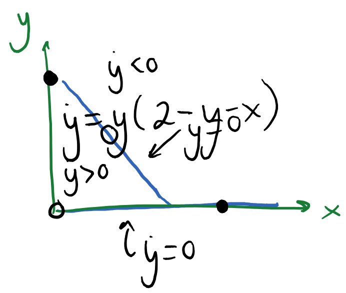


partial phase portrait using linearisations at fixed points

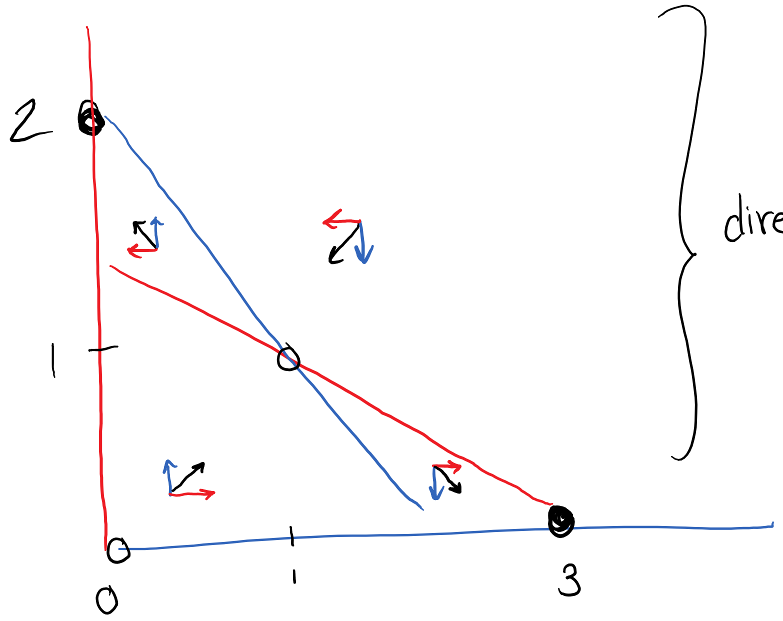


direct information from nullclines

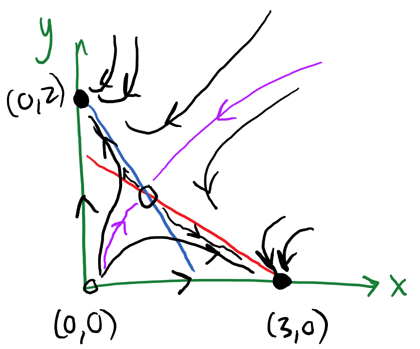




$\dot{x} = x(3-x-2y)$



direction field by hand



global phase portrait

- the purple line is called the stable manifold (1,1)
- it is a curve tangent to the stable eigenvector $\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$
- stable manifolds are special trajectories
- model predicts "competitive exclusion"
- one or the other species becomes extinct as $t \to \infty$ depending on where the initial value (x_0, y_0) is located relative to the stable manifold of (1,1)
- this system is "bistable": it has 2 stable fixed points

The effect of small nonlinear terms

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \ , \ \vec{x} \in \mathbb{R}^2$$

- fixed point $\vec{x}^*$, put $\vec{x} = \vec{x}^* + \vec{u}$

$$\dot{\vec{u}} = A\vec{u} + \underbrace{O(\|\vec{u}\|^2)}_{\text{small nonlinear terms, if } \vec{u} \simeq 0 \ (\text{i.e. } \vec{x} \simeq \vec{x}^*)}, \quad A = D\vec{f}(\vec{x}^*)$$

- linearisation

$$\dot{\vec{u}} = A\vec{u}$$

- the fixed point $\vec{x}^*$ is hyperbolic if the linearisation at $\vec{x}^*$ is hyperbolic
- if $\vec{x}^*$ is hyperbolic, then the small nonlinear terms do not affect topological type or stability (saddle point, attractor, repeller)
  i) If $\vec{u} = \vec{0}$ is a saddle point, node, or spiral for the linearisation, then $\vec{x}^*$ has the same type
  ii) If $\vec{u} = \vec{0}$ is a star or degenerate node for the linearisation, then $\vec{x}^*$ could be a star or degenerate node or node or spiral (topological type does not change)
- if $\vec{x}^*$ is nonhyperbolic, the small nonlinear terms could change the topological type and stability

Exercise For the fixed point $\vec{0}$, determine the linearised stability and then the stability for

$$\begin{cases} \dot{x} = ax^3 \\ \dot{y} = -y \end{cases} \quad \text{with i) } a<0 \quad \text{ii) } a>0$$

$$\boxed{\text{Example 2.4}}$$

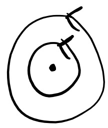
$$\dot{x} = -y + ax^3 + axy^2 \qquad \text{i) } a<0$$
$$\dot{y} = x + ax^2y + ay^3 \qquad \text{ii) } a>0$$

$\vec{0}$ is a fixed point

linearisation: $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

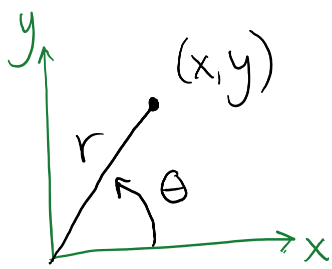
$\Delta = 1, \ \tau = 0$

purely imaginary eigenvalues ($\pm i$)

Non hyperbolic "linear centre": if there were no nonlinear terms, this would be neutrally stable with phase portrait 

there are nonlinear nonlinear terms, so no conclusions can be made

to see the effect of nonlinear terms, transform to polar coordinates



$x = r\cos\theta, \ y = r\sin\theta$

$r \in [0,\infty), \ \theta \in S^1$

$r^2 = x^2 + y^2$

$2r\dot{r} = 2x\dot{x} + 2y\dot{y}$

$$r\dot{r} = x(-y + ax^3 + axy^2) + y(x + ax^2y + ay^3)$$
$$= a(x^2+y^2)^2 = ar^4$$
$$\dot{r} = ar^3$$

<u>Exercise</u> use $\tan\theta = \frac{y}{x}$ to show that $\dot{\theta} = 1$