Introduction to Algorithms
Lecture 5: Dynamic Programming (DP)

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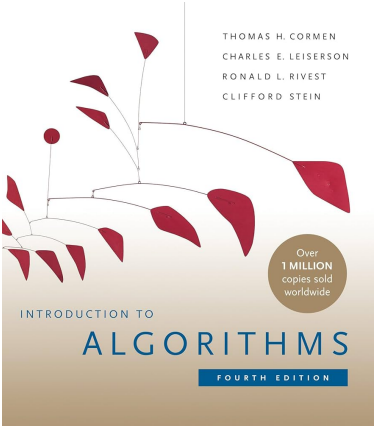
September 30, 2025

Introduction to Algorithms



Content has been extracted from *Introduction to Algorithms*, Fourth Edition, by Cormen, Leiserson, Rivest, and Stein. MIT Press. 2022.

Visit https://mitpress.mit.edu/9780262046305/introduction-to-algorithms/.

Original slides from *Introduction to Algorithms 6.046J/18.401J*, Fall 2005 Class by Prof. Charles Leiserson and Prof. Erik Demaine. MIT OpenCourseWare Initiative available at https://ocw.mit.edu/courses/6-046j-introduction-to-algorithms-sma-5503-fall-2005/.

Plan

Dynamic Programming* (DP)

- Invented by Richard Bellman in 1950s.
- Desing technique, like Divide & Conquer.
- Applies when the subproblems overlap (that is, when subproblems share subsubproblems).
- It solves each subsubproblem just once and then saves its answer in a table.
- DP typically applies to optimization problems:
  - have many possible solutions.
  - find a solution with the optimal (min or max) value.
  - *an* optimal solution, not *the* optimal solution.

Programming in this context refers to a tabular method, not to writing computer code.

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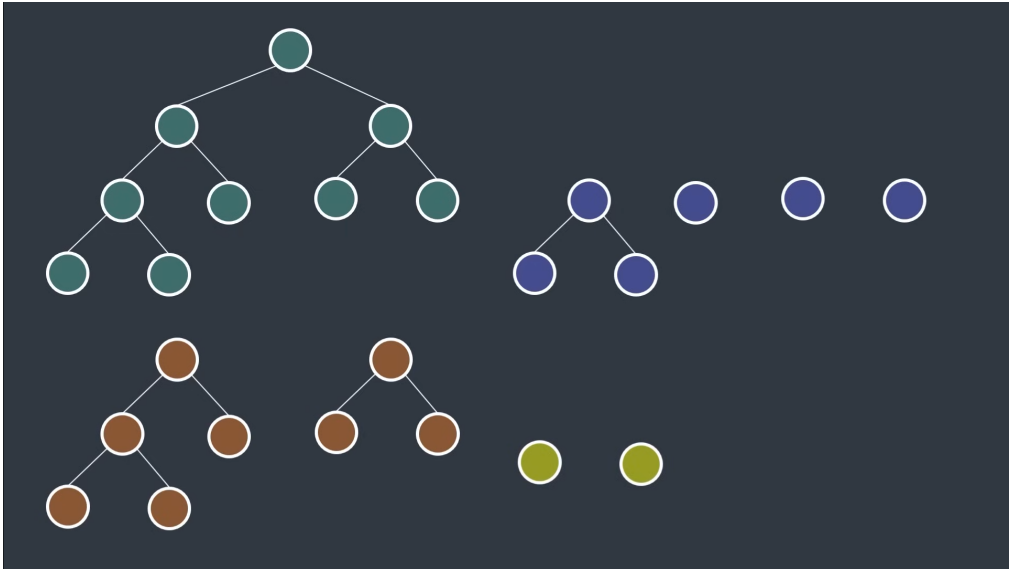
DP = Recursion + Memoization

DP notions



Plan

N-th Fibonacci Number[†]

Write a function that returns the n-th Fibonacci number.

$$F_1 = F_2 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

n	1	2	3	4	5	6	7
F_n	1	1	2	3	5	8	13

[†] *Mastering Dynamic Programming - How to solve any interview problem (Part 1).* Tech With Nikola Channel, 2024. YouTube, available at https://youtu.be/Hdr64lKQ3e4?si=ycTe-hoyfaICRWXt

Naive Approach

```
1    def fib(n):
2        if n <= 2:
3            result = 1
4        else:
5            result = fib(n - 1) + fib(n - 2)
6        return result
```

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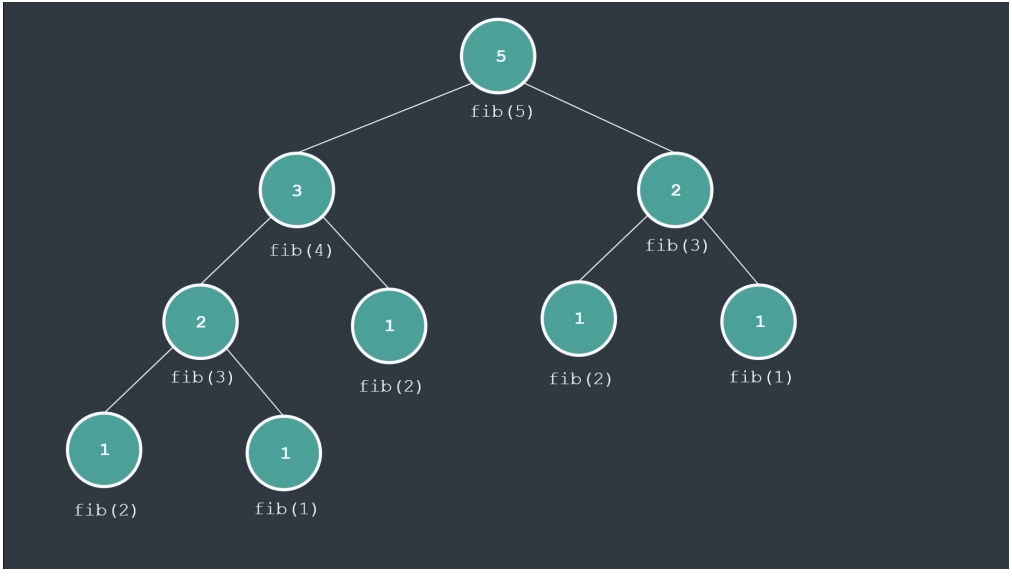
```
print(fib(7))
Output: 13
print(fib(50))
Output: ???
```

Naive Approach



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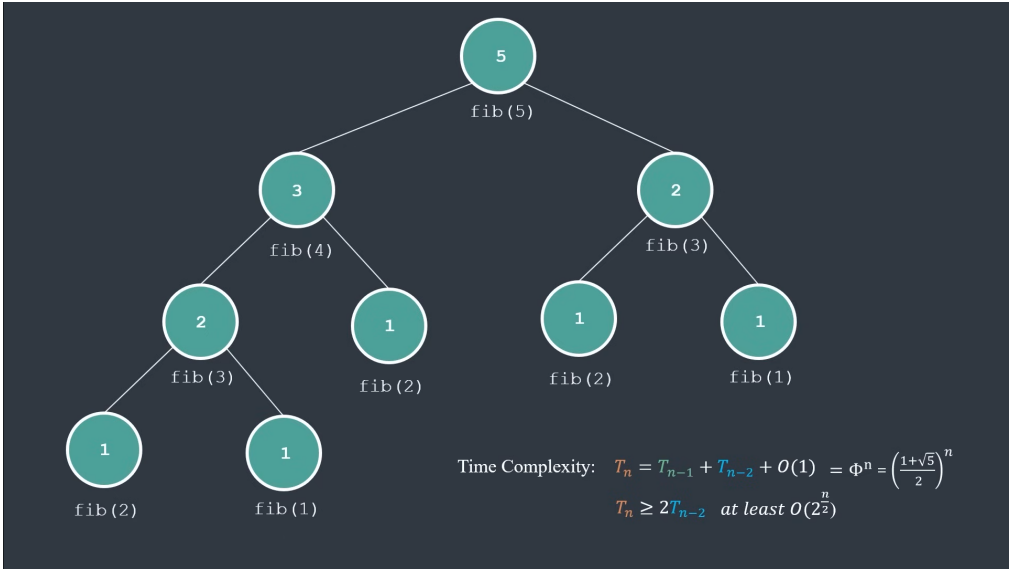

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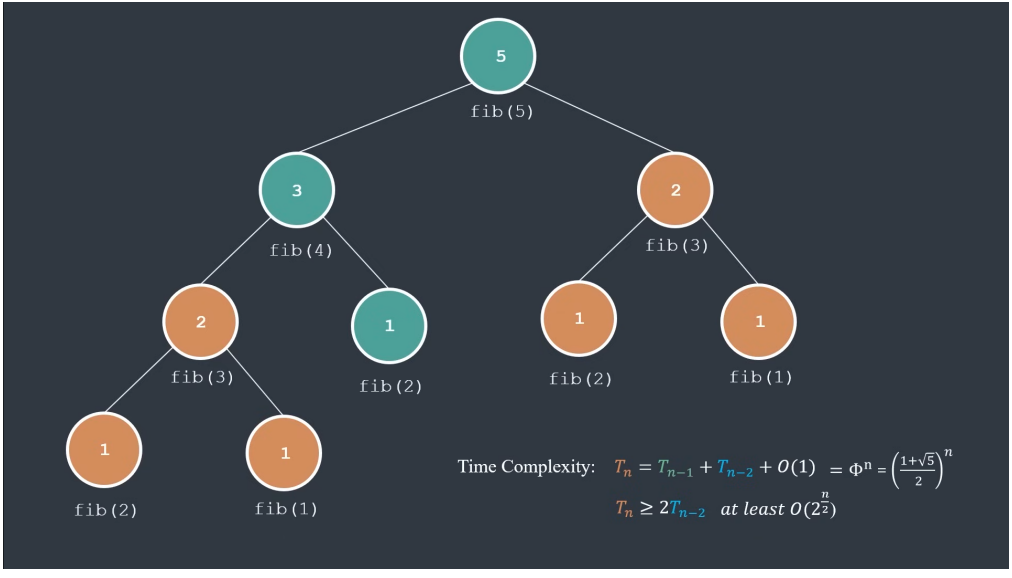
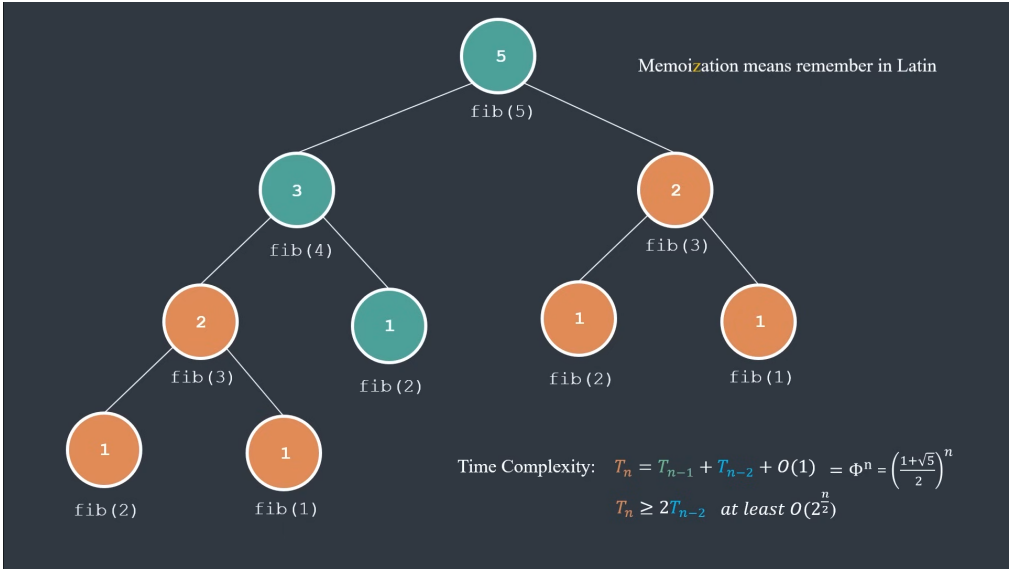


Naive Approach



Memoization Approach

```
memo = {} # adding a dictionary...
def fib(n):
    if n in memo: # asking if n is in dict...
        return memo[n] # if so, we are done!
    if n <= 2:
        result = 1
    else:
        result = fib(n - 1) + fib(n - 2)
    return result
```

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Memoization Approach

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print(fib(7))
Output: 13
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Output: 12586269025
```

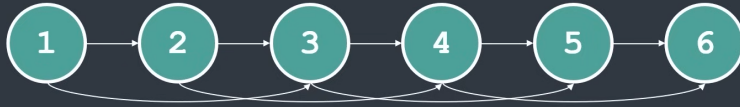
Time Complexity: $O(n)$.

$$\textbf{DP} = \textbf{Recursion} + \textbf{Memoization}$$

Bottom-up Approach

```
1    def fib(n):
2        memo = {}
3        for i in range(1, n + 1):
4            if i <= 2:
5                result = 1
6            else:
7                result = memo[n - 1] + memo[n - 2]
8            memo[i] = result
9        return memo[n]
```

Topological sort order



Plan

Longest Palindromic Sequence

Definition:

A palindrome is a string that is unchanged when reversed.

- Examples: `radar`, `civic`, `t`, `bb`, `redder`.
- Given: A string $X[1 \ldots n]$, $n \geq 1$.
- To find: Longest palindrome that is a subsequence.
- Example: Given "c h a r a c t e r".
- Output: "c a r a c".
- Answer will be ≥ 1 in length.

Strategy

$L(i, j)$: length of longest palindromic subsequence of $X[i \dots j]$ for $i \leq j$.

```
1     def L(i, j):
2         if i == j:
3             return 1
4         if X[i] == X[j]:
5             if i + 1 == j:
6                 return 2
7             else:
8                 return 2 + L(i + i, j - 1)
9         else:
10            return max( L(i + 1, j), L(i, j - 1) )
```

Analysis

As written, program can run in exponential time: suppose all symbols $X[i]$ are distinct.

$$T(n) = \text{running time on input of length } n$$

$$T(n) = \begin{cases} 1 & n = 1 \\ 2T(n-1) & n > 1 \end{cases}$$

$$= 2^{n-1}$$

What is missing?

- Complexity is exponential... why?

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- There is **recursion** but there is not **Memoization**...

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- Complexity is exponential... why?
- We are still not completing all the DP notions...
- There is **recursion** but there is not **Memoization**...
- Cache is missing!
- There is a single line of code that will fix it...

Understanding the Subproblems

The problem typically involves checking whether a substring of a given string is a palindrome. If the input string has a length of n, then the natural way to define subproblems is:

- Consider all possible substrings $s[i : j]$ of the string.
- For each substring, determine whether it is a palindrome.

Counting the Subproblems

A substring is defined by two indices, i (starting index) and j (ending index), where $0 \leq i \leq j < n$. This means:

- i can take values from 0 to $n - 1$.
- For each i, j can take values from i to $n - 1$.

Total Number of Subproblems

The total number of such (i, j) pairs (i.e., total subproblems) is given by the summation:

$$\sum_{i=0}^{n-1}(n-i) = n + (n-1) + (n-2) + \cdots + 1$$

Using the formula for the sum of the first n natural numbers:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Thus, the number of subproblems is:

$$n^2$$

Formula for Computing the Complexity of a DP

`# of subproblems` $\times$ `time to solve each subproblem`

Given that smaller ones are already solved[‡].

So,

- Given n^2 distinct subproblems...
- By solving each subproblem only once...
- Running time reduces to:

$$\Theta(n^2) \cdot \Theta(1) = \Theta(n^2)$$

[‡]lookup is $\Theta(1)$

New Strategy

- Memoize $L(i, j)$, hash inputs to get output value, and lookup hash table to see if the subproblem is already solved, else recurse.

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```

Look at `L[i, j]` and don't recurse if `L[i, j]` is already computed.

Memoizing Vs. Iterating

1. Memoizing uses a dictionary for $L(i, j)$ where value of L is looked up by using i, j as a key. Could just use a 2-D array here where null entries signify that the problem has not yet been solved.
2. Can solve subproblems in order of increasing $j - i$ so smaller ones are solved first.

Plan

- Given two sequences $x[1 \cdots m]$ and $y[1 \cdots n]$, find a[†] longest subsequence common to them both.

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`LCS(x, y)` as notation **not function**...

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- Checking $= O(n)$ time per subsequence.
- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).

Worst-case running time $= O(n2^m)$

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Worst-case running time $= O(n2^m)$ **Exponential time!**

Towards a Better Algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
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Towards a Better Algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

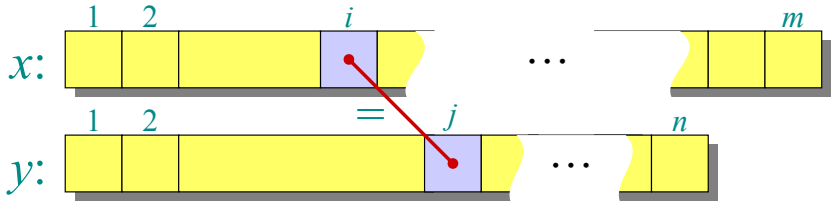
- **Notation:** Denote the length of a sequence s as $|s|$.
- **Strategy:** Consider **prefixes** of x and y:
  - Define $c[i, j] = |LCS(x[1 \cdots i], y[1 \cdots j])|$.
  - Then, $c[m, n] = |LCS(x, y)|$.

Recursive formulation

Theorem

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0, \\ c[i-1, j-1] & \text{if } x[i] = y[j], \\ \max\{c[i, j-1], c[i-1, j]\} & \text{otherwise.} \end{cases}$$

Proof: Case $x[i] = y[j] \ldots$



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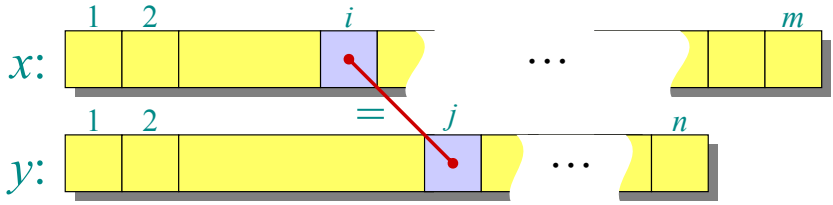

- Let $z[1 \cdots k] = LCS(x[1 \cdots i], y[1 \cdots j])$, where $c[i, j] = k$.

Recursive formulation

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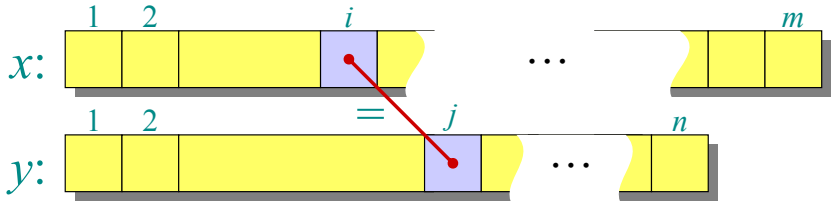

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- Then, $z[k] =$

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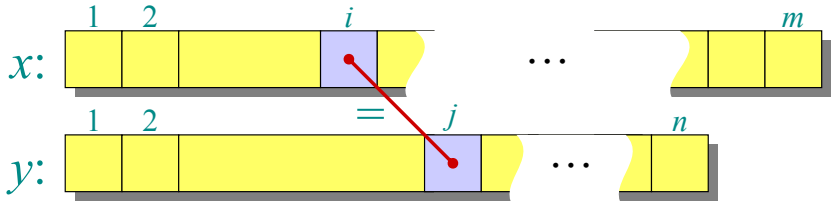

- Let $z[1 \cdots k] = LCS(x[1 \cdots i], y[1 \cdots j])$, where $c[i, j] = k$.
- Then, $z[k] = x[i] (= y[j])$

Recursive formulation

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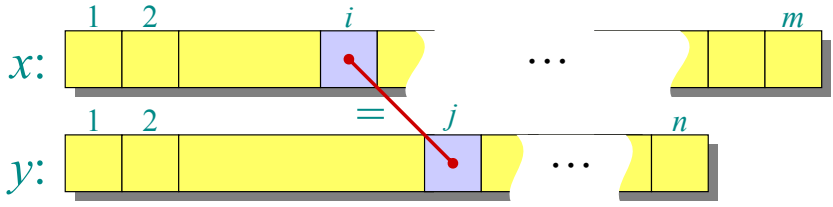

- Let $z[1 \cdots k] = LCS(x[1 \cdots i], y[1 \cdots j])$, where $c[i, j] = k$.
- Then, $z[k] = x[i](= y[j])$, or else z could be extended.

Recursive formulation

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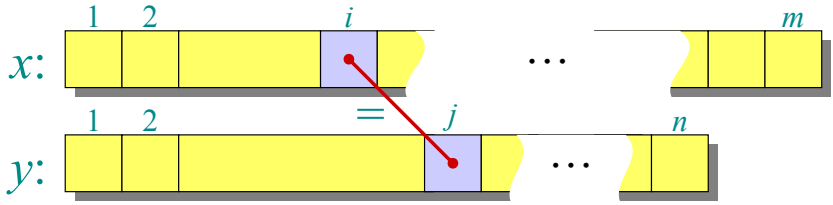

- Let $z[1 \cdots k] = LCS(x[1 \cdots i], y[1 \cdots j])$, where $c[i, j] = k$.
- Then, $z[k] = x[i] (= y[j])$, or else z could be extended.
- Thus, $z[1 \cdots k–1]$ is CS of $x[1 \cdots i - 1]$ and $y[1 \cdots j - 1]$.

Step-by-Step Example of LCS Algorithm

We will use the dynamic programming (DP) approach to compute the LCS for the following strings:

$$X = \text{“ACDBE”}$$

$$Y = \text{“ABCDE”}$$

Step 1: Create a DP Table

- We construct a $(m+1) \times (n+1)$ table, where m and n are the lengths of X and Y, respectively. The table will store the LCS length at each step.
- **Initial Table (before computation):** We initialize the first row and first column with 0s (LCS of empty strings is 0).

Step 1: Create a DP Table

	∅	A	B	C	D	E
∅	0	0	0	0	0	0
A	0					
C	0					
D	0					
B	0					
E	0					

Step 2: Fill the DP Table Using Recurrence

We iterate over each character of X and Y and apply the recurrence:

$$L[i,j] = \begin{cases} L[i-1,j-1]+1 & \text{if } x[i] = y[j], \\ \max\{c[i,j-1], c[i-1,j]\} & \text{otherwise.} \end{cases}$$

	$\varnothing$	A	B	C	D	E
$\varnothing$	0	0	0	0	0	0
A	0					
C	0					
D	0					
B	0					
E	0					

Step 2: Fill the DP Table Using Recurrence

Filling the table...

$$L[i,j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i, j-1], c[i-1, j]\} & \text{otherwise.} \end{cases}$$

- Compare 'A' ($X[1]$) with each character in Y.
- 'A' matches 'A' $\longrightarrow L[1,1] = L[0,0] + 1 = 1$.
- Other cells get the max of left or top.

	∅	A	B	C	D	E
∅	0	0	0	0	0	0
A	0	1	1	1	1	1
C	0					
D	0					
B	0					
E	0					

Step 2: Fill the DP Table Using Recurrence

Filling the table...

$$L[i,j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i, j-1], c[i-1, j]\} & \text{otherwise.} \end{cases}$$

- Compare 'C' ($X[2]$) with each character in Y.
- 'C' matches 'C' $\longrightarrow L[2,3] = L[1,2] + 1 = 2$.
- Other cells get the max of left or top.

	∅	A	B	C	D	E
∅	0	0	0	0	0	0
A	0	1	1	1	1	1
C	0	1	1	2	2	2
D	0					
B	0					
E	0					

Step 2: Fill the DP Table Using Recurrence

Filling the table...

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- Compare 'D' ($X[3]$) with each character in Y.
- 'D' matches 'D' $\longrightarrow L[3,4] = L[2,3] + 1 = 3$.
- Other cells get the max of left or top.

	∅	A	B	C	D	E
∅	0	0	0	0	0	0
A	0	1	1	1	1	1
C	0	1	1	2	2	2
D	0	1	1	2	3	3
B	0					
E	0					

Step 2: Fill the DP Table Using Recurrence

Filling the table...

$$L[i,j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i, j-1], c[i-1, j]\} & \text{otherwise.} \end{cases}$$

- Compare 'B' ($X[4]$) with each character in Y.
- 'B' matches 'B' $\longrightarrow L[4,2] = L[3,1] + 1 = 2$.
- Other cells get the max of left or top.

	∅	A	B	C	D	E
∅	0	0	0	0	0	0
A	0	1	1	1	1	1
C	0	1	1	2	2	2
D	0	1	1	2	3	3
B	0	1	2	2	3	3
E	0					

Step 2: Fill the DP Table Using Recurrence

Filling the table...

$$L[i,j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i, j-1], c[i-1, j]\} & \text{otherwise.} \end{cases}$$

- Compare 'E' ($X[5]$) with each character in Y.
- 'E' matches 'E' $\longrightarrow L[5,5] = L[4,4] + 1 = 4$.
- Other cells get the max of left or top.

	∅	A	B	C	D	E
∅	0	0	0	0	0	0
A	0	1	1	1	1	1
C	0	1	1	2	2	2
D	0	1	1	2	3	3
B	0	1	2	2	3	3
E	0	1	2	2	3	4

Step 3: Extract the LCS

- The **LCS** length is in the bottom-right cell, $L[5,5] = 4$.
- To trace back the **LCS**, we start from $L[5,5]$ and follow:
  - If $X[i] = Y[j]$, include it in the **LCS**.
  - Otherwise, move to the maximum value from the left or top.

Tracing Back from L[5,5]:

1. E ($X[5] = Y[5]$) $\longrightarrow$ Add 'E'
2. D ($X[3] = Y[4]$) $\longrightarrow$ Add 'D'
3. C ($X[2] = Y[3]$) $\longrightarrow$ Add 'C'
4. A ($X[1] = Y[1]$) $\longrightarrow$ Add 'A'

Thus, the LCS = "ACDE".

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

- You can solve the LCS problem with a brute-force approach:
  1. Enumerate all subsequences of X.
  2. Check each subsequence to see weather it is also subsequence of Y.
  3. Keep track of the longest subsequence.
- Because X has 2^m possible subsequence, time is exponential and, ergo, impractical.

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

- The LCS problem has an optimal-substructure property.
- the natural classes of subproblems correspond to pairs of *"prefixes"* of the two input sequences:
  - Given the sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$:
  - We can define the *i-th* **prefix** of X, for $i = 0, 1, \ldots, m$, as $X_i = \langle x_1, x_2, \ldots, x_i \rangle$.
  - For instance, if $X = \langle ABCBDAB \rangle$, then $X_4 = \langle ABCB \rangle$ and X_0 in the empty sequence.

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

Theorem 14.1 (Optimal substructure of an LCS)

Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is a LCS of X_{m-1} and Y_{n-1}.

Proof in CLRS page 395.

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

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Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is a LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$ and $z_k \neq x_m$, then Z is a LCS of X_{m-1} and Y.

Proof in CLRS page 395.

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

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Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is a LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$ and $z_k \neq x_m$, then Z is a LCS of X_{m-1} and Y.
3. If $x_m \neq y_n$ and $z_k \neq y_n$, then Z is a LCS of X and Y_{n-1}.

Proof in CLRS page 395.

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

- Theorem 14.1 says that an LCS of two sequences contains within it an LCS of prefixes of the two sequences.
- Thus, the LCS problem has an optimal-substructure property.
- A recursive solution also has the overlapping-subproblems property (as we'll see in a moment).

Longest Common Subsequence – Formal Steps of DP

Step 2: A recursive solution

- To find an LCS of X and Y, you might need to find the LCSs of X and Y_{n-1} and of X_{m-1} and Y (overlapping property).
- Each of these subproblems has the subsubproblem of finding an LCS of X_{m-1} and Y_{n-1}.
- Many other subproblems share subsubproblems.

Longest Common Subsequence – Formal Steps of DP

Step 2: A recursive solution

The optimal substructure of the LCS problem gives the recursive formula:

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0, \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x[i] = y[j], \\ \max\{c[i, j-1], c[i-1, j]\} & \text{if } i, j > 0 \text{ and } x[i] \neq y[j]. \end{cases}$$

Longest Common Subsequence – Formal Steps of DP

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- Define $c[i,j]$ = length of LCS of X_i and Y_j. Want $c[m,n]$.

Longest Common Subsequence – Formal Steps of DP

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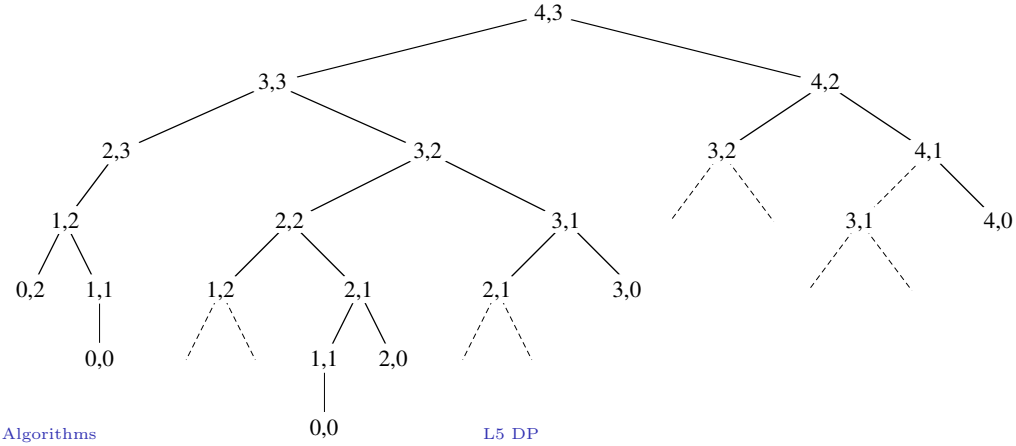
- Define $c[i,j]$ = length of LCS of X_i and Y_j. Want $c[m,n]$.
- Again, could write a recursive algorithm based on this formulation, but...

Longest Common Subsequence – Formal Steps of DP

Step 2: A recursive solution

- Try with $X = \langle a, t, o, m \rangle$ and $Y = \langle a, n, t \rangle$.
- Numbers in nodes are values of i, j in each recursive call.
- Dashed lines indicate subproblems already computed.



Longest Common Subsequence – Formal Steps of DP

Step 3: Computing the length of an LCS

```
LCS-LENGTH(X, Y, m, n)
 1  let b[1 : m, 1 : n] and c[0 : m, 0 : n] be new tables
 2  for i = 1 to m
 3      c[i, 0] = 0
 4  for j = 0 to n
 5      c[0, j] = 0
 6  for i = 1 to m              // compute table entries in row-major order
 7      for j = 1 to n
 8          if x_i == y_j
 9              c[i, j] = c[i − 1, j − 1] + 1
10              b[i, j] = "↖"
11          elseif c[i − 1, j] ≥ c[i, j − 1]
12              c[i, j] = c[i − 1, j]
13              b[i, j] = "↑"
14          else c[i, j] = c[i, j − 1]
15              b[i, j] = "←"
16  return c and b
```

Longest Common Subsequence – Formal Steps of DP

Step 3: Computing the length of an LCS

```
PRINT-LCS(b, X, i, j)
1  if i == 0 or j == 0
2       return                  // the LCS has length 0
3  if b[i, j] == "↖"
4       PRINT-LCS(b, X, i − 1, j − 1)
5       print x_i               // same as y_j
6  elseif b[i, j] == "↑"
7       PRINT-LCS(b, X, i − 1, j)
8  else PRINT-LCS(b, X, i, j − 1)
```

Longest Common Subsequence – Formal Steps of DP

Step 3: Computing the length of an LCS

Thomas H. Cormen
Emeritus Professor, Department of Computer Science
New Hampshire State Representative, Grafton 15 (Lebanon Ward 3)
CormenForNH.com

I retired on January 1, 2022. I am no longer taking any new students or interns at any level.

Ph.D., Massachusetts Institute of Technology, 1992.

- **Papers**
- **FG**
- **Other software**
- **Vita**

I taught my final course in Fall 2019. You can view a video of my last lecture, which was not about computer science.

Khan Academy now carries algorithms tutorials for which Devin Balkcom and I produced content.

Are you looking for solutions to exercises and problems in *Introduction to Algorithms*?

If you are, then see the frequently asked question and answer below.

If you request solutions from me, I will not respond.

From July 2004 through June 2008, I was the director of the Dartmouth Institute for Writing and Rhetoric.

I occasionally taught a graduate Computer Science course on how to write papers and how to give talks. I publish a list of usage rules that I required my students to observe.

In 2015, PRI's "The World" ran a story on mentoring women in computer science in which a couple of my students and I were interviewed.

I was interviewed for the *Command Line Heroes* podcast "Learning the BASICs."
You can also hear me spout off in "Philosophical Trials #7."
I talked about writing *Introduction to Algorithms* in an episode of "Frank Stajano Explains."
And an interview with a Brazilian vlogger.

Graduate Student Alumni

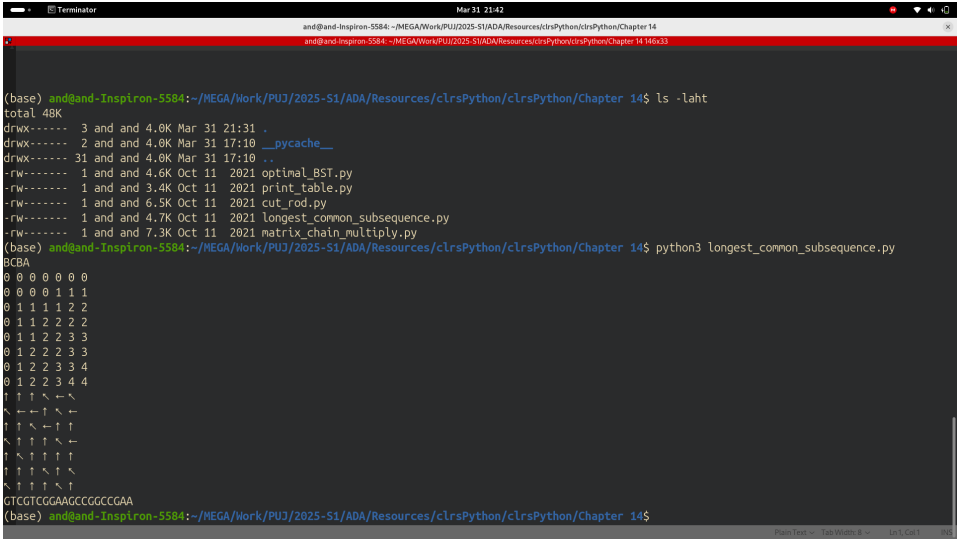
- Alex Colvin, Ph.D. 1999
- Peter C. Johnson
- Priya Natarajan, Ph.D. 2011 [Photo of Priya and me at 2012 Dartmouth graduation]
- Geeta Chaudhry Petrovic, Ph.D. 2004 [Photo of Geeta and me at 2004 Dartmouth graduation]
- Elena Riccio Strange (formerly Elena Riccio Davidson), Ph.D. 2006
- Len Wisniewski, Ph.D. 1996
- Melissa Hirschl Chawla, M.S. 1997
- Michael Ringenburg, M.S. 2001
- Georgi Vassilev, M.S. 1994

https://www.cs.dartmouth.edu/~thc/

Implementations **here**

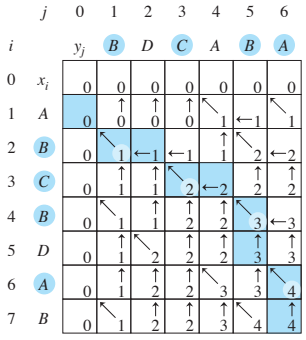
Longest Common Subsequence – Formal Steps of DP

Step 3: Computing the length of an LCS



Longest Common Subsequence – Formal Steps of DP

Step 4: Constructing an LCS



i		j	0	1	2	3	4	5	6
			y_j	B	D	C	A	B	A
0	x_i		0	0	0	0	0	0	0
1	A		0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B		0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C		0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B		0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D		0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A		0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B		0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

Figure 14.8 The c and b tables computed by LCS-LENGTH on the sequences $X = \langle A, B, C, B, D, A, B \rangle$ and $Y = \langle B, D, C, A, B, A \rangle$. The square in row i and column j contains the value of $c[i, j]$ and the appropriate arrow for the value of $b[i, j]$. The entry 4 in $c[7, 6]$—the lower right-hand corner of the table—is the length of an LCS $\langle B, C, B, A \rangle$ of X and Y. For $i, j > 0$, entry $c[i, j]$ depends only on whether $x_i = y_j$ and the values in entries $c[i - 1, j]$, $c[i, j - 1]$, and $c[i - 1, j - 1]$, which are computed before $c[i, j]$. To reconstruct the elements of an LCS, follow the $b[i, j]$ arrows from the lower right-hand corner, as shown by the sequence shaded blue. Each "↖" on the shaded-blue sequence corresponds to an entry (highlighted) for which $x_i = y_j$ is a member of an LCS.

Homework

Determine an LCS of $\langle 1, 0, 0, 1, 0, 1, 0, 1 \rangle$ and $\langle 0, 1, 0, 1, 1, 0, 1, 1, 0 \rangle$.

Plan

Introduction

Different types of algorithms can be used to solve the all-pairs shortest paths problem:

- Dynamic programming
- Matrix multiplication
- Floyd-Warshall algorithm
- Johnson's algorithm
- Difference constraints

Single-Source Shortest Paths

- Given directed graph $G = (V, E)$, vertex $s \in V$ and edge weights $w : E \to \mathbb{R}$
- Find $\delta(s, v)$, equal to the shortest-path weight $s \to v, \forall v \in V$ (or $-\infty$ if negative weight cycle along the way, or ∞ if no path)

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Situtation	**Algorithm**	**Time**
unweigthed ($w = 1$)	BFS	

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unweigthed ($w = 1$)	BFS	$O(V + E)$
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general	Bellman-Ford	$O(VE)$
acyclic graph (DAG)	Topological sort + one pass of B-F	$O(V + E)$

All of the above results are the best known. We achieve a $O(E + V \lg V)$ bound on Dijkstra's algorithm using **Fibonacci heaps**.

All-Pairs Shortest Paths (APSP)

- Given edge-weighted graph, $G = (V, E, w)$.
- Find $\delta(u, v)$ for all $u, v \in V$.
- A simple way of solving APSP problems is by running a single-source shortest path algorithm from each of the V vertices in the graph.

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Situtation	**Algorithm**	**Time**	$E = \Theta(V^2)$
unweigthed ($w = 1$)	$\|V\|$+ BFS	$O(VE)$	$O(V^3)$
non-negative edge weights	$\|V\|$+ Dijkstra	$O(VE + V^2 \lg V)$	$O(V^3)$
general	$\|V\|$+ Bellman-Ford	$O(V^2E)$	$O(V^4)$
general	Johnson's	$O(VE + V^2 \lg V)$	$O(V^3)$

These results (apart from the third) are also best known –don't know how to beat $|V| \times Dijkstra$.

Algorithms to solve APSP

Dynamic Programming (attempt 1):

1. **Sub-problems**:
2. **Guessing**:
3. **Recurrence**:
4. **Topological ordering**:
5. **Original problem**:

Algorithms to solve APSP[§]

Dynamic Programming (attempt 1):

1. **Sub-problems**: $d_{uv}^{(m)}$ = weight of shortest path $u \to v$ using $\leq m$ edges.

[§]For all the algorithms described, we assume that $w(u, v) = \infty$ if $(u, v) \notin E$

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1. **Sub-problems**: $d_{uv}^{(m)}$ = weight of shortest path $u \to v$ using $\leq m$ edges.
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$$d_{uv}^{(m)} = min(d_{ux}^{(m-1)} + w(x, v) \quad \text{for } x \in V)$$

$$d_{uv}^{(0)} = \begin{cases} 0 & \text{if } u = v. \\ \infty & \text{otherwise.} \end{cases}$$

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5. **Original problem**: If graph contains no negative-weight cycles (by Bellman-Ford analysis), then shortest path is simple $\Rightarrow \delta(u,v) = d_{uv}^{(n)} = d_{uv'}^{(n-1)} = \cdots$

[§]For all the algorithms described, we assume that $w(u,v) = \infty$ if $(u,v) \notin E$

Bottom-up via Relaxation Steps[¶]

```
1  for m ← 1 to n by 1
2      for u in V
3          for v in V
4              for x in V
5                  if d_uv > d_ux + d_xv
6                      d_uv = d_ux + d_xv
```

[¶] In the above pseudocode, we omit superscripts because more relaxation can never hurt.

Time complexity

- In this Dynamic Program, we have $O(V^3)$ total sub-problems.
- Each sub-problem takes $O(V)$ time to solve, since we need to consider V possible choices.
- This gives a total runtime complexity of $O(V^4)$.
- Note that this is no better than $|V|\times$ Bellman-Ford.

Matrix Multiplication

Recall the task of standard matrix multiplication:

- Given $n \times n$ matrices A and B, compute $C = A \cdot B$, such that $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$.

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  1. $O(n^3)$ using standard algorithm.
  2. $O(n^{2.807})$ using Strassen algorithm.
  3. $O(n^{2.376})$ using Coppersmith-Winograd algorithm.
  4. $O(n^{2.3728})$ using Vassilevska-Williams algorithm.

Connection to Shortest Paths

- Let's define $\oplus = \min$ and $\odot = +$.
- Then, $C = A \odot B$ produces $c_{ij} = \min_k(a_{ik} + b_{kj})$.
- Define $D^{(m)} = (d_{ij}^{(m)})$, $W = (w(i,j))$, $V = \{1, 2, \ldots, n\}$

With the above definitions, we see that $D^{(m)}$ can be expressed as $D^{(m-1)} \odot W$. In other words, $D^{(m)}$ can be expressed as the circle-multiplication of W with itself m times.

Matrix Multiplication Algorithm

- $\Rightarrow O(n^4)$ time (still no better).

Matrix Multiplication Algorithm

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Matrix Multiplication Algorithm

- $\Rightarrow O(n^4)$ time (still no better).
- Repeated squaring: $((W^2)^2)^{2^{\cdots}} = W^{2^{\lg n}} = W^{n-1} = (\delta(i,j))$ if no negative-weight cycles.
- Time complexity of this algorithm is now $O(n^3 \lg n)$.

Floyd-Warshall Algorithms

Dynamic Programming (attempt 2):

1. **Sub-problems**: $c_{uv}^{(k)}$ = weight of shortest path $u \to v$ whose intermediate vertices $\in \{1, 2, \ldots, k\}$

Floyd-Warshall Algorithms

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Floyd-Warshall Algorithms

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2. **Guessing**: Does shortest path use vertex k?

Floyd-Warshall Algorithms

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2. **Guessing**: Does shortest path use vertex k?
3. **Recurrence**:

$$c_{uv}^{(0)} = w(u, v)$$
$$c_{uv}^{(k)} = \min(c_{uv}^{(k-1)}, c_{ux}^{(k-1)} + c_{xv}^{(k-1)})$$

Floyd-Warshall Algorithms

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3. **Recurrence**:
$$c_{uv}^{(0)} = w(u, v)$$
$$c_{uv}^{(k)} = \min(c_{uv}^{(k-1)}, c_{ux}^{(k-1)} + c_{xv}^{(k-1)})$$
4. **Topological order**: for k: for u and v in V:
5. **Original problem**: $\delta(u, v) = c_{uv}^{(n)}$. Negative weight cycle $\Leftrightarrow$ negative $c_{uu}^{(n)}$

Time Complexity

This Dynamic Program contains $O(V^3)$ problems as well. However, in this case, it takes only $O(1)$ time to solve each sub-problem, which means that the total runtime of this algorithm is $O(V^3)$.

Bottom-up via Relaxation

```
1  C = (w(u, v))
2  for k ← 1 to n
3      for u in V
4          for v in V
5              if c_uv > c_uk + c_kv
6                  c_uv = c_uk + c_kv
```

Johnson's algorithm

1. Find function $h : V \to \mathbb{R}$ such that $w_h(u, v) = w(u, v) + h(u) - h(v) \geq 0$ for all $u, v \in V$ or determine that a negative-weight cycle exists.
2. Run Dijkstra's algorithm on (V, E, w_h) from every source vertex $s \in V \Rightarrow$ get $\delta_h(u, v)$ for all $u, v \in V$.
3. Given $\delta_h(u, v)$, it is easy to compute $\delta(u, v)$.

Proof

Claim. $\delta(u, v) = \delta_h(u, v) - h(u) + h(v)$.

Proof. Look at any $u \to v$ path p in the graph G:

- Say p is $v_0 \to v_1 \to v_2 \to \cdots \to v_k$, where $v_0 = u$ and $v_k = v$.

$$\begin{aligned} w_h(p) &= \sum_{i=1}^{k} w_h(v_{i-1}, v_i) \\ &= \sum_{i=1}^{k} [w(v_{i-1}, v_i) + h(v_{i-1}) - h(v_i)] \\ &= \sum_{i=1}^{k} w(v_{i-1}, v_i) + h(v_0) - h(v_k) \\ &= w(p) + h(u) - h(v) \end{aligned}$$

- Hence all $u \to v$ paths change in weight by the same offset $h(u) - h(v)$, which implies that the shortest path is preserved. □

How to find h?

We know that

$$w_h(u, v) = w(u, v) + h(u) - h(v) \geq 0$$

This is equivalent to,

$$h(v) - h(u) \leq w(u, v)$$

for all $(u, v) \in V$. This is called a **system of difference constraints**.

Theorem.

If (V, E, w) has a negative-weight cycle, then there exists no solution to the above system of difference constraints.

Proof

Say $v_0 \to v_1 \to \cdots \to v_k \to v_0$ is a negative weight cycle.
Let us assume to the contrary that the system of difference constraints has a solution; let's call it h.
This gives us the following system of equations,

$$\begin{aligned}
h(v_1) - h(v_0) &\leq w(v_0, v_1) \\
h(v_2) - h(v_1) &\leq w(v_1, v_2) \\
h(v_3) - h(v_2) &\leq w(v_2, v_3) \\
&\vdots \\
h(v_k) - h(v_{k-1}) &\leq w(v_{k-1}, v_k) \\
h(v_0) - h(v_k) &\leq w(v_k, v_0)
\end{aligned}$$

Summing all these equations gives us

$$0 \leq w(cycle) < 0$$

which is obviously not possible.
From this, we can conclude that no solution to the above system of difference constraints exists if the graph (V, E, w) has a negative weight cycle. □

Theorem.

If (V, E, w) has no negative-weight cycle, then we can find a solution to the difference constraints.

Proof. Add a new vertex s to G, and add edges (s, v) of weight 0 for all $v \in V$.

- Clearly, these new edges do not introduce any new negative weight cycles to the graph.
- Adding these new edges ensures that there now exists at least one path from s to v. This implies that $\delta(s, v)$ is finite for all $v \in V$
- We now claim that $h(v) = \delta(s, v)$. This is obvious from the triangle inequality: $\delta(s, u) + w(u, v) \geq \delta(s, v) \Leftrightarrow \delta(s, v) - \delta(s, u) \leq w(u, v) \Leftrightarrow h(v) - h(u) \leq w(u, v)$. □

Time Complexity

1. The first step involves running Bellman-Ford from s, which takes $O(VE)$ time. We also pay a pre-processing cost to reweight all the edges ($O(E)$).

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The total running time of this algorithm is $O(VE + V^2 \lg V)$.

Plan

Rod Cutting Problem

Problem Statement:
Given a rod of length n inches and a table of prices p_i for $i = 1, 2, \ldots, n$, determine the maximum revenue obtainable by cutting the rod and selling the pieces.

Example Price Table:

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

Problem Statement

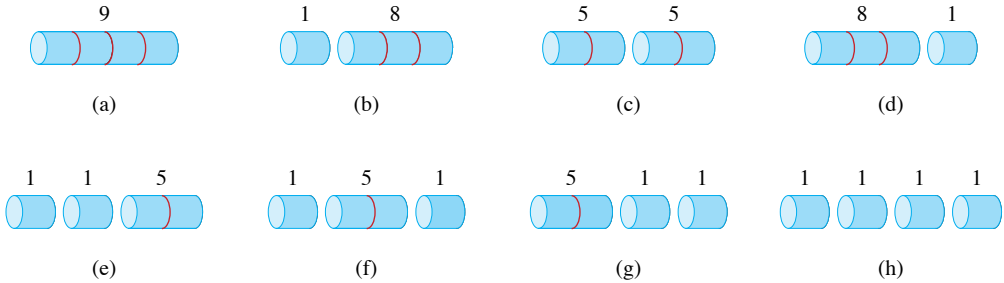


Figure 14.2 The 8 possible ways of cutting up a rod of length 4. Above each piece is the value of that piece, according to the sample price chart of Figure 14.1. The optimal strategy is part (c)—cutting the rod into two pieces of length 2—which has total value 10.

Recursive Formulation

Let r_n be the maximum revenue for a rod of length n. Then:

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

Base case: $r_0 = 0$

Exponential Time Algorithm:

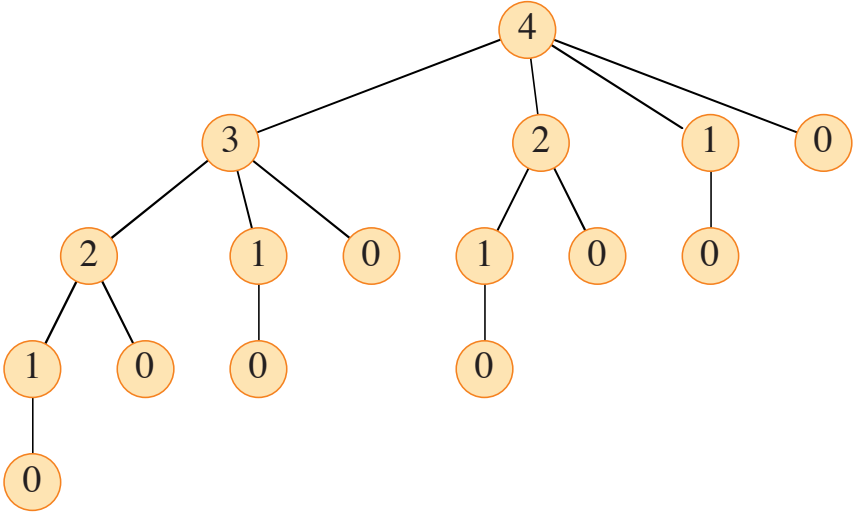
```
CUT-ROD(p, n)
1  if n == 0
2       return 0
3  q = -∞
4  for i = 1 to n
5       q = max {q, p[i] + CUT-ROD(p, n - i)}
6  return q
```

Recursive Formulation

- Let $T(n)$ denote the total number of calls made to CUT-ROD(p, n) for a particular value of n.
- The number of nodes in a subtree whose root is labeled n in the recursion tree.

$$T(n) = 1 + \sum_{j=0}^{n-1} T(j).$$

Recursive Formulation



Memoized Version

Avoids recomputation by storing already computed values.

```
MEMOIZED-CUT-ROD(p, n)
1  let r[0:n] be a new array          // will remember solution values in r
2  for i = 0 to n
3      r[i] = -∞
4  return MEMOIZED-CUT-ROD-AUX(p, n, r)

MEMOIZED-CUT-ROD-AUX(p, n, r)
1  if r[n] ≥ 0                       // already have a solution for length n?
2      return r[n]
3  if n == 0
4      q = 0
5  else q = -∞
6      for i = 1 to n                // i is the position of the first cut
7          q = max{q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n - i, r)}
8  r[n] = q                          // remember the solution value for length n
9  return q
```

Bottom-Up Dynamic Programming

```
BOTTOM-UP-CUT-ROD(p, n)
1  let r[0:n] be a new array          // will remember solution values in r
2  r[0] = 0
3  for j = 1 to n                     // for increasing rod length j
4      q = -∞
5      for i = 1 to j                 // i is the position of the first cut
6          q = max{q, p[i] + r[j - i]}
7      r[j] = q                       // remember the solution value for length j
8  return r[n]
```

Time complexity: $\mathcal{O}(n^2)$

Subproblem graphs

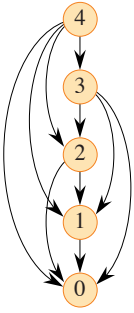


Figure 14.4 The subproblem graph for the rod-cutting problem with $n = 4$. The vertex labels give the sizes of the corresponding subproblems. A directed edge (x, y) indicates that solving subproblem x requires a solution to subproblem y. This graph is a reduced version of the recursion tree of Figure 14.3, in which all nodes with the same label are collapsed into a single vertex and all edges go from parent to child.

Reconstructing the Solution

Store the optimal cuts as well:

- Maintain a second array $s[1..n]$.
- $s[j]$ stores the length of the first piece to cut from a rod of length j.

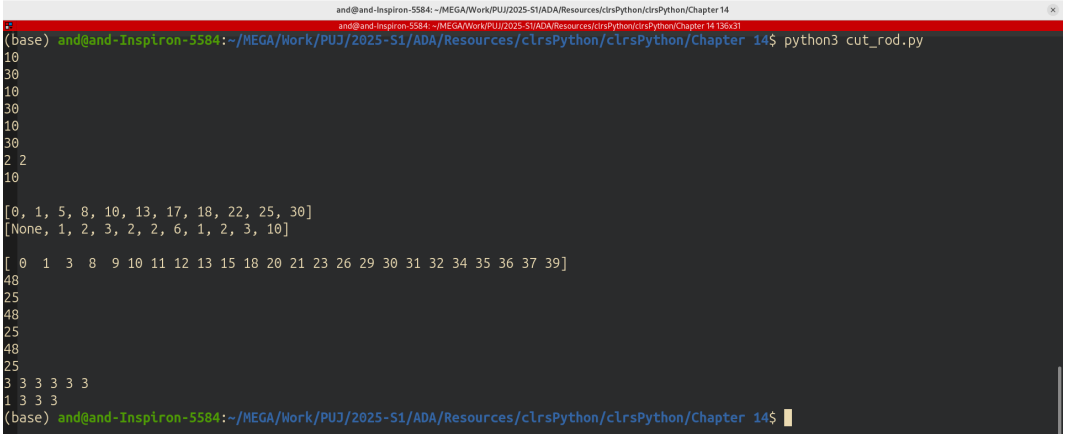
Then backtrack using $s[n]$ to print the actual cuts.

i	0	1	2	3	4	5	6	7	8	9	10
$r[i]$	0	1	5	8	10	13	17	18	22	25	30
$s[i]$		1	2	3	2	2	6	1	2	3	10

Reconstructing a solution

```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
 1  let r[0 : n] and s[1 : n] be new arrays
 2  r[0] = 0
 3  for j = 1 to n                  // for increasing rod length j
 4      q = −∞
 5      for i = 1 to j              // i is the position of the first cut
 6          if q < p[i] + r[j − i]
 7              q = p[i] + r[j − i]
 8              s[j] = i            // best cut location so far for length j
 9      r[j] = q                    // remember the solution value for length j
10  return r and s
```

Reconstructing a solution



Learned lessons

- Naive recursive solution has exponential time complexity.
- Dynamic programming reduces it to $\mathcal{O}(n^2)$.
- Optimal substructure and overlapping subproblems make DP suitable.

Plan

Matrix-chain multiplication

Problem: Given a sequence (chain) $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices, compute the product $A_1 A_2 \cdots A_n$ using standard matrix multiplication (not Strassen's method) while minimizing the number of scalar multiplications.

How to parenthesize the product to minimize the number of scalar multiplications?

Matrix-chain multiplication

- Suppose multiplying matrices A and B: $C = A \cdot B$.
- The matrices must be compatible: number of columns of A equals number of rows of B.
- If A is $p \times q$ and B is $q \times r$, then C is $p \times r$ and takes pqr scalar multiplications.

Example

$A_1 : 10 \times 100$, $A2 : 100 \times 5$, $A3 : 5 \times 50$. Compute $A_1A_2A_3$, which is 10×50.

- Try parenthesizing by $((A_1A_2)A_3)$. First perform $10 \times 100 \times 5 = 5000$ multiplications, then perform $10 \times 5 \times 50 = 2500$, for a total of 7500.
- Try parenthesizing by $(A_1(A_2A_3))$. First perform $100 \times 5 \times 50 = 25,000$ multiplications, then perform $10 \times 100 \times 50 = 50,000$, for a total of 75,000.
- The first way is 10 times faster.

Input to the Problem

- Let A_i be $p_{i-1} \times p_i$. The input is the sequence of dimensions $\langle p_0, p_1, p_2, \ldots, p_n \rangle$.
- **Note**: Not actually multiplying matrices. Just deciding an order with the lowest cost.

Counting the Number of Parenthesizations

- Let $P(n)$ denote the number of ways to parenthesize a product of n matrices. $P(1) = 1$.
- When $n \geq 2$, can split anywhere between A_k and A_{k+1} for $k = 1, 2, \ldots, n-1$. Then have to split the subproducts.
- Get

$$P(n) = \begin{cases} 1 & \text{if } n = 1, \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2. \end{cases}$$

- The solution is $P(n) = \Omega\left(\frac{4^n}{n^{\frac{3}{2}}}\right)$. So brute force is a bad strategy.

Step 1: Structure of an optimal solution

- Let $A_{i:j}$ be the matrix product $A_i A_{i+1} \cdots A_j$.
- If $i < j$, then must split between A_k and A_{k+1} for some $i \leq k < j \Rightarrow$ compute $A_{i:k}$ and $A_{k+1:j}$ and then multiply them together.
- Cost is

$$\begin{aligned} &\text{cost of computing } A_{i:k} \\ +\ &\text{cost of computing } A_{k+1:j} \\ +\ &\text{cost of multiplying them together.} \end{aligned}$$

Optimal substructure

- Suppose that optimal parenthesization of $A_{i:j}$ splits between A_k and A_{k+1}.
- Then the parenthesization of $A_{i:k}$ must be optimal.
- Otherwise, if there's a less costly way to parenthesize it, you'd use it and get a parenthesization of $A_{i:j}$ with a lower cost. Same for $A_{k+1:j}$.
- Therefore, to build an optimal solution to $A_{i:j}$:
  - split it into how to optimally parenthesize $A_{i:k}$ and $A_{k+1:j}$,
  - find optimal solutions to these subproblems,
  - and then combine the optimal solutions.
- Need to consider all possible splits.

Step 2: A recursive solution

- Define the cost of an optimal solution recursively in terms of optimal subproblem solutions.
- Let $m[i, j]$ be the minimum number of scalar multiplications to compute $A_{i:j}$. For the full problem, want $m[1, n]$.
  - If $i = j$, then just one matrix $m[i, i] = 0$ for $i = 1, 2, \ldots, n$.
  - If $i < j$, then suppose the optimal split is between A_k and A_{k+1}, where $i \leq k < j$.
  - Then $m[i, j] = m[i, k] + m[k + 1, j] + p_{i-1} p_i p_j$.

Step 2: A recursive solution

- But that's assuming you know the value of k. Have to try all possible values and pick the best, so that

$$m[i,j] = \begin{cases} 0 & \text{if } i = j, \\ min\{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j : i \leq k < j\} & \text{if } i < j. \end{cases}$$

- That formula gives the cost of an optimal solution, but not how to construct it.
  - Define $s[i,j]$ to be a value of k to split $A_{i:j}$ in an optimal parenthesization.
  - Then $s[i;j] = k$ such that $m[i,j] = m[i,k] + m[k+1,j] + p_{i-1}p_k p_j$.

Step 3: Compute the optimal costs

- Could implement a recursive algorithm based on the above equation for $m[i, j]$ but it would take exponential time.
- There are not all many subproblems:
  - just one for each i, j such that $1 \leq i \leq j \leq n$.
  - there are $\binom{n}{2} + n = \Theta(n^2)$ of them.
  - thus, a recursive algorithm would solve the same subproblem over and over.
- In other words, this problem has overlapping subproblems.

Step 3: Compute the optimal costs

- Here is a tabular, bottom-up method to solve the problem.
- It solves subproblems in order of increasing chain length.

```
MATRIX-CHAIN-ORDER(p, n)
 1  let m[1:n, 1:n] and s[1:n−1, 2:n] be new tables
 2  for i = 1 to n                          // chain length 1
 3      m[i, i] = 0
 4  for l = 2 to n                          // l is the chain length
 5      for i = 1 to n − l + 1              // chain begins at A_i
 6          j = i + l − 1                   // chain ends at A_j
 7          m[i, j] = ∞
 8          for k = i to j − 1              // try A_{i:k} A_{k+1:j}
 9              q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
10              if q < m[i, j]
11                  m[i, j] = q             // remember this cost
12                  s[i, j] = k             // remember this index
13  return m and s
```

Step 3: Compute the optimal costs

- Here is a tabular, bottom-up method to solve the problem.
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 6          j = i + l − 1                   // chain ends at A_j
 7          m[i, j] = ∞
 8          for k = i to j − 1              // try A_{i:k} A_{k+1:j}
 9              q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
10              if q < m[i, j]
11                  m[i, j] = q             // remember this cost
12                  s[i, j] = k             // remember this index
13  return m and s
```

Time: $O(n^3)$, from triply nested loops. Also $\Omega(n^3) \Rightarrow \Theta(n^3)$.

Step 4: Construct an optimal solution

- With the s table filled in, recursively print an optimal solution.
- Initial call is PRINT-OPTIMAL-PARENS$(s, 1, n)$.

```
PRINT-OPTIMAL-PARENS(s, i, j)
1  if i == j
2       print "A"_i
3  else print "("
4       PRINT-OPTIMAL-PARENS(s, i, s[i, j])
5       PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
6       print ")"
```

Plan

Optimal binary search trees

- Given sequence $K = \langle k_1, k_2, \ldots, k_n \rangle$ of n distinct keys, sorted ($k_1 < k_2 < \ldots < k_n$).
- Want to build a binary search tree from the keys.
- For k_i, have probability p_i that a search is for k_i.
- Want BST with minimum expected search cost.

BST Cost

Actual cost = # of items examined.

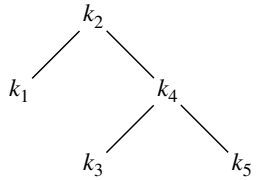
For k_i, $cost = depth_T(k_i) + 1$, where $depth_T(k_i) =$ depth k_i in BST T.

$E[$ search cost in $T]$

$$
\begin{aligned}
&= \sum_{i=n}^{n} (depth_T(k_i) + 1) \cdot p_i \\
&= \sum_{i=n}^{n} depth_T(k_i) \cdot p_i + \sum_{i=1}^{n} p_i \\
&= 1 + \sum_{i=n}^{n} depth_T(k_i) \cdot p_i
\end{aligned}
$$

Example

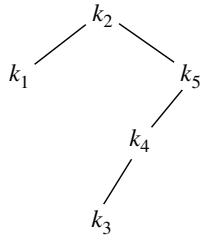
i	1	2	3	4	5
p_i	.25	.2	.05	.2	.3



i	$\text{depth}_T(k_i)$	$\text{depth}_T(k_i) \cdot p_i$
1	1	.25
2	0	0
3	2	.1
4	1	.2
5	2	.6
		1.15

Example

i	1	2	3	4	5
p_i	.25	.2	.05	.2	.3



i	$\text{depth}_T(k_i)$	$\text{depth}_T(k_i) \cdot p_i$
1	1	.25
2	0	0
3	3	.15
4	2	.4
5	1	.3
		1.10

Observations

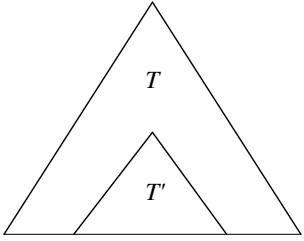
- Optimal BST might not have smallest height.
- Optimal BST might not have highest-probability key at root.

Build by exhaustive checking?

- Construct each n-node BST.
- For each, put in keys.
- Then compute expected search cost.
- But there are $\Omega\left(\frac{4^n}{n^{\frac{3}{2}}}\right)$ different BSTs with n nodes.

Step 1: The structure of an optimal BST

Consider any subtree of a BST. It contains keys in a contiguous range $k_i, \ldots, k_j$ for some $1 \leq i \leq j \leq n$.

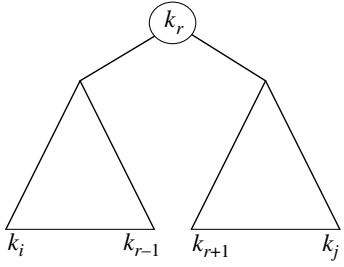


If T is an optimal BST and T contains subtree T' with keys $k_i, \ldots, k_j$, then T' must be an optimal BST for keys $k_i, \ldots, k_j$.

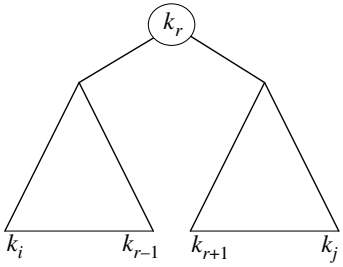
Step 1: The structure of an optimal BST

Use optimal substructure to construct an optimal solution to the problem from optimal solutions to subproblems:

- Given keys $k_i, \ldots, k_j$ (the problem).
- One of them, k_r, where $i \leq r \leq j$, must be the root.
- Left subtree of k_r contains $k_i, \ldots, k_{r-1}$.
- Right subtree of k_r contains $k_{r+1}, \ldots, k_j$.



Step 1: The structure of an optimal BST



- If
  - you examine all candidate roots k_r, for $i \leq r \leq j$, and
  - you determine all optimal BSTs containing $k_i, \ldots, k_{r-1}$ and containing $k_{r+1}, \ldots, k_j$,

  then you're guaranteed to find an optimal BST for $k_i, \ldots, k_j$.

Step 2: Recursive solution

Subproblem domain:

- Find optimal BST for $k_i, \ldots, k_j$, where $i \geq 1$, $j \leq n$, $j \geq i - 1$.
- When $j = i - 1$, the tree is empty.

Define $e[i, j]$ = expected search cost of optimal BST for $k_i, \ldots, k_j$.

Step 2: Recursive solution

- If $j = i - 1$, then $e[i, j] = 0$.
- If $j \geq i$,
  - Select root k_r, for some $i \leq r \leq j$.
  - Make an optimal BST with $k_i, \ldots, k_{r-1}$ as the left subtree.
  - Make an optimal BST with $k_{r+1}, \ldots, k_j$ as the right subtree.
  - Note: when $r = i$, left subtree is $k_i, \ldots, k_{i-1}$; when $r = j$, right subtree is $k_{j+1}, \ldots, k_j$. These subtrees are empty.

Step 2: Recursive solution

When a subtree becomes a subtree of a node:

- Depth of every node in subtree goes up by 1.
- Expected search cost increases by

$$w(i,j) = \sum_{l=i}^{j} p_l$$

If k_r is the root of an optimal BST for $k_i, \ldots, k_j$:

$$e[i,j] = p_r + (e[i,r-1] + w(i,r-1)) + (e[r+1,j] + w(r+1,j))$$

- But $w(i,j) = w(i,r-1) + p_r + w(r+1,j)$.
- Therefore, $e[i,j] = e[i,r-1] + e[r+1,j] + w(i,j)$

Step 2: Recursive solution

That equation assumes that we already know which key is k_r. We don't.

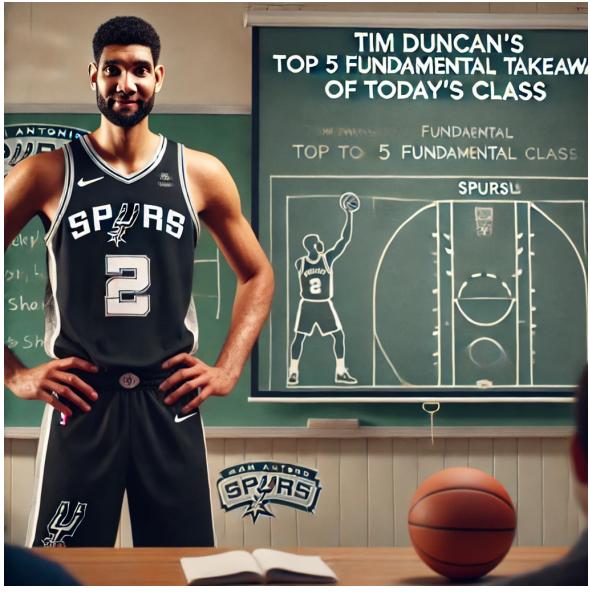
- Try all candidates, and pick the best one:

$$e[i,j] = \begin{cases} 0 & \text{if } j = i-1, \\ \min\{e[i,r-1] + e[r+1,j] + w(i,j) : i \le r \le j\} & \text{if } i \le j. \end{cases}$$

- Could write a recursive algorithm. . .

End of Lecture 5.

TDT5FTOTTC


TIM DUNCAN'S
TOP 5 FUNDAMENTAL TAKEAW
OF TODAY'S CLASS
FUNDAMENTAL
TOP TO 5 FUNDAMENTAL CLASS
SPURS
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2
SPURS

Top 5 Fundamental Takeaways

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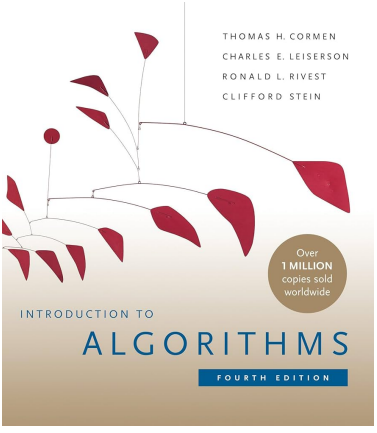
1 Dynamic Programming = recursion + memoization.

Introduction to Algorithms



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Original slides from *Introduction to Algorithms 6.046J/18.401J*, Fall 2005 Class by Prof. Charles Leiserson and Prof. Erik Demaine. MIT OpenCourseWare Initiative available at https://ocw.mit.edu/courses/6-046j-introduction-to-algorithms-sma-5503-fall-2005/.