The Zone Theorem

1. Arrangements of Lines
   - an incremental algorithm
   - the zone of a line
   - the zone theorem
2. Levels and Discrepancy
   - counting the number of lines
   - computing the level at vertices
3. Proof of the Zone Theorem
   - induction on the number of lines

MCS 481 Lecture 25
Computational Geometry
Jan Verschelde, 15 March 2019

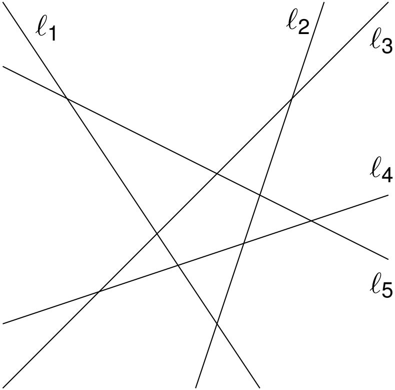
The Zone Theorem

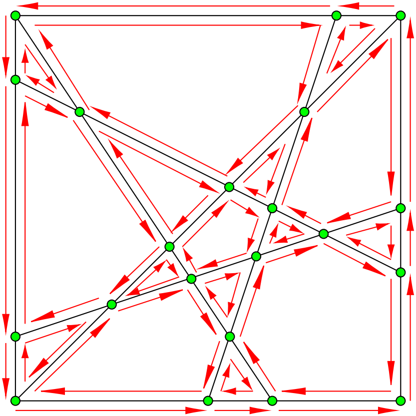
1. Arrangements of Lines
   - an incremental algorithm
   - the zone of a line
   - the zone theorem

2. Levels and Discrepancy
   - counting the number of lines
   - computing the level at vertices

3. Proof of the Zone Theorem
   - induction on the number of lines

five lines in the plane induce a subdivision





The arrangement of lines is stored in a doubly connected edge list, within a bounding box.

an incremental algorithm

Algorithm ConstructArrangement(L)

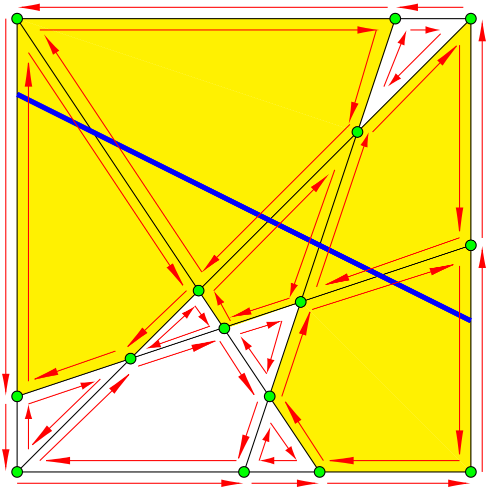
Input: a set L of n lines.
Output: $A(L)$, stored in doubly connected edge list,
within a bounding box $B(L)$.

1. compute $B(L)$ enclosing all vertices of $A(L)$
2. construct a doubly connected edge list $\mathcal{D}$ to store $B(L)$
3. for i from 1 to n do
4.   find the edge e on $B(L)$
       that contains leftmost intersection point of ℓ_i and A_{i-1}
5.   let f be the bounded face incident to e
6.   while f is not outside $B(L)$ do
7.   split f, update $\mathcal{D}$
8.   set f to the next face intersected by ℓ_i

The Zone Theorem

1. Arrangements of Lines
   - an incremental algorithm
   - the zone of a line
   - the zone theorem
2. Levels and Discrepancy
   - counting the number of lines
   - computing the level at vertices
3. Proof of the Zone Theorem
   - induction on the number of lines

the zone of a line – an example



The blue line intersects 5 of the 11 faces.

the zone of a line – definitions

Definition (the closure of a face in a subdivision)

Let f be a face in a subdivision.
The *closure $\overline{f}$ of the face f* is f and all its vertices and edges.

Definition (the zone of a line in an arrangement)

Let $A(L)$ be a line arrangement and ℓ be a line.
The *zone of the line ℓ in the arrangement $A(L)$* is

$$\{\ f \text{ face of } A(L) \mid \overline{f} \cap \ell \neq \emptyset\ \}.$$

Definition (the zone complexity)

Let $A(L)$ be a line arrangment and ℓ be a line.
The *zone complexity of ℓ in $A(L)$* is the sum of the number of vertices, the number of edges, and the number of faces in the zone of ℓ in $A(L)$.

The Zone Theorem

the zone theorem

Theorem (the zone theorem)

Let L be a set of m lines and ℓ be some line.
The zone complexity of a line ℓ in the arrangement $A(L)$ is $O(m)$.

By the zone theorem,
the cost of the incremental algorithm is quadratic.

Theorem (cost of ConstructArrangement)

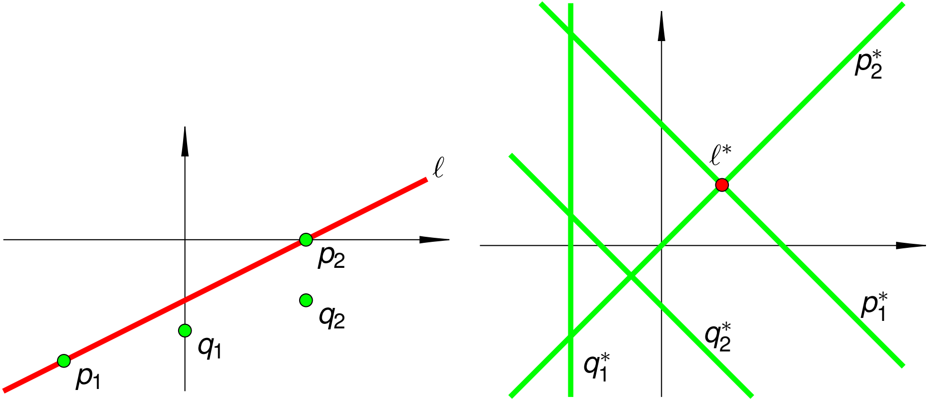
A doubly connected edge list for the arrangement induced by a set of n lines in the plane can be constructed in $O(n^2)$ time.

The Zone Theorem

1. Arrangements of Lines
   - an incremental algorithm
   - the zone of a line
   - the zone theorem
2. Levels and Discrepancy
   - counting the number of lines
   - computing the level at vertices
3. Proof of the Zone Theorem
   - induction on the number of lines

the dual of the discrepancy problem

Given a line ℓ, we want to count all points below ℓ.

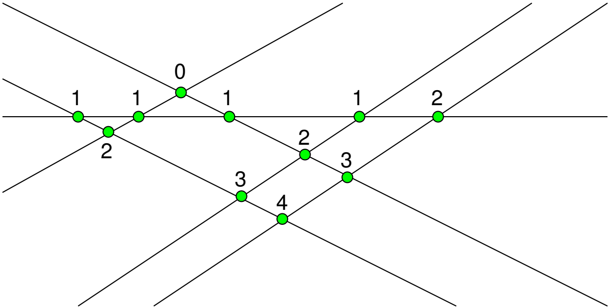


Given the point ℓ^*, count the lines below ℓ^*.

the level of a point in an arrangement

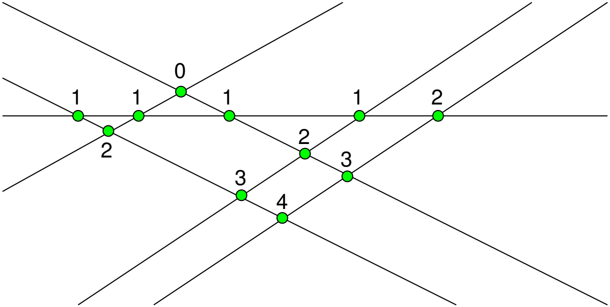
Definition (level of a point in an arrangement)

Given an arrangement $A(L)$ of lines and a point p,
the *level of the point p in $A(L)$* is the number of lines strictly above p.



The Zone Theorem

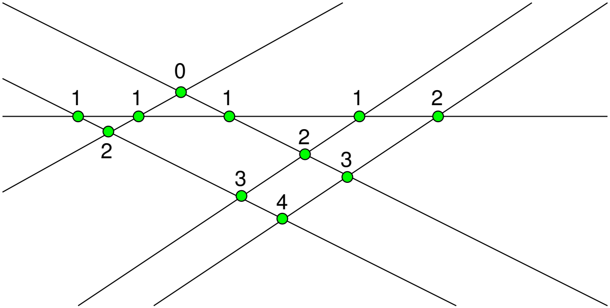
walking a line and computing levels



For any line ℓ do the following:

1. compute the level at the leftmost vertex,
2. while not at the rightmost vertex on ℓ do
3. walk to the next vertex v on ℓ and compute the level of v.

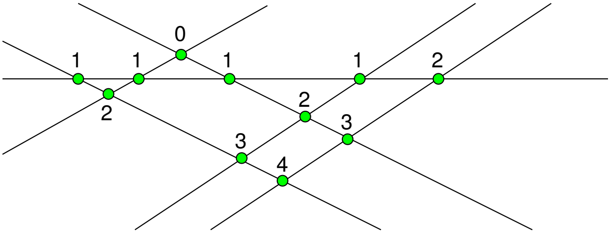
computing the level of the leftmost vertex



In an arrangement of n lines, and a given line ℓ,
computing the level of the leftmost vertex on ℓ runs in $O(n)$.

$\rightarrow$ for the vertex v on ℓ with the smallest x-coordinate,
check all other $n-1$ lines to see whether v lies below.

computing the level of the next vertex



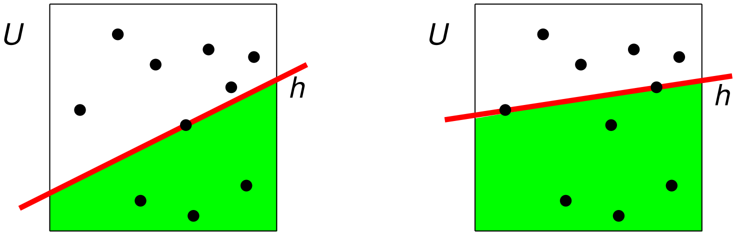
In an arrangement of n lines, and a given line ℓ, computing the level of the next vertex on ℓ also runs in $O(n)$.

$\rightarrow$ in the walk from one vertex to the next, we follow the edges in the doubly connected edge list and update the level as follows:

$$\begin{array}{rl} +1 & \text{if the edge we follow goes down,} \\ -1 & \text{if the edge we follow goes up.} \end{array}$$

computing the discrete measure in quadratic time

The discrete measure of S in U is $\mu_S(h) = \#(S \cap U)/\#S$.



The dual of the sample set S of points is the set of lines S^*.
We count the levels of the vertices in the arrangement $A(S^*)$.

Theorem (cost of half plane discrepancy)

The half plane discrepancy of a set S of n points in the unit square U can be computed in $O(n^2)$ time.

The Zone Theorem

proving the zone theorem

Theorem (the zone theorem)

Let L be a set of m lines and ℓ be some line.
The zone complexity of a line ℓ in the arrangement $A(L)$ is $O(m)$.

Outline of the proof:

- Choose the coordinate system so that ℓ is the x-axis.
- Each edge in $A(L)$ bounds two faces.
  An edge is a *left bounding face* for the face to its *right*.
  An edge is a *right bounding face* for the face to its *left*.
- In the zone of ℓ, the number of left bounding edges $\leq 5m$.

The theorem follows from the last statement.

the number of left bounding edges

Lemma (the number of left bounding edges)

Let L be a set of m lines and ℓ be the x-axis.
In the zone of ℓ in $A(L)$, the number of left bounding edges $\leq 5m$.

The lemma is proven by induction on m.

- The base case: $m = 1$, only one line in L,
  5 is indeed an upper bound to the number of left bounding edges.

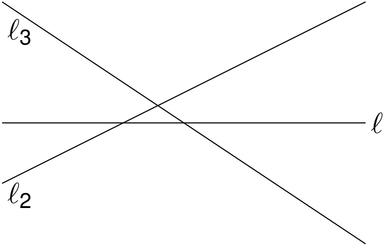
- The general case.
  Let ℓ_1 be the line in L that has the rightmost intersection with ℓ.

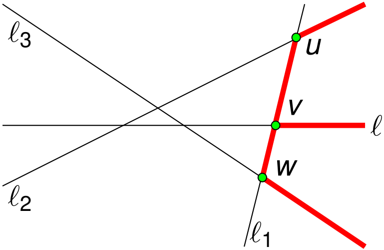
  We apply the induction hypothesis to $A(L \setminus \{\ell_1\})$:
  in $A(L \setminus \{\ell_1\})$, the number of left bounding edges $\leq 5(m-1)$.

  Need to show:
  no more than 5 new left bounding edges when ℓ_1 is added.

the general case

We first assume ℓ_1 intersects ℓ only at one point v:



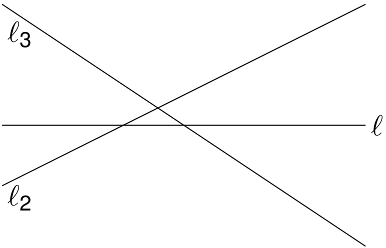


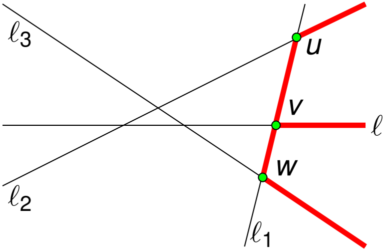
We see 5 new edges.

1. The edge on ℓ_1, spanned by (u, v).
2. The edge on ℓ_1, spanned by (v, w).
3. The edge on ℓ_2, starting at u.
4. The edge on ℓ, starting at v.
5. The edge on ℓ_3, starting at w.

the general case – continued

We first assume ℓ_1 intersects ℓ only at one point v:





The 5 new edges may not the only new edges.
However, other new edges are above the vertex u or below w and therefore do not belong to the zone of ℓ.

outline of the proof continued

We first assumed ℓ_1 intersects ℓ only at one point v, but the degree of the vertex v may be much higher, for example: u and/or w may collide with v.

Exercise 1: Examine the case u collides with v. How many new edges appear in this case?

Exercise 2: Examine the case *u and w* collide with v. How many new edges appear in this case?

recommended assignments

We finished chapter 8 in the textbook.

Consider the following activities, listed below.

1. Write the solutions to exercises 1 and 2.
2. Consult the CGAL documentation and example code on arrangements of lines.
3. Consider the exercises 10, 12, 13 in the textbook.