10 February - MATH 345

February 10, 2015 1:56 PM

- midterm Tuesday Feb. 24
- HW 3 due Thursday Feb. 26
- HW 3 (analytical) questions good practice for midterm
- practice midterm & solutions may be put on website

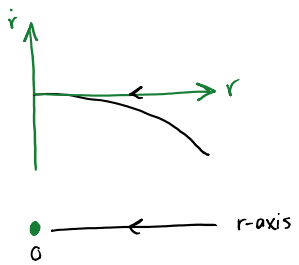
$$\begin{cases} \dot{x} = -y + ax^3 + axy^2 \\ \dot{y} = x + ax^2y + ay^3 \end{cases}$$

- in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$
- the system becomes

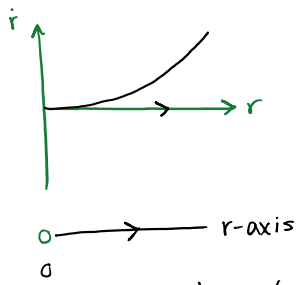
$$\begin{cases} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{cases} \quad \left(\text{Hint: } \tan\theta(t) = \frac{y(t)}{x(t)} \to r^2\dot{\theta} = x\dot{y} - \dot{x}y\right)$$

- plot $\dot{r}$ vs r $(r \geq 0)$

i) $a < 0$

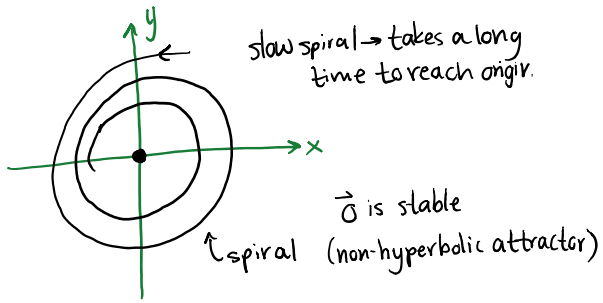


ii) $a > 0$

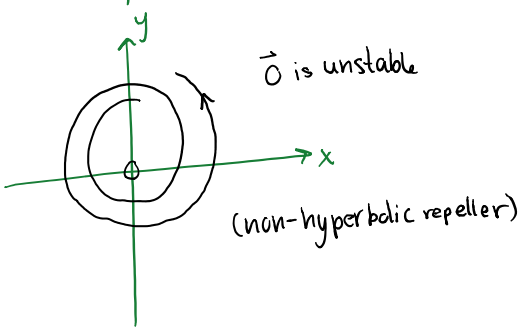


- solution of $\dot{\theta} = 1$, $\theta(t) = t + \theta_0 \pmod{2\pi}$
- phase portrait in $\mathbb{R}^2$

i) $a < 0$



ii) $a > 0$



- needed to look at non-linear terms because the linear stability is the same
- non-hyperbolic fixed point means we need to look at higher order terms
- at a non-hyperbolic fixed point, small nonlinear terms can affect type and stability
- Note: any decay/growth associated with eigenvalue λ, $Re(\lambda)=0$ is slower than exponential decay/growth associated with λ, $Re(\lambda)\neq 0$

*midterm covers up to here

Conservative systems

$$\dot{\vec{x}} = \vec{f}(\vec{x}), \quad \vec{x}\in\mathbb{R}^2$$

- a conserved quantity is any real-valued continuous (i.e. a scalar) function $E(\vec{x})$ that is nonconstant on every open set and is constant on all trajectories $\vec{x}(t)$ of the system:

$$0 = \frac{d}{dt}\left(E(\vec{x}(t))\right) = \vec{\nabla}E(\vec{x}(t))\cdot\dot{\vec{x}}(t) = \vec{\nabla}E(\vec{x}(t))\cdot\vec{f}(\vec{x})$$

dot product

- i.e. $\vec{\nabla}E\cdot\vec{f}=0$
- a system that has a conserved quantity is called conservative
- trajectories in a conservative system must remain on level sets of E

$$E(\vec{x}(t)) = \text{const}$$

- therefore a conservative system cannot have attractors or repellors

Example 2.5(a)

undamped mass-spring system

$$m\ddot{x} = -kx \quad \text{or} \quad m\ddot{x}+kx=0$$

potential energy of the spring is

$$V(x) = \frac{1}{2}kx^2$$

restoring force is

$$F = -kx = -V'(x)$$

$$V(x) = -\int kx\,dx$$

total energy is

$$E = \frac{1}{2}mv^2 + V(x)$$
$$= \frac{1}{2}m(\dot{x})^2 + V(x)$$

let $y = \dot{x} \rightarrow \dot{y} = \ddot{x}$

the equivalent system is

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{k}{m}x \end{cases}$$

*make sure to do this step properly on exams

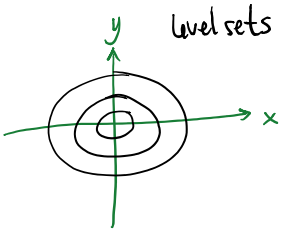
$$E(x,y) = \frac{1}{2}my^2 + \frac{1}{2}kx^2$$

verify E is conserved:

$$\vec{\nabla}E \cdot \vec{f} = \frac{\partial E}{\partial x}\dot{x} + \frac{\partial E}{\partial y}\dot{y}$$
$$= (kx)(y) + (my)\left(-\frac{k}{m}x\right)$$
$$= 0 \checkmark$$

Therefore the trajectories must lie on level sets

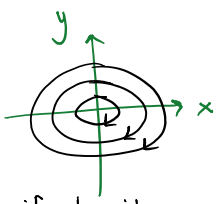
$$\frac{1}{2}my^2 + \frac{1}{2}kx^2 = \text{const}$$



level sets

at $y=0$, $\dot{x}=0$ and $\dot{y}$ has opposite sign of x

phase portrait (look at direction field)



easily verified with explicit solution

$\boxed{\text{Example 2.5(b)}}$

$$\begin{cases} \dot{x} = y & \text{depends only on } y \\ \dot{y} = x - x^3 & \text{depends only on } x \end{cases}$$

suspect $E(x,y)$ is conserved

$$0 = \frac{\partial E}{\partial x}(\dot{x}) + \frac{\partial E}{\partial y}(\dot{y})$$

$$= \frac{\partial E}{\partial x}(y) + \frac{\partial E}{\partial y}(x - x^3)$$

want $\frac{\partial E}{\partial x} = -x + x^3$

↑ negative sign here to correspond with physical energy

$$\frac{\partial E}{\partial y} = y$$

arbitrary function

$$\frac{\partial E}{\partial y} = y \rightarrow E = \int y\,dy = \frac{1}{2}y^2 + f(x)$$

$$\frac{\partial E}{\partial x} = -x + x^3 \rightarrow E = \int (-x + x^3)\,dx = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + g(y)$$

$$\frac{1}{2}y^2 + f(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + g(y)$$

arbitrary constant

so

$$E(x,y) = \underbrace{\frac{1}{2}y^2}_{\text{"kinetic energy"}} \underbrace{- \frac{1}{2}x^2 + \frac{1}{4}x^4}_{\text{"potential energy" } V(x)} \; (+\,C) \text{ is conserved}$$

fixed points

$$\begin{cases} 0 = y \\ 0 = x - x^3 = x(1 - x^2) = x(1+x)(1-x) \end{cases}$$

$$(x^*, y^*) = (0,0), (1,0), (-1,0)$$

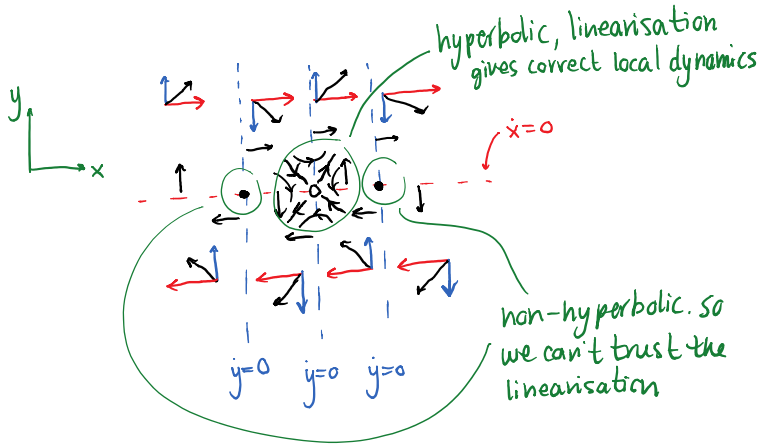
x^* are critical points of $V(x)$

linearisation (check) (note: be able to do linearisations for midterm!)

$(0,0)$ is hyperbolic saddle point

$(\pm 1, 0)$ are non-hyperbolic with purely imaginary eigen values (linear centres)

nullclines and direction field (by hand), linearisations: get limited information



now use fact that E is a conserved quantity

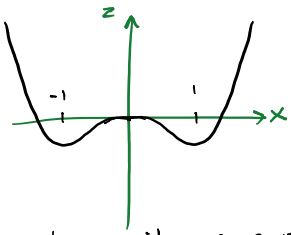
$$\frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = \text{const}$$

consider graph in $\mathbb{R}^3$

$$z = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$$

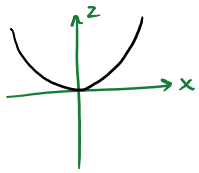
intersection with $y=0$ plane:

$$z = -\frac{1}{2}x^2 + \frac{1}{4}x^4$$



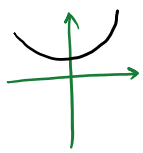
intersection with $x=0$ plane:

$$z = \frac{1}{2}y^2$$

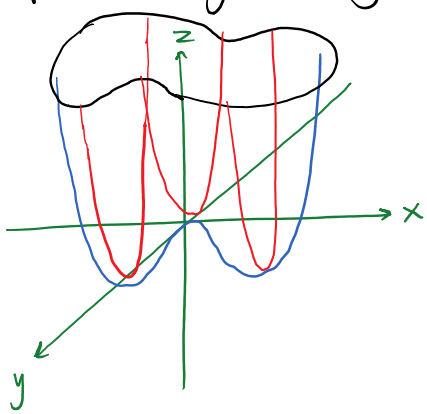


intersection with $x=x_0$ plane:

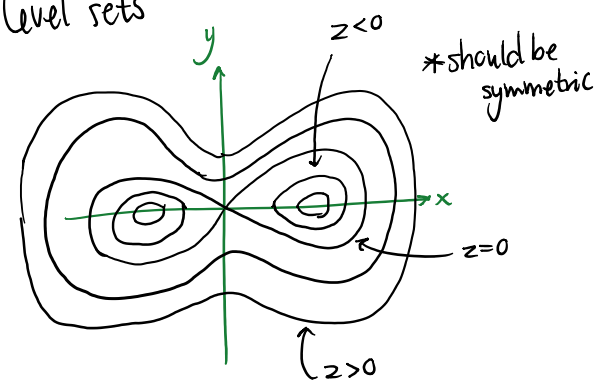
$$z = \frac{1}{2}y^2 + \text{const}$$

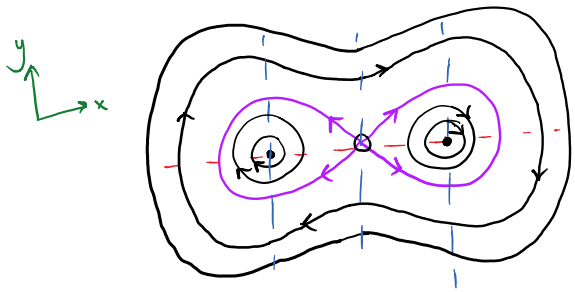


graph $z = E(x,y)$ "energy surface"



Level sets





double well potential