Lecture 21 – Independence and Conditional Independence

DSC 40A, Spring 2023

Announcements

- Homework 6 is due **tomorrow at 11:59pm**.
  - This homework has some tricky problems — come to office hours for help!
- Great source of practice problems for recent content: stat88.org/textbook.
- Also check out the Probability Roadmap on the resources tab of the course website.
- Consider applying for the HDSI Undergrad Scholarship Program!

Agenda

- Recap of Lecture 20.
- Independence.
- Conditional independence.

Last time

Last time

- A set of events $E_1, E_2, ..., E_k$ is a **partition** of S if each outcome in S is in exactly one E_i.
- The **Law of Total Probability** states that if A is an event and $E_1, E_2, ..., E_k$ is a partition of S, then

$$= P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k)$$

$$P(A) = P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + \ldots + P(E_k) \cdot P(A|E_k)$$

$$= \sum_{i=1}^{k} P(E_i) \cdot P(A|E_i)$$

$E_1 = B$

$E_2 = \bar{B}$

- **Bayes' Theorem** states that

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

- We often re-write the denominator $P(A)$ in Bayes' Theorem using the Law of Total Probability.

Discussion Question

Consider any two events A and B. Choose the expression that's equivalent to

not B

$$P(B|A) + P(\bar{B}|A).$$

$$= \frac{P(A \cap B)}{P(A)} + \frac{P(A \cap \bar{B})}{P(A)}$$

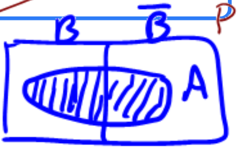
$$= \frac{P(A \cap B) + P(A \cap \bar{B})}{P(A)}$$

law of total prob.

$$= \frac{P(A)}{P(A)}$$

$$= 1$$

a) $P(A)$
b) $1 - P(B)$
c) $P(B)$
d) $P(\bar{B})$
e) 1



Example: prosecutor's fallacy

A bank was robbed yesterday by one person. Consider the following facts about the crime:

- The person who robbed the bank wore Nikes.
- Of the 10,000 other people who came to the bank yesterday, only 10 of them wore Nikes.

The prosecutor finds the prime suspect, and states that "given this evidence, the chance that the prime suspect was not at the crime scene is 1 in 1,000". $\rightarrow \dfrac{10}{10{,}000} = P(\text{Nikes} \mid \text{innocent})$

(innocent)

1. What is wrong with this statement?

   $P(\text{Nikes} \mid \text{innocent}) \neq P(\text{innocent} \mid \text{Nikes})$

2. Find the probability that the prime suspect is guilty given only the evidence in the exercise.

	guilty	innocent	
Nikes	1	10	} 11 Nike wearers
no Nikes	0	9,990	

$\underbrace{9{,}990}_{10{,}000 \text{ total}}$ → 10,001 people

$$P(\text{innocent} \mid \text{Nikes}) = \frac{P(\text{innocent} \cap \text{Nikes})}{P(\text{Nikes})}$$

$$= \frac{10/10{,}001}{11/10{,}001} = \boxed{\frac{10}{11}}$$

≈ 90%

$$P(\text{innocent}) = \boxed{\frac{10{,}000}{10{,}001}} \approx 99.9\%$$

→ update opinion based on new info

Independence

Updating probabilities

B = innocent
A = Nikes

- Bayes' theorem describes how to update the probability of one event, given that another event has occurred.

new   old

$$\boxed{P(B|A)} = \frac{\boxed{P(B)} \cdot P(A|B)}{P(A)}$$

multiply by this ratio to update prob.

  - $P(B)$ can be thought of as the "prior" probability of B occurring, before knowing anything about A.
  - $P(B|A)$ is sometimes called the "posterior" probability of B occurring, given that A occurred.
- What if knowing that A occurred doesn't change the probability that B occurs? In other words, what if

now   old

$$\boxed{P(B|A)} = \boxed{P(B)}$$

$\Rightarrow$ ratio $\frac{P(A|B)}{P(A)} = 1$

Independent events

- A and B are **independent events** if one event occurring does not affect the chance of the other event occurring.

with extra knowledge of A, P(B) is unchanged

$$\boxed{P(B|A) = P(B)} \qquad \boxed{P(A|B) = P(A)}$$

2 equivalent conditions

- Otherwise, A and B are **dependent events**.

- Using Bayes' theorem, we can show that if one of the above statements is true, then so is the other.

Suppose $\underline{P(B|A) = P(B)}$, let's show $P(A|B) = P(A)$

Bayes: $P(B|A) = \dfrac{P(B) \cdot P(A|B)}{P(A)} \Rightarrow P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$

by assumption $\underline{P(A)} \cdot P(B) = P(B) \cdot \underline{P(A|B)} \Longrightarrow P(A) = P(A|B)$

Independent events

- **Equivalent definition:** A and B are independent events if

$$P(A \cap B) = P(A) \cdot P(B)$$

$\leftarrow$ 3rd equivalent def'n of ind.

- To check if A and B are independent, use whichever is easiest:
  - $P(B|A) = P(B)$.
  - $P(A|B) = P(A)$.
  - $\boxed{P(A \cap B) = P(A) \cdot P(B).}$

by mult rule

$$P(A \cap B) = P(A) \cdot P(B|A)$$

$$\boxed{= P(A) \cdot P(B)}$$

when A,B independent only

Mutual exclusivity and independence

don't overlap
can't happen at same time
disjoint

① $P(B|A) = P(B)$
$P(A|B) = P(A)$
$P(A \cap B) = P(A) \cdot P(B)$

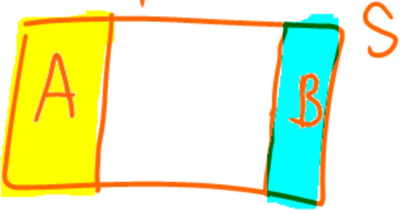
Discussion Question

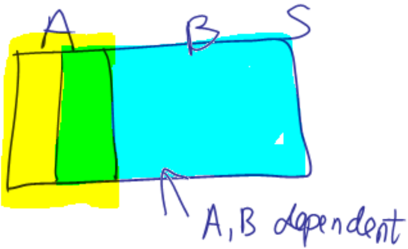
Suppose A and B are two events with non-zero probabilities. Is it possible for A and B to be both mutually exclusive and independent?

a) Yes
b) No

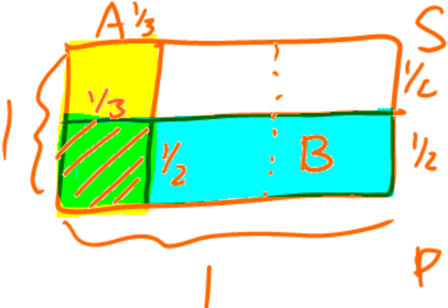
① if A,B ind ⇒ $P(B|A) = P(B)$
but if A,B mutually exclusive, $P(B|A) = 0$
told $P(B) \neq 0$

mutually exclusive





independent



$P(B|A) = 1/2$
$P(B) = 1/2$ > same

$P(A|B) = 1/3$
$P(A) = 1/3$

$P(A \cap B) = \frac{1}{6} = \frac{1}{3} \cdot \frac{1}{2} = P(A) \cdot P(B)$

Example: Venn diagrams

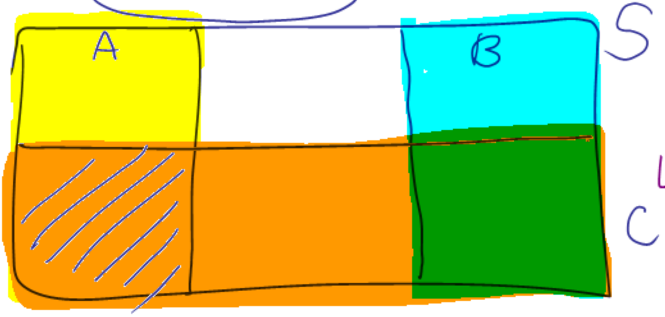
For three events A, B, and C, we know that

- A and C are independent,
- B and C are independent,
- A and B are mutually exclusive,
- $\boxed{P(A \cup C) = \frac{2}{3}}$, $\boxed{P(B \cup C) = \frac{3}{4}}$, $P(A \cup B \cup C) = \frac{11}{12}$.

Find $P(A)$, $P(B)$, and $P(C)$.

$P(A \cup C) = P(A) + P(C) - P(A \cap C)$

$= P(A) + P(C) - P(A) \cdot P(C)$



Notation

Let

$a = P(A)$

$b = P(B)$

$c = P(C)$

$P(A \cup C) = 2/3 \Rightarrow a + c - a \cdot c = 2/3$

$P(B \cup C) = 3/4 \Rightarrow b + c - b \cdot c = 3/4$

3 eqns in 3 vars

$P(A \cup B \cup C) = 11/12 \Rightarrow a + b + c - a \cdot c - b \cdot c = 11/12$

$$\cancel{a} + \cancel{b} + c - (\underbrace{\cancel{a} + c - 2/3}_{ac \text{ from blue}}) - (\underbrace{\cancel{b} + c - 3/4}_{bc \text{ from yellow}}) = 11/12$$

$$-c + \frac{2}{3} + \frac{3}{4} = \frac{11}{12}$$

$$c = \frac{8}{12} + \frac{9}{12} - \frac{11}{12} = \frac{6}{12} = \boxed{\frac{1}{2} = c}$$

now, use c to find a in blue!

$$a + \frac{1}{2} - a \cdot \frac{1}{2} = \frac{2}{3} \Rightarrow \frac{1}{2} \cdot a = \frac{1}{6} \Rightarrow \boxed{a = 1/3}$$

use c to find b in yellow:

$$b + \frac{1}{2} - b \cdot \frac{1}{2} = 3/4$$

$$\frac{1}{2} \cdot b = 1/4$$

$$\boxed{b = 1/2}$$

Example: cards

$P(B|A) = \frac{13}{51}$

♥: 2, 3, 4, 5, 6, 7, ~~8~~, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

$P(B) = \frac{13}{52}$

- Suppose you draw two cards, one at a time.
  - A is the event that the first card is a heart.
  - B is the event that the second card is a club.

think about it

- If you draw the cards **with** replacement, are A and B independent?

independent $P(B|A) = \frac{13}{52} = P(B)$

- If you draw the cards **without** replacement, are A and B independent?

once you remove Heart,

dependent - remaining cards less likely to be Hearts

Example: cards

B

A ♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

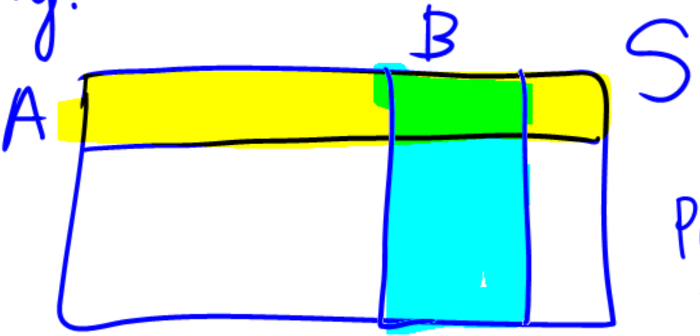
- Suppose you draw one card from a deck of 52.
  - *A* is the event that the card is a heart.
  - *B* is the event that the card is a face card (J, Q, K).

- Are *A* and *B* independent? yes

same fraction of face cards within the Hearts as within the full deck

$P(B/A) = \frac{3}{13}$, $P(B) = \frac{12}{52} = \frac{3}{13}$

Visualizing independence when outcomes are equally likely:



$P(A|B) = P(A)$

$P(B|A) = P(B)$

Example: cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw two cards, one at a time.
  - A is the event that the first card is a heart.
  - B is the event that the second card is a club.

- If you draw the cards **with** replacement, are A and B independent?

- If you draw the cards **without** replacement, are A and B independent?

Example: cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw one card from a deck of 52.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).

- Are A and B independent?

Assuming independence

- Sometimes we assume that events are independent to make calculations easier.
- Real-world events are almost never exactly independent, but they may be close.

Example: breakfast

1% of UCSD students are DSC majors. 25% of UCSD students eat avocado toast for breakfast. Assuming that being a DSC major and eating avocado toast for breakfast are independent:

1. What percentage of DSC majors eat avocado toast for breakfast?

2. What percentage of UCSD students are DSC majors who eat avocado toast for breakfast?

Conditional independence

Conditional independence

- Sometimes, events that are dependent *become* independent, upon learning some new information.

- Or, events that are independent can become dependent, given additional information.

Example: cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).
- Are A and B independent?

Example: cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).

- Suppose you learn that the card is red. Are A and B independent given this new information?

Conditional independence

- Recall that A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

- A and B are **conditionally independent** given C if

$$P((A \cap B)|C) = P(A|C) \cdot P(B|C)$$

- Given that C occurs, this says that A and B are independent of one another.

Assuming conditional independence

- Sometimes we assume that events are conditionally independent to make calculations easier.
- Real-world events are almost never exactly conditionally independent, but they may be close.

Example: Harry Potter and Discord

Suppose that 50% of UCSD students like Harry Potter and 80% of UCSD students use Discord. What is the probability that a random UCSD student likes Harry Potter and uses Discord, assuming that these events are conditionally independent given that a person is a UCSD student?

Independence vs. conditional independence

- Is it reasonable to assume conditional independence of
  - liking Harry Potter
  - using Discord

  given that a person is a UCSD student?
- Is it reasonable to assume independence of these events in general, among all people?

Discussion Question

Which assumptions do you think are reasonable?

a) Both
b) Conditional independence only
c) Independence (in general) only
d) Neither

Summary

Summary

- Two events A and B are **independent** when knowledge of one event does not change the probability of the other event.
  - Equivalent conditions: $P(B|A) = P(B)$, $P(A|B) = P(A)$, $P(A \cap B) = P(A) \cdot P(B)$.
- Two events A and B are **conditionally independent** if they are independent given knowledge of a third event, C.
  - Condition: $P((A \cap B)|C) = P(A|C) \cdot P(B|C)$.