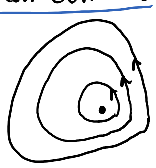
12 February - MATH 345

February 12, 2015 1:59 PM

- midterm Tuesday, February 24 (doesn't include conservative systems)
- HW3 due Thursday, February 26 (covers conservative systems)

- for a conservative 2-D system, local max/min of E that correspond to isolated fixed points of $\dot{\vec{x}} = \vec{f}(\vec{x})$ give nonlinear centres and they are neutrally stable

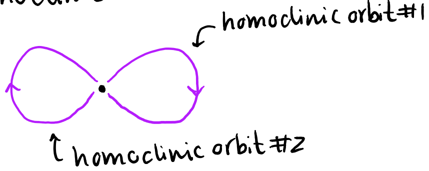
e.g.



a linear centre would be exact ellipses

- almost all solutions are periodic
- exceptions:
  i) fixed points
  ii) homoclinic orbits

e.g.

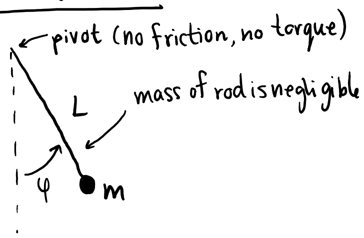


trajectory that approaches the same fixed point as $t \to \infty$ and $t \to -\infty$

  iii) heteroclinic orbits

trajectory that approaches different fixed points as $t \to \infty$ and $t \to -\infty$

Pendulum



- no external forcing or torque, no damping

$$mL\frac{d^2\varphi}{dt^2} = -mg\sin\varphi \quad \text{(see Jan 27)}$$

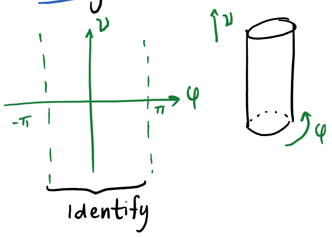
$$\frac{d^2\varphi}{dt^2} + \frac{g}{L}\sin\varphi = 0, \quad \varphi \in S^1$$

Exercise: Show that dimensional analysis and scaling give

$$\frac{d^2\varphi}{d\tau^2} + \sin\varphi = 0, \quad \varphi \in S^1$$

- let $\dot{} = \frac{d}{d\tau}$, $\nu = \dot{\varphi} \in \mathbb{R}^1$
- nondimensionalised equation is equivalent to system

$$\begin{cases} \dot{\varphi} = \nu \\ \dot{\nu} = -\sin\varphi \end{cases}, \quad (\varphi, \nu) \in \mathbb{S}^1 \times \mathbb{R}^1 \text{ the cylinder}$$



- calculate as if $\varphi \in \mathbb{R}^1$, but interpret results for $\varphi \in \mathbb{S}^1$
- fixed points:

$$0 = \nu, \quad 0 = -\sin\varphi$$

$$\nu^* = 0, \quad \varphi^* = n\pi, \quad n \in \mathbb{Z} \quad (\text{i.e. } \varphi^* = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots)$$

- but for $\varphi \in \mathbb{S}^1$, there are only two fixed points
  - i) $(\varphi^*, \nu^*) = (0 \pmod{2\pi}, 0)$ - mass hanging directly below pivot
  - ii) $(\varphi^*, \nu^*) = (\pi \pmod{2\pi}, 0)$ - mass balanced directly above pivot

- linearise:

$$\vec{f}(\varphi, \nu) = \begin{pmatrix} \nu \\ -\sin\varphi \end{pmatrix}, \quad D\vec{f}(\varphi, \nu) = \begin{pmatrix} 0 & 1 \\ -\cos\varphi & 0 \end{pmatrix}$$

i) at $(0 \pmod{2\pi}, 0)$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Delta = 1, \tau = 0$$

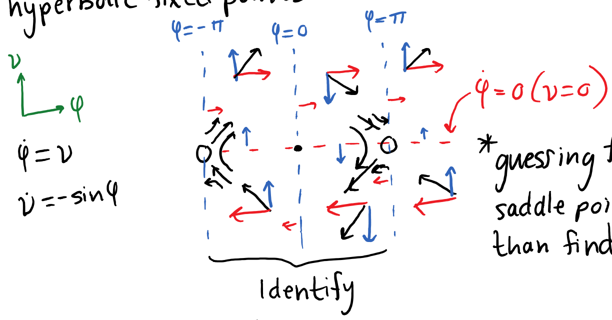
nonhyperbolic with eigenvalues $\pm i$ (linear centre)

ii) at $(\pi \pmod{2\pi}, 0)$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Delta = -1, \tau = 0$$

hyperbolic saddlepoint

- nullclines, direction field, local phase portraits at hyperbolic fixed points



*guessing form of saddle point is faster than finding eigenvectors

- find conserved quantity E

$$\dot{E} = \frac{\partial E}{\partial \varphi}\dot{\varphi} + \frac{\partial E}{\partial \nu}\dot{\nu} = \underbrace{\frac{\partial E}{\partial \varphi}}_{= \sin\varphi\,?} \cdot (\nu) + \underbrace{\frac{\partial E}{\partial \nu}}_{= \nu ?} (-\sin\varphi) = 0$$

$$\frac{\partial E}{\partial \nu} = \nu \rightarrow E = \tfrac{1}{2}\nu^2 + f(\varphi)$$

$$\frac{\partial E}{\partial \varphi} = \sin\varphi \rightarrow E = -\cos\varphi + g(\nu)$$

$$\Bigg\} \; E = \tfrac{1}{2}\nu^2 - \cos\varphi + 1$$

any constant would give a conserved quantity

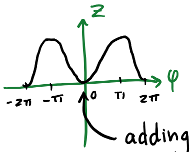
chosen for physics

$E(\varphi,\nu) = \frac{1}{2}\nu^2 + 1 - \cos\varphi$ is conserved (by construction)

- energy surface $z = E(\varphi,\nu)$

- intersect with plane $\nu = 0$

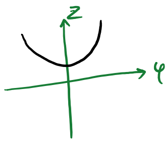
  $z = 1 - \cos\varphi$

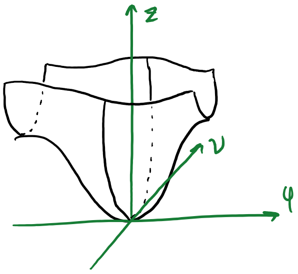


adding 1 makes potential energy zero at $\varphi = 0$

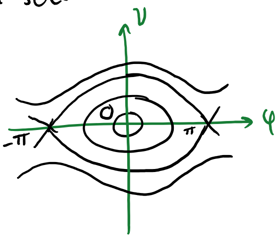
- intersect with plane $\varphi = \text{const.}$

  $z = \frac{1}{2}\nu^2 + \text{const}$

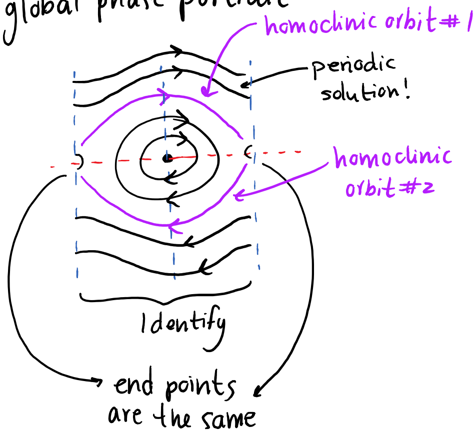


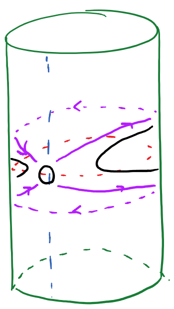


- level sets



- global phase portrait





Damping

$\boxed{\text{Example 2.5(a) with damping}}$

$$m\ddot{x} + b\dot{x} + kx = 0 \ , \quad b>0 \quad (m>0, k>0)$$

as system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{k}{m}x - \frac{b}{m}y \end{cases}$$

the "old energy", now call it $V(x,y) = \frac{1}{2}my^2 + \frac{1}{2}kx^2$, is no longer a conserved quantity

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = (kx)(y) + (my)\left(-\frac{k}{m}x - \frac{b}{m}y\right)$$

$$= -by^2 \leq 0$$

this implies $V(\vec{x}(t))$ is nonincreasing and in fact is decreasing for every trajectory except a fixed point

trajectories that are not fixed points move "downhill" on the contour map of V