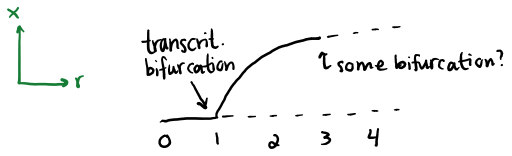
24 March - MATH 345

March 24, 2015 2:00 PM

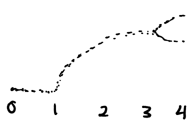
- logistic map $f:[0,1]\to[0,1]$ for $0<r\le 4$

$$x_{n+1} = f(x_n) = r x_n (1-x_n)$$

- bifurcation diagram (partial)

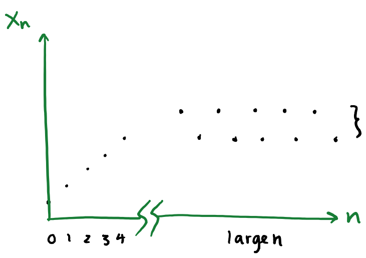


- "numerical bifurcation diagram"
  ↳ see course website



each dot represents $n \sim 350-500$ iterates for a particular value of r

- for fixed $r=3.1$, look at x_n vs. n



} iterates appear to converge to two values on alternate iterates

- a cycle is a periodic orbit for a map

- in the example at $r=3.1$, iterates appear to converge to a stable cycle of period 2 (or period-2 cycle, or 2-cycle)

Logistic map: analytics (2)

- bifurcation at $r_{c2}=3$, $x^*_{c2} = 1-\frac{1}{r_{c2}} = \frac{2}{3}$

- $x^*_{c2}=\frac{2}{3}$ is a non-hyperbolic fixed point with multiplier $\lambda=-1$

- linearisation of map at $x^*=\frac{2}{3}$ when $r=3$ is

$$\eta_{n+1} = (-1)\eta_n$$

- orbit $\{\eta_0, -\eta_0, \eta_0, -\eta_0, \eta_0, -\eta_0, \dots\}$

- every orbit is a 2-cycle (the fixed point $\eta^*=0$ is a trivial 2-cycle)

- for the nonlinear map f itself:

x_0 , $x_1 = f(x_0)$, $x_2 = f(x_1) = \underbrace{f^2(x_0)}_{f(f(x_0))}$

- if there is a 2-cycle, then $x_2 = x_0$

  $x_0 = f^2(x_0)$

- so to look for 2-cycles of f, we look for fixed points of f^2 (2nd iterate)

  $f(x) = rx(1-x)$

  $f^2(x) = f(f(x))$

  $= r f(x)[1 - f(x)]$

  $= r \cdot rx(1-x)[1 - rx(1-x)]$

- fixed point:

  $f^2(x) = x$

  $\underbrace{r^2 x(1-x)[1-rx(1-x)]}_{\text{this is a quartic equation}} - x = 0$

- factor as much as possible

  $x\{r^2(1-x)[1-rx(1-x)] - 1\} = 0$

- $x^* = 0$ is a fixed point of original map (also a trivial 2-cycle)

- every fixed point of f is a trivial 2-cycle: $x^* = 0$, $x^* = 1 - \frac{1}{r}$

- problem should have factors

  (x) and $(x - 1 + \frac{1}{r}$ or $rx - r + 1)$

- expand $r^2(1-x)[1-rx(1-x)] - 1$, $\overbrace{\text{divide by}}^{\text{long division or symbolically (e.g. wolfram alpha)}}$ $rx - r + 1$

  $f^2(x) - x = x(rx - r + 1)\underbrace{[-r^2x^2 + (r^2+r)x - r - 1]}_{\text{find roots of this}}$

- these roots are fixed points of f^2 that are not fixed points of f
  i.e. non-trivial 2-cycles

- $x = \dfrac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r}$ } Exercise: check this

- $(r-3)(r+1) \geq 0$ is required

- since $0 < r \leq 4$, these roots exist for

  $r \geq 3$

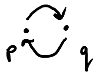
- let $p = p(r) = \dfrac{r+1-\sqrt{(r-3)(r+1)}}{2r}$

  $q = q(r) = \dfrac{r+1+\sqrt{(r-3)(r+1)}}{2r}$

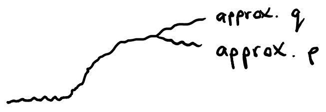
  $\Bigg\}$ p, q are different as long as $r>3$

- as $r \to 3^+$ (from >3 direction), $p(r), q(r) \to \frac{2}{3}$ the nonhyperbolic fixed point

- $\{p, q\}$ is a 2-cycle: $f(p)=q$, $f(q)=p$ for $r>3$



- in numerical bifurcation diagram



- apparently stable (otherwise wouldn't expect to see it with arbitrary choice of x_0)

- linear stability analysis: linearise f^2 at p (or q)

$$\lambda = (f^2)'(p) = \frac{d}{dx} f(f(x)) \Big|_p \quad \text{evaluate at } p$$

$$= f'(f(x)) \cdot f'(x) \Big|_{x=p} \quad \text{from chain rule}$$

$$= f'(f(p)) \cdot f'(p)$$

$$= f'(q) \cdot f'(p)$$

Exercise: show $\lambda = f'(p(r)) f'(q(r))$

$$= 4 + 2r - r^2 \quad (r>3)$$

- as $r \to 3^+$, $\lambda \to 1$

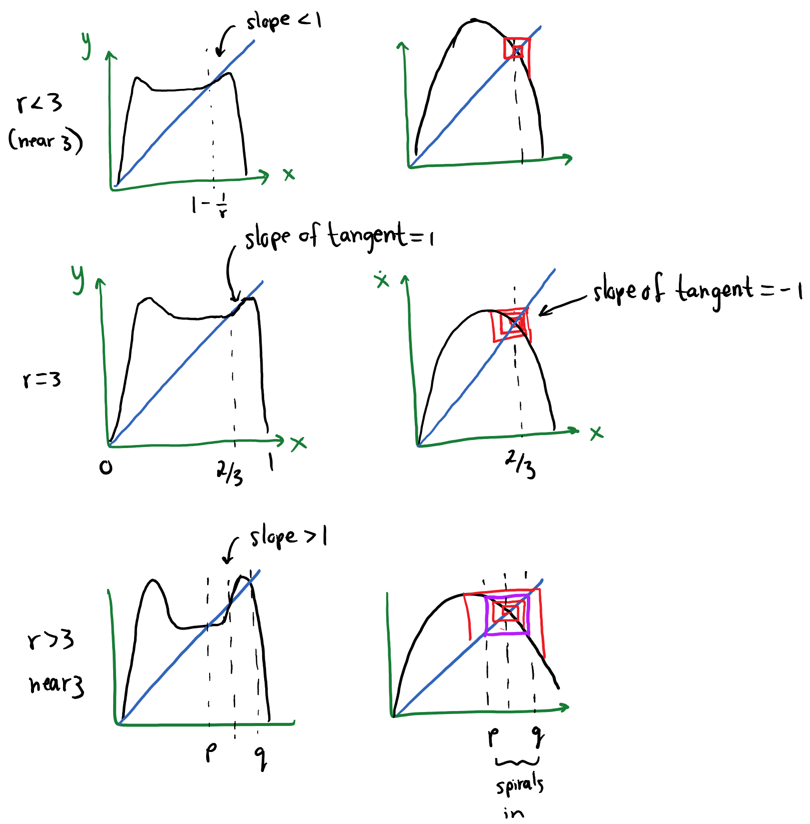
- in fact, if $r>3$ and sufficiently close to 3, then $-1 < \lambda < 1$

  i.e. $|\lambda| < 1$ and the 2-cycle is stable

- in this case, p, q are hyperbolic attracting fixed points for f^2

- $\{p, q\}$ is a hyperbolic attracting 2-cycle for f

- the bifurcation at $r_{c2} = 3$, $x^*_{c2} = \frac{2}{3}$ is called a flip (or period doubling) bifurcation (associated with multiplier passing through -1 as r is changed)



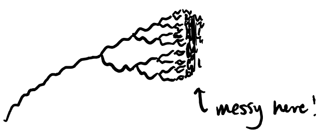
Exercise: solve $\lambda = 4 + 2r - r^2 = -1$, $r > 3$

show this gives $r = 1 + \sqrt{6}$

- at $r_{c_3} = 1 + \sqrt{6}$, f^2 has a flip bifurcation
- if $r > r_{c_3}$, f^2 has a 2-cycle, f has a 4 cycle

Logistic map: numerics (2)

- "numeric bifurcation diagram" done carefully



see p.364 in text

- $r_{c_1} = 1$ } stable fixed point (2^0-cycle)
  $r_{c_2} = 3$ } stable 2^1-cycle
  $r_{c_3} = 1 + \sqrt{6}$ } stable 2^2-cycle
  $r_{c_4} \simeq 3.54409$ } stable 2^3-cycle
  $r_{c_5} \simeq 3.56440$ } stable 2^4-cycle
  $r_{c_6} \simeq 3.568759$