DSC 40A - Homework 6

Due: Wednesday, Feb 28 at 11:59pm

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Homeworks are due to Gradescope by 11:59pm on the due date. You can use a slip day to extend the deadline by 24 hours.

Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should **always explain and justify** your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.

Homeworks should be written up and turned in by each student individually. You may talk to other students in the class about the problems and discuss solution strategies, but you should not share any written communication and you should not check answers with classmates. You can tell someone how to do a homework problem, but you cannot show them how to do it.

For each problem you submit, you should **cite your sources** by including a list of names of other students with whom you discussed the problem. Instructors do not need to be cited.

This homework will be graded out of 50 points. The point value of each problem or sub-problem is indicated by the number of avocados shown.

Problem 1. Reflection and Feedback Form

Make sure to fill out this Reflection and Feedback Form, linked here for two points on this homework! This form is primarily for your benefit; research shows that reflecting and summarizing knowledge helps you understand and remember it.

Problem 2. Baseball

a) Suppose that your baseball team has 12 players but you only have 8 matching team hats. How many ways are there for you to select who gets a hat and who doesn't?

> **Solution:** $C(12, 4) = C(12, 8) = 495$.
>
> An unordered selection of 8 people to get the hats (or equivalently, 4 people to not get them) determines who has a hat and who doesn't. There are $C(12, 4) = C(12, 8)$ many ways to make this selection.

b) Suppose that your baseball team has 12 players and you have 16 jerseys, numbered 1 through 16. You will give out one jersey to each player, and four will be leftover. How many ways are there for you to assign jerseys to players?

> **Solution:** $P(16, 12) \approx 8.718 \times 10^{11}$.
>
> We need an ordered selection of 12 jerseys to determine who gets which jersey. There are $P(16, 12) = \frac{16!}{4!}$ many ways to do this. Alternatively, we can see this as follows: the first person could get any of 16 jerseys, the next person could get any of 15 jerseys, and so on.

c) As before, your baseball team has 12 players and you have 16 jerseys, numbered 1 through 16. You will give out one jersey to each player, and four will be leftover. How many ways are there for you to assign jerseys to players if you must give out jersey number 7?

Solution: $12 * P(15, 11) = 6.538 \times 10^{11}$.

Let's first figure out which of the 12 players gets jersey number 7. There are 12 ways to do that. Then, we need an ordered selection of 11 jerseys from among the remaining 15 to assign the other jerseys to players. There are $P(15, 11) = \frac{15!}{4!}$ many ways to do this. This results in $12 * P(15, 11)$ ways.

d) Suppose that at the batting cage, where a machine pitches the ball, each time you swing the bat, your probability of hitting the ball is 3/5. If you swing 9 times, what is the probability that you hit the ball exactly 7 times?

Solution: $\binom{9}{7} * \left(\frac{3}{5}\right)^7 * \left(\frac{2}{5}\right)^2 \approx 0.161$.

We can think of your results as a sequence of length 9, where each component is either Y for yes or N for no, corresponding to whether you hit the ball. Note that not all such sequences are equally likely. For example, you are far more likely to hit the ball all 9 times than you are to never hit the ball, since your success rate is large, so $P(YYYYYYYYY) > P(NNNNNNNNN)$. However, every sequence with exactly 7 Y's has the same probability, which is $\left(\frac{3}{5}\right)^7 * \left(\frac{2}{5}\right)^2$. So the total probability of all such outcomes comes from multiplying this value times the number of sequences with exactly 7 Y's. There are $\binom{9}{7}$ many sequences with exactly 7 Y's, because we need to pick any set of 7 positions for the Y's to occupy. Therefore, the total probability is $\binom{9}{7} * \left(\frac{3}{5}\right)^7 * \left(\frac{2}{5}\right)^2$.

e) Suppose that when a pitcher is throwing the ball, your probability of hitting the ball depends on the kind of pitch. Your probability of hitting the ball is

- $\frac{1}{2}$ for a fastball,
- $\frac{1}{3}$ for a breaking ball, and
- $\frac{1}{4}$ for a changeup

Suppose that at practice, a pitcher throws you three fastballs, three breaking balls, and three changeups. What is the probability that you miss one breaking ball and one changeup, but hit all 7 other balls?

Solution: $9 * \left(\frac{1}{2}\right)^3 * \left(\frac{1}{3}\right)^2 * \left(\frac{2}{3}\right)^1 * \left(\frac{1}{4}\right)^2 * \left(\frac{3}{4}\right)^1 \approx 0.00391$.

We can think of your performance as a sequence of length 9, where each component is either Y for yes or N for no, corresponding to whether you hit the ball. Let each element of the sequence represent, from left to right:

- the first fastball
- the second fastball
- the third fastball
- the first breaking ball
- the second breaking ball
- the third breaking ball
- the first changeup
- the second changeup

- the third changeup

There are $\binom{3}{1} * \binom{3}{1} = 9$ sequences with an N in one of the middle three positions and an N in one of the last three positions, which are the sequences that correspond to missing one breaking ball and one changeup. For each such sequence, the probability of that sequence occurring is

$$\left(\frac{1}{2}\right)^3 * \left(\frac{1}{3}\right)^2 * \left(\frac{2}{3}\right)^1 * \left(\frac{1}{4}\right)^2 * \left(\frac{3}{4}\right)^1,$$

so the probability of missing exactly one breaking ball and one changeup is

$$9 * \left(\frac{1}{2}\right)^3 * \left(\frac{1}{3}\right)^2 * \left(\frac{2}{3}\right)^1 * \left(\frac{1}{4}\right)^2 * \left(\frac{3}{4}\right)^1 \approx 0.00391.$$

Problem 3. NBA Draft

In an NBA draft, there are two categories of players to be selected from: college players and international players. The probability that a college player is selected is 0.7, while the probability that an international player is selected is 0.3.

Answer the following sub questions based on information above. Calculations by hand or by code are both acceptable as long as sufficient work is shown.

a) Suppose that 100 players declared for the draft this year. There are 60 players selected from each year's draft. How many draft classes of 60 students can be selected from the 100 players who declared for the draft?

Solution:

Since order does not matter, the number of ways to choose a class is as follows:

$\binom{100}{60} = 1.375e28$ ways

b) How many draft classes of 60 students can be selected from the 100 players who declared for the draft if order does matter?

Solution:

Since order does matter, which means that 2 different orders count as 2 different ways of selecting a draft class, the number of ways to choose a class is as follows:

$${}_{100}P_{60} = \frac{100!}{(100-60)!} = 1.144e110 \text{ ways}$$

c) If the probability that a draft pick from the college players will be a starter in their rookie year is 0.5, and the probability for an international player to achieve the same is 0.2, what is the overall probability that the draft pick will not be a starter in their rookie year?

Solution:

Let P(A) = the probability that the draft pick is from college Let P(B) = the probability that the draft pick is international.

We have P(A)=0.7, P(B)=0.3, $P(Starter|A) = 0.5$, $P(Starter|B) = 0.2$.
Using the the law of total probability, we have:

P(Starter) = P(A) * $P(Starter|A)$ + P(B) * $P(Starter|B)$
$= 0.7 * 0.5 + 0.3 * 0.2$
$= 0.41$

Since the event of not being a starter and being a starter are mutually exclusive, $P(NotStarter) = 1 - P(Starter) = 1 - 0.41 = 0.59$.

d) After the draft, it's found that the likelihood of a player being a first-round pick given that they are a college player is 0.8, and the likelihood of being a first-round pick given that they are an international player is 0.4. During the draft, a player is announced as a first-round pick. What is the probability that this player is international?

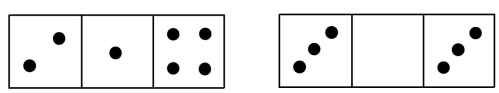
Solution: Let P(A) = the probability that the draft pick is from college Let P(B) = the probability that the draft pick is international. Given that:
We have P(A)=0.7, P(B)=0.3, $P(FirstRound|A) = 0.8$, $P(FirstRound|B) = 0.4$.
To get the overall probability of being a first-round pick, we use the law of total probability.

$P(FirstRound) = P(A) * P(FirstRound|A) + P(B) * P(FirstRound|B)$
$= 0.7 * 0.8 + 0.3 * 0.4$
$= 0.68$

Using Bayes' Theorem, we have:
$P(B|FirstRound) = \frac{P(FirstRound|B) \cdot P(B)}{P(FirstRound)}$
$= \frac{0.4*0.3}{0.68}$
$= 0.176$
THe probability of that the player is international is 0.176.

Problem 4. From Dominoes to Trominoes

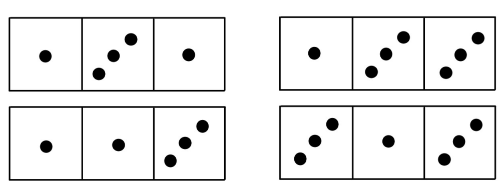
A *tromino* is a rectangular tile divided into three sections. On each section, there is some number of dots between 0 and 6, inclusive. Two example trominoes are shown below.



A complete set of trominoes consists of every possible combination of dots on each section.

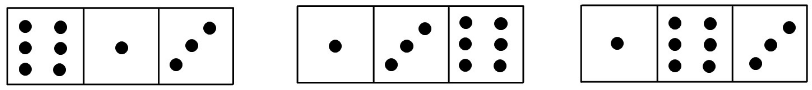
a) Draw a picture of all the trominoes that have at least one 1, at least one 3, and only 1s and 3s (no other numbers). No explanation needed.

Solution:



b) Draw a picture of all the trominoes that have a 1, a 3, and a 6. No explanation needed.

Solution:



c) How many trominoes are in a complete set?

Hint: The answers to parts (a) and (b) should help you.

Solution: $C(7,1) + 4 * C(7,2) + 3 * C(7,3) = 196$

We are separately counting the number of trominoes with 1, 2, and 3 distinct numbers of dots.

If there is just one distinct number of dots, we must have that number of dots on all three sections of the tromino.

If there are two different numbers of dots, we need to first choose which number should be duplicated (in one of 2 ways), and then determine whether the duplicated numbers go in adjacent sections or in the outermost sections (2 more options). There are 4 such trominoes, as we can see from the example in part (a).

If there are three different numbers of dots, we need to determine which number goes in the middle (in one of 3 ways). This is like the example in part (b).

Therefore, when we incorporate the number of ways to select 1, 2, or 3 distinct numbers of dots and the number of ways to arrange those dots on a tile, the total number of trominoes is given by $C(7,1) + 4 * C(7,2) + 3 * C(7,3) = 196$.

d) In class, we calculated that the number of dominoes in a complete set is 28. How does your answer to part (c) relate to the number 28? Explain why this makes sense.

Solution: Our answer was $196 = 7*28$. So there are seven times as many trominoes as dominoes. One way to interpret this is to organize the trominoes by what number they have in the middle section. There are 28 trominoes with no dots in the middle section, because every possible pair of dots can be on the outer sections, and we know there are 28 possible pairs of dots. Similarly, there are 28 trominoes with one dot in the middle, and so on. There are 7 choices for the number of dots in the middle, and once we determine that, there are 28 choices for how to complete the remaining two sections, which form a domino.

Problem 5. Book Club

A book club includes p people, and there are b books that the book club is considering reading this month. Before deciding on which book to read this month, the book club president asks each person which of the b books they have already read.

A *book club description* is a description of who, among p people, has already read each of b books. For example, if a book club has $p = 3$ people (Ben, Tunan, Pallavi) and there are $b = 2$ books (Book 1, Book 2), one possible book club description is as follows:

- Ben has read Book 1.
- Tunan has read Book 1 and Book 2.
- Pallavi has read neither Book 1 nor Book 2.

a) How many book club descriptions are possible? Give your answer as a formula involving p and b, with explanation of where the formula comes from.

Solution: 2^{bp}

For one particular person and one particular book, there are two options that describe the relationship between the person and the book: either they read the book or they did not. Therefore, each person has 2^b ways to describe which of the b books they read. Since there are p people, this means there are $(2^b)^p = 2^{bp}$ possible book club descriptions.

We can equivalently say that each book has 2^p many ways to describe which people have read it, and since there are b books, this means there are $(2^p)^b = 2^{pb} = 2^{bp}$ possible book club descriptions.

b) How many book club descriptions are such that nobody has read Book 1? Give your answer as a formula involving p and b, with explanation of where the formula comes from.

Solution: $2^{(b-1)p}$

Each person's relationship with Book 1 is now fixed, so there are 2^{b-1} ways to describe which of the other $b-1$ books each person has read. Since there are p people, there are $(2^{b-1})^p = 2^{(b-1)p}$ book club descriptions in which nobody has read Book 1.

c) How many book club descriptions are such that there is at least one book that nobody has read? Give your answer as a formula involving p and b, with explanation of where the formula comes from.

Solution: $2^{bp} - (2^p - 1)^b$

We are looking for book club descriptions in which there is at least one book that nobody has read. Using the complement, we'll count this as the total number of book club descriptions minus the number of book club descriptions in which there are zero books that nobody has read. We've already calculated in part (a) that the total number of book club descriptions is 2^{bp} so we just need to find the number of book club descriptions in which there are zero books that nobody has read. That is, every book needs to have been read by at least one person.

Think of one particular book. This book has two potential relationships with each person: it's been read by that person, or it hasn't. Therefore, there are 2^p many ways to describe the relationship of a given book to each person. Only one of these corresponds to the case where nobody has read the book. So there are $2^p - 1$ ways in which at least one person has read the book. Since we need this to be the case for each of the b books, the number of book club descriptions in which every book is read by at least one person is given by $(2^p - 1)^b$. Note this cannot be simplified!

Therefore, using the complement rule as outlined above, the number of book club descriptions in which there is at least one book that nobody has read is given by $2^{bp} - (2^p - 1)^b$.

Problem 6. Hockey-stick identity

You've learned in lectures that:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

which means the number of combinations of picking k elements out of n unique elements.

a) Prove Pascal's rule:

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

You may assume $n > k$. Then based on the meaning of $\binom{n}{k}$, give intuitive explanation for this rule.

Hint: You may find the following properties useful:

- $(n+1)-(k+1) = n-k$
- $(n+1-(k+1))! = (n+1-(k+1))(n-(k+1))!$

In general, there is no restriction on the relative sizes of n and k. If $n < k$, the value of the binomial coefficient is zero and the identity remains valid.

Solution:

$$\begin{aligned}\frac{n!}{(n-(k+1))!(k+1)!} + \frac{n!}{(n-k)!k!} &= n!\left(\frac{1}{(n-(k+1))!(k+1)!} + \frac{1}{(n-k)!k!}\right)\\ &= n!\left(\frac{n+1-(k+1)}{(n+1-(k+1))!(k+1)!} + \frac{k+1}{\underbrace{(n-k)!}_{=(n+1-(k+1))!}\ (k+1)!}\right)\\ &= \frac{(n+1)!}{(n+1-(k+1))!(k+1)!}\end{aligned}$$

Explanation:

Picking $k+1$ elements out of $n+1$ elements is equivalent to Case 1: Skip location 1, picking $k+1$ elements out of the other n locations, plus Case 2: Pick one element at location 1, then picking the rest k elements out of the other n locations. That is:

$$\binom{n+1}{k+1} = \binom{1}{0}\binom{n}{k+1} + \binom{1}{1}\binom{n}{k} = \binom{n}{k+1} + \binom{n}{k}$$

b) Use Pascal's rule, show Hockey-stick identity:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \ldots + \binom{k}{k}$$

The Hockey-stick name comes from the visualized pattern in Pascal's triangle, in which each element's value equals the sum of its left-upper and right-upper neighbors:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

$$\binom{4}{2} = \binom{3}{1} + \binom{2}{1} + \binom{1}{1}$$

Solution:

$$\begin{aligned}
\binom{n+1}{k+1} &= \underbrace{\binom{n}{k+1}}_{expand} + \binom{n}{k} \\
&= \underbrace{\binom{n-1}{k+1}}_{expand} + \binom{n-1}{k} + \binom{n}{k} \\
&= \underbrace{\binom{n-2}{k+1}}_{expand} + \binom{n-2}{k} + \binom{n-1}{k} + \binom{n}{k} \\
&\text{keep expanding} \\
&= \underbrace{\binom{k+1}{k+1}}_{=1=\binom{k}{k}} + \ldots + \binom{n-2}{k} + \binom{n-1}{k} + \binom{n}{k}
\end{aligned}$$