Introduction to Algorithms
Bonus Lecture: Proof by Induction

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March 6, 2025

Proof by Induction

- A powerful mathematical technique.
- Prove that a statement is true for all natural numbers (or some sequence of numbers).
- It's like knocking over a line of dominoes...

How Induction Works

Principle of Mathematical Induction:

- **Base Case:** Show the statement holds for the first value (usually $n = 1$).
- **Inductive Hypothesis:** Assume the statement holds for some arbitrary $n = k$.
- **Inductive Step:** Prove it holds for $n = k + 1$.

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- **Inductive Step:** Prove it holds for $n = k + 1$.

If those steps hold, the statement is true for all n.

Example

Prove that:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for all $n \geq 1$.

Example

Step 1: Base Case

For $n = 1$:

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

✓**True!**

Example

Step 2: Inductive Hypothesis

Assume that for some $n = k$, the formula holds:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

(This is our assumption or "*inductive hypothesis*".)

Example

Step 3: Inductive Step

We must prove it holds for $n = k + 1$, meaning:

$$1 + 2 + 3 + \cdots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}$$

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Using the inductive hypothesis:

$$\left(\frac{k(k+1)}{2}\right) + (k+1)$$

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Using the inductive hypothesis:

$$\left(\frac{k(k+1)}{2}\right) + (k+1)$$

Factor $k + 1$ out:

$$\frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Example

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$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Using the inductive hypothesis:

$$\left(\frac{k(k+1)}{2}\right) + (k+1)$$

Factor $k + 1$ out:

$$\frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

This matches the formula for $n = k + 1$, so the statement holds!

$\blacksquare$

Conclusion

By induction, the formula is true for all natural numbers n.

Why Does This Work?

Think of induction like climbing an infinite ladder:

- The **base case** puts your foot on the first rung.
- The **inductive hypothesis** and the **inductive step** shows that if you can reach one step, you can reach the next.

Since they are true, you can climb forever!

■[1]

[1] ■ = Q.E.D. which means "*quod erat demonstrandum*".

Example

Proof by Induction:

$$2^n \geq n + 1$$

for all $n \geq 1$.

Step 1: Base Case

For $n = 1$:

$$
\begin{aligned}
2^1 &= 2, && \text{left side} \\
1 + 1 &= 2. && \text{right side}
\end{aligned}
$$

Since $2 \geq 2$, the base case holds. ✓

Step 2: Inductive Hypothesis

Assume the statement is true for $n = k$:

$$2^k \geq k + 1.$$

This assumption is the *inductive hypothesis*.

Step 3: Inductive Step

We need to prove the statement holds for $n = k + 1$:

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Start with the left-hand side:

$$2^{k+1} = 2 \cdot 2^k.$$

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Using the inductive hypothesis $2^k \geq k + 1$:

$$2^{k+1} \geq 2 \cdot (k+1) = 2k + 2.$$

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Using the inductive hypothesis $2^k \geq k + 1$:

$$2^{k+1} \geq 2 \cdot (k + 1) = 2k + 2.$$

Since $2k + 2 \geq k + 2$, the statement holds for $n = k + 1$.

$\blacksquare$

Conclusion

By mathematical induction, we have proven that:

$$2^n \geq n + 1 \quad \text{for all } n \geq 1.$$

Induction helps us prove statements for infinitely many cases!

Another Example

We will prove that the sum of the first n odd numbers is given by:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Step 1: Base Case

For $n = 1$:

$$1 = 1^2.$$

Since both sides are equal, the base case holds. ✓

Step 2: Inductive Hypothesis

Assume the statement is true for $n = k$:

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

The *inductive hypothesis*.

Step 3: Inductive Step

We need to prove the statement holds for $n = k + 1$:

$$1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2.$$

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Using the *inductive hypothesis*:

$$k^2 + (2(k + 1) - 1).$$

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Expanding the term:

$$k^2 + (2k + 1) = (k + 1)^2.$$

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Expanding the term:

$$k^2 + (2k + 1) = (k + 1)^2.$$

Since both sides match, the statement holds for $n = k + 1$.

$\blacksquare$

One More

Prove that:

$$\sum_{i=0}^{n} 3^i = \frac{3^{n+1} - 1}{2}$$

for all non negative integer n.

Step 1: Base Case

For $n = 0$:

$$\begin{aligned}
\sum_{i=0}^{0} 3^i &= \frac{3^{0+1} - 1}{2} \\
3^0 &= \frac{3^1 - 1}{2} \\
1 &= \frac{3 - 1}{2} \\
1 &= \frac{2}{2} \\
1 &= 1
\end{aligned}$$

Since both sides are equal, the base case holds. ✓

Step 1: Base Case

For $n = 1$:

$$\sum_{i=0}^{1} 3^i = \frac{3^{1+1} - 1}{2}$$

$$3^0 + 3^1 = \frac{3^2 - 1}{2}$$

$$1 + 3 = \frac{9 - 1}{2}$$

$$4 = \frac{8}{2}$$

$$4 = 4$$

Since both sides are equal, the base case holds. ✓

Step 2: Inductive Hypothesis

Assume the statement is true for $n = k$:

$$\sum_{i=0}^{k} 3^i = \frac{3^{k+1} - 1}{2}$$

The *inductive hypothesis*.

Step 3: Inductive Step

We need to prove the statement holds for $n = k + 1$:

$$\sum_{i=0}^{k+1} 3^i = \frac{3^{(k+1)+1} - 1}{2}$$

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By $\sum$ definition:

$$\sum_{i=0}^{k} 3^i + 3^{k+1} = \frac{3^{(k+1)+1} - 1}{2}$$

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Using the *inductive hypothesis*:

$$\frac{3^{k+1} - 1}{2} + 3^{k+1} = \frac{3^{(k+1)+1} - 1}{2}$$

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Using the *inductive hypothesis*:

$$\frac{3^{k+1} - 1}{2} + 3^{k+1} = \frac{3^{(k+1)+1} - 1}{2}$$

Expanding the term:

$$\frac{3^{k+1}}{2} - \frac{1}{2} + \frac{2 \cdot 3^{k+1}}{2} = \frac{3^{(k+1)+1} - 1}{2}$$

$$\frac{3^{k+1} - 1 + 2 \cdot 3^{k+1}}{2} = \frac{3^{(k+1)+1} - 1}{2}$$

Step 3: Inductive Step

Using the *inductive hypothesis*:

$$\frac{3^{k+1}-1}{2}+3^{k+1}=\frac{3^{(k+1)+1}-1}{2}$$

Expanding the term:

$$\begin{aligned}
\frac{3^{k+1}}{2}-\frac{1}{2}+\frac{2\cdot 3^{k+1}}{2} &= \frac{3^{(k+1)+1}-1}{2}\\
\frac{3^{k+1}-1+2\cdot 3^{k+1}}{2} &= \frac{3^{(k+1)+1}-1}{2}\\
\frac{3\cdot 3^{k+1}-1}{2} &= \frac{3^{k+1+1}-1}{2}\\
\frac{3^{k+1+1}-1}{2} &= \frac{3^{k+1+1}-1}{2}\\
\frac{3^{k+2}-1}{2} &= \frac{3^{k+2}-1}{2}
\end{aligned}$$

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Expanding the term:

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\frac{3\cdot 3^{k+1}-1}{2} &= \frac{3^{k+1+1}-1}{2}\\
\frac{3^{k+1+1}-1}{2} &= \frac{3^{k+1+1}-1}{2}\\
\frac{3^{k+2}-1}{2} &= \frac{3^{k+2}-1}{2}
\end{aligned}$$

Since both sides match, the statement holds for $n=k+1$.

■

More Examples

Use mathematical induction to show that:

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all non negative integers n.

1. **Basis case:** For $n = 0$, $2^0 = 1 = 2^1 - 1$ ✓

More Examples

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1. **Basis case:** For $n = 0$, $2^0 = 1 = 2^1 - 1$ ✓
2. **Inductive hypothesis:** Let $n = k$, so

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   holds...

More Examples

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3. **Inductive step:** Let's solve for $n = k + 1$,

   $$1 + 2 + 2^2 + \cdots + 2^{k+1} = 2^{(k+1)+1} - 1$$

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$$2 \cdot 2^{k+1} - 1 \stackrel{?}{=} 2^{k+2} - 1$$

$$2^{k+2} - 1 = 2^{k+2} - 1$$

$\blacksquare$

Another One

Prove the following statement by induction:

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

1. **Basis step:** For $n = 1$, $1 = \frac{1 \times 2 \times 3}{6}$ is true!

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   holds...

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   holds...
3. **Inductive step:** Let's solve for $n = k + 1$,

   $$1 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1) \cdot ((k+1)+1) \cdot (2 \cdot (k+1)+1)}{6}$$

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$$(k+1) \cdot (k \cdot (2k+1) + 6(k+1)) \stackrel{?}{=} (k+1) \cdot (k+2) \cdot (2k+3)$$

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$$(k+1) \cdot (2k^2 + 7k + 6) \stackrel{?}{=} (k+1) \cdot (k+2) \cdot (2k+3)$$

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$$(k+1)\cdot(k\cdot(2k+1) + 6(k+1)) \stackrel{?}{=} (k+1)\cdot(k+2)\cdot(2k+3)$$

$$(k+1)\cdot(2k^2+7k+6) \stackrel{?}{=} (k+1)\cdot(k+2)\cdot(2k+3)$$

$$(k+1)\cdot(k+2)\cdot(2k+3) = (k+1)\cdot(k+2)\cdot(2k+3)$$

$\blacksquare$

Theorems

Theorem

Let b be a positive real number and x and y real numbers. Then,

1. $b^{x+y} = b^x \cdot b^y$, *and*
2. $(b^x)^y = b^{x \cdot y}$.

Theorems

Theorem

Let b be a real number greater than 1. Then,

1. $\log_b(xy) = \log_b x + \log_b y$ *whenever x and y are positive real numbers, and*
2. $\log_b(x^y) = y \log_b x$ *whenever x is a positive real number and y is a real number.*

Theorems

Theorem

Let a and b be real numbers greater than 1, and let x be a positive real number. Then,

1. $\log_a x = \frac{\log_b x}{\log_b a}$.

Substitution Method

Example: Solving $T(n) = 2T(\frac{n}{2}) + O(n)$

The **substitution method** is used to solve recurrence relations by guessing a solution and then proving it using mathematical induction.

Let's solve the recurrence:

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

Substitution Method

Step 1: Guess the solution

We assume $T(n) = O(n \log n)$. We will prove this by induction.

Substitution Method

Step 2: Expand the recurrence

Expand the recurrence for a few levels:

$$T(n) = 2T\left(\frac{n}{2}\right) + cn$$

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + c\left(\frac{n}{2}\right)$$

Substituting $T\left(\frac{n}{2}\right)$ into $T(n)$:

$$\begin{aligned} T(n) &= 2\left[2T\left(\frac{n}{4}\right) + c\left(\frac{n}{2}\right)\right] + cn \\ &= 4T\left(\frac{n}{4}\right) + cn + cn \\ &= 4T\left(\frac{n}{4}\right) + 2cn \end{aligned}$$

Continuing this process for k steps:

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + kcn$$

Substitution Method

Step 3: Base Case

The recursion stops when $\frac{n}{2^k} = 1$, so $k = \lg n$. At this point, $T(1) = O(1)$, so:

$$T(n) = 2^{\lg n} T(1) + (\lg n)cn$$

Since $2^{\lg n} = n$, we get:

$$\begin{aligned} T(n) &= nO(1) + cn \lg n \\ &= O(n \lg n) \end{aligned}$$

Substitution Method

Step 4: Inductive Proof

We assume $T(n) \leq dn \lg n$ holds for all smaller values and prove it for n:

$$T(n) = 2T\left(\frac{n}{2}\right) + cn$$

Using the inductive hypothesis:

$$\begin{aligned} T(n) &\leq 2\left[d\left(\frac{n}{2}\right)\lg\left(\frac{n}{2}\right)\right] + cn \\ &= dn\lg\left(\frac{n}{2}\right) + cn \\ &= dn(\lg n - 1) + cn \\ &= dn\lg n - dn + cn \end{aligned}$$

For a sufficiently large d, we can choose $d \geq c$, so:

$$T(n) \leq dn\lg n$$

Thus, $T(n) = O(n \log n)$, which matches our guess.

Substitution Method

Conclusion

By using substitution and induction, we confirmed that:

$$T(n) = O(n \lg n)$$

Another Example

An example of a recurrence relation that **cannot** be solved using the **Master Theorem** but can be solved using the **substitution method** is:

$$T(n) = T(n-1) + O(n)$$

Step 1: Expand the Recurrence

Expanding the recurrence iteratively:

$$\begin{aligned} T(n) &= T(n-1) + O(n) \\ &= (T(n-2) + O(n-1)) + O(n) \\ &= T(n-2) + O(n-1) + O(n) \\ &= T(n-3) + O(n-2) + O(n-1) + O(n) \end{aligned}$$

Repeating this expansion until we reach the base case $T(1)$, we get:

$$T(n) = T(1) + O(2) + O(3) + \cdots + O(n)$$

Step 2: Approximate the Summation

The summation of the first n natural numbers is:

$$\sum_{k=1}^{n} O(k) = O(1 + 2 + 3 + \cdots + n) = O\left(\frac{n(n+1)}{2}\right) = O(n^2)$$

Thus, the recurrence simplifies to:

$$T(n) = O(n^2)$$

Why Can't Master Theorem Be Used?

The **Master Theorem** applies to recurrences of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + O(n^d)$$

where a, b, and d are constants. However, our recurrence $T(n) = T(n-1) + O(n)$ does **not** fit this form because:

- There is **no division** of n (i.e., no factor like $\frac{n}{b}$).
- The recurrence is **not a divide-and-conquer** structure.

Thus, the **Master Theorem does not apply**, and we must use the **substitution method** or an iterative expansion approach.

Final Answer

$$T(n) = O(n^2)$$

This example demonstrates a **linear recurrence** that grows quadratically, and it is a case where the **Master Theorem fails**, but substitution (expanding and summing the terms) works effectively.

Why $T(n) = T(n-1) + O(n)$ is Not Divide & Conquer

Key Differences:

1. **No Division of the Problem Size**
   - In divide & conquer, we break the problem into smaller parts of size $\frac{n}{b}$, where b is usually a constant.
   - Here, we only reduce n by **one** in each step ($n-1$ instead of $\frac{n}{b}$). This means we are reducing the problem by a fixed amount rather than dividing it into subproblems of proportional size.

Why $T(n) = T(n-1) + O(n)$ is Not Divide& Conquer

Key Differences:

2. **Only One Subproblem** ($a = 1$)
   - In divide & conquer, there are typically multiple recursive calls (e.g., **Merge Sort** has two recursive calls, so $a = 2$).
   - Here, there is **only one recursive call** to $T(n-1)$, so it follows a linear recurrence pattern rather than a branching recursive structure.

Why $T(n) = T(n-1) + O(n)$ is Not Divide & Conquer

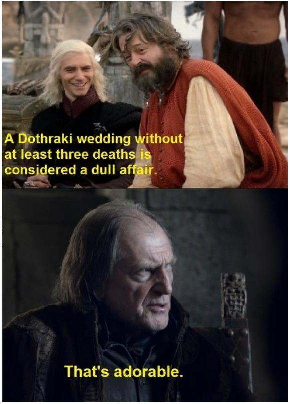
Key Differences:

3 **Linear Reduction Instead of Exponential**

- In divide & conquer, the problem size shrinks **exponentially** (e.g., $\frac{n}{2}$, $\frac{n}{4}$, etc.), leading to logarithmic depth recursion trees.
- Here, the problem size decreases **linearly** $(n-1, n-2, n-3, \dots)$, leading to a **deep recursion tree of depth** $O(n)$.

Conclusion

Since this recurrence follows a **linear** reduction pattern instead of an **exponential** divide & conquer structure, it **does not** fit into the Master Theorem framework, which applies to problems that **divide** into multiple subproblems. Instead, it is better solved using expansion (substitution method) or summation techniques.



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