Introduction to Algorithms
Lecture 3: Divide and Conquer

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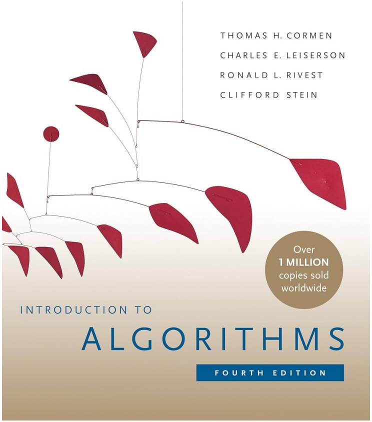
Introduction to Algorithms



Content has been extracted from *Introduction to Algorithms*, Fourth Edition, by Cormen, Leiserson, Rivest, and Stein. MIT Press. 2022.
Visit https://mitpress.mit.edu/9780262046305/introduction-to-algorithms/.
Original slides from *Introduction to Algorithms 6.046J/18.401J*, Fall 2005 Class by Prof. Charles Leiserson and Prof. Erik Demaine. MIT OpenCourseWare Initiative available at https://ocw.mit.edu/courses/6-046j-introduction-to-algorithms-sma-5503-fall-2005/.

The Divide & Conquer Design Paradigm

1. **Divide** the problem (instance) into subproblems.
2. **Conquer** the subproblems by solving them recursively.
3. **Combine** subproblems solutions.

Merge-sort

1. **Divide:** Trivial.
2. **Conquer:** Recursively sort 2 subarrays.
3. **Combine:** Linear-time merge.

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$$f(n) = O\left(n^{\log_b a - \varepsilon}\right), \text{ constant } \varepsilon > 0$$
$$\implies T(n) = \Theta(n^{log_b a}).$$

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Case 3

$$f(n) = \Omega\left(n^{lob_b(a)+\varepsilon}\right), \text{ constant } \varepsilon > 0, \text{ and regularity condition}$$
$$\implies T(n) = \Theta(f(n)).$$

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$$\textsc{Merge-sort}:\ a = 2, b = 2 \implies n^{log_b a} = n^{log_2 2} = n$$
$$\implies \text{Case 2 } (k = 0) \implies T(n) = \Theta(n \lg n).$$

Plan

Binary Search

Powering a Number

Fibonacci Numbers

Matrix Multiplication

Strassen's Algorithm

VLSI Tree Layout

Binary Search

Find an element in a sorted array:

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Find 9

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$$T(n) = \underbrace{1}_{\text{Number of subproblems.}} T\left(\underbrace{\frac{n}{2}}_{\text{Subproblem size.}}\right) + \underbrace{\Theta(1)}_{\text{Work dividing and combining.}}$$

BINARY SEARCH: $a = 1, b = 2 \implies n^{log_b a} = n^{log_2 1} = n^0 = 1$

$\implies$ Case 2 $(k = 0) \implies T(n) = \Theta(\lg n)$.

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Powering a Number

Problem:

Compute a^n, where $n \in \mathbb{N}$.

Naive algorithm:

$\Theta(n)$.

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$$a^n = \begin{cases} a^{\frac{n}{2}} \cdot a^{\frac{n}{2}} & \text{if } n \text{ is even;} \\ a^{\frac{n-1}{2}} \cdot a^{\frac{n-1}{2}} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

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$$T(n) = T(\frac{n}{2}) + \Theta(1) \ldots$$

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$$T(n) = T(\frac{n}{2}) + \Theta(1) \implies T(n) = \Theta(lgn).$$

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$$0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad \ldots$$

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$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

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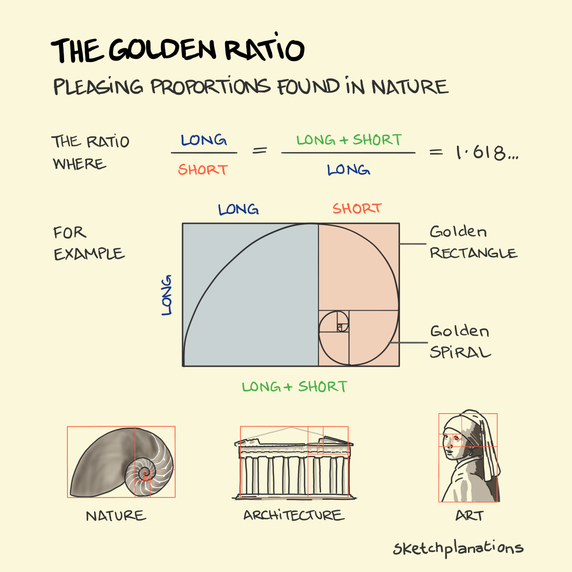
where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

The Golden Ratio (1.61803398875...)



sketchplanations.com/the-golden-ratio

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(exponential time!)

Computing Fibonacci Numbers

Bottom-up:

- Compute F_0, F_1, F_2, …, F_n in order, forming each number by summing the two previous.
- Running time: $\Theta(n)$.

[1] Computer Floating-Point Arithmetic and round-off errors, Kaluarachchi, 2022.

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Naive recursive squaring:

$F_n = \frac{\phi^n}{\sqrt{5}}$ rounded to the nearest integer.

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- Recursive squaring: $\Theta(\lg n)$ time.
- This method is unreliable, since floating-point arithmetic is prone to round-off errors[1].

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Recursive Squaring

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Recursive squaring.
Time $= \Theta(\lg n)$.

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Inductive Hypothesis ($n = k$).

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>
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Theorem:

$$A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, n \geq 1.$$

Proof of theorem. (Induction on n.)

Base case ($n = 1$):

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Proof of theorem. (Induction on n.)

Base case ($n = 1$):

$$A^1 = A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}$$

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Inductive Hypothesis ($n = k$):

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Inductive Step ($n = k + 1$):

$$A^{k+1} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}$$

$$A^k \cdot A = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}$$

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Recursive Squaring

$$F_0, F_1, F_2, F_3, \ldots, F_{k-2}, F_{k-1}, F_k, F_{k+1}, F_{k+2}, \ldots$$

Proof of theorem. (Induction on n.)

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$$(1,1)\text{: } F_{k+1} + F_k = F_{k+2}$$

$$(1,2)\text{: } F_{k+1} + 0 = F_{k+1}$$

$$(2,1)\text{: } F_k + F_{k-1} = F_{k+1}$$

$$(2,2)\text{: } F_k + 0 = F_k \qquad \blacksquare$$

Plan

Matrix Multiplication

$$\left.\begin{array}{rl}\textbf{Input:} & A = [a_{ij}], B = [b_{ij}]. \\ \textbf{Output:} & C = [c_{ij}] = A \cdot B.\end{array}\right\} i, j = 1, 2, \ldots, n.$$

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$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

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$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Standard Algorithm

$$
\begin{array}{l}
\textbf{for } i \leftarrow 1 \textbf{ to } n \textbf{ do} \\
\quad \textbf{for } j \leftarrow 1 \textbf{ to } n \textbf{ do} \\
\quad\quad c_{ij} \leftarrow 0 \\
\quad\quad \textbf{for } k \leftarrow 1 \textbf{ to } n \textbf{ do} \\
\quad\quad\quad c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj} \\
\quad\quad \textbf{end for} \\
\quad \textbf{end for} \\
\textbf{end for}
\end{array}
$$

Running time $= \Theta(n^3)$

Divide-and-Conquer Algorithm

IDEA:

$n \times n$ matrix $= 2 \times 2$ matrix of $\left(\frac{n}{2}\right) \times \left(\frac{n}{2}\right)$ submatrices:

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = A \cdot B$$

$$\left.\begin{array}{rcccc} r & = & ae & + & bg \\ s & = & af & + & bh \\ t & = & ce & + & dg \\ u & = & cf & + & dh \end{array}\right\}$$

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$$\left.\begin{array}{rcrcl} r & = & ae & + & bg \\ s & = & af & + & bh \\ t & = & ce & + & dg \\ u & = & cf & + & dh \end{array}\right\} \begin{array}{l} \overset{\text{recursive}}{\uparrow} \\ 8 \text{ mults of } \left(\frac{n}{2}\right) \times \left(\frac{n}{2}\right) \textit{ submatrices.} \\ 4 \text{ adds of } \left(\frac{n}{2}\right) \times \left(\frac{n}{2}\right) \textit{ submatrices.} \end{array}$$

Analysis of D&C Algorithm

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Work adding submatrices.

Submatrix size.

$$n^{\log_b a} = n^{\log_2 8} = n^3 \implies \text{ Case 1 } \implies T(n) = \Theta(n^3).$$

No better than the ordinary algorithm.

Plan

Strassen's Idea

- Multiply 2 matrices with only 7 recursive multiplications.

$$
\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
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\end{aligned}
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
s &= P_1 + P_2 \\
t &= P_3 + P_4 \\
u &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

7 mults, 18 adds/subs.

NOTE:
No reliance on commutativity of multiplication!

Strassen's Idea

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\end{aligned}
$$

$$
\begin{aligned}
r =& P_5 + P_4 - P_2 + P_6 \\
=& (a + d)(e + h) \\
& + d(g - e) - (a + b)h \\
& + (b - d)(g + h) \\
=& ae + ah + de + dh \\
& + dg - de - ah - bh \\
& + bg + bh - dg - dn \\
=& ae + bg
\end{aligned}
$$

Strassen's Algorithm

1. **Divide:** Partition A and B into $\frac{n}{2} \times \frac{n}{2}$ submatrices. Form terms to be multiplied using $+$ and $-$.
2. **Conquer:** Perform 7 multiplications of $\frac{n}{2} \times \frac{n}{2}$ submatrices recursively.
3. **Combine:** Form C using $+$ and $-$ on $\frac{n}{2} \times \frac{n}{2}$ submatrices.

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$$n^{\log_b a} = n^{\log_2 7} = n^{2.81} \implies \text{Case 1} \implies T(n) = \Theta(n^{\lg 7}).$$

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Note:
The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 32$ or so.

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Best to date (of theoretical interest only): $\Theta(n^{2.376\cdots})$.

Plan

Binary Search

Powering a Number

Fibonacci Numbers

Matrix Multiplication

Strassen's Algorithm

VLSI Tree Layout

VLSI Layout

Problem:
Embed a complete binary tree with n leaves in a grid using minimal area.

VLSI Layout

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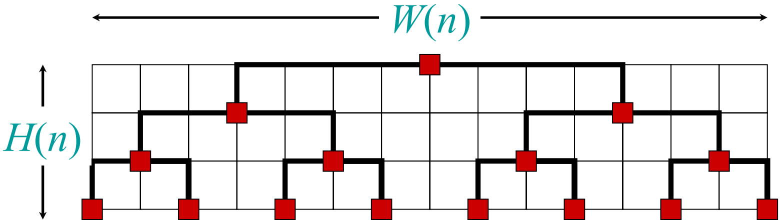

VLSI Layout

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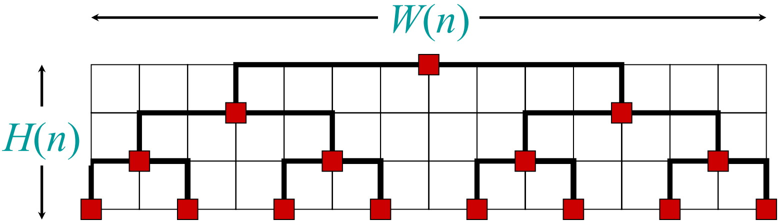

$$
\begin{aligned}
H(n) &= H\left(\frac{n}{2}\right) + \Theta(1) \\
&= \Theta(\lg n)
\end{aligned}
$$

VLSI Layout

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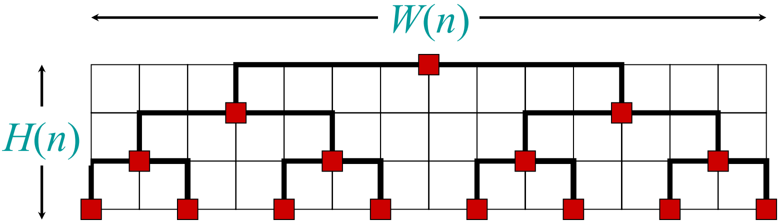

$$H(n) = H\left(\frac{n}{2}\right) + \Theta(1) = \Theta(\lg n)$$

$$W(n) = 2W\left(\frac{n}{2}\right) + \Theta(1) = \Theta(n)$$

VLSI Layout

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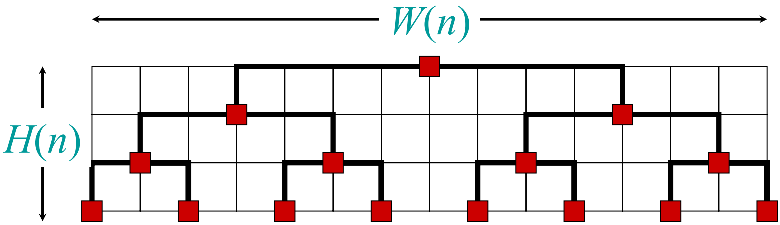

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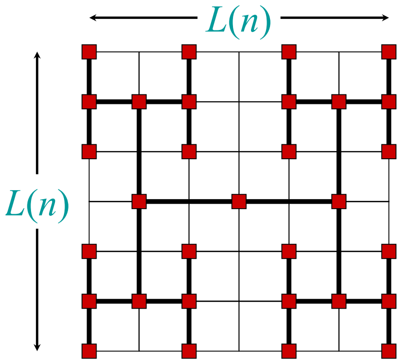
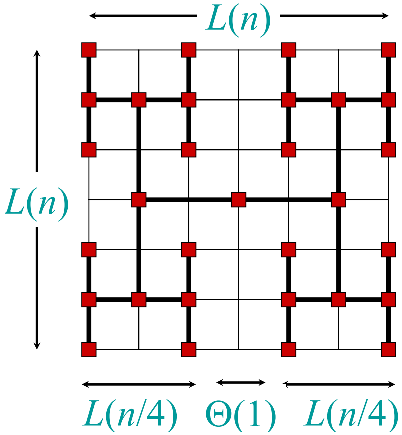
Area: $= \Theta(n \ln n)$

H-tree Embedding



H-tree Embedding



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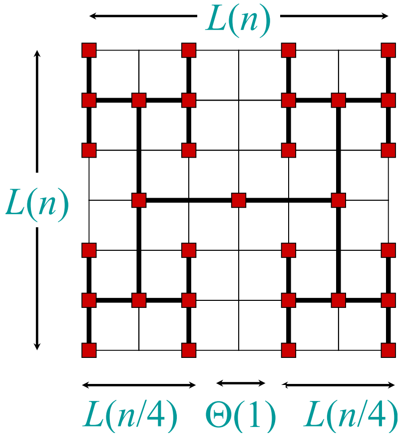
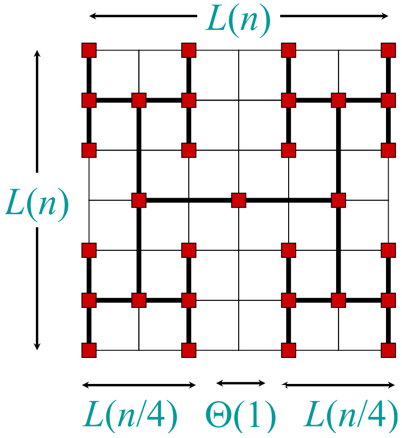


$$
\begin{aligned}
L(n) =& 2L\left(\frac{n}{4}\right) + \Theta(1) \\
=& \Theta(\sqrt{n})
\end{aligned}
$$

H-tree Embedding



$$L(n) = 2L\left(\frac{n}{4}\right) + \Theta(1)$$
$$= \Theta(\sqrt{n})$$

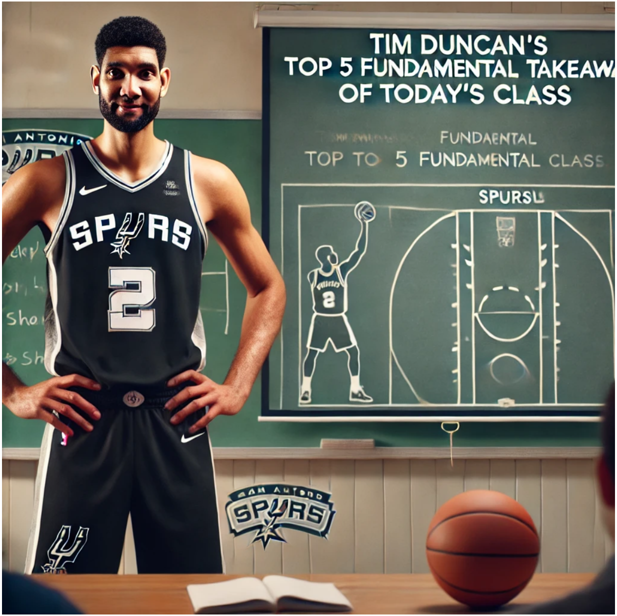
Area: $= \Theta(n)$

Conclusions

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- The divide-and-conquer strategy often leads to efficient algorithms.

End of Lecture 3.

TDT5FTOTTC



TIM DUNCAN'S
TOP 5 FUNDAMENTAL TAKEAW
OF TODAY'S CLASS
FUNDAENTAL
TOP TO 5 FUNDAMENTAL CLASS
SPURSL
SPURS
2
SPURS

Top 5 Fundamental Takeaways

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1 **Divide and Conquer Paradigm**: This algorithmic approach breaks a problem into smaller subproblems, solves them recursively, and combines the results efficiently.

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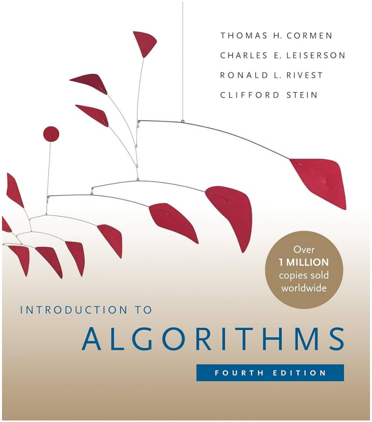
Introduction to Algorithms



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