29 January - MATH 345

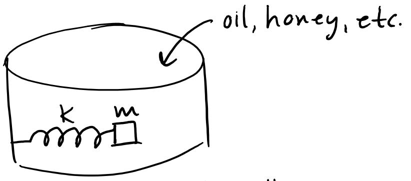
January 29, 2015 1:55 PM

- HW 2 due February 5 (next Thursday)

- 2nd-order equations (even non-linear) can be expressed as systems of 1st-order equations

$\boxed{\text{Example 2.1}}$

mass-spring system with viscous damping



Hooke's "law"

$$\underbrace{m\frac{d^2x}{dt^2}}_{\substack{\text{inertia}\\ \text{term}}} = \underbrace{-kx}_{\substack{\text{spring}\\ \text{restoring}\\ \text{force}}} - \underbrace{b\frac{dx}{dt}}_{\substack{\text{viscous}\\ \text{damping}}}$$

define $y = \frac{dx}{dt}$

$$\frac{dx}{dt} = y \quad \text{(by definition!)}$$

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} = -\frac{k}{m}x - \frac{b}{m}\overset{y}{\frac{dx}{dt}}$$

get a system:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{k}{m}x - \frac{b}{m}y \end{cases}$$

OR

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}$$

- in general, consider

$$\dot{\vec{x}} = A\vec{x}, \quad \vec{x} \in \mathbb{R}^2$$

- fixed points:

$\vec{0} = A\vec{x}$

- $\vec{x}^* = \vec{0}$ is a fixed point (there could be others)

<u>Classification of 2-dim linear systems</u>

- are there solutions of the form $\vec{x}(t) = e^{\lambda t}\vec{v}$?

$$\dot{\vec{x}}(t) = A\vec{x}(t)$$
$$\lambda e^{\lambda t}\vec{v} = Ae^{\lambda t}\vec{v}$$
$$\lambda\vec{v} = A\vec{v}$$

plug in $\vec{x} = e^{\lambda t}\vec{v}$ & take derivative

- if λ is an eigenvalue and $\vec{v}$ is an eigenvector, then $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a solution

- if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the characteristic equation (to find eigenvalues) is

identity matrix

$$\underbrace{(A - \lambda I)}_{\text{matrix must be singular (i.e. determinant is zero)}}\vec{v} = \vec{0} \quad , \vec{v} \neq 0$$

$$0 = \det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc)$$

$$a + d = \mathrm{trace}(A) = \tau$$

$$ad - bc = \det(A) = \Delta$$

$$\lambda^2 - \tau\lambda + \Delta = 0$$

- roots of this are eigen values

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

<u>Exercise</u> show that $\lambda_1 + \lambda_2 = \tau$

$\lambda_1\lambda_2 = \Delta$

- diagonalised matrix is $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
- trace & determinant are invariant
- eigenvalues determine the stability
- trace & determinant of A determine stability

· recall: if eigenvalues are real and distinct, then the general solution is

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

Exercise show that if $\Delta < 0$ then both eigenvalues are real and have opposite signs

$\boxed{\text{Example 2.2}}$

$$\dot{\vec{x}} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}$$

$$\Delta = (1)(-2) - (1)(4) = -6 < 0$$

so λ_1, λ_2 will be real and have opposite signs

$$\tau = 1 - 2 = -1$$

$$\lambda^2 - (-1)\lambda - 6 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda - 2)(\lambda + 3) = 0$$

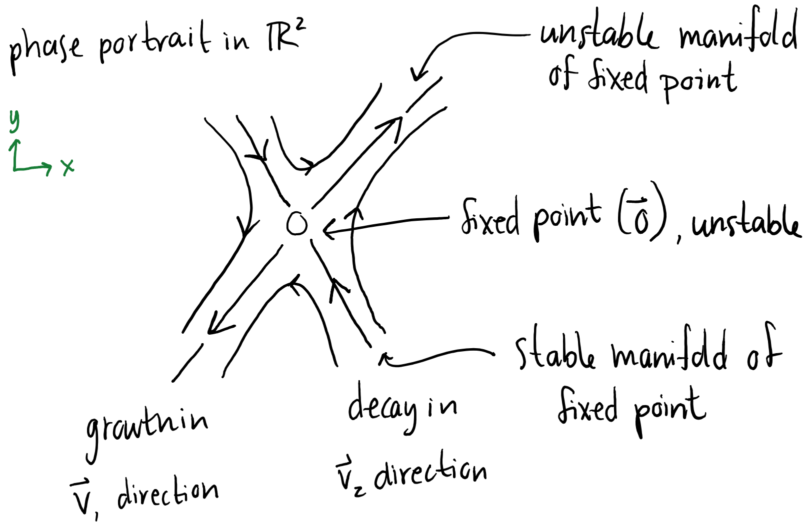
$$\lambda_1 = 2, \quad \lambda_2 = -3$$

Exercise find eigenvectors $\vec{v}_1, \vec{v}_2$ for λ_1, λ_2

general solution:

$$\vec{x}(t) = \underbrace{c_1 e^{2t} \vec{v}_1}_{\text{growth}} + \underbrace{c_2 e^{-3t} \vec{v}_2}_{\text{decay}}$$

phase portrait in $\mathbb{R}^2$

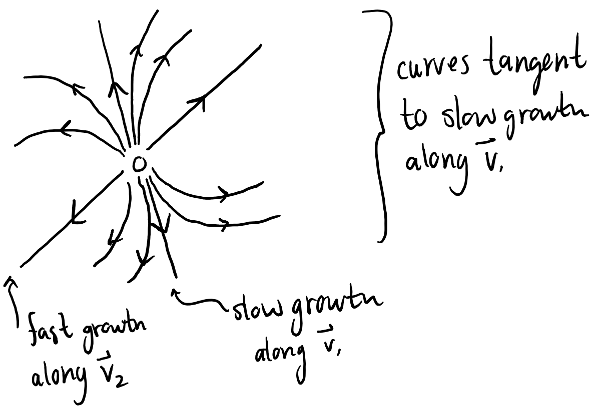


- this type of fixed point is called a <u>saddle point</u> and it is unstable

- If $\tau^2 - 4\Delta > 0$ and $\Delta > 0$ then λ_1, λ_2 are real, distinct and both have the same sign as τ. In this case, the fixed point $\vec{0}$ is called a <u>node</u>

Exercise Show i) $\tau > 0$ is stable
ii) $\tau < 0$ is unstable

e.g. $0 < \lambda_1 < \lambda_2$



Exercise Carefully plot phase portrait of

$$\dot{\vec{x}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}$$

- if $\tau^2 - 4\Delta < 0$, then the eigenvalues are complex

$$\lambda_1, \lambda_2 = \frac{\tau \pm i\sqrt{4\Delta - \tau^2}}{2} = \alpha + i\omega$$

$$\alpha = \mathrm{Re}(\lambda_1) = \tau/2$$

$$\omega = \mathrm{Im}(\lambda_1) = \frac{\sqrt{4\Delta - \tau^2}}{2}$$

$$e^{\lambda_1 t} = e^{(\alpha + i\omega)t} = e^{\alpha t} e^{i\omega t} = e^{\alpha t}(\cos\omega t + i\sin\omega t)$$

- components of (real) general solution $\vec{x}(t)$ are linear combinations of $e^{\alpha t}\cos\omega t$ and $e^{\alpha t}\sin\omega t$
- α has the same sign as τ
- if $\tau \neq 0$ then the fixed point $\vec{0}$ in this case is called a spiral

  stable if $\tau < 0$

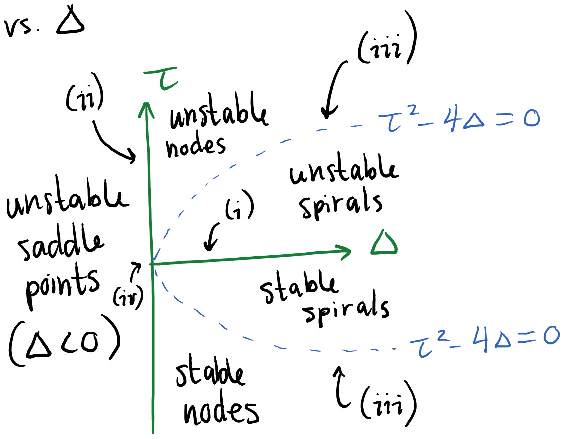
  unstable if $\tau > 0$

e.g.

$\tau > 0$ $\qquad\qquad$ $\tau < 0$

- classification:

  τ vs. Δ

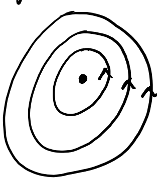


- "borderline" cases

  (i) $\tau = 0$, $\Delta > 0$

  purely imaginary eigenvalues $\pm i\omega$

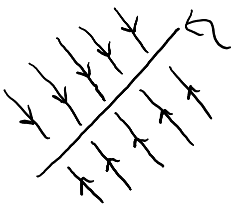
  "linear centre"



  (ii) $\Delta = 0$, $\tau \neq 0$

  simple zero eigenvalue

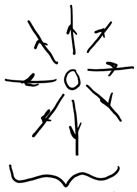
  "non-isolated fixed points"



(iii) $\tau^2 - 4\Delta = 0$, $\tau \neq 0$

double non-zero eigenvalue $\lambda_{1,2} = \tau/2$

"stars, degenerate nodes"



(iv) $\tau = 0$, $\Delta = 0$

double zero eigenvalue

Stability

$\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{x} \in \mathbb{R}^2$, i.e. $\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$ or $\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$

- fixed point $\vec{x}^*$ is a solution of $\vec{0} = \vec{f}(\vec{x})$
- Lyapunov stable if for all $\vec{x}_0 = \vec{x}(0)$ sufficiently close to $\vec{x}^*$, $\vec{x}(t)$ remains arbitrarily close to $\vec{x}^*$ for all $t \geq 0$
- Unstable if not Lyapunov stable (equivalent to definition from Jan. 8 for 1D)
- Attracting if for all $\vec{x}_0$ sufficiently close to $\vec{x}^*$, $\vec{x}(t) \to \vec{x}^*$ as $t \to \infty$
- Stable (or asymptotically stable) if Lyapunov stable and attracting
- Neutrally stable if Lyapunov stable but not attracting