6 January - MATH 345

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- website: www.math.ubc.ca/~nagata/m345
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- need a program called XPPAUT
- can download onto own machine
- available on computers in lab in LSK
- textbook (optional) by Strogatz
- background reading ch. 1 & sections 2.0-2.3

A geometric way of thinking

Flows on the line $\mathbb{R}$ (or an interval)

$$\dot{x} = f(x), \quad x \in \mathbb{R} \qquad \left(\dot{x} = \frac{dx}{dt} \right)$$

a continuously differentiable function of t defined on some open interval

- 'Analytic' viewpoint: find solution as an explicit function of t

$$\frac{dx}{dt} = f(x) \rightarrow \frac{dx}{f(x)} = dt \rightarrow \int \frac{dx}{f(x)} = \int dt$$

$\boxed{\text{Example 1.1a}}$

$$\dot{x} = x(1-x) \rightarrow \frac{dx}{x(1-x)} = dt \rightarrow \int \frac{dx}{x(1-x)} = \int dt$$

- integrate using partial fractions method

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

$$\int \frac{dx}{x(1-x)} = \int \frac{dx}{x} + \int \frac{dx}{1-x}$$

$$\int \frac{dx}{x(1-x)} = \ln|x| - \ln|1-x| \quad (+\text{ arbitrary constant})$$

$$\int dt = t \quad (+\text{ arb. const.})$$

combined constants

$$\ln|x| - \ln|1-x| = t + \alpha$$

- solve for x as a function of t

• solve for x as a function ...

$$\ln \frac{|x|}{|1-x|} = t + \alpha$$

↑ use log law

$$\left| \frac{x}{1-x} \right| = e^{t+\alpha} = e^{\alpha} e^{t}$$

• want to get rid of abs. val.

$$\frac{x}{1-x} = \underbrace{\pm e^{\alpha}}_{} e^{t} = C e^{t}$$

↑ $\pm e^{\alpha}$

• typically there is an initial condition $x_0 = x(0)$

for $t=0$, $e^{t} = 1 \rightarrow \frac{x_0}{1-x_0} = C$

• C is solved for, giving x in terms of x_0

$$x = x(t) = \frac{x_0}{x_0 + (1-x_0)e^{-t}}$$

• solution is valid in an open interval of t-values that includes $t=0$

Exercise: what is the open interval?

it depends on initial value x_0

• can check solution for $t=0$ (should be x_0)

• 'Geometric' viewpoint: instead of finding $x(t)$ explicitly, get qualitative information about it

• under certain conditions, a unique continuously differentiable solution exists

theorem on p.27 in older version of textbook

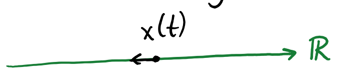
• the dependent variable (x here) is called the phase variable

• the space that the phase variable belongs to ($\mathbb{R}$ here) is called the phase space

• think of $x(t)$ as the position in phase space of an imaginary particle called a phase point at time t (e.g. a point moving on a line)

• the function $x(t)$ describes the trajectory (or orbit) of the phase point

e.g. if $x(t)$ is a decreasing function of t



- $\dot{x}(t)$ is the phase velocity of the phase point (it is a vector)
- f is a vector field: at every point x, it specifies the phase velocity vector

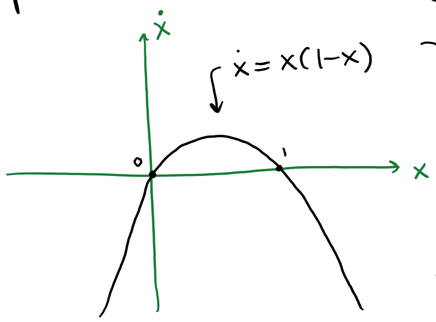
$$\dot{x}(t) = f(x(t))$$

- the collection of all trajectories $x(t)$ (corresponding to all initial conditions) is called the flow
- we say f generates a one-dimensional flow in $\mathbb{R}$
- the collection of all distinct trajectories is called the phase portrait

$\boxed{\text{Example 1.1b}}$

$$\dot{x} = x(1-x), \quad x \in \mathbb{R}$$

- plot $\dot{x}$ vs. x (think of $\dot{x}$ as 'y')



height above x-axis gives phase velocity (vector) at the point x

- from the plot,

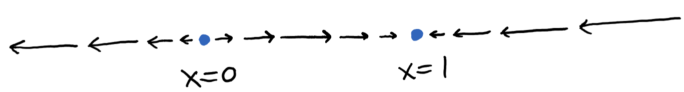
$\dot{x} < 0$ if $-\infty < x < 0$ i.e. $x(t)$ is decreasing

$\dot{x} > 0$ if $0 < x < 1$ i.e. $x(t)$ is increasing

$\dot{x} < 0$ if $1 < x < \infty$ i.e. $x(t)$ is decreasing

$\dot{x} = 0$ if $x = 0, 1$ i.e. $x(t)$ is constant

- sketch of the vector field:

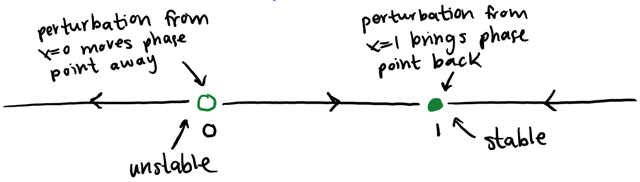


- tells you how fast it's moving & direction
- $\dot{x} = 0$ means $x(t) = \text{const}$: an equilibrium solution that stays for all t
- $\dot{x} = 0$ if $x = 0, 1$ (x-intercept of $\dot{x}$ vs. x plot)

Exercise: $x(t) = \dfrac{x_0}{x_0 + (1-x_0)e^{-t}}$

substitute $x_0 = 0, 1$ and check that $x(t) = 0, 1$

- the points $x=0,1$ are fixed points (in general, solve $f(x)=0$ for x)
- denoted $x^*=0,1$ using textbook's notation
- sketch of the phase portrait:



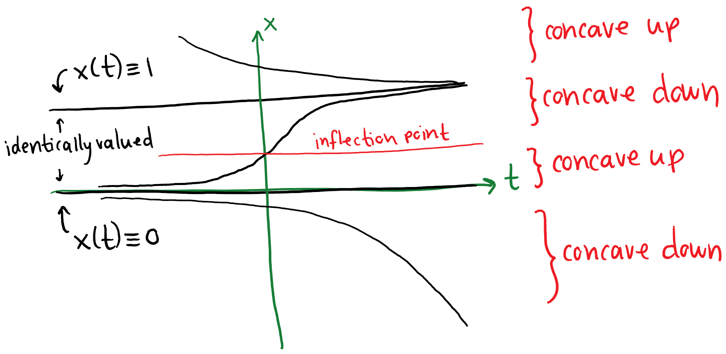
- there are 5 distinct trajectories:

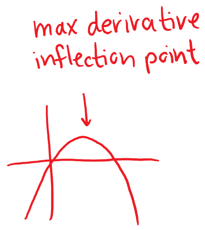
$$(-\infty,0),\ \{0\},\ (0,1),\ \{1\},\ (1,\infty)$$

- can deduce shapes of graph $x(t)$ vs. t

$$\lim_{t\to\infty} x(t)=1 \quad \text{if } x_0>0$$

$$\lim_{t\to-\infty} x(t)=0 \quad \text{if } x_0<1$$

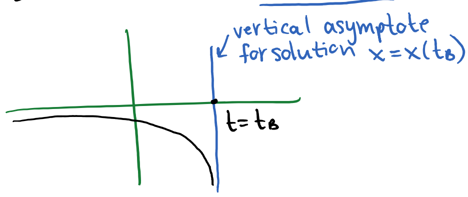




Exercise: confirm asymptotic behaviour as $t\to\pm\infty$ and graph with explicit solution

$$x(t)=\frac{x_0}{x_0+(1-x_0)e^{-t}}$$

- one detail the geometric viewpoint misses is that some solutions blow up in finite time

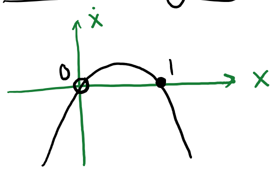


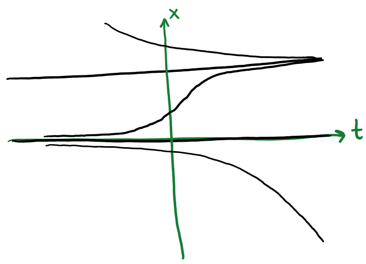
- t_B depends on x_0

Exercise: for what t is solution $x(t)$ defined if

i) $x_0 = 0$ ii) $x_0 = 1$ iii) $0 < x < 1$ iv) $x_0 < 0$ v) $x_0 > 1$

Summary:





qualitative viewpoint of solution

- sometimes we denote the solution as

$$x = x(t, x_0)$$

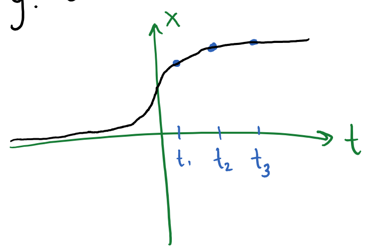
i.e. explicitly include the initial condition as a variable

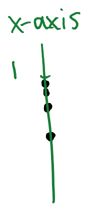
- if $x(t, x_0)$ is defined and bounded for all $t \in [0, \infty)$ then a point p is an omega limit point (ω-limit point) of x_0 if

$$p = \lim_{k \to \infty} x(t_k, x_0)$$

for some sequence of times $t_k \to \infty$ as $k \to \infty$

e.g. $0 < x_0 < 1$





$p = 1$ is an ω-limit point

- α-limit point goes backwards in time