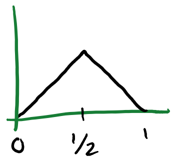
2 April - MATH 345

April 2, 2015 2:01 PM

- tent map:

$$f(x) = \begin{cases} rx & \text{if } 0 \le x \le 1/2 \\ r(1-x) & \text{if } 1/2 \le x \le 1 \end{cases}$$

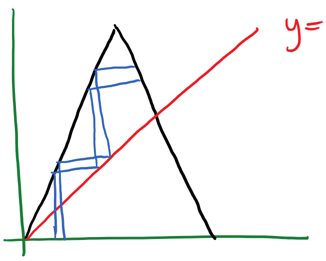


- Lyapunov exponent for a "typical" orbit $\{x_0, x_1, x_2, x_3 \dots\}$ ($x_n \neq 0, 1/2, 1$ for any n)

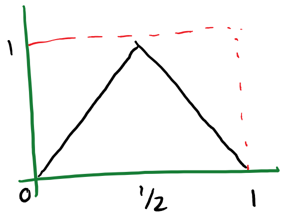
$$f'(x_i) = r \quad \text{if} \quad 0 < x_i < 1/2$$
$$f'(x_i) = -r \quad \text{if} \quad 1/2 < x_i < 1$$

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| = \lim_{n\to\infty} \frac{1}{n} n \ln r = \ln r$$

- if $r > 1$ there is SDIC

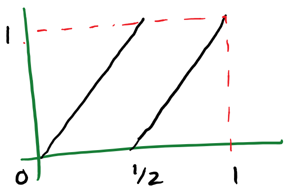


- when $r=2$, the tent map of slope 2 has similarities both to the doubling/binary shift map and to the logistic map with $r=4$



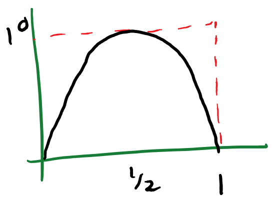
tent map
$y = 2\min\{x, 1-x\}$

$\lambda(x_0) = \ln 2$



doubling
$y = 2x \pmod 1$

$\lambda(x_0) = \ln 2$

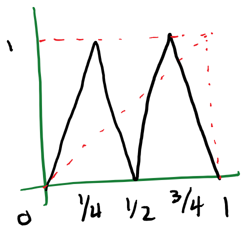


logistic
$y = 4x(1-x)$

$\lambda(x_0) = \ln 2$

for "typical" orbits

- for example, graph of 2nd iterate of $f(x) = 2\min\{x, 1-x\}$ is



$y = f^2(x)$

Exercise: Find formula for $f^2(x)$

find 2-cycle $\{p, q\}$ for f

Lorenz equations

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

- phase space $(x,y,z) \in \mathbb{R}^3$, parameters $\underbrace{\sigma, r, b}_{\text{physical parameters}} > 0$
- originally derived as ODE approx. to a system of PDEs describing convection in a layer of fluid
- as a model, only valid for (x,y,z) near $(0,0,0)$ and r near 1
- but the ODEs themselves have interesting behaviours
- symmetry: eqns are unchanged (equivariant) under "reflection"

$$(x,y,z) \to (-x,-y,z)$$

- this implies if $(x(t), y(t), z(t))$ is a solution, then $(-x(t), -y(t), z(t))$ is a solution. So individual solutions are either symmetric themselves or have a symmetric partner
- fixed point: $\vec{x}^* = (0,0,0)$ is a fixed point (an example of a symmetric solution) (for all $\sigma, r, b > 0$)
- linear stability:

$$D\vec{f}(\vec{x}) = \begin{pmatrix} -\sigma & -\sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

at $\vec{x}=\vec{0}$

block

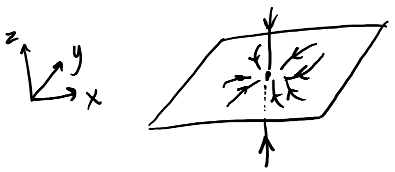
$$A=\begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

block

· eigenvalues: $\lambda_3=-b$ and eigenvalues λ_1, λ_2 of

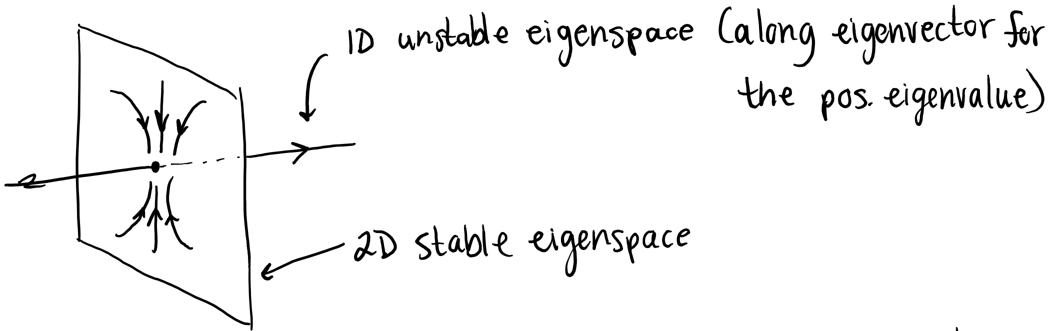
$$\left.\begin{aligned} \hat{A} &= \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \\ \hat{\Delta} &= \sigma(1-r) \\ \hat{\tau} &= -\sigma-1 \end{aligned}\right\} \quad \lambda_{1,2}=\frac{\hat{\tau}\pm\sqrt{\hat{\tau}^2-4\hat{\Delta}}}{2}$$

· if $0<r<1$, $\vec{0}$ is a hyperbolic attractor:
all 3 eigenvalues have negative real parts
(in fact, they are negative and real)



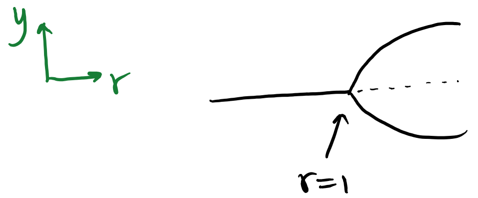
· if $r=1$, $\vec{0}$ is non hyperbolic, with a simple zero eigenvalue
(eigenvalues $0, -\sigma-1, -b$)

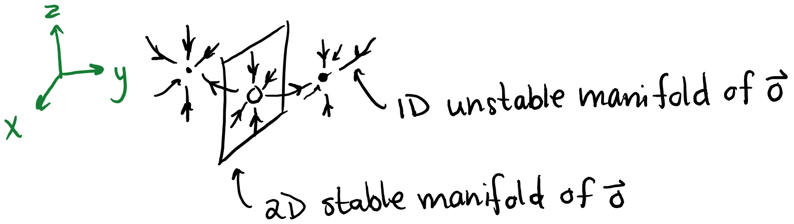
· if $r>1$, $\vec{0}$ is hyperbolic and unstable
(for r near 1 at least, one eigenvalue is positive, others negative)



· as you pass through $r=1$, $\vec{0}$ loses its stability and you expect a bifurcation. Since there is the reflection symmetry, can guess that it is a pitchfork bifurcation

- there is a bifurcation at $r_c = 1$, $\vec{x}_c^* = \vec{0}$
- it turns out to be a supercritical pitchfork bifurcation in 3D phase space



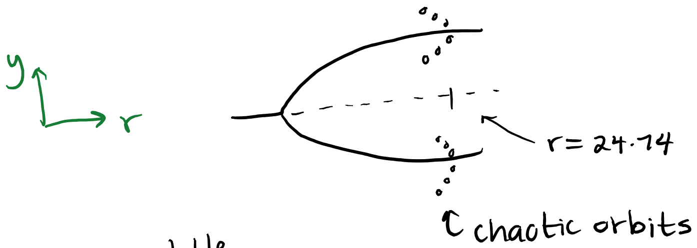


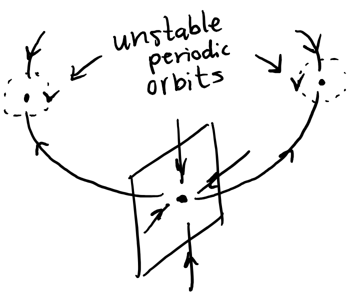
- further analysis shows there is a pair of Hopf bifurcations at

$$r_{c_H} = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \quad \text{if } \sigma - b - 1 > 0$$

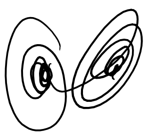
- for $\sigma = 10$, $b = \frac{8}{3}$ (physically reasonable values for the fluid convection model)

$r_{c_H} = 24.74$ and Hopf bifurcations are subcritical





- chaos? for $\sigma = 10$, $b = \frac{8}{3}$, $r = 28$ E. Lorenz (1963) found numerically what looked like a chaotic attractor
- Lorenz attractor: Google, Youtube, p. 326 plate 2



} looks something like this

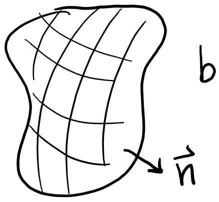
- is there really a chaotic attractor?
- phase volume contraction:

$$\vec{\nabla}\cdot\vec{f} = \frac{\partial}{\partial x}\left[\sigma(y-x)\right] + \frac{\partial}{\partial y}\left[rx-y-z\right] + \frac{\partial}{\partial z}\left[xy-bz\right]$$

$$= -\sigma - 1 - b < 0$$

implies the phase space is shrinking

- consider an open set ("blob") of initial values V_0 in $\mathbb{R}^3$



boundary surface S_0

- solving the IVP

$$\vec{x} = \vec{f}(\vec{x}), \quad \vec{x}(0) = \vec{x}_0 \text{ for every } \vec{x}_0 \in V_0$$

- we generate a volume $V(t)$ of phase points $\vec{x}(t)$ at each time t with boundary surface $S(t)$ and outward unit normal vectors $\vec{n}$

$$\frac{dV}{dt}(t) = \iint_{S(t)} \vec{f}(\vec{x}(t))\cdot\vec{n}\, dA \quad \Big\} \text{ use divergence theorem}$$

$$= \iiint_{V(t)} \underbrace{\vec{\nabla}\cdot\vec{f}}_{\text{const}}\, dV$$

$$= -(\sigma+1+b)\iiint_{V(t)} dV$$

$$= -(\sigma+1+b)V(t)$$

- the volume satisfies a very simple ODE

$$\underline{dV} = -(\sigma+1+b)\,V$$

$$V(t) = V_0 e^{-(\sigma+1+b)t}$$

- $V(t) \to 0$ as $t \to \infty$, this can be shown to imply there is an attractor