Introduction to Algorithms
Lecture 5: Dynamic Programming (DP)

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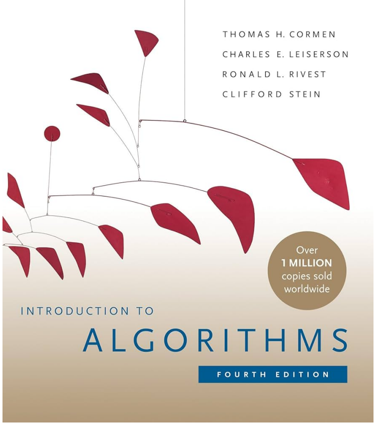
Introduction to Algorithms



Content has been extracted from *Introduction to Algorithms*, Fourth Edition, by Cormen, Leiserson, Rivest, and Stein. MIT Press. 2022.
Visit https://mitpress.mit.edu/9780262046305/introduction-to-algorithms/.
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https://ocw.mit.edu/courses/6-046j-introduction-to-algorithms-sma-5503-fall-2005/.

Plan

Dynamic Programming

N-th Fibonacci Number

Longest Palindromic Sequence

Longest Common Subsequence

Dynamic Programming* (DP)

- Invented by Richard Bellman in 1950s.
- Desing technique, like Divide & Conquer.
- Applies when the subproblems overlap (that is, when subproblems share subsubproblems).
- It solves each subsubproblem just once and then saves its answer in a table.
- DP typically applies to optimization problems:
  - have many possible solutions.
  - find a solution with the optimal (min or max) value.
  - *an* optimal solution, not *the* optimal solution.

* *Programming* in this context refers to a tabular method, not to writing computer code.

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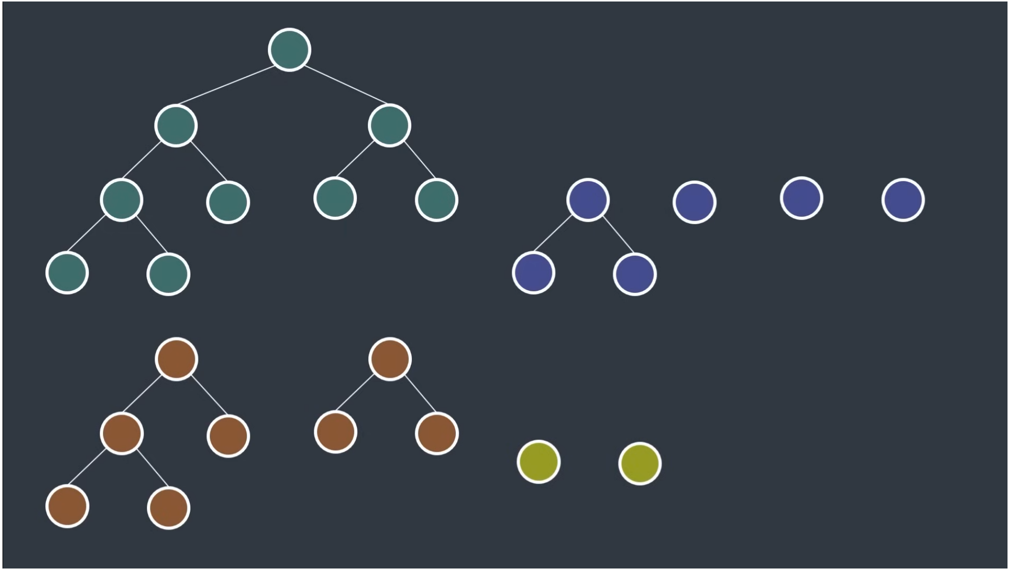
DP = Recursion + Memoization

DP notions



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N-th Fibonacci Number[†]

Write a function that returns the n-th Fibonacci number.

$$F_1 = F_2 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

n	1	2	3	4	5	6	7
F_n	1	1	2	3	5	8	13

[†] *Mastering Dynamic Programming - How to solve any interview problem (Part 1)*. Tech With Nikola Channel, 2024. YouTube, available at https://youtu.be/Hdr64lKQ3e4?si=ycTe-hoyfaICRWXt

Naive Approach

```python
def fib(n):
    if n <= 2:
        result = 1
    else:
        result = fib(n - 1) + fib(n - 2)
    return result
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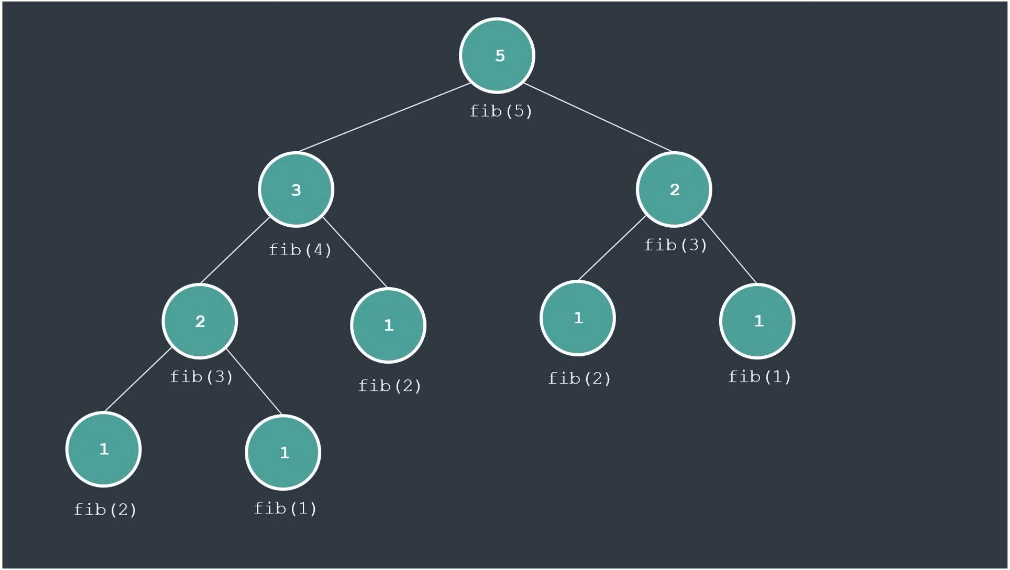
```
print(fib(7))
Output: 13
print(fib(50))
Output: ???
```

Naive Approach



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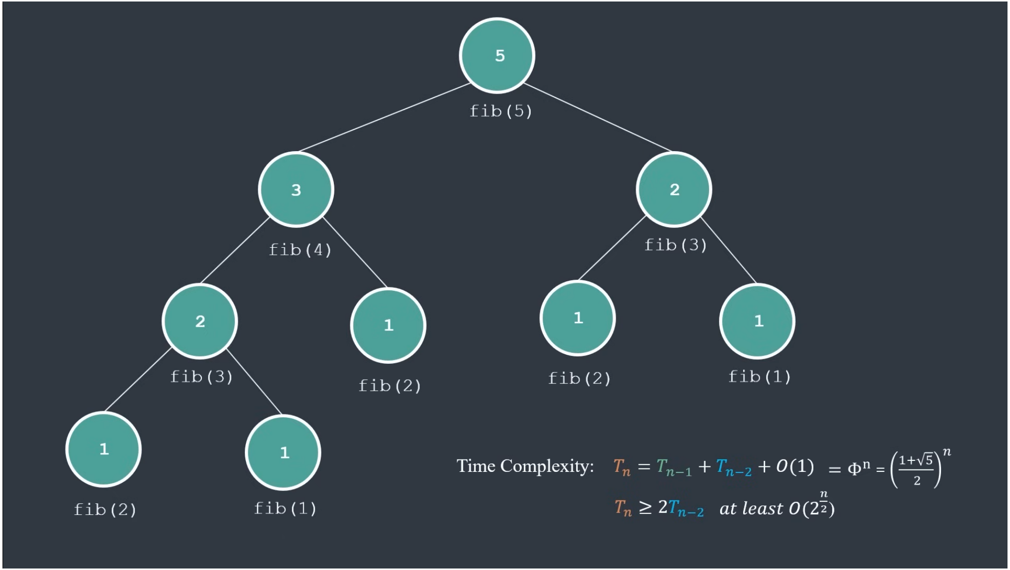

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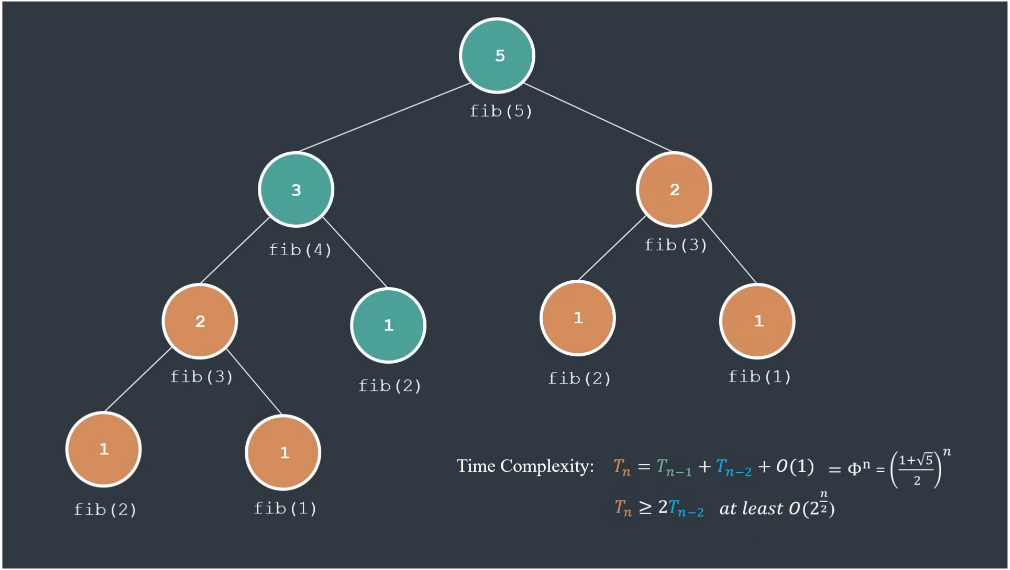

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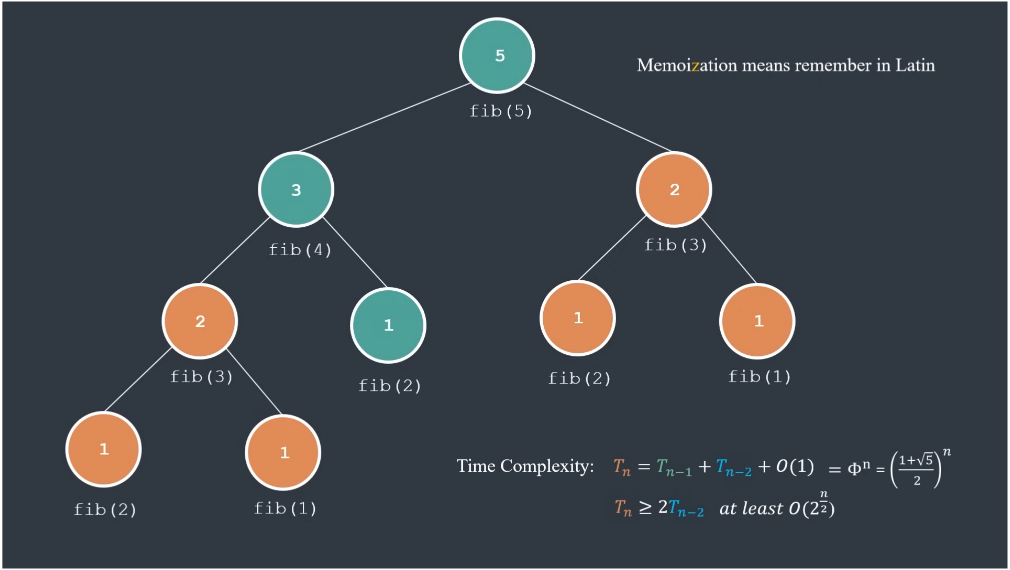

Memoization Approach

```python
memo = {} # adding a dictionary...
def fib(n):
    if n in memo: # asking if n is in dict...
        return memo[n] # if so, we are done!
    if n <= 2:
        result = 1
    else:
        result = fib(n - 1) + fib(n - 2)
    return result
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```
print(fib(7))
Output: 13
print(fib(50))
Output: 12586269025
```

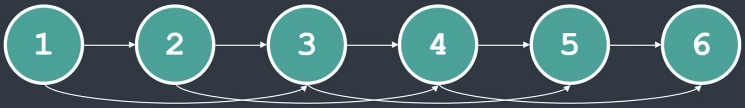
Time Complexity: $O(n)$.

DP = **Recursion** + **Memoization**

Bottom-up Approach

```python
def fib(n):
    memo = {}
    for i in range(1, n + 1):
        if i <= 2:
            result = 1
        else:
            result = memo[n - 1] + memo[n - 2]
        memo[i] = result
    return memo[n]
```

Topological sort order



Plan

Dynamic Programming

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Longest Palindromic Sequence

Longest Common Subsequence

Longest Palindromic Sequence

Definition:

A palindrome is a string that is unchanged when reversed.

- Examples: `radar`, `civic`, `t`, `bb`, `redder`.
- Given: A string $X[1 \ldots n]$, $n \geq 1$.
- To find: Longest palindrome that is a subsequence.
- Example: Given "c h a r a c t e r".
- Output: "c a r a c".
- Answer will be ≥ 1 in length.

Strategy

$L(i, j)$: length of longest palindromic subsequence of $X[i \ldots j]$ for $i \leq j$.

```python
def L(i, j):
    if i == j:
        return 1
    if X[i] == X[j]:
        if i + 1 == j:
            return 2
        else:
            return 2 + L(i + i, j - 1)
    else:
        return max( L(i + 1, j), L(i, j - 1) )
```

Analysis

As written, program can run in exponential time: suppose all symbols $X[i]$ are distinct.

$$\begin{aligned} T(n) &= \text{running time on input of length } n \\ T(n) &= \begin{cases} 1 & n = 1 \\ 2T(n-1) & n > 1 \end{cases} \\ &= 2^{n-1} \end{aligned}$$

What is missing?

- Complexity is exponential... why?

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- There is **recursion** but there is not **Memoization**...

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- Complexity is exponential... why?
- We are still not completing all the DP notions...
- There is **recursion** but there is not **Memoization**...
- Cache is missing!
- There is a single line of code that will fix it...

Understanding the Subproblems

The problem typically involves checking whether a substring of a given string is a palindrome. If the input string has a length of n, then the natural way to define subproblems is:

- Consider all possible substrings $s[i:j]$ of the string.
- For each substring, determine whether it is a palindrome.

Counting the Subproblems

A substring is defined by two indices, i (starting index) and j (ending index), where $0 \leq i \leq j < n$. This means:

- i can take values from 0 to $n - 1$.
- For each i, j can take values from i to $n - 1$.

Total Number of Subproblems

The total number of such (i, j) pairs (i.e., total subproblems) is given by the summation:

$$\sum_{i=0}^{n-1}(n-i) = n + (n-1) + (n-2) + \cdots + 1$$

Using the formula for the sum of the first n natural numbers:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Thus, the number of subproblems is:

$$n^2$$

Formula for Computing the Complexity of a DP

`# of subproblems` $\times$ `time to solve each subproblem`

Given that smaller ones are already solved[‡].

So,

- Given n^2 distinct subproblems...
- By solving each subproblem only once...
- Running time reduces to:

$$\Theta(n^2) \cdot \Theta(1) = \Theta(n^2)$$

[‡] lookup is $\Theta(1)$

New Strategy

- Memoize $L(i, j)$, hash inputs to get output value, and lookup hash table to see if the subproblem is already solved, else recurse.

```python
def L(i, j):
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```

Look at `L[i, j]` and don't recurse if `L[i, j]` is already computed.

Memoizing Vs. Iterating

1. Memoizing uses a dictionary for $L(i, j)$ where value of L is looked up by using i, j as a key. Could just use a 2-D array here where null entries signify that the problem has not yet been solved.
2. Can solve subproblems in order of increasing $j - i$ so smaller ones are solved first.

Plan

- Given two sequences $x[1 \cdots m]$ and $y[1 \cdots n]$, find a[†] longest subsequence common to them both.

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`LCS(x, y)` as notation **not function**...

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Analysis

- Checking $= O(n)$ time per subsequence.
- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).

Worst-case running time $= O(n2^m)$

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Worst-case running time $= O(n2^m)$ **Exponential time!**

Towards a Better Algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
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- **Notation:** Denote the length of a sequence s as $|s|$.

Towards a Better Algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

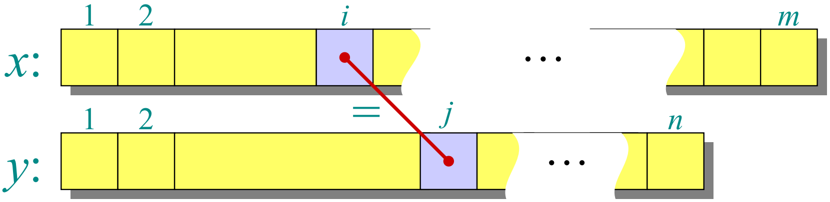
- **Notation:** Denote the length of a sequence s as $|s|$.
- **Strategy:** Consider **prefixes** of x and y:
  - Define $c[i,j] = |LCS(x[1 \cdots i], y[1 \cdots j])|$.
  - Then, $c[m,n] = |LCS(x,y)|$.

Recursive formulation

Theorem

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0, \\ c[i-1, j-1] & \text{if } x[i] = y[j], \\ \max\{c[i, j-1], c[i-1, j]\} & \text{otherwise.} \end{cases}$$

Proof: Case $x[i] = y[j]$ ...



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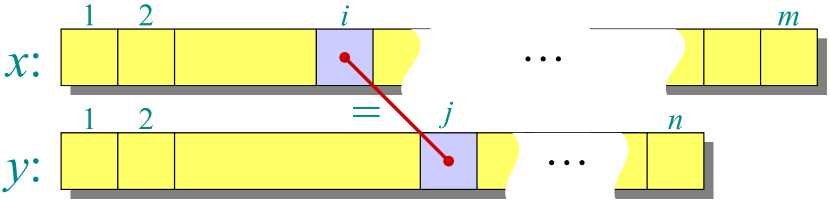
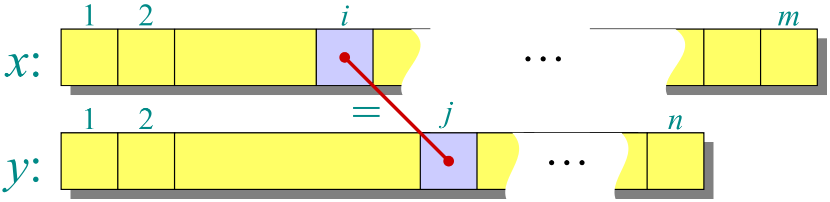


- Let $z[1 \cdots k] = LCS(x[1 \cdots i], y[1 \cdots j])$, where $c[i, j] = k$.

Recursive formulation

Proof: Case $x[i] = y[j] \ldots$



- Let $z[1 \cdots k] = LCS(x[1 \cdots i], y[1 \cdots j])$, where $c[i, j] = k$.
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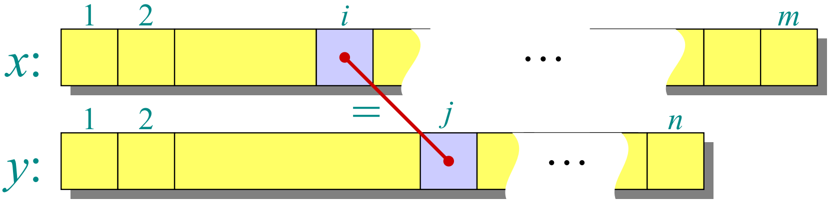

- Let $z[1 \cdots k] = LCS(x[1 \cdots i], y[1 \cdots j])$, where $c[i, j] = k$.
- Then, $z[k] = x[i] (= y[j])$

Recursive formulation

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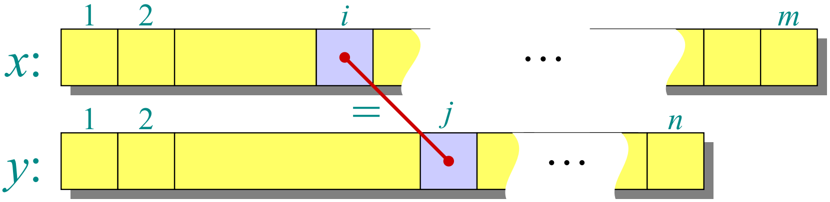

- Let $z[1 \cdots k] = LCS(x[1 \cdots i], y[1 \cdots j])$, where $c[i, j] = k$.
- Then, $z[k] = x[i] (= y[j])$, or else z could be extended.

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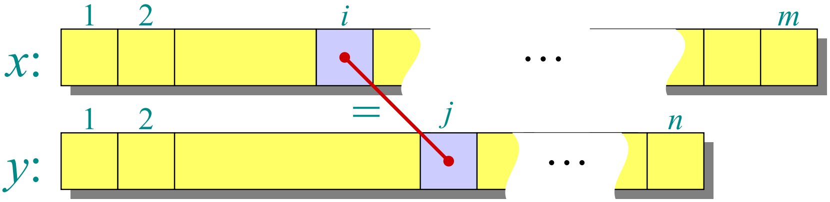

- Let $z[1 \cdots k] = LCS(x[1 \cdots i], y[1 \cdots j])$, where $c[i, j] = k$.
- Then, $z[k] = x[i] (= y[j])$, or else z could be extended.
- Thus, $z[1 \cdots k\text{–}1]$ is CS of $x[1 \cdots i - 1]$ and $y[1 \cdots j - 1]$.

Step-by-Step Example of LCS Algorithm

We will use the dynamic programming (DP) approach to compute the LCS for the following strings:

$$X = \text{``ACDBE''}$$

$$Y = \text{``ABCDE''}$$

Step 1: Create a DP Table

- We construct a $(m+1) \times (n+1)$ table, where m and n are the lengths of X and Y, respectively. The table will store the LCS length at each step.
- **Initial Table (before computation):** We initialize the first row and first column with 0s (LCS of empty strings is 0).

Step 1: Create a DP Table

	$\varnothing$	A	B	C	D	E
$\varnothing$	0	0	0	0	0	0
A	0					
C	0					
D	0					
B	0					
E	0					

Step 2: Fill the DP Table Using Recurrence

We iterate over each character of X and Y and apply the recurrence:

$$L[i,j] = \begin{cases} L[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max\{c[i,j-1], c[i-1,j]\} & \text{otherwise.} \end{cases}$$

	∅	A	B	C	D	E
∅	0	0	0	0	0	0
A	0					
C	0					
D	0					
B	0					
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Step 2: Fill the DP Table Using Recurrence

Filling the table...

$$L[i,j] = \begin{cases} L[i-1,j-1]+1 & \text{if } x[i] = y[j], \\ \max\{c[i,j-1], c[i-1,j]\} & \text{otherwise.} \end{cases}$$

- Compare 'A' ($X[1]$) with each character in Y.
- 'A' matches 'A' $\longrightarrow L[1,1] = L[0,0]+1 = 1$.
- Other cells get the max of left or top.

	∅	A	B	C	D	E
∅	0	0	0	0	0	0
A	0	1	1	1	1	1
C	0					
D	0					
B	0					
E	0					

Step 2: Fill the DP Table Using Recurrence

Filling the table...

$$L[i,j] = \begin{cases} L[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max\{c[i,j-1], c[i-1,j]\} & \text{otherwise.} \end{cases}$$

- Compare 'C' ($X[2]$) with each character in Y.
- 'C' matches 'C' $\longrightarrow L[2,3] = L[1,2]+1 = 2$.
- Other cells get the max of left or top.

	$\varnothing$	A	B	C	D	E
$\varnothing$	0	0	0	0	0	0
A	0	1	1	1	1	1
C	0	1	1	2	2	2
D	0					
B	0					
E	0					

Step 2: Fill the DP Table Using Recurrence

Filling the table...

$$L[i,j] = \begin{cases} L[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max\{c[i,j-1], c[i-1,j]\} & \text{otherwise.} \end{cases}$$

- Compare 'D' ($X[3]$) with each character in Y.
- 'D' matches 'D' $\longrightarrow L[3,4] = L[2,3]+1 = 3$.
- Other cells get the max of left or top.

	∅	A	B	C	D	E
∅	0	0	0	0	0	0
A	0	1	1	1	1	1
C	0	1	1	2	2	2
D	0	1	1	2	3	3
B	0					
E	0					

Step 2: Fill the DP Table Using Recurrence

Filling the table...

$$L[i,j] = \begin{cases} L[i-1,j-1]+1 & \text{if } x[i] = y[j], \\ \max\{c[i,j-1], c[i-1,j]\} & \text{otherwise.} \end{cases}$$

- Compare 'B' ($X[4]$) with each character in Y.
- 'B' matches 'B' $\longrightarrow L[4,2] = L[3,1] + 1 = 2$.
- Other cells get the max of left or top.

	∅	A	B	C	D	E
∅	0	0	0	0	0	0
A	0	1	1	1	1	1
C	0	1	1	2	2	2
D	0	1	1	2	3	3
B	0	1	2	2	3	3
E	0					

Step 2: Fill the DP Table Using Recurrence

Filling the table...

$$L[i,j] = \begin{cases} L[i-1,j-1]+1 & \text{if } x[i] = y[j], \\ \max\{c[i,j-1], c[i-1,j]\} & \text{otherwise.} \end{cases}$$

- Compare 'E' ($X[5]$) with each character in Y.
- 'E' matches 'E' $\longrightarrow L[5,5] = L[4,4] + 1 = 4$.
- Other cells get the max of left or top.

	$\varnothing$	A	B	C	D	E
$\varnothing$	0	0	0	0	0	0
A	0	1	1	1	1	1
C	0	1	1	2	2	2
D	0	1	1	2	3	3
B	0	1	2	2	3	3
E	0	1	2	2	3	4

Step 3: Extract the LCS

- The **LCS** length is in the bottom-right cell, $L[5,5] = 4$.
- To trace back the **LCS**, we start from $L[5,5]$ and follow:
  - If $X[i] = Y[j]$, include it in the **LCS**.
  - Otherwise, move to the maximum value from the left or top.

Tracing Back from L[5,5]:

1. E $(X[5] = Y[5]) \longrightarrow$ Add ‘E’
2. D $(X[3] = Y[4]) \longrightarrow$ Add ‘D’
3. C $(X[2] = Y[3]) \longrightarrow$ Add ‘C’
4. A $(X[1] = Y[1]) \longrightarrow$ Add ‘A’

Thus, the LCS = “ACDE”.

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

- You can solve the LCS problem with a brute-force approach:
  1. Enumerate all subsequences of X.
  2. Check each subsequence to see weather it is also subsequence of Y.
  3. Keep track of the longest subsequence.
- Because X has 2^m possible subsequence, time is exponential and, ergo, impractical.

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

- The LCS problem has an optimal-substructure property.
- the natural classes of subproblems correspond to pairs of *"prefixes"* of the two input sequences:
  - Given the sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$:
  - We can define the *i-th* **prefix** of X, for $i = 0, 1, \ldots, m$, as $X_i = \langle x_1, x_2, \ldots, x_i \rangle$.
  - For instance, if $X = \langle ABCBDAB \rangle$, then $X_4 = \langle ABCB \rangle$ and X_0 in the empty sequence.

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

Theorem 14.1 (Optimal substructure of an LCS)

Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is a LCS of X_{m-1} and Y_{n-1}.

Proof in CLRS page 395.

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

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2. If $x_m \neq y_n$ and $z_k \neq x_m$, then Z is a LCS of X_{m-1} and Y.

Proof in CLRS page 395.

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

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3. If $x_m \neq y_n$ and $z_k \neq y_n$, then Z is a LCS of X and Y_{n-1}.

Proof in CLRS page 395.

Longest Common Subsequence – Formal Steps of DP

Step 1: Characterizing a longest common subsequence

- Theorem 14.1 says that an LCS of two sequences contains within it an LCS of prefixes of the two sequences.
- Thus, the LCS problem has an optimal-substructure property.
- A recursive solution also has the overlapping-subproblems property (as we'll see in a moment).

Longest Common Subsequence – Formal Steps of DP

Step 2: A recursive solution

- To find an LCS of X and Y, you might need to find the LCSs of X and Y_{n-1} and of X_{m-1} and Y (overlapping property).
- Each of these subproblems has the subsubproblem of finding an LCS of X_{m-1} and Y_{n-1}.
- Many other subproblems share subsubproblems.

Longest Common Subsequence – Formal Steps of DP

Step 2: A recursive solution

The optimal substructure of the LCS problem gives the recursive formula:

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0, \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x[i] = y[j], \\ \max\{c[i, j-1], c[i-1, j]\} & \text{if } i, j > 0 \text{ and } x[i] \neq y[j]. \end{cases}$$

Longest Common Subsequence – Formal Steps of DP

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- Define $c[i,j]$ = length of LCS of X_i and Y_j. Want $c[m,n]$.

Longest Common Subsequence – Formal Steps of DP

Step 2: A recursive solution

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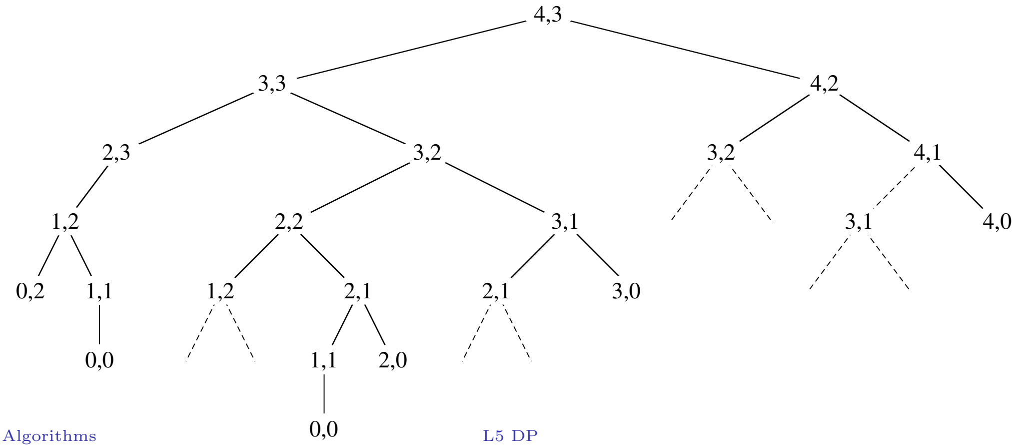
- Define $c[i,j]$ = length of LCS of X_i and Y_j. Want $c[m,n]$.
- Again, could write a recursive algorithm based on this formulation, but...

Longest Common Subsequence – Formal Steps of DP

Step 2: A recursive solution

- Try with $X = \langle a, t, o, m \rangle$ and $Y = \langle a, n, t \rangle$.
- Numbers in nodes are values of i, j in each recursive call.
- Dashed lines indicate subproblems already computed.



Longest Common Subsequence – Formal Steps of DP

Step 3: Computing the length of an LCS

```
LCS-LENGTH(X, Y, m, n)
let b[1:m, 1:n] and c[0:m, 0:n] be new tables
for i = 1 to m
    c[i,0] = 0
for j = 0 to n
    c[0, j] = 0
for i = 1 to m              // compute table entries in row-major order
    for j = 1 to n
        if x_i == y_j
            c[i, j] = c[i − 1, j − 1] + 1
            b[i, j] = "↖"
        elseif c[i − 1, j] ≥ c[i, j − 1]
            c[i, j] = c[i − 1, j]
            b[i, j] = "↑"
        else c[i, j] = c[i, j − 1]
            b[i, j] = "←"
return c and b
```

Longest Common Subsequence – Formal Steps of DP

Step 3: Computing the length of an LCS

```
PRINT-LCS(b, X, i, j)
if i == 0 or j == 0
    return                    // the LCS has length 0
if b[i, j] == "↖"
    PRINT-LCS(b, X, i − 1, j − 1)
    print x_i                 // same as y_j
elseif b[i, j] == "↑"
    PRINT-LCS(b, X, i − 1, j)
else PRINT-LCS(b, X, i, j − 1)
```

Longest Common Subsequence – Formal Steps of DP

Step 3: Computing the length of an LCS

Thomas H. Cormen
Emeritus Professor, Department of Computer Science
New Hampshire State Representative, Grafton 15 (Lebanon Ward 3)
CormenForNH.com

I retired on January 1, 2022. I am no longer taking any new students or interns at any level.

Ph.D., Massachusetts Institute of Technology, 1992.

- **Papers**
- **FG**
- **Other software**
- **Vita**

I taught my final course in Fall 2019. You can view a video of my last lecture, which was not about computer science.

Khan Academy now carries algorithms tutorials for which Devin Balkcom and I produced content.

Are you looking for solutions to exercises and problems in *Introduction to Algorithms*?

If you are, then see the frequently asked question and answer below.

If you request solutions from me, I will not respond.

From July 2004 through June 2008, I was the director of the Dartmouth Institute for Writing and Rhetoric.

I occasionally taught a graduate Computer Science course on how to write papers and how to give talks. I publish a list of usage rules that I required my students to observe.

In 2015, PRI's "The World" ran a story on mentoring women in computer science in which a couple of my students and I were interviewed.

I was interviewed for the *Command Line Heroes* podcast "Learning the BASICs."
You can also hear me spout off in "Philosophical Trials #7."
I talked about writing *Introduction to Algorithms* in an episode of "Frank Stajano Explains."
And an interview with a Brazilian vlogger.

Graduate Student Alumni

- Alex Colvin, Ph.D. 1999
- Peter C. Johnson
- Priya Natarajan, Ph.D. 2011 [Photo of Priya and me at 2012 Dartmouth graduation]
- Geeta Chaudhry Petrovic, Ph.D. 2004 [Photo of Geeta and me at 2004 Dartmouth graduation]
- Elena Riccio Strange (formerly Elena Riccio Davidson), Ph.D. 2006
- Len Wisniewski, Ph.D. 1996
- Melissa Hirschl Chawla, M.S. 1997
- Michael Ringenburg, M.S. 2001
- Georgi Vassilev, M.S. 1994

`https://www.cs.dartmouth.edu/~thc/`

Implementations **here**

Longest Common Subsequence – Formal Steps of DP

Step 3: Computing the length of an LCS

```
Terminator                                   Mar 31 21:42
and@and-Inspiron-5584: ~/MEGA/Work/PUJ/2025-S1/ADA/Resources/clrsPython/clrsPython/Chapter 14
and@and-Inspiron-5584: ~/MEGA/Work/PUJ/2025-S1/ADA/Resources/clrsPython/clrsPython/Chapter 14 146x33

(base) and@and-Inspiron-5584:~/MEGA/Work/PUJ/2025-S1/ADA/Resources/clrsPython/clrsPython/Chapter 14$ ls -laht
total 48K
drwx------  3 and and 4.0K Mar 31 21:31 .
drwx------  2 and and 4.0K Mar 31 17:10 __pycache__
drwx------ 31 and and 4.0K Mar 31 17:10 ..
-rw-------  1 and and 4.6K Oct 11  2021 optimal_BST.py
-rw-------  1 and and 3.4K Oct 11  2021 print_table.py
-rw-------  1 and and 6.5K Oct 11  2021 cut_rod.py
-rw-------  1 and and 4.7K Oct 11  2021 longest_common_subsequence.py
-rw-------  1 and and 7.3K Oct 11  2021 matrix_chain_multiply.py
(base) and@and-Inspiron-5584:~/MEGA/Work/PUJ/2025-S1/ADA/Resources/clrsPython/clrsPython/Chapter 14$ python3 longest_common_subsequence.py
BCBA
0 0 0 0 0 0 0
0 0 0 0 1 1 1
0 1 1 1 1 2 2
0 1 1 2 2 2 2
0 1 1 2 2 3 3
0 1 2 2 2 3 3
0 1 2 2 3 3 4
0 1 2 2 3 4 4
↑ ↑ ↑ ↖ ← ↖
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GTCGTCGGAAGCCGGCCGAA
(base) and@and-Inspiron-5584:~/MEGA/Work/PUJ/2025-S1/ADA/Resources/clrsPython/clrsPython/Chapter 14$
Plain Text  Tab Width: 8  Ln 1, Col 1  INS
```

Longest Common Subsequence – Formal Steps of DP

Step 4: Constructing an LCS

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

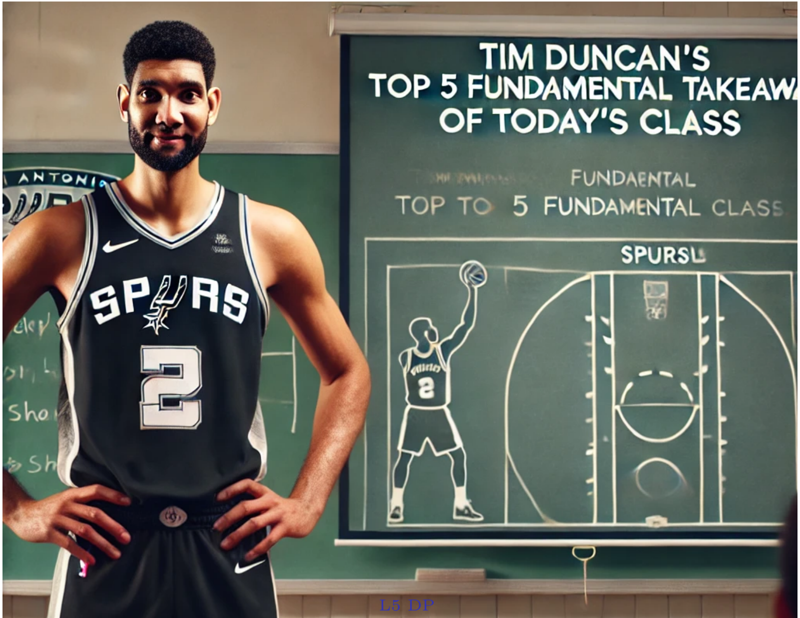
Figure 14.8 The c and b tables computed by LCS-LENGTH on the sequences $X = \langle A, B, C, B, D, A, B \rangle$ and $Y = \langle B, D, C, A, B, A \rangle$. The square in row i and column j contains the value of $c[i, j]$ and the appropriate arrow for the value of $b[i, j]$. The entry 4 in $c[7, 6]$—the lower right-hand corner of the table—is the length of an LCS $\langle B, C, B, A \rangle$ of X and Y. For $i, j > 0$, entry $c[i, j]$ depends only on whether $x_i = y_j$ and the values in entries $c[i-1, j]$, $c[i, j-1]$, and $c[i-1, j-1]$, which are computed before $c[i, j]$. To reconstruct the elements of an LCS, follow the $b[i, j]$ arrows from the lower right-hand corner, as shown by the sequence shaded blue. Each "↖" on the shaded-blue sequence corresponds to an entry (highlighted) for which $x_i = y_j$ is a member of an LCS.

Exercise

Determine an LCS of $\langle 1, 0, 0, 1, 0, 1, 0, 1 \rangle$ and $\langle 0, 1, 0, 1, 1, 0, 1, 1, 0 \rangle$.

End of Lecture 5.

TDT5FTOTTC


TIM DUNCAN'S
TOP 5 FUNDAMENTAL TAKEAW
OF TODAY'S CLASS
FUNDAENTAL
TOP TO 5 FUNDAMENTAL CLASS
SPURSL
SPURS
2

Top 5 Fundamental Takeaways

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5. Computing Fibonacci numbers or finding the longest palindromic subsequence becomes efficient when subproblem results are cached to prevent repeated computation.

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4 The LCS problem uses a table-based approach to find the length and content of the longest common subsequence in quadratic time.

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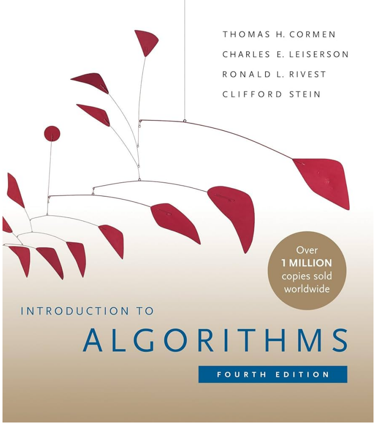
1 Dynamic Programming = recursion + memoization.

Introduction to Algorithms



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