26 February - MATH 345

February 26, 2015 1:58 PM

- HW 4 due March 12

Pendulum with damping

$$mL\frac{d^2\varphi}{dt^2} = -mg\sin\varphi - b\frac{d\varphi}{dt}, \quad \varphi \in \mathbb{S}^1, \quad b>0$$

- nondimensionalised (Exercise: show this)

$$\frac{d^2\varphi}{d\tau^2} + \delta\frac{d\varphi}{d\tau} + \sin\varphi = 0, \quad \varphi \in \mathbb{S}^1, \quad \delta > 0$$

- or a system:

$$\begin{cases} \dot\varphi = v \\ \dot v = -\sin\varphi - \delta v \end{cases} \qquad (\varphi, v) \in \mathbb{S}^1 \times \mathbb{R}^1$$

- fixed points (Exercise: show this)

$$\text{for all } \delta \geq 0: (\varphi^*, v^*) = (0 \pmod{2\pi}, 0)$$
$$= (\pi \pmod{2\pi}, 0)$$

- linearisation:

$$D\vec{f}(\varphi, v) = \begin{pmatrix} 0 & 1 \\ -\cos\varphi & -v \end{pmatrix}$$

i) at $(0 \pmod{2\pi}, 0)$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -\delta \end{pmatrix}, \quad \Delta = 1 > 0, \quad \tau = -\delta < 0$$

fixed point is a hyperbolic attractor

$\delta^2 < 4$: spiral, $\delta^2 > 4$: node

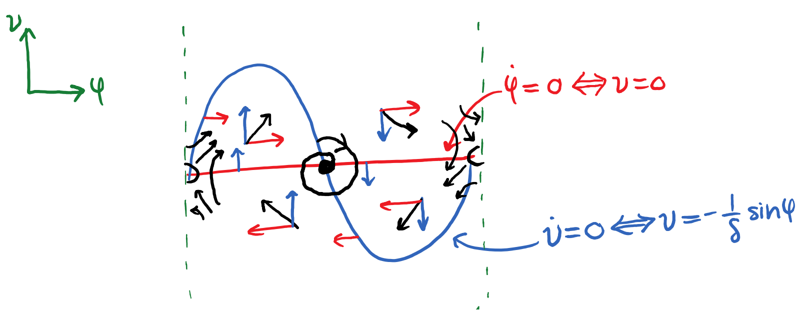
ii) at $(\pi \pmod{2\pi}, 0)$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -\delta \end{pmatrix}, \quad \Delta = -1 < 0$$

fixed point is a hyperbolic saddle point

- nullclines, direction field, local phase portraits at hyperbolic fixed points

(assume $\delta < 2$)



$\varphi=-\pi$ identify $\varphi=\pi$

↑ these are the same point ↑

- for global information: let $V(\varphi,\upsilon)=\frac{1}{2}\upsilon^2-\cos\varphi+1$
- V would be a conserved quantity if there were no damping
- along trajectories of the system

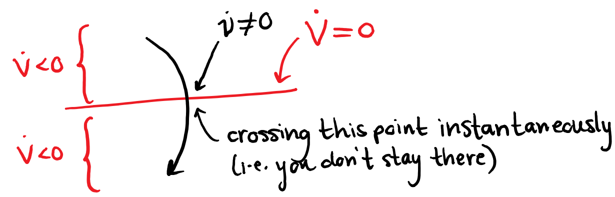
$$V=V(\varphi(\tau),\upsilon(\tau))$$

$$\dot{V}=\frac{\partial V}{\partial\varphi}\dot{\varphi}+\frac{\partial V}{\partial\upsilon}\dot{\upsilon}=(\sin\varphi)(\upsilon)+(\upsilon)(-\sin\varphi-\delta\upsilon)$$

$$\dot{V}=-\delta\upsilon^2\leq 0 \text{ always}$$
$$<0 \text{ if } \upsilon\neq 0$$

if $\upsilon=0$, then $\dot{\upsilon}\neq 0$ except at a fixed point

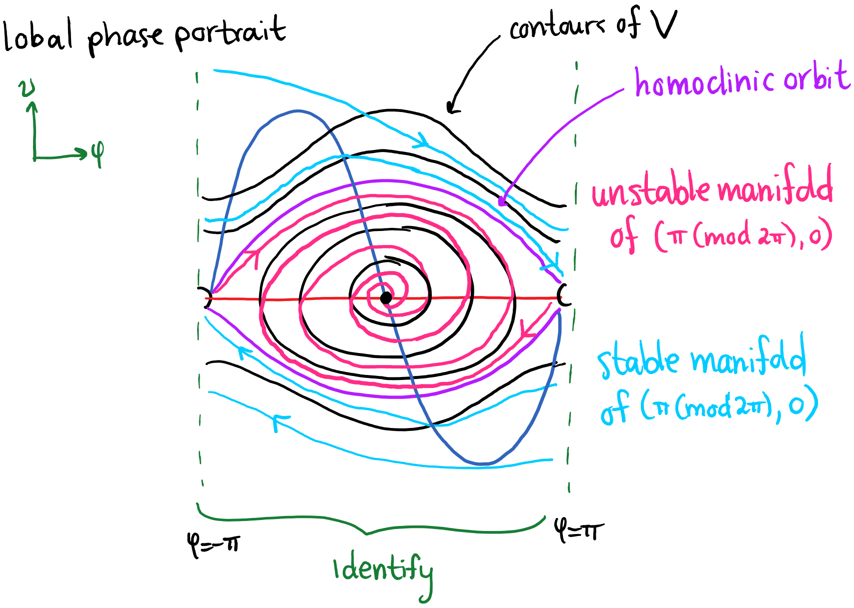
Exercise: show that $V(\varphi(\tau),\upsilon(\tau))$ is decreasing (not just nonincreasing) along trajectories, even if the trajectory crosses $\upsilon=0$, except at fixed points

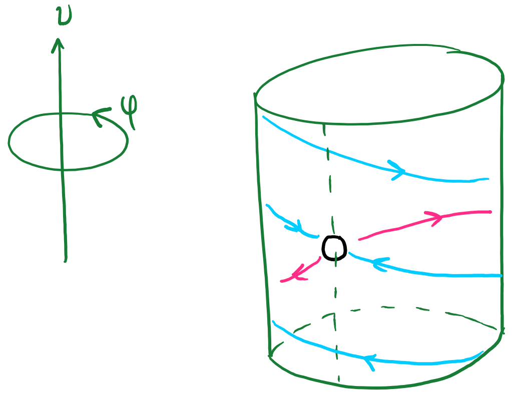


- trajectories that are not fixed points always move "downhill" on the contour map of V

Exercise: show that this implies that closed orbits are impossible

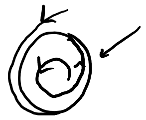
- global phase portrait

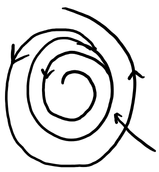


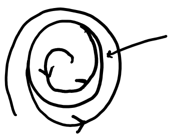


Limit cycles

· a limit cycle is an isolated closed orbit (trajectory)

stable (attracting) e.g. 

unstable (repelling) e.g. 

half-stable (semistable) e.g. 

$\boxed{\text{Example 2.6}}$

$$\begin{cases} \dot{x} = x - y - x^3 - xy^2 \\ \dot{y} = x + y - x^2 y - y^3 \end{cases}$$

fixed point $(x^*, y^*) = (0,0)$

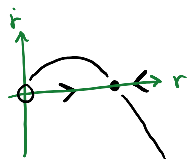
linearising: $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. $\Delta = 2$, $\tau = 2$

hyperbolic repeller (unstable spiral)

Exercise: show that in polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

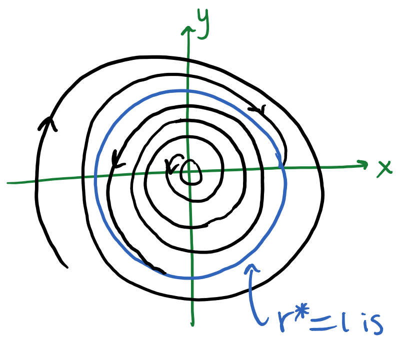
$$\begin{cases} \dot{r} = r - r^3 \\ \dot{\theta} = 1 \end{cases} \qquad (r,\theta) \in [0,\infty) \times S^1$$



fixed points
$r^* = 0$
$r^* = 1$

angular equation: $\dot{\theta} = 1 \rightarrow \theta(t) = t + \theta_0 \pmod{2\pi}$
radial equation: $\dot{r} = r - r^3$

phase portrait in $\mathbb{R}^2$ (xy-plane)



$r^*=1$ is stable limit cycle

also ω-limit set for all trajectories except origin } see notes 6 January

Ruling out closed orbits

(a) sometimes the system can be written in the form

$\dot{\vec{x}} = -\nabla V(\vec{x})$ for some C^1 scalar function V

↑ continuous & 1st derivative continuous

this is called a gradient system

fixed points of the system ($\dot{\vec{x}}=0$) are the same as critical points of V ($\vec{\nabla} V = 0$)