Introduction to Algorithms
Bonus Lecture: Proof by Induction

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Proof by Induction

- A powerful mathematical technique.
- Prove that a statement is true for all natural numbers (or some sequence of numbers).
- It's like knocking over a line of dominoes...

How Induction Works

Principle of Mathematical Induction:

- **Base Case:** Show the statement holds for the first value (usually $n = 1$).
- **Inductive Hypothesis:** Assume the statement holds for some arbitrary $n = k$.
- **Inductive Step:** Prove it holds for $n = k + 1$.

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- **Inductive Hypothesis:** Assume the statement holds for some arbitrary $n = k$.
- **Inductive Step:** Prove it holds for $n = k + 1$.

If those steps hold, the statement is true for all n.

Example

Prove that:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for all $n \geq 1$.

Example

Step 1: Base Case

For $n = 1$:

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

$\checkmark$ **True!**

Example

Step 2: Inductive Hypothesis

Assume that for some $n = k$, the formula holds:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

(This is our assumption or “*inductive hypothesis*”.)

Example

Step 3: Inductive Step

We must prove it holds for $n = k + 1$, meaning:

$$1 + 2 + 3 + \cdots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}$$

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$$\left(\frac{k(k+1)}{2}\right) + (k+1)$$

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Factor $k + 1$ out:

$$\frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Example

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Using the inductive hypothesis:

$$\left(\frac{k(k + 1)}{2}\right) + (k + 1)$$

Factor $k + 1$ out:

$$\frac{k(k + 1) + 2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}$$

This matches the formula for $n = k + 1$, so the statement holds!

$\blacksquare$

Conclusion

By induction, the formula is true for all natural numbers n.

WHY DOES THIS WORK?

Think of induction like climbing an infinite ladder:

- The **base case** puts your foot on the first rung.
- The **inductive hypothesis** and the **inductive step** shows that if you can reach one step, you can reach the next.

Since they are true, you can climb forever!

$\blacksquare$[1]

[1] $\blacksquare$ = Q.E.D. which means "*quod erat demonstrandum*".

Example

Proof by Induction:

$$2^n \geq n + 1$$

for all $n \geq 1$.

Step 1: Base Case

For $n = 1$:

$$\begin{aligned} 2^1 &= 2, && \text{left side} \\ 1 + 1 &= 2. && \text{right side} \end{aligned}$$

Since $2 \geq 2$, the base case holds. ✓

Step 2: Inductive Hypothesis

Assume the statement is true for $n = k$:

$$2^k \geq k + 1.$$

This assumption is the *inductive hypothesis*.

Step 3: Inductive Step

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$$2^{k+1} \geq (k + 1) + 1.$$

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Start with the left-hand side:

$$2^{k+1} = 2 \cdot 2^k.$$

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Using the inductive hypothesis $2^k \geq k + 1$:

$$2^{k+1} \geq 2 \cdot (k+1) = 2k + 2.$$

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We need to prove the statement holds for $n = k + 1$:

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Start with the left-hand side:

$$2^{k+1} = 2 \cdot 2^k.$$

Using the inductive hypothesis $2^k \geq k + 1$:

$$2^{k+1} \geq 2 \cdot (k + 1) = 2k + 2.$$

Since $2k + 2 \geq k + 2$, the statement holds for $n = k + 1$.

∎

Conclusion

By mathematical induction, we have proven that:

$$2^n \geq n + 1 \quad \text{for all } n \geq 1.$$

Induction helps us prove statements for infinitely many cases!

Another Example

We will prove that the sum of the first n odd numbers is given by:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Step 1: Base Case

For $n = 1$:

$$1 = 1^2.$$

Since both sides are equal, the base case holds. ✓

Step 2: Inductive Hypothesis

Assume the statement is true for $n = k$:

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

The *inductive hypothesis.*

Step 3: Inductive Step

We need to prove the statement holds for $n = k + 1$:

$$1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2.$$

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Using the *inductive hypothesis*:

$$k^2 + (2(k + 1) - 1).$$

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Expanding the term:

$$k^2 + (2k + 1) = (k + 1)^2.$$

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Expanding the term:

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Since both sides match, the statement holds for $n = k + 1$.

$\blacksquare$

One More

Prove that:

$$\sum_{i=0}^{n} 3^i = \frac{3^{n+1} - 1}{2}$$

for all non negative integer n.

Step 1: Base Case

For $n = 0$:

$$\begin{aligned}
\sum_{i=0}^{0} 3^i &= \frac{3^{0+1} - 1}{2} \\
3^0 &= \frac{3^1 - 1}{2} \\
1 &= \frac{3 - 1}{2} \\
1 &= \frac{2}{2} \\
1 &= 1
\end{aligned}$$

Since both sides are equal, the base case holds. ✓

Step 1: Base Case

For $n = 1$:

$$\begin{aligned}
\sum_{i=0}^{1} 3^i &= \frac{3^{1+1} - 1}{2} \\
3^0 + 3^1 &= \frac{3^2 - 1}{2} \\
1 + 3 &= \frac{9 - 1}{2} \\
4 &= \frac{8}{2} \\
4 &= 4
\end{aligned}$$

Since both sides are equal, the base case holds. ✓

Step 2: Inductive Hypothesis

Assume the statement is true for $n = k$:

$$\sum_{i=0}^{k} 3^i = \frac{3^{k+1} - 1}{2}$$

The *inductive hypothesis*.

Step 3: Inductive Step

We need to prove the statement holds for $n = k + 1$:

$$\sum_{i=0}^{k+1} 3^i = \frac{3^{(k+1)+1} - 1}{2}$$

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By $\sum$ definition:

$$\sum_{i=0}^{k} 3^i + 3^{k+1} = \frac{3^{(k+1)+1} - 1}{2}$$

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Using the *inductive hypothesis*:

$$\frac{3^{k+1} - 1}{2} + 3^{k+1} = \frac{3^{(k+1)+1} - 1}{2}$$

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$$\frac{3^{k+1} - 1}{2} + 3^{k+1} = \frac{3^{(k+1)+1} - 1}{2}$$

Expanding the term:

$$\frac{3^{k+1}}{2} - \frac{1}{2} + \frac{2 \cdot 3^{k+1}}{2} = \frac{3^{(k+1)+1} - 1}{2}$$

$$\frac{3^{k+1} - 1 + 2 \cdot 3^{k+1}}{2} = \frac{3^{(k+1)+1} - 1}{2}$$

Step 3: Inductive Step

Using the *inductive hypothesis*:

$$\frac{3^{k+1}-1}{2} + 3^{k+1} = \frac{3^{(k+1)+1}-1}{2}$$

Expanding the term:

$$\begin{aligned}
\frac{3^{k+1}}{2} - \frac{1}{2} + \frac{2\cdot 3^{k+1}}{2} &= \frac{3^{(k+1)+1}-1}{2}\\
\frac{3^{k+1}-1+2\cdot 3^{k+1}}{2} &= \frac{3^{(k+1)+1}-1}{2}\\
\frac{3\cdot 3^{k+1}-1}{2} &= \frac{3^{k+1+1}-1}{2}\\
\frac{3^{k+1+1}-1}{2} &= \frac{3^{k+1+1}-1}{2}\\
\frac{3^{k+2}-1}{2} &= \frac{3^{k+2}-1}{2}
\end{aligned}$$

Step 3: Inductive Step

Using the *inductive hypothesis*:

$$\frac{3^{k+1}-1}{2} + 3^{k+1} = \frac{3^{(k+1)+1}-1}{2}$$

Expanding the term:

$$\begin{aligned}
\frac{3^{k+1}}{2} - \frac{1}{2} + \frac{2\cdot 3^{k+1}}{2} &= \frac{3^{(k+1)+1}-1}{2} \\
\frac{3^{k+1}-1+2\cdot 3^{k+1}}{2} &= \frac{3^{(k+1)+1}-1}{2} \\
\frac{3\cdot 3^{k+1}-1}{2} &= \frac{3^{k+1+1}-1}{2} \\
\frac{3^{k+1+1}-1}{2} &= \frac{3^{k+1+1}-1}{2} \\
\frac{3^{k+2}-1}{2} &= \frac{3^{k+2}-1}{2}
\end{aligned}$$

Since both sides match, the statement holds for $n = k + 1$.

$\blacksquare$