22 January - MATH 345

January 22, 2015 2:01 PM

- HW 2 due February 5
- midterm Tuesday, February 24

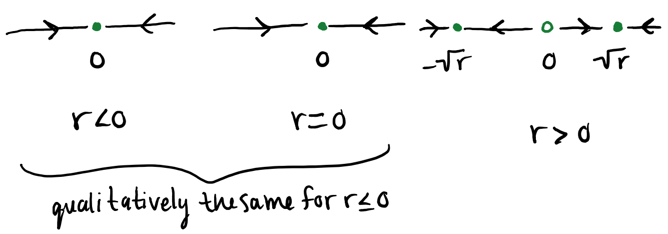
$\boxed{\text{Example 1.7}}$ (cont)

$$\dot{x} = f(x,r) = rx - x^3$$

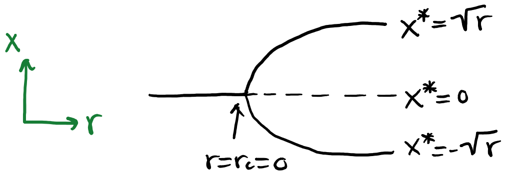
fixed points $x^* = 0$, $x^* = \sqrt{r}$, $x^* = -\sqrt{r}$

Exercise: Calculate linearised stability. Plot $\dot{x}$ vs x (cubic, odd, slope of tangent line at $x=0$ is $\frac{\partial f}{\partial x}(0,r) = r$)

Phase portraits



Bifurcation Diagram

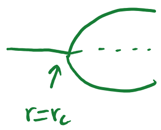


Exercise: Sketch bifurcation diagrams for the normal form

$$\dot{x} = a(r - r_c)x + bx^3, \quad a \neq 0, b \neq 0$$

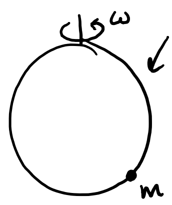
(Four cases)

e.g. $a>0$, $b<0$



- called supercritical if the bifurcating non-zero solutions are stable
- called subcritical if the bifurcating non-zero solutions are unstable

Overdamped bead on a rotating hoop



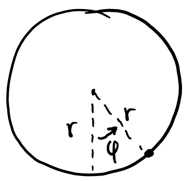
rigid circular wire hoop of radius r rotating about vertical axis with constant angular velocity ω

$r = [m]$, $\omega = [s^{-1}]$

bead of mass m free to slide but subject to viscous damping (force)

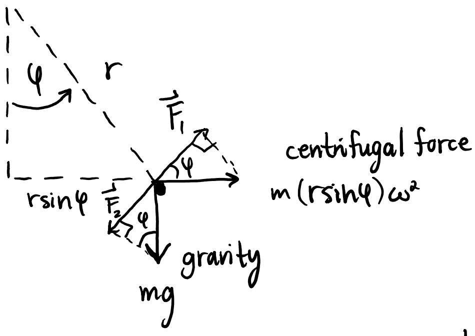
$m = [kg]$

- angle from vertical axis (at bottom) $= \varphi(t)$ radians



φ measured positive in counter clockwise direction

- in frame of reference co-rotating with the hoop, balance forces



- since wire is rigid, we only care about tangential components

$F_1 = |\vec{F}_1|$ = tangential component of centrifugal force

$F_2 = |\vec{F}_2|$ = tangential component of weight (gravity)

$$\underbrace{m\overset{\text{tangential acceleration}}{a}}_{\text{inertia term}} = F_1 - F_2 - \text{damping force}$$

- damping always opposes acceleration

$$mr\frac{d^2\varphi}{dt^2} = m(r\sin\varphi)\omega^2\cos\varphi - mg\sin\varphi - b\frac{d\varphi}{dt} \quad \Big\} \text{ units: Newtons } kg\,m\,s^{-2}$$

- physically, $\varphi = \pi$ is the "same" as $\varphi = -\pi$
  $\varphi = 0$ is the "same" as $\varphi = 2\pi$ } etc.

- we say φ belongs to the circle $\mathbb{S}^1$ where φ_1 and $\varphi_2 \in \mathbb{R}^1$ are identified iff $\varphi_2 - \varphi_1$ is an integer multiple of 2π
- in symbols: $\mathbb{S}^1 = \mathbb{R}^1 / (2\pi\mathbb{Z})$
- our model:

$$mr\frac{d^2\varphi}{dt^2} = mr\omega^2 \sin\varphi\cos\varphi - mg\sin\varphi - b\frac{d\varphi}{dt}, \quad \varphi \in \mathbb{S}^1 \quad (*)$$

- assume heavy damping: b is "large"
- dimensional analysis and scaling:

  φ - radians (no units)

  t - time has units s

- define dimensionless time

$$\tau = \frac{t}{B} \quad \text{where } B \text{ is a constant with units s}$$

- change of variables $\varphi = \varphi(\tau(t))$

$$\frac{d\varphi}{dt} = \frac{d\varphi}{d\tau}\frac{d\tau}{dt} = \frac{d\varphi}{d\tau}\cdot\frac{1}{B}$$

Exercise: show

$$\frac{d^2\varphi}{dt^2} = \frac{d^2\varphi}{d\tau^2}\cdot\frac{1}{B^2}$$

$(*)$ becomes

$$\frac{mr}{B^2}\frac{d^2\varphi}{d\tau^2} = mr\omega^2\sin\varphi\cos\varphi - mg\sin\varphi - \frac{b}{B}\frac{d\varphi}{dt} \Bigg\} \text{units of force } \mathrm{Kg\,m\,s^{-2}}$$

- to get non dimensional DE, divide by some quantity with units of $\mathrm{kg\,m\,s^{-2}}$, e.g. divide by mg

$$\underbrace{\frac{r}{gB^2}\frac{d^2\varphi}{d\tau^2}}_{\text{inertia term}} = \frac{r\omega^2}{g}\sin\varphi\cos\varphi - \sin\varphi - \underbrace{\frac{b}{mgB}}_{\text{damping term}}\frac{d\varphi}{d\tau} \Bigg\} \text{no units}$$

· "heavily damped" means $\dfrac{b}{mgB} \gg \dfrac{r}{gB^2}$

↑ much larger than e.g. $1 \gg 10^{-4}$

· choose B so that $\dfrac{b}{mgB} = 1 \rightarrow B = \dfrac{b}{mg}$ units of s

· this is the "natural" or "characteristic" timescale for this problem

$$1 \gg \underbrace{\frac{m^2 g r}{b^2}}_{\varepsilon}$$

$$1 \gg \varepsilon$$

· heavily damped $\Leftrightarrow \varepsilon \ll 1 \Leftrightarrow b^2 \gg m^2 g r$

· we get

$$\varepsilon \frac{d^2\varphi}{d\tau^2} = \underbrace{\frac{r\omega^2}{g}}_{\gamma} \sin\varphi\cos\varphi - \sin\varphi - \frac{d\varphi}{d\tau}$$

$$\varepsilon \frac{d^2\varphi}{d\tau^2} = \gamma \sin\varphi\cos\varphi - \sin\varphi - \frac{d\varphi}{d\tau}, \quad \varphi \in S^1, \quad 0 < \varepsilon \ll 1$$

· "overdamped" system: take $\varepsilon = 0$

$$0 = \gamma \sin\varphi\cos\varphi - \sin\varphi - \frac{d\varphi}{d\tau}$$

$$\frac{d\varphi}{d\tau} = f(\varphi, \gamma) = \sin\varphi(\gamma\cos\varphi - 1), \quad \varphi \in S^1$$

· observe $f(\varphi + 2\pi, \gamma) = f(\varphi, \gamma)$ for all φ, γ

· this equation is well defined for $\varphi \in S^1$

· observe $f(-\varphi, \gamma) = -f(\varphi, \gamma)$ for all φ, γ, so (PF1) is satisfied

- find γ_c so (PF2) is true

$$f(\varphi,\gamma) = \gamma\sin\varphi\cos\varphi - \sin\varphi = \frac{\gamma}{2}\sin 2\varphi - \sin\varphi$$

$$\frac{\partial f}{\partial \varphi}(\varphi,\gamma) = \gamma\cos 2\varphi - \cos\varphi$$

$$\frac{\partial f}{\partial \varphi}(0,\gamma) = \gamma - 1 = 0 \rightarrow \gamma_c = 1$$

- (PF2) is true if $\gamma_c = 1$

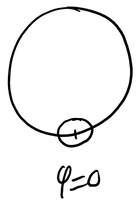
Exercise: verify (PF3) $\frac{\partial^2 f}{\partial\varphi\partial\gamma}(0,\gamma_c) = 1$

(PF4) $\frac{1}{6}\frac{\partial^3 f}{\partial\varphi^3}(0,\gamma_c) = -\frac{1}{2}$

- so there is a pitchfork bifurcation at $\varphi = 0$, $\gamma = 1$
- normal form is

$$\frac{d\varphi}{d\tau} = (\gamma - 1)\varphi + \frac{1}{2}\varphi^3$$

- this is valid locally, near $\varphi = 0$, $\gamma = 1$ (Taylor series)



- locally, it doesn't matter if $\varphi \in S^1$ or $\varphi \in \mathbb{R}^1$