COMPUTING 3-DIMENSIONAL GROUPS: CROSSED SQUARES AND CAT²-GROUPS

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ABSTRACT. The category **XSq** of crossed squares is equivalent to the category **Cat2** of cat²-groups. Functions for computing with these structures have been developed in the package **XMod** written using the **GAP** computational discrete algebra programming language. This paper includes details of the algorithms used. It contains tables listing the 1,000 isomorphism classes of cat²-groups on groups of order at most 30.

1. Introduction

This paper ^{1 2} is concerned with the latest developments in the general programme of "computational higher-dimensional group theory" which forms part of the "higher-dimensional group theory" programme described, for example, by Brown in [8].

The 2-dimensional part of these programmes is concerned with group objects in the categories of groups or groupoids, and these objects may equivalently be considered as crossed modules or cat¹-groups. A summary of the definitions of these objects, with some examples, is contained in §2.

The initial computational part of this programme was described in Alp and Wensley [2]. The output from this work was the package XMod [1] for GAP [16] which, at the time, contained functions for constructing crossed modules and cat¹-groups of groups, and their morphisms, and conversions from one to another. It also contained functions for computing the monoid of derivations of a crossed module, and the equivalent monoid of sections of a cat¹-group. The next development of XMod used the package groupoids [19] to compute crossed modules of groupoids. Later still, a GAP package XModAlg [3] was written to compute cat¹-algebras and crossed modules of algebras, as described in [4].

The 3-dimensional part of the higher-dimensional group theory programme is concerned with objects in the category **XSq** of crossed squares and the equivalent cat²-groups category **Cat2**. The mathematical basis of these structures is described in §3, and some computational details are included in §4. In §5 we enumerate the 1,000 isomorphism classes of cat²-group structures on the 92 groups of order at most 30.

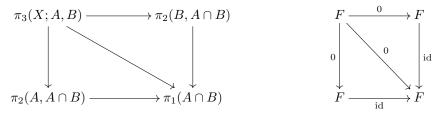
There are many other ways of viewing crossed squares and cat²-groups. Conduché in [12] defined the equivalent notion of 2-crossed module. Brown and Gilbert in [10] introduced braided, regular crossed modules as an alternative algebraic

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model of homotopy 3-types. They also proved that these structures are equivalent to simplicial groups with Moore complex of length 2. In [5] Arvasi and Ulualan explore the algebraic relationship between these structures and also the quadratic modules of Baues [6], and the homotopy equivalences between them.

The impetus for the study of higher-dimensional groups comes from algebraic topology [9]. Crossed modules are algebraic models of connected (weak homotopy) 2-types, while crossed squares model connected 3-types. The principal topological example of a crossed module arises from a pointed pair of spaces $A \subseteq X$ where the boundary map is $\partial: \pi_2(X,A) \to \pi_1(A)$. Similarly, given a triad of pointed spaces $A \subseteq X$, $B \subseteq X$ we obtain a crossed square as shown in the left-hand diagram below. A simple case, when X is a 2-sphere and A, B are the upper and lower hemispheres, results in the square on the right. Here F is a free group on one generator x, the boundaries are the trivial and identity homomorphisms, and the crossed pairing is given by $h: F \times F \to F$, $(x^i, x^j) \mapsto x^{ij}$ (see Ellis [13]).



The XMod package follows a purely algebraic approach, and does not compute any specifically topological results. The interested reader may wish to investigate the GAP package HAP [14] which also computes with cat¹-groups.

2. Crossed Modules and Cat¹-Groups

The notion of crossed module, generalizing the notion of a G-module, was introduced by Whitehead [22] in the course of his studies on the algebraic structure of the second relative homotopy group.

A crossed module consists of a group homomorphism $\partial: S \to R$, endowed with a left action R on S (written by $(r,s) \to {}^r s$ for $r \in R$ and $s \in S$) satisfying the following conditions:

$$\begin{array}{rcl} \partial(^rs) & = & r(\partial s)r^{-1} & \quad \forall \ s \in S, \ r \in R; \\ (\partial s_2)s_1 & = & s_2s_1s_2^{-1} & \quad \forall \ s_1,s_2 \in S. \end{array}$$

The first condition is called the *pre-crossed module property* and the second one the *Peiffer identity*. We will denote such a crossed module by $\mathcal{X} = (\partial : S \to R)$.

A morphism of crossed modules $(\sigma, \rho) : \mathcal{X}_1 \to \mathcal{X}_2$, where $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ and $\mathcal{X}_2 = (\partial_2 : S_2 \to R_2)$, consists of two group homomorphisms $\sigma : S_1 \to S_2$ and $\rho : R_1 \to R_2$ such that

$$\partial_2 \circ \sigma = \rho \circ \partial_1$$
, and $\sigma(rs) = (\rho r) \sigma s \quad \forall s \in S, r \in R$.

Standard constructions for crossed modules include the following.

- (1) A conjugation crossed module is an inclusion of a normal subgroup $N \subseteq R$, where R acts on N by conjugation.
- (2) An automorphism crossed module has as range a subgroup R of the automorphism group $\operatorname{Aut}(S)$ of S which contains the inner automorphism group

 $\operatorname{Inn}(S)$ of S. The boundary maps $s \in S$ to the inner automorphism of S by s.

- (3) A zero boundary crossed module has a R-module as source and $\partial = 0$.
- (4) Any homomorphism $\partial: S \to R$, with S abelian and im ∂ in the centre of R, provides a crossed module with R acting trivially on S.
- (5) A central extension crossed module has as boundary a surjection $\partial: S \to R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$.
- (6) The direct product of $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ and $\mathcal{X}_2 = (\partial_2 : S_2 \to R_2)$ is $\mathcal{X}_1 \times \mathcal{X}_2 = (\partial_1 \times \partial_2 : S_1 \times S_2 \to R_1 \times R_2)$ with direct product action $(r_1,r_2)(s_1,s_2) = (r_1s_1,r_2s_2).$

Loday reformulated the notion of crossed module as a cat¹-group. Recall from [18] that a cat^1 -group is a triple (G;t,h) consisting of a group G with two endomorphisms: the tail map t and the head map h, having a common image R and satisfying the following axioms.

(2.1)
$$t \circ h = h, \quad h \circ t = t, \quad \text{and} \quad [\ker t, \ker h] = 1.$$

When only the first two of these axioms are satisfied, the structure is a $pre-cat^1$ group. It follows immediately that $t \circ t = t$ and $h \circ h = h$. We picture (G; t, h)as

$$G \xrightarrow{t,h} R$$

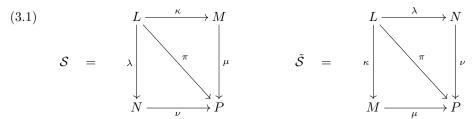
A morphism of cat^1 -groups $(G_1; t_1, h_1) \to (G_2; t_2, h_2)$ is a group homomorphism $f: G_1 \to G_2$ such that

$$f \circ t_1 = t_2 \circ f$$
 and $f \circ h_1 = h_2 \circ f$.

Crossed modules and cat¹-groups are equivalent two-dimensional generalisations of a group. It was shown in [18, Lemma 2.2] that, on setting $S = \ker t$, $R = \operatorname{im} t$ and $\partial = h|_{S}$, the conjugation action makes $(\partial : S \to R)$ into a crossed module. Conversely, if $(\partial : S \to R)$ is a crossed module, then setting $G = S \times R$ and defining t, h by t(s, r) = (1, r) and $h(s, r) = (1, (\partial s)r)$ for $s \in S, r \in R$, produces a cat^1 -group (G; t, h).

3. Crossed Squares and Cat²-Groups

The notion of a crossed square is due to Guin-Walery and Loday [17]. A crossed square of groups S is a commutative square of groups



together with left actions of P on L, M, N and a crossed pairing map $\boxtimes : M \times N \to$ L. Then M acts on N and L via P and N acts on M and L via P. The diagram illustrates an oriented crossed square, since a choice of where to place M and Nhas been made. The transpose \hat{S} of S is obtained by making the alternative choice. Since crossed pairing identities are similar to those for commutators, the crossed pairing for S is \boxtimes where $(n \boxtimes m) = (m \boxtimes n)^{-1}$. Transposition gives an equivalence 4

relation on the set of oriented crossed squares, and a crossed square is an equivalence class.

The structure of an oriented crossed square must satisfy the following axioms for all $l \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

- (1) With the given actions, the homomorphisms $\kappa, \lambda, \mu, \nu$ and $\pi = \mu \circ \kappa = \nu \circ \lambda$ are crossed modules, and both κ, λ are P-equivariant,
- (2) $(mm' \boxtimes n) = (^mm' \boxtimes ^mn) (m \boxtimes n)$ and $(m \boxtimes nn') = (m \boxtimes n) (^nm \boxtimes ^nn'),$
- (3) $\kappa(m \boxtimes n) = m(^n m^{-1})$ and $\lambda(m \boxtimes n) = (^m n)n^{-1}$,
- (4) $(\kappa l \boxtimes n) = l(^n l^{-1})$ and $(m \boxtimes \lambda l) = (^m l) l^{-1}$,
- $(5) p(m \boxtimes n) = (pm \boxtimes pn).$

Note that axiom 1. implies that $(id, \mu), (id, \nu), (\kappa, id)$ and (λ, id) are morphisms of crossed modules.

Standard constructions for crossed squares include the following.

(1) If M, N are normal subgroups of the group P then the diagram of inclusions

$$\begin{array}{ccc}
M \cap N & \longrightarrow M \\
\downarrow & & \downarrow \\
N & \longrightarrow P
\end{array}$$

together with the actions of P on M, N and $M \cap N$ given by conjugation, and the commutator map

$$[\ ,\]: M \times N \to M \cap N, \quad (m,n) \mapsto [m,n] = mnm^{-1}n^{-1},$$

is a crossed square. We call this an inclusion crossed square.

(2) The diagram

$$\begin{array}{ccc}
M & \xrightarrow{\alpha} & \operatorname{Inn} M \\
\downarrow^{\alpha} & & \downarrow^{\iota} \\
\operatorname{Inn} M & \xrightarrow{\iota} & \operatorname{Aut} M
\end{array}$$

is a crossed square, where α maps $m \in M$ to the inner automorphism

$$\beta_m: M \to M, \quad m' \mapsto mm'm^{-1},$$

and where ι is the inclusion of Inn M in Aut M; the actions are standard; and the crossed pairing is

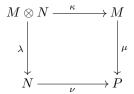
$$\boxtimes$$
: $\operatorname{Inn} M \times \operatorname{Inn} M \to M$, $(\beta_m, \beta_{m'}) \mapsto [m, m']$.

(3) If P is a group and M, N are ordinary P-modules, and if A is an arbitrary abelian group on which P is assumed to act trivially, then there is a crossed

square

$$\begin{array}{ccc}
A & \longrightarrow & M \\
\downarrow & & \downarrow & \downarrow \\
\downarrow & & \downarrow & \downarrow \\
N & \longrightarrow & P
\end{array}$$

(4) Given two crossed modules, $(\mu: M \to P)$ and $(\nu: N \to P)$, there is a universal crossed square



where $M \otimes N$ is constructed using the nonabelian tensor product of groups

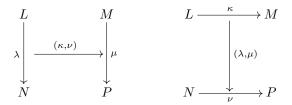
(5) The direct product of crossed squares S_1, S_2 is

$$\begin{array}{c|c} L_1 \times L_2 & \xrightarrow{\kappa_1 \times \kappa_2} & M_1 \times M_2 \\ \downarrow & & \downarrow \\ \lambda_1 \times \lambda_2 & & \downarrow \\ N_1 \times N_2 & \xrightarrow{\nu_1 \times \nu_2} & P_1 \times P_2 \end{array}$$

with actions $(p_1, p_2)(l_1, l_2) = (p_1 l_1, p_2 l_2), (p_1, p_2)(m_1, m_2) = (p_1 m_1, p_2 m_2)$ and $(p_1,p_2)(n_1,n_2)=(p_1,p_1,p_2,p_2)$, and with crossed pairing

$$\boxtimes ((m_1, m_2), (n_1, n_2)) = (\boxtimes_1 (m_1, n_1), \boxtimes_2 (m_2, n_2)).$$

The crossed square S in (3.1) can be thought of as a horizontal or vertical crossed module of crossed modules:



where (κ, ν) is the boundary of the crossed module with domain $(\lambda : L \to N)$ and codomain $(\mu: M \to P)$. (See also section 9.2 of [21].)

There is an evident notion of morphism of crossed squares which preserves all the structure, so that we obtain a category **XSq**, the category of crossed squares.

Although, when first introduced by Loday and Walery [17], the notion of crossed square of groups was not linked to that of cat²-groups, it was in this form that Loday gave their generalisation to an n-fold structure, cat^n -groups (see [18]). When n=1this is the notion of cat¹-group given earlier.

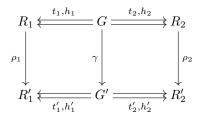
When n = 2 we obtain a cat²-group. Again we have a group G, but this time with two *independent* cat¹-group structures on it. So a cat^2 -group is a 5-tuple, $(G; t_1, h_1; t_2, h_2)$, where $(G; t_i, h_i)$, i = 1, 2 are cat¹-groups, and

$$t_1 \circ t_2 \ = \ t_2 \circ t_1, \quad h_1 \circ h_2 \ = \ h_2 \circ h_1, \quad t_1 \circ h_2 \ = \ h_2 \circ t_1, \quad t_2 \circ h_1 \ = \ h_1 \circ t_2.$$

To emphasise the relationship with crossed squares we may illustrate an oriented cat²-group by the diagram

where R_{12} is the image of $t_1 \circ t_2 = t_2 \circ t_1$.

A morphism of cat²-groups is a triple (γ, ρ_1, ρ_2) , as shown in the diagram



where $\gamma: G \to G'$, $\rho_1 = \gamma|_{R_1}$ and $\rho_2 = \gamma|_{R_2}$ are homomorphisms satisfying:

$$\rho_1 \circ t_1 = t_1' \circ \gamma, \qquad \rho_1 \circ h_1 = h_1' \circ \gamma, \qquad \rho_2 \circ t_2 = t_2' \circ \gamma, \qquad \rho_2 \circ h_2 = h_2' \circ \gamma.$$

We thus obtain a category Cat2, the category of cat²-groups.

Notice that, unlike the situation with crossed squares where the diagonal is a crossed module, it is *not* required that the diagonal in (3.2) is a cat^1 -group – it may just be a pre-cat¹-group. The simplest example of this situation is described in Example 3.1 below.

Loday, in [18] proved that there is an equivalence of categories between the category **Cat2** and the category **XSq**. We now consider the sketch proof of this result (see also [20]).

The cat²-group $(G; t_1, h_1; t_2, h_2)$ determines a diagram of homomorphisms

where each morphism is a crossed module for the natural action, conjugation in G. The required crossed pairing is given by the commutator in G since, if $x \in$

 $\operatorname{im} t_1 \cap \ker t_2$ and $y \in \ker t_1 \cap \operatorname{im} t_2$ then $[x,y] \in \ker t_1 \cap \ker t_2$. It is routine to check the crossed square axioms.

Conversely, if

$$\begin{array}{ccc}
L & \xrightarrow{\kappa} & M \\
\downarrow^{\lambda} & & \downarrow^{\mu} \\
N & \xrightarrow{\iota} & P
\end{array}$$

is a crossed square, then we consider it as a morphism of crossed modules (κ, ν) : $(\lambda: L \to N) \to (\mu: M \to P)$. Using the equivalence between crossed modules and cat¹-groups this gives a morphism

$$\partial: (L \rtimes N, t, h) \longrightarrow (M \rtimes P, t', h')$$

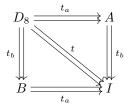
of cat¹-groups. There is an action of $(m,p) \in M \times P$ on $(l,n) \in L \times N$ given by

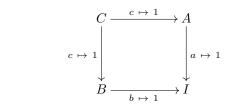
$$^{(m,p)}(l,n) = (^m(^pl)(m \boxtimes {}^pn), {}^pn).$$

Using this action, we form its associated cat²-group with source $(L \times N) \times (M \times P)$ and induced endomorphisms t_1, h_1, t_2, h_2 .

Example 3.1. Let $D_8 = \langle a, b \mid a^2, b^2, (ab)^4 \rangle$ be the dihedral group of order 8, and let $c = [a, b] = (ab)^2$ so that $a^b = ac$ and $b^a = bc$. (The standard permutation representation is given by a = (1, 2)(3, 4), b = (1, 3), ab = (1, 2, 3, 4), c = (1, 3)(2, 4).

Define $t_a, t_b: D_8 \to D_8$ by $t_a: [a,b] \mapsto [a,1]$ and $t_b: [a,b] \mapsto [1,b]$. Construct cat¹-groups $C_a = (D_8, t_a, t_a)$ and $C_b = (D_8, t_b, t_b)$. Diagrams (3.2) and (3.3) become





where $A = \langle a \rangle$, $B = \langle b \rangle$, $C = \langle c \rangle$ and I is the trivial group. The crossed pairing is given by $\boxtimes (a,b) = c$. The diagonal map $t = t_a \circ t_b$ has kernel D_8 , and $[\ker t, \ker t] =$ C, so the diagonal is not a cat¹-group.

Definition 3.2. A cat^n -group consists of a group G with n independent cat^1 -group structures $(G; t_i, h_i), 1 \le i \le n$, such that for all $i \ne j$

$$t_i t_j = t_i t_i$$
, $h_i h_j = h_j h_i$ and $t_i h_j = h_j t_i$.

A generalisation of crossed square to higher dimensions was given by Ellis and Stenier (cf. [15]). It is called a "crossed n-cube". We only use this construction for n=2.

4. Computer Implementation

GAP [16] is an open-source system for discrete computational algebra. The system consists of a library of implementations of mathematical structures: groups, vector spaces, modules, algebras, graphs, codes, designs, etc.; plus databases of groups of small order, character tables, etc. The system has world-wide usage in the area of education and scientific research. GAP is free software and user contributions to the system are supported. These contributions are organized in a form of GAP packages and are distributed together with the system. Contributors can submit additional packages for inclusion after a reviewing process.

The Small Groups library by Besche, Eick and O'Brien in [7] provides access to descriptions of the groups of small order. The groups are listed up to isomorphism. The library contains all groups of order at most 2000 except 1024.

4.1. **2-Dimensional Groups.** The XMod package for GAP contains functions for computing with crossed modules, cat^1 -groups and their morphisms, and was first described in [1]. A more general notion of cat^1 -group is implemented in XMod, where the tail and head maps are no longer required to be endomorphisms on G. Instead it is required that t and h have a common image R, and an *embedding* $e: R \to G$ is added. The axioms in (2.1) then become:

```
t \circ e \circ h = h, h \circ e \circ t = t, and [\ker t, \ker h] = 1,
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and again it follows that $t \circ e \circ t = t$ and $h \circ e \circ h = h$. We denote such a cat¹-group by $(e; t, h : G \to R)$.

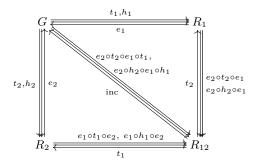
This package may be used to select a cat¹-group from a data file. All cat¹-structures on groups of size up to 70 (ordered according to the GAP numbering of small groups) are stored in a list in the file cat1data.g. The function Cat1Select may be used in three ways. Cat1Select(size) returns the names of the groups with this size, while Cat1Select(size, gpnum) prints a list of cat¹1-structures for this chosen group. Cat1Select(size, gpnum, num) returns the chosen cat¹1-group. XModOfCat1Group produces the associated crossed module.

The following GAP session illustrates the use of these functions.

```
gap> Cat1Select( 12 );
Usage: Cat1Select( size, gpnum, num );
[ "C3 : C4", "C12", "A4", "D12", "C6 x C2" ]
gap> Cat1Select( 12, 3 );
Usage: Cat1Select( size, gpnum, num );
There are 2 cat1-structures for the group A4.
Using small generating set [ f1, f2 ] for source of homs.
[ [range gens], [tail genimages], [head genimages] ] :-
(1)  [ [ f1 ], [ f1, <identity> of ... ], [ f1, <identity> of ... ] ]
(2)  [ [ f1, f2 ], tail = head = identity mapping ]
2
gap> C1 := Cat1Select( 12, 3, 2 );
[A4=>A4]
gap> X1 := XModOfCat1Group( C1 );
[triv->A4]
```

4.2. **3-dimensional Groups.** We have developed new functions for XMod which construct (pre-)cat²-groups, (pre-)cat³-groups, and their morphisms. Functions for (pre-)cat²-groups include **PreCat2Group**, **Cat2Group**, **PreCat2GroupByPreCat1Groups**, **IsPreCat2Group** and **IsCat2Group**. Attributes of a (pre)cat²-group constructed in this way include **GeneratingCat1Groups**, **Size**, **Name** and **Edge2DimensionalGroup** where 'Edge' is one of {**Up**, **Left**, **Right**, **Down**, **Diagonal**}.

As with cat¹-groups, we use a more general notion for cat²-groups. An oriented cat²group has the form



where R_1, R_2 need not be subgroups of G, but R_{12} is taken to be the common image of $e_2 \circ t_2 \circ e_1 \circ t_1$ and $e_1 \circ t_1 \circ e_2 \circ t_2$, a subgroup of G.

Generalizing these functions, we have introduced ${f Cat3Group}$ and ${f Higher Dimension}$ which construct cat^3 -groups. Functions for cat^n -groups of higher dimension will be added as time permits. An orientation of a cat^3 -group on G displays a cube whose six faces (ordered as front; up, left, right, down, back) are cat^2 -groups. The group G is positioned where the front, up and left faces meet. The following GAP session illustrates the use of these functions. Notice that the cat²-group C2ab is the second example with a diagonal which is only a pre-cat¹-group.

```
gap> a := (1,2,3,4)(5,6,7,8);;
gap> b := (1,5)(2,6)(3,7)(4,8);;
gap> c := (2,6)(4,8);;
gap> G := Group( a, b, c );;
gap> SetName( G, "c4c2:c2" );
gap> t1a := GroupHomomorphismByImages( G, G, [a,b,c], [(),(),c] );;
gap> C1a := PreCat1GroupByEndomorphisms( t1a, t1a );;
gap> t1b := GroupHomomorphismByImages( G, G, [a,b,c], [a,(),()] );;
gap> C1b := PreCat1GroupByEndomorphisms( t1b, t1b );;
gap> C2ab := Cat2Group( C1a, C1b );
(\verb"pre-")" cat 2- \verb"group" with generating (\verb"pre-")" cat 1- \verb"groups":
1 : [c4c2:c2 \Rightarrow Group([(), (), (2,6)(4,8)])]
2: [c4c2:c2 \Rightarrow Group([(1,2,3,4)(5,6,7,8), (), ()])]
gap> IsCat2Group( C2ab );
true
gap> Size( C2ab );
[ 16, 2, 4, 1 ]
gap> IsCat1Group( Diagonal2DimensionalGroup( C2ab ) );
false
gap> t1c := GroupHomomorphismByImages( G, G, [a,b,c], [a,b,c] );;
gap> C1c := PreCat1GroupByEndomorphisms( t1c, t1c );;
gap> C3abc := Cat3Group( C1a, C1b, C1c );
(pre-)cat3-group with generating (pre-)cat1-groups:
1 : [c4c2:c2 \Rightarrow Group([(), (), (2,6)(4,8)])]
2 : [c4c2:c2 \Rightarrow Group([(1,2,3,4)(5,6,7,8), (), ()])]
3 : [c4c2:c2 => Group([(1,2,3,4)(5,6,7,8), (1,5)(2,6)(3,7)(4,8),
(2,6)(4,8) ])]
gap> IsPreCat3Group( C3abc );
true
gap> HigherDimension( C3abc );
```

```
4
gap> Front3DimensionalGroup( C3abc );
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [c4c2:c2 => Group( [ (), (), (2,6)(4,8) ] )]
2 : [c4c2:c2 => Group( [ (1,2,3,4)(5,6,7,8), (), () ] )]
```

4.3. Morphisms of 3-Dimensional Groups. The function MakeHigherDimensionalGroupMorphism defines morphisms of higher dimensional groups, such as cat²-groups and crossed squares. In the cat²-group case these include IsCat2GroupMorphism, Cat2GroupMorphism and Cat2GroupMorphismByCat1GroupMorphisms. The function AllCat2GroupMorphisms is used to find all morphisms between two cat²-groups.

In the following GAP session, we obtain a cat²-group morphism using these functions. gap> C2_82 := Cat2Group(Cat1Group(8,2,1), Cat1Group(8,2,2));

```
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [C4 x C2 => Group( [ <identity> of ..., <identity> of ...,
<identity> of ... ] )]
2 : [C4 x C2 => Group( [ <identity> of ..., f2 ] )]
gap> C2_83 := Cat2Group( Cat1Group(8,3,2), Cat1Group(8,3,3) );
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [D8 => Group( [ f1, f1 ] )]
2 : [D8=>D8]
gap> up1 := GeneratingCat1Groups( C2_82 )[1];;
gap> lt1 := GeneratingCat1Groups( C2_82 )[2];;
gap> up2 := GeneratingCat1Groups( C2_83 )[1];;
gap> lt2 := GeneratingCat1Groups( C2_83 )[2];;
gap> G1 := Source( up1 );; R1 := Range( up1 );; Q1 := Range( lt1 );;
gap> G2 := Source( up2 );; R2 := Range( up2 );; Q2 := Range( lt2 );;
gap> homG := AllHomomorphisms( G1, G2 );;
gap> homR := AllHomomorphisms( R1, R2 );;
gap> homQ := AllHomomorphisms( Q1, Q2 );;
gap> upmor := Cat1GroupMorphism( up1, up2, homG[1], homR[1] );;
gap> IsCat1GroupMorphism( upmor );
true
gap> ltmor := Cat1GroupMorphism( lt1, lt2, homG[1], homQ[1] );;
gap> mor2 := PreCat2GroupMorphism( C2_82, C2_83, upmor, 1tmor );
<mapping: (pre-)cat2-group with generating (pre-)cat1-groups:</pre>
1 : [C4 x C2 => Group( [ <identity> of ..., <identity> of ...,
 <identity> of ... ] )]
2 : [C4 x C2 => Group( [ <identity> of ..., f2 ] )] -> (pre-)cat
2-group with generating (pre-)cat1-groups:
1 : [D8 => Group( [ f1, f1 ] )]
2 : [D8=>D8] >
gap> IsCat2GroupMorphism( mor2 );
gap> mor8283 := AllCat2GroupMorphisms( C2_82, C2_83 );;
gap> Length( mor8283 );
```

4.4. Natural Equivalence. The equivalence between the categories XSq and Cat2 is implemented by the functions CrossedSquareOfCat2Group, which constructs a crossed square from a cat²-group, and Cat2GroupOfCrossedSquare which performs the opposite operation.

The following GAP session illustrates the use of these functions. The dihedral group D_{20} has two normal subgroups D_{10} whose intersection is the cyclic C_5 . We construct the crossed square of normal subgroups, and then use the conversion functions to obtain the associated cat²-group. We then obtain the crossed square Xab associated to the cat²-group C2ab obtained earlier.

```
gap> d20 := DihedralGroup( IsPermGroup, 20 );;
gap> gend20 := GeneratorsOfGroup( d20 );
[(1,2,3,4,5,6,7,8,9,10), (2,10)(3,9)(4,8)(5,7)]
gap> p1 := gend20[1];; p2 := gend20[2];; p12 := p1*p2;
(1,10)(2,9)(3,8)(4,7)(5,6)
gap> d10a := Subgroup( d20, [ p1\^2, p2 ] );;
gap> d10b := Subgroup( d20, [ p1\^2, p12 ] );;
gap> c5d := Subgroup( d20, [ p1\^2 ] );;
gap> SetName( d20, "d20" ); SetName( d10a, "d10a" );
gap> SetName( d10b, "d10b" ); SetName( c5d, "c5d" );
gap> XS1 := CrossedSquareByNormalSubgroups( c5d, d10a, d10b, d20 );
[ c5d -> d10a ]
  1 1
Γ
         7
[ d10b -> d20 ]
gap> IsCrossedSquare( XS1 );
gap> C2G1 := Cat2GroupOfCrossedSquare( XS1 );
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [((d20 | X d10a) | X (d10b | X c5d))=>(d20 | X d10a)]
2 : [((d20 | X d10a) | X (d10b | X c5d)) = > (d20 | X d10b)]
gap> IsCat2Group( C2G1 );
true
gap> Xab := CrossedSquareOfCat2Group( C2ab );
crossed square with crossed modules:
up = [Group([(1,5)(2,6)(3,7)(4,8)]) -> Group([(2,6)(4,8)])]
left = [Group([(1,5)(2,6)(3,7)(4,8)]) \rightarrow Group(
[(1,2,3,4)(5,6,7,8),(),()]
right = [Group( [ ( 2, 6)( 4, 8) ] ) -> Group( () )]
down = [Group([(1,2,3,4)(5,6,7,8), (), ()]) \rightarrow Group(())]
gap> IdGroup( Xab );
[[2,1],[2,1],[4,1],[1,1]]
```

5. Table of cat²-groups

A list L_G of all n_G cat²-groups $(G; t_1, h_1; t_2, h_2)$ over G is constructed by the function **AllCat2Groups**(**G**). Then the function **AreIsomorphicCat2Groups** is used to check whether or not pairs of cat²-groups are isomorphic. Finally, a list of representatives of the isomorphism classes is returned by **AllCat2GroupsUpToIsomorphism**. The function **AllCat2GroupFamilies** returns a list of the positions $[1 \dots n_G]$ partitioned according to these classes.

In the following GAP session, we compute all 47 cat^2 -groups on $C_4 \times C_2$; representatives of the 14 isomorphism classes; and the list of lists of positions in the families. So the eighth class consists of cat^2 -group numbers [31, 34, 35, 38], and iso82[8]=all82[31].

```
gap> c4c2 := SmallGroup( 8, 2 );;
gap> all82 := AllCat2Groups( c4c2 );;
gap> Length( all82 );
47
gap> iso82 := AllCat2GroupsUpToIsomorphism( c4c2 );;
```

```
gap> Length( iso82 );
14
gap> AllCat2GroupFamilies( c4c2 );
[ [ 1 ], [ 2, 5, 8, 11 ], [ 3, 4, 9, 10 ], [ 14, 17, 22, 25 ],
       [ 15, 16, 23, 24 ], [ 30 ], [ 6, 7, 12, 13 ], [ 31, 34, 35, 38 ],
       [ 32, 33, 36, 37 ], [ 18, 19, 26, 27 ], [ 20, 21, 28, 29 ],
       [ 39, 42, 43, 46 ], [ 40, 41, 44, 45 ], [ 47 ] ]
gap> iso82[8];
(pre-)cat2-group with generating (pre-)cat1-groups:
1 : [Group( [ f1, f2, f3 ] ) => Group( [ f2, f2 ] )]
2 : [Group( [ f1, f2, f3 ] ) => Group( [ f2, f1 ] )]
gap> IsomorphismCat2Groups( all82[31], all82[34] ) = fail;
false
```

In the following tables the groups of size at most 30 are ordered by their GAP number. For each group G we list the number |IE(G)| of idempotent endomorphisms; the number $|\mathcal{C}^1(G)|$ of cat^1 -groups on G, followed by the number of their isomorphism classes; and then the number $|\mathcal{C}^2(G)|$ of cat^2 -groups on G, and the number of their isomorphism classes. The number of isomorphism classes $\mathcal{C}^1(G)$ of cat^1 -groups is given in [2].

We may reduce the size of the table by noting the results for cyclic groups. When $G=C_{p^k}$ is cyclic, with p prime, the only idempotent endomorphisms are the identity and zero maps. In this case all the cat¹-groups have equal tail and head maps, and all isomorphism classes are singletons. Similarly, when $G=C_{p_1^{k_1}p_2^{k_2}...p_m^{k_m}}$ is cyclic, and its order is the product of m distinct primes p_i having multiplicities k_i , there are 2^m idempotent endomorphisms and cat¹-groups. The results up to m=4 are shown in Table 1. The rows headed "groups" list, for each cat²-group, its four groups $[G,R_1,R_2,R_{12}]$ where, for example, $2\times[G,I,C_{p^k},I]$ denotes $\{[G,I,C_{p^{k_1}},I],[G,I,C_{p^{k_2}},I]\}$.

Table 1. Results for families of cyclic groups.

G		$ \mathrm{IE}(G) $	$ \mathcal{C}^1(G) $	$ \mathcal{C}^1(G)/\cong $	$ \mathcal{C}^2(G) $	$ \mathcal{C}^2(G)/\cong $	
groups	[G, I, I]	,I], [G,I]	[G,G,I], [G]	[G,G,G,G]			
$C_{p_1^k}$	1	2	2	2	3	3	
$C_{p_1^{k_1}}$		4	4	4	10	10	
groups	[G, I, I]	[,I], [G,I]	[G,G,I], [G]	$[G,G,G,G], \ 2$	$\times [G, I, C_{p^l}]$	$[\epsilon,I],$	
	$2 \times [G]$	C_{p^k}, G, C	$[C_{p^k}], \ 2 \times$	$[G, C_{p^k}, C_{p^k}, C_{p^k}, C_{p^k}]$	$[C_{p^k}], [G, C]$	$C_{p_1^{k_1}}, C_{p_2^{k_2}}, I]$	
$\overline{C_{p_1^{k_1}p_2^k}}$	$^{2}p_{3}^{k_{3}}$	8	8	8	36	36	
groups	[G, I, I]	[,I], [G,I]	[G,G,I], [G]	$[G,G,G,G], \ 3$	$\times [G, I, C_{p^l}]$	[k, I],	
$3 \times [G, C_{p^k}, G, C_{p^k}], \ 3 \times [G, C_{p^k}, C_{p^k}, C_{p^k}], \ 3 \times [G, C_{p^k}, C_{q^j}, I],$							
	$3 \times [G]$	$I, C_{p^k q^j},$	I], $3 \times [6]$	$G, C_{p^k q^j}, G, C_p$	$_{pk_{q^j}}], 3 \times [$	$G, C_{p^kq^j}, C_{p^kq^j}, C_{p^kq^j}],$	
	$6 \times [G]$	C_{p^k}, C_{p^k}	$_{q^j}, C_{p^k}],$	$3 \times [G, C_{r^i}, C_p]$	$_{o^kq^j},I],\ 3$	$\times \left[G, C_{p^kq^j}, C_{p^kr^i}, C_{p^k}\right]$	
$C_{\mathfrak{p}^{k_1}\mathfrak{p}^{k_2}}$	ko kı	16	16	16	136	136	

When m=1 there are 16 cyclic groups of order at most 30; when m=2 there are 12 such groups; and when m=3 there is just the small group $30/4=C_{30}$.

Table 2 contains the results for those G which are not cyclic. The 1,000 isomorphism classes contain just 13 cat²-groups whose diagonal is *not* a cat¹-group: one each for groups [8/3, 16/3, 16/13, 27/3], three for 24/10 and six for 16/11.

Table 2: Results for families of non-cyclic groups.

GAP #	G	IE(G)	$ \mathcal{C}^1(G) $	$ \mathcal{C}^1(G)/\cong $	$ \mathcal{C}^2(G) $	$ \mathcal{C}^2(G)/\cong $
1/1	I	1	1	1	1	1
4/2	$K_4 = C_2 \times C_2$	8	14	4	36	9
6/1	S_3	5	4	2	7	3
8/2	$C_4 \times C_2$	10	18	6	47	14
8/3	D_8	10	9	3	21	6
8/4	Q_8	2	1	1	1	1
8/5	$C_2 \times C_2 \times C_2$	58	226	6	1,711	23
9/2	$C_3 \times C_3$	14	38	4	93	9
10/1	D_{10}	7	6	2	11	3
12/1	$C_3 \ltimes C_4$	5	4	2	7	3
12/3	A_4	6	5	2	9	3
12/4	D_{12}	21	12	4	41	10
12/5	$C_3 \times K_4$	16	28	8	136	32
14/1	D_{14}	9	8	2	15	3
16/2	$C_4 \times C_4$	26	98	5	231	11
16/3	$(C_4 \times C_2) \ltimes C_2$	18	25	4	57	7
16/4	$C_4 \ltimes C_4$	10	17	3	25	4
16/5	$C_8 \times C_2$	10	18	6	47	14
16/6	$C_8 \ltimes C_2$	6	5	2	9	3
16/7	D_{16}	18	9	2	17	3
16/8	QD_{16}	10	5	2	9	3
16/9	Q_{16}	2	1	1	1	1
16/10	$C_4 \times K_4$	82	322	12	$2,\!875$	53
16/11	$C_2 \times D_8$	82	97	9	649	29
16/12	$C_2 \times Q_8$	18	17	3	25	4
16/13	$(C4 \times C2) \ltimes C_2$	26	13	2	37	4
16/14	$K_4 imes K_4$	382	4,162	9	$298,\!483$	53
18/1	D_{18}	11	10	2	19	3
18/3	$C_3 \times S_3$	12	8	4	24	10
18/4	$(C_3 \times C_3) \ltimes C_2$	47	118	4	541	9
18/5	$C_6 \times C_3$	28	76	8	358	32
20/1	Q_{20}	7	6	2	11	3
20/3	$C_4 \ltimes C_5$	7	6	2	11	3
20/4	D_{20}	31	18	4	65	10
20/5	$C_5 \times K_4$	16	28	8	136	32
21/1	$C_3 \ltimes C_7$	9	8	2	15	3
22/1	D_{22}	13	12	2	23	3
24/1	$C_3 \ltimes C_8$	5	4	2	7	3
24/3	SL(2,3)	6	1	1	1	1
24/4	Q_{24}	5	4	2	7	3
24/5	$S_3 \times C_4$	27	12	4	41	10
24/6	D_{24}	33	20	4	75	10
24/7	$Q_{12} \times C_2$	25	36	6	115	14
24/8	$D_8 \ltimes C_3$	23	12	4	41	10
24/9	$C_{12} \times C_2$	20	36	12	178	52
24/10	$D_8 \times C_3$	20	18	6	75	20
24/11	$Q_8 \times C_3$	4	2	2	3	3
$\frac{24}{12}$	S_4	12	5	2	9	3
$\frac{24}{13}$	$A_4 \times C_2$	15	10	4	31	10
24/14	$S_3 \times K_4$	157	116	8	999	32

GAP #	G	IE(G)	$ \mathcal{C}^1(G) $	$ \mathcal{C}^1(G)/\cong $	$ \mathcal{C}^2(G) $	$ \mathcal{C}^2(G)/\cong $
24/15	$C_6 \ltimes K_4$	116	452	12	6,786	84
25/2	$C_5 \times C_5$	32	152	4	348	9
26/1	D_{26}	15	14	2	27	3
27/2	$C_9 \times C_3$	20	56	6	138	14
27/3	$(C_3 \times C_3) \ltimes C_3$	38	37	2	127	4
27/4	$C_9 \ltimes C_3$	11	10	2	19	3
27/5	$C_3 \times C_3 \times C_3$	236	2,108	6	24,222	16
28/1	Q_{28}	9	8	2	15	3
28/3	D_{28}	41	24	4	89	10
28/4	$C_7 \times K_4$	16	28	8	136	32
30/1	$S_3 \times C_5$	10	8	4	24	10
30/2	$D_{10} \times C_3$	14	12	4	38	10
30/3	D_{30}	25	24	4	92	10

Table 2: Results for families of non-cyclic groups.

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