Introduction to Algorithms
Lecture 1: Analysis of Algorithms

Prof. Charles E. Leiserson and Prof. Erik Demaine
Massachusetts Institute of Technology

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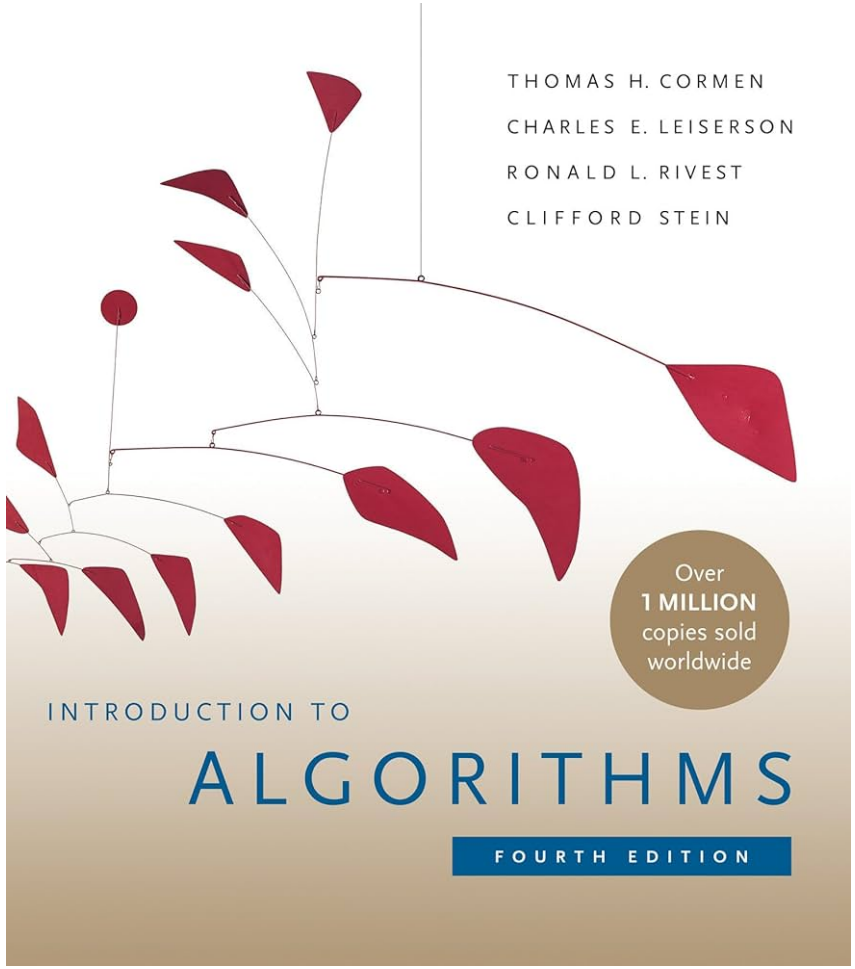
Introduction to Algorithms



Content has been extracted from *Introduction to Algorithms*, Fourth Edition, by Cormen, Leiserson, Rivest, and Stein. MIT Press. 2022.
Visit `https://mitpress.mit.edu/9780262046305/introduction-to-algorithms/`.
Original slides from *Introduction to Algorithms 6.046J/18.401J*, Fall 2005 Class by Prof. Charles Leiserson and Prof. Erik Demaine. MIT OpenCourseWare Initiative available at `https://ocw.mit.edu/courses/6-046j-introduction-to-algorithms-sma-5503-fall-2005/`.

Plan

Analysis of Algorithms

Insertion-sort

Asymptotic Analysis

Merge-sort

Recurrences

Analysis of Algorithms

- The theoretical study of computer-program performance and resource usage.
- But... What's more important than performance?

Analysis of Algorithms

- The theoretical study of computer-program performance and resource usage.
- But... What's more important than performance?

- modularity
- correctness
- maintainability
- functionality
- robustness
- user-friendliness
- programmer time
- simplicity
- extensibility
- reliability

Why study algorithms and performance?

- Algorithms help us to understand scalability.
- Performance often draws the line between what is feasible and what is impossible.
- Algorithmic mathematics provides a language for talking about program behavior.
- Performance is the currency of computing.
- The lessons of program performance generalize to other computing resources.
- Speed is fun!

Plan

The problem of sorting

Input: sequence $<a_1, a_2, \cdots, a_n>$ of numbers.
Output: permutation $<a'_1, a'_2, \cdots, a'_n>$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Example:

Input: 8 2 4 9 3 6
Output: 2 3 4 6 8 9

Insertion Sort

Algorithm 1 Insertion Sort

```
1: INSERTION-SORT(A, n)        ▷ A[1 . . . n]
2:     for j ← 2 to n do
3:         key ← A[j]
4:         i ← j − 1
5:         while i > 0 and A[i] > key do
6:             A[i + 1] ← A[i]
7:             i ← i − 1
8:         A[i + 1] ← key
```

"pseudocode"

Insertion Sort

Algorithm 2 Insertion Sort

```
1: INSERTION-SORT(A, n)        ▷ A[1 . . . n]
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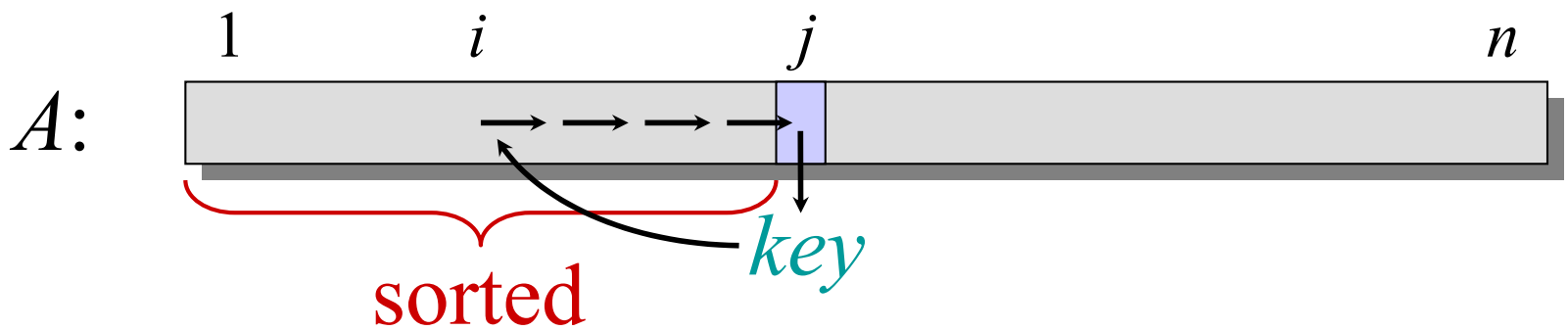

Example of insertion sort

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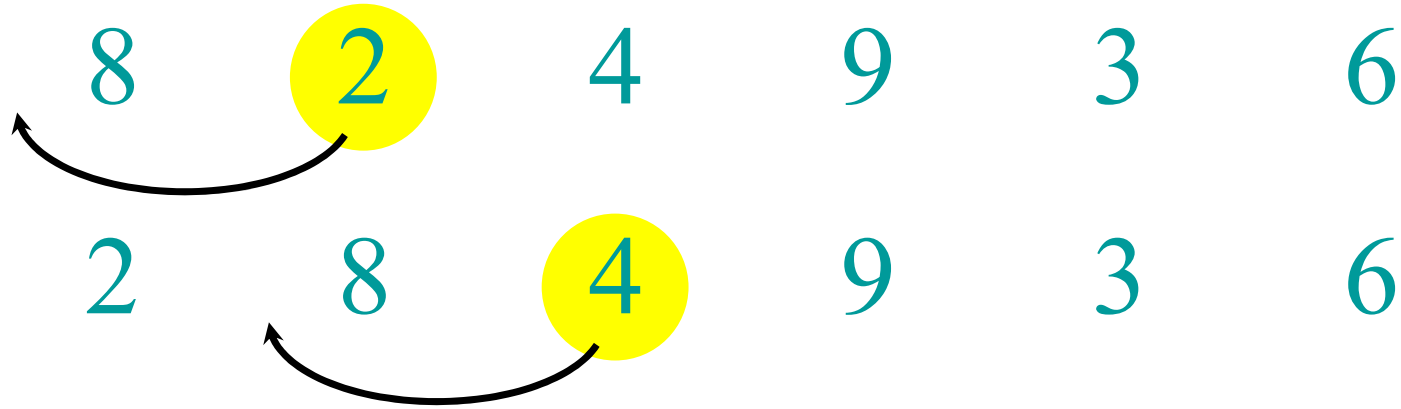

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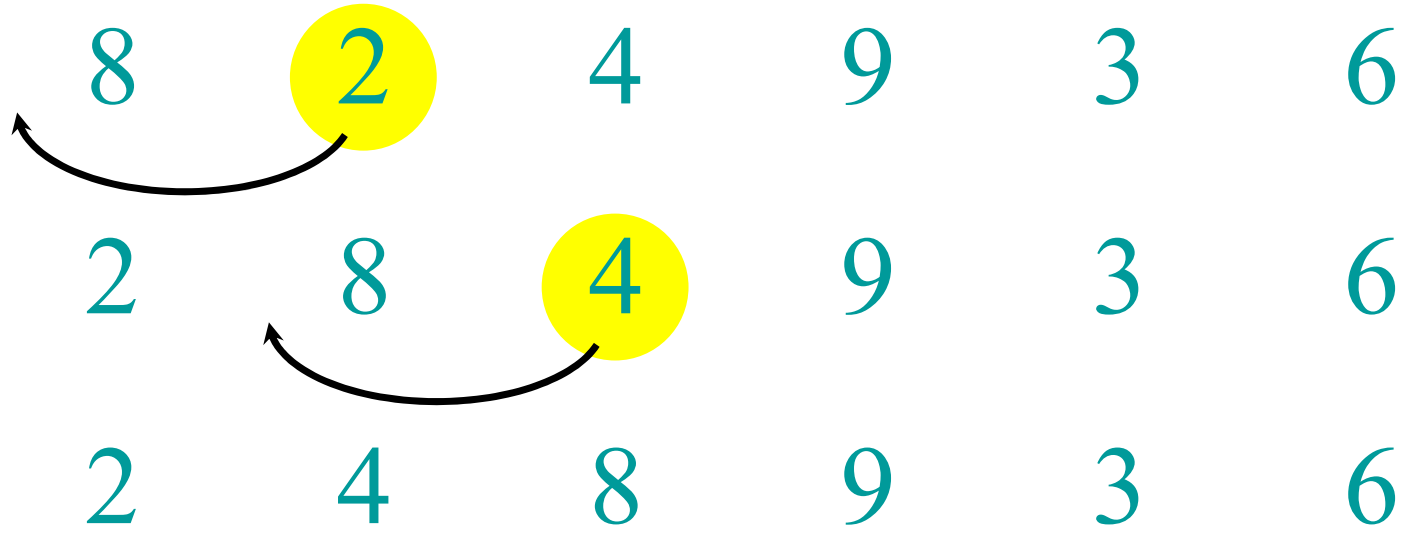

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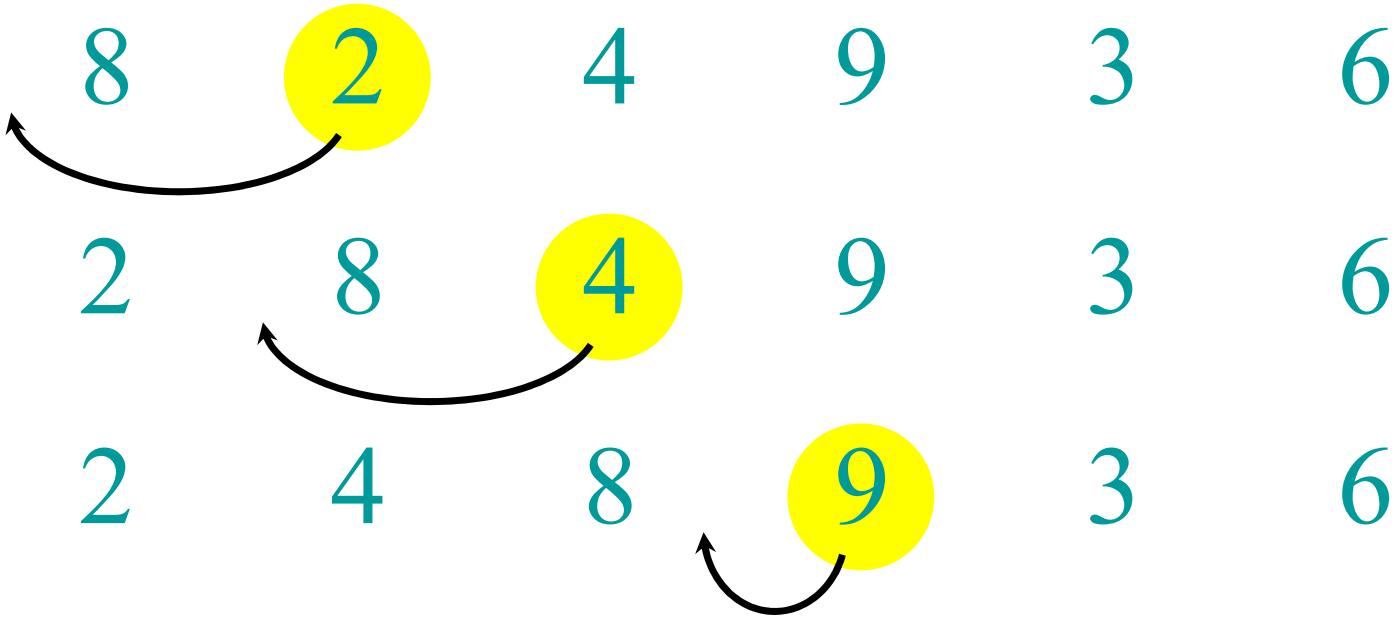

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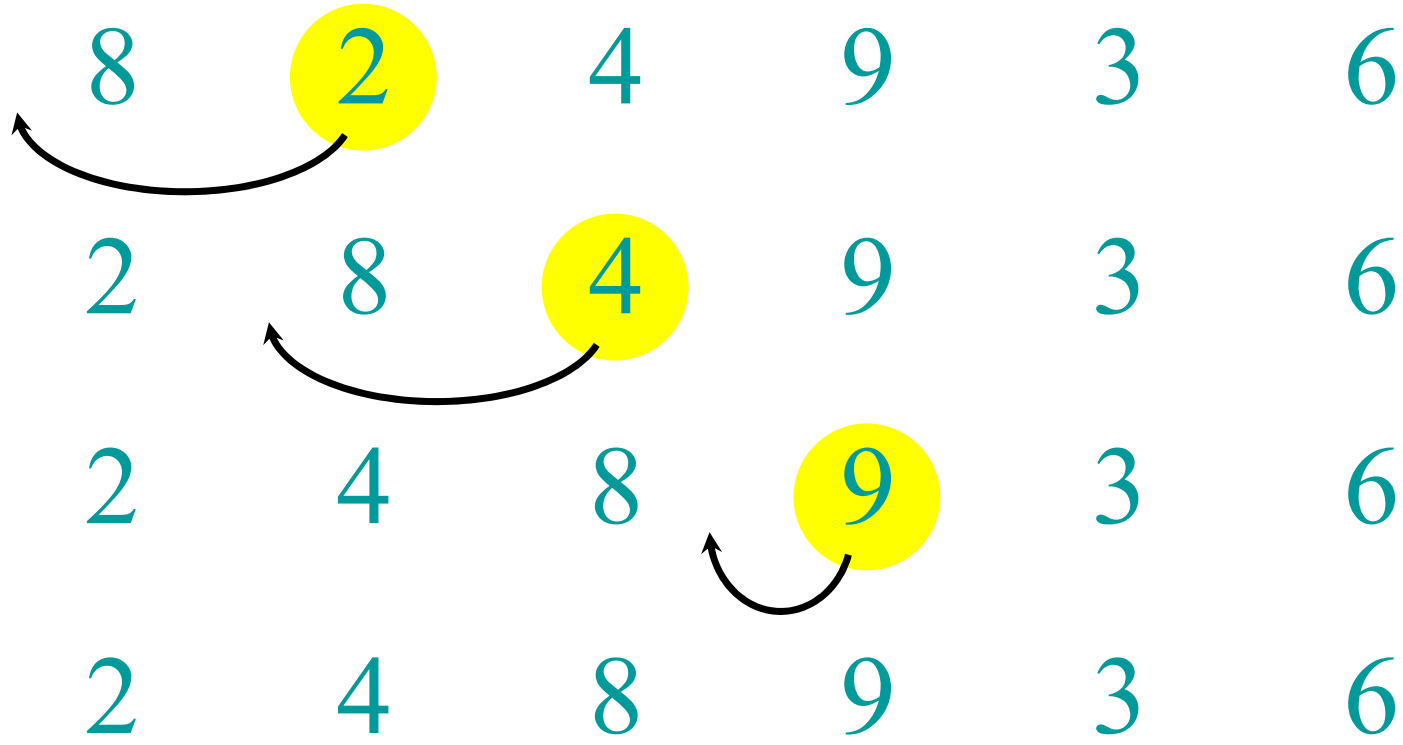

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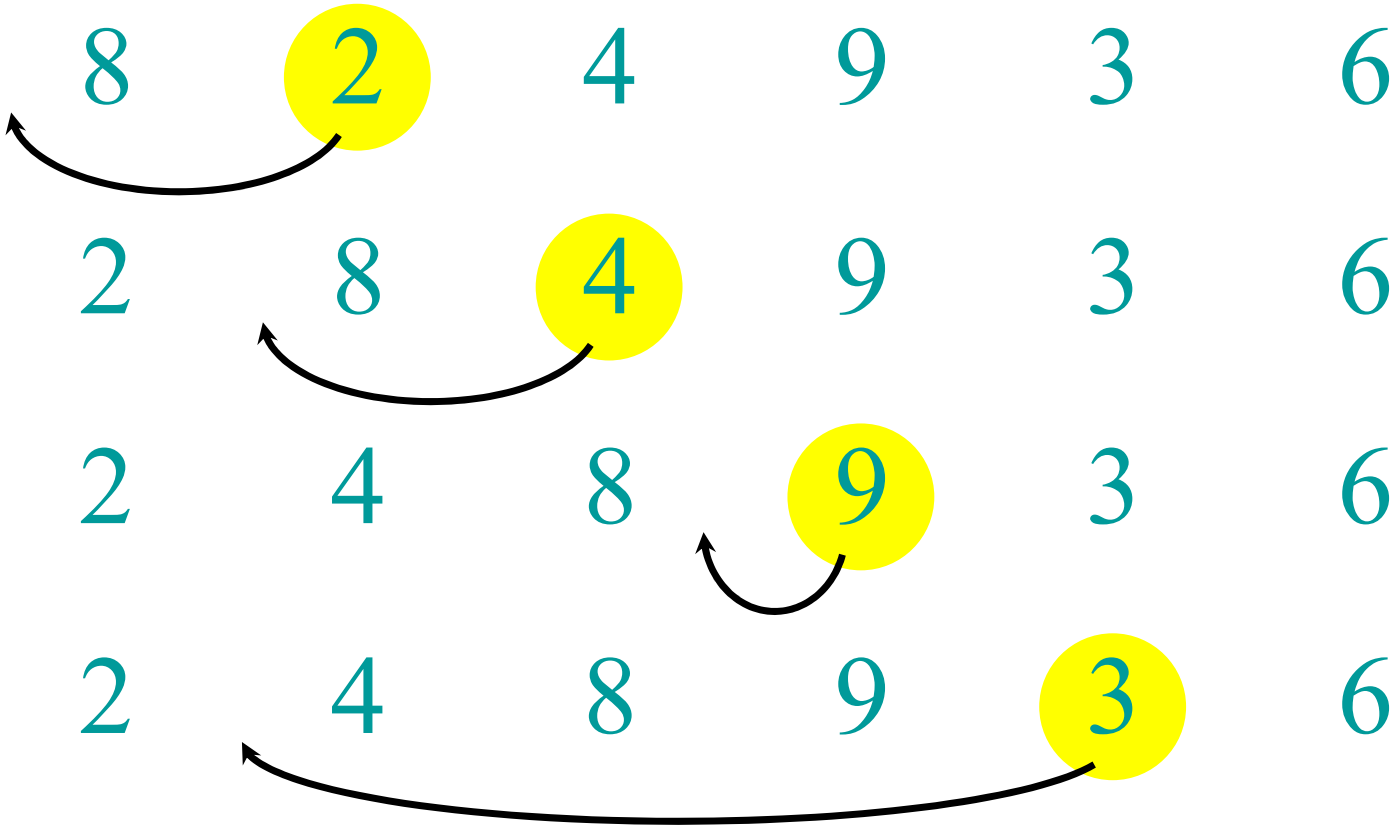

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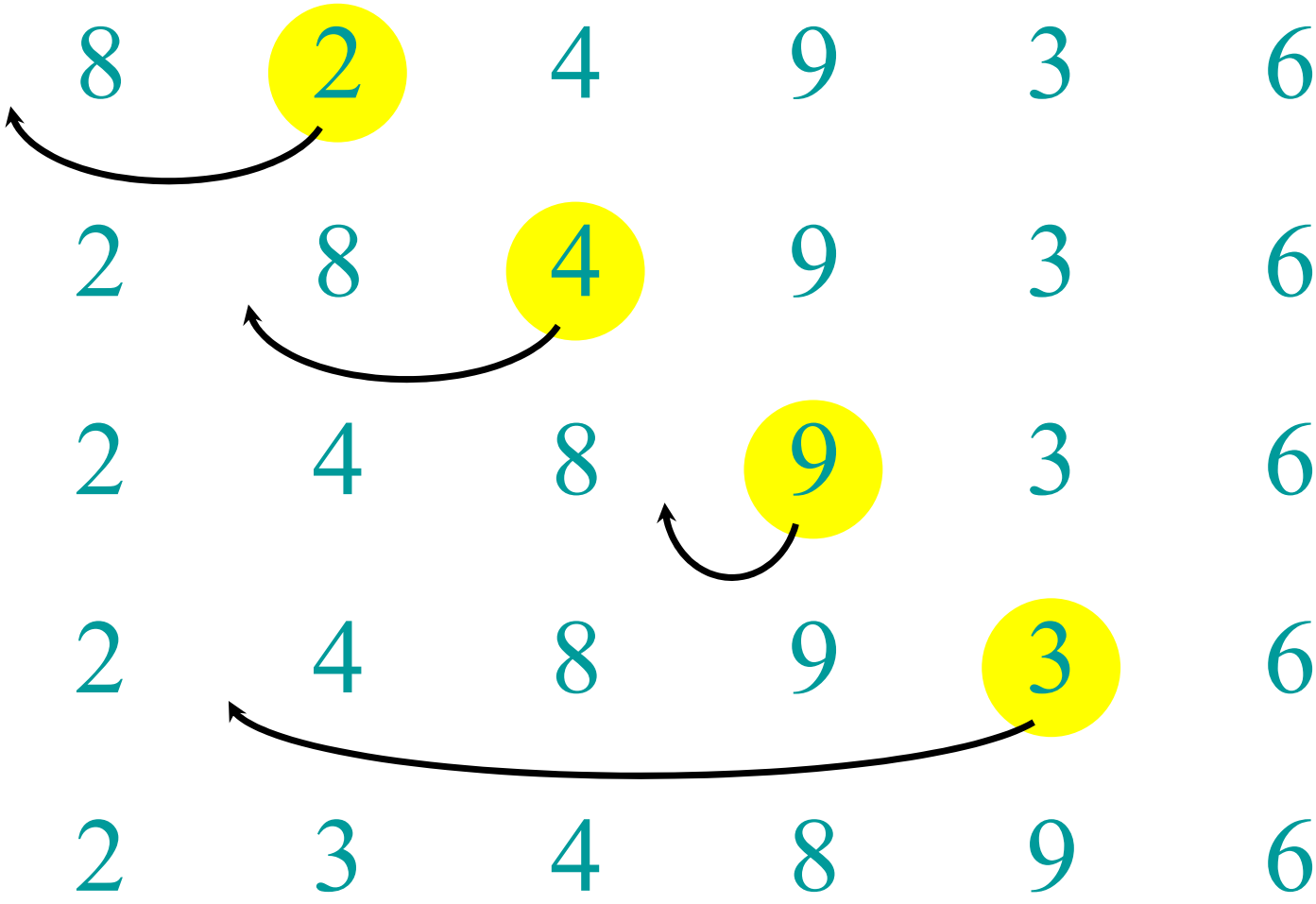

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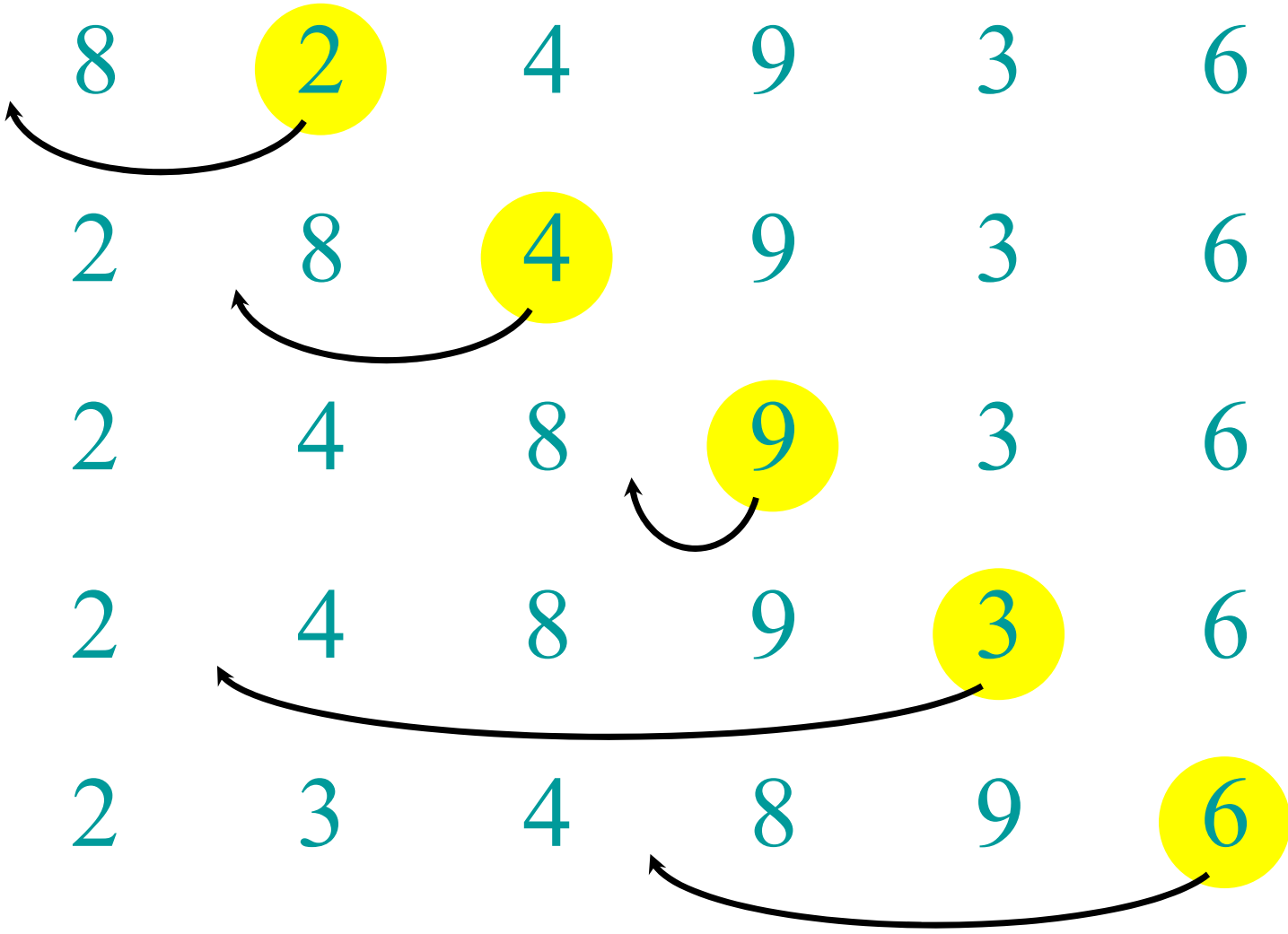

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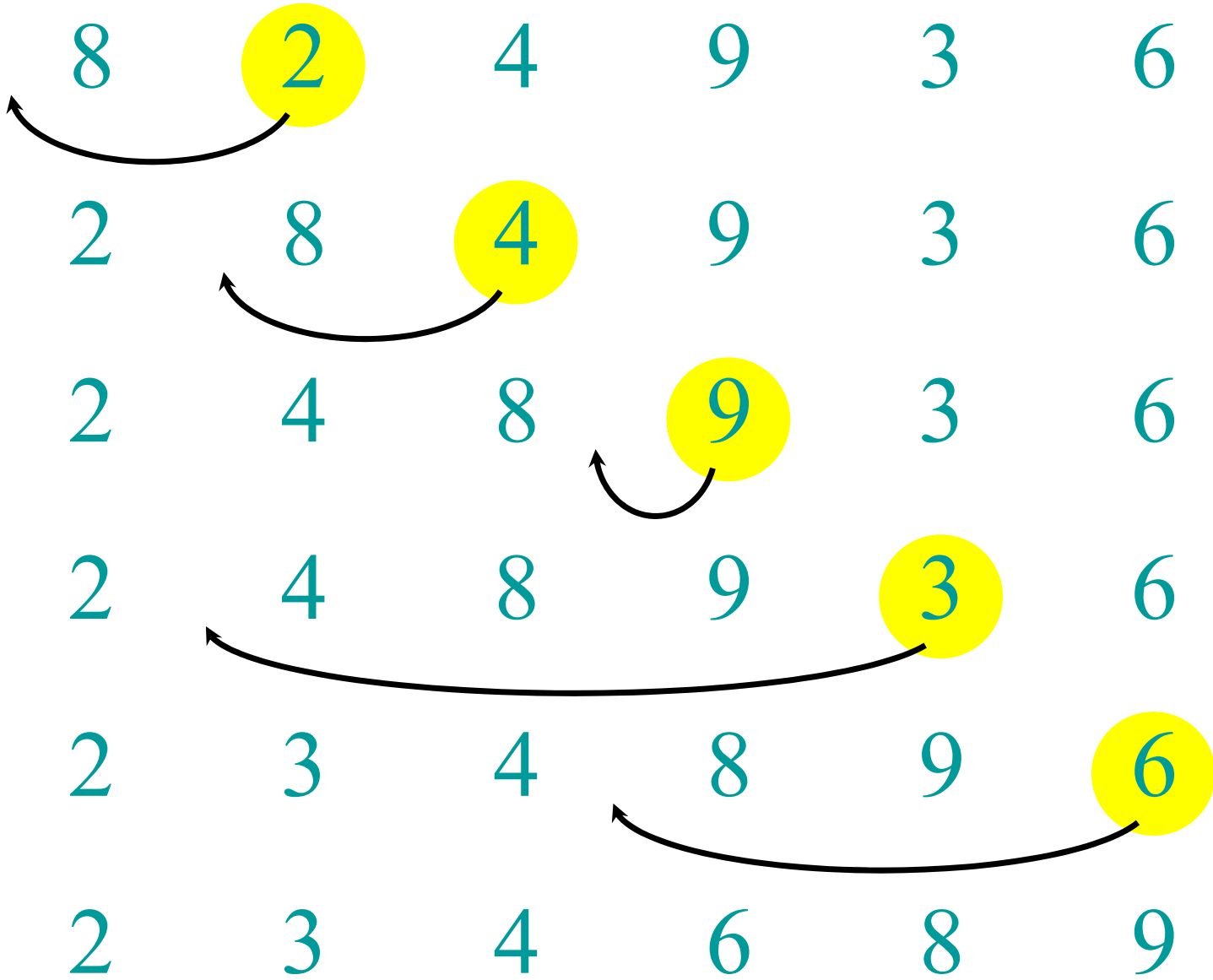

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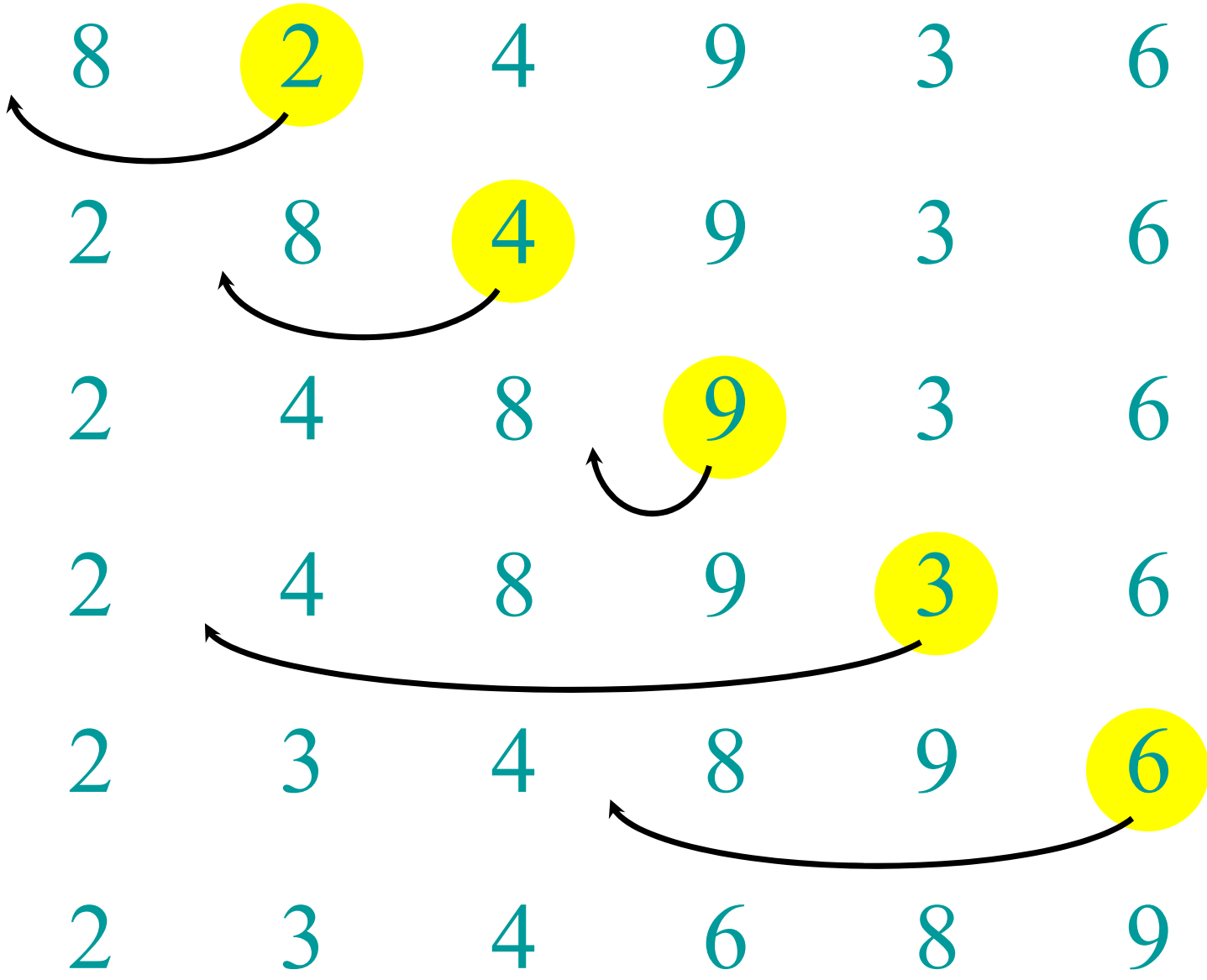


Done!

Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
- Generally, we seek upper bounds on the running time, because everybody likes a guarantee.

Plan

Analysis of Algorithms

Insertion-sort

Asymptotic Analysis

Merge-sort

Recurrences

Kinds of analyses

- Worst-case: (usually)
  - $T(n) =$ *maximum time* of algorithm on any input of size n.
- Average-case: (sometimes)
  - $T(n) =$ *expected time* of algorithm over all inputs of size n.
  - Need assumption of statistical distribution of inputs.
- Best-case: (Bogus!)
  - Cheat with a slow algorithm that works fast on *some* input.

Machine-independent time

¿What is insertion sort's worst-case time?

- It depends on the speed of our computer:
  - relative speed (on the same machine).
  - absolute speed (on different machines).

Big Idea!

- Ignore machine-dependent constants.
- Look at **growth** of $T(n)$ as $n \to \infty$

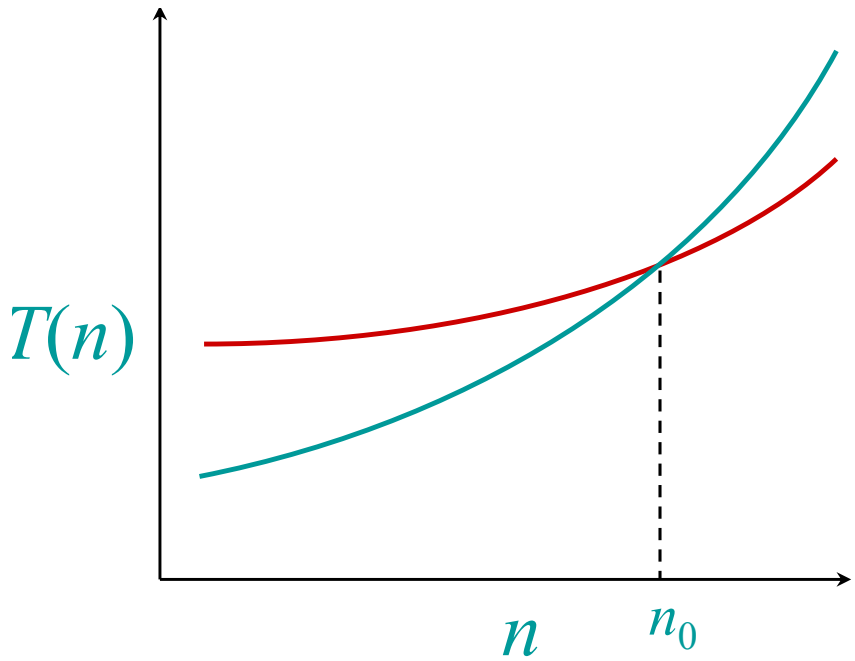
"Asymptotic Analysis"

Θ-notation

- Math:
  - $\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$
- Engineering:
  - Drop low-order terms; ignore leading constants.
  - For instance: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

Asymptotic Performance

When n get large enough, a $\Theta(n^2)$ algorithm **always** beats a $\Theta(n^3)$ algorithm.



- We should not ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing of engineering objectives.
- Asymptotic analysis is a useful tool to help to structure our thinking.

Insertion sort Analysis

- **Worst case:**
  - Input reverse sorted.
  - $T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$.
  - Arithmetic series!
- **Average case:**
  - All permutations equally likely.
  - $T(n) = \sum_{j=2}^{n} \Theta(\frac{j}{2}) = \Theta(n^2)$.
- *¿Is insertion sort a fast sorting algorithm?*
  - Moderately so, for small n.
  - Not at all, for large n.

Plan

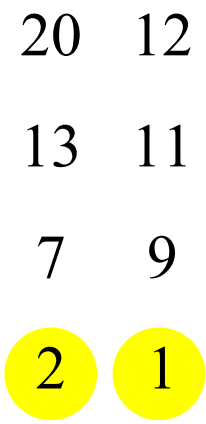
Merge-sort

Algorithm 3 Insertion Sort

1: MERGE-SORT(A, n) $\quad \triangleright A[1 \ldots n]$
2: **if** $n = 1$, done.
3: Recursively sort $A[1 \ldots \lceil \frac{n}{2} \rceil]$ and $A[\lceil \frac{n}{2} \rceil + 1 \ldots n]$
4: MERGE the 2 sorted list.

Key subroutine: MERGE

Merging two sorted arrays



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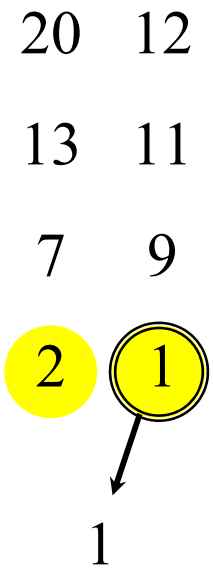

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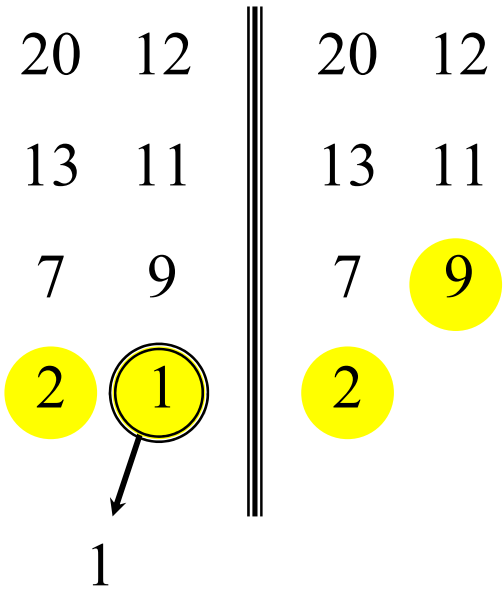

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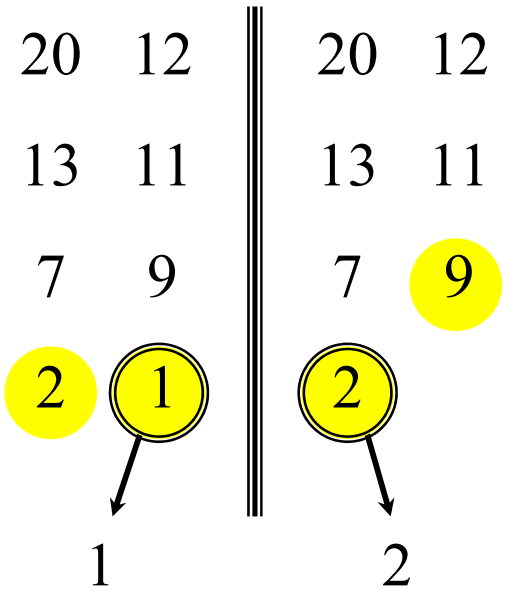

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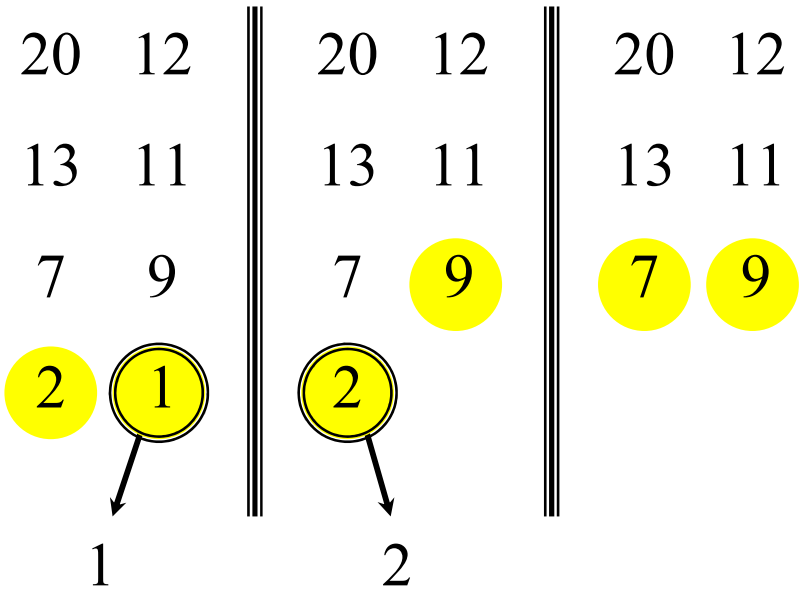

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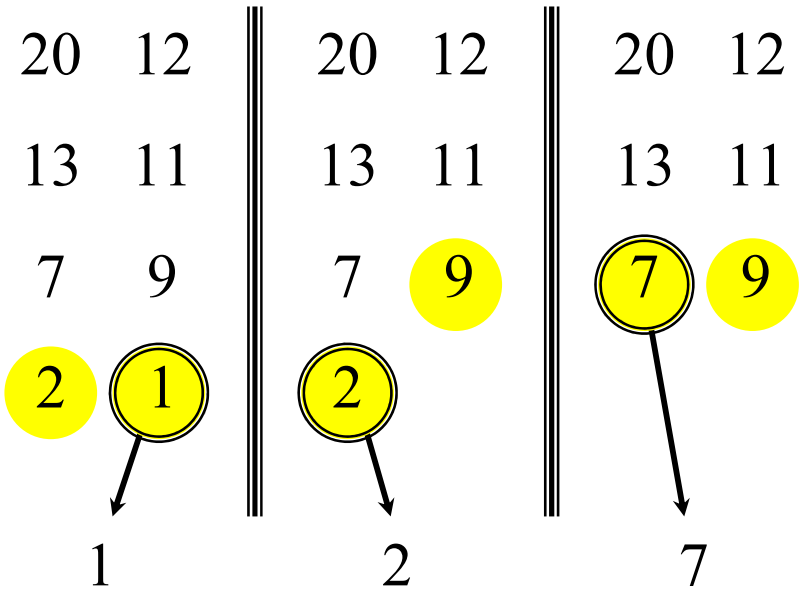

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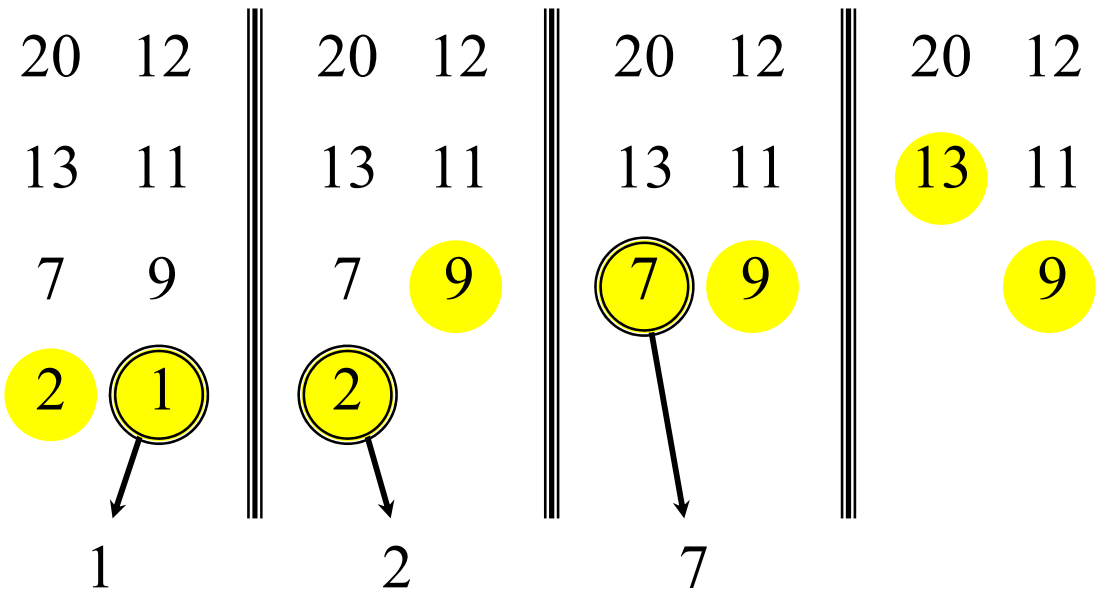

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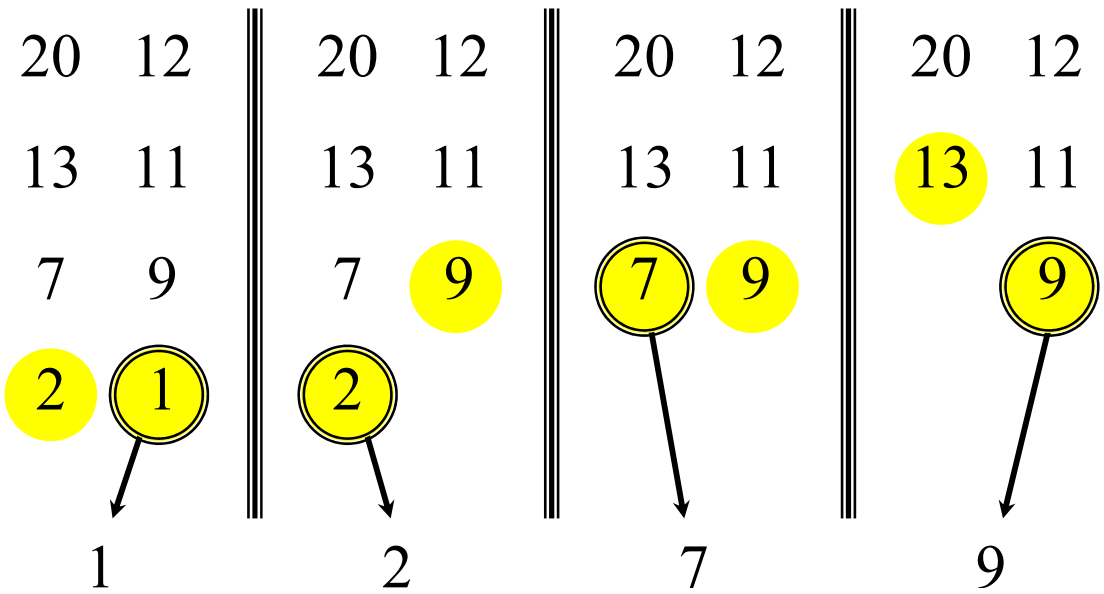

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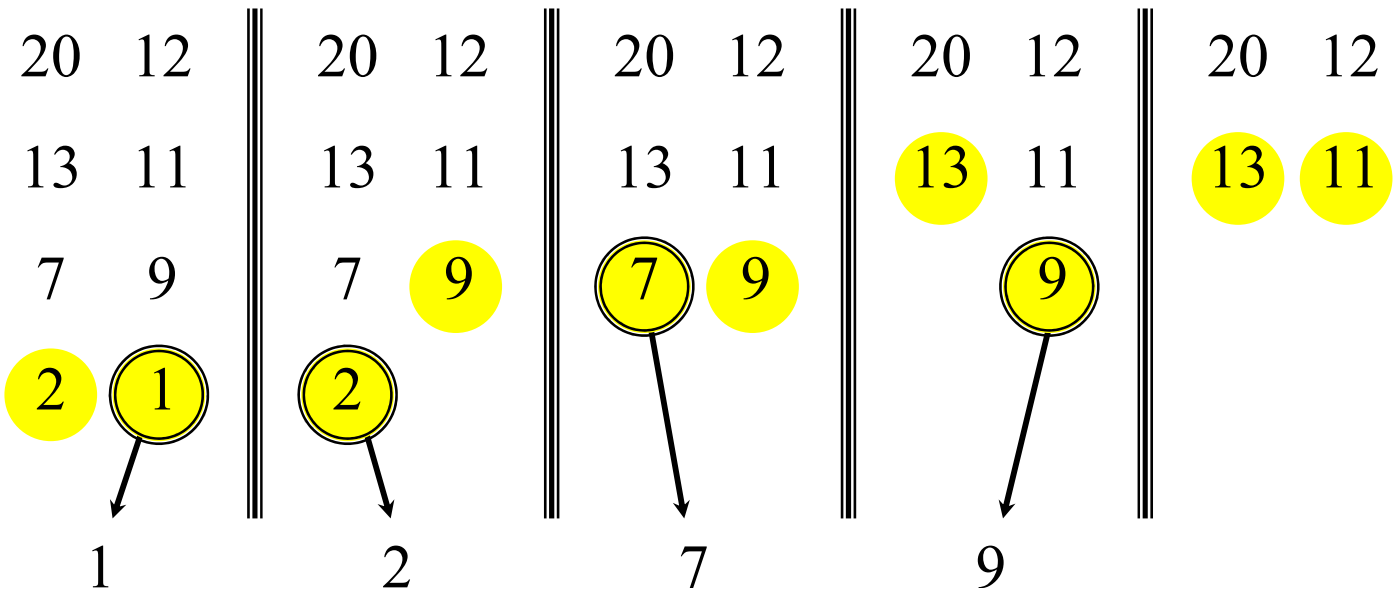

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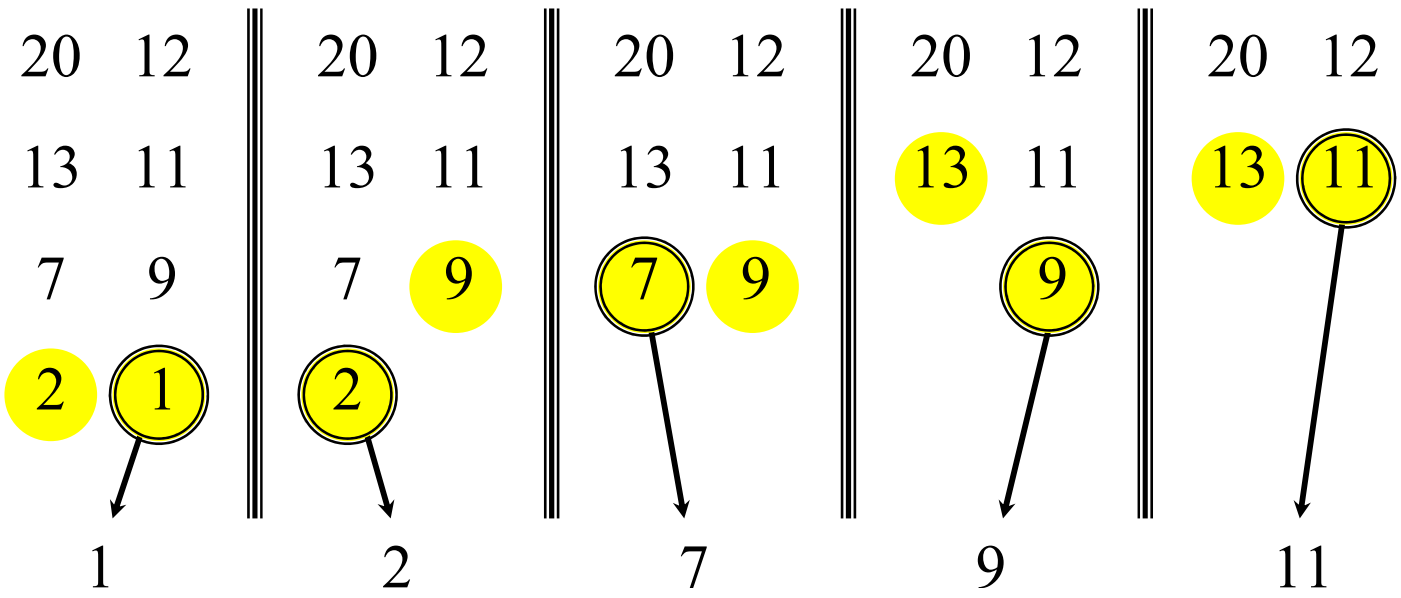

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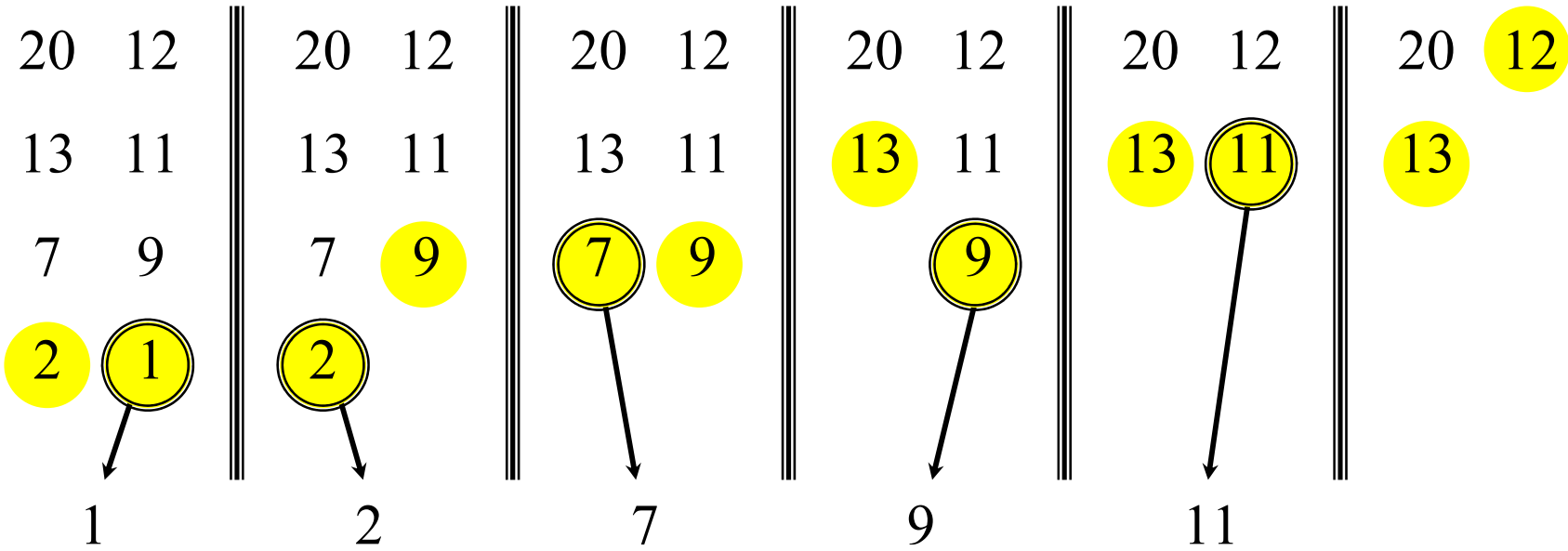

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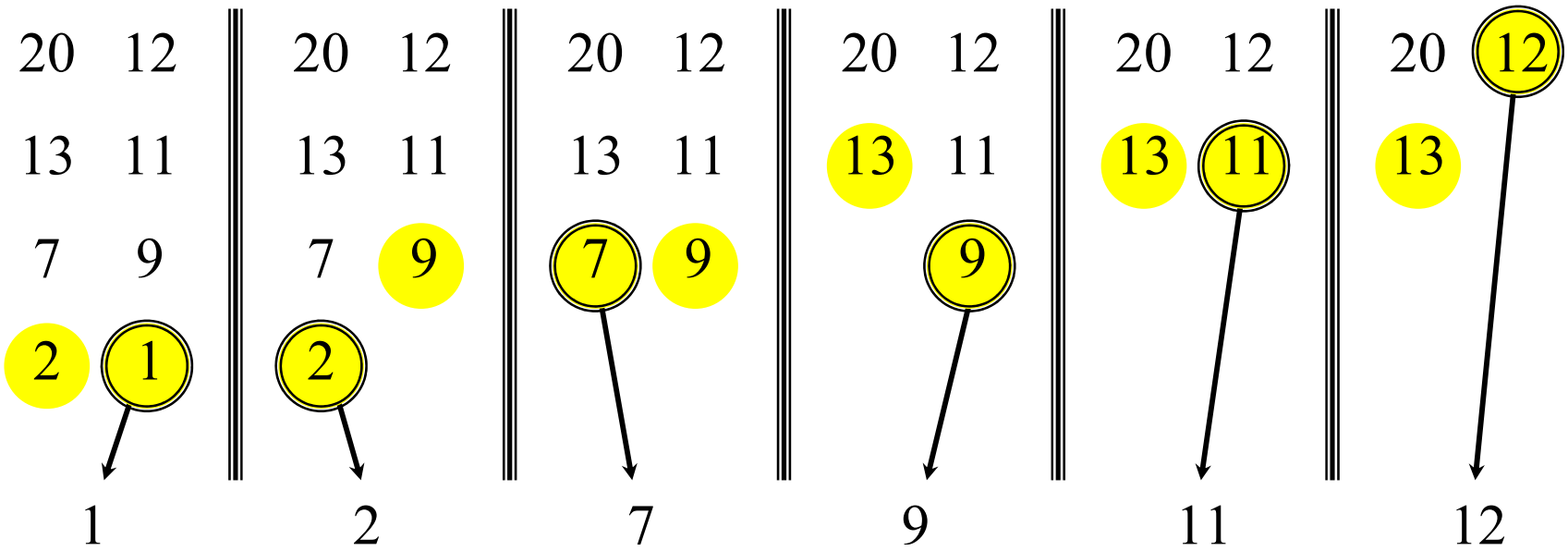

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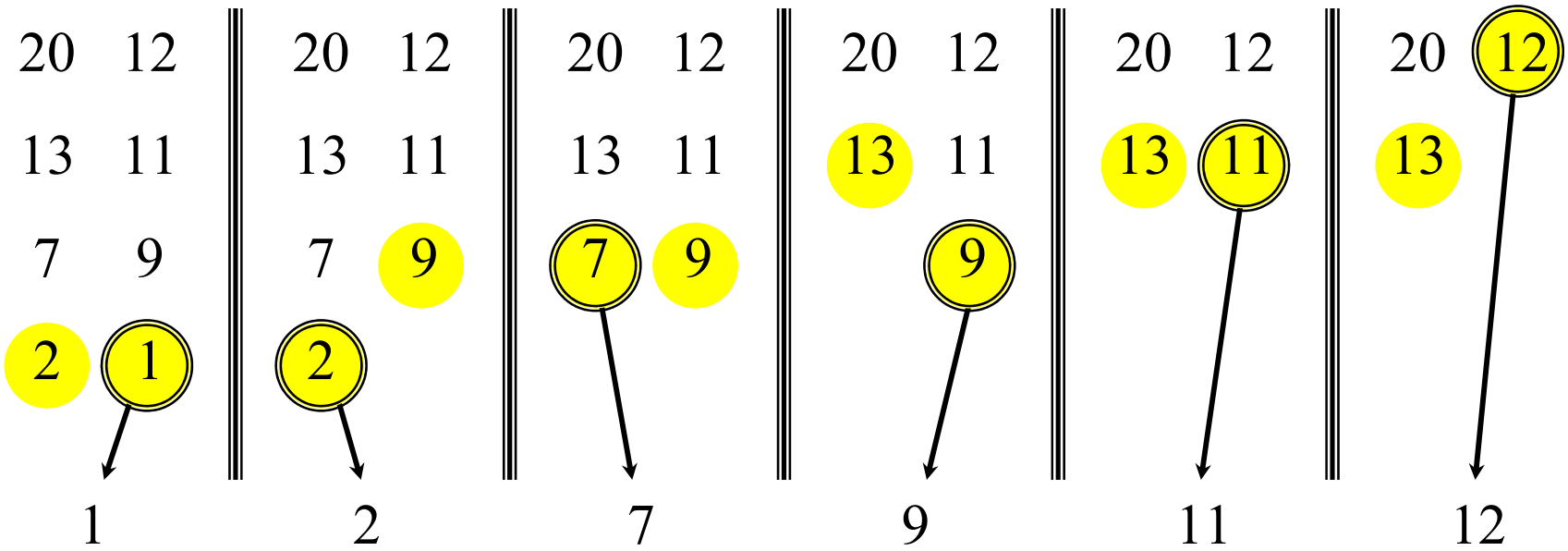

Time= $\Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge-sort

Abuse!

$T(n)$	MERGE-SORT $A[1 \ldots n]$
$\Theta(1)$	1. **if** $n = 1$, done!
$2T(\frac{n}{2})$	2. Recursively sort $A[1 \ldots \lceil \frac{n}{2} \rceil]$ and $A[\lceil \frac{n}{2} \rceil + 1 \ldots n]$.
$\Theta(n)$	3. MERGE the 2 sorted lists.

Sloppiness: Should be $T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor)$, but it turns out not matter asymptotically.

Plan

Recurrence for merge-sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1 \\ 2T(\frac{n}{2}) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.
- CLRS[1] and Lecture 2 provide several ways to find a good upper bound on $T(n)$.

[1] The book

Recursion Tree

Solve $T(n) = 2T(\frac{n}{2}) + cn$, where $c > 0$ is constant.

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$T(n)$

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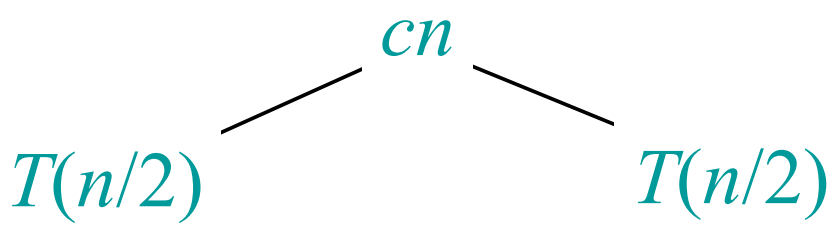

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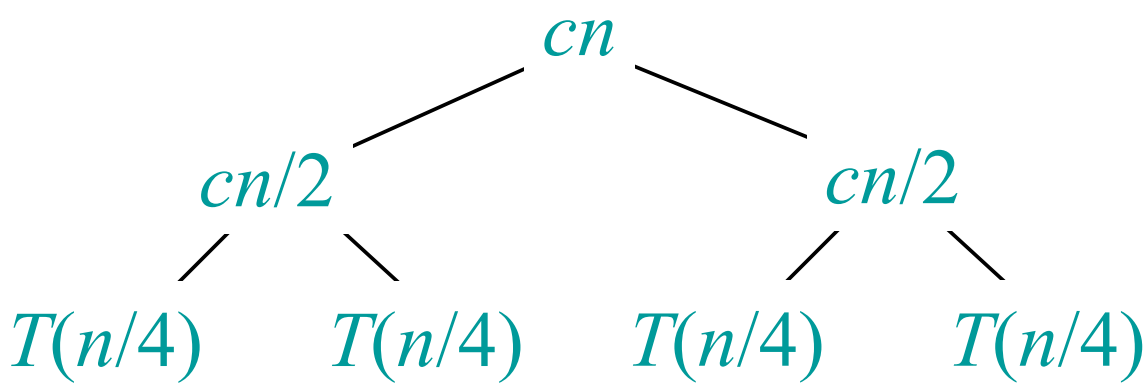

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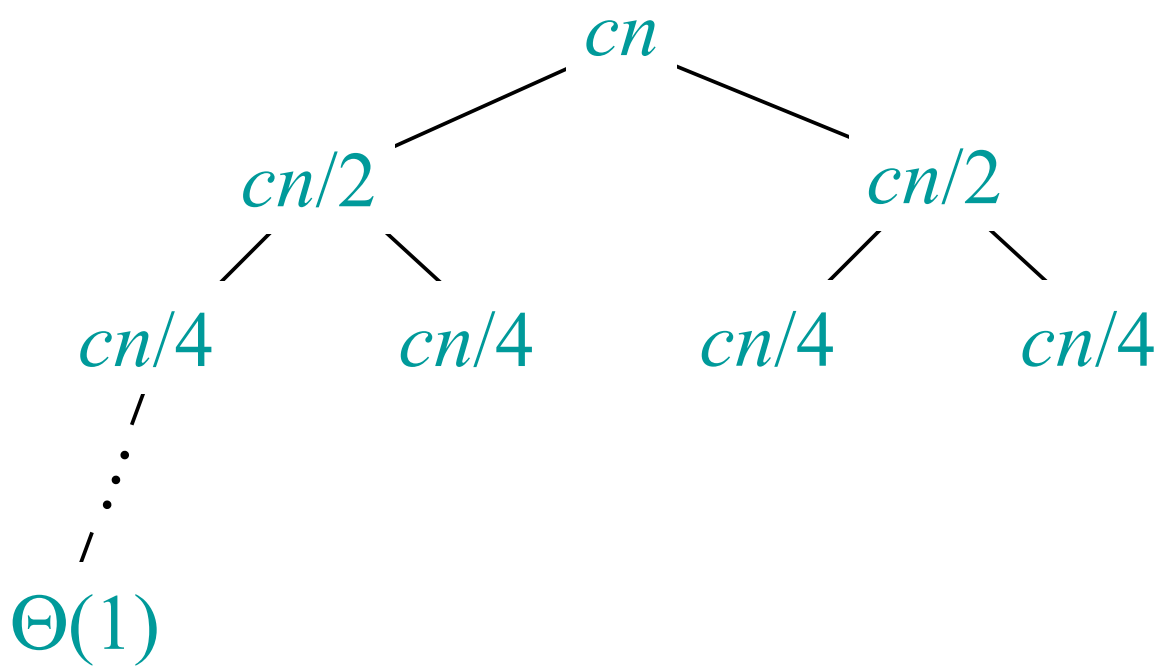

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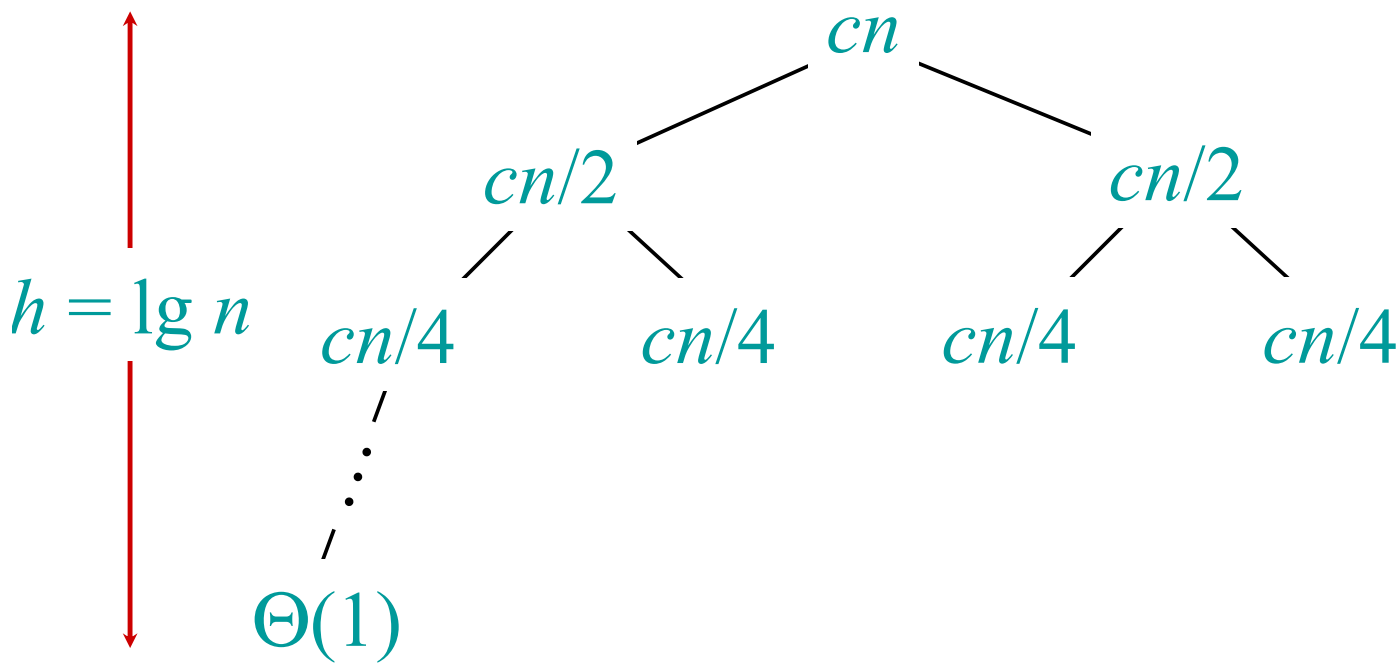

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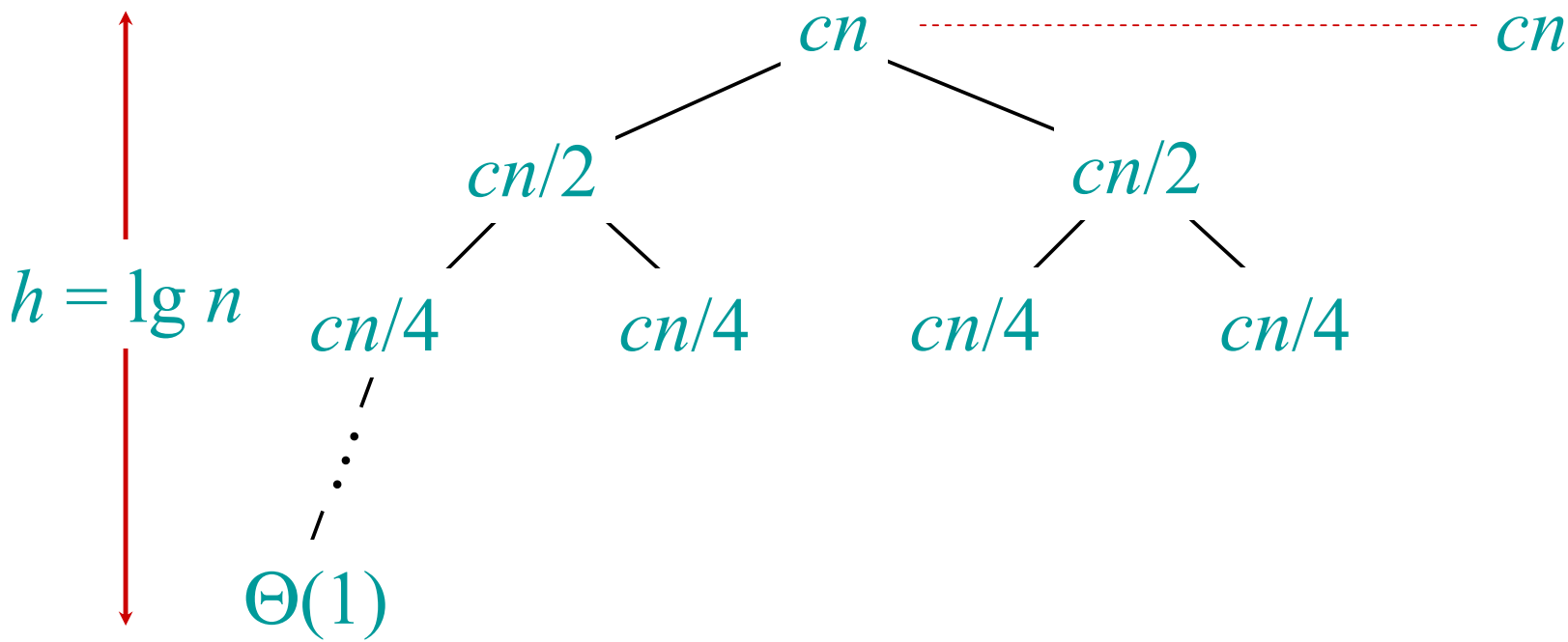

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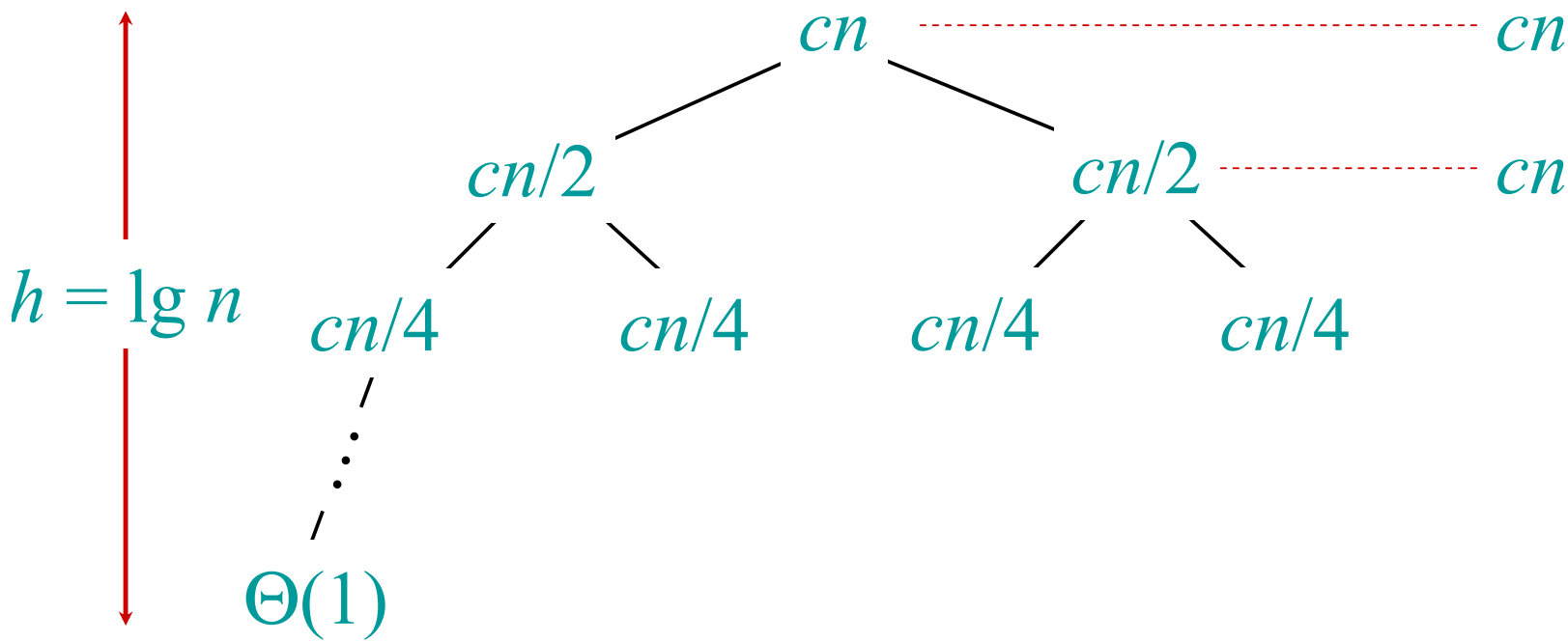

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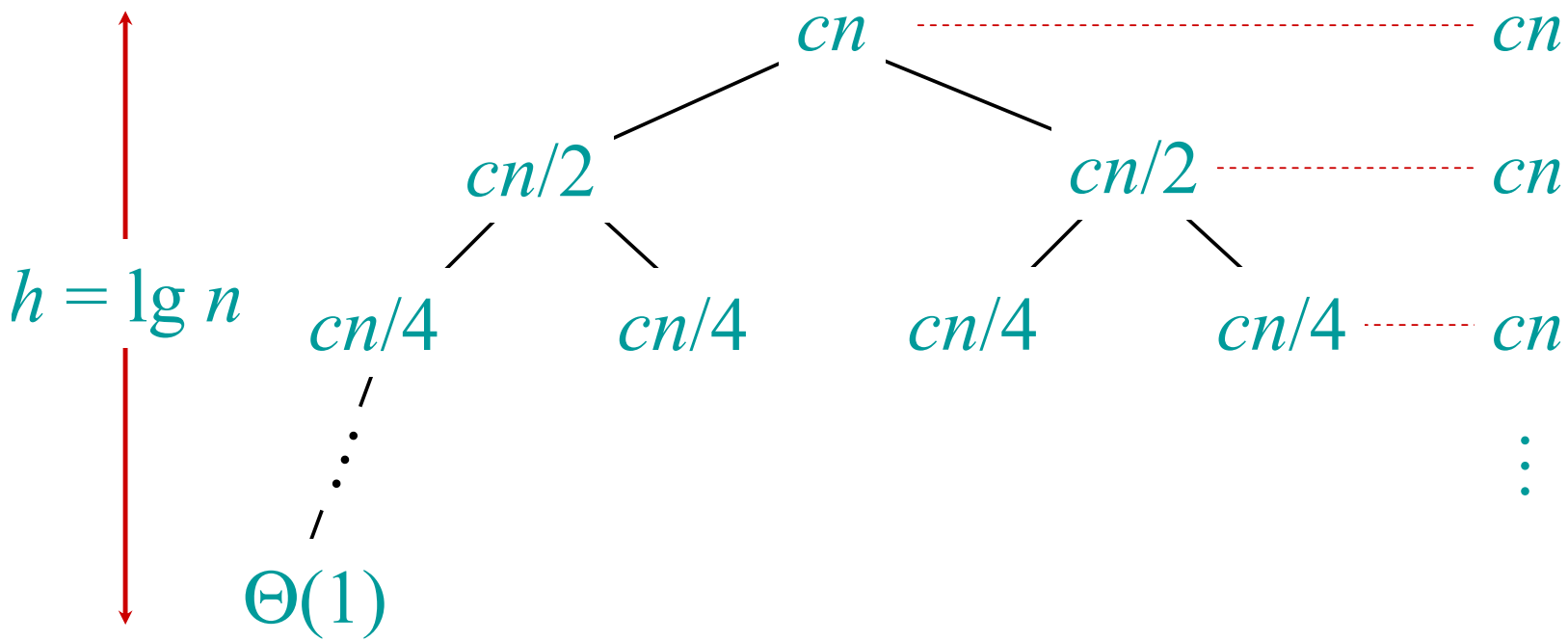

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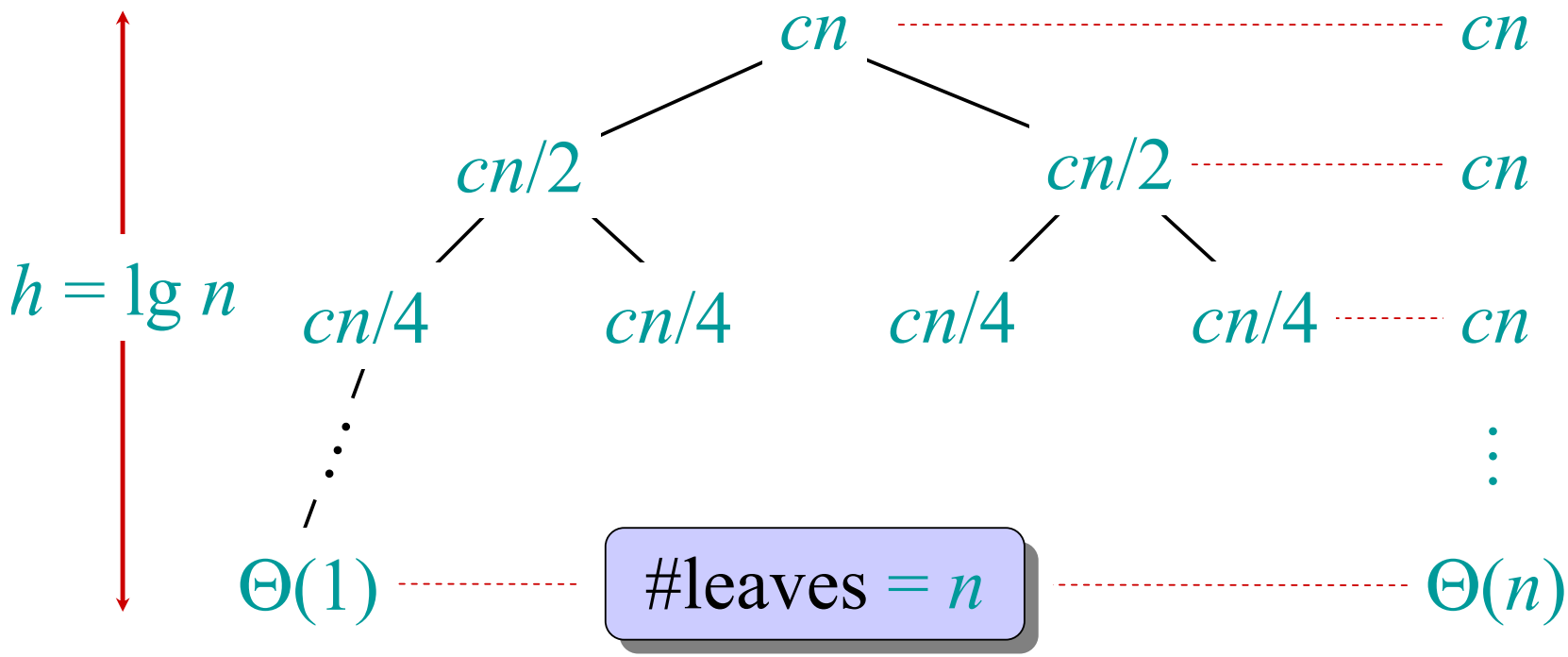

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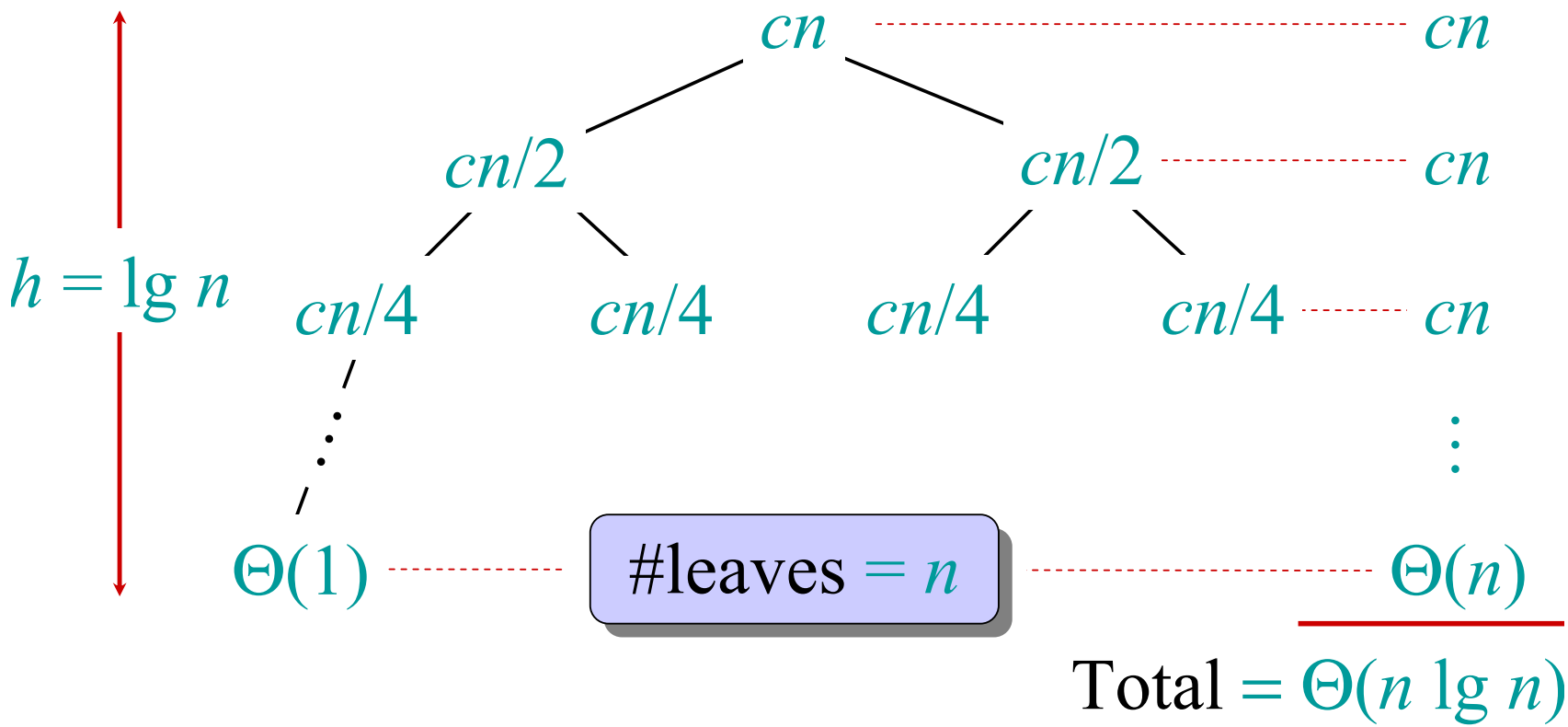

Conclusions

- $\Theta(nlogn)$ grows more slowly that $\Theta(n^2)$.
- Therefore, merge-sort asymptotically beats insertion-sort in the worst case.
- In practice, merge-sort beats insertion-sort for $n > 30$ or so.
- Go test it out for yourself!