Introduction to Algorithms
Lecture 4: Quicksort

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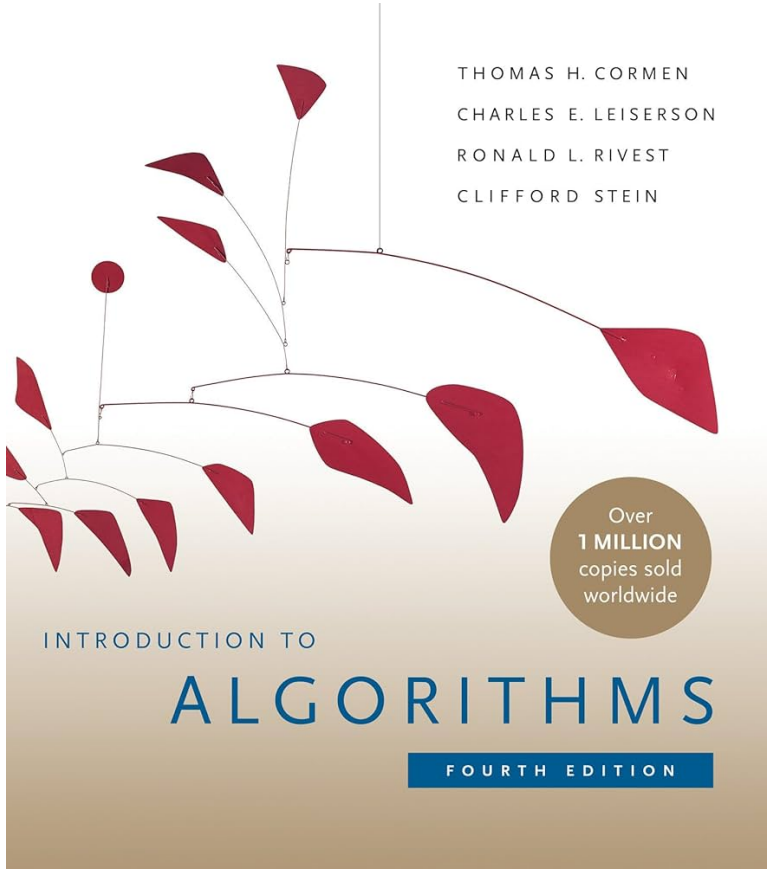
Introduction to Algorithms



Content has been extracted from *Introduction to Algorithms*, Fourth Edition, by Cormen, Leiserson, Rivest, and Stein. MIT Press. 2022.
Visit `https://mitpress.mit.edu/9780262046305/introduction-to-algorithms/`.
Original slides from *Introduction to Algorithms 6.046J/18.401J*, Fall 2005 Class by Prof. Charles Leiserson and Prof. Erik Demaine. MIT OpenCourseWare Initiative available at `https://ocw.mit.edu/courses/6-046j-introduction-to-algorithms-sma-5503-fall-2005/`.

Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts 'in place' (like insertion sort, but not like merge sort).
- Very practical (with tuning).

Divide and Conquer

Quicksort an n-element array:

1. **Divide:** Partition the array into two subarrays around a **pivot** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.

2. **Conquer:** Recursively sort the two subarrays.
3. **Combine:** Trivial.

Key:

Linear-time partitioning subroutine.

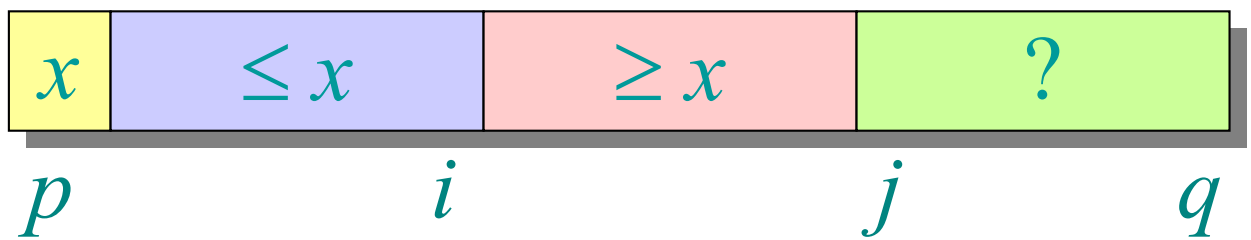
Partitioning Subroutine

```
 1: procedure PARTITION(A, p, q)
 2:     x ← A[p]
 3:     i ← p
 4:     for j ← p + 1 to q do
 5:         if A[j] ≤ x then
 6:             i ← i + 1
 7:             exchange A[i] ↔ A[j]
 8:         end if
 9:     end for
10:     exchange A[p] ↔ A[i]
11:     return i
12: end procedure
```

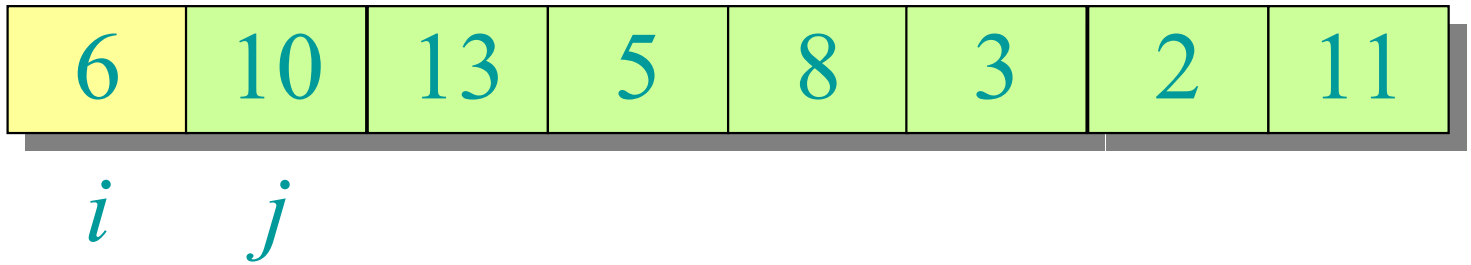
Running time:

$O(n)$ for n elements.

Invariant:



Example of partitioning



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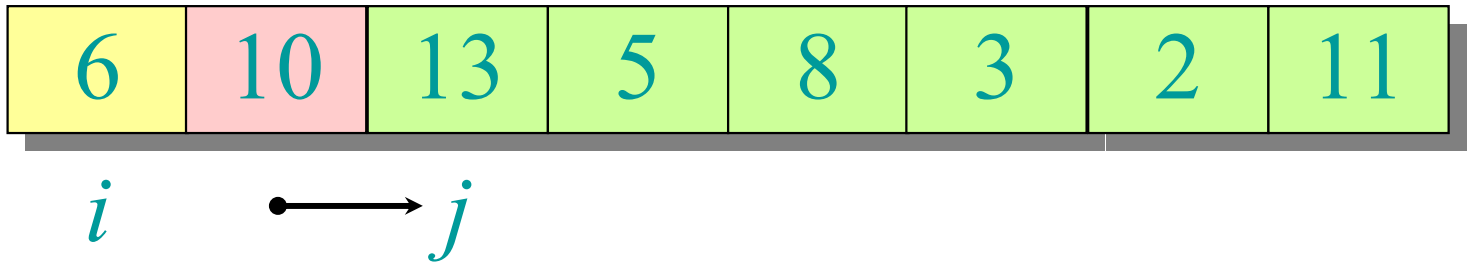

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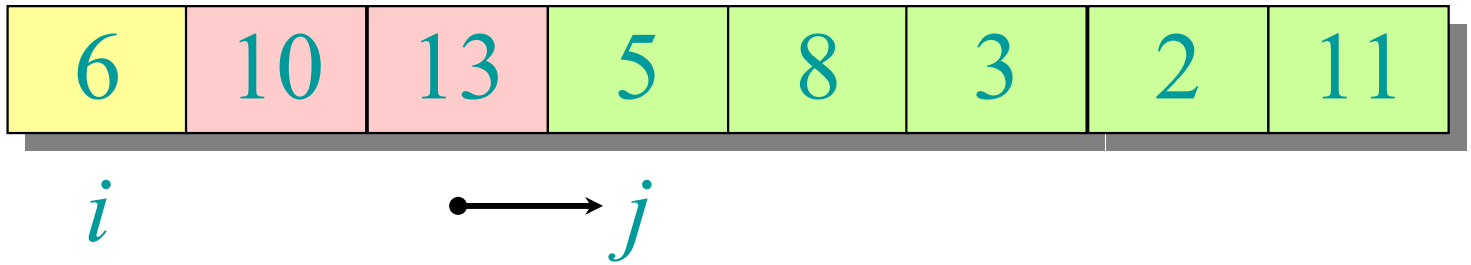

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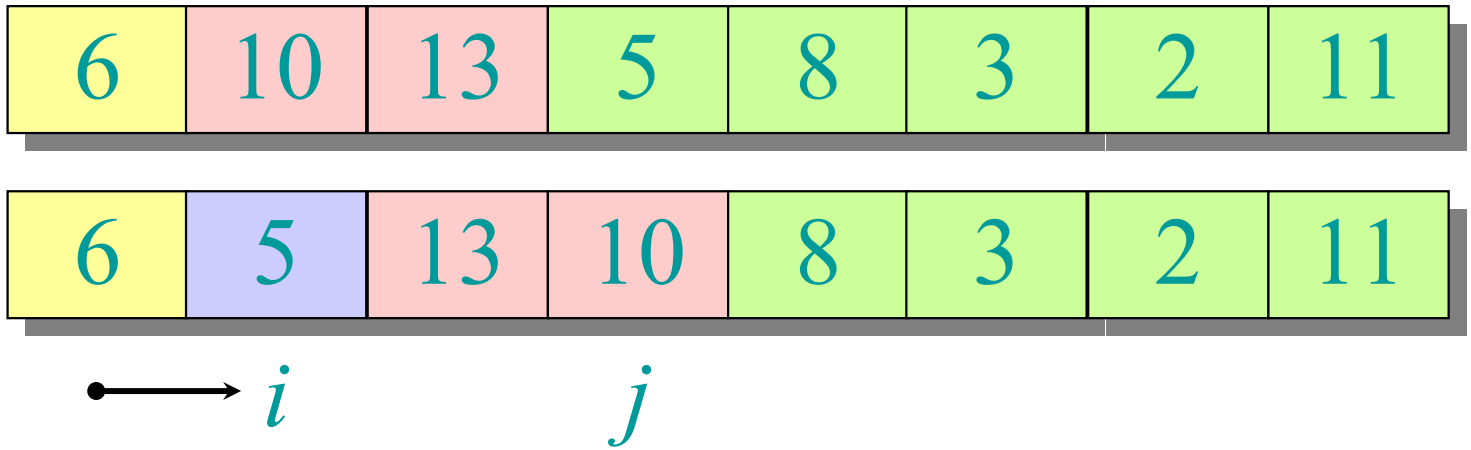
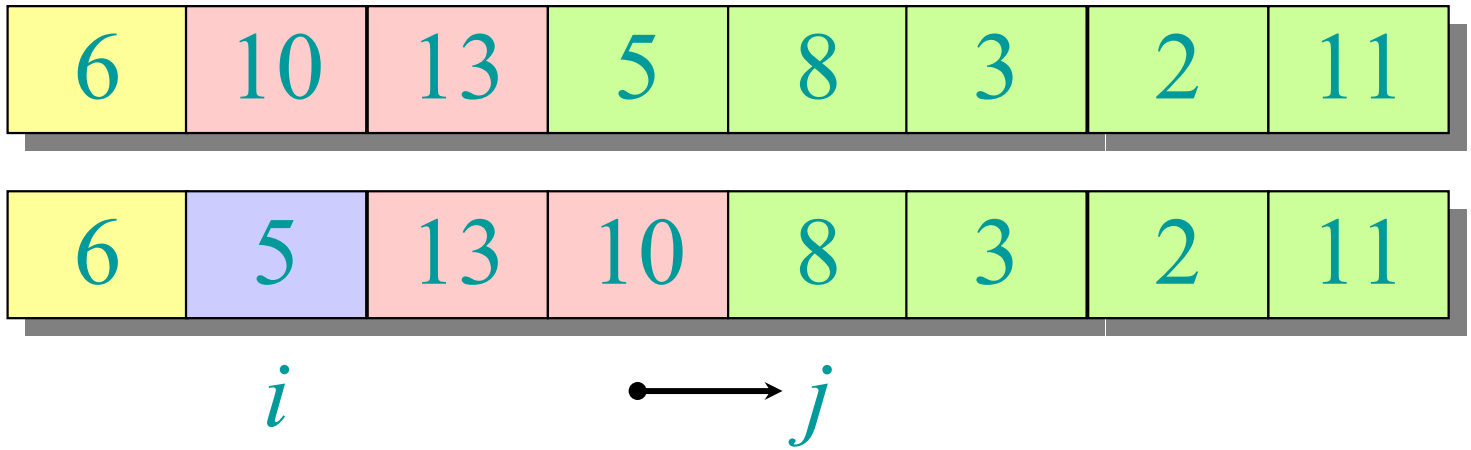


Example of partitioning



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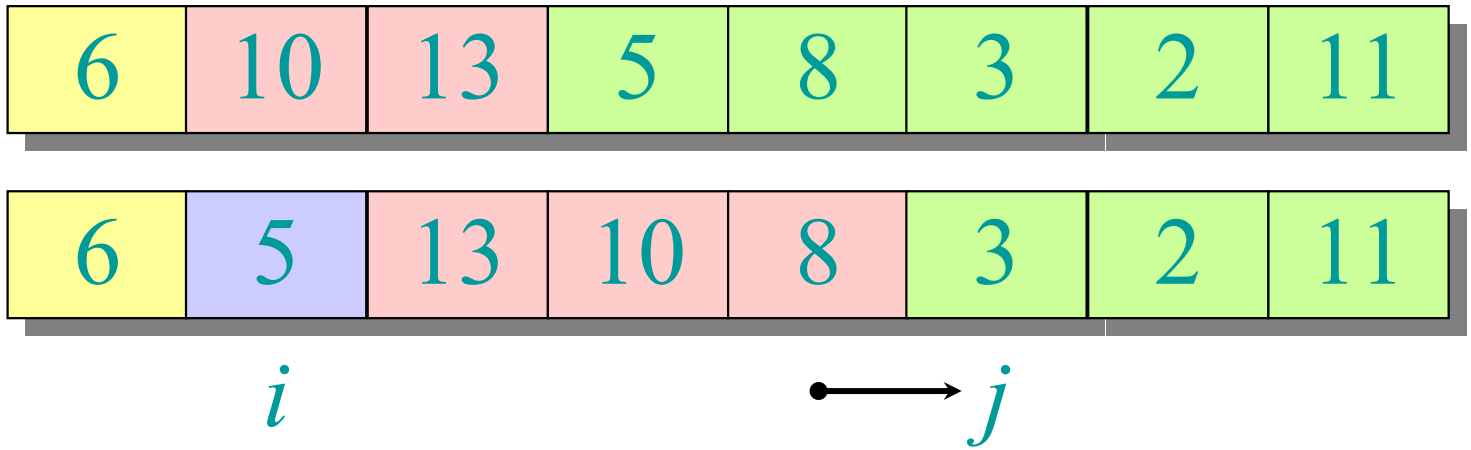

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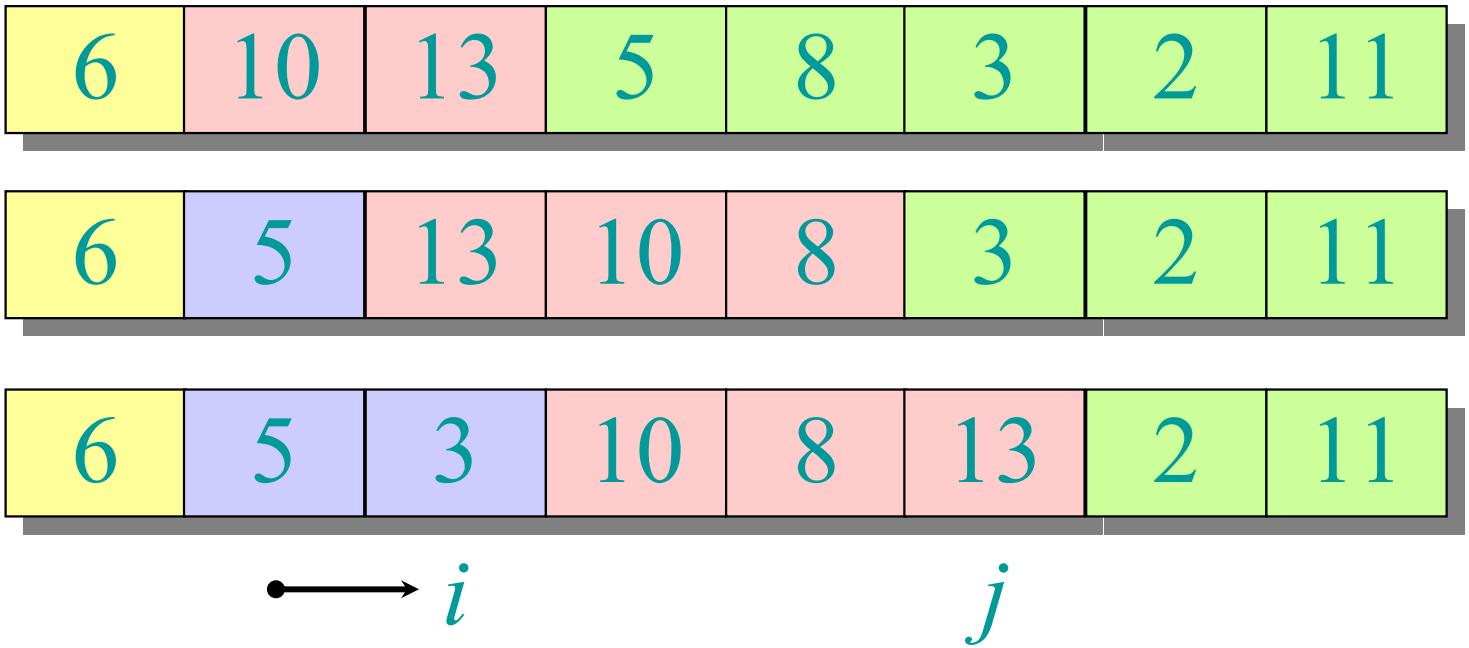

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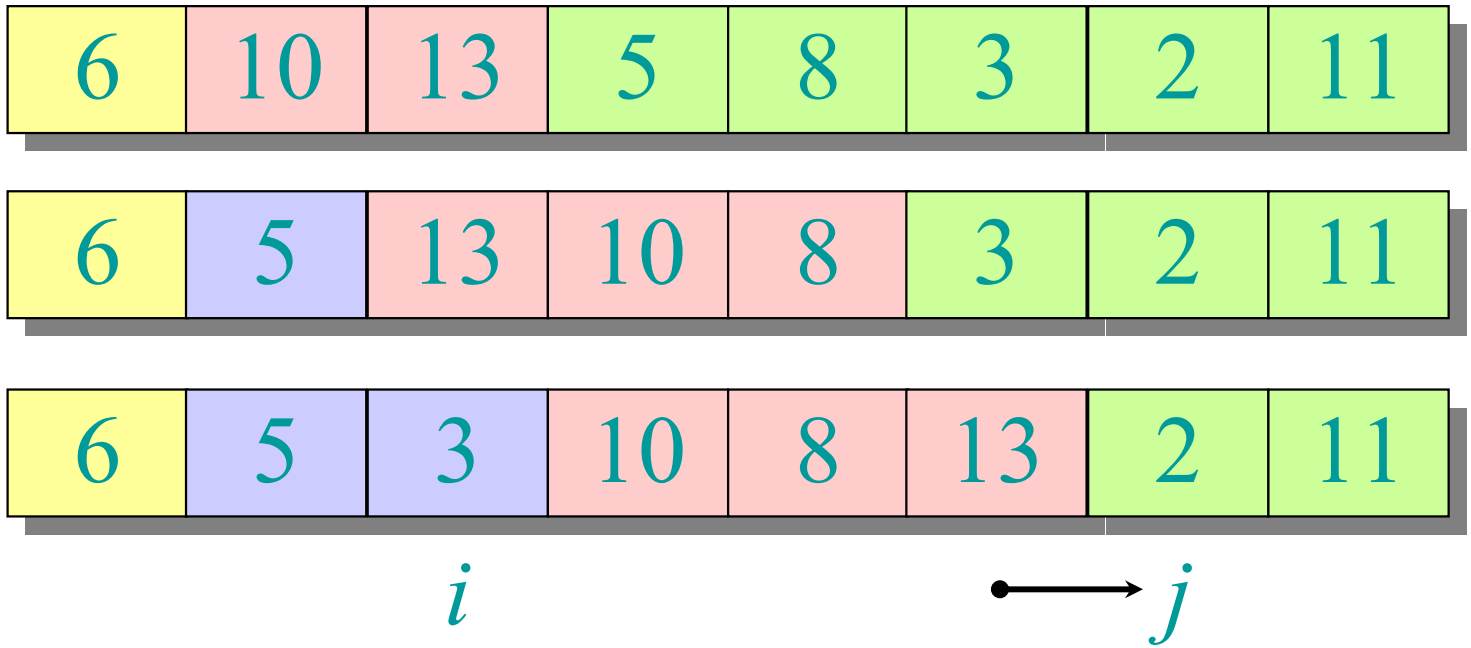

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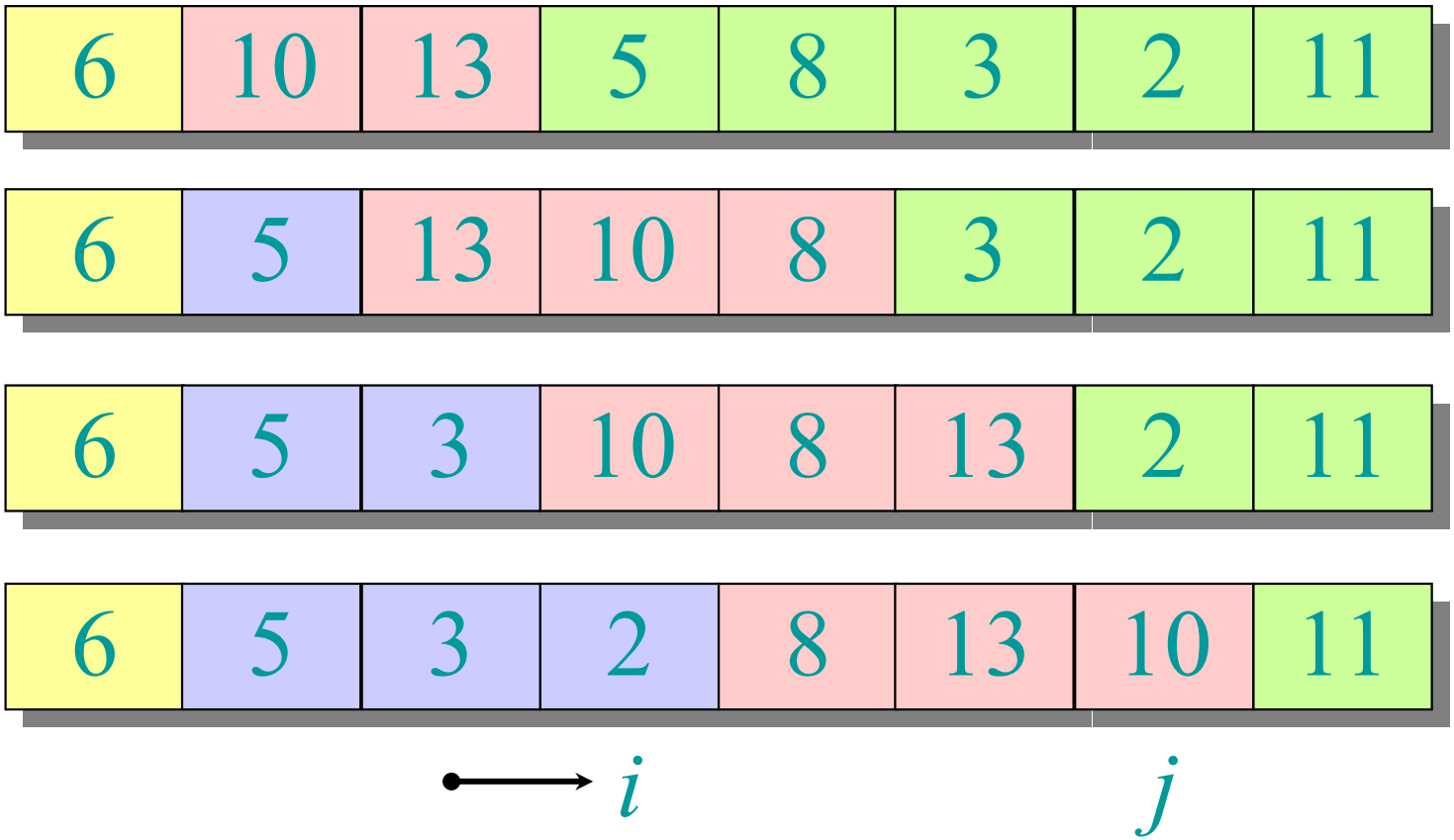

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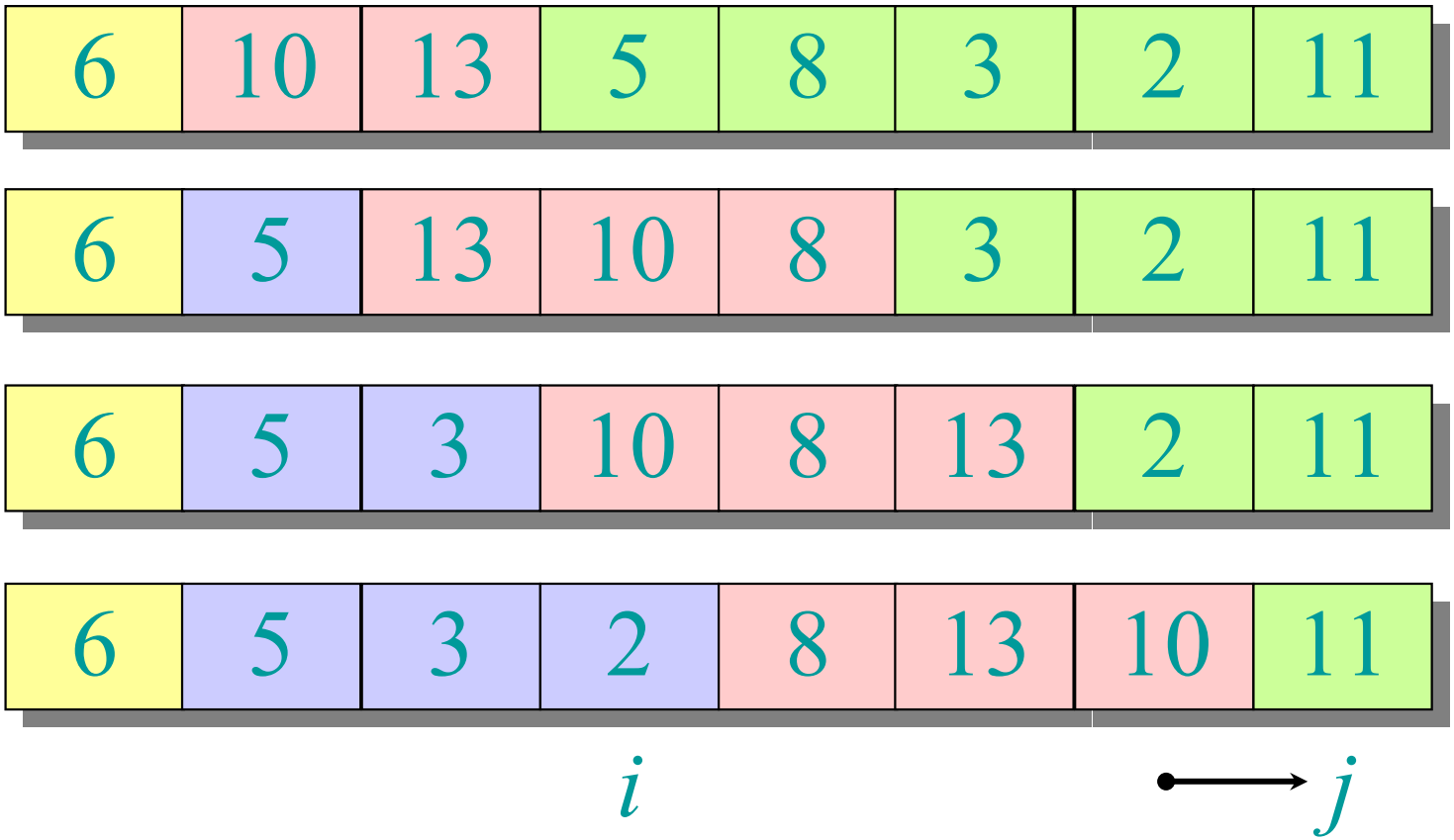

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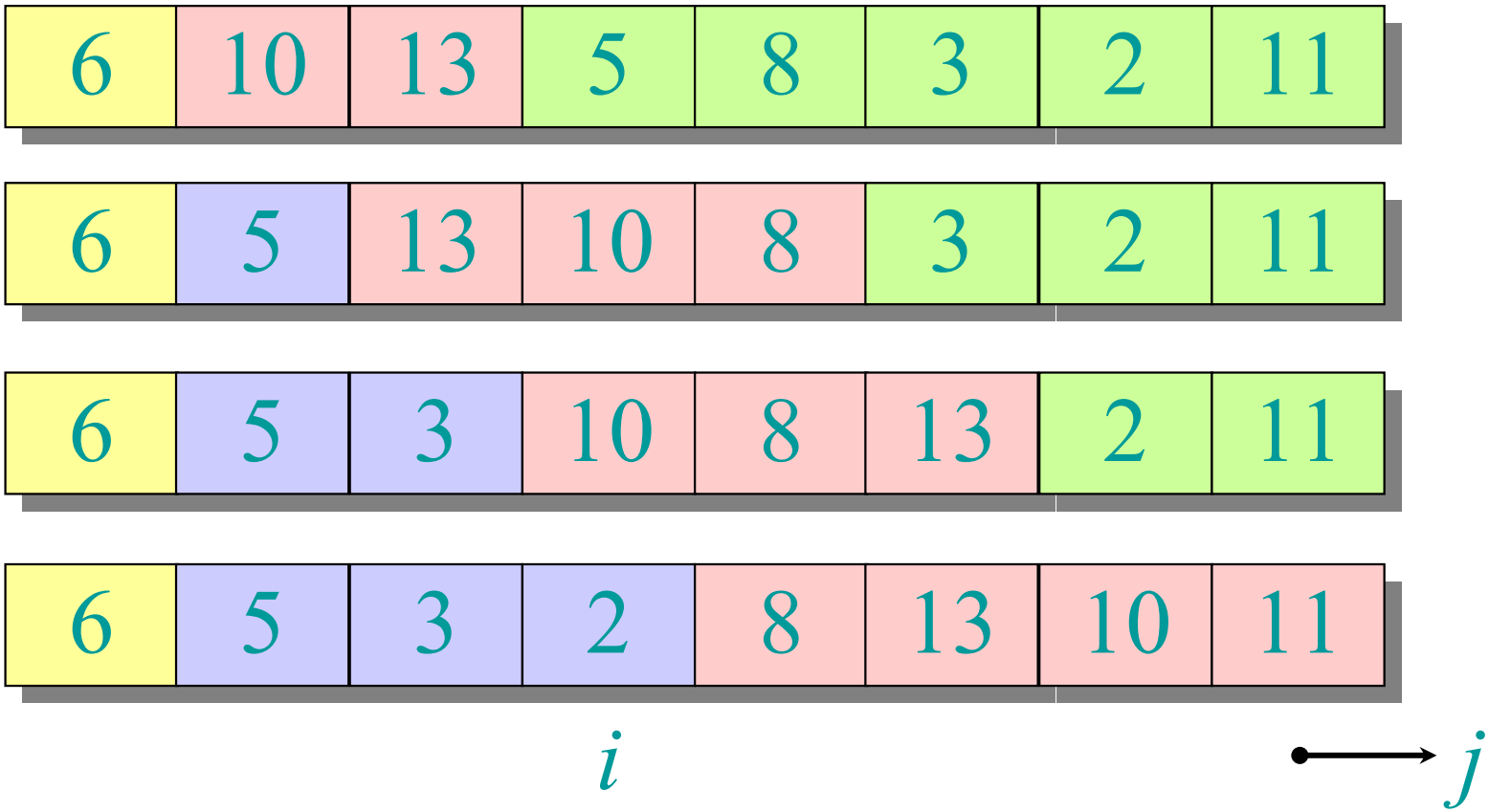

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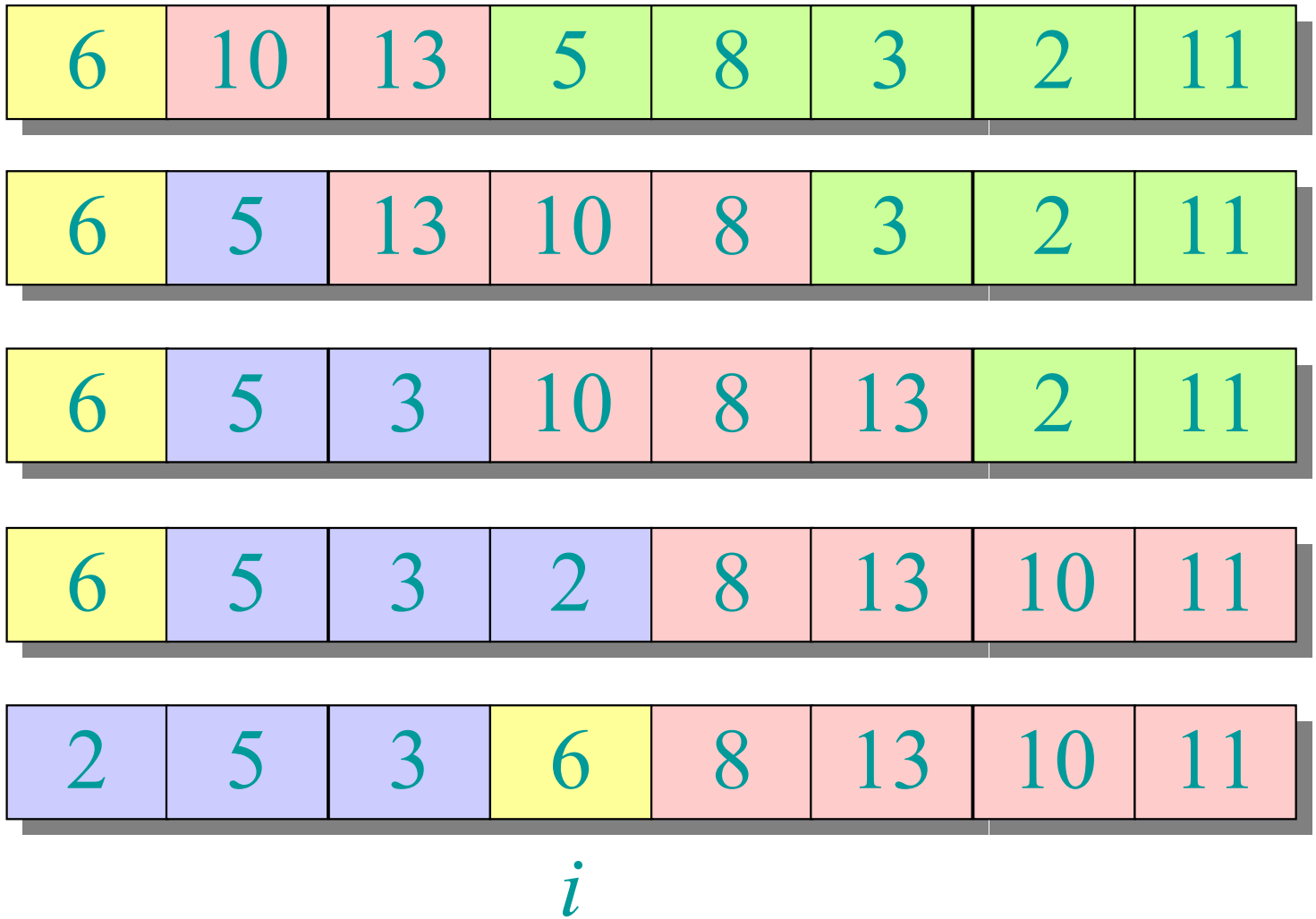

Pseudocode for Quicksort

```
1: procedure QUICKSORT(A, p, r)
2:     if p < r then
3:         q ← PARTITION(A,p, r)
4:         QUICKSORT(A, p, q − 1)
5:         QUICKSORT(A, q + 1, r)
6:     end if
7: end procedure
```

Initial call:

QUICKSORT$(A, 1, n)$

Analysis of Quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n) =$ worst-case running time on an array of n elements.

Worst-case of Quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \end{aligned}$$

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Arithmetic Series!

Worst-case Recursion Tree

$$T(n) = T(0) + T(n-1) + cn$$

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$T(n)$

Worst-case Recursion Tree

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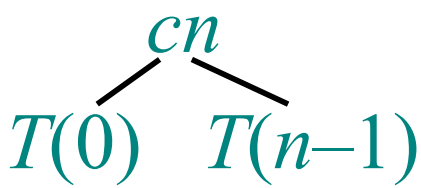

Worst-case Recursion Tree

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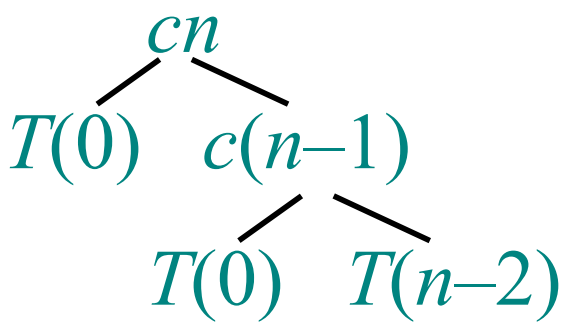

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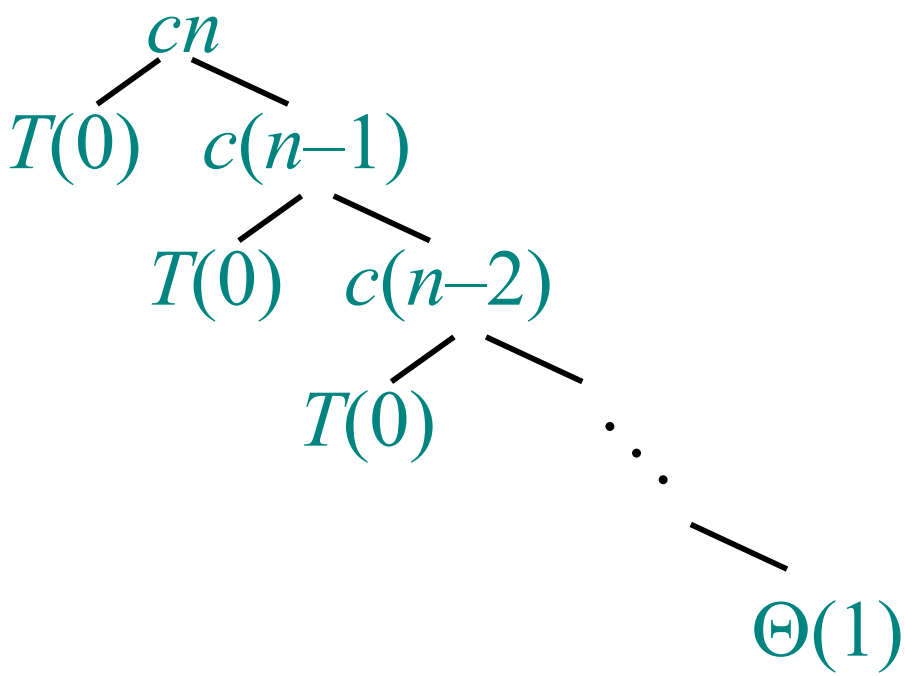

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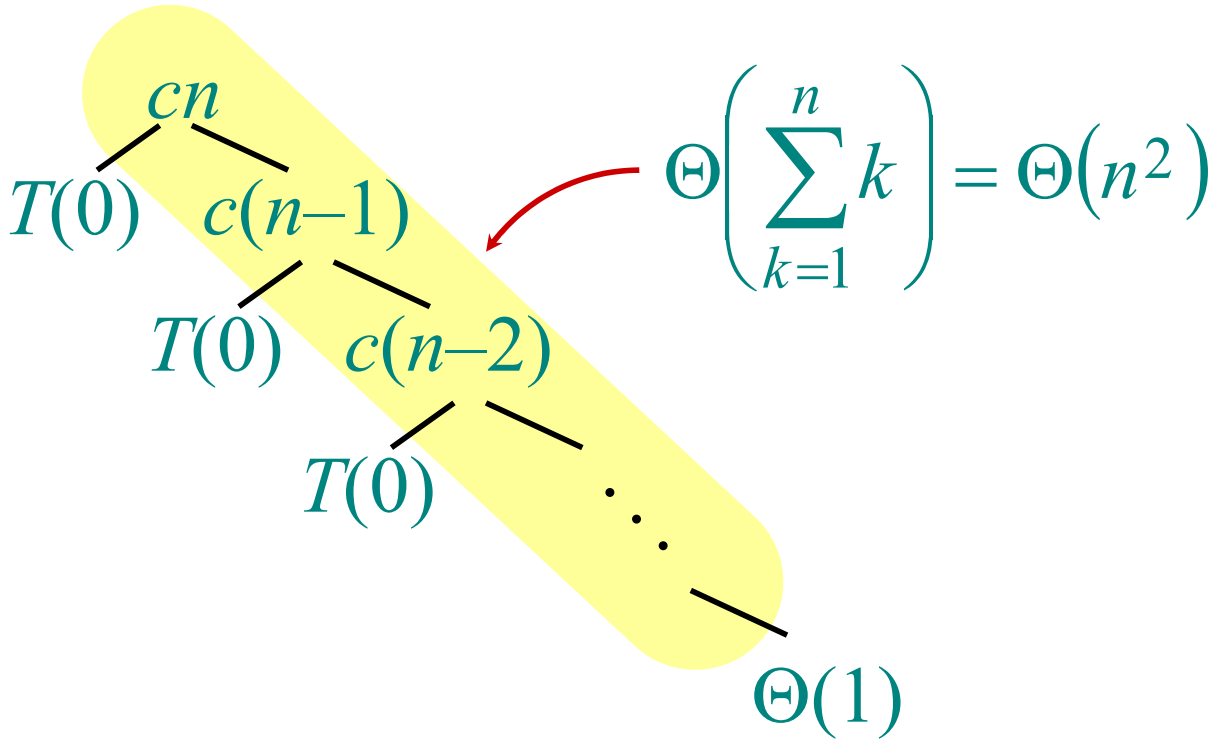

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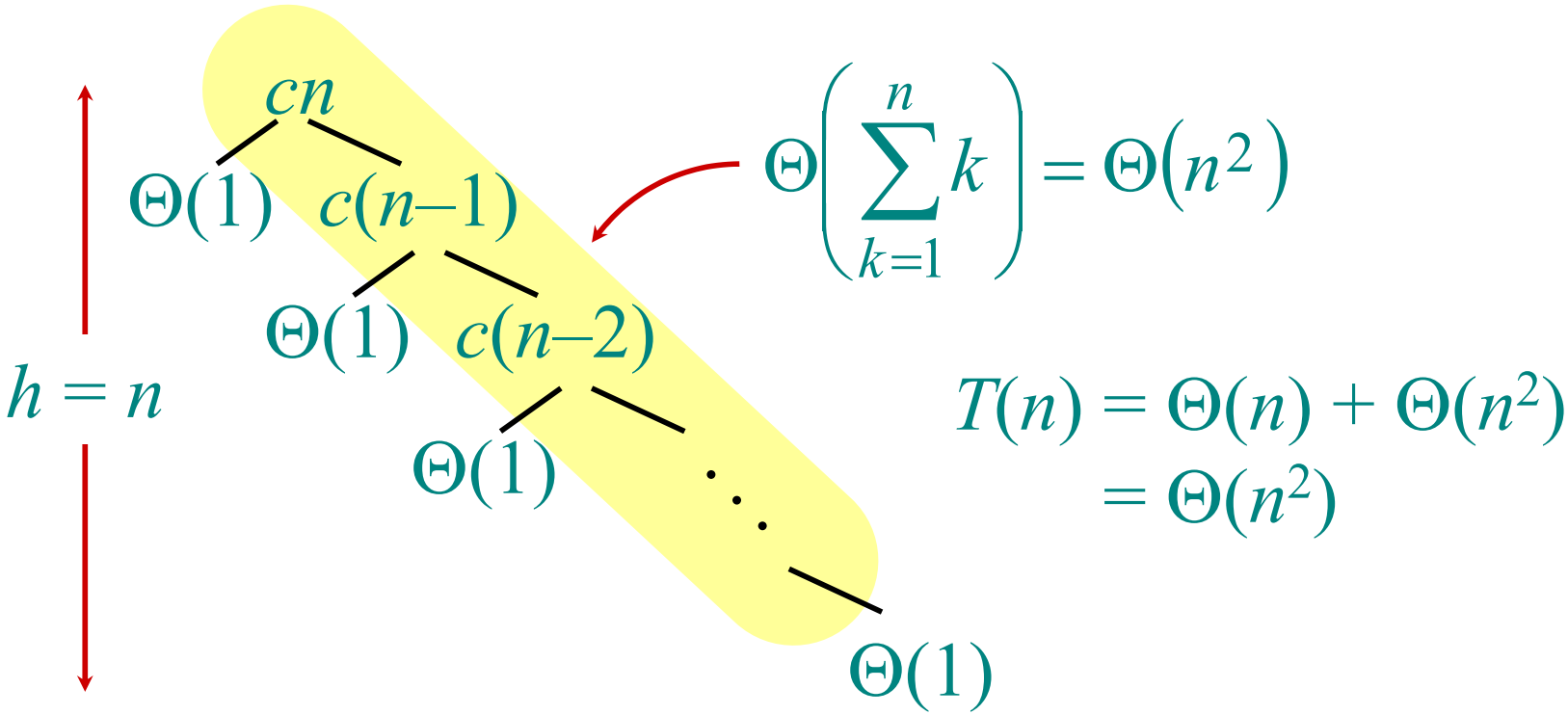


Best-case Analysis

For intuition only!

If we're lucky, PARTITION splits the array evenly:

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + \Theta(n) \\ &= \Theta(n \ln n) \qquad \text{(same as merge-sort)} \end{aligned}$$

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

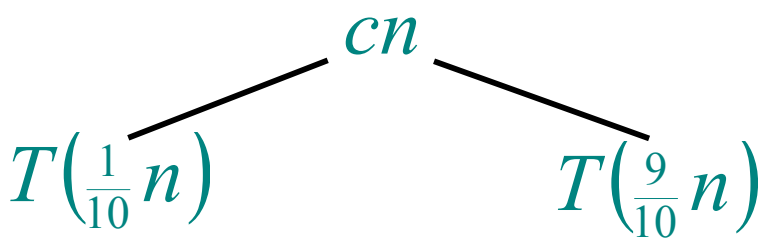
$$T(n) = T\left(\frac{1}{10}n\right) + T\left(\frac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

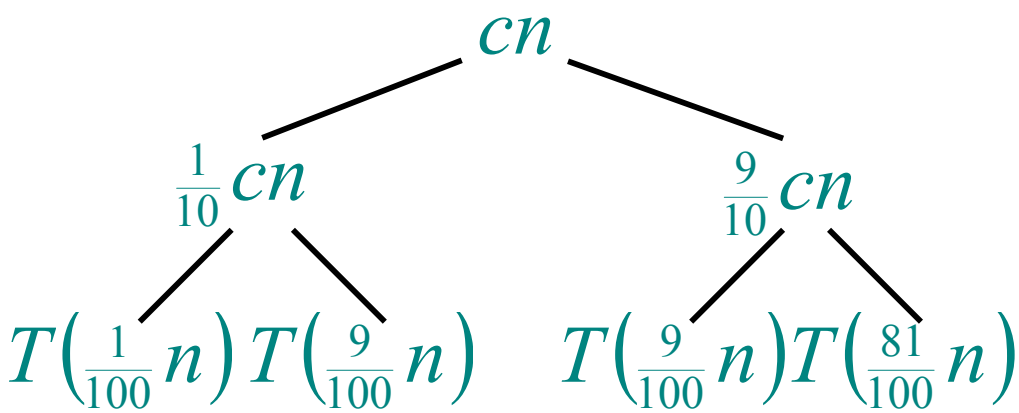
Analysis of "Almost-best" Case

$$T(n)$$

Analysis of "Almost-best" Case



Analysis of "Almost-best" Case



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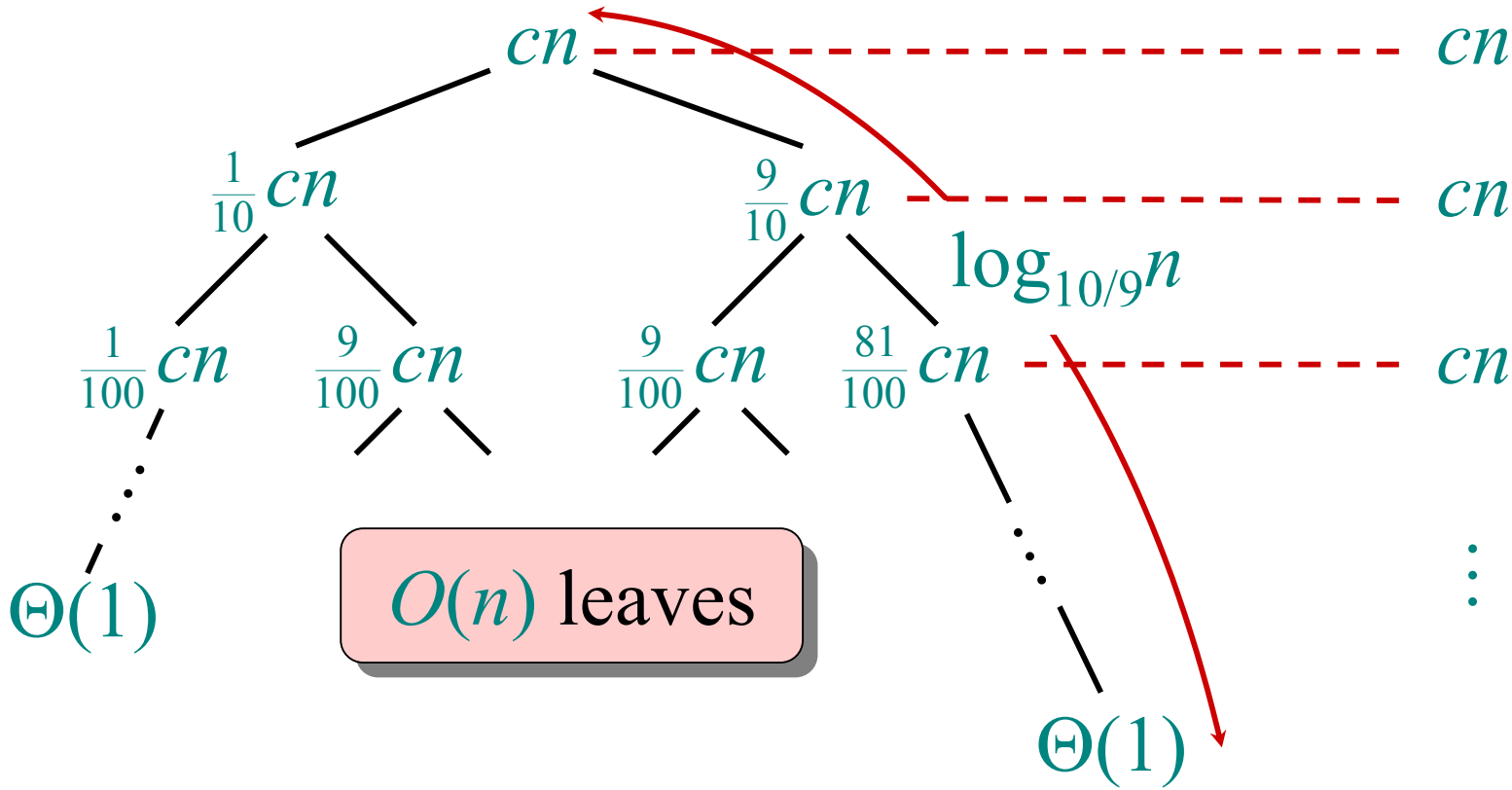

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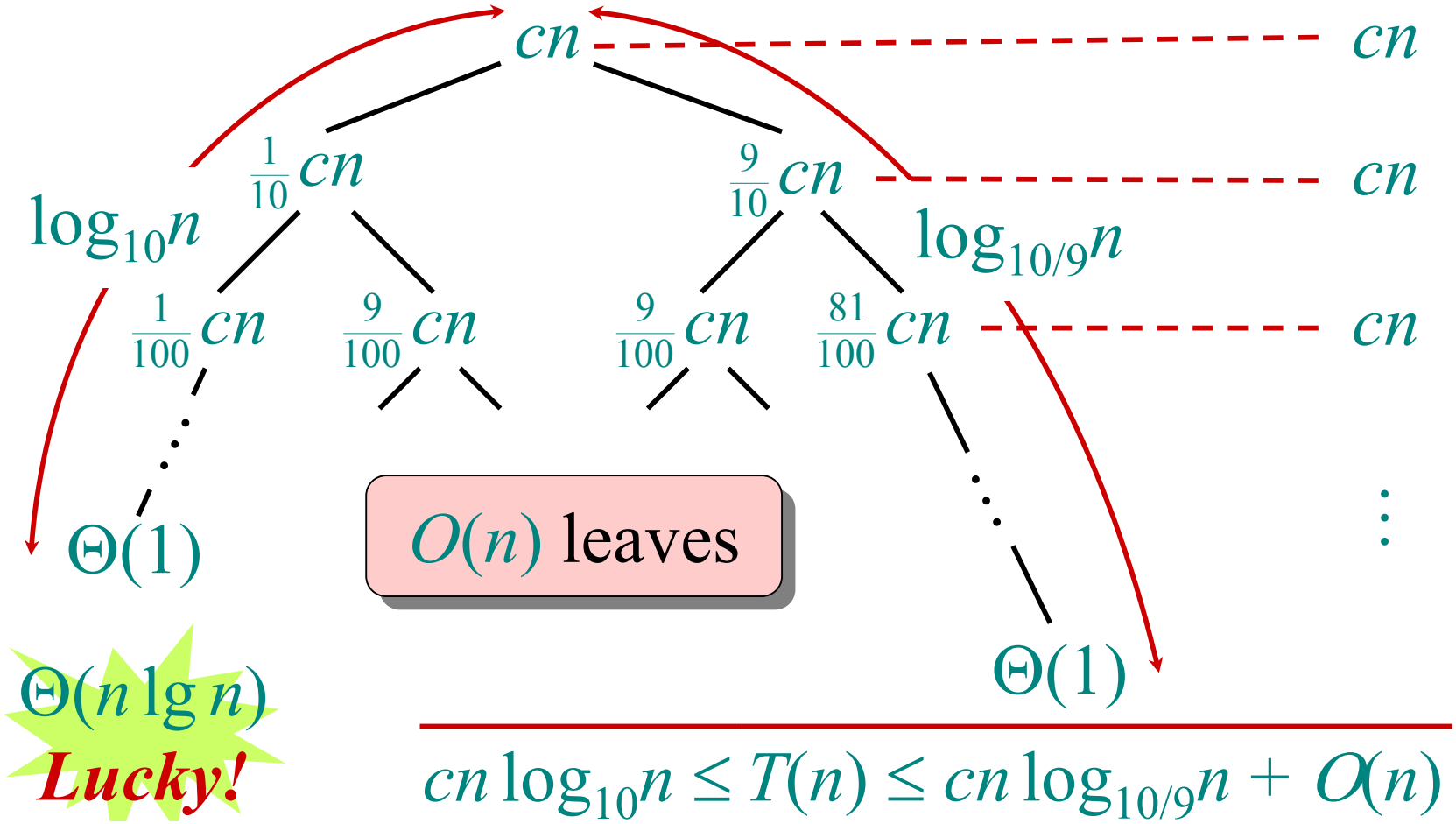


More Intuition

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ...

$$\begin{aligned} L(n) &= 2U\left(\frac{n}{2}\right) + \Theta(n) && \text{luckcy} \\ U(n) &= L\left(n-1\right) + \Theta(n) && \text{unluckcy} \end{aligned}$$

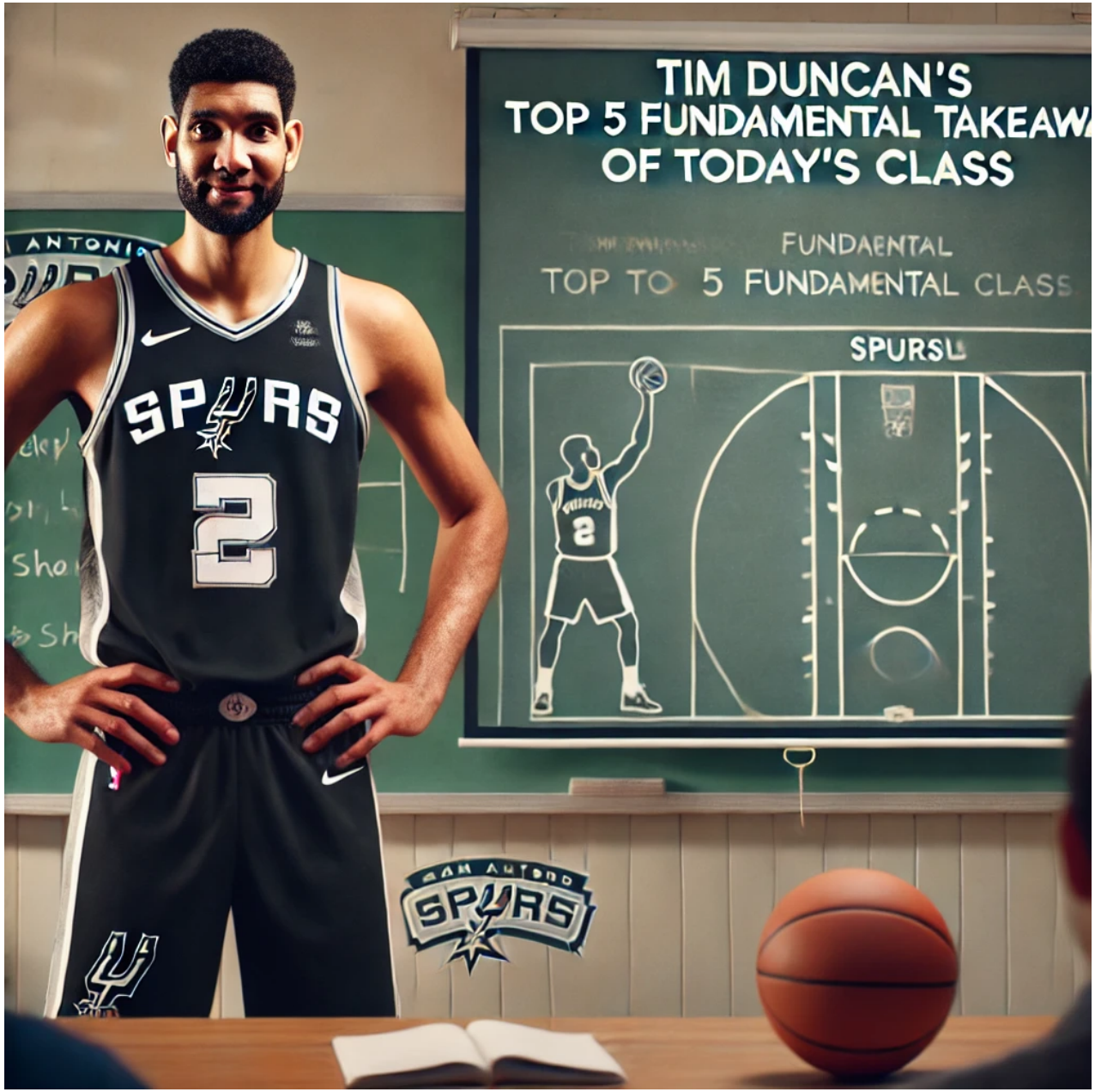
Solving:

$$\begin{aligned} L(n) &= 2\left(L\left(\frac{n}{2}-1\right) + \Theta(\frac{n}{2})\right) + \Theta(n) \\ &= 2L\left(\frac{n}{2}-1\right) + \Theta(n) \\ &= \Theta(n \ln n) && \textbf{lucky!} \end{aligned}$$

How can we make sure we are usually lucky?

End of Lecture 4.

TDT5FTOTTC


TIM DUNCAN'S
TOP 5 FUNDAMENTAL TAKEAW
OF TODAY'S CLASS
FUNDAENTAL
TOP TO 5 FUNDAMENTAL CLASS
SPURS
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Top 5 Fundamental Takeaways

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5

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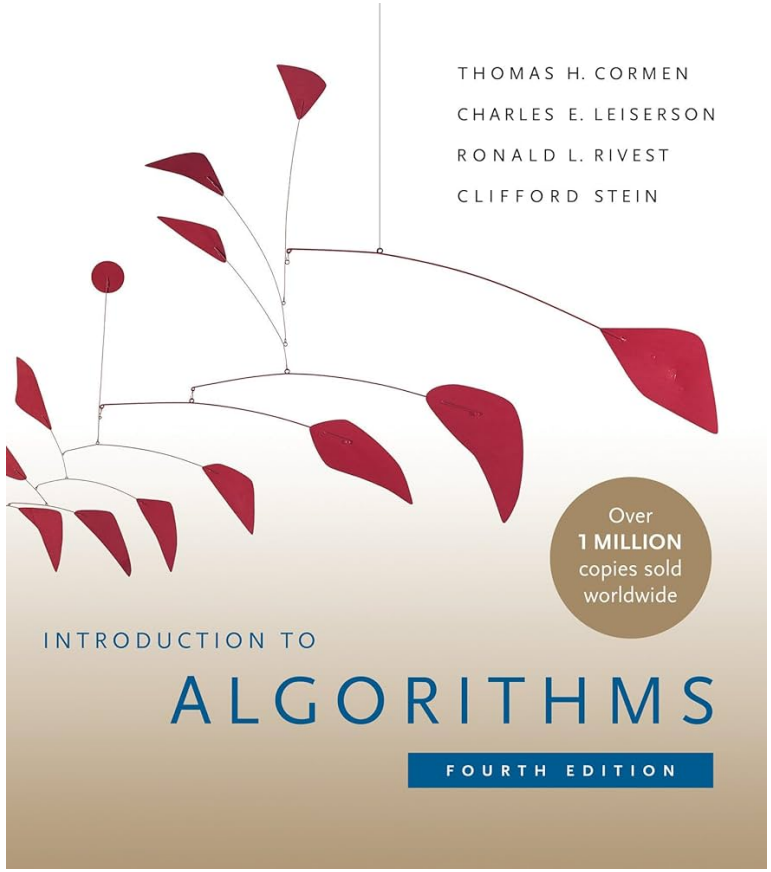

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