Introduction to Algorithms
Bonus Lecture: Proof by Induction

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March 5, 2025

Proof by Induction

- A powerful mathematical technique.
- Prove that a statement is true for all natural numbers (or some sequence of numbers).
- It's like knocking over a line of dominoes...

How Induction Works

Principle of Mathematical Induction:

- **Base Case:** Show the statement holds for the first value (usually $n = 1$).
- **Inductive Hypothesis:** Assume the statement holds for some arbitrary $n = k$.
- **Inductive Step:** Prove it holds for $n = k + 1$.

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- **Inductive Step:** Prove it holds for $n = k + 1$.

If those steps hold, the statement is true for all n.

Example

Prove that:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for all $n \geq 1$.

Example

Step 1: Base Case

For $n = 1$:

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

✓**True!**

Example

Step 2: Inductive Hypothesis

Assume that for some $n = k$, the formula holds:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

(This is our assumption or "*inductive hypothesis*".)

Example

Step 3: Inductive Step

We must prove it holds for $n = k + 1$, meaning:

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

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$$\left(\frac{k(k+1)}{2}\right) + (k+1)$$

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Using the inductive hypothesis:

$$\left(\frac{k(k + 1)}{2}\right) + (k + 1)$$

Factor $k + 1$ out:

$$\frac{k(k + 1) + 2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}$$

Example

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$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Using the inductive hypothesis:

$$\left(\frac{k(k+1)}{2}\right) + (k+1)$$

Factor $k + 1$ out:

$$\frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

This matches the formula for $n = k + 1$, so the statement holds!

$\blacksquare$

Conclusion

By induction, the formula is true for all natural numbers n.

Why Does This Work?

Think of induction like climbing an infinite ladder:

- The **base case** puts your foot on the first rung.
- The **inductive hypothesis** and the **inductive step** shows that if you can reach one step, you can reach the next.

Since they are true, you can climb forever!

$\blacksquare$[1]

[1] $\blacksquare$ = Q.E.D. which means "*quod erat demonstrandum*".

Example

Proof by Induction:

$$2^n \geq n + 1$$

for all $n \geq 1$.

Step 1: Base Case

For $n = 1$:

$$\begin{aligned} 2^1 &= 2, && \text{left side} \\ 1 + 1 &= 2. && \text{right side} \end{aligned}$$

Since $2 \geq 2$, the base case holds. ✓

Step 2: Inductive Hypothesis

Assume the statement is true for $n = k$:

$$2^k \geq k + 1.$$

This assumption is the *inductive hypothesis*.

Step 3: Inductive Step

We need to prove the statement holds for $n = k + 1$:

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Start with the left-hand side:

$$2^{k+1} = 2 \cdot 2^k.$$

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Start with the left-hand side:

$$2^{k+1} = 2 \cdot 2^k.$$

Using the inductive hypothesis $2^k \geq k + 1$:

$$2^{k+1} \geq 2 \cdot (k+1) = 2k + 2.$$

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We need to prove the statement holds for $n = k + 1$:

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Start with the left-hand side:

$$2^{k+1} = 2 \cdot 2^k.$$

Using the inductive hypothesis $2^k \geq k + 1$:

$$2^{k+1} \geq 2 \cdot (k+1) = 2k + 2.$$

Since $2k + 2 \geq k + 2$, the statement holds for $n = k + 1$.

$\blacksquare$

Conclusion

By mathematical induction, we have proven that:

$$2^n \geq n + 1 \quad \text{for all } n \geq 1.$$

Induction helps us prove statements for infinitely many cases!

Another Example

We will prove that the sum of the first n odd numbers is given by:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Step 1: Base Case

For $n = 1$:

$$1 = 1^2.$$

Since both sides are equal, the base case holds. ✓

Step 2: Inductive Hypothesis

Assume the statement is true for $n = k$:

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

The *inductive hypothesis.*

Step 3: Inductive Step

We need to prove the statement holds for $n = k + 1$:

$$1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2.$$

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Using the *inductive hypothesis*:

$$k^2 + (2(k + 1) - 1).$$

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Expanding the term:

$$k^2 + (2k + 1) = (k + 1)^2.$$

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Expanding the term:

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Since both sides match, the statement holds for $n = k + 1$.

$\blacksquare$

One More

Prove that:

$$\sum_{i=0}^{n} 3^i = \frac{3^{n+1} - 1}{2}$$

for all non negative integer n.

Step 1: Base Case

For $n = 0$:

$$\sum_{i=0}^{0} 3^i = \frac{3^{0+1} - 1}{2}$$
$$3^0 = \frac{3^1 - 1}{2}$$
$$1 = \frac{3 - 1}{2}$$
$$1 = \frac{2}{2}$$
$$1 = 1$$

Since both sides are equal, the base case holds. ✓

Step 1: Base Case

For $n = 1$:

$$\sum_{i=0}^{1} 3^i = \frac{3^{1+1} - 1}{2}$$

$$3^0 + 3^1 = \frac{3^2 - 1}{2}$$

$$1 + 3 = \frac{9 - 1}{2}$$

$$4 = \frac{8}{2}$$

$$4 = 4$$

Since both sides are equal, the base case holds. ✓

Step 2: Inductive Hypothesis

Assume the statement is true for $n = k$:

$$\sum_{i=0}^{k} 3^i = \frac{3^{k+1} - 1}{2}$$

The *inductive hypothesis*.

Step 3: Inductive Step

We need to prove the statement holds for $n = k + 1$:

$$\sum_{i=0}^{k+1} 3^i = \frac{3^{(k+1)+1} - 1}{2}$$

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By $\sum$ definition:

$$\sum_{i=0}^{k} 3^i + 3^{k+1} = \frac{3^{(k+1)+1} - 1}{2}$$

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Using the *inductive hypothesis*:

$$\frac{3^{k+1} - 1}{2} + 3^{k+1} = \frac{3^{(k+1)+1} - 1}{2}$$

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Using the *inductive hypothesis*:

$$\frac{3^{k+1} - 1}{2} + 3^{k+1} = \frac{3^{(k+1)+1} - 1}{2}$$

Expanding the term:

$$\frac{3^{k+1}}{2} - \frac{1}{2} + \frac{2 \cdot 3^{k+1}}{2} = \frac{3^{(k+1)+1} - 1}{2}$$

$$\frac{3^{k+1} - 1 + 2 \cdot 3^{k+1}}{2} = \frac{3^{(k+1)+1} - 1}{2}$$

Step 3: Inductive Step

Using the *inductive hypothesis*:

$$\frac{3^{k+1}-1}{2}+3^{k+1}=\frac{3^{(k+1)+1}-1}{2}$$

Expanding the term:

$$\begin{aligned}
\frac{3^{k+1}}{2}-\frac{1}{2}+\frac{2\cdot 3^{k+1}}{2} &= \frac{3^{(k+1)+1}-1}{2}\\
\frac{3^{k+1}-1+2\cdot 3^{k+1}}{2} &= \frac{3^{(k+1)+1}-1}{2}\\
\frac{3\cdot 3^{k+1}-1}{2} &= \frac{3^{k+1+1}-1}{2}\\
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\frac{3^{k+2}-1}{2} &= \frac{3^{k+2}-1}{2}
\end{aligned}$$

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Using the *inductive hypothesis*:

$$\frac{3^{k+1}-1}{2} + 3^{k+1} = \frac{3^{(k+1)+1}-1}{2}$$

Expanding the term:

$$\begin{aligned}
\frac{3^{k+1}}{2} - \frac{1}{2} + \frac{2 \cdot 3^{k+1}}{2} &= \frac{3^{(k+1)+1}-1}{2} \\
\frac{3^{k+1} - 1 + 2 \cdot 3^{k+1}}{2} &= \frac{3^{(k+1)+1}-1}{2} \\
\frac{3 \cdot 3^{k+1} - 1}{2} &= \frac{3^{k+1+1}-1}{2} \\
\frac{3^{k+1+1} - 1}{2} &= \frac{3^{k+1+1}-1}{2} \\
\frac{3^{k+2} - 1}{2} &= \frac{3^{k+2}-1}{2}
\end{aligned}$$

Since both sides match, the statement holds for $n = k + 1$.

$\blacksquare$

More Examples

Use mathematical induction to show that:

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all non negative integers n.

1. **Basis case:** For $n = 0$, $2^0 = 1 = 2^1 - 1$ ✓

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1. **Basis case:** For $n = 0$, $2^0 = 1 = 2^1 - 1$ ✓
2. **Inductive hypothesis:** Let $n = k$, so

   $$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$$

   holds...

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3. **Inductive step:** Let's solve for $n = k + 1$,

   $$1 + 2 + 2^2 + \cdots + 2^{k+1} = 2^{(k+1)+1} - 1$$

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$$2 \cdot 2^{k+1} - 1 \stackrel{?}{=} 2^{k+2} - 1$$

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$$2 \cdot 2^{k+1} - 1 \stackrel{?}{=} 2^{k+2} - 1$$

$$2^{k+2} - 1 = 2^{k+2} - 1$$

$\blacksquare$

Example 2

Prove the following statement by induction:

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

1. **Basis step:** For $n = 1$, $1 = \frac{1 \times 2 \times 3}{6}$ is true!

Example 2

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1. **Basis step:** For $n = 1$, $1 = \frac{1 \times 2 \times 3}{6}$ is true!
2. **Assumption step:** Let $n = k$, so

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   holds...
3. **Inductive step:** Let's solve for $n = k + 1$,

   $$1 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1) \cdot ((k+1)+1) \cdot (2 \cdot (k+1)+1)}{6}$$

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$$\frac{k \cdot (k+1) \cdot (2k+1) + 6(k+1)^2}{6} \stackrel{?}{=} \frac{(k+1) \cdot (k+2) \cdot (2k+3)}{6}$$

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$$(k+1)\cdot(2k^2 + 7k + 6) \stackrel{?}{=} (k+1)\cdot(k+2)\cdot(2k+3)$$

$$(k+1)\cdot(k+2)\cdot(2k+3) = (k+1)\cdot(k+2)\cdot(2k+3)$$

$\blacksquare$

Theorems

Theorem

Let b be a positive real number and x and y real numbers. Then,

1. $b^{x+y} = b^x \cdot b^y$, *and*
2. $(b^x)^y = b^{x \cdot y}$.

Theorems

Theorem

Let b be a real number greater than 1. Then,

1. $\log_b(xy) = \log_b x + \log_b y$ *whenever x and y are positive real numbers, and*
2. $\log_b(x^y) = y \log_b x$ *whenever x is a positive real number and y is a real number.*

Theorems

Theorem

Let a and b be real numbers greater than 1, and let x be a positive real number. Then,

1. $\log_a x = \frac{\log_b x}{\log_b a}$.