20 January - MATH 345

January 20, 2015 1:54 PM

- HW1 due Thursday, 22 January (2PM / start of class)

$\boxed{\text{Example 1.5 (cont.)}}$

$\dot{x} = x - 1 + r\ln x$, $x > 0$

$x^* = 1$ is a fixed point for all r

$u = x - 1$, $x = 1 + u$

↑ deviation from x^*

$\dot{u} = f(u,r) = u + r\ln(1+u)$, $u > -1$

Check (TC1) $f(0,r) = 0$ ✓

↳ $u^* = 0$ is a fixed point for all r

Note: cannot be a saddle-node bifurcation at $u^* = 0$

Find r_c so (TC2) is satisfied

$$\frac{\partial f}{\partial u}(u,r) = 1 + \frac{r}{1+u}$$

$$\frac{\partial f}{\partial u}(0,r) = 1 + r = 0 \text{ if } r = -1$$

so taking $r_c = -1$ satisfies (TC2)

verify (TC3):

$$\frac{\partial^2 f}{\partial u \partial r}(u,r) = \frac{1}{1+u}$$

$$a = \frac{\partial^2 f}{\partial u \partial r}(0,-1) = 1 \neq 1 \checkmark$$

verify (TC4):

$$\frac{\partial^2 f}{\partial u^2}(u,r) = \frac{-r}{(1+u)^2}$$

$$b = \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(0,-1) = \frac{1}{2}\cdot \frac{1}{1} = \frac{1}{2} \neq 0 \checkmark$$

(TC1)–(TC4) $\Rightarrow$ there is a transcritical bifurcation at $(u_c^*, r_c) = (0,-1)$

corresponds to $(x_c^*, r_c) = (1,-1)$

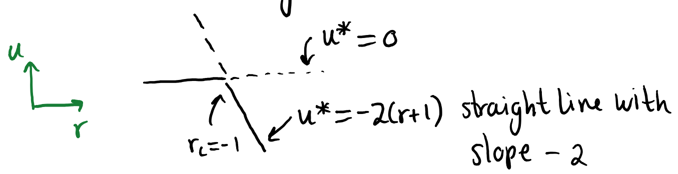
Normal form is

$$\dot{u} = u\left[(r+1) + \tfrac{1}{2}u\right]$$

Fixed points: $u^* = 0$, $u^* = -2(r+1)$

Exercise: Determine linearised stability, sketch phase portraits

Bifurcation diagram:



Compare with original vector field

$$\dot{x} = x - 1 + r\ln x = (x-1)\left[1 + r\frac{\ln x}{x-1}\right], \quad x > 0$$

we know $x=1$ is a fixed point, so we factor it out

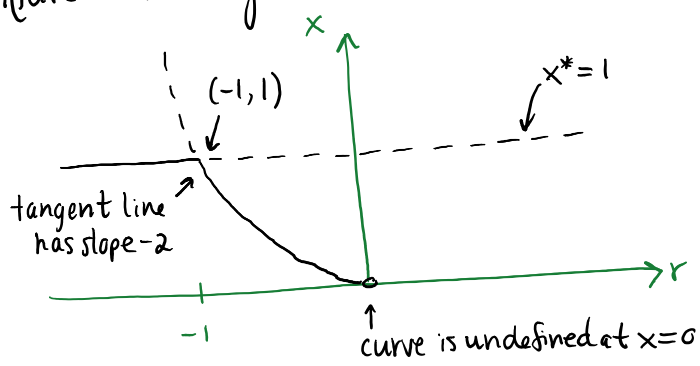
Fixed points:

$$x^* = 1, \quad 1 + r\frac{\ln x}{x-1} = 0 \longrightarrow r = \frac{1-x^*}{\ln x^*}$$

x^* as a function of r is inconvenient
r as a function of x^*

Exercise: Determine linearised stability, sketch phase portraits

Bifurcation diagram



Normal form gives qualitatively correct dynamics near $x=1, r=-1$ and is a good approximation near $x=1, r=-1$

$\boxed{\text{Example 1.6}}$ Chemical kinetics: a catalysed reaction

$$\overbrace{BrO_3^-}^{A} + \overbrace{HBrO_2}^{X} + H^+ \rightleftharpoons 2BrO_2 + H_2O$$

$$2BrO_2 + 2H^+ + 2Ce^{3+} \rightleftharpoons 2Ce^{4+} + \underbrace{2HBrO_2}_{2X}$$

summarise by

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$$

lower case $\quad$ upper case

$x(t) =$ concentration of X, ≥ 0 $\quad$ $mol\,L^{-1}$ or $kg\,L^{-1}$

$a =$ concentration of A, ≥ 0

assume there is so much of A that its concentration can be assumed constant

"law" (i.e. model) of mass action: reaction rate is proportional to product of concentration of reactants

$\dot{x} = k_1 a x - k_{-1} x^2, \quad x \geq 0$

where k_1, k_{-1} are positive numbers called "rate constants"

now consider

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X, \quad X + B \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} C$$

and assume $k_{-2} = 0$ (i.e. back reaction is negligible)

$$\dot{x} = k_1 a x - k_{-1} x^2 - k_2 x b$$

$$\dot{x} = \underbrace{x[(k_1a - k_2b) - k_{-1}x]}_{f(x,a)} \;, \; x \geq 0$$

$f(x,a) \to$ think of b as fixed

↑ a is a constant but its value can be changed (e.g. different concentrations for different experiments)

This is already a normal form for a transcritical bifurcation

(TC1) $f(0,a) = 0$ for all $a \geq 0$ ✓

(TC2) $\frac{\partial f}{\partial x}(x,a) = k_1a - k_2b - 2k_{-1}x$

$\frac{\partial f}{\partial x}(0,a) = k_1a - k_2b = 0$ if $a = \frac{k_2b}{k_1}$

$\frac{\partial f}{\partial x}(0,a_c) = 0$ if $a_c = \frac{k_2b}{k_1}$ ✓

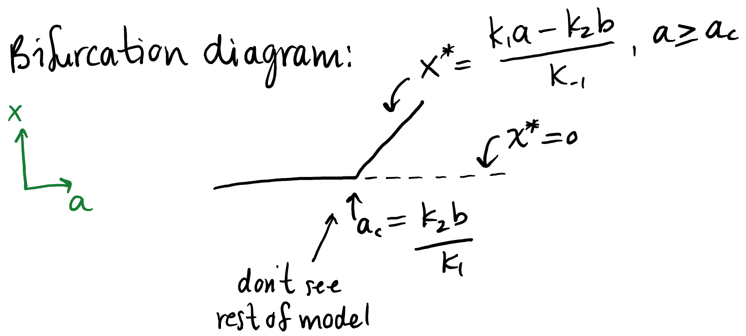
(TC3) $\frac{\partial^2 f}{\partial x \partial a}(0,a_c) = k_1 \neq 0$ ✓

(TC4) $\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,a_c) = -k_{-1} \neq 0$ ✓

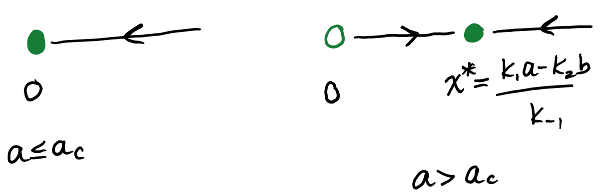
so there is a transcritical bifurcation at $x_c^* = 0$, $a_c = \frac{k_2b}{k_1}$

but we only "see" the part with $x \geq 0$

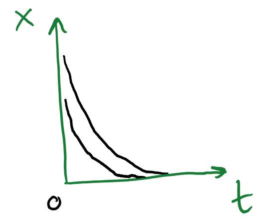
Bifurcation diagram:

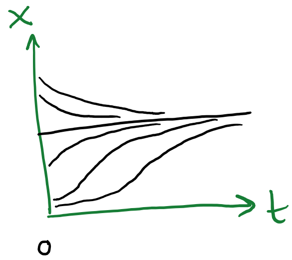


phase portraits



x vs t

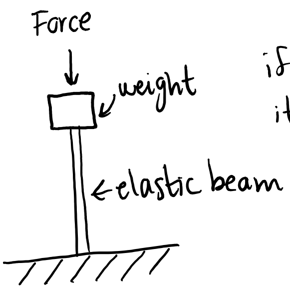




Pitchfork Bifurcations

. models sometimes have symmetry

e.g.



if force is big enough, it will buckle in either direction

$\dot{x} = f(x,r)$, $x \in \mathbb{R}$, $r \in \mathbb{R}$

. suppose (due to your model) f is odd in x

(PF1) $f(-x,r) = -f(x,r)$ for all x, r

Exercise: Show that (PF1) implies $f(0,r) = 0$ for all r

. assume

(PF2) $\dfrac{\partial f}{\partial x}(0,r) = 0$

. is this transcritical? No

Exercise (PF1) $\Rightarrow \dfrac{\partial^2 f}{\partial x^2}(0,r) = 0$ for all r

. in fact,

$$\frac{\partial^k f}{\partial x^k}(0, r_c) = 0 \text{ for all } r \text{ and all even } k = 2, 4, 6, \ldots$$

· if, in addition we have

$$(\text{PF3}) \quad a = \frac{\partial^2 f}{\partial x \partial r}(0, r_c) \neq 0$$

$$(\text{PF4}) \quad b = \frac{1}{6}\frac{\partial^3 f}{\partial x^3}(0, r_c) \neq 0$$

then the local dynamics at $(0, r_c)$ are correctly determined (and well approximated by) the normal form for the pitchfork bifurcation

$$\dot{x} = a(r - r_c)x + bx^3 = x[a(r - r_c) + bx^2]$$

$$\boxed{\text{Example 1.7}}$$

$$\dot{x} = \underbrace{rx - x^3}_{f(x,r)}$$

this is already in normal form

$r_c = 0, \ a = 1, \ b = -1$

fixed points:

$$0 = rx - x^3 = x(r - x^2)$$

$$x^* = 0, \ x^* = \pm\sqrt{r}$$

$\uparrow r \geq 0$