7 April - MATH 345

April 7, 2015 2:08 PM

Lorenz eqns, cont.

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

- for $\sigma = 10$, $b = 8/3$, $r = 28$ there appears in numerical simulations what appears to be a chaotic attractor
- seems to be essentially (except for some "unimportant" details) independent of step size
- numerical method used

$$\vec{\nabla} \cdot \vec{f}(\vec{x}) = -\sigma - 1 - b < 0 \rightarrow \text{(it can be shown) there is an attractor}$$

- is the attractor chaotic?
- the maximal Lyapunov exponent $\lambda(\vec{x}_0)$ for an initial value is defined as follows:

i) let $\vec{x}(t)$ be the solution of IVP $\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{x}(0) = \vec{x}_0$

let $\vec{x} = \vec{x}^*(t)$ be the solution of $\dot{\vec{x}} = \vec{f}(\vec{x})$, $\underbrace{\vec{x}(0)}_{\vec{x}^*(0)} = \vec{x}_0 + \vec{u}_0$

$\|\vec{u}_0\|$ small

ii) to approximate $\vec{x}^*(t) - \vec{x}(t)$, linearise the system about the (moving) trajectory $\vec{x}(t)$ and solve the IVP

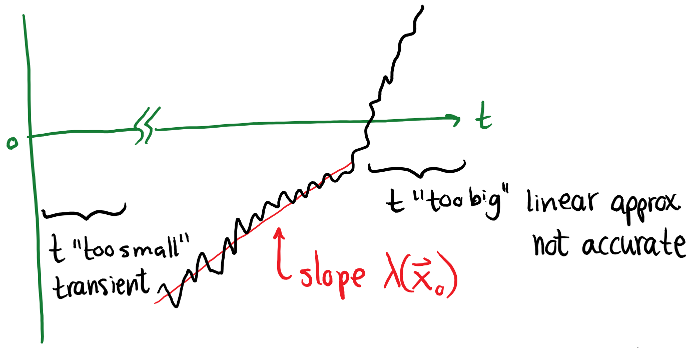
$$\dot{\vec{u}} = D\vec{f}(\vec{x})\vec{u}, \quad \vec{u}(0) = \vec{u}_0$$

then $\vec{x}^*(t) - \vec{x}(t) = \vec{u}(t) + O(\|\vec{u}_0\|^2)$ for finite t

iii) $$\lambda(\vec{x}_0) = \limsup_{t \to \infty} \max_{\vec{u}_0 \neq \vec{0}} \frac{1}{t} \ln\left(\frac{\|\vec{u}(t)\|}{\|\vec{u}_0\|}\right)$$

if $\lambda(\vec{x}_0) > 0$ then there is SDIC

- the typical $\ln\left(\frac{\|\vec{x}^*(t) - \vec{x}(t)\|}{\|\vec{u}_0\|}\right)$ vs t



- XPP (or other software) can approximate the maximal Lyapunov exponent with more or less accuracy depending (in part) or skill and knowledge of user

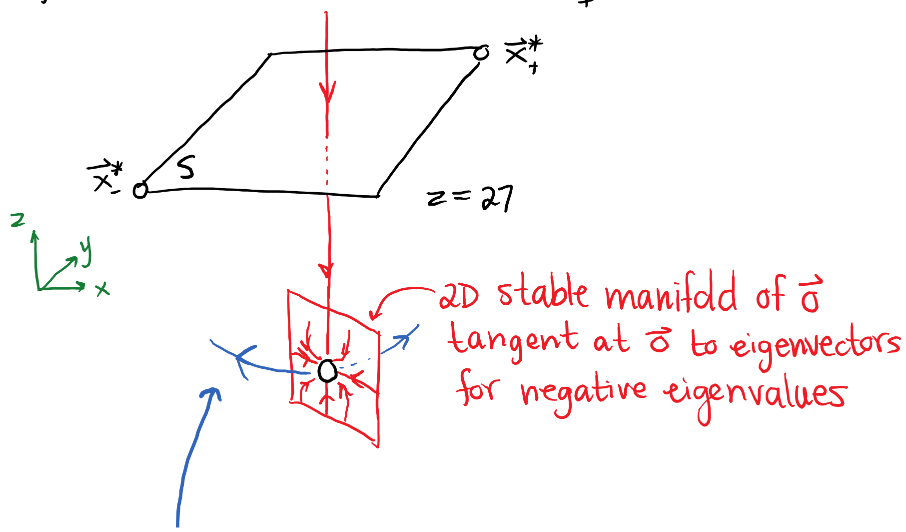
- to rigorously prove chaos, need to show SDIC for <u>all</u> $\vec{x}_0$ in the basin of attraction

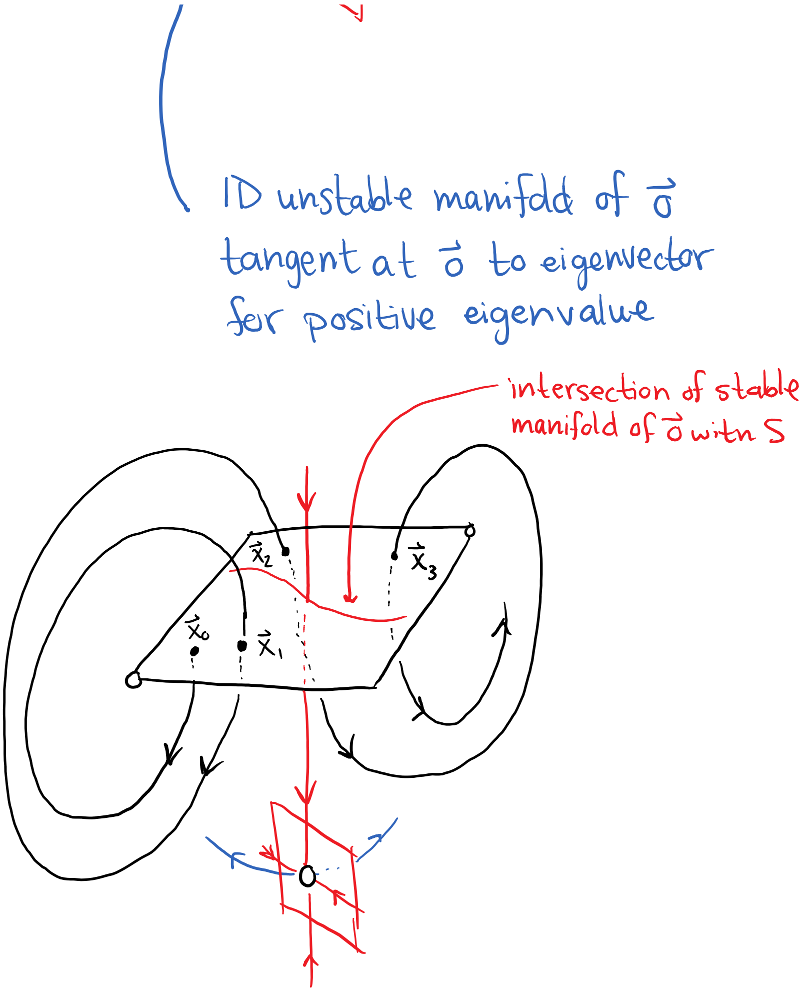
- careful numerical experiments with Lyapunov exponents indicate that the Lorenz attractor is very likely to be chaotic

- another approach: Poincaré map

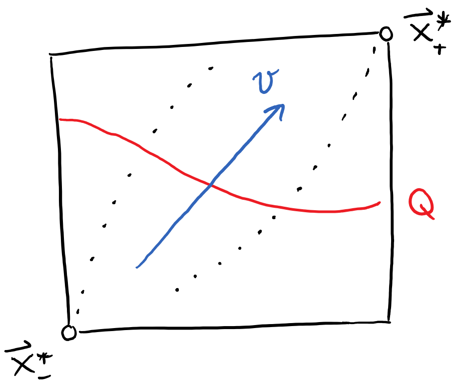
- recall for $r > 1$ there are 3 fixed points: $\vec{0}$ (whose linearisation has 2 negative eigenvalues and 1 positive eigenvalue) and $\vec{x}^*_{\pm} = (\pm 8.48, \pm 8.48, 27)$

- define a section S between $\vec{x}^*_+$ and $\vec{x}^*_-$ with $z = 27$





· result (symmetric)

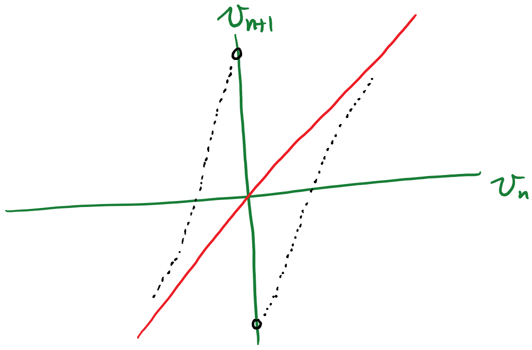


iterates (x_n, y_n) appear to lie on 2 curves

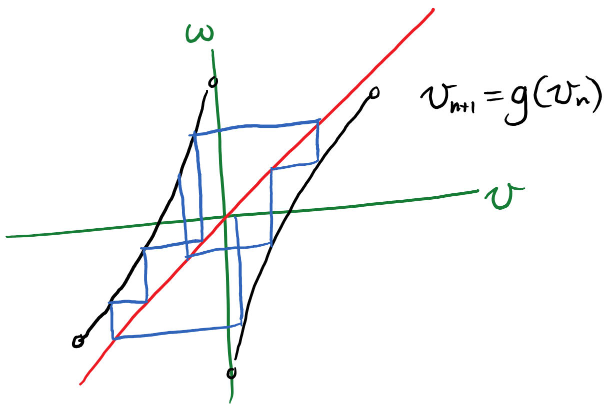
the 2D Poincaré map is approximately 1D

· for the flow in $\mathbb{R}^3$, the attractor is thin and flat, approx 2D, recall it has volume 0

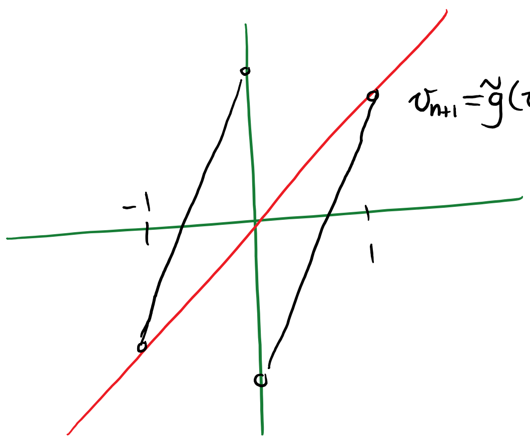
· measuring the signed distance from Q by some coordinate v, make a Ruelle plot v_{n+1} vs. v_n



· approximate 1D map



· simplified 1D map – piecewise linear



$$v_{n+1} = \tilde{g}(v_n) = \begin{cases} 2v_n + 1 & \text{if } -1 < v_n < 0 \\ 2v_n - 1 & \text{if } 0 < v_n < 1 \end{cases}$$

looks like binary shift map

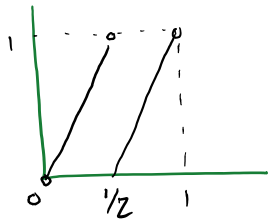
· do coordinate transformation $v = \alpha u + \beta$

$$\alpha u_{n+1} + \beta = \tilde{g}(\alpha u_n + \beta) \qquad (\text{from } v_{n+1} = \tilde{g}(v_n))$$

$$u_{n+1} = \frac{1}{\alpha}\left[\tilde{g}(\alpha u_n + \beta) - \beta\right] = \tilde{f}(u_n)$$

$$= \begin{cases} 2u_n & \text{if } 0 < u_n < 1/2 \\ 2u_n - 1 & \text{if } 1/2 < u_n < 1 \end{cases}$$

choose α, β to get this



this is the doubling (binary shift) map

- we know some of its properties
  - no fixed point
  - countable infinity of periodic points
  - uncountable infinity of non-periodic points
  - SDIC ($\lambda = \ln 2 > 0$)

- this suggests (but does not prove) similar properties for the Poincaré map and Lorenz attractor

- work backwards:

$\tilde{f} \to \tilde{g}$: no problem, just a smooth coord. change

$\tilde{g} \to g$: harder, only make general assumptions about g (e.g. $g' > 1$), but okay

$g \to$ actual Poincaré map : ? need to clarify what "approx 1D" means

Poincaré map $\to$ Lorenz flow : numerical solutions "probably" okay