12 March - MATH 345

March 12, 2015 2:02 PM

- HW 4 due March 19

Hopf bifurcations

- bifurcations of limit cycles

$$\dot{\vec{x}} = \vec{f}(\vec{x}, \mu), \ \vec{x} \in \mathbb{R}^2, \ \mu \in \mathbb{R}$$

- suppose there is μ_c, $\vec{x}_c^*$ such that

(1) $\vec{f}(\vec{x}_c^*, \mu_c) = \vec{0}$

(2) $A = D\vec{f}(\vec{x}_c^*, \mu_c)$ has purely imaginary eigenvalues

$\lambda_{1,2} = \pm i\omega$, $\omega = \sqrt{\Delta}$, $\omega > 0$, $\tau = 0$

$\boxed{\text{Example 2.12}}$

$$\begin{cases} \dot{x} = \mu x - \omega y + ax^3 + ax^2 y \\ \dot{y} = \omega x + \mu y + ax^2 y + ay^3 \end{cases}$$

fix $a \neq 0$, $\omega > 0$, consider μ as parameter

- fixed point $(x^*, y^*) = (0,0)$
- linearisation at $(0,0)$

$$A(\mu) = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

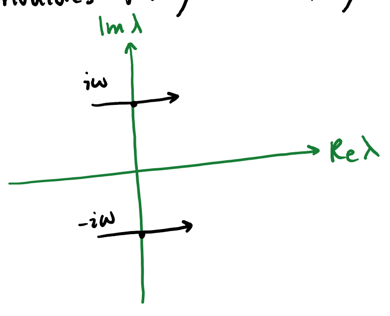
$\Delta = \mu^2 + \omega^2 > 0$, $\tau = 2\mu$

i) $\mu < 0$: $\vec{0}$ is a hyperbolic attractor (stable spiral $\tau^2 - 4\Delta < 0$)

ii) $\mu = 0$: $\vec{0}$ is a nonhyperbolic fixed point with purely imaginary eigenvalues

iii) $\mu > 0$: $\vec{0}$ is a hyperbolic repeller (unstable spiral)

- eigenvalues of $A(\mu)$ are $\lambda_{1,2}(\mu) = \mu \pm i\omega$



(3) $\left. \dfrac{d}{d\mu} \operatorname{Re} \lambda_{1,2}(\mu) \right|_{\mu = \mu_c = 0} = 1 \neq 0$

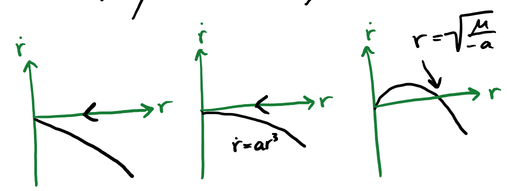
- eigenvalues cross imaginary axis with nonzero "speed" as μ increases
- in polar coordinates (Exercise: show this)

$$\begin{cases} \dot{r} = \mu r + a r^3 \\ \dot{\theta} = \omega \end{cases}$$

- $\theta(t) = \omega t + \theta_0 \pmod{2\pi}$
- $\dot{r}$ equation can be considered as 1D flow, $r \geq 0$

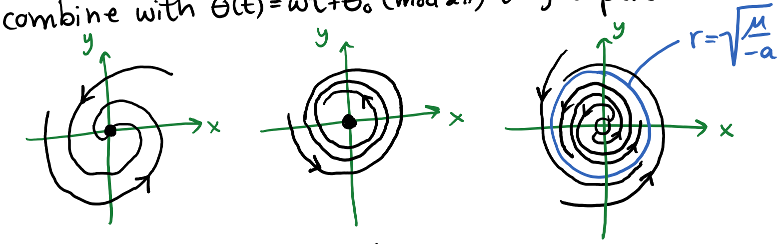
Case 1: $a < 0$

$$\dot{r} = \mu r + a r^3 = r(\mu + a r^2)$$



$\mu < 0$ $\mu = 0$ $\mu > 0$

- combine with $\theta(t) = \omega t + \theta_0 \pmod{2\pi}$ to get portraits in $\mathbb{R}^2$

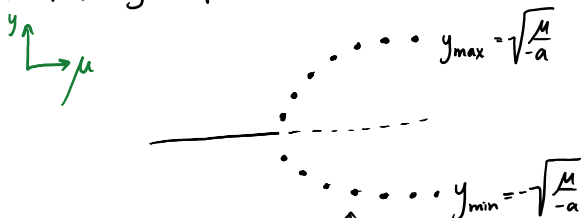


$\mu < 0$: $\vec{0}$ is hyperbolic, stable spiral

$\mu = 0$: $\vec{0}$ is nonhyperbolic, stable (weak) spiral

$\mu > 0$: $\vec{0}$ is hyperbolic, unstable spiral; periodic orbit at $r = \sqrt{\frac{\mu}{-a}}$, stable limit cycle

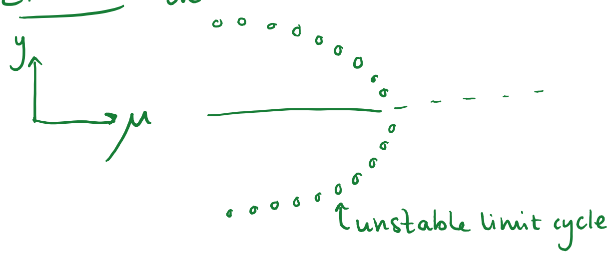
- this is a supercritical Hopf bifurcation - stable limit cycles
- bifurcation diagram: y (or x or $\sqrt{x^2+y^2}$ etc.) vs μ (like AUTO)
- for limit cycle, plot max and min values of $y(t)$ for each μ



stable limit cycle

Case 2: $a > 0$

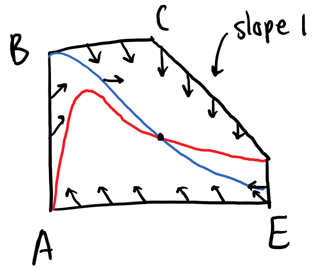
Exercise: determine the subcritical Hopf bifurcation



Example 2.13 (continuation of 2.9)

$$\begin{cases} \dot{x} = -x + ay + x^2 y \\ \dot{y} = b - ay - x^2 y \end{cases} \qquad \text{fixed point } (x^*, y^*) = \left(b, \frac{b}{a+b^2}\right)$$

- trapping region $\bar{R}$ contains fixed point



HW hint:
look where parameter is increasing/decreasing as a fn of x^*, y^*

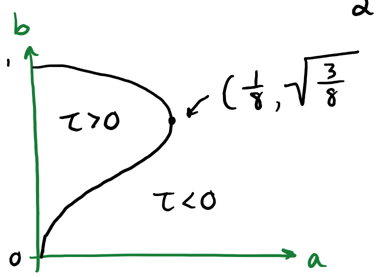
- linearised stability

$$A = \begin{pmatrix} -1 + \frac{2b}{a+b^2} & a+b^2 \\ -\frac{2b^2}{a+b^2} & -a-b^2 \end{pmatrix}, \quad \Delta = a+b^2 > 0$$

- trace

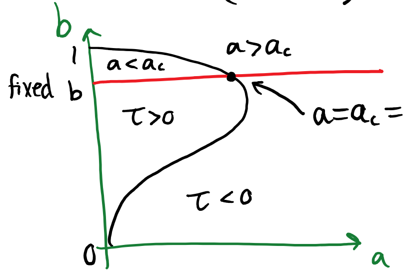
$$\tau = -1 + \frac{2b^2}{a+b^2} - a - b^2 = -\frac{a^2 + (1+2b^2)a + (b^4 - b^2)}{a+b^2}$$

$$\tau = 0 \rightarrow a = \frac{-(1+2b^2) \pm \sqrt{1+8b^2}}{2}$$



- fix b, $0 < b < 1$; consider $a > 0$ as bifurcation parameter

$$\vec{x}^*(a) = \left(b, \frac{b}{a+b^2}\right), \quad b \text{ is fixed}$$

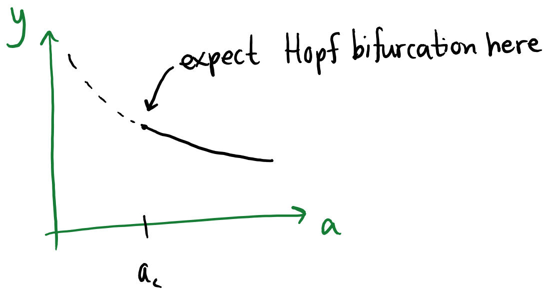


$$a = a_c = \frac{-(1+2b^2) + \sqrt{1+8b^2}}{2}$$

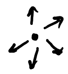
$a < a_c$: $\Delta > 0$, $\tau > 0$, $\vec{x}^*(a)$ is hyperbolic repeller (spiral)

$a = a_c$: $\Delta > 0$, $\tau = 0$, $\vec{x}^*(a_c)$ is nonhyperbolic with purely imaginary eigenvalues

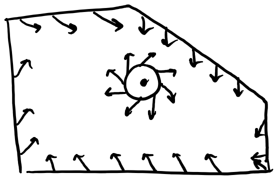
$a > a_c$: $\Delta > 0$, $\tau < 0$, $\vec{x}^*(a)$ is hyperbolic attractor



- expect a periodic orbit close to fixed point, but we want the fixed point to be a repeller
- for $a < a_c$, we can use PBT to prove there is a closed orbit
- $\vec{x}^*(a)$ is a hyperbolic repeller → near $\vec{x}^*(a)$, the vector field points away from the fixed point



- let $\bar{R}_1$ be the closed and bounded region obtained by "puncturing" $\bar{R}$ (i.e. remove a sufficiently small open set that contains the fixed point)



- $\bar{R}_1$ is also trapping, but it contains no fixed point
- PBT gives a closed orbit (at least one) in $\bar{R}_1$

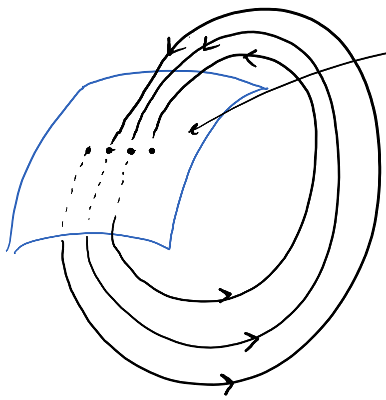
- need to use XPP and AUTO to show (numerically, but not rigourously) that there is a unique closed orbit for $a < a_c$, it is stable, and that there are no closed orbits for $a > a_c$

Poincaré maps

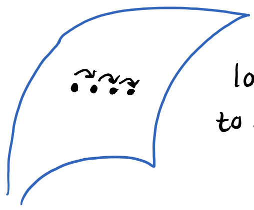
- for flows that repeatedly visit some region of phase space e.g. near a closed orbit

$$\dot{\vec{x}} = \vec{f}(\vec{x}), \; \vec{x} \in \mathbb{R}^n \quad (n \geq 2)$$

- S : an $(n-1)$ dimensional manifold (1D curve, 2D surface, ...) called a section transverse to the flow



puncture this sheet & draw a dot



looks like it hops from dot to dot as time passes