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COMPUTING 3-DIMENSIONAL GROUPS: CROSSED SQUARES AND CAT^2-GROUPS

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The category **XSq** of crossed squares is equivalent to the category **Cat2** of cat^2-groups. Functions for computing with these structures have been developed in the package XMod written using the GAP computational discrete algebra programming language. This paper includes details of the algorithms used. It contains tables listing the $1,000$ isomorphism classes of cat^2-groups on groups of order at most 30.

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1. Introduction

This paper is concerned with the latest developments in the general programme of "computational higher-dimensional group theory" which forms part of the "higher-dimensional group theory" programme described, for example, by Brown in [8].

The 2-dimensional part of these programmes is concerned with group objects in the categories of groups or groupoids, and these objects may equivalently be considered as crossed modules or cat^1-groups. A summary of the definitions of these objects, with some examples, is contained in §2.

The initial computational part of this programme was described in Alp and Wensley [2]. The output from this work was the package XMod [1] for GAP [16] which, at the time, contained functions for constructing crossed modules and cat^1-groups of groups, and their morphisms, and conversions from one to another. It also contained functions for computing the monoid of derivations of a crossed module, and the equivalent monoid of sections of a cat^1-group. The next development of XMod used the package groupoids [19] to compute crossed modules of groupoids. Later still, a GAP package XModAlg [3] was written to compute cat^1-algebras and crossed modules of algebras, as described in [4].

The 3-dimensional part of the higher-dimensional group theory programme is concerned with objects in the category **XSq** of crossed squares and the equivalent cat^2-groups category **Cat2**. The mathematical basis of these structures is described in §3, and some computational details are included in §4. In §5 we enumerate the $1,000$ isomorphism classes of cat^2-group structures on the 92 groups of order at most 30.

There are many other ways of viewing crossed squares and cat^2-groups. Conduché in [12] defined the equivalent notion of 2-crossed module. Brown and Gilbert in [10] introduced braided, regular crossed modules as an alternative algebraic model of homotopy 3-types. They also proved that these structures are equivalent to simplicial groups with Moore complex of length 2. In [5] Arvasi and Ulualan explore the algebraic relationship between these structures and also the quadratic modules of Baues [6], and the homotopy equivalences between them.

The impetus for the study of higher-dimensional groups comes from algebraic topology [9]. Crossed modules are algebraic models of connected (weak homotopy) 2-types, while crossed squares model connected 3-types. The principal topological example of a crossed module arises from a pointed pair of spaces $A \subseteq X$ where the boundary map is $\partial : \pi_2(X, A) \to \pi_1(A)$. Similarly, given a triad of pointed spaces $A \subseteq X,\ B \subseteq X$ we obtain a crossed square as shown in the left-hand diagram below. A simple case, when X is a 2-sphere and A, B are the upper and lower hemispheres, results in the square on the right. Here F is a free group on one generator x, the boundaries are the trivial and identity homomorphisms, and the crossed pairing is given by $h : F \times F \to F,\ (x^i, x^j) \mapsto x^{ij}$ (see Ellis [13]).

$$\begin{tikzcd}
\pi_3(X; A, B) \arrow[r] \arrow[d] \arrow[dr] & \pi_2(B, A \cap B) \arrow[d] \\
\pi_2(A, A \cap B) \arrow[r] & \pi_1(A \cap B)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F \arrow[r, "0"] \arrow[d, "0"'] \arrow[dr, "0"] & F \arrow[d, "\mathrm{id}"] \\
F \arrow[r, "\mathrm{id}"'] & F
\end{tikzcd}$$

The XMod package follows a purely algebraic approach, and does not compute any specifically topological results. The interested reader may wish to investigate the GAP package HAP [14] which also computes with cat^1-groups.

2. Crossed Modules and Cat^1-Groups

The notion of crossed module, generalizing the notion of a G-module, was introduced by Whitehead [22] in the course of his studies on the algebraic structure of the second relative homotopy group.

A *crossed module* consists of a group homomorphism $\partial : S \to R$, endowed with a left action R on S (written by $(r,s) \to {}^r s$ for $r \in R$ and $s \in S$) satisfying the following conditions:

$$\begin{array}{rcll} \partial({}^r s) &=& r(\partial s)r^{-1} & \forall\ s \in S,\ r \in R; \\ {}^{(\partial s_2)}s_1 &=& s_2 s_1 s_2^{-1} & \forall\ s_1, s_2 \in S. \end{array}$$

The first condition is called the *pre-crossed module property* and the second one the *Peiffer identity*. We will denote such a crossed module by $\mathcal{X} = (\partial : S \to R)$.

A *morphism of crossed modules* $(\sigma, \rho) : \mathcal{X}_1 \to \mathcal{X}_2$, where $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ and $\mathcal{X}_2 = (\partial_2 : S_2 \to R_2)$, consists of two group homomorphisms $\sigma : S_1 \to S_2$ and $\rho : R_1 \to R_2$ such that

$$\partial_2 \circ \sigma \ = \ \rho \circ \partial_1, \quad \text{and} \quad \sigma({}^r s) \ = \ {}^{(\rho r)}\sigma s \qquad \forall s \in S,\ r \in R.$$

Standard constructions for crossed modules include the following.

(1) A *conjugation crossed module* is an inclusion of a normal subgroup $N \trianglelefteq R$, where R acts on N by conjugation.
(2) An *automorphism crossed module* has as range a subgroup R of the automorphism group $\mathrm{Aut}(S)$ of S which contains the inner automorphism group $\mathrm{Inn}(S)$ of S. The boundary maps $s \in S$ to the inner automorphism of S by s.
(3) A *zero boundary crossed module* has a R-module as source and $\partial = 0$.
(4) Any homomorphism $\partial : S \to R$, with S abelian and $\mathrm{im}\,\partial$ in the centre of R, provides a crossed module with R acting trivially on S.
(5) A *central extension crossed module* has as boundary a surjection $\partial : S \to R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$.
(6) The *direct product* of $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ and $\mathcal{X}_2 = (\partial_2 : S_2 \to R_2)$ is $\mathcal{X}_1 \times \mathcal{X}_2 = (\partial_1 \times \partial_2 : S_1 \times S_2 \to R_1 \times R_2)$ with direct product action ${}^{(r_1,r_2)}(s_1,s_2) = ({}^{r_1}s_1, {}^{r_2}s_2)$.

Loday reformulated the notion of crossed module as a cat^1-group. Recall from [18] that a *cat^1-group* is a triple $(G; t, h)$ consisting of a group G with two endomorphisms: the *tail map* t and the *head map* h, having a common image R and satisfying the following axioms.

$$t \circ h = h, \quad h \circ t = t, \quad \text{and} \quad [\ker t, \ker h] = 1. \tag{2.1}$$

When only the first two of these axioms are satisfied, the structure is a *pre-cat^1-group*. It follows immediately that $t \circ t = t$ and $h \circ h = h$. We picture $(G; t, h)$ as

$$G \overset{t,h}{\rightrightarrows} R$$

A *morphism of cat^1-groups* $(G_1; t_1, h_1) \to (G_2; t_2, h_2)$ is a group homomorphism $f : G_1 \to G_2$ such that

$$f \circ t_1 \;=\; t_2 \circ f \quad \text{and} \quad f \circ h_1 \;=\; h_2 \circ f.$$

Crossed modules and cat^1-groups are equivalent two-dimensional generalisations of a group. It was shown in [18, Lemma 2.2] that, on setting $S = \ker t,\ R = \operatorname{im} t$ and $\partial = h|_S$, the conjugation action makes $(\partial : S \to R)$ into a crossed module. Conversely, if $(\partial : S \to R)$ is a crossed module, then setting $G = S \rtimes R$ and defining t, h by $t(s, r) = (1, r)$ and $h(s, r) = (1, (\partial s)r)$ for $s \in S,\ r \in R$, produces a cat^1-group $(G; t, h)$.

3. Crossed Squares and Cat^2-Groups

The notion of a crossed square is due to Guin-Walery and Loday [17]. A *crossed square of groups* $\mathcal{S}$ is a commutative square of groups

$$\mathcal{S} \;=\; \begin{tikzcd} L \arrow[r, "\kappa"] \arrow[d, "\lambda"'] \arrow[dr, "\pi"] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd} \qquad\qquad \tilde{\mathcal{S}} \;=\; \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\kappa"'] \arrow[dr, "\pi"] & N \arrow[d, "\nu"] \\ M \arrow[r, "\mu"'] & P \end{tikzcd} \tag{3.1}$$

together with left actions of P on L, M, N and a *crossed pairing* map $\boxtimes : M \times N \to L$. Then M acts on N and L via P and N acts on M and L via P. The diagram illustrates an *oriented crossed square*, since a choice of where to place M and N has been made. The *transpose* $\tilde{\mathcal{S}}$ of $\mathcal{S}$ is obtained by making the alternative choice. Since crossed pairing identities are similar to those for commutators, the crossed pairing for $\tilde{\mathcal{S}}$ is $\tilde{\boxtimes}$ where $(n \mathbin{\tilde{\boxtimes}} m) = (m \boxtimes n)^{-1}$. Transposition gives an equivalence relation on the set of oriented crossed squares, and a crossed square is an equivalence class.

The structure of an oriented crossed square must satisfy the following axioms for all $l \in L,\ m, m' \in M,\ n, n' \in N$ and $p \in P$.

(1) With the given actions, the homomorphisms $\kappa, \lambda, \mu, \nu$ and $\pi = \mu \circ \kappa = \nu \circ \lambda$ are crossed modules, and both κ, λ are P-equivariant,
(2) $(mm' \boxtimes n) \;=\; ({}^{m}m' \boxtimes {}^{m}n)\,(m \boxtimes n)$
and $\quad (m \boxtimes nn') \;=\; (m \boxtimes n)\,({}^{n}m \boxtimes {}^{n}n')$,
(3) $\kappa(m \boxtimes n) \;=\; m({}^{n}m^{-1}) \quad$ and $\quad \lambda(m \boxtimes n) \;=\; ({}^{m}n)n^{-1}$,
(4) $(\kappa l \boxtimes n) \;=\; l({}^{n}l^{-1}) \quad$ and $\quad (m \boxtimes \lambda l) \;=\; ({}^{m}l)l^{-1}$,
(5) ${}^{p}(m \boxtimes n) \;=\; ({}^{p}m \boxtimes {}^{p}n)$.

Note that axiom 1. implies that $(\mathrm{id}\,, \mu), (\mathrm{id}\,, \nu), (\kappa, \mathrm{id}\,)$ and $(\lambda, \mathrm{id}\,)$ are morphisms of crossed modules.

Standard constructions for crossed squares include the following.

(1) If M, N are normal subgroups of the group P then the diagram of inclusions

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & P \end{tikzcd}$$

together with the actions of P on M, N and $M \cap N$ given by conjugation, and the commutator map

$$[\ ,\ ] \ : \ M \times N \to M \cap N, \quad (m,n) \mapsto [m,n] \ = \ mnm^{-1}n^{-1},$$

is a crossed square. We call this an *inclusion crossed square.*

(2) The diagram

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & \operatorname{Inn} M \arrow[d, "\iota"] \\ \operatorname{Inn} M \arrow[r, "\iota"'] & \operatorname{Aut} M \end{tikzcd}$$

is a crossed square, where α maps $m \in M$ to the inner automorphism

$$\beta_m : M \to M, \quad m' \mapsto mm'm^{-1},$$

and where ι is the inclusion of $\operatorname{Inn} M$ in $\operatorname{Aut} M$; the actions are standard; and the crossed pairing is

$$\boxtimes \ : \ \operatorname{Inn} M \times \operatorname{Inn} M \to M, \quad (\beta_m, \beta_{m'}) \ \mapsto \ [m, m'] \ .$$

(3) If P is a group and M, N are ordinary P-modules, and if A is an arbitrary abelian group on which P is assumed to act trivially, then there is a crossed square

$$\begin{tikzcd} A \arrow[r, "0"] \arrow[d, "0"'] & M \arrow[d, "0"] \\ N \arrow[r, "0"'] & P \end{tikzcd}$$

(4) Given two crossed modules, $(\mu : M \to P)$ and $(\nu : N \to P)$, there is a universal crossed square

$$\begin{tikzcd} M \otimes N \arrow[r, "\kappa"] \arrow[d, "\lambda"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

where $M \otimes N$ is constructed using the nonabelian tensor product of groups [11].

(5) The *direct product* of crossed squares $\mathcal{S}_1, \mathcal{S}_2$ is

$$\begin{tikzcd} L_1 \times L_2 \arrow[r, "\kappa_1 \times \kappa_2"] \arrow[d, "\lambda_1 \times \lambda_2"'] & M_1 \times M_2 \arrow[d, "\mu_1 \times \mu_2"] \\ N_1 \times N_2 \arrow[r, "\nu_1 \times \nu_2"'] & P_1 \times P_2 \end{tikzcd}$$

with actions

$${}^{(p_1,p_2)}(l_1, l_2) = ({}^{p_1}l_1, {}^{p_2}l_2)\,, \quad {}^{(p_1,p_2)}(m_1, m_2) = ({}^{p_1}m_1, {}^{p_2}m_2)\,, \quad {}^{(p_1,p_2)}(n_1, n_2) = ({}^{p_1}n_1, {}^{p_2}n_2)\,,$$

and crossed pairing

$$\boxtimes\,((m_1, m_2), (n_1, n_2)) \;=\; (\boxtimes_1(m_1, n_1), \boxtimes_2(m_2, n_2))\,.$$

The crossed square $\mathcal{S}$ in (3.1) can be thought of as a horizontal or vertical crossed module of crossed modules:

$$\begin{tikzcd} L \arrow[d, "\lambda"'] & M \arrow[d, "\mu"] \\ N \arrow[r, phantom, "\xrightarrow{(\kappa,\nu)}"] & P \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, "\kappa"] \arrow[d, phantom, "\downarrow(\lambda,\mu)"] & M \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

where (κ, ν) is the boundary of the crossed module with domain $(\lambda : L \to N)$ and codomain $(\mu : M \to P)$. (See also section 9.2 of [21].)

There is an evident notion of morphism of crossed squares which preserves all the structure, so that we obtain a category **XSq**, the category of crossed squares.

Although, when first introduced by Loday and Walery [17], the notion of crossed square of groups was not linked to that of cat^2-groups, it was in this form that Loday gave their generalisation to an n-fold structure, cat^n-groups (see [18]). When $n = 1$ this is the notion of cat^1-group given earlier.

When $n = 2$ we obtain a cat^2-group. Again we have a group G, but this time with two *independent* cat^1-group structures on it. So a *cat^2-group* is a 5-tuple, $(G; t_1, h_1; t_2, h_2)$, where $(G; t_i, h_i),\ i = 1, 2$ are cat^1-groups, and

$$t_1 \circ t_2 \;=\; t_2 \circ t_1, \quad h_1 \circ h_2 \;=\; h_2 \circ h_1, \quad t_1 \circ h_2 \;=\; h_2 \circ t_1, \quad t_2 \circ h_1 \;=\; h_1 \circ t_2.$$

To emphasise the relationship with crossed squares we may illustrate an *oriented* cat^2-group by the diagram

$$
\begin{tikzcd}
G \arrow[r, shift left, "{t_1,h_1}"] \arrow[r, shift right] \arrow[d, shift right, "{t_2,h_2}"'] \arrow[d, shift left] \arrow[dr, shift left, "t_1\circ t_2"] \arrow[dr, shift right, "h_1\circ h_2"'] & R_1 \arrow[d, shift left, "{t_2,h_2}"] \arrow[d, shift right] \\
R_2 \arrow[r, shift left] \arrow[r, shift right, "{t_1,h_1}"'] & R_{12}
\end{tikzcd}
\tag{3.2}
$$

where R_{12} is the image of $t_1 \circ t_2 = t_2 \circ t_1$.

A morphism of cat^2-groups is a triple (γ, ρ_1, ρ_2), as shown in the diagram

$$
\begin{tikzcd}
R_1 \arrow[d, "\rho_1"'] & G \arrow[l, shift right, "{t_1,h_1}"'] \arrow[l, shift left] \arrow[r, shift left, "{t_2,h_2}"] \arrow[r, shift right] \arrow[d, "\gamma"] & R_2 \arrow[d, "\rho_2"] \\
R_1' & G' \arrow[l, shift right] \arrow[l, shift left, "{t_1',h_1'}"] \arrow[r, shift left] \arrow[r, shift right, "{t_2',h_2'}"'] & R_2'
\end{tikzcd}
$$

where $\gamma : G \to G'$, $\rho_1 = \gamma|_{R_1}$ and $\rho_2 = \gamma|_{R_2}$ are homomorphisms satisfying:

$$\rho_1 \circ t_1 = t_1' \circ \gamma, \qquad \rho_1 \circ h_1 = h_1' \circ \gamma, \qquad \rho_2 \circ t_2 = t_2' \circ \gamma, \qquad \rho_2 \circ h_2 = h_2' \circ \gamma.$$

We thus obtain a category **Cat2**, the category of cat^2-groups.

Notice that, unlike the situation with crossed squares where the diagonal is a crossed module, it is *not* required that the diagonal in (3.2) is a cat^1-group – it may just be a pre-cat^1-group. The simplest example of this situation is described in Example 3.1 below.

Loday, in [18] proved that there is an equivalence of categories between the category **Cat2** and the category **XSq**. We now consider the sketch proof of this result (see also [20]).

The cat^2-group $(G; t_1, h_1; t_2, h_2)$ determines a diagram of homomorphisms

$$
\begin{tikzcd}
\ker t_1 \cap \ker t_2 \arrow[r, "{(\partial_1, \mathrm{id})}"] \arrow[d, "{(\mathrm{id}, \partial_2)}"'] & \operatorname{im} t_1 \cap \ker t_2 \arrow[d, "{(\mathrm{id}, \partial_2)}"] \\
\ker t_1 \cap \operatorname{im} t_2 \arrow[r, "{(\partial_1, \mathrm{id})}"'] & \operatorname{im} t_1 \cap \operatorname{im} t_2
\end{tikzcd}
\tag{3.3}
$$

where each morphism is a crossed module for the natural action, conjugation in G. The required crossed pairing is given by the commutator in G since, if $x \in \operatorname{im} t_1 \cap \ker t_2$ and $y \in \ker t_1 \cap \operatorname{im} t_2$ then $[x, y] \in \ker t_1 \cap \ker t_2$. It is routine to check the crossed square axioms.

Conversely, if

$$\begin{tikzcd} L \arrow[r, "\kappa"] \arrow[d, "\lambda"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

is a crossed square, then we consider it as a morphism of crossed modules $(\kappa, \nu) : (\lambda : L \to N) \to (\mu : M \to P)$. Using the equivalence between crossed modules and cat^1-groups this gives a morphism

$$\partial : (L \rtimes N, t, h) \longrightarrow (M \rtimes P, t', h')$$

of cat^1-groups. There is an action of $(m, p) \in M \rtimes P$ on $(l, n) \in L \rtimes N$ given by

$$^{(m,p)}(l, n) \;=\; (^{m}(^{p}l)(m \boxtimes {}^{p}n),\; {}^{p}n)\,.$$

Using this action, we form its associated cat^2-group with source $(L \rtimes N) \rtimes (M \rtimes P)$ and induced endomorphisms t_1, h_1, t_2, h_2.

Example 3.1. Let $D_8 = \langle a, b \mid a^2, b^2, (ab)^4 \rangle$ be the dihedral group of order 8, and let $c = [a, b] = (ab)^2$ so that $a^b = ac$ and $b^a = bc$. (The standard permutation representation is given by $a = (1, 2)(3, 4), b = (1, 3), ab = (1, 2, 3, 4), c = (1, 3)(2, 4)$.)

Define $t_a, t_b : D_8 \to D_8$ by $t_a : [a, b] \mapsto [a, 1]$ and $t_b : [a, b] \mapsto [1, b]$. Construct cat^1-groups $C_a = (D_8, t_a, t_a)$ and $C_b = (D_8, t_b, t_b)$. Diagrams (3.2) and (3.3) become

$$\begin{tikzcd} D_8 \arrow[r, shift left, "t_a"] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right, "t_b"'] \arrow[dr, shift left, "t"] \arrow[dr, shift right] & A \arrow[d, shift left, "t_b"] \arrow[d, shift right] \\ B \arrow[r, shift left] \arrow[r, shift right, "t_a"'] & I \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "c \mapsto 1"] \arrow[d, "c \mapsto 1"'] & A \arrow[d, "a \mapsto 1"] \\ B \arrow[r, "b \mapsto 1"'] & I \end{tikzcd}$$

where $A = \langle a \rangle$, $B = \langle b \rangle$, $C = \langle c \rangle$ and I is the trivial group. The crossed pairing is given by $\boxtimes(a, b) = c$. The diagonal map $t = t_a \circ t_b$ has kernel D_8, and $[\ker t, \ker t] = C$, so the diagonal is *not* a cat^1-group.

Definition 3.2. *A* cat^n-group *consists of a group G with n independent cat^1-group structures $(G; t_i, h_i)$, $1 \leq i \leq n$, such that for all $i \neq j$*

$$t_i t_j = t_j t_i, \quad h_i h_j = h_j h_i \quad \textit{and} \quad t_i h_j = h_j t_i.$$

A generalisation of crossed square to higher dimensions was given by Ellis and Stenier (cf. [15]). It is called a "crossed n-cube". We only use this construction for $n = 2$.

4. Computer Implementation

GAP [16] is an open-source system for discrete computational algebra. The system consists of a library of implementations of mathematical structures: groups, vector spaces, modules, algebras, graphs, codes, designs, etc.; plus databases of groups of small order, character tables, etc. The system has world-wide usage in the area of education and scientific research. GAP is free software and user contributions to the system are supported. These contributions are organized in a form of GAP packages and are distributed together with the system. Contributors can submit additional packages for inclusion after a reviewing process.

The Small Groups library by Besche, Eick and O'Brien in [7] provides access to descriptions of the groups of small order. The groups are listed up to isomorphism. The library contains all groups of order at most 2000 except 1024.

4.1. *2-Dimensional Groups*

The XMod package for GAP contains functions for computing with crossed modules, cat^1-groups and their morphisms, and was first described in [1]. A more general notion of cat^1-group is implemented in XMod, where the tail and head maps are no longer required to be endomorphisms on G. Instead it is required that t and h have a common image R, and an *embedding* $e : R \to G$ is added. The axioms in (2.1) then become:

$$t \circ e \circ h = h, \quad h \circ e \circ t = t, \quad \text{and} \quad [\ker t, \ker h] = 1,$$

and again it follows that $t \circ e \circ t = t$ and $h \circ e \circ h = h$. We denote such a cat^1-group by $(e; t, h : G \to R)$.

This package may be used to select a cat^1-group from a data file. All cat^1-structures on groups of size up to 70 (ordered according to the GAP numbering of small groups) are stored in a list in the file `cat1data.g`. The function **Cat1Select** may be used in three ways. **Cat1Select(size)** returns the names of the groups with this size, while **Cat1Select(size, gpnum)** prints a list of cat^{1}1-structures for this chosen group. **Cat1Select(size, gpnum, num)** returns the chosen cat^{1}1-group. **XModOfCat1Group** produces the associated crossed module.

The following GAP session illustrates the use of these functions.

```
gap> Cat1Select( 12 );
Usage:  Cat1Select( size, gpnum, num );
[ "C3 : C4", "C12", "A4", "D12", "C6 x C2" ]
gap> Cat1Select( 12, 3 );
Usage:  Cat1Select( size, gpnum, num );
There are 2 cat1-structures for the group A4.
Using small generating set [ f1, f2 ] for source of homs.
[ [range gens], [tail genimages], [head genimages] ] :-
(1)  [ [ f1 ], [ f1, <identity> of ... ], [ f1, <identity> of ... ] ]
(2)  [ [ f1, f2 ],  tail = head = identity mapping ]
2
```

```
gap> C1 := Cat1Select( 12, 3, 2 );
[A4=>A4]
gap> X1 := XModOfCat1Group( C1 );
[triv->A4]
```

4.2. *3-dimensional Groups*

We have developed new functions for XMod which construct (pre-)cat^2-groups, (pre-)cat^3-groups, and their morphisms. Functions for (pre-)cat^2-groups include **PreCat2Group**, **Cat2Group**, **PreCat2GroupByPreCat1Groups**, **IsPreCat2Group** and **IsCat2Group**. Attributes of a (pre)cat^2-group constructed in this way include **GeneratingCat1Groups**, **Size**, **Name** and **Edge2DimensionalGroup** where '**Edge**' is one of {**Up, Left, Right, Down, Diagonal**}.

As with cat^1-groups, we use a more general notion for cat^2-groups. An *oriented cat^2-group* has the form

$$\begin{tikzcd}
G \arrow[rr, shift left, "t_1{,}h_1"] \arrow[dd, shift right, "t_2{,}h_2"'] \arrow[ddrr, shift left, "e_2\circ t_2\circ e_1\circ t_1{,}\ e_2\circ h_2\circ e_1\circ h_1"] & & R_1 \arrow[ll, shift left, "e_1"] \arrow[dd, shift left, "e_2\circ t_2\circ e_1 \ e_2\circ h_2\circ e_1"] \\
 & & \\
R_2 \arrow[uu, shift right, "e_2"'] \arrow[rr, shift left, "e_1\circ t_1\circ e_2{,}\ e_1\circ h_1\circ e_2"] & & R_{12} \arrow[ll, shift left, "t_1"] \arrow[uu, shift left, "t_2"] \arrow[uull, shift left, "\mathrm{inc}"]
\end{tikzcd}$$

where R_1, R_2 need not be subgroups of G, but R_{12} is taken to be the common image of $e_2 \circ t_2 \circ e_1 \circ t_1$ and $e_1 \circ t_1 \circ e_2 \circ t_2$, a subgroup of G.

Generalizing these functions, we have introduced **Cat3Group** and **HigherDimension** which construct cat^3-groups. Functions for cat^n-groups of higher dimension will be added as time permits. An orientation of a cat^3-group on G displays a cube whose six faces (ordered as front; up, left, right, down, back) are cat^2-groups. The group G is positioned where the front, up and left faces meet. The following GAP session illustrates the use of these functions. Notice that the cat^2-group `C2ab` is the second example with a diagonal which is only a pre-cat^1-group.

```
gap> a := (1,2,3,4)(5,6,7,8);;
gap> b := (1,5)(2,6)(3,7)(4,8);;
gap> c := (2,6)(4,8);;
gap> G := Group( a, b, c );;
gap> SetName( G, "c4c2:c2" );
gap> t1a := GroupHomomorphismByImages( G, G, [a,b,c], [(),(),c] );;
gap> C1a := PreCat1GroupByEndomorphisms( t1a, t1a );;
gap> t1b := GroupHomomorphismByImages( G, G, [a,b,c], [a,(),()] );;
gap> C1b := PreCat1GroupByEndomorphisms( t1b, t1b );;
gap> C2ab := Cat2Group( C1a, C1b );
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [c4c2:c2 => Group( [ (), (), (2,6)(4,8) ] )]
2 : [c4c2:c2 => Group( [ (1,2,3,4)(5,6,7,8), (), () ] )]
gap> IsCat2Group( C2ab );
```

```
true
gap> Size( C2ab );
[ 16, 2, 4, 1 ]
gap> IsCat1Group( Diagonal2DimensionalGroup( C2ab ) );
false
gap> t1c := GroupHomomorphismByImages( G, G, [a,b,c], [a,b,c] );;
gap> C1c := PreCat1GroupByEndomorphisms( t1c, t1c );;
gap> C3abc := Cat3Group( C1a, C1b, C1c );
(pre-)cat3-group with generating (pre-)cat1-groups:
1 : [c4c2:c2 => Group( [ (), (), (2,6)(4,8) ] )]
2 : [c4c2:c2 => Group( [ (1,2,3,4)(5,6,7,8), (), () ] )]
3 : [c4c2:c2 => Group( [ (1,2,3,4)(5,6,7,8), (1,5)(2,6)(3,7)(4,8),
(2,6)(4,8) ] )]
gap> IsPreCat3Group( C3abc );
true
gap> HigherDimension( C3abc );
4
gap> Front3DimensionalGroup( C3abc );
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [c4c2:c2 => Group( [ (), (), (2,6)(4,8) ] )]
2 : [c4c2:c2 => Group( [ (1,2,3,4)(5,6,7,8), (), () ] )]
```

4.3. *Morphisms of 3-Dimensional Groups*

The function **MakeHigherDimensionalGroupMorphism** defines morphisms of higher dimensional groups, such as cat^2-groups and crossed squares. In the cat^2-group case these include **Cat2GroupMorphismByCat1GroupMorphisms**, **Cat2GroupMorphism** and **IsCat2GroupMorphism**. The function **AllCat2GroupMorphisms** is used to find all morphisms between two cat^2-groups.

In the following GAP session, we obtain a cat^2-group morphism using these functions.

```
gap> C2_82 := Cat2Group( Cat1Group(8,2,1), Cat1Group(8,2,2) );
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [C4 x C2 => Group( [ <identity> of ..., <identity> of ...,
<identity> of ... ] )]
2 : [C4 x C2 => Group( [ <identity> of ..., f2 ] )]
gap> C2_83 := Cat2Group( Cat1Group(8,3,2), Cat1Group(8,3,3) );
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [D8 => Group( [ f1, f1 ] )]
2 : [D8=>D8]
gap> up1 := GeneratingCat1Groups( C2_82 )[1];;
gap> lt1 := GeneratingCat1Groups( C2_82 )[2];;
gap> up2 := GeneratingCat1Groups( C2_83 )[1];;
gap> lt2 := GeneratingCat1Groups( C2_83 )[2];;
gap> G1 := Source( up1 );; R1 := Range( up1 );; Q1 := Range( lt1 );;
gap> G2 := Source( up2 );; R2 := Range( up2 );; Q2 := Range( lt2 );;
gap> homG := AllHomomorphisms( G1, G2 );;
gap> homR := AllHomomorphisms( R1, R2 );;
gap> homQ := AllHomomorphisms( Q1, Q2 );;
gap> upmor := Cat1GroupMorphism( up1, up2, homG[1], homR[1] );;
gap> IsCat1GroupMorphism( upmor );
```

```
true
gap> ltmor := Cat1GroupMorphism( lt1, lt2, homG[1], homQ[1] );;
gap> mor2 := PreCat2GroupMorphism( C2_82, C2_83, upmor, ltmor );
<mapping: (pre-)cat2-group with generating (pre-)cat1-groups:
1 : [C4 x C2 => Group( [ <identity> of ..., <identity> of ...,
  <identity> of ... ] )]
2 : [C4 x C2 => Group( [ <identity> of ..., f2 ] )] -> (pre-)cat
2-group with generating (pre-)cat1-groups:
1 : [D8 => Group( [ f1, f1 ] )]
2 : [D8=>D8] >
gap> IsCat2GroupMorphism( mor2 );
true
gap> mor8283 := AllCat2GroupMorphisms( C2_82, C2_83 );;
gap> Length( mor8283 );
2
```

4.4. *Natural Equivalence*

The equivalence between categories **XSq** and **Cat2** is implemented by the functions **CrossedSquareOfCat2Group** and **Cat2GroupOfCrossedSquare** which construct crossed squares and cat^2-groups from the given cat^2-groups and crossed squares, respectively.

The following GAP session illustrates the use of these functions. The dihedral group D_{20} has two normal subgroups D_{10} whose intersection is the cyclic C_5. We construct the crossed square of normal subgroups, and then use the conversion functions to obtain the associated cat^2-group. We then obtain the crossed square `Xab` associated to the cat^2-group `C2ab` obtained earlier.

```
gap> d20 := DihedralGroup( IsPermGroup, 20 );;
gap> gend20 := GeneratorsOfGroup( d20 );
[ (1,2,3,4,5,6,7,8,9,10), (2,10)(3,9)(4,8)(5,7) ]
gap> p1 := gend20[1];;  p2 := gend20[2];;  p12 := p1*p2;
(1,10)(2,9)(3,8)(4,7)(5,6)
gap> d10a := Subgroup( d20, [ p1\^2, p2 ] );;
gap> d10b := Subgroup( d20, [ p1\^2, p12 ] );;
gap> c5d := Subgroup( d20, [ p1\^2 ] );;
gap> SetName( d20, "d20" );  SetName( d10a, "d10a" );
gap> SetName( d10b, "d10b" );  SetName( c5d, "c5d" );
gap> XS1 := CrossedSquareByNormalSubgroups( c5d, d10a, d10b, d20 );
[  c5d -> d10a ]
[   |     |    ]
[ d10b -> d20  ]
gap> IsCrossedSquare( XS1 );
true
gap> C2G1 := Cat2GroupOfCrossedSquare( XS1 );
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [((d20 |X d10a) |X (d10b |X c5d))=>(d20 |X d10a)]
2 : [((d20 |X d10a) |X (d10b |X c5d))=>(d20 |X d10b)]
gap> IsCat2Group( C2G1 );
true
gap> Xab := CrossedSquareOfCat2Group( C2ab );
```

```
crossed square with crossed modules:
up = [Group( [ (1,5)(2,6)(3,7)(4,8) ] ) -> Group( [ ( 2, 6)( 4, 8) ] )]
left = [Group( [ (1,5)(2,6)(3,7)(4,8) ] ) -> Group(
[ (1,2,3,4)(5,6,7,8), (), () ] )]
right = [Group( [ ( 2, 6)( 4, 8) ] ) -> Group( () )]
down = [Group( [ (1,2,3,4)(5,6,7,8), (), () ] ) -> Group( () )]
gap> IdGroup( Xab );
[ [ 2, 1 ], [ 2, 1 ], [ 4, 1 ], [ 1, 1 ] ]
```

5. Table of cat^2-groups

A list L_G of all n_G cat^2-groups $(G; t_1, h_1; t_2, h_2)$ over G is constructed by the function **AllCat2Groups(G)**. Then the function **AreIsomorphicCat2Groups** is used to check whether or not pairs of cat^2-groups are isomorphic. Finally, a list of representatives of the isomorphism classes is returned by **AllCat2GroupsUpToIsomorphism**. The function **AllCat2GroupFamilies** returns a list of the positions $[1 \dots n_G]$ partitioned according to these classes.

In the following GAP session, we compute all 47 cat^2-groups on $C_4 \times C_2$; representatives of the 14 isomorphism classes; and the list of lists of positions in the families. So the eighth class consists of cat^2-group numbers $[31, 34, 35, 38]$, and `iso82[8]=all82[31]`.

```
gap> c4c2 := SmallGroup( 8, 2 );;
gap> all82 := AllCat2Groups( c4c2 );;
gap> Length( all82 );
47
gap> iso82 := AllCat2GroupsUpToIsomorphism( c4c2 );;
gap> Length( iso82 );
14
gap> AllCat2GroupFamilies( c4c2 );
[ [ 1 ], [ 2, 5, 8, 11 ], [ 3, 4, 9, 10 ], [ 14, 17, 22, 25 ],
  [ 15, 16, 23, 24 ], [ 30 ], [ 6, 7, 12, 13 ], [ 31, 34, 35, 38 ],
  [ 32, 33, 36, 37 ], [ 18, 19, 26, 27 ], [ 20, 21, 28, 29 ],
  [ 39, 42, 43, 46 ], [ 40, 41, 44, 45 ], [ 47 ] ]
gap> iso82[8];
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [Group( [ f1, f2, f3 ] ) => Group( [ f2, f2 ] )]
2 : [Group( [ f1, f2, f3 ] ) => Group( [ f2, f1 ] )]
gap> IsomorphismCat2Groups( all82[31], all82[34] ) = fail;
false
```

In the following tables the groups of size at most 30 are ordered by their GAP number. For each group G we list the number $|IE(G)|$ of idempotent endomorphisms; the number $|\mathcal{C}^1(G)|$ of cat^1-groups on G, followed by the number of their isomorphism classes; and then the number $|\mathcal{C}^2(G)|$ of cat^2-groups on G, and the number of their isomorphism classes. The number of isomorphism classes $\mathcal{C}^1(G)$ of cat^1-groups is given in [2].

We may reduce the size of the table by noting the results for cyclic groups. When $G = C_{p^k}$ is cyclic, with p prime, the only idempotent endomorphisms are the identity and zero maps. In this case all the cat^1-groups have equal tail and head maps, and all isomorphism classes are singletons. Similarly, when $G = C_{p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}}$ is cyclic, and its order is the product of m distinct primes p_i having multiplicities k_i, there are 2^m idempotent endomorphisms and cat^1-groups. The results up to $m = 4$ are shown in Table 1. The rows headed "groups" list, for each cat^2-group, its four groups $[G, R_1, R_2, R_{12}]$ where, for example, $2 \times [G, I, C_{p^k}, I]$ denotes $\{[G, I, C_{p_1^{k_1}}, I], [G, I, C_{p_2^{k_2}}, I]\}$.

Table 1. Results for families of cyclic groups.

G	$\lvert \mathrm{IE}(G) \rvert$	$\lvert \mathcal{C}^1(G) \rvert$	$\lvert \mathcal{C}^1(G)/\cong \rvert$	$\lvert \mathcal{C}^2(G) \rvert$	$\lvert \mathcal{C}^2(G)/\cong \rvert$
groups	$[G,I,I,I],\ [G,I,G,I],\ [G,G,G,G]$				
$C_{p_1^{k_1}}$	2	2	2	3	3
$C_{p_1^{k_1} p_2^{k_2}}$	4	4	4	10	10
groups	$[G,I,I,I],\ [G,I,G,I],\ [G,G,G,G],\ 2 \times [G,I,C_{p^k},I],$ $2 \times [G,C_{p^k},G,C_{p^k}],\ 2 \times [G,C_{p^k},C_{p^k},C_{p^k}],\ [G,C_{p_1^{k_1}},C_{p_2^{k_2}},I]$				
$C_{p_1^{k_1} p_2^{k_2} p_3^{k_3}}$	8	8	8	36	36
groups	$[G,I,I,I],\ [G,I,G,I],\ [G,G,G,G],\ 3 \times [G,I,C_{p^k},I],$ $3 \times [G,C_{p^k},G,C_{p^k}],\ 3 \times [G,C_{p^k},C_{p^k},C_{p^k}],\ 3 \times [G,C_{p^k},C_{q^j},I],$ $3 \times [G,I,C_{p^k q^j},I],\ 3 \times [G,C_{p^k q^j},G,C_{p^k q^j}],\ 3 \times [G,C_{p^k q^j},C_{p^k q^j},C_{p^k q^j}],$ $6 \times [G,C_{p^k},C_{p^k q^j},C_{p^k}],\ 3 \times [G,C_{r^i},C_{p^k q^j},I],\ 3 \times [G,C_{p^k q^j},C_{p^k r^i},C_{p^k}]$				
$C_{p_1^{k_1} p_2^{k_2} p_3^{k_3} p_4^{k_4}}$	16	16	16	136	136

When $m = 1$ there are 16 cyclic groups of order at most 30; when $m = 2$ there are 12 such groups; and when $m = 3$ there is just the small group $30/4 = C_{30}$.

Table 2 contains the results for those G which are not cyclic.

Table 2. Results for families of non-cyclic groups.

GAP #	G	$\lvert \mathrm{IE}(G) \rvert$	$\lvert \mathcal{C}^1(G) \rvert$	$\lvert \mathcal{C}^1(G)/\cong \rvert$	$\lvert \mathcal{C}^2(G) \rvert$	$\lvert \mathcal{C}^2(G)/\cong \rvert$
1/1	I	1	1	1	1	1
4/2	$K_4 = C_2 \times C_2$	8	14	4	36	9
6/1	S_3	5	4	2	7	3
8/2	$C_4 \times C_2$	10	18	6	47	14
8/3	D_8	10	9	3	21	6
8/4	Q_8	2	1	1	1	1
8/5	$C_2 \times C_2 \times C_2$	58	226	6	1,711	23
9/2	$C_3 \times C_3$	14	38	4	93	9
10/1	D_{10}	7	6	2	11	3
12/1	$C_3 \ltimes C_4$	5	4	2	7	3
12/3	A_4	6	5	2	9	3
12/4	D_{12}	21	12	4	41	10
12/5	$C_3 \times K_4$	16	28	8	136	32
14/1	D_{14}	9	8	2	15	3
16/2	$C_4 \times C_4$	26	98	5	231	11
16/3	$(C_4 \times C_2) \ltimes C_2$	18	25	4	57	7
16/4	$C_4 \ltimes C_4$	10	17	3	25	4
16/5	$C_8 \times C_2$	10	18	6	47	14
16/6	$C_8 \ltimes C_2$	6	5	2	9	3
16/7	D_{16}	18	9	2	17	3
16/8	QD_{16}	10	5	2	9	3
16/9	Q_{16}	2	1	1	1	1

Table 3. Table 2. (Continued)

GAP #	G	$\lvert\mathrm{IE}(G)\rvert$	$\lvert\mathcal{C}^1(G)\rvert$	$\lvert\mathcal{C}^1(G)/\cong\rvert$	$\lvert\mathcal{C}^2(G)\rvert$	$\lvert\mathcal{C}^2(G)/\cong\rvert$
16/10	$C_4 \times K_4$	82	322	12	2,875	53
16/11	$C_2 \times D_8$	82	97	9	649	29
16/12	$C_2 \times Q_8$	18	17	3	25	4
16/13	$(C4 \times C2) \ltimes C_2$	26	13	2	37	4
16/14	$K_4 \times K_4$	382	4,162	9	298,483	53
18/1	D_{18}	11	10	2	19	3
18/3	$C_3 \times S_3$	12	8	4	24	10
18/4	$(C_3 \times C_3) \ltimes C_2$	47	118	4	541	9
18/5	$C_6 \times C_3$	28	76	8	358	32
20/1	Q_{20}	7	6	2	11	3
20/3	$C_4 \ltimes C_5$	7	6	2	11	3
20/4	D_{20}	31	18	4	65	10
20/5	$C_5 \times K_4$	16	28	8	136	32
21/1	$C_3 \ltimes C_7$	9	8	2	15	3
22/1	D_{22}	13	12	2	23	3
24/1	$C_3 \ltimes C_8$	5	4	2	7	3
24/3	$SL(2,3)$	6	1	1	1	1
24/4	Q_{24}	5	4	2	7	3
24/5	$S_3 \times C_4$	27	12	4	41	10
24/6	D_{24}	33	20	4	75	10
24/7	$Q_{12} \times C_2$	25	36	6	115	14
24/8	$D_8 \ltimes C_3$	23	12	4	41	10
24/9	$C_{12} \times C_2$	20	36	12	178	52
24/10	$D_8 \times C_3$	20	18	6	75	20
24/11	$Q_8 \times C_3$	4	2	2	3	3
24/12	S_4	12	5	2	9	3
24/13	$A_4 \times C_2$	15	10	4	31	10
24/14	$S_3 \times K_4$	157	116	8	999	32
24/15	$C_6 \ltimes K_4$	116	452	12	6,786	84
25/2	$C_5 \times C_5$	32	152	4	348	9
26/1	D_{26}	15	14	2	27	3
27/2	$C_9 \times C_3$	20	56	6	138	14
27/3	$(C_3 \times C_3) \ltimes C_3$	38	37	2	127	4
27/4	$C_9 \ltimes C_3$	11	10	2	19	3
27/5	$C_3 \times C_3 \times C_3$	236	2,108	6	24,222	16
28/1	Q_{28}	9	8	2	15	3
28/3	D_{28}	41	24	4	89	10
28/4	$C_7 \times K_4$	16	28	8	136	32
30/1	$S_3 \times C_5$	10	8	4	24	10
30/2	$D_{10} \times C_3$	14	12	4	38	10
30/3	D_{30}	25	24	4	92	10

The $1,000$ isomorphism classes contain just 13 cat^2-groups whose diagonal is *not* a cat^1-group: one each for groups [8/3, 16/3, 16/13, 27/3], three for 24/10 and six for 16/11.

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References

[1] M. Alp, A. Odabaş, E. O. Uslu and C. D. Wensley, Crossed modules and cat^1-groups, manual for the XMod package for GAP, version 2.77 (2019).

[2] M. Alp and C. D. Wensley, Enumeration of cat^1-groups of low order, *Int. J. Algebra Comput.* **10** (2000) 407-424.

[3] Z. Arvasi and A. Odabaş, Crossed Modules and cat^1-algebras, manual for the XModAlg share package for GAP, version 1.12 (2015).

[4] Z. Arvasi and A, Odabaş, Computing 2-dimensional algebras: Crossed modules and Cat^1-algebras, *J. Algebra Appl.* **15** (2016) 165-185.

[5] Z. Arvasi and E. Ulualan, On Algebraic Models for Homotopy 3-types, *J. Homotopy and Related Structures* **1** (2006) 1-27.

[6] H. J. Baues, *Combinatorial homotopy and* 4*-dimensional complexes* (Walter de Gruyter Expositions in Mathematics **2** (1991)).

[7] H. U. Besche, B. Eick and E. A. O'Brien, A millennium project: constructing Small Groups, *Internat. J. Algebra Comput.* **12** (2002) 623-644.

[8] R. Brown, Higher Dimensional Group Theory, in: *Low Dimensional Topology*, London Math. Soc. Lecture Note Series **48** (1982) 215-238.

[9] R. Brown, Modelling and computing homotopy types, *Indag. Math.* **29** (2018) 459-482.

[10] R. Brown and N. D. Gilbert, Algebraic models of 3-types and automorphism structures for crossed modules, *Proc. London Math. Soc.* (3) **59** (1989) 51-73.

[11] R. Brown and J.-L. Loday, Van Kampen theorems for diagram of spaces, *Topology* **26**(3) (1987) 311-335.

[12] D. Conduché, Modules croisés généralisés de longueur 2, *J. Pure Appl. Algebra* **34** (1984) 155-178.

[13] G. J. Ellis, Crossed Squares and Combinatorial Homotopy, *Math. Z.* **214** (1993) 93-110.

[14] G. J. Ellis, Homological Algebra Programming, manual for the HAP package for GAP, version 1.19 (2019).

[15] G. J. Ellis and R. Steiner, Higher dimensional crossed modules and the homotopy groups of $(n+1)$-ads., *J. Pure and Applied Algebra* **46** (1987) 117-136.

[16] The GAP Group, GAP – Groups, Algorithms, and Programming, version 4.10.2 (2019) (https://www.gap-system.org).

[17] D. Guin-Walery and J.-L. Loday, Obstructions à l'excision en K-théorie algébrique, in: *Evanston conference on algebraic K-Theory, 1980*, eds. E. M. Friedlander and M. R. Stein, Lecture Notes in Math., Vol. 854, (Berlin Heidelberg New York : Springer, 1981), pp. 179–216.

[18] J.-L. Loday, Spaces with finitely many non-trivial homotopy groups, *J. Pure and Applied Algebra* **24** (1982) 179-202.

[19] E. J. Moore and C. D. Wensley, Calculations with finite groupoids and their homomorphisms, manual for the groupoids package for GAP, version 1.68 (2019).

[20] A. Mutlu and T. Porter, Crossed squares and 2-crossed modules, *arXiv:math/0210462* (2002) (https://arxiv.org/pdf/math/0210462v1.pdf).

[21] C. D. Wensley, *Notes on higher dimensional groups and groupoids and related topics* (2019) (https://github.com/cdwensley/xmod-notes/blob/master/notes.pdf).

[22] J. H. C. Whitehead, Combinatorial homotopy II , *Bull. Amer. Math. Soc.* **55** (1949) 453-496.