10 March - MATH 345

March 10, 2015 1:59 PM

- HW4 due March 19 (extended by one week)

Example 2.10, cont

$$\begin{cases} \dot{x} = \mu - x^2 \\ \dot{y} = -y \end{cases}$$

- fixed points

$(x^*, y^*) = (\pm\sqrt{\mu}, 0)$ if $\mu \geq 0$

- x & y are decoupled $\to$ diagonal matrices
- linearised stability

$$D\vec{f}(x,y) = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix}$$

a) $\mu < 0 \to$ no fixed points (only care about real fixed points)

b) $\mu = 0 \to$ one fixed point $(x^*, y^*) = (0,0)$

$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, $\tau = -1$, $\Delta = 0$

fixed point is non-hyperbolic with a simple zero eigenvalue

c) $\mu > 0 \to$ two fixed points

i) $(x^*, y^*) = (-\sqrt{\mu}, 0)$

$A = \begin{pmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{pmatrix}$, $\Delta < 0$

hyperbolic saddle point

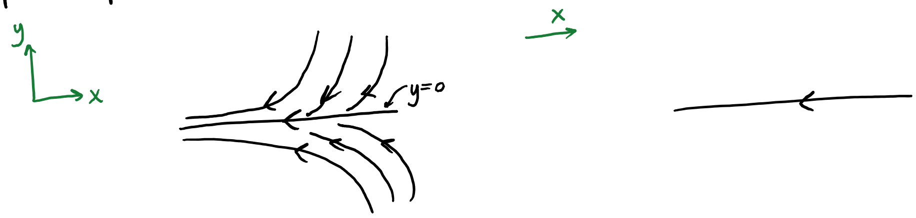
ii) $(x^*, y^*) = (\sqrt{\mu}, 0)$

$A = \begin{pmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{pmatrix}$ $\Delta > 0$, $\tau < 0$

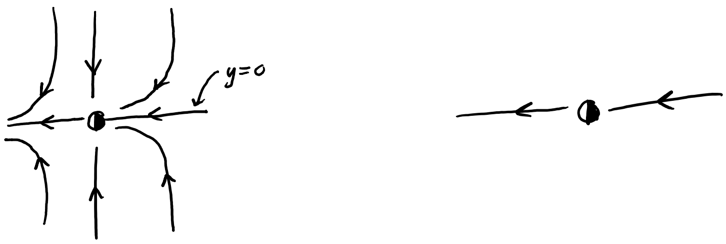
hyperbolic attractor, node

- at $\mu = \mu_c = 0$, the hyperbolic fixed point appears and splits into a saddle point and a node: this is a saddle-node bifurcation in $\mathbb{R}^2$

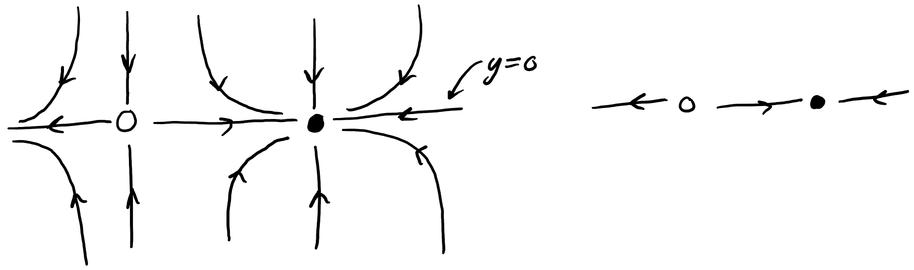
phase portraits in $\mathbb{R}^2$



$\mu < 0$



$\mu = 0$



$\mu > 0$

- along a 1D curve (in this case, the x-axis), we see the phase portraits of a saddle-node bifurcation in a 1D flow

Exercise: Find fixed points, determine linear stability, and sketch phase portraits for $\mu < \mu_c$, $\mu = \mu_c$, $\mu > \mu_c$

(a) $\begin{cases} \dot{x} = \mu x - x^2 \\ \dot{y} = -y \end{cases}$

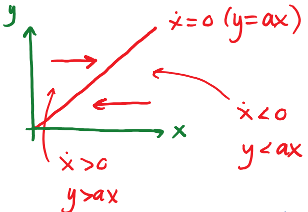
(b) $\begin{cases} \dot{x} = \mu x - x^3 \\ \dot{y} = -y \end{cases}$

Example 2.11 genetic or other kind of "switch"

$$\begin{cases} \dot{x} = -ax + y \\ \dot{y} = \dfrac{x^2}{1+x^2} - y \end{cases}, \quad 0 < x < \infty, \quad 0 < y < \infty, \quad a > 0$$

- nullclines, fixed points, and direction field:

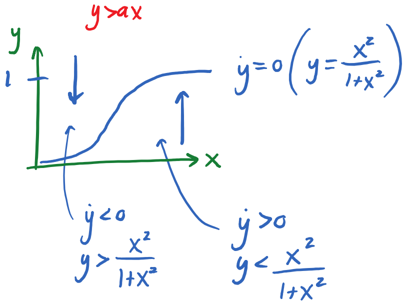
$\dot{x} = 0 \rightarrow y = ax$



$\dot{y} = 0 \rightarrow y = \dfrac{x^2}{1+x^2}$

for small x, $y \sim x^2$

for big x, $y \sim 1$



- fixed points are where both curves intersect
- number of fixed points depends on value of a
- graphically there are 0, 1, or 2 fixed points
- analytically

$$ax = \frac{x^2}{1+x^2} \rightarrow ax + ax^3 = x^2$$

$$x(a - x + ax^2) = 0$$

$x = 0$, but not in domain

or

$$x = \frac{1 \pm \sqrt{1-4a^2}}{2a}$$

- if $a > \frac{1}{2}$ there are no real solutions
- if $a = \frac{1}{2}$, there is one solution $x^* = 1$
- if $a < \frac{1}{2}$, there are two solutions

$$x_1^* = \frac{1 - \sqrt{1-4a^2}}{2a}$$

$$x_2^* = \frac{1 + \sqrt{1+4a^2}}{2a}$$

Exercise: show that $x_1^* < 1 < x_2^*$ for $0 < a < \frac{1}{2}$

note that $x_1^* \rightarrow 1^-$, $x_2^* \rightarrow 1^+$ as $a \rightarrow \frac{1}{2}^-$ ← from direction of smaller values

· linearised stability:

$$D\vec{f}(x,y,a)=\begin{pmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -1 \end{pmatrix}$$

· $a=a_c=\frac{1}{2}$, $(x^*,y^*)=(1,\frac{1}{2})$

$$A=D\vec{f}(1,\tfrac{1}{2},\tfrac{1}{2})=\begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & -1 \end{pmatrix}$$

$$\Delta_c=0,\quad \tau_c=-\frac{3}{2}<0$$

non-hyperbolic simple zero eigenvalue

· $a<\frac{1}{2}$

$$A_{1,2}=D\vec{f}(x^*_{1,2},y^*_{1,2},a)$$

$$=\begin{pmatrix} -a & 1 \\ \frac{2x^*_{1,2}}{[1+(x^*_{1,2})^2]^2} & -1 \end{pmatrix}$$

$$\Delta_{1,2}=a-\frac{2x^*_{1,2}}{[1+(x^*_{1,2})^2]^2}\Bigg\}\text{want to simplify this expression}$$

$$\tau_{1,2}=-a-1$$

· $x^*_{1,2}$ satisfy $a-x^*_{1,2}+a(x^*_{1,2})^2=0$

$$a[1+(x^*_{1,2})^2]=x^*_{1,2}$$

$$[1+(x^*_{1,2})^2]=\frac{x^*_{1,2}}{a}$$

$$\Delta_{1,2}=a-\frac{2x^*_{1,2}}{[1+(x^*_{1,2})^2]\frac{x_{1,2}}{a}}$$

$$=a-\frac{2a}{1+(x^*_{1,2})^2}$$

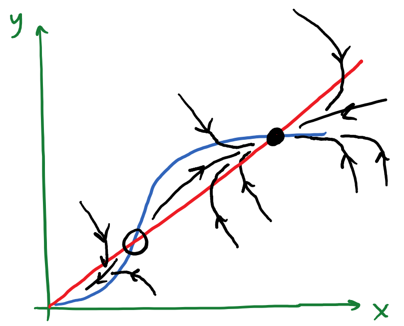
$$=a\frac{(x^*_{1,2})^2-1}{1+(x^*_{1,2})^2}$$

· using the results of the previous exercise

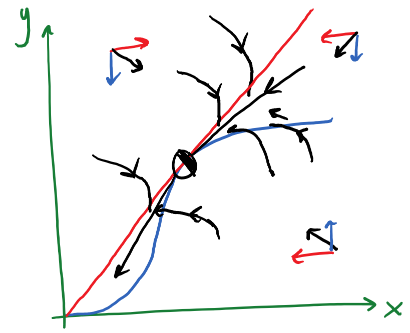
$x^*_1<1 \rightarrow \Delta_1<0 \rightarrow$ hyperbolic saddlepoint

$X_2^* > 1 \to \Delta_2 > 0$, $\tau_2 < 0 \to$ hyperbolic attractor (node at least when a is near $\frac{1}{2}$)

- phase portraits



$a < \frac{1}{2}$



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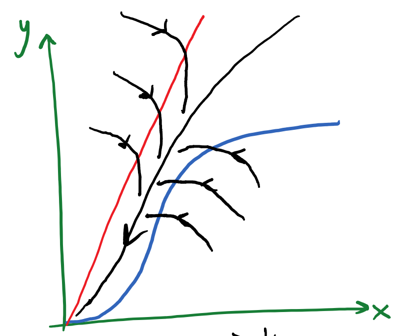
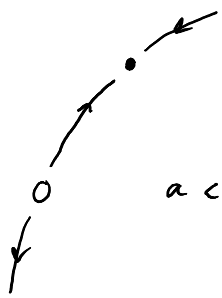


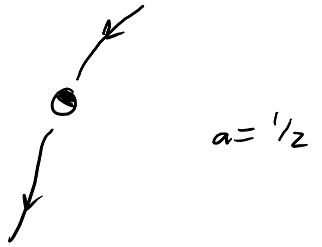
$a > 1/2$

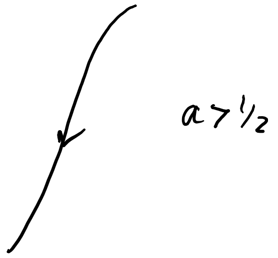
cross nullclines horizontally or vertically ($\dot{y}=0$, $\dot{x}=0$)

- on a 1D curve (centre manifold)



$a < 1/2$





$a < 1/2$

- bifurcation diagram plot y^* (or x^* or $\sqrt{(x^*)^2 + (y^*)^2}$ etc.) vs a
- AUTO can do this