

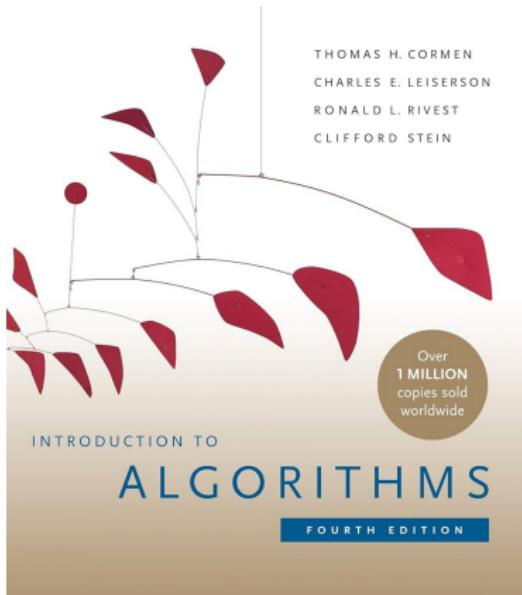
Introduction to Algorithms

Lecture 4: Quicksort

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Massachusetts Institute of Technology

February 25, 2025

Introduction to Algorithms



Content has been extracted from *Introduction to Algorithms*, Fourth Edition, by Cormen, Leiserson, Rivest, and Stein. MIT Press. 2022.

Visit <https://mitpress.mit.edu/9780262046305/introduction-to-algorithms/>.

Original slides from *Introduction to Algorithms* 6.046J/18.401J, Fall 2005 Class by Prof. Charles Leiserson and Prof. Erik Demaine. MIT OpenCourseWare Initiative available at <https://ocw.mit.edu/courses/6-046j-introduction-to-algorithms-sma-5503-fall-2005/>.

Quicksort

- ▶ Proposed by C.A.R. Hoare in 1962.
- ▶ Divide-and-conquer algorithm.
- ▶ Sorts ‘in place’ (like insertion sort, but not like merge sort).
- ▶ Very practical (with tuning).

Divide and Conquer

Quicksort an n -element array:

1. **Divide:** Partition the array into two subarrays around a **pivot** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.



2. **Conquer:** Recursively sort the two subarrays.
3. **Combine:** Trivial.

Key:

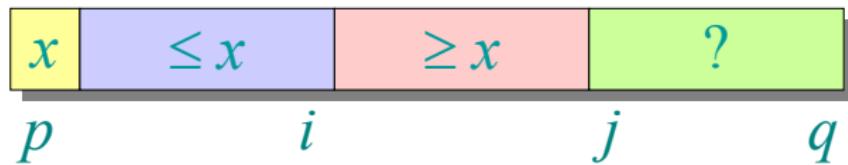
Linear-time partitioning subroutine.

Partitioning Subroutine

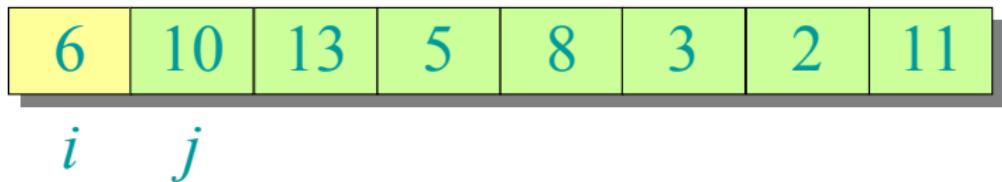
```
1: procedure PARTITION( $A, p, q$ )
2:    $x \leftarrow A[p]$ 
3:    $i \leftarrow p$ 
4:   for  $j \leftarrow p + 1$  to  $q$  do
5:     if  $A[j] \leq x$  then
6:        $i \leftarrow i + 1$ 
7:       exchange  $A[i] \leftrightarrow A[j]$ 
8:     end if
9:   end for
10:  exchange  $A[p] \leftrightarrow A[i]$ 
11:  return  $i$ 
12: end procedure
```

Running time:
 $O(n)$ for n elements.

Invariant:



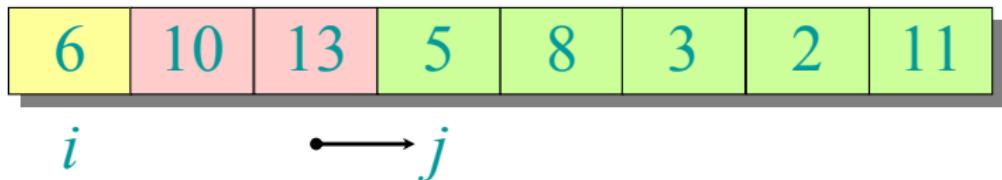
Example of partitioning



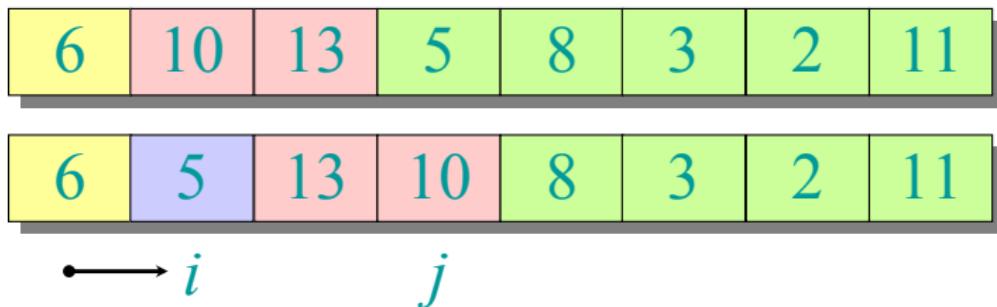
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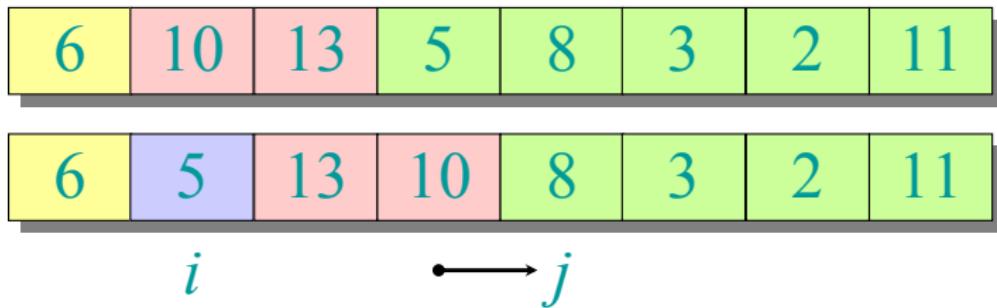
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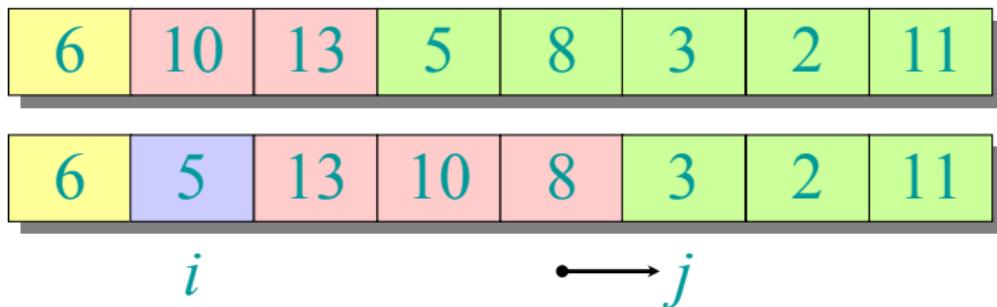
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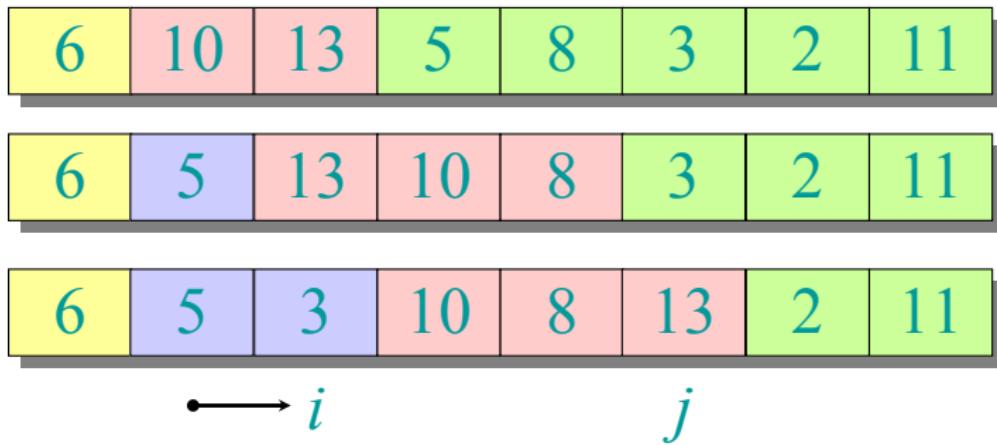
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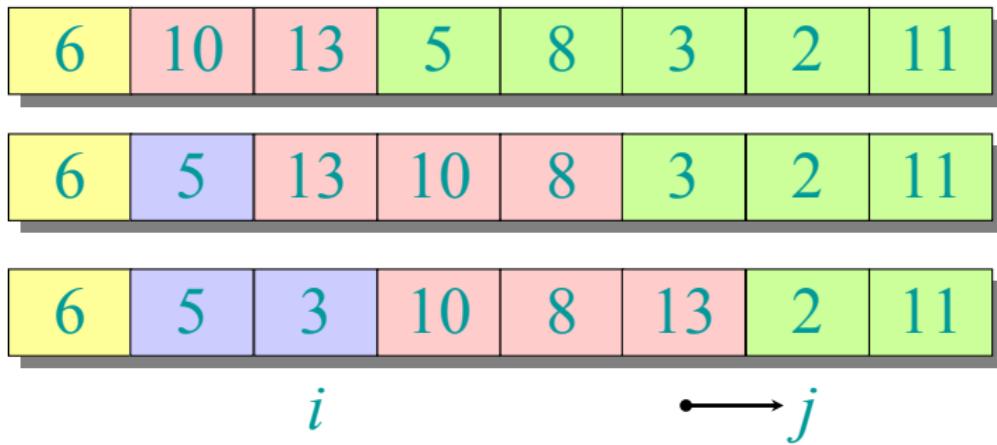
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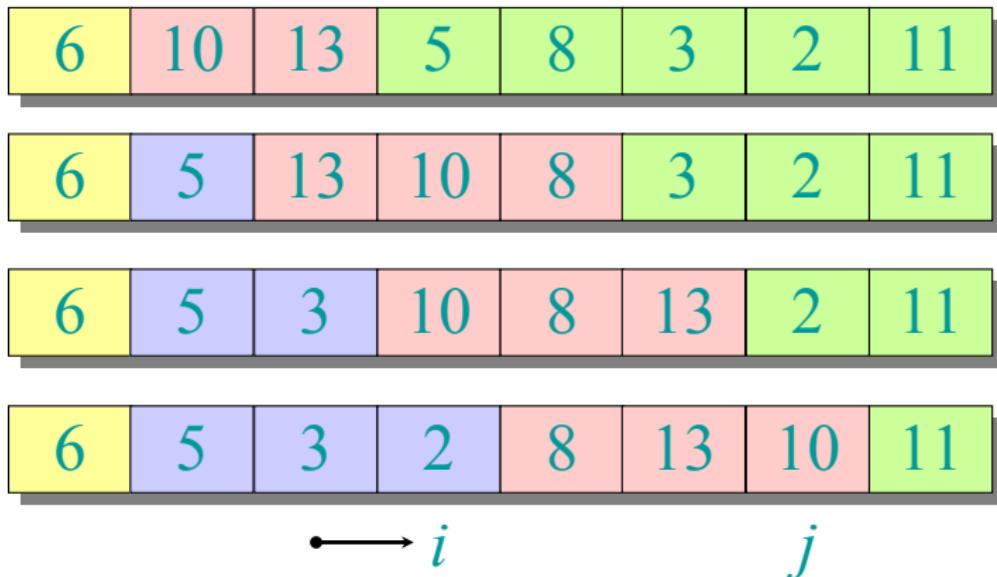
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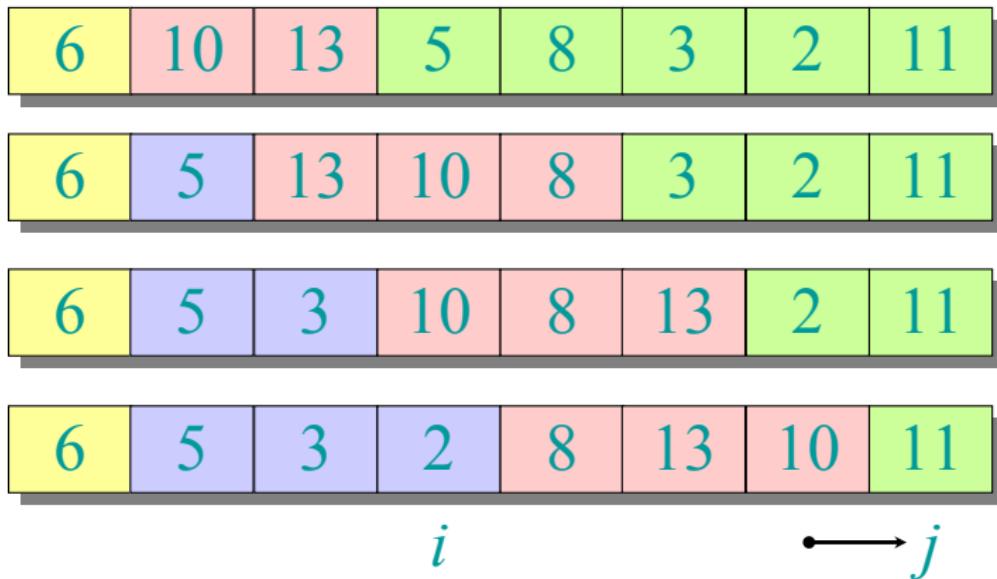
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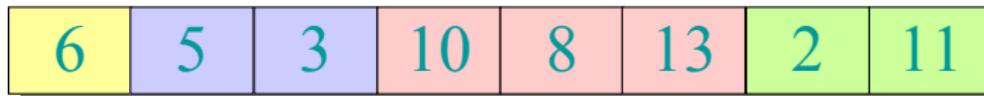
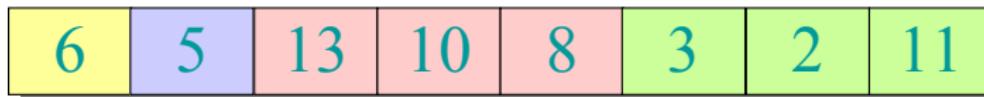
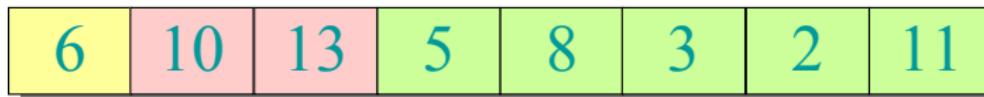
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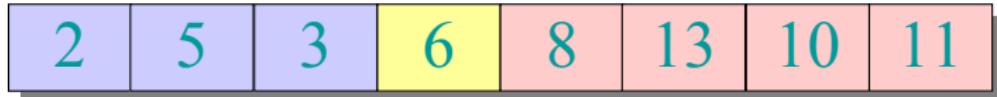
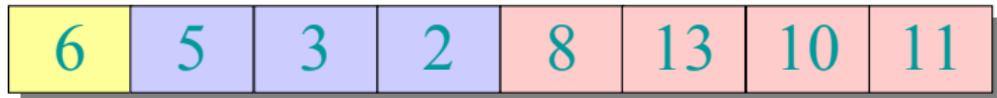
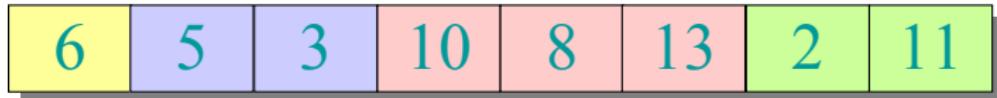
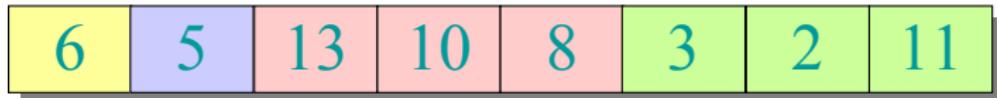
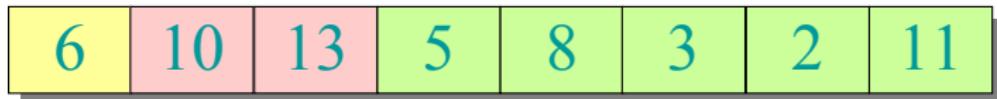
Example of partitioning



i

→ j

Example of partitioning



i

Pseudocode for Quicksort

```
1: procedure QUICKSORT( $A, p, r$ )
2:   if  $p < r$  then
3:      $q \leftarrow \text{PARTITION}(A, p, r)$ 
4:     QUICKSORT( $A, p, q - 1$ )
5:     QUICKSORT( $A, q + 1, r$ )
6:   end if
7: end procedure
```

Initial call:

QUICKSORT($A, 1, n$)

Analysis of Quicksort

- ▶ Assume all input elements are distinct.
- ▶ In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- ▶ Let $T(n) =$ worst-case running time on an array of n elements.

Worst-case of Quicksort

- ▶ Input sorted or reverse sorted.
- ▶ Partition around min or max element.
- ▶ One side of partition always has no elements.

$$\begin{aligned}T(n) &= T(0) + T(n - 1) + \Theta(n) \\&= \Theta(1) + T(n - 1) + \Theta(n) \\&= T(n - 1) + \Theta(n) \\&= \Theta(n^2)\end{aligned}$$

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Arithmetic Series!

Worst-case Recursion Tree

$$T(n) = T(0) + T(n - 1) + cn$$

Recurrence relation for worst-case recursion tree

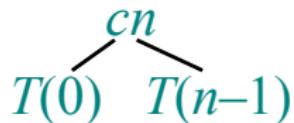
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$$T(n)$$

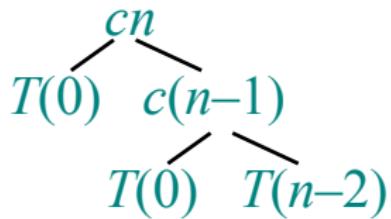
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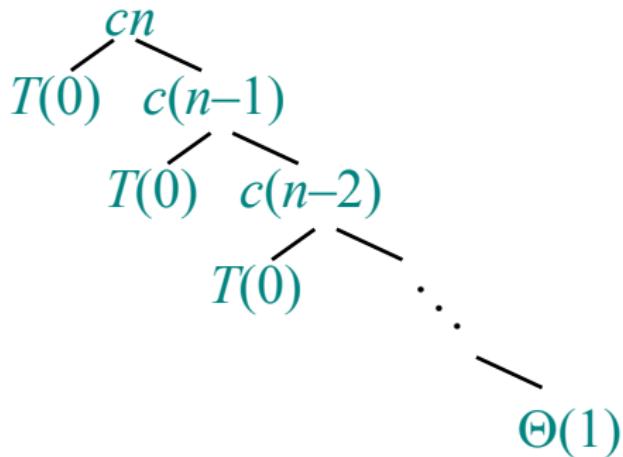
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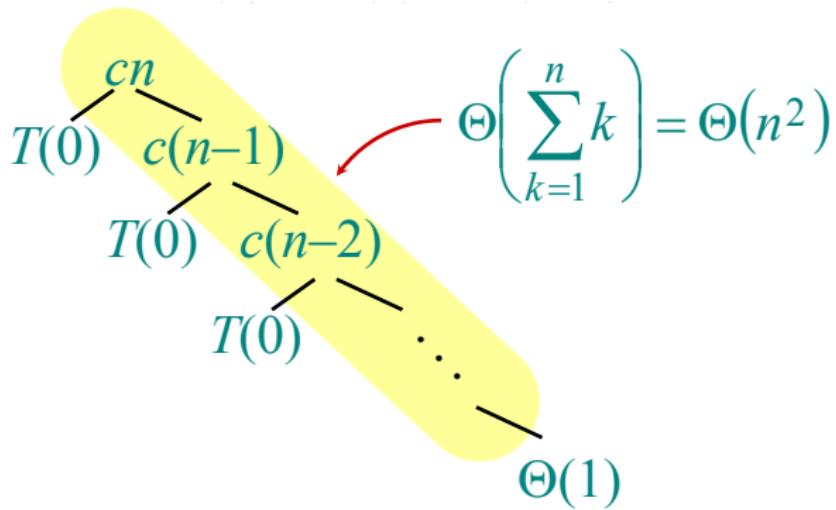
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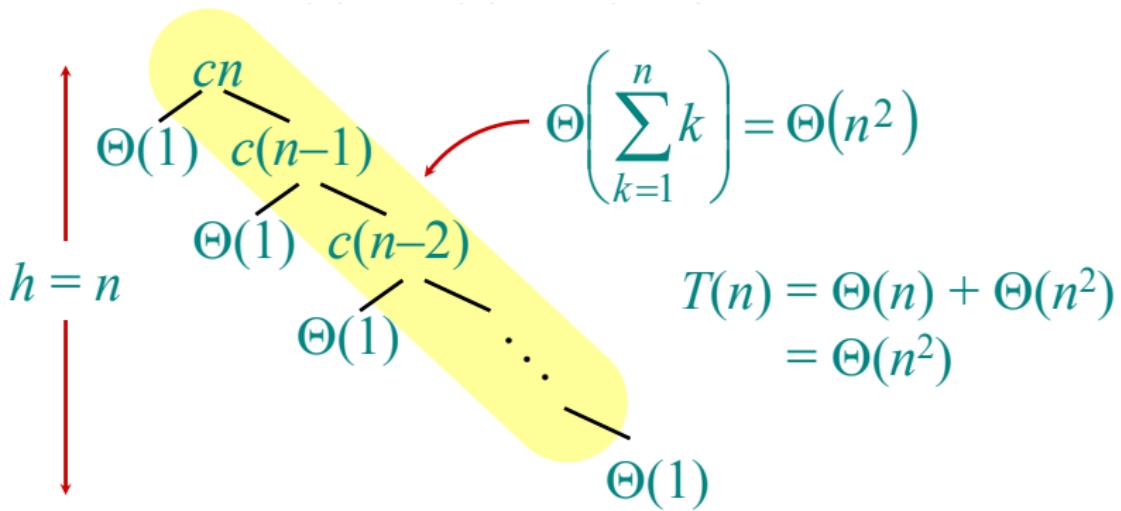
Worst-case Recursion Tree

$$T(n) = T(0) + T(n - 1) + cn$$



Worst-case Recursion Tree

$$T(n) = T(0) + T(n-1) + cn$$



Best-case Analysis

For intuition only!

If we're lucky, PARTITION splits the array evenly:

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + \Theta(n) \\ &= \Theta(n \ln n) \quad (\text{same as merge-sort}) \end{aligned}$$

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\frac{1}{10}n\right) + T\left(\frac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

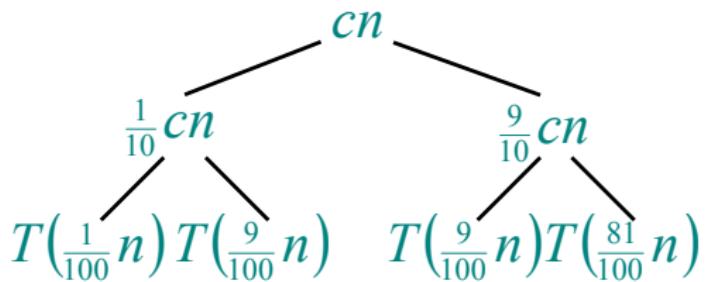
Analysis of “Almost-best” Case

$$T(n)$$

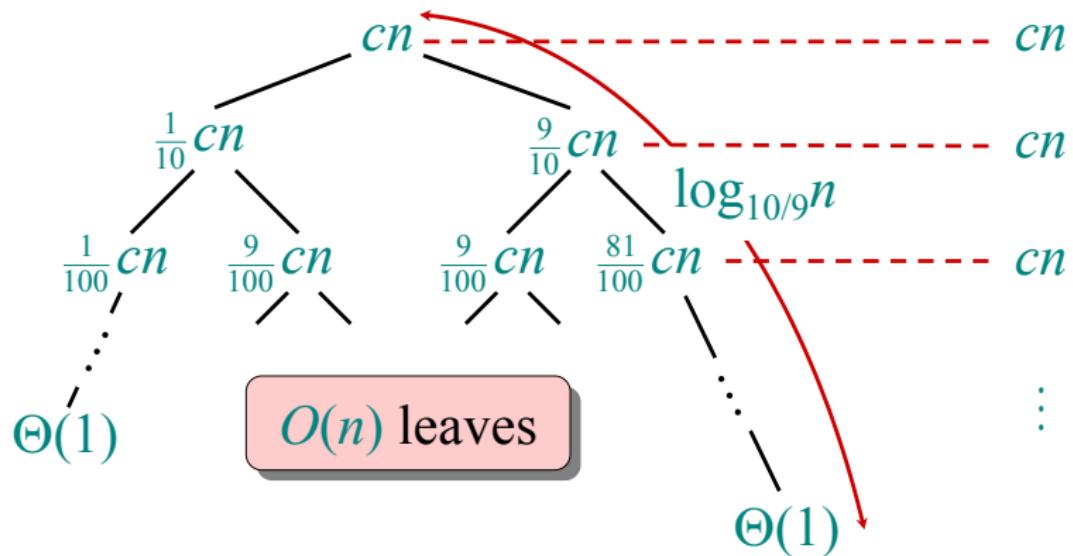
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$$\begin{array}{c} cn \\ / \quad \backslash \\ T\left(\frac{1}{10}n\right) \quad T\left(\frac{9}{10}n\right) \end{array}$$

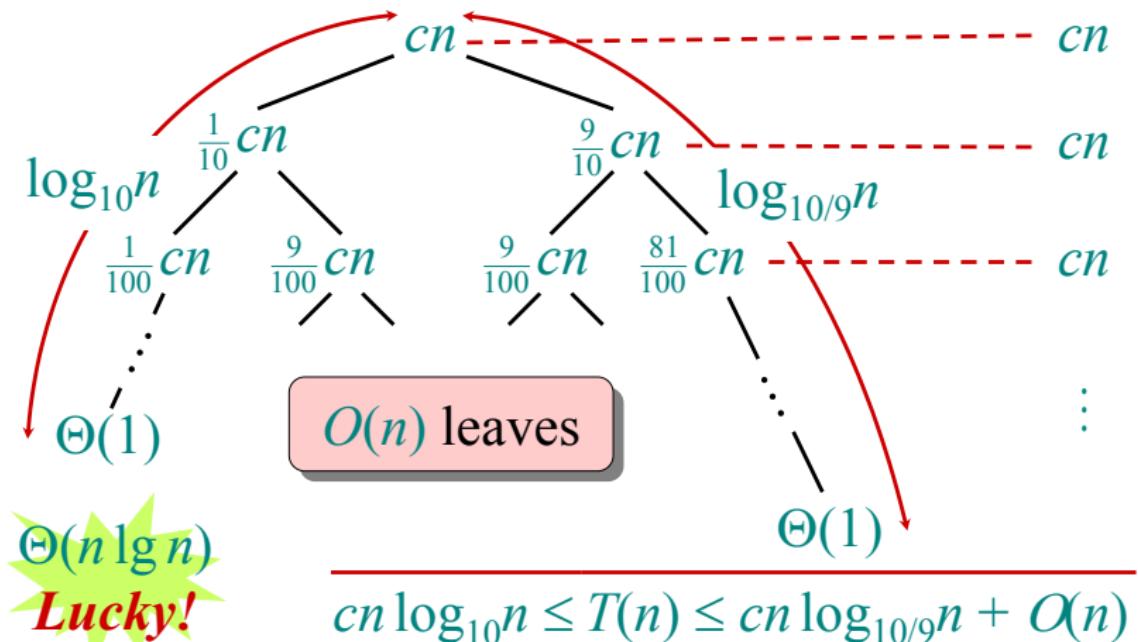
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Analysis of “Almost-best” Case



More Intuition

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ...

$$L(n) = 2U\left(\frac{n}{2}\right) + \Theta(n) \quad \textit{lucky}$$

$$U(n) = L(n-1) + \Theta(n) \quad \textit{unlucky}$$

Solving:

$$\begin{aligned} L(n) &= 2 \left(L\left(\frac{n}{2} - 1\right) + \Theta\left(\frac{n}{2}\right) \right) + \Theta(n) \\ &= 2L\left(\frac{n}{2} - 1\right) + \Theta(n) \\ &= \Theta(n \ln n) \quad \textbf{Lucky!} \end{aligned}$$

How can we make sure we are usually lucky?

Plan

Randomized Quicksort

Randomized Quicksort

IDEA:

Partition around a **random** element.

- ▶ Running time is independent of the input order.
- ▶ No assumptions need to be made about the input distribution.
- ▶ No specific input elicits the worst-case behavior.
- ▶ The worst case is determined only by the output of a random-number generator.

Randomized Quicksort Analysis

Let $T(n) =$ the random variable for the running time of randomized quicksort on an input of size n , assuming random numbers are independent.

For $k = 0, 1, \dots, n - 1$, define the **indicator random variable**.

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n - k - 1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_k] = \Pr\{X_k = 1\} = \frac{1}{n}$, since all splits are equally likely, assuming elements are distinct.

Analysis (Cont.)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \vdots \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split.} \end{cases}$$
$$= \sum_{k=0}^{n-1} X_k(T(k) + T(n-k-1) + \Theta(n))$$

Calculating expectation

Take expectations of both sides.

$$E[T(n)] = E \left[\sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n)) \right]$$

Calculating expectation

Linearity of expectation.

$$\begin{aligned} E[T(n)] &= E \left[\sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n)) \right] \\ &= \sum_{k=0}^{n-1} E [X_k (T(k) + T(n-k-1) + \Theta(n))] \end{aligned}$$

Calculating expectation

Independence of X_k from other random choices.

$$\begin{aligned} E[T(n)] &= E \left[\sum_{k=0}^{n-1} X_k(T(k) + T(n-k-1) + \Theta(n)) \right] \\ &= \sum_{k=0}^{n-1} E[X_k(T(k) + T(n-k-1) + \Theta(n))] \\ &= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)] \end{aligned}$$

Calculating expectation

Linearity of expectation: $E[X_k] = \frac{1}{n}$.

$$\begin{aligned} E[T(n)] &= E \left[\sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n)) \right] \\ &= \sum_{k=0}^{n-1} E [X_k (T(k) + T(n-k-1) + \Theta(n))] \\ &= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)] \\ &= \frac{1}{n} \sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n} \sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n} \sum_{k=0}^{n-1} \Theta(n) \end{aligned}$$

Calculating expectation

Summations have identical terms.

$$\begin{aligned} E[T(n)] &= E \left[\sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n)) \right] \\ &= \sum_{k=0}^{n-1} E [X_k (T(k) + T(n-k-1) + \Theta(n))] \\ &= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)] \\ &= \frac{1}{n} \sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n} \sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n} \sum_{k=0}^{n-1} \Theta(n) \\ &= \frac{2}{n} \sum_{k=0}^{n-1} E[T(k)] + \Theta(n) \end{aligned}$$

Hairy recurrence

$$E[T(n)] = \frac{2}{n} \sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove:

$$E[T(n)] \leq an \lg n \text{ for constant } a > 0.$$

- ▶ Choose a large enough so that $an \lg n$ dominates $E[T(n)]$ for sufficiently small $n \geq 2$.

Use fact:

$$\sum_{k=2}^{n-1} k \lg k \leq \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2 \text{ (exercise).}$$

Substitution method

Substitute inductive hypothesis.

$$E[T(n)] \leq \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

Substitution method

Use fact.

$$\begin{aligned} E[T(n)] &\leq \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n) \\ &\leq \frac{2a}{n} \left(\frac{1}{2} n^2 \lg n - \frac{1}{8} n^2 \right) + \Theta(n) \end{aligned}$$

Substitution method

$$\begin{aligned} E[T(n)] &\leq \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n) \\ &\leq \frac{2a}{n} \left(\frac{1}{2} n^2 \lg n - \frac{1}{8} n^2 \right) + \Theta(n) \\ &= an \lg n - \left(\frac{an}{4} - \Theta(n) \right) \\ &\leq an \lg n, \end{aligned}$$

if a is chosen large enough so that

$\frac{an}{4}$ dominates the $\Theta(n)$.

Quicksort in practice

- ▶ Quicksort is a great general-purpose sorting algorithm.
- ▶ Quicksort is typically over twice as fast as merge sort.
- ▶ Quicksort can benefit substantially from **code tuning**.
- ▶ Quicksort behaves well even with caching and virtual memory.

End of Lecture 4.

TDT5FTOTTC



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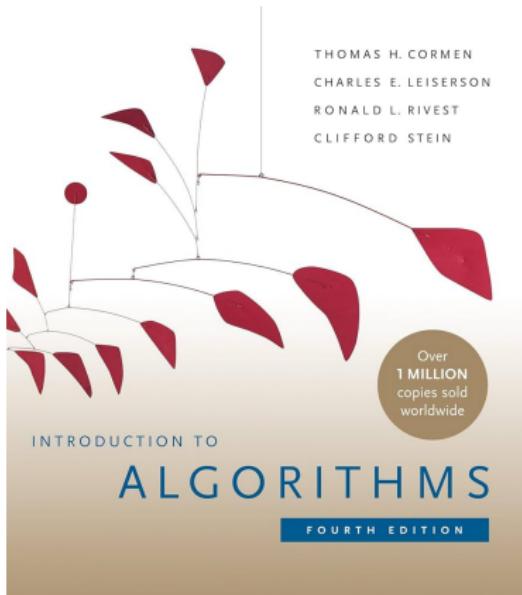
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- 2 **Randomized Quicksort Helps Avoid Worst-case Behavior** – Choosing a random pivot prevents consistently bad splits and ensures an expected $O(n \log n)$ runtime.
- 1 **Quicksort is Highly Efficient in Practice** – It outperforms merge sort in most cases and benefits from hardware optimizations.

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