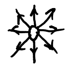
3 February - MATH 345

February 3, 2015 2:01 PM

HW2 due February 5 (Thursday)

- addition to January 29 notes:
- a fixed point $\vec{x}^*$ is <u>repelling</u> if for all $\vec{x}(0)=\vec{x}_0$ sufficiently close to $\vec{x}^*$, $\vec{x}(t) \to \vec{x}^*$ as $t \to -\infty$

  e.g. 

Linear systems

$$\dot{\vec{x}} = A\vec{x}, \quad \vec{x} \in \mathbb{R}^2$$

- topological classification of fixed point $\vec{0}$

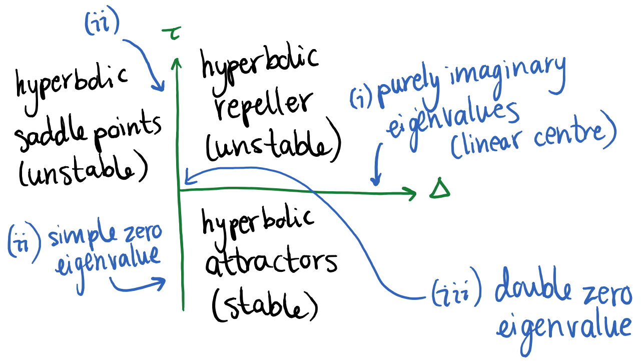
  i) <u>Hyperbolic</u> cases: $Re(\lambda) \neq 0$ for all eigenvalues

  these cases are "robust" - their topological type will not change with all sufficiently small perturbations of A

  ii) <u>Non-hyperbolic</u> cases: $Re(\lambda) = 0$ for at least one eigenvalue

  these cases are "marginal" - their topological type can change with arbitrarily small change in A

topological classification diagram

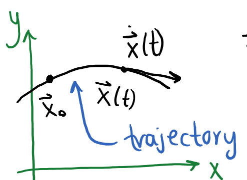


easy classification should know / be able to reproduce for exam

Phase portraits

$$\dot{\vec{x}} = \vec{f}(\vec{x}), \ \vec{x} \in \mathbb{R}^2 \quad \text{or} \quad \begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}, \ (x,y) \in \mathbb{R}^2 \ \text{or} \ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

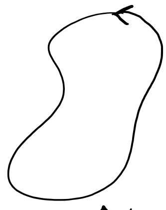
- vector field does not depend explicitly on t
- solutions $\vec{x}(t)$ trace out <u>trajectories</u> (or <u>orbits</u>) in $\mathbb{R}^2$



$\dot{\vec{x}}(t)$ is always tangent to the trajectory

- trajectory shows no information about velocity

- the phase portrait is the collection of all trajectories
- the flow is the collection of all solutions $\vec{x}(t,\vec{x}_0)$
- closed orbits correspond to periodic solutions: there exists $T>0$ such that $\vec{x}(t+T)=\vec{x}(t)$ for all $t\in\mathbb{R}$



↑ closed orbit

- often useful to plot the vector field (phase velocity vector at each point $\vec{x}$) or direction field (normalised phase velocity vector field)
- null clines are the implicitly defined curves

  $f(x,y)=0$ (on this curve $\dot{x}=0$, trajectories have vertical tangents)

  $g(x,y)=0$ (on this curve $\dot{y}=0$, trajectories have horizontal tangents)
- trajectories can not cross since there is no explicit time dependence
- crossing trajectories violate existence & uniqueness theorem
- numerical methods (e.g. RK4, Gear's method) give approximate solutions

Fixed points and linearisation

$$\dot{\vec{x}}=\vec{f}(\vec{x}),\quad \vec{x}\in\mathbb{R}^2$$

note: these are not trajectories!

- a fixed point $\vec{x}^*=(x^*,y^*)$ or $\begin{pmatrix}x^*\\y^*\end{pmatrix}$ is a solution of $\begin{cases}f(x,y)=0\\g(x,y)=0\end{cases}$
- to describe dynamics near a fixed point

$$u(t)=x(t)-x^*,\quad v(t)=y(t)-y^*$$

$$\dot{u}=\dot{x}=f(x,y)=f(x^*+u,\,y^*+v)$$

norm

$$\dot{u}=\underbrace{f(x^*,y^*)}_{0}+\frac{\partial f}{\partial x}(x^*,y^*)u+\frac{\partial f}{\partial y}(x^*,y^*)v+O(\|(u,v)\|^2)$$

↑ higher order terms

$$\dot{u}=\frac{\partial f}{\partial x}(x^*,y^*)u+\frac{\partial f}{\partial y}(x^*,y^*)v+O(\|(u,v)\|^2)$$

- similarly

$$\dot{v}=\frac{\partial g}{\partial y}(x^*,y^*)u+\frac{\partial g}{\partial y}(x^*,y^*)v+O(\|(u,v)\|^2)$$

$$\begin{pmatrix}\dot{u}\\\dot{v}\end{pmatrix}=\begin{pmatrix}f_x(x^*,y^*) & f_y(x^*,y^*)\\ g_x(x^*,y^*) & g_y(x^*,y^*)\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}+O(\|(u,v)\|^2)$$

↑ $\frac{\partial g}{\partial y}$

$$\dot{\vec{u}} = A\vec{u} + O(\|(u,v)\|^2) \text{ where } A = D\vec{f}(\vec{x}^*) = \begin{pmatrix} f_x(x^*,y^*) & f_y(x^*,y^*) \\ g_x(x^*,y^*) & g_y(x^*,y^*) \end{pmatrix}$$

- $Df(\vec{x}^*)$ is the Jacobian matrix (or derivative) of $\vec{f}$ at $\vec{x}^*$
- approximation: the linearisation at $\vec{x}^*$ is

$$\dot{\vec{u}} = A\vec{u}, \ \vec{u} \in \mathbb{R}^2 \text{ where } A = D\vec{f}(\vec{x}^*)$$

↑dropped higher order terms

- if $\vec{x}^*$ is hyperbolic (i.e. the linearisation at $\vec{x}^*$ is hyperbolic), then the linearisation at $\vec{x}^*$ correctly predicts the topological type (attractor, repeller, or saddle point) of $\vec{x}^*$ and its stability

Rabbits vs. sheep

$N_1(t), N_2(t)$ populations of two species that compete for the same resources (e.g. rabbits, sheep)

- per capita growth rate of species 1:

maximum per capita growth rate

$$\frac{\dot{N}_1}{N_1} = \underbrace{r_1\left(1 - \frac{N_1}{k_1}\right)}_{\text{logistic model}} \underbrace{- b_1 N_2}_{\text{reduced per capita growth rate if species 2 is present}}$$

$$\dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{k_1}\right) - b_1 N_1 N_2$$

$$= N_1\left(r_1 - \frac{r_1}{k_1}N_1 - b_1 N_2\right)$$

- similarly

$$\dot{N}_2 = N_2\left(r_2 - \frac{r_2}{k_2}N_2 - b_2 N_1\right)$$

- doing dimensional analysis and scaling (a specific example)

$$\frac{dx}{d\tau} = x(3 - x - 2y)$$

x = dimensionless rabbit population
$x \geq 0$ (biological system)

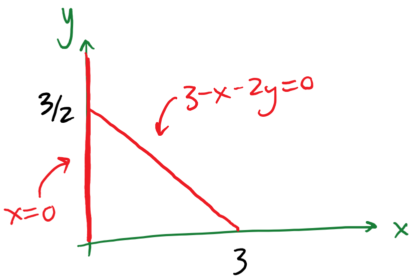
$$\frac{dy}{d\tau} = y(2 - y - x)$$

y = dimensionless sheep population
$y \geq 0$

- nullclines:

$$\frac{dx}{d\tau} = 0 \rightarrow x(3 - x - 2y) = 0 \longrightarrow x = 0 \text{ OR } 3 - x - 2y = 0$$

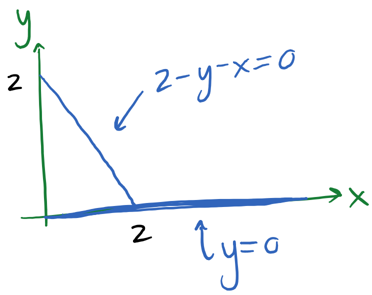
$$y = -\frac{1}{2}x + \frac{3}{2}$$



$$\frac{dy}{d\tau}=0 \rightarrow y(2-y-x)=0 \longrightarrow y=0 \text{ or } 2-y-x=0$$

$$y=-x+2$$

red & blue intersections are the fixed points



· fixed points (analytically)

$$\vec{x}^* = \underbrace{(0,0)}_{\text{both extinct}},\ \underbrace{(3,0)}_{\text{sheep extinct}},\ \underbrace{(0,2)}_{\text{rabbits extinct}},\ \underbrace{(1,1)}_{\text{coexistance}}$$