- Math deals with finite objects or processes in finite steps.

— Kronecker

- "Gods created the integers, all else is the work of man."
- Some irrational numbers are constructible, like $\sqrt{2}$, but π is not until Newton found it can be obtained by an infinite series.
- So <u>analysis is wrestling with infinity</u>.

<u>Sets & Relations</u>

- A set is <u>collection</u> of objects.

$$S = \{1, ☹, \square, \{1, ☺\}\}.$$

Or: $S = \{x : P(x) \text{ is true}\}$.

$:$ such that

P is some statement of x. e.g. "$x < 2$".

<u>Shorthand</u>: $x \in S$ "x is in S"

$x \notin S$ "x is not in S".

ϕ is the empty set.

$A \subset B$ means "A is a <u>subset</u> of B", which means

if $x \in A$ then $x \in B$.

$(x \in A \Rightarrow x \in B)$

If $A \subset B$ and $B \not\subset A$, then call A a <u>proper</u> subset of B.

If $A \subset B$ and $B \subset A$, then $A = B$, else $A \neq B$.

More sets: (from old sets)

union: $A \cup B = \{x : x \in A \text{ } \underline{\text{or}} \text{ } x \in B\}$.

intersection: $A \cap B = \{x : \cdots \text{ } \underline{\text{AND}} \text{ } \cdots\}$.

complement: $A^c = \{x : x \notin A\}$.

minus: $A \setminus B = \{x : x \in A \text{ AND } x \notin B\}$.

product: $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

(a, b): order pair

- A (binary) <u>relation</u> R is a <u>subset</u> of $A \times B$.
  If $(a,b) \in R$, <u>write</u> aRb.

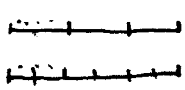
EX. A "is an ancester of" is a ~~pre~~ rel'n on $P \times P$ (people)
L "likes" ... $P \times P$.
S "is a sibling of" ... $P \times P$.
$<$ "less than" ... $Z \times Z$ (integers)

- A <u>equivalence rel'n</u> R on set S is
  a <u>relation</u> on $S \times S$ s.t.
  ① aRa <u>reflexies</u>
  ② $aRb \Rightarrow bRa$ <u>symmetry</u>
  ③ aRb and $bRc \Rightarrow aRc$ <u>transitive</u>

  often write $\sim$, $\approx$, $\cong$, etc. but avoid $=$.

- Aside, A <u>function</u> from A to B is a <u>relation</u> s.t.
  <u>if</u> aFb and aFb' then $b = b'$.
  (rule that assigns to each $a \in A$ a unique $b \in B$)

Construction of $\mathbb{Q}$, the rational numbers

- Assume Z, the <u>integers</u>, their <u>multiplications</u> & <u>sums</u>.
- What is $\mathbb{Q}$? Perhaps it's that $\{\frac{m}{n} : m, n \in Z,\ n \neq 0\}$
  not quite enough, what does it mean by $\frac{m}{n}$?
- Motivation  one out of six is equivalent to two out of six

  <u>Write</u> $(1,3) \sim (2,6)$ as equivalent ordered pairs. $(Z \times Z) \times (Z \times Z)$
- <u>Idea</u> these belong to <u>equivalent class</u>, we'll call "$\frac{1}{3}$".
  Let $\mathbb{Q}$ = set of all such <u>equivalent class</u> of ordered pairs in $Z \times (Z \setminus \{0\})$.
- extend pairs to include Z such that "$\frac{n}{1}$" $\in \mathbb{Q}$ corresponds to $n \in Z$.

- See that $\mathbb{Q} = \{ \frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0 \}$ (set of classes)

  where $\frac{m}{n}$ is an equivalence class of (m, n)

  with relation $(p, q) \sim (m, n)$ if $mq = pn$ and $q, n \neq 0$.

- Check : $\sim$ is equivalence rel'n.

  ① check $(p,q) \sim (p,q)$: $pq = pq$

  ② check $(p,q) \sim (m,n) \Rightarrow (m,n) \sim (p,q)$

  ③ check $(p,q) \sim (m,n)$ and $(m,n) \sim (a,b) \Rightarrow (p,q) \sim (a,b)$

  The cancellation law in $\mathbb{Z}$: if $ab = ac$ and $a \neq 0$ then $b = c$.

Given $pn = qm$, $mb = an$

① $m = 0 \Rightarrow p = a = 0$, $pq = pb = qa$

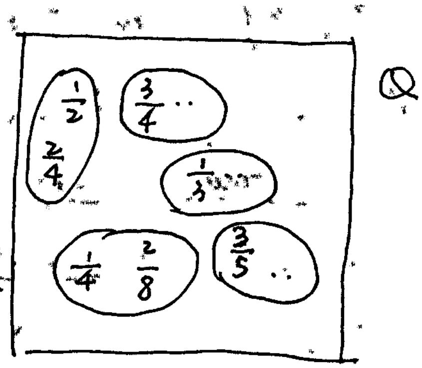
② $m \neq 0 \Rightarrow p\, mnb = qmna$, $pb = qa$ by cancellation law.

PROPERTIES of $\mathbb{Q}$: ARITHMETIC, ORDER

- Addition (on classes)

(BAD) Def'n: $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$

The definition should not depend on the representatives you pick. (NOT WELL-DEFINED)



EX. $\frac{1}{2} + \frac{1}{3} = \frac{2}{5} \not\sim \frac{2}{4} + \frac{1}{3} = \frac{3}{7}$.

Want: notion that does not depend on representative chosen!

addition of classes

- How about $\frac{a}{b} + \frac{c}{d} = \frac{0}{1}$. Check : it's well-defined, but boring.

- Good def'n : $\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}}$

To show it's well-def'd, must show

if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ then $(ad+bc, bd) \sim (a'd'+b'c', b'd')$.

Proof. $ab' = a'b$, $cd' = c'd$

$(ad+bc)b'd' = a'bdd' + bb'c'd = (a'd'+b'c')bd$

• <u>Multiplication</u>: $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$. check:

Given $(a,b)\sim(a',b')$ and $(c,d)\sim(c',d')$, we have $ab'=a'b$, $cd'=c'd$.

Show $(ac,bd)\sim(a'c',b'd')$.

<u>Proof</u>. $acb'd'=a'bc'd=bda'c'$

• In what sense does $\mathbb{Q}$ <u>extends</u> $\mathbb{Z}$? Check that $\{\frac{n}{1}:n\in\mathbb{Z}\}$ behaves like $\mathbb{Z}$.

The <u>correspondence</u> is $\frac{n}{1}\longleftrightarrow n$.

• $\mathbb{Z}$ has an <u>order</u>. Does $\mathbb{Q}$?

• <u>Def'n</u>. An <u>order</u> on set S is a <u>relation</u> $<$ satisfying

① (trichotomy) If $x,y\in S$, <u>exactly</u> one of these is true:
$x<y$, $x=y$, $y<x$.

② (transitivity) If $x,y,z\in S$, ~~&~~ and if $x<y$, $y<z$, then $x<z$.

call S an ordered set if it has an order.

<u>EX</u>. in $\mathbb{Z}$, say $m<n$ if $n-m$ is <u>positive</u>, i.e., in the set $\{1,2,3,4,\cdots\}$.

<u>EX</u>. in $\mathbb{Z}\times\mathbb{Z}$, say $(a,b)<(c,d)$ if $[a<c]$ <u>or</u> $[a=c$ and $b<d]$.
(<u>dictionary order</u>)

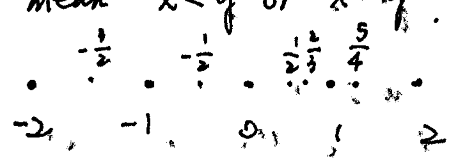
<u>EX</u>. in $\mathbb{Q}$, say $\frac{m}{n}$ is <u>positive</u> if $mn>0$.

<u>Check</u>: well-defined: if $(m,n)\sim(p,q)$ and $mn>0$, then
$mq=np$ and both m,n are positive or negative
$\Rightarrow p,q$ are both positive or negative. $pq>0$.

Then say $\frac{m}{n}<\frac{m'}{n'}$ if $\frac{m'}{n'}-\frac{m}{n}$ is <u>positive</u>.

- Write "$y>x$" for $x<y$.
- Write "$x\le y$" to mean "$x<y$ or $x=y$".

New picture of $\mathbb{Q}$:



$\mathbb{Q}$: Good enough to solve: $5x=3$. See $x=\frac{3}{5}$ solves this.

not <u>good enough</u> to solve: $x^2=2$.

Thm. $x^2=2$ has no <u>solution</u> in $\mathbb{Q}$.

Proof. (by contradiction).

Assume $x^2=2$ has a solution in $\mathbb{Q}$, i.e., say $x=\frac{p}{q}$ where $p, q \in \mathbb{Z}$, and assume p, q are in "Lowest terms", i.e., have no common factors.

So $(\frac{p}{q})^2=2$, hence $p^2=2q^2$ (cross product).

Then p^2 is even (divisible by 2).

Then p is even (because if p were odd, p^2 would be odd.).

(because p^2 has a factor 2, p has a factor 2).

So $p=2m$, $m \in \mathbb{Z}$, hence $p^2=4m^2$ and $4m^2=2q^2$. Then $2m^2=q^2$.

Then q^2 is <u>even</u>, hence q is <u>even</u>.

This contradicts p, q are in lowest form.

[So, $x^2=2$ must have no solution in $\mathbb{Q}$] ∎

- $\mathbb{Q}$ is a <u>field</u>.

  in $\mathbb{Q}$, 0 element is $\frac{0}{1}$.

  1 element is $\frac{1}{1}$.

- check these <u>axioms</u> hold.
- $\mathbb{Z}$ is <u>not</u> a field because does not satisfy (M5).
- $\mathbb{Q}$ is an <u>ordered field</u>: <u>field</u> with an <u>order</u> so that:

  order is <u>preserved</u> by <u>field ops</u>

  ① $y < z \Rightarrow x+y < x+z$,

  ② $y < z$, $x > 0 \Rightarrow xy < xz$.

- We cannot solve certain equations before as $5x=3$, but now we can solve it in $\mathbb{Q}$ because $\mathbb{Q}$ is a field.

CONSTRUCT THE REAL NUMBERS

- No $x \in \mathbb{Q}$ exists such that $x^2 = 2$. There are holes on the umber line.

LEAST UPPER BOUND

(The notations are not appropriate for formal writing.)

- <u>Def'n</u>: Say $E \subset S$ <u>ordered</u>.

  If <u>there exists</u> $\exists$ $\beta \in S$ s.t. <u>for all</u> $\forall$ $x \in E$ we have $x \le \beta$, then call β an <u>upper bound</u> for E, say E is <u>bounded above</u>.

<u>Def'n</u> of <u>lower bound</u>, replace $\le$ with $\ge$.

<u>EX</u>. 2 is an upper bound for $A = \{x \mid x^2 < 2\}$.

$\frac{3}{2}$ is an <u>u.b.</u> for A. (WHY? If not, $\exists x \in A$ s.t. $x > \frac{3}{2}$, then $x^2 > (\frac{3}{2})^2 > 2$.)

(formal: imply ; informal: $\Rightarrow$)

- <u>Def'n</u>. If $\alpha \in S$ s.t.:

  ① α is an <u>u.b.</u> of E and

  ② if $\gamma < \alpha \Rightarrow \gamma$ is not an <u>u.b.</u> for E

  then α is the <u>least upper bound</u> (lub) of E or <u>supremum</u> of E, called <u>$\alpha = \sup E$</u>.

<u>EX</u>. ($S = \mathbb{Q}$) $E = \{\frac{1}{2}, 1, 2\}$. $\sup E = 2$. (finite sets have "sup".)

$E = \mathbb{Q}_-$, the <u>negative rationals</u>. $\sup E = 0$.

$E = \mathbb{Q}$. $\sup E$ <u>does not exist</u>. (unbounded above) "$\sup E = +\infty$"

$E = A$. $\sup A$ <u>does not exist</u> though it's <u>bounded</u>.

(hint: $\forall$ u.b. α, $\exists \gamma < \alpha$ is also an u.b.)

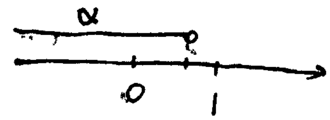
CONSTRUCT $\mathbb{R}$ AND PROVE

<u>THM</u>. (1.19) $\mathbb{R}$ is an <u>ordered field</u>, with <u>l.ub property</u>, and $\mathbb{R}$ ~~extends~~ contains $\mathbb{Q}$ as a subfield.

<u>Def'n</u> A set S has <u>lub property</u> (satisfies the <u>completeness</u> axiom) if every <u>non-empty</u> subset of S that has an upper bound, also has a lub (sup) in S.

If E is empty, then there's no need to check the condition, which is automatically true.

<u>Dedekind</u>: A <u>cut</u> α is a subset of $\mathbb{Q}$ s.t.



① $\alpha \neq \phi, \mathbb{Q}$, (<u>nontrivial</u>) and

② If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$ (<u>closed downwards</u>)

③ If $p \in \alpha$, then $p < r$ for some $r \in \alpha$. (<u>no largest number</u>).

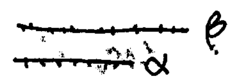
<u>EX</u>. set A is not a cut (fails ②).

<u>EX</u>. $\alpha = \mathbb{Q}_-$ is a cut.

<u>EX</u>. $\beta = \{x \in \mathbb{Q} \mid x \leq 2\}$ is not a cut (fails ③).

Let <u>$\mathbb{R} \overset{\Delta}{=} \{\alpha : \alpha \text{ is a cut}\}$</u>. <u>Def'n</u> some set, show it has <u>structure</u>!

• <u>Define</u> <u>order</u> <u>say $\alpha < \beta$ means $\alpha \subsetneqq \beta$</u>. (proper subset)



<u>Check</u> order (trichotomy, transitivity)

• Addition:

define $\alpha + \beta := \{r + s : r \in \alpha \text{ and } s \in \beta\}$. (a collection of rationals.)

<u>check</u> is a <u>cut</u>: ① u.b. $m+n$, m for α and n for β. (non-trivial)

② (closed downs): If $p \in \alpha + \beta$, say $q < p = r + s$.

Note $q - s < r$, so $q - s \in \alpha$.

Then $q = (q-s) + s \in \alpha + \beta$, <u>as desired</u>.

③ (no largest number): If $q \in \alpha + \beta$, $q = r + s$ where $r \in \alpha$ and $s \in \beta$.

So $\exists\, r' \in \alpha$ and $\cancel{p \in}$ $s' \in \beta$ s.t. $r < r'$ and $s < s'$.

Then $q < s' + r' = p \in \alpha + \beta$.

• Show axioms <u>A1 – A5</u>.

A1: $\alpha + \beta = \beta + \alpha$ by communativity of rational fields.

A2: add identity $0^* = \mathbb{Q}_-$ <u>neg. rat'ls</u>. check $\alpha + 0^* = \alpha$.

so verify $\alpha + 0^* \subset \alpha$ and $\alpha \subset \alpha + 0^*$

If $q \in \alpha$ and $p \in 0^*$ s.t. $p < 0$, then $q + p < q$.

Then $q + p \in \alpha$ by closed downs. $\Rightarrow \alpha + 0^* \subset \alpha$.

$\forall\, p \in \alpha$, $\exists\, r \in \alpha$ s.t. $r > p$. Similarly, $\exists\, q \in \alpha$ s.t. $q > r$.

Note $\cancel{p-r}_{\,r-q} < 0$, so $\cancel{p-r}_{\,r-q} \in 0^*$. Then $p < q + (r - q) = r$. $\Rightarrow p \in \alpha + 0^* \Rightarrow \alpha \subset \alpha + 0^*$

A3: add inverse for α

$\beta = \{ p : \exists r > 0 \text{ s.t. } -p-r \notin \alpha \}$

show $\alpha + \beta = 0^*$

<u>Multiplication</u>: be careful of <u>neg rat's</u>

<u>Define</u>: if $\alpha, \beta \in R_+$ $(\alpha\beta > 0^*)$

$\alpha\beta := \{ p : p < rs \text{ for some } r \in \alpha,\ s \in \beta,\ r,s > 0 \}$

let $1^* = \{ q < 1 : q \in \mathbb{Q} \}$

- Given a set A of cuts, let $\gamma = \bigcup \{ \alpha : \alpha \in A \}$.
  Then γ is a cut and $\gamma = \sup A$.

- Also, $\mathbb{R}$ <u>contains</u> $\mathbb{Q}$ as a <u>subfield</u>?
  Associate to <u>$q \in \mathbb{Q}$</u> the q^* ~~set~~ cut $=$ <u>$\{ r \in \mathbb{Q} : r < q \}$</u>

  <u>check</u>: $f : \mathbb{Q} \to \mathbb{R}$ preserves $+, \times, <$.
  $q \to q^*$

  If $p^* = \{ r \in \mathbb{Q} : r < p \}$, $q^* = \{ r \in \mathbb{Q} : r < q \}$, $p, q \in \mathbb{Q}$.

① then $p^* + q^* = \{ r + s : r \in p^* \text{ and } s \in q^* \} = \{ r \in \mathbb{Q} : r < p + q \}$

② Then $p^* q^* = \{ m : m < rs \text{ for some } r \in p^*;\ s \in q^*,\ r,s > 0 \}$

If $p, q \in \mathbb{Q}_+$, $= \{ m : m < rs \text{ for some } r < p,\ s < q,\ r,s > 0 \}$

$= \{ m : m < pq \}$

③ ~~Then~~ If $p^* < q^*$, so $p^* \subseteq q^*$. Then ~~$\alpha \in p^* \Rightarrow \alpha \in q^*$~~ $p < q$.

It's also a one-~~to~~-one injection, because for two different $p, q \in \mathbb{Q}$, p^* and q^* are different.

Then $\mathbb{Q}' = \{ p^* : p \in \mathbb{Q} \}$ is a ~~subj~~ subfield of $\mathbb{R}$.

Notice: Length "$\sqrt{2}$" sits in $\mathbb{R}$ as

$\gamma = \{ q : q^2 < 2 \text{ or } q < 0 \}$.

Check: use def'n of multiplication, that $\gamma^2 = 2^*$.

- $\mathbb{R}$ has the lub property!

If A is a collection of cuts with ub β.

Let $\gamma = \bigcup \{\alpha : \alpha \in A\}$, a subset of $\mathbb{Q}$.

Check: γ is a cut & $\gamma = \sup A$.

- nontrivial (because it's a union of cuts and bounded above).
  closed downwards ($\forall x \in \gamma$, $\exists y \in$ some $\alpha \in A$, s.t. $x < y$)
  no largest number ($\forall x \in \gamma$, $\exists \alpha \in A$, $\exists y \in \alpha$, s.t. $x < y$, $y \in \gamma$).
- γ is an u.b. clearly, since γ contains all α in A.
- γ is lub, because if $\delta < \gamma$, $\exists x \in \gamma \setminus \delta$. Then $x \in$ some α in A, not in δ. So δ is not u.b. for A.

Then (1.19): $\mathbb{R}$ is an ordered field, extends $\mathbb{Q}$, has lub property.

FACT: $\mathbb{R}$ is the only ordered field with the lub property.

Consequence: Length "$\sqrt{2}$" $= \sup \{1, 1.4, 1.41, 1.414, 1.4142, \cdots\}$. Def'n of $\sqrt{2}$.
$= 1.4142135\cdots$ (decimal representation)

- More generally: $a^{1/n} \triangleq \sup \{ r \in \mathbb{Q} : r^n < a \}$. check: $(a^{1/n})^n = a$
  root exists

GREATEST LOWER BOUND (glb) as infimum, write $\inf A$.

$\inf A = -\sup(-A)$

- In $\mathbb{R}$, $\inf A$ exists if set is bounded below. ($\mathbb{R}$ satisfies glb property.)

CONSEQS OF LUB PROP

- Archimedean property of real numbers $\mathbb{R}$.

If $x, y \in \mathbb{R}$, $x > 0$, then $\exists$ positive integer n s.t. $nx > y$.

Equivalently, if $x > 0$, then $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < x$.

proof. $A = \{nx : n \in \mathbb{N}\}$. (by contradiction)

If A were bounded by y (e.g. $nx < y \ \forall n \in \mathbb{N}$).

So A has a lub, call it α.

Then $\alpha - x$ is not u.b. for A, hence $\alpha - x < mx$ for some $m \in \mathbb{N}$.

So $\alpha < (m+1)x$, so α is not an u.b. for A. by contradiction. ∎

• Thm. Between $x, y \in \mathbb{R}$, $x < y$, $\exists q \in \mathbb{Q}$ s.t. $x < q < y$.

($\mathbb{Q}$ is dense in $\mathbb{R}$.)

Proof. Choose n s.t. $\frac{1}{n} < y - x$. (ARCHIMEDEAN PROPERTY)

Consider multiples of $\frac{1}{n}$, these are unbdd. ( )

Choose first multiple $\frac{m}{n} > x$.

Claim $\frac{m}{n} < y$. If not, then $\frac{m-1}{n} < x$ and $\frac{m}{n} > y$.

But this implies $\frac{1}{n} > y - x$. ∎

PROPERTIES of sup:

(a) γ is an u.b for $A \Leftrightarrow \sup A \leq \gamma$.

(b) $\forall a \in A$, $a \leq \gamma \Leftrightarrow \sup A \leq \gamma$.

(c) $\forall a \in A$, $a < \gamma \Rightarrow \sup A \leq \gamma$.

(d) $\gamma < \sup A \Rightarrow \exists a \in A$ s.t. $\gamma < a \leq \sup A$.

(e) If $A \subset B$, then $\sup A \leq \sup B$.

($\forall a \in A$, $a \in B$. so $a \leq \sup B$. By (b), $\sup A \leq \sup B$)

two ways showing supremum:

① γ is an u.b., smaller ones are not.

② γ is an u.b., all the other u.b. is larger than γ.

(f) To show $\sup A = \sup B$.

one strategy: show $\forall a \in A$, $\exists b \in B$. s.t. $a = b$. Thus $\sup A \leq \sup B$.

Then the other way around.

- EXTENDED REALS

$\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$

Put order: $\forall x \in \mathbb{R}, -\infty < x < +\infty$.

and arithmetic: $x + (+\infty) = +\infty$; $x + (-\infty) = -\infty$.

if $x > 0$, $x \cdot (+\infty) = +\infty$

if $x < 0$, $x \cdot (+\infty) = -\infty$, etc.

This may be not a field! Because of not well-defined..

- Why care? convenient, e.g.,
  every subset in $\bar{\mathbb{R}}$ a sup (possibly $+\infty$).

- Euclidean space $\mathbb{R}^k := \{(x_1, x_2, \cdots, x_k) : x_i \in \mathbb{R} \ \forall i\}$

Define: arithmetic $\underbrace{(x_1, \cdots, x_k)}_{\vec{x}} + \underbrace{(y_1, \cdots, y_k)}_{\vec{y}} = (x_1 + y_1, \cdots, x_k + y_k)$ elementwise addition.

It doesn't have some multiplication definition to turn this into a field.

scalar mult: $\alpha(x_1, \cdots, x_k) = (\alpha x_1, \cdots, \alpha x_k)$.

α ↑ scalar in $\mathbb{R}$ (associative, communitive, distributive)

- Also, $\mathbb{R}^k$ has an "inner product" (dot product)

$\vec{x} \cdot \vec{y} = \sum_i x_i y_i$ (summation)

norm / length $|\vec{x}| := (\vec{x} \cdot \vec{x})^{\frac{1}{2}}$.

- Complex number field.

$\mathbb{R}^2$ can be given a field structure.

$(a, b) + (c, d) = (a+c, b+d)$,

$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

Here, the zero is $(0, 0)$, and the 1 is $(1, 0)$. (check)

additive identity — multiplicative identity

Write $\mathbb{C}$, the set $\mathbb{R}^2$ with $+, \times$ as ~~of~~ above.

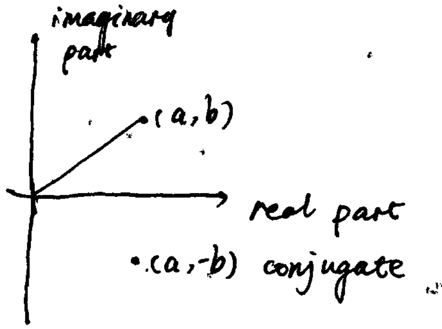
- $\mathbb{C}$ extends $\mathbb{R}$: $\{(a, 0) : a \in \mathbb{R}\}$ behaves like $\mathbb{R}$, is a subfield isomorphic to $\mathbb{R}$.

• Note: $\underbrace{(0,1)}_{i} \cdot (0,1) = \underbrace{(-1,0)}_{\text{in } \mathbb{R}}$. zero element in the first coordinate produces something in the 1st coordinate.

see $i^2 = -1$, a real number by extention.

Write $a+bi$ for (a,b).

If $z = a+bi$, let $\bar{z} = a-bi$, the conjugate of z. Def'n



Check: $\overline{z+\omega} = \bar{z} + \bar{\omega}$.

$\overline{z\cdot\omega} = \bar{z}\cdot\bar{\omega}$.

$z + \bar{z} = 2\mathrm{Re}(z)$, and $z - \bar{z} = 2i\,\mathrm{Im}(z)$.

$z\cdot\bar{z} = a^2 + b^2$ real ≥ 0

Define $|z| = (z\cdot\bar{z})^{1/2}$, same as length in $\mathbb{R}^2$.
abs. value

• Suggests, a $\mathbb{C}^k = \{(z_1, \cdots, z_k),\ z_i \in \mathbb{C}\}$ (twice size of $\mathbb{R}^k$), the inner product

$$\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{k} x_i \bar{y}_i$$

Prop's. $|z| \geq 0$, $|\bar{z}| = |z|$, $|z\omega| = |z||\omega|$, $\mathrm{Re}(z) \leq |z|$, and $|z+\omega| \leq |z| + |\omega|$ (triangle inequality)

based on $(ac-bd)^2 + (ad+bc)^2 = (a^2+b^2)(c^2+d^2)$

Why? $|z+\omega|^2 = (z+\omega)\cdot(\bar{z}+\bar{\omega})$

$= z\cdot\bar{z} + z\cdot\bar{\omega} + \omega\cdot\bar{z} + \bar{\omega}\cdot\omega$

$= |z|^2 + 2\mathrm{Re}(z\bar{\omega}) + |\omega|^2 \leq |z|^2 + 2|z||\omega| + |\omega|^2 = (|z| + |\omega|)^2$

This yields desired inequality.

• Cauchy-Schwarz inequality.

If $a_1, \cdots, a_n, b_1, \cdots, b_n$ are complex numbers, then

$$\left|\sum_{i=1}^{n} a_i \bar{b}_i\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \cdot \sum_{i=1}^{n} |b_i|^2$$

In $\mathbb{R}^n$. $|\vec{v}\cdot\vec{\omega}| \leq |\vec{v}|\cdot|\vec{\omega}|$.

$|\langle \vec{v}\cdot\vec{\omega}\rangle|^2 \leq \langle \vec{v}\cdot\vec{v}\rangle\langle\vec{\omega}\cdot\vec{\omega}\rangle$

<u>proof</u>. Let $\vec{a}, \vec{b} \in \mathbb{C}^n$, note $0 \le |\vec{a} - y\vec{b}|^2 = \langle a - yb, a - yb \rangle = \sum (a_i - yb_i)(\bar{a}_i - \bar{y}\bar{b}_i)$

$$= \langle a,a \rangle - \bar{y}\langle a,b \rangle - y\langle b,a \rangle + |y|^2 \langle b,b \rangle$$

Choose $y = \dfrac{\langle a,b \rangle}{\langle b,b \rangle}$ $\longrightarrow$ $= \langle a,a \rangle - \dfrac{|\langle a,b \rangle|^2}{\langle b,b \rangle}$

Mutiply both sides with $\langle b,b \rangle$ then is the desired inequality. ∎

<u>INDUCTION</u>

- Let $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$, natural numbers.
- <u>Well-ordering property</u> of $\mathbb{N}$ (WOP):

  (Axiom)

  $\mathbb{N}$ is <u>well-ordered</u>: every <u>nonempty</u> subset of $\mathbb{N}$ has a <u>least element</u>.

<u>Remark</u> can take WOP ~~to~~ be an <u>axiom</u> of $\mathbb{N}$.

- <u>Principle of Induction</u>: (POI)

  Let S be a subset of $\mathbb{N}$ such that:

  ① $1 \in S$,

  ② If $k \in S$ then $k+1 \in S$,

  then $S = \mathbb{N}$.

<u>FACT</u>: WOP $\Leftrightarrow$ POI.

<u>Proof</u>. WOP $\Rightarrow$ POI

(by contradiction) Suppose S exists with given <u>properties</u> of POI, but $S \ne \mathbb{N}$.

Then $A = \mathbb{N} \setminus S$ is nonempty, has a <u>least element</u> (by WOP), call it n.

Notice $n > 1$ because $1 \in S$ so $1 \notin A$, by prop. ①.

Consider $n-1$, it is not in A, so it is in S, By ~~prop~~ prop. ②, $(n-1)+1 \in S$, implying $n \in S$, <u>contradicts</u> by $n \in A$.

Therefore, POI holds. ∎

- PROOFS OF INDUCTION

Let $P(x)$ be statement indexed by $x \in \mathbb{N}$.

Idea: to show $P(n)$ is true for all n,

we'll show ⓐ $P(1)$ is true (base case)

ⓑ If $P(k)$ is true, then $P(k+1)$ is true. (inductive step)

inductive hypothesis

By the principle of induction, $P(n)$ is true for all n.

We're really considering $S = \{ n : P(n) \text{ is true} \}$ by showing $S = \mathbb{N}$.

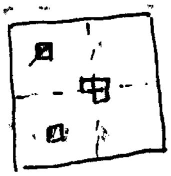
STRONG INDUCTION: use ⓑ' if $P(1), P(2), \cdots, P(k)$ is true, then $P(k+1)$ is true.

- STYLE: at start, tell reader (proof by induction)
  - tell the reader when you're doing base case* & ind. step*.
  - assume terms are understood.
  - remind reader of conclusion at end.

- EX. Every $2^n \times 2^n$ chessboard with one square removed can be tiled by .

Proof. (by induction on n)

- For base case, see can be filled by .
- For ind. step, we can assume any $2^n \times 2^n$ board (with a square removed)



Consider $2^{n+1} \times 2^{n+1}$

Thm. Prove $S_n = 1+3+5+\cdots+(2n-1)$ is a perfect square.

proof. base case $(n=1)$ holds, since $S_1 = 1 = 1^2$, a square desired.

For ind. step, we $\boxed{\text{assume } S_n \text{ is a square of } k^2}$ for some $k \in \mathbb{N}$.

too weak

$\to n^2$ strengthen

We wish to show S_{n+1} is a square. Let $S_{n+1} = 1+3+\cdots+(2n+1)$

$= S_n + (2n+1)$

$= k^2 + 2n+1$???

$= n^2+2n+1 = (n+1)^2$ (instead)

It's a square desired ∎

thm. All natural numbers are even. (any n)

proof. (by induction)

By strong induction: assume all numbers $\leq n$ are even.

Notice: $n+1 = (n-1)+2$, by ind. step, $n-1$ is even.

SO $n+1$ is an addition of two even numbers. It's even desired.

NO! The base case doesnt hold.

Thm. All horses are all the same color.

proof. (by induction on # horses in a given set).

For base case, with set of one horse, statement holds.

For ind step, let $S = \{h_1, \cdots, h_{n+1}\}$.

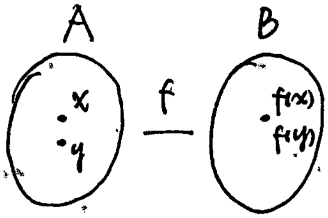
then $S' = \{h_1, \cdots, h_n\}$ and $S'' = \{h_2, \cdots, h_{n+1}\}$ have n horses

and by the ind. hypothesis, has the same color as ~~them~~ h_2.

So all horses have the same color as h_2 as desired. ∎

NO! The inductive step fails when $n=2$. There are no common horses.

HOW TO COUNT



- Recall $f: A \to B$
  domain co-domain

  "maps" $x \mapsto f(x)$

- If $C \subset A$, ~~$B \subset D$~~ $D \subset B$, define $f(C) = \{f(x): x \in C\}$ the image of C.

  define $f^{-1}(D) \doteq \{x : f(x) \in D\}$, the inverse image of D.

  B can ~~be~~ have multiple correspondences in A.

- when $\underbrace{f(A)}_{\text{range}} = B$, say f is onto (a surjection). $\twoheadrightarrow$

  when $f(x) = f(y)$ implies $x = y$, say f is 1-1 (an injection). $\hookrightarrow$

  When f is 1-1 and onto, call f a bijection, $\hookrightarrow\!\!\!\!\rightarrow$

  and say A and B are in "1-1 correspondence". Write $A \sim B$.

- Counting (Elementary): use $A = J_n = \{1, 2, \cdots, n\}$ or $J_0 = \Phi$.

EX. $\{1,$ ☺, ☺, $\pi\}$; $1 \leftrightarrow 4$, ☺ $\leftrightarrow 1$, ☺ $\leftrightarrow 3$, $\pi \leftrightarrow 2$; $A \leftrightarrow J_4$. Say $|A| = 4$.

- Def'n. Call A finite if $A \sim J_n$, else A infinite.
  Call A countable if $A \sim \mathbb{N}$.

- EX. $\mathbb{N}$ is countable: use $f: \mathbb{N} \to \mathbb{N}$ where $f(n) = n$.

- EX. A sequence $x_1, x_2, x_3, \cdots$ of distinct terms is countable: $f(n) = x_n$, $f: \mathbb{N} \to \{x_i\}$ set

  MORE: A set that can be "listed" in a sequence is countable!

- EX. $\{2, 3, 4, 5, \cdots\}$ is countable; use $f(n) = n+1$.

  $\{1, 2, 3, \cdots, k-1, k+1, k+2, \cdots\}$ countable; use $\begin{bmatrix} f(n) = n & \text{if } n < k \\ f(n) = n+1 & \text{if } n \geq k \end{bmatrix}$.

- Thm. $\mathbb{N}$ is in fact infinite.

proof (sketch). by induction n and show $\mathbb{N} \not\sim J_n$.

Base case: if $\mathbb{N} \xleftrightarrow{j} \{1\}$, then consider $\mathbb{N} \setminus \{j_1\}$ is not empty, so j is not onto.

ind. step: We'll show if $J_n \not\sim \mathbb{N}$ then $J_{n+1} \not\sim \mathbb{N}$.

If there were bijection $\mathbb{N} \xleftrightarrow{h} J_{n+1} = \{1, \dots, n, n+1\}$

then $\exists$ bijection $\mathbb{N} \setminus \{h(n+1)\} \xleftarrow{\hat{h}} J_n = \{1, \dots, n\}$

$$\begin{tikzcd}
\mathbb{N} \setminus \{h(n+1)\} \arrow[d, "f"] & J_n = \{1, \dots, n\} \arrow[l, "\hat{h}"'] \arrow[dl, dashed] \\
\mathbb{N} &
\end{tikzcd}$$

$\exists$ bijection f (previous example)

so $\exists$ bijection $\mathbb{N} \sim J_n$.

- EX. $\mathbb{N} \sim 2\mathbb{N}$ (even natural numbers) $= \{2, 4, 6, 8, \dots\}$, use $f(n) = 2n$.

  $\mathbb{N}$, $2\mathbb{N}$ has same cardinality (size).

- EX. $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is countable.

$$\begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ 7 & 5 & 3 & 1 & 2 & 4 & 6 \end{array}$$

Thm. Every inf. subset E of countable set A is countable.

proof idea let $n_1 = \inf\{i : x_i \in E\}$ ← exists by WOP.

$n_2 = \inf\{i : x_i \in E, i > n_1\}$

$\vdots$

$n_k = \inf\{ \quad \cdots \quad \cdots, i > n_{k-1}\}$.

Then $E = \{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\}$. ■

~~EX. $\mathbb{Q}$~~

Thm. $\mathbb{Q}^+$ is countable. (extends $\mathbb{Q}$.)

$$\begin{array}{ccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \end{array}$$

Then by the previous theorem, we take the subset of the sequence, it's still countable.

Thm. A countable $\Rightarrow$ $A \times A$ countable.

Is $\mathbb{R}$ countable?

- proposed listing

$\mathbb{N}$	$\overset{f}{\longleftrightarrow}$	$\mathbb{R}$
1	$\longleftrightarrow$	0.1234567890123
2	$\longleftrightarrow$	0.3041597653589
3	$\longleftrightarrow$	0.1404213511111
4	$\longleftrightarrow$	0.777077712357
5	$\longleftrightarrow$	0.4132400899245

propose a number that's different from $f(n)$ in the n^{th} decimal place. x^* is not on the list.

Sup. **Thm.** $\mathbb{R}$ is not countable.

x^* (claim x^* is not $f(n)$ for any n) $= 0.77717\cdots$

proof (by contradiction)

Suppose $\exists$ bijection f, s.t. we'll show f is not onto.

Why? x^* differs from $f(n)$ in the n^{th} decimal place! x^* not in image of f. ∎

- Have sets of different cardinalities $J_0, J_1, J_2, \cdots, J_n, \cdots, \underset{\mathbb{R}\,\mathbb{Q}}{\overset{}{\mathbb{N}}}, \mathbb{R}, ??$

Thm. For any A, $A \nsim 2^A$, the power set of A. (the set of all subsets of A)

proof. (diagonalization argument)

Thm. The countable union of countable sets is countable.

Proof. Say each $A_1, A_2, A_3, \cdots$ are cntable (indexed by $\mathbb{N}$)

Then
$$A_1 = \{a_{11}, a_{12}, a_{13}, \cdots\}$$
$$A_2 = \{a_{21}, a_{22}, a_{23}, \cdots\}$$
$$A_3 = \{a_{31}, a_{32}, a_{33}, \cdots\}$$
$$\vdots$$

(some elements may repeat)

If there is a A that's finite, fill blocks to make it ~~cnt~~ infinite.

So $\bigcup_{i=1}^{\infty} A_i$ is cntable.

Notation: Use $\bigcup_{\alpha \in I} A_\alpha$ for possibly uncntable ~~set~~ collection. I = index set.

- EX. The set of computer programs is countable!

Recall $\mathbb{R}$ is ~~a~~ uncountable (infinite).

so amazing: there are real numbers that are not "computable". (can't be specified to an arbitrary ~~decision~~ precision).

- Given set A, the power set 2^A is the set of all subsets of A.

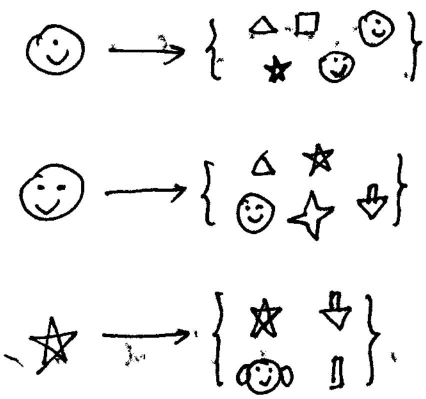
  EX. $\{☺, □, △\}$ these $D = \{☺, △\}$; $E = \{△\}$, $\emptyset$ are elements of 2^A, which has in fact 2^3 element.

Cantor's Thm. (diagonal argument) For any set A, we have $A \not\sim 2^A$.

Proof. (by contradiction) Suppose $\exists$ bijection $f: A \twoheadrightarrow 2^A$.

Then $a \mapsto f(a)$, a subset of A.

Idea:



construct a new set that differs everything in the right hand side.

What we want is a subset B that's not in the image of f.

Let $B = \{\alpha : \alpha \notin f(\alpha)\}$

- So, if $B = f(x)$ for some $x \in A$,
  consider x. Is $x \in B$? NO. If $x \in B$, Because then $x \notin f(x) = B$, contradiction.
  ~~if B~~ Then $x \notin B$, but then $x \notin f(x) = B$. So $x \in B$, contradiction.
  by definition of B

So, B is not $f(x)$ for any $x \in A$. ■

EX. $2^A \sim \{\text{all } f: A \to \{0,1\}\}$ by membership.

$2^{\mathbb{R}} \sim$ all functions from $\mathbb{R} \to \{0,1\}$.

- Consequencies: There are infinitely many cardinalities.

$$0, 1, 2, 3, 4, \dots, \aleph_0, \aleph_1, \aleph_2, \dots, \aleph_n, \dots$$

card of $\mathbb{Z}$ ($\aleph_0$); card of $\mathbb{R} = c$ ($\aleph_1$); ordinal numbers ($\aleph_n$)

- Continuum Hypothesis: $\aleph = c$ is undecidable (independent ZFC)

METRIC SPACES

ⓐ How to measure distance? (eg. in $\mathbb{R}^n$, in genomic seqs?)

Def'n A set X is a metric space if $\exists$ metric $d: X \times X \to \mathbb{R}$ s.t.

$\forall p, q \in X$. ⓐ $d(p,q) \geq 0$. ($= 0$ iff $p = q$) (non-negative)

ⓑ $d(p,q) = d(q,p)$ (symmetric)

ⓒ $d(p,q) \leq d(p,r) + d(r,q)$ $\forall r \in X$ (triangle inequality)

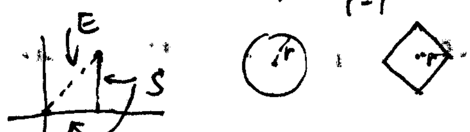
EX. $\mathbb{R}$ with $d(x,y) = |x-y|$. Write $(\mathbb{R}, d)$

$\mathbb{R}^n$ with $d(\vec{x}, \vec{y}) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$

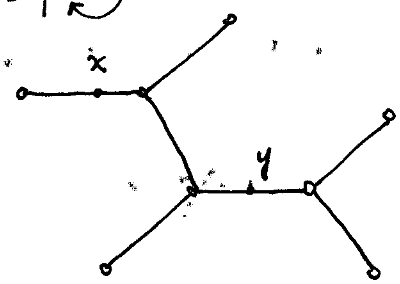
} Euclidean metric on $\mathbb{R}^n$

↑ usual metric (if not specified)

EX. $\mathbb{R}^n$ with $d_s(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$ (stair-case metric)



EX.



tree

$d_T(x,y)$ = shortest path between them.

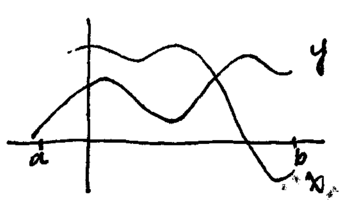
EX. $X = \{$ ~~geom~~ genomic seqs $\}$

$\vec{x}$ = GATTACA

$\vec{y}$ = AGATCAT

$d_g(\vec{x}, \vec{y})$ = # letters of difference.

EX. space of functions



$d_I(f,g) = \int_a^b |f-g| \, dx$ on continuous functions on $[a,b]$

Write $C([a,b])$.

or $d_{sup}(f,g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$ on space continuous, bounded functions on $\mathbb{R}$

$C_b(\mathbb{R})$

sup norm

sometimes maximum may not exist.

<u>OPEN BALL</u> $N_r(x) = \{y : d(x,y) < r\}$

N_r neighborhood

<u>CLOSED BALL</u> $\overline{N_r(x)} = \{y : d(x,y) \leq r\}$.

<u>Def'n</u> Say $p \in X$ is a <u>limit point</u> of E if <u>every neighborhood</u> of p contains a point $q \neq p$ s.t. $q \in E$

<u>Recall</u> : (X, d) metric space ① non-negativity ② symmetricity ③ triangle inequality.

X: set, d: metric

<u>EX</u>. $(\mathbb{R}^n$, Euclidean$)$.

<u>EX</u>. $(X$, discrete metric $\delta)$ $\delta(p,q) = \begin{cases} 0, & p=q \\ 1, & \text{otherwise} \end{cases}$. Δ inequality: right-hand side is either 1 or 2.

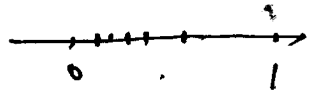
<u>Recall</u>. "open ball" or "nbhd" $N_r(x)$ tell us which points are "close".

<u>EX</u>. $(X$, discrete$)$, open ball are single pts (if $r \leq 1$) or all X (if $r > 1$).

Ⓠ When does set E "approach" a point p?

<u>Def'n</u> : A point $p \in X$ is a <u>limit point</u> of E if <u>every</u> nbhd of p <u>contains</u> a point $q \in E$, $q \neq p$.

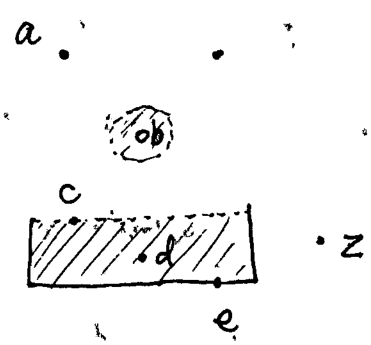
<u>EX</u>. in $\mathbb{R}$, $C = \{\frac{1}{n} : n \in \mathbb{N}\}$.



C has no interior points.

nbhd of $p = (p-\epsilon, p+\epsilon)$ 0 is a limit point.

<u>EX</u>. in $\mathbb{R}^2$, consider B:



b, c, d, e are limit points.

a and z are not.

a is <u>isolated</u>.

d is an interior point.

- So a point p is <u>not</u> a limit point of E if $\exists$ nbhd N of p s.t. N does not contain any <u>other</u> point of E.

- **Def'n** $p \in E$ is an <u>isolated points</u> of E if <u>$p \in E$</u> and p is <u>not limit point</u> of E.
- **Def'n** $p \in E$ is an <u>interior points</u> of E if $\exists$ nbhd N of p s.t. $N \subset E$.

<u>EX</u>. in $\mathbb{R}$, consider ϕ, $\mathbb{R}$, $\mathbb{Q}$] limit points? interior points? isolated points?
in ($\mathbb{R}$, discrete)

E in ($\mathbb{R}$, discrete) has no limit points and every point is an interior point.

- The metric gives what the geometry should look like.

<u>EX</u>. $E = \mathbb{Q}$ in $\mathbb{R}$ has all points as limit points. ($\mathbb{Q}$, discrete) is a subset of ($\mathbb{R}$, discrete), so it also doesnt contain limit points.

- <u>Thm</u>. If p is a l.p. of E, then every <u>nbhd</u> of p <u>contains</u> infinitely many pts of E.

<u>proof</u>. (by contradiction)

$\exists$ nbhd N of p with only finitely many points of E: $e_1, \cdots, e_n$.

let $r = \min_{i=1}^{n} \{ d(p, e_i) \}$

e_i are finite, so minimum is achieved.

So the nbhd $N_r(p)$ doesnt contain points of E. It contradicts p is a l.p. ∎

- <u>**Def'n**</u> A set E is <u>open</u> if every point of E is an <u>interior pt</u> of E.

<u>EX</u>. none of B is <u>open</u>.

<u>EX</u>. in $\mathbb{R}$ (open) interval $(a,b) = \{x : a < x < b\}$ is <u>open</u>.

<u>EX</u>. in $\mathbb{R}$, ϕ is <u>open</u> and $\mathbb{R}$ is <u>open</u>. (for ϕ, the condition is vacuously true)

- <u>**Def'n**</u> A set E is <u>closed</u> if E contains all of its <u>limit points</u>.

<u>EX</u>. in $\mathbb{R}$, single points $\{P\}$ is <u>closed</u>. (no limit points)

(closed) interval $[a,b] = \{x :$ ~~x<c~~ $a < x < b\}$ is <u>closed</u>.

$(a,b]$
$[a,b)$ half open interval: they aren't <u>open</u> nor <u>closed</u>.

in $\mathbb{R}$, ϕ and $\mathbb{R}$ ~~is~~ are <u>closed</u>. ("clopen")

EX. Mouth of B is not closed. Can we close it?

- Def'n Let E' = set of l.p's of E. The closure of E is $\bar{E} = E \cup E'$.
- The concept of open set is important because it allows you to ~~pur~~ perturb the points inside it a little bit.
- The concept of closed set is important because it allows to define limit.

- Thm. $\bar{A}$ is a closed set.

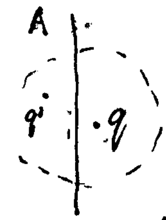
Proof. Consider p a limit point of $\bar{A}$. Want to show $p \in \bar{A}$.
($p \in A$ trivially true).

Consider a nbhd N of p. Assume $p \notin A$, we'll show N contains a pt (q') of A.

Since p is l.p. of $\bar{A}$, N contains a point q of $\bar{A}$.

If q is in A, we've found desired point $q' = q$.

If $q \notin A$, then q is a limit pt of A. ($q \in A'$)



Consider nbhd N' of q s.t. $N' \subset N$ (because nbhd is open).

So $q' \in N$, a desired point. ($q' \in N' \cap A$)

$\hookrightarrow \in A$ to be a l.p. of A.

- Lemma. Nbhd is open.

Note $a \stackrel{\text{def}}{=} d(p,q) < r$

let $r' = r - a$

claim: $N_{r'}(q) \subset N_r(p)$

If $d(x,q) < r'$ then $d(x,p) \leq d(x,q) + d(q,p) < r' + a = r$.

- Thm. E closed $\Leftrightarrow E = \bar{E}$.

proof. $(\Rightarrow) E' \subset E \Rightarrow E' \cup E \subset E$, so $\bar{E} \subset E$. Since $E \subset \bar{E}$, $E = \bar{E}$.

$(\Leftarrow)$ $E = \bar{E} \Rightarrow E$ contains all its l.p's.

- Once you learn a new thing, ask yourself if its intuitive.
- Thm. If $E \subset$ closed set F, then $\bar{E} \subset F$. } Moral: $\bar{E}$ is the smallest closed set containing E.

Proof. p is a l.p. of E $\Rightarrow p$ is l.p. of F.

For F contains its l.p.'s. So F contains all l.p.'s of E. $\bar{E}$ ∎

- RELATIONSHIP BETWEEN OPEN AND CLOSED SET.

- Thm. E is open $\Leftrightarrow$ E^c is closed.

  (E^c is the complement of E: $E^c = X \setminus E = \{p \in X : p \notin E\}$

  proof. E open $\Leftrightarrow$ any point $x \in E$ is an interior point.

  $\Leftrightarrow$ $\forall x \in E$, $\exists$ nbhd N of x s.t. $N \subset E$ and N is disjoint to E^c.

  $\Leftrightarrow$ $\forall x \in E$, x is not l.p. of E^c.

  $\Leftrightarrow$ E^c contains all its limit points. ▨

EX. $\mathbb{Q}$ is neither closed nor open.

- Unions & Intersections

  $K_i = [-1+\frac{1}{i}, 1-\frac{1}{i}]$ closed $\qquad \bigcup_{i=1}^{\infty} K_i = (-1, 1)$ not closed

- Lemma. $\{E_\alpha\}$ collection of sets.

  $$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$$

  proof. $x \in LHS \Leftrightarrow x \notin$ any E_α

  $\Leftrightarrow x \in E_\alpha^c, \forall \alpha \in A$

  $\Leftrightarrow x \in \bigcap_\alpha E_\alpha^c$

- THM. (a) Arbitrary $\cup$ of open sets is open.

  (b) $\quad$ " $\quad \cap \quad$ closed " $\quad$ closed. $\qquad \bigcap_i^\infty (-\frac{1}{i}, \frac{1}{i}) = \{0\}$ (open $\quad$ not open)

  (c) Finite $\cap$ open $\quad$ open.

  (d) $\quad$ " $\quad \cup \quad$ closed " $\quad$ closed

proof. (a) $x \in \bigcup_\alpha U_\alpha$ ← open $\Rightarrow x \in$ some U_α $\exists N$ of x, $N \subset U_\alpha$, $N \subset \bigcup_\alpha U_\alpha$.

(b) Say B_α closed, then $U_\alpha = B_\alpha^c$ is open.

use Lemma above: $\bigcup_\alpha B_\alpha^c$ is open $\Rightarrow (\bigcap_\alpha B_\alpha)^c$ is open $\Rightarrow \bigcap_\alpha B_\alpha$ is closed by Thm.

(c) $\exists N_{r_i}(x)$ for each U_i. Let $r = \min(r_1, \dots, r_n)$. (Infinite $\not\Rightarrow$ min)

Then $N_r(x)$ shows x interior to $\bigcap_{i=1}^{n} U_i$.

(d) similarly.

• Def'n E is dense in metric X

if every point of X is a limit point in E or in E.

OR $\bar{E} = X$,

OR every open set of X contains a point in E.

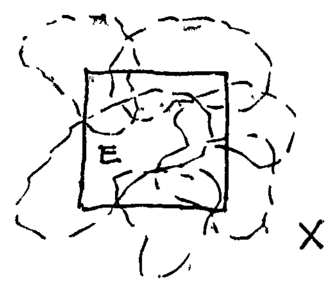
EX. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Is a subset of functions (e.g., polynomial functions) dense in continuous functions?

• Finite sets are nice: • small, bounded; • closed; • in $\mathbb{R}$, they contain sup and inf (max/min); • doing things, the process ends.

• Compact sets are the next best thing to finiteness.

• Def'n: An open cover of E in X (in metric space) is a collection of open sets $\{G_\alpha\}$ whose union "covers" (contains) E.



• Def'n A subcover of $\{G_\alpha\}$ is a sub-collection $\{G_{\alpha_\gamma}\}$ that still covers E.

EX. In $\mathbb{R}$, $[\frac{1}{2}, 1)$ has cover $\{V_n\}_{n=3}^{\infty}$ where $V_n = (\frac{1}{n}, 1 - \frac{1}{n})$.

Also $\{(0,2)\}$ is an open cover. $\{W_x\}_{x \in [1/2, 1)}$ where $W_x = N_{\frac{1}{4}}(x)$

• Given a cover, do we need all these sets to still cover?

• ~~$\{V_n\}^{\infty}$~~ EX. $[0,1]$ in $\mathbb{R}$ has cover by $\{V_n\} \cup \{W_0, W_1\}$

A finite subset: $\{W_0, W_1, V_{11}\}$.

• <u>Def'n</u> say a set K is <u>compact</u> (in X) if ~~any~~ every <u>open cover of K</u> contains a <u>finite subcover</u>.

(So K not compact : $\exists$ some <u>open cover</u> w/o <u>finite subcover</u>.)

WARNING: not saying: there is a finite cover.

<u>EX</u>. $[\frac{1}{2}, 1)$ not <u>compact</u> (see $\{V_n\}$).

$\mathbb{Z}$ (in $\mathbb{R}$) is not <u>compact</u>.

<u>EX</u>. $[0,1]$ <u>may</u> be <u>compact</u> but I'll need to <u>check every</u> open cover.

• <u>Thm</u>. <u>Finite sets</u> are <u>compact</u>.

<u>proof</u>. Consider an open cover $\{G_\alpha\}$ covering $x_1, \cdots, x_N$.

$\forall x_i$, choose <u>one</u> G_{α_i} that contains x_i.

Then $\{G_{\alpha_i}\}_{i=1}^N$ covers the set. ■

• <u>Thm</u>. <u>Compact sets</u> are <u>bounded</u>.

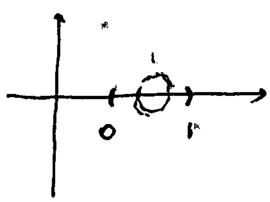
<u>Def'n</u>. A set K is <u>bdd</u> if $K \subset N_r(x)$ for some $x \in X$.

<u>proof</u>. Let K be compact. Let $B(x) = N_1(x)$ ball with radius 1.

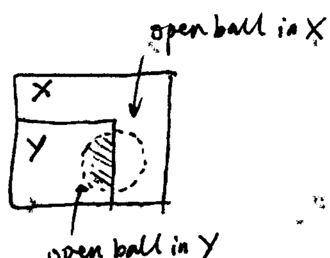
$\{B(x)\}_{x \in K}$ is an <u>open cover</u> of K, by <u>compactness</u> of K, $\exists$ finite ~~cover~~ subcover $\{B(x_i)\}_{i=1}^N$.

Let $R = \max\{d(x_i, x_j)\}$; this max exists because set $\{x_1, \cdots, x_N\}$ is finite.

Then $N_{R\cdot 2}(x_1)$ <u>contains</u> all K. ■ ($\triangle$ inequality) $d(x_1,p) \leq \underbrace{d(x_1,q) + d(q,p)}_{< 2R}$

<u>EX</u>.  open in $\mathbb{R}$, not open in $\mathbb{R}^2$. The $\left\{\begin{array}{l}\text{nbhd}\\ \text{open ball}\end{array}\right.$ changes.

• "<u>RELATIVE</u>" OPEN SETS



If $Y \subset X_{\text{metric}}$, Y "inherits" metric from X.

• A <u>set</u> U is <u>open</u> in Y ("relative to Y")

if <u>every point</u> ~~in~~ of U is <u>an interior pt</u> of U. (using metric in Y)

<u>(Def'n)</u>

- Thm. $E \subset Y \subset X$. E open in $Y \Leftrightarrow E = Y \cap G$ for some G open in X.

  (proof ind. idea) ($\Leftarrow$) Use $N_r(x) \subset G$ then $N_r(x) \cap Y$ is nbhd of x in Y.

  ($\Rightarrow$) Every pt. x has $N_r(x) \subset Y \cap E$.

  Then $\bigcup_{x \in E} N_{r_x}(x)$ in X is open, call it G. ∎

- open in $Y \Rightarrow$ perturb a little bit only in Y.

- Thm. $Y \subset X$. K is compact in $Y \Leftrightarrow K$ is compact in X.

  proof. ($\Rightarrow$) (Assume K compact in Y.)

  Consider open cover $\{U_\alpha\}$ of K in X.

  Let $V_\alpha = U_\alpha \cap Y$. Then $\{V_\alpha\}$ cover K in Y.

  By compactness of K in Y, $\exists$ finite subcover $\{V_{\alpha_1}, \cdots, V_{\alpha_N}\}$. finite

  The use of $\cancel{\alpha_1, \cdots, \alpha_N}$ Then $\{U_{\alpha_1}, \cdots, U_{\alpha_N}\}$ covers K in X as desired.

  ($\Leftarrow$) Consider open cover $\{V_\alpha\}$ of K in Y.

  $V_\alpha \to U_\alpha$. $\exists U_\alpha$ s.t. $U_\alpha \cap Y = V_\alpha$.

  $\{U_\alpha\}$ cover K in X, so $\exists$ finite subcover $\{U_{\alpha_1}, \cdots, U_{\alpha_N}\}$.

  Then $\{V_{\alpha_1}, \cdots, V_{\alpha_N}\}$ is finite subcover of K.

- Moral: compactness is an intrinsic prop of a set (doesn't depend on metric)

- Thm. Compact sets are closed.

  Proof. K compact, consider $p \notin K$. We'll show p has nbhd not intersecting K.

  $\forall q \in K$, let $V_q = N_{\frac{r}{2}}(q)$, $U_q = N_{\frac{r}{2}}(p)$ where $r = d(p,q)$. (p is interior to K^c, K^c open).

  Notice $\{V_q\}$ cover K, so by compactness of K, $\exists$ finite subcover $\{V_{q_1}, \cdots, V_{q_N}\}$.

  Then $W = U_{q_1} \cap \cdots \cap U_{q_N}$ is open (ball of radius $\min d(p,q)$).

  ~~W~~ claim $W \cap V_{q_i} = \phi$ for each i, because $W \subset U_{q_i}$ and $U_{q_i} \cap V_{q_i} = \phi$. ∎

EX. $(0,1)$ is not cpt.

EX. $\mathbb{R}$ (in $\mathbb{R}$) is not cpt ~~we noticed~~ because not bdd ~~though~~ closed.

So closed sets are not necessarily compact.

- **Thm.** A closed subset B of cpt set K is cpt.

proof. Let $\{U_\alpha\}$ be open covers of B.

Notice: B^c is open (because B closed).

So $\{U_\alpha\} \cup \{B^c\}$ is open cover of ~~B~~ K.

By cptness, $\exists$ finite subcover $\{\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}, B^c\}$.

Notice $B^c \cap B = \phi$, so $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ covers B, which is finite open cover set. ∎

- **Cor.** If F is closed and K is compact, then $F \cap K$ is compact. (closed subset of compact set)
- **Thm.** Nested closed intervals in $\mathbb{R}$ are not empty. (closed sets are not true: $[n, +\infty)$)

$I_n = [a_n, b_n]$

Nested: (if $m > n$, $a_n \le a_m \le b_m \le b_n$)

proof. Let $x = \sup\{a_i\}$, exists because they're bdd by b_1.

Clearly $x \ge a_i$ for all i, write sup, and $x \le b_n$ for all n because b_n is an u.b. for all a_m by ~~x~~.

ASIDE. PROP. $\mathbb{R}$ is uncountable.

Suppose $\mathbb{R} = \{x_1, \cdots, x\}$. Choose I_1 misses x_1. $I_2 \subset I_1$ misses x_2, $I_3 \subset I_2$ misses x_3.

By the theorem, $\exists$ point $x \in \bigcap_{\alpha}^{\infty} I_\alpha$ that's not in $\mathbb{R}$.

- **Thm.** $[a, b]$ is compact (in $\mathbb{R}$).
  (k-cell) (in $\mathbb{R}^k$).

proof (by contradiction) Suppose not. Then $\exists$ open cover $\{G_\alpha\}$ that has no finite subcovers. Then $\{G_\alpha\}$ covers $[a, c_1]$ and $[c_1, b]$, at least one has no finite subcover (FS) (Because if they had, their union is finite), call it I_1.

WLOG, ~~any~~ say $I_1 = [a_1, c_1]$ has no FS. Then subdivide (half) using c_2; note $[a, c_2]$, $[c_2, c_1]$ at least one of them has no finite subcover.

Continue, obtain sequence $I_1 \supset I_2 \supset I_3 \supset \cdots$ nested, closed intervals,

① each halved at each step;
② with no FS of $\{G_\alpha\}$

By nested interval then $\exists x \in I_n$ for all n.

By $x \in$ some $G_{\hat{\alpha}}$ of open cover, so $\exists r > 0$ s.t. $N_r(x) \subset G_{\hat{\alpha}}$. By ①, some $I_n \subset N_r(x)$, which says $G_{\hat{\alpha}}$ cover I_n, finite cover. contradicts ②.

Now we can show:

<u>Heine-Borel Thm.</u> In $\mathbb{R}$ (or $\mathbb{R}^n$) K <u>compact</u> $\Leftrightarrow$ K is <u>closed</u> and <u>bounded</u>.

<u>proof</u>. ($\Rightarrow$) already satisfied.

($\Leftarrow$) <u>not true in arbitrary metric space!</u>

K is <u>bdd</u> $\Rightarrow$ $K \subset [-r,r]$ for some $r>0$, but K is <u>closed</u> and $[-r,r]$ is <u>compact</u> (by Thm.) so K is compact (by Thm).

subset of compact. ■

(For $\mathbb{R}^n$, replace $[-r,r]$ by k-cell.)

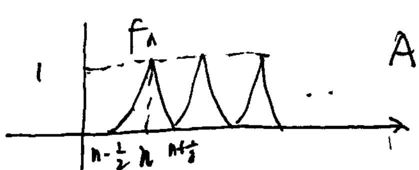
<u>EX.</u> Discrete metric on <u>inf.</u> set A. $(k-\frac{1}{2}, k+\frac{1}{2})$ open ball

A is <u>closed</u>, <u>bdd</u>, but <u>not cpt</u>. $G = \{ \cancel{\ } \mid k \in K \}$

<u>EX.</u> $C_b(\mathbb{R})$ = set of all continuous bdd functions $f: \mathbb{R} \to \mathbb{R}$.

$$d(f,g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$$

<u>neighborhood of functions</u>



$A = \{f_n\}_{n \in \mathbb{Z}}$. This set contains no limit points, so it's closed.

<u>bdd</u>: $d(f_i, f_j) = 1$. $\Rightarrow$ compact

• Thm. K cpt $\Leftrightarrow$ every infinite subset E of K has a l.p. in K.
(points should accumulate somewhere)

proof. ($\Rightarrow$) If no point of K is l.p. of E, then each point $q \in \cancel{K}$ E has nbhd V_q, containing exactly one point q of $\cancel{K}$ E.

$\{V_q\}$ cover E with no finite subcover.

($\Leftarrow$) [proof is true for $\mathbb{R}^n$, but it's true for all metric spaces]

We'll show K is closed & bdd.

Suppose K not bdd, choose a sequence x_n s.t. $|x_n| > n$.

These have no l.p.

Suppose K not closed, $\exists\, p \notin K$ ~~exists~~ is a l.p. of E, choose a sequence x_n s.t. $d(x_n, p) < \frac{1}{n}$.

$\{x_n\}$ has a l.p. at p and only one l.p. of p. (no other) (If there's another one, a distance exists and triangle inequality would break.)

(triangle inequality).

no limit point in $\cancel{E}$ K. ($p \notin K$)

• Thm. (Weierstrass) Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof. $\exists$ k-cell B, $E \subset B$, so has l.p. in k-cell. (k-cell is compactness). ■

- <u>Thm</u>. (Cantor) $\{K_\alpha\}$ compact subsets of subspace X. If any finite ~~number~~ subcollection has non-emtpy <u>intersection</u>, then the intersection of all of $K_\alpha \neq \phi$.

<u>proof</u>. let $U_\alpha = K_\alpha^c$, open.

Fix an K in $\{K_\alpha\}$. (by contradiction)

If $\bigcap_\alpha K_\alpha = \phi$, then $\{U_\alpha\}$ <u>cover</u> K (compact)

$\Rightarrow \exists$ finite subcover $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ cover K.

so $K \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \phi$. ∎ contradiction

(FIP)
Finite Intersection Prop.

- <u>COR</u>. $\{K_n\}$ a sequence <u>cpt</u> sets, nested, then $\bigcap_\alpha K_\alpha$ is non-empty.

- <u>Thm</u>. X is <u>cpt</u> $\Leftrightarrow$ Any collection of <u>closed sets</u> $\{D_\alpha\}$ satisfies the FIP.

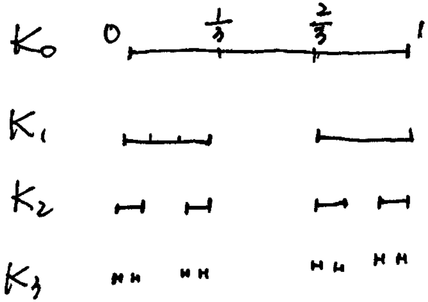
<u>proof</u>. $(\Rightarrow)$ Given $\{D_\alpha\}$, these are closed subsets of a compact space, so they're <u>cpt</u>. Applying prev. thm.

$(\Leftarrow)$ $\{U_\alpha\}$ is an open cover for $X \Leftrightarrow \bigcup_\alpha U_\alpha = X \Leftrightarrow (\bigcup_\alpha U_\alpha)^c = \bigcap_\alpha U_\alpha^c = \phi$

$\Leftrightarrow U_\alpha^c$ is closed sets.

So X is compact if and only if every collection of closed sets $\{D_\alpha\}$ with ~~an empty i~~ $\bigcap_\alpha D_\alpha = \phi$, $\{D_\alpha\}$ has a finite subcollection $\{D_{\alpha_1}, \cdots, D_{\alpha_n}\}$ whose $\bigcap_{\alpha_i} D_{\alpha_i} = \phi$. (by def. of <u>cpt</u>). Take contraposition $\Rightarrow$ conclusion. ∎

- Cantor set



Note: K_n are closed, cpt, nested.

Cantor set $C = \bigcap_n^{\infty} K_n$ (non-emtpy)

- C is closed because it's intersection of closed sets.
- Def'n. C is perfect: it's closed and every point is a limit point.
- C consists of real numbers whose ternary (base 3) expansion contains only 0's or 2's.
- ternary: $\sum_{k=-\infty}^{\infty} a_k 3^k$ write $\cdots a_2 a_1 a_0 . a_{-1} a_{-2} \cdots$

  $\frac{1}{3} = .022\overline{2}$

- Shows C is uncountable ~~and C~~ (use Diagonalization argument) countable endpoints

  C has non-end points of K_n point: $.020202\cdots$ (uncountable)

  Every point is a limit point:
  .0
  .02
  .022
  .0222

see book

- A base (basis) for topology is $\{V_\alpha\}$ open collection s.t. $\forall x \in$ open $\mathcal{U}$, $\exists V_\alpha$ s.t. $x \in V_\alpha \subset \mathcal{U}$.

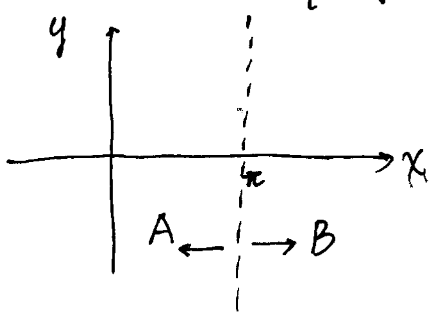
  So any open set is the <u>union</u> of its <u>basis</u> elements. (The set is small.)

- <u>CONNECTED SETS</u>

- <u>Def'n</u> Say A, B in X are <u>separated</u> if <u>both</u> $A \cap \bar{B}$ and $\bar{A} \cap B$ are <u>empty</u>.

- Say E is <u>connected</u> if E is <u>not</u> union of 2 <u>separated sets</u>.

- <u>EX</u>. in $\mathbb{R}^2$, $E = \{(x,y) : x, y \in \mathbb{Q}\}$



- E is <u>connected</u> $\Leftrightarrow$ if E is not union of 2 relatively open sets <u>in E</u>.

  $\Leftrightarrow$ 〃 〃 closed 〃

• **Thm** $[a, b]$ is connected.

proof (by contradiction). If not, $\exists$ a separation A and B, with $a \in A$.

Let $s = \sup A$, then $s \in \bar{A}$. (by theorem 2.28) So $s \notin B$. $(A \cup B = [a, b])$

Then $s \in A$. So $s \notin \bar{B}$.

Then $\exists (s-\varepsilon, s+\varepsilon)$ ~~not~~ containing no point of ~~in~~ B, so $(s-\varepsilon, s+\varepsilon)$ all in A.

This contradicts s is a supremum. $\blacksquare$

• SEQUENCES

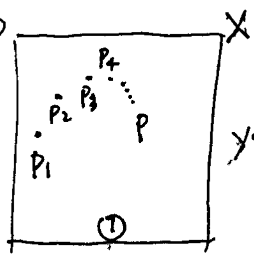
(metric space)

• Recall. A sequence $\{p_n\}$ in X is a function $f: \mathbb{N} \to X$

maps $n \to p_n$, a point in X.

• Sequences are not merely real numbers.

EX. ①



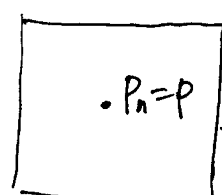
yes

Q. What does it mean the sequence converges?

EX. ② $\mathbb{R}^2$

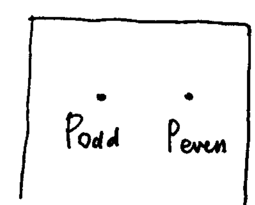
NO

EX. ③



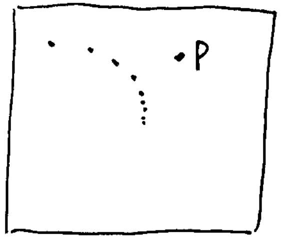
yes

EX. ④



NO

EX. 5



• Def'n $\{p_n\}$ converges if $\exists p \in X$ s.t. $[\forall \varepsilon > 0 \ \exists N$ s.t. $n > N$ implies $d(p_n, p) < \varepsilon.]$

Write. $p_n \to p$ or $\lim_{n\to\infty} p_n = p$

say "p_n converges to p", "p is the limit of sequence p_n".

EX. $p_n = \frac{n+1}{n}$ in $\mathbb{R}$. scratch:

Claim $p_n = 1$. (Given ε, task: find an N that works for the definition.)

~~Proof.~~ We have to bound $\left|\frac{n+1}{n} - 1\right| < \varepsilon$ when? to what N?

$$\Leftrightarrow \left|\frac{1}{n}\right| < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n$$

proof. Given $\varepsilon > 0$, choose $N = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$.

For if $n \geq N$ then $n > \frac{1}{\varepsilon}$, hence $\frac{1}{n} < \varepsilon$.

So $|p_n - p| = \left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| < \varepsilon$. ∎

TRUE OR FALSE ?

(A) $p_n \to p$ & $p_n \to p' \Rightarrow p = p'$ (T)

(B) $\{p_n\}$ bdd. (range of p_n) $\Rightarrow p_n$ converges (F, see (4))

(C) p_n converges $\Rightarrow \{p_n\}$ bdd. (T)

(D) $p_n \to p \Rightarrow p$ is limit point of range of $\{p_n\}$ (F, see (5)) p is a limit of $\{p_n\}$ but not a limit point of the range of $\{p_n\}$

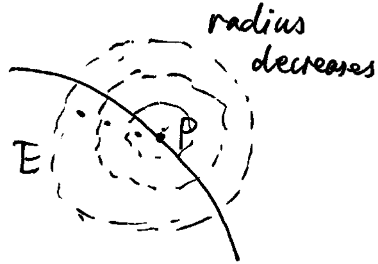
(E) ~~If~~ p is l.p. of $E \subset X \Rightarrow \exists$ seq. $\{p_n\}$ in E s.t. $p_n \to p$ (T)

(F) $p_n \to p \Rightarrow$ every nbhd of p contains all but finitely many p_n. (T)

(A) Assume $p_n \to p$, $p_n \to q$, let $\varepsilon = d(p,q)$.
Then $\exists N_p$ s.t. $n > N_p$ implies $d(p_n, p) < \frac{\varepsilon}{2}$.
Also $\exists N_q$ s.t. $n > N_q$ implies $d(p_n, q) < \frac{\varepsilon}{2}$.
Let $N = \max\{N_p, N_q\}$ then $n > N$ implies $\varepsilon = d(p,q) \le d(p_n, p) + d(p_n, q) < \varepsilon$,
hence a contradiction. ∎

(C) Use $\varepsilon = 1$, then $\exists N$ s.t. $n > N \Rightarrow d(p_n, p) < 1$.
Let $R = \max\{1, d(p_1, p), \cdots, d(p_N, p)\}$
So All $\{p_n\}$ are in $B_R(p)$ ∎

Ⓓ Ⓔ



Choose sequence $p_n \in B_{\frac{1}{n}}(p)$, then $p_n \to p$.

Ⓕ By definition.

The converse is also true. (Also by definition)

- Warm-up: consider seq: $\{s_n\}, \{t_n\} \in \mathbb{C}$ & $s_n \to s$, $t_n \to t$

- <u>Thm</u>. $\lim\limits_{n\to\infty} (s_n + t_n) = s+t$

<u>IMPORTANT IDEA OF DEF'N</u>: to show convergence, we must find an N.

<u>proof</u>. (IDEA: bound $|(s_n+t_n)-(s+t)| \leq |s_n - s| + |t_n - t|$)

Given $\varepsilon > 0$, $\exists N_1, N_2$, s.t.

$$n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2},$$

$$n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$, we have

$$|(s_n+t_n)-(s+t)| \leq |s_n - s| + |t_n - t| < \varepsilon \text{ as desired.} \quad ■$$

Thm. $\lim_{n\to\infty} cS_n = Sc$, $\lim_{n\to\infty} (c+S_n) = c+S$ $|c+S_n - c-S| = |S_n - S|$

(IDEA: $|cS_n - cS| = |c||S_n - S|$)

proof. For $\varepsilon > 0$. Then $\exists N$ s.t. $n > N \Rightarrow |S_n - S| < \frac{\varepsilon}{|c|}$

Then for this N, $n > N \Rightarrow |cS_n - cS| = |c||S_n - S| < \varepsilon$.

So $cS_n \to cS$. ∎

Thm. $\lim_{n\to\infty} S_n t_n = st$.

Idea. $|S_n t_n - st| = |(S_n - S)(t_n - t) + S(t_n - t) + t(S_n - S)|$

proof. Given $\varepsilon > 0$, let $K = \max\{s, t, 1, \varepsilon\}$

$\exists N_1, N_2$ s.t. $n > N_1 \Rightarrow |S_n - S| < \frac{\varepsilon}{3K}$

$n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{3K}$

Let $N = \max\{N_1, N_2\}$.

Then $|S_n t_n - st| < \frac{\varepsilon^2}{9K^2} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$

$\left(\frac{\varepsilon}{K} < 1\right) \to \; < \frac{\varepsilon}{9K} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$ as desired. ∎

SUBSEQUENCES

- $\{p_n\}$ is a seq. Let $n_1 < n_2 < \cdots$ in $\mathbb{N}$ an increasing sequence.
  Then $\{p_{n_i}\}$ is a subsequence. (Def'n)

EX. $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots\}$
$n_1 = 2 \quad n_2 = 4 \quad n_3 = 5$

Q If $p_n \to p$, must any subseq. converge to p?
– Yes. Because every nbhd. of p contains all but finitely many points of p_n.

EX. $\{1, \pi, \frac{1}{2}, \pi, \frac{1}{3}, \pi, \frac{1}{4}, \pi, \cdots\}$ doesnt converge.
but a subseq. converges (to "subsequential limit")

Q Must every seq. contain a conv. subseq.?
– NO. $\{1, 2, 3, 4, \cdots\}$

- If, ~~say~~ seq. bdd. must contain a convergent subsequence? NO, in $\mathbb{Q}$ $\{3, 3.1, 3.14, 3.141, \cdots\}$

- Def'n A metric space's sequentially compact if every sequence has a conv. subseq.
- Thm. If X is compact, then X is seq. cpt.
  (In cpt space, every seq. has a subseq. converging to a point in X.)

- FACT seq. cpt. $\Leftrightarrow$ cpt.

- COR. Every bdd seq. in $\mathbb{R}^k$ has a conv. subseq..
  (Because lives in a compact subspace of $\mathbb{R}^k$, $\Rightarrow$ seq. cpt. $\Rightarrow$ conv. subseq.)

- Proof. Let $R =$ range $\{p_n\}$.
  - If R is finite, then some p in $\{p_n\}$ is achieved ∞ many times.
  - If R infinite, then by §2.37 thm, since X cpt, R has a limit point p,
    Form $B_{\frac{1}{n}}(p)$, choose a seq. of $\{p_n\}$ where $p_n \in B_{\frac{1}{n}}(p)$.
    $\{p_n\}$ is a subseq. ∎

CAUCHY SEQUENCE

Q How to tell $\{p_n\}$ converges if I don't know it's limit?

IDEA. If they do converge, then p_n must get close to each other.

- Def'n $\{p_n\}$ is Cauchy seq. means $\forall \varepsilon > 0$, $\exists N$ s.t.
  $m, n \geq N$ implies $d(p_n, p_m) < \varepsilon$.

- Thm. If $\{p_n\}$ conv. then $\{p_n\}$ Cauchy.
  Proof. (Idea: bound $d(p_n, p_m) \leq d(p_n, p) + d(p_m, p)$ Δ inequality)

Given $\varepsilon>0$, $\exists N$ s.t. $n>N \Rightarrow d(p,p_n)<\frac{\varepsilon}{2}$.

So for this N, $n,m>N \Rightarrow d(p_n,p_m) \le d(p,p_n)+d(p,p_m)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. ∎

- Converse is not true, in $\mathbb{Q}$ $\{3, 3.1, 3.14, 3.141, \cdots\}$ is Cauchy but not converges.

- Def'n A metric space X is complete if every Cauchy seq. converges to point of X.

  $\mathbb{Q}$ is not complete as the above example.

  Is $\mathbb{R}$ complete?

  - In a complete space, cauchy and convergence are the same

- Thm. Compact metric spaces are complete.

proof. let $\{x_n\}$ be a Cauchy seq. in X.

Since X is compact, it's seq. cpt.

So $\exists$ subseq $\{x_{n_k}\}$ converging to point $x \in X$.

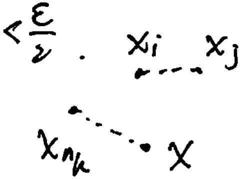
For $\varepsilon>0$, so $\{x_n\}$ Cauchy implies $\exists N_1$ s.t. $i,j > N_1 \Rightarrow d(x_i,x_j)<\frac{\varepsilon}{2}$.

$\{x_{n_k}\}\to x$ implies $\exists N_2$ s.t. $n_k>N_2 \Rightarrow d(x_{n_k},x)<\frac{\varepsilon}{2}$.

Let $N=\max\{N_1,N_2\}$

If $n>N$, then $d(x_n,x)\le d(x_n,x_{n_k})+d(x_{n_k},x)<\varepsilon$.

So given $\varepsilon>0$, we've found N. $\{x_n\}\to x$. ∎



make $x_{n_k}=x_i$

• COR. $[0,1]$ is complete, k-cells in $\mathbb{R}^k$ are complete, closed subset of cpt space is complete, like cantor set.

• COR. $\mathbb{R}^n$ is complete.

proof. If $\{x_n\}$ is Cauchy, it is bounded.

(Why? Let $\varepsilon = 17$, $\exists N$ s.t. $n, m \geq N$ $d(x_n, x_m) < 17$.
Let $R = \max\{d(x_1, x_N), \ldots, d(x_{N-1}, x_N), 17\}$. Seq. bdd. by $B_R(x_N)$.)

So $B_R(x) \subset$ some k-cell in $\mathbb{R}^n$, so $\{x_n\}$ converges because k-cell is complete. ∎

• EX. Does $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ converge?

Consider $|x_n - x_m| = \left(\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}\right) \geq \frac{n-m}{n} = 1 - \frac{m}{n}$.

Let $n > 2m$, $|x_n - x_m| > \frac{1}{2}$, so seq not Cauchy, it doesn't converge.

• EX. $x_1 = 1$, $x_2 = 2$, $\ldots$, $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$. Easy show Cauchy, so converge!

Q If X is not complete, can it be embeded in one that is?
EX. $\mathbb{Q}$ $\mathbb{R}$

Thm. Every metric space (X, d) has a completion (X^*, d).

IDEA. Given X, let X^* = the set of all Cauchy seq. in X under equivalence rel'n. $\sim$
where $\underset{P}{\{p_n\}} \sim \underset{Q}{\{q_n\}}$ if $\lim_{n \to \infty} d(p_n, q_n) = 0$.

• Let $\Delta(P,Q) = \lim_{n\to\infty} d(p_n, q_n)$ where $\{p_n\}, \{q_n\}$ are representatives of P, Q.

Then X^* is complete with X isometrically embedded in X^*.

[Another way to construct $\mathbb{R}$ from $\mathbb{Q}$.]

BOUNDED SEQS in $\mathbb{R}$

monotonically increasing seq: $s_n \leq s_{n+1}$

decreasing seq: $s_n \geq s_{n+1}$

Thm Bounded monotonic seqs converge. (to sup or inf)

proof: Given $\{s_n\}$, let $s = \sup \text{range}\{s_n\}$.

$\forall \varepsilon > 0, \exists N$ s.t. $s - \varepsilon < s_N \leq s$

but then $n > N$, $s - \varepsilon < s_n \leq s$. So this N works. ∎

• Write $s_n \to +\infty$, if $\forall M \in \mathbb{R}, \exists N$ s.t. $n \geq N \Rightarrow s_n \geq M$.

Similarly $s_n \to -\infty$, $\quad$ " $\quad$ " $\quad$ " $\quad$ $s_n \leq M$.

• Given $\{s_n\}$, let $E = \{\text{subseq. limits}\}$ ← allow $+\infty, -\infty$.

let $s^* = \sup E$. ← $\limsup_{n\to\infty} s_n$, "upper limit" of $\{s_n\}$.

$s_* = \inf E$. ← $\liminf_{n\to\infty} s_n$, "lower limit" of $\{s_n\}$.

Alternative $\limsup s_n = \lim_{n\to\infty} \left(\sup_{k\geq n} s_k \right)$

$\liminf s_n = \lim_{n\to\infty} \left(\inf_{k\geq n} s_k \right)$

EX. $s_n \to s$, then $\liminf s_n = \limsup s_n = s$.

EX. $s_n = \{ .1, \frac{3}{2}, .11, \frac{4}{3}, .111, \frac{5}{4}, \cdots \}$

$s^* = 1, \quad s_* = \frac{1}{9}$.

SERIES

Q What does this mean?

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

Or this?

$$1 - 1 + 1 - 1 + \cdots$$

$$(1-1) + (1-1) + \cdots = 0$$

$$1 + (-1+1) + (-1+1) + \cdots = 1$$

Some have learned

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{1-\frac{1}{3}}$$

as a special case

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \sim \text{why?}$$

$$(1 + x + x^2 + \cdots)(1-x) = 1$$

$$\left.\begin{aligned} 1 + 2 + 4 + 8 + \cdots &= \frac{1}{1-2} = -1 \\ 1 - 2 + 4 - 8 + \cdots &= \frac{1}{1-(-2)} = \frac{1}{3} \\ 1 - 1 + 1 - 1 + \cdots &= \frac{1}{1-(-1)} = \frac{1}{2} \end{aligned}\right\} \text{Euler}$$

• Define *series*:

Given $\{a_n\}$, then $\sum_{n=p}^{q} a_n = a_p + \cdots + a_q$

let $S_n = \sum_{k=1}^{n} a_k$, the n-th partial sum.

• Then $\{S_n\}$ is a sequence, sometimes written $\sum_{n=1}^{\infty} a_n$, called an infinite ~~sequence~~ series.

This may not converge, but if it does, write $\sum_{n=1}^{\infty} a_n = S$.

Note: $\sum_{n=1}^{\infty} a_n = S \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k\right)$, not simply by addition.

Q when does a series converge?

Q when seq. is partial sum? when is that?

EX. $a_n = \frac{1}{n}$, Does $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ converge?

Is $\{S_n\}$ Cauchy?

$S_m - S_n = a_{n+1} + \cdots + a_m > \frac{1}{2}$. So $\{S_n\}$ not Cauchy, does not converge.

• The Cauchy criterion (for series)

Thm. $\sum a_n$ converges $\Leftrightarrow \forall \epsilon > 0\ \exists N$ s.t. $m, n > N \Rightarrow \left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon$.

Let $m = n$, get

COR. $\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$ (terms $\to 0$) Converse is not true.

• **Thm.** (non-neg series)

If $a_n \geq 0$, then Σa_n converges $\Leftrightarrow$ partial sums are bounded.

Proof. If $a_n \geq 0$, p sums mon. incr. $\Leftarrow$, and bounded, seqs converge. ∎

• **Thm.** (comparison test).

(a) If $|a_n| \leq c_n$, for if n larger enough, and Σc_n conv., then Σa_n conv.

(b) If $a_n \geq d_n \geq 0$, for n large enough,

If Σd_n diverges, then Σa_n diverges.

proof. (a) Since Σc_n conv., $\forall \varepsilon > 0$, $\exists N_1$ s.t. $m, n > N_1 \Rightarrow \left|\sum_{k=m}^{n} c_k\right| < \varepsilon$.

~~let~~ For the same m, n, $\left|\sum_{k=m}^{n} a_k\right| \leq \Sigma |a_k| \leq \Sigma c_k < \varepsilon$.

(b) Similar to (a). ∎

• what to compare to?

Geometric Series

Thm. If $|x| < 1$, then $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$.

If $|x| \geq 1$, then Σx^n diverge.

proof. If $|x| < 1$, let $S_n = 1 + x + \cdots + x^n$. Then $S_n = \frac{1-x^{n+1}}{1-x}$

So $\lim_{n\to\infty} S_n = \frac{1}{1-x} \lim_{n\to\infty} (1-x^{n+1}) = \frac{1}{1-x}$

If $|x| \geq 1$, then $\lim\limits_{n\to\infty} a_n \neq 0$, so S_n diverges. $\blacksquare$

Q. $\sum \frac{1}{n^p}$ conv. & diverge? For which p?

• Thm. (Cauchy) If $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ mono. decr.

Then $\sum a_n$ conv. $\Longleftrightarrow$ $\sum 2^n a_{2^n}$ conv.

proof. Compare $s_n = a_1 + a_2 + \cdots + a_n = (a_1) + (a_2 + a_3) + (a_4 + \cdots + a_n)$.

$t_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k} = (a_1) + (a_2 + a_2) + (a_4 + \cdots + a_{2^k})$.

If $n < 2^k$, then $S_n \leq t_k$. use comp.

$2S_n = 2a_1 + 2a_2 + 2(a_3 + a_4) + \cdots + 2(a \cdots a_n)$

$t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$

If $n > 2^k$, then $t_k \leq 2S_n$. use comp. $\blacksquare$

• Applications: $\sum \frac{1}{n^p}$ conv. if $p > 1$, div if $p \leq 1$.

proof. If $p < 0$, $\frac{1}{n^p} \not\to 0$, the series diverges.

If $p > 0$, use $\sum\limits_{k\geq 0}^{\infty} 2^k \cdot \frac{1}{2^{kp}} = \sum\limits_{k\geq 0}^{\infty} \cancel{\frac{1}{2^p}}\, 2^{(1-p)k}$ a geometric series.

$\cancel{k \leq 1}$ $p \leq 1$, $p > 1$ $\blacksquare$

• $\sum\limits_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ (converges to e) (Taylor series) why? It's p sums are bdd. by 3, & for non-neg series.

$(1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots \leq 3.)$

- conv. ~~rapiq~~ rapid! $\quad e - S_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$

$$\leq \frac{1}{(n+1)!}\underbrace{\left(1+\frac{1}{(n+1)}+\frac{1}{(n+1)^2}+\cdots\right)}_{\frac{n+1}{n}} \qquad \frac{1}{n!\,n}$$

- see e is ~~trat~~ irrational.

If $e = \frac{m}{n}$, then $0 < \underbrace{n!(e-S_n)}_{\text{integer}} < \frac{1}{n}$.

There's no integer between 0 to $\frac{1}{n}$. ■

- Thm. (Root test) Given $\sum a_n$

Let $\alpha = \limsup \sqrt[n]{|a_n|}$ $\quad$ ($\alpha^n \longleftrightarrow a_n$: geometric series)

Then $\alpha < 1 \Rightarrow$ series conv.

$\alpha > 1 \Rightarrow$ .. div.

$\alpha = 1 \Rightarrow$ test ~~inclusive~~ inconclusive.

proof. by comparison with geometric series.

- If $\alpha < 1$, choose β s.t. $\alpha < \beta < 1$.

Then $\exists N$ s.t. $n \geq N \Rightarrow \sqrt[n]{|a_n|} < \beta$. (by definition of lim sup)

So $|a_n| < \beta^n$ for $n \geq N$.

But $\sum \beta^n$ conv., so $\sum \{a_n\}$ conv. as well.

- If $\alpha > 1$, $\exists$ sub seq $\sqrt[n]{|a_{n_k}|} \to \alpha > 1$

  so $|a_{n_k}| > 1$ for ∞ many terms, so terms $\not\to 0$, so series div.

- If $\alpha = 1$, note $\sum \frac{1}{n}$ div. $\sum \frac{1}{n^2}$ conv, both $\alpha = 1$. $\blacksquare$

- Thm. (Ratio Test) $\sum a_n$ conv. if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

  div. if $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for n large enough.

$\boxed{\text{ROOT POWERFUL} \\ \text{RATIO EASIER}}$

$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \cdots$

RATIO FAILS

Proof. (comparison) Have $\left|\frac{a_{n+1}}{a_n}\right| < \beta < 1$ for $n > N$.

$|a_{n+1}| < \beta |a_n| < \beta^2 |a_{n-1}| < \beta^3 |a_{n-2}| < \cdots$

$|a_{N+k}| < \cdots < \beta^k a_N$

$\sum_{k=0}^{\infty} a_{N+k} \leq a_N \underline{\sum_{k=0}^{\infty} \beta^k}$ converges!

For div. see terms $\not\to 0$. $\blacksquare$

- POWER SERIES

If c_n complex, $\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots$ power series.

For what z does it converge?

• <u>Thm</u> If $\alpha = \limsup\limits_{n\to\infty} \sqrt[n]{|c_n|}$, let $R = \frac{1}{\alpha}$, then $\sum c_n z^n$ conv. if $|z| < R$
(radius of convergence) div. $|z| > R$.

<u>proof</u>. $\sqrt[n]{|a_n|} = |z| \cdot \sqrt[n]{|c_n|}$ $\left(|z| \limsup \sqrt[n]{|c_n|} \quad 1 \right)$

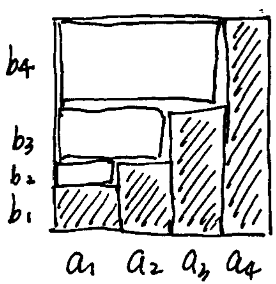
• Two seqs $\{a_n\}$, $\{b_n\}$, say something about $\sum a_n b_n$?

<u>SUMMATION BY PARTS</u>

• let $A_n = \sum\limits_{k=0}^{n} a_k$ for $n = 0$. Set $A_{-1} = 0$,
Then $\sum\limits_{n=p}^{q} a_n b_n = \sum\limits_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$.

$A \rightsquigarrow v \quad b \rightsquigarrow u$
$a \rightsquigarrow dv$
$\int_p^q u\,dv = -\int_p^q v\,du + uv\Big|$

<u>proof</u>.



The idea of integration by parts.

Total square - ~~bl~~blank square = shadow square

• <u>Thm</u>. If A_n <u>bdd</u>, b_n <u>decr.</u> $\to 0$, then $\sum a_n b_n$ converges.

<u>proof</u>. Say $|A_n| \le M$. Given $\varepsilon > 0$, $\exists N$, s.t. $b_N \le \frac{\varepsilon}{2M}$.

For $q \ge p \ge N$, $\left| \sum\limits_{n=p}^{q} a_n b_n \right| \le M \left| \sum\limits_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p \right| \le 2M b_p \le 2M b_N \le \varepsilon$. ∎

• COR. $|c_1| \geq |c_2| \geq \cdots$ c_i is alternating in signs $\to 0$ Then $\sum c_n$ conv.

proof. $a_n = (-1)^{n+1}$, $b_n = |c_n|$.

• SUMS OF SERIES $\sum a_n + \sum b_n = \sum (a_n + b_n)$

PRODUCTS? motivation: power series

$$(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_0 z + a_2 z^2 + \cdots)$$

$$= (a_0 b_0) + (a_0 b_1 + a_1 b_0) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + \cdots$$

• Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, product series: $\sum c_n$ (Def'n)

problem. $\sum c_n$ may not converge even if $\sum a_n$ $\sum b_n$ does!

But Thm. If $\underset{\to A}{\sum a_n}$, $\underset{\to B}{\sum b_n}$ converge absolutely then $\sum c_n$ conv. ~~to~~ to AB.

ABSOLUTE CONVERGENCE

Def'n $\sum a_n$ conv. absolutely, if $\sum |a_n|$ converge

EX. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ Conv, but not absolutely.

Thm. If $\sum a_n$ conv. absolutely, then $\sum a_n$ converges.

proof. $\left| \sum_{k=n}^{M} a_k \right| \leq \sum_{k=n}^{m} |a_k|$ small by Cauchy criterion. ∎

REARRANGEMENTS

ⓐ Say $\sum a_n = A$. If I rearrange the terms, must it converge? <u>NO</u>. to A? No!

<u>EX</u>. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$

- <u>Remind</u>. If $\sum a_n$ converges (not abs), then a rearrangement can have any lim sup, lim inf you <u>like</u>.

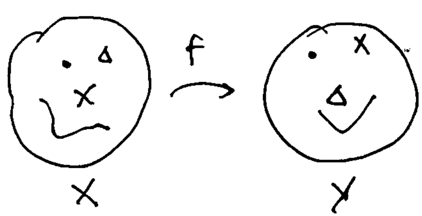
- If <u>conv abs</u> all rearrangement has the same <u>limit</u>!

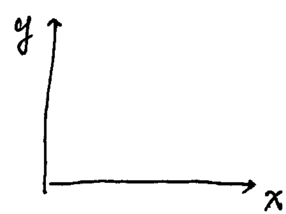
$$\underbrace{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots}_{\pi} \underbrace{- \frac{1}{2} - \frac{1}{4} - \frac{1}{8}}_{<\pi} + \cdots$$

- <u>FUNCTIONS</u>

- Let X, Y be metric spaces, $f: X \to Y$.

- visualize: f as mapping | f as a graph

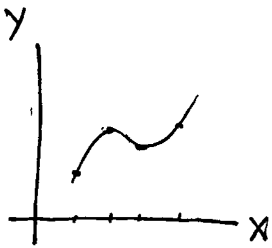




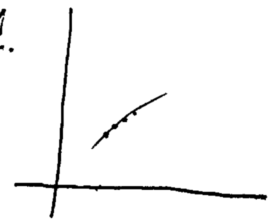
- <u>Recall</u>; Know what this means $\lim_{n\to\infty} x_n = x$

  What's this? $\lim_{x\to p} f(x) = q$

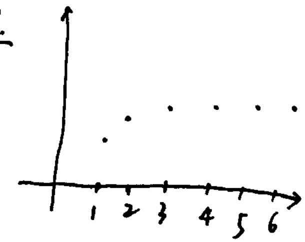
- <u>EX</u>.



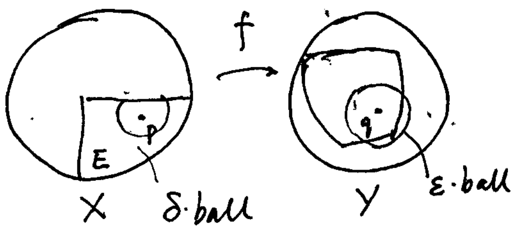
<u>EX</u>.



<u>EX</u>.



<u>Def'n</u>. X, Y <u>metric</u> $E \subset X$, p is <u>limit point</u> of E. Let $f: E \to Y$.



To say "$f(x) \to q$ as $x \to p$" or "$\lim_{x\to p} f(x) = q$".

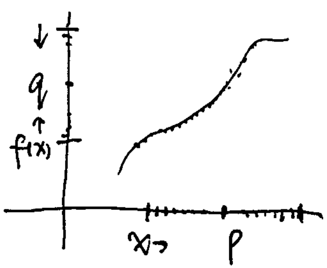
means: $\exists\, q \in Y$ s.t. in E.

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t.

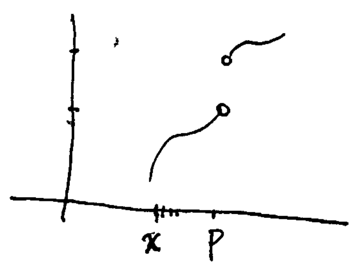
$\forall x \in E$, $0 < d(x,p) < \delta \Rightarrow d(f(x), q) < \varepsilon$

This excludes $x = p$ (δ characterization)

<u>EX</u>.

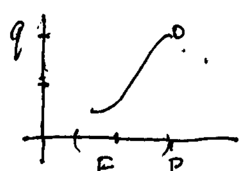


<u>EX</u>.



no limit as $x \to p$

<u>EX</u>.



p does not have to be in E.

• To show convergence; Given $\varepsilon>0$; find a δ "that meets".

• Thm. $\lim_{x\to p} f(x) = q$ iff $\forall$ seq. $\{p_n\}$ in E, ~~s.t.~~ $p_n \neq p$, $p_n \to p$ we have $f(p_n) \to q$.

(SEQUENCE CHARACTERIZATION) Seq. conv.

proof. ($\Rightarrow$) Given $\varepsilon>0$ (GOAL: find an N)

$\exists \delta>0$ s.t. $0<d(x,p)<\delta \Rightarrow$ ~~f~~ $d(f(x),q)<\varepsilon$.

So for a given $\{p_n\}$ as above, $\exists N$ s.t. $d(p_n,p)<\delta$.

so $n\geq N$ implies $d(f(p_n),q)<\varepsilon$.

($\Leftarrow$) If $\lim_{x\to p} f(x) \not\to q$ then $\exists \varepsilon>0$, s.t. $\forall \delta>0$ $\exists x\in E$ s.t. $0<d(x,p)<\delta$ but $d(f(x),q)\geq\varepsilon$.

We propose bad sequence: use $\delta_n = \frac{1}{n}$. choose x_n by , but $d(f(x_n),q)\geq\varepsilon$.

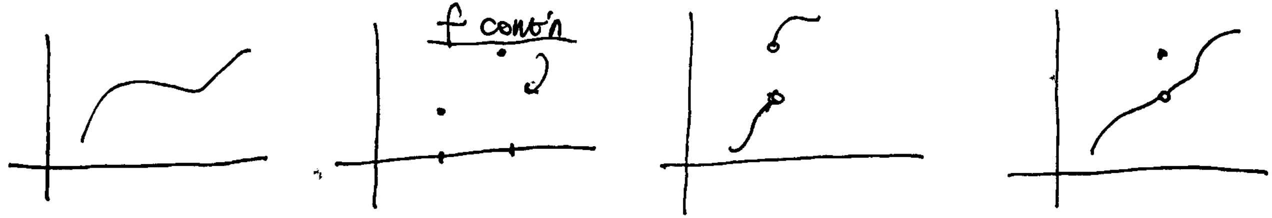
then $x_n \to p$,

So $f(x_n) \not\to q$. ∎ (contradiction)

• From thms on seqs: ① limits are unique, ② limits of sums (prods) are sums (prods) of limits.

$\left(\lim_{x\to p} f(x)+g(x) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x) \right)$ $(Y=\mathbb{R})$

CONTINUOUS FUNCTIONS



- **Def'n** X, Y metric spaces. $p \in E \subset X$, $f: E \to Y$. Say f is <u>continuous</u> at p if

  (x may be p)

  $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x \in E$, $d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon$.

- **Thm.** ① If p is a limit point of E, then f <u>continuous</u> at p $\overset{\Leftrightarrow}{\text{means}}$ $\boxed{\lim_{x \to p} f(x) = f(p).}$

  ② Also if x_n conv. $\overset{\text{seq}}{}$ then $\boxed{f \text{ continuous}} \Leftrightarrow \boxed{\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)}$ ☆

- **COR.** Sums, prods, of <u>cont'n</u> function are <u>cont'n</u>.

  quotient $\frac{f}{g}$ $(g \neq 0)$.

- **COR.** $f, g: X \to \mathbb{R}^k$ s.t. $f = (f_1, \cdots, f_k)$

  a) f <u>cont'n</u> $\Leftrightarrow$ each f_i cont'n.

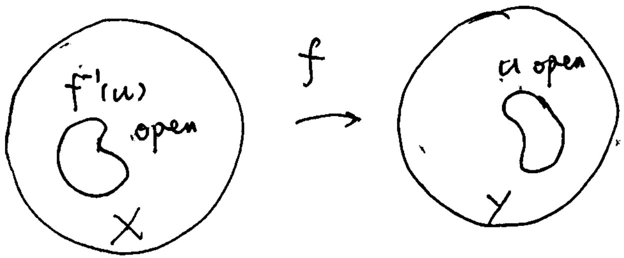
  b) $f+g$, $f \cdot g$ <u>cont'n</u>. (dot prod.)

<u>proof</u> (a) $|f_i(x) - f_i(y)| \le \|f(x) - f(y)\| = \sqrt{\sum_{i=1}^{n} |f_i(x) - f_i(y)|^2}$

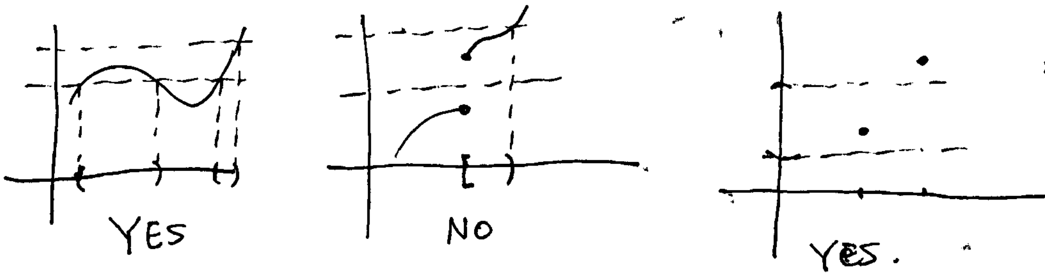
(b) use components; part (a).

Thm ③ $f: X \to Y$ is <u>cont'n</u> $\Leftrightarrow$ $\forall$ <u>open sets</u> U in Y, $f^{-1}(U)$ is <u>open</u> in X.

(BEST DEFINITION)

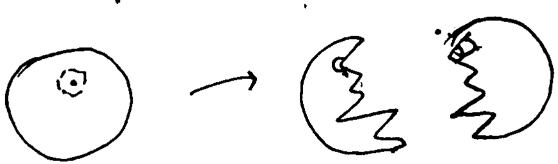


<u>EX.</u>



A single point set is an open set under discrete metric of two-point set.

- If f is not continuous at p, intuitively, two close points in domain, like x, y, their images fall apart.



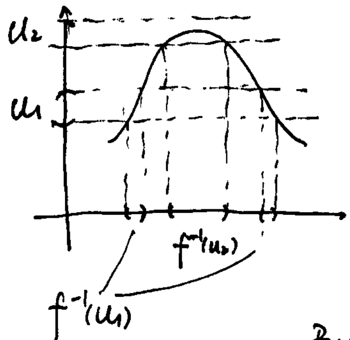
- For every convergent sequence $\{x_n\}$, f cont'n, ~~$f(x_n)$~~ $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$.

  (Continuous preserves limit.)

One <u>very useful</u> characterization of continuous function. (open set)

<u>Thm</u>. $f: X \to Y$ <u>cont'n</u> $\iff$ $\forall$ open set $U \subset Y$, $f^{-1}(U)$ is open.

<u>EX</u>.



($\Rightarrow$)

<u>Proof</u>. Given U <u>open</u> in Y, $f^{-1}(U)$

Consider $x \in f^{-1}(U)$, we'll show x is an <u>interior</u> point of $f^{-1}(U)$.

Note $f(p) \in U$ open, so $\exists$ ball $N_\varepsilon(f(p)) \subset U$.

By <u>continuity</u> of f, $\exists$ δ-ball $N_\delta(p)$ that maps into $N_\varepsilon(f(p)) \subset U$.

This means $N_\delta(p) \subset f^{-1}(U)$.

So x is an interior point of $f^{-1}(U)$.

($\Leftarrow$) Fix $p \in X$, $\varepsilon > 0$.

Let $B = \varepsilon$-ball about $f(p)$.

Then $p \in f^{-1}(B)$, which is open by assumption.

Since p is an <u>interior</u> point of $f^{-1}(B)$, $\exists$ some ball $N_\delta(p) \subset f^{-1}(B)$.

This δ has the required property, means $f(N_\delta(p)) \subset B$,

so f is <u>continuous</u>. ■

• <u>CONSEQS</u>. <u>Thm</u>. $X \xrightarrow{f} Y \xrightarrow{g} Z$, f, g <u>cont'n</u>, $\Rightarrow g \circ f$ is cont'n.

<u>proof</u>. Given $\mathcal{U}$ open in Z, $\Rightarrow g^{-1}(\mathcal{U})$ is open in Y. (g cont'n)

$$\Rightarrow f^{-1}(g^{-1}(\mathcal{U})) = (g \circ f)^{-1}(\mathcal{U}) \text{ open in } X. \quad (f \text{ cont'n}) \quad \blacksquare$$

• <u>Thm</u>. $f: X \to Y$ <u>closed</u> $\iff \forall$ <u>closed</u> K in Y, $f^{-1}(K)$ is <u>closed</u> in X.

<u>proof</u>. $f^{-1}(K) = [\underbrace{f^{-1}(K^c)}_{\text{open.}}]^c$

• <u>Thm</u>. $f: X \to Y$ <u>cont'n</u>, X <u>cpt</u> $\Rightarrow f(X)$ <u>cpt</u>.

<u>proof</u>. Let $\{V_\alpha\}$ cover of $f(X)$, $\mathcal{U}_\alpha = f^{-1}(V_\alpha)$.

By <u>cpt</u> of X, $\exists$ finite cover $\{\mathcal{U}_1, \cdots, \mathcal{U}_N\}$.

Then $\{V_1, \cdots, V_N\}$ <u>covers</u> $f(X)$. ($x \in \mathcal{U}_\alpha$, $f(x) \in V_\alpha$) $\blacksquare$

• <u>EX</u>. $(\;\;) \xrightarrow{f} \mathbb{R}$ $\checkmark$ f exists.

$[\;\;] \overset{f}{\not\to} \mathbb{R}$ $\times$ f doesnt exist.

• <u>COR</u> $f: X_{cpt} \to \mathbb{R}^k$, then $f(X)$ <u>closed</u> and <u>bdd</u>. ($f(X)$ cpt ; in $\mathbb{R}^k$; closed an. bdd)

• <u>COR</u> $f: X_{cpt} \to \mathbb{R}$, then f achieves <u>max</u> and <u>min</u>. (bdd $\Rightarrow$ sup. inf ; closed $\Rightarrow$ max, min)

- **Thm.** If $f: X \to Y$ bijection (1-1, onto), cont'n, X cpt $\Rightarrow f^{-1}$ is cont'n.

  proof. U is open in $X \Rightarrow U^c$ closed in X cpt $\Rightarrow U^c$ cpt $\Rightarrow f(U^c)$ cpt

  $\Rightarrow f(U^c)$ closed $\Rightarrow f(U)$ open.

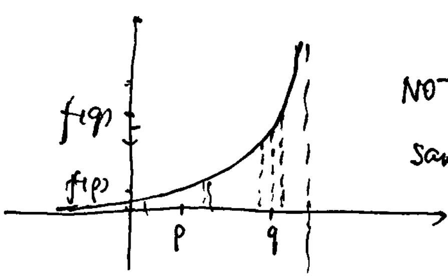
Cont'n.

- close enough points map to close points.
- The function preserves limits of sequences.
- Inverse images of open sets are open.
- Inverse images of closed sets are closed.

- ~~open~~ composition ; compact ; compact $\to$ min, max.

- **Def'n** (UNIFORM CONTINUITY) Call $f: X \to Y$ uniformly continuous on X if ~~for~~ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x$ and p in X, $d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon$.

  (same δ works for all p in X.)



NOT UNIFORMLY CONTINUOUS.

Same ε, different δ.

• **Thm.** $f: X \to Y$ contin, X cpt, then f is uniformly contin on X.

proof. Give $\varepsilon > 0$. (Goal: find a δ that works for all p.)
Each point x has δ_x-ball s.t. $d(y,x) < \delta \Rightarrow d(f(y), f(x)) < \frac{\varepsilon}{2}$.
These cover X.

Q Can I find δ s.t. if $d(p,q) < \delta$, then p, q in the same cover set?
For then ~~$d(p,q) < d$~~ $d(f(p), f(q)) \le d(f(p), f(x)) + d(f(q), f(x)) \le \varepsilon$
$(d(p,x) < \delta_x,\ d(q,x) < \delta_x)$

• **Lebesgue covering lemma.** If $\{U_\alpha\}$ is an open cover of cpt X,
then $\exists \delta > 0$ s.t. $\forall x \in X$, $B_\delta(x)$ is contained in some U_α.
δ — Lebesgue number of cover.

• This is not true for open interval. cannot find such an δ.

proof. Since X is cpt, $\exists$ finite subcover $\{U_{\alpha_i}\}_{i=1}^{n}$.
If K closed, define $d(x, K) = \inf_{y \in K} d(x,y)$.
Claim: $d(x, K)$ is cont'n function of x.
Then $f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, U_{\alpha_i}^c)$ is cont'n fun on cpt set, so it allows min δ.
So if $f(x) \ge \delta$ then at least one of $d(x, U_{\alpha_i}^c) \ge \delta$, so for this δ,
$B_\delta(x) \subset U_{\alpha_i}$. ■

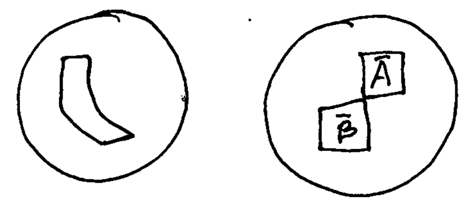
• Thm. $f: X \to Y$ cont'n, E connected $\subset X$, then $f(E)$ conn.

proof. Suppose $f(E)$ is not conn, then $f(E) \to A \cup B$, a separation (non ϕ, $\bar{A} \cap B = \bar{B} \cap A = \phi$)

Notice: $K_A = f^{-1}(\bar{A})$ $K_B = f^{-1}(\bar{B})$ are closed (since f ~~closed~~ cont'n).

Let $\left.\begin{array}{l} E_1 = f^{-1}(A) \cap E. \\ E_2 = f^{-1}(B) \cap E. \end{array}\right\}$ disjoint, non-empty.

claim: they separate E.



Notice $E_1 \subset K_A$ closed. so $\bar{E}_1 \subset K_A$

$E_2 \subset K_B$ closed. So $\bar{E}_2 \subset K_B$

& $K_A \cap E_2 = \phi$ (why? $f^{-1}(\bar{A}) \cap f^{-1}(B) = \phi$)

$K_B \cap E_1 = \phi$. So E is separated (contradiction). ■

• Thm (Intermediate Value Thm) If $f: [a,b] \to \mathbb{R}$ cont'n, & $f(a) < c < f(b)$, then $\exists\, x \in (a,b)$ s.t. $f(x) = c$.

proof. $[a,b]$ conn $\Rightarrow$ $f([a,b])$ conn

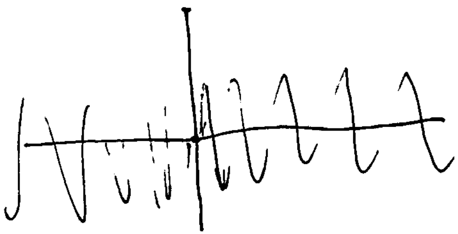
but if c not achieved, then c would disconnect $f([a,b])$. ■

Converse false.

$$f(x)=\begin{cases} 0 & x=0 \\ \sin(\frac{1}{x}) & x\neq 0 \end{cases}$$

topologst's sine curve

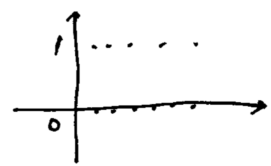
not cont'n but satisfies IV property.



· DISCONTINUOUS FUNCTIONS

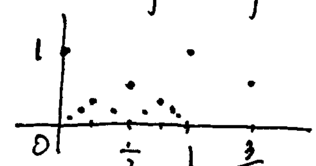
· Dirichlet function

$$f(x)=\begin{cases} 1 & \text{if } x\in\mathbb{Q} \\ 0 & \text{if } x\notin\mathbb{Q} \end{cases}$$



f is not continuous at any p. (NOT satisfy the def'n of continuity)

EX. (HW) $f(x)=\begin{cases} \frac{1}{q} & \text{if } \frac{p}{q}=x \text{ lowest form.} \\ 0 & \text{if } x\notin\mathbb{Q}. \end{cases}$

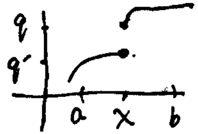


discontinuous at all rationals but continuous at all irrationals.

(Finitely many points ~~above~~ beyond any ε-ball around 0 for y ·)·

• <u>DISCONTINUITIES</u>

$f:(a,b)\to \cancel{X}\,\mathbb{R}$



<u>Note</u>: For all $\{t_n\}$ in (x,b) with $t_n\to x$, have $f(t_n)\to q$. Write $\begin{cases} f(x^+)\to q. \\ \lim_{\cancel{t_n}\to x^+} f(t) = q. \end{cases}$

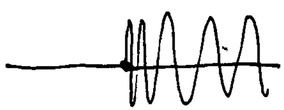
Similarly, say $f(x^-)=q'$, $\lim_{t\to x^-} f(t) = q'$

$\lim_{t\to x} f(t)$ exists $\iff$ $\lim_{t\to x^-} f(t) = \lim_{\cancel{x}\to x^+} f(x) = \lim_{t\to x} f(t)$

If they <u>exist</u>, but <u>not</u> equal, say f has <u>discontinuity</u> of the $1^{\underline{st}}$ kind.

Else we say $2^{\underline{nd}}$ kind.

• $2^{\underline{nd}}$ <u>kind</u>:

① 

at $x=0$

$$f(x)=\begin{cases} 0 & \text{if } x\le 0 \\ \sin(\frac{1}{x}) & \text{if } x>0. \end{cases}$$

② Ori. Dirichlet function is 2^{nd} kind.

③ <u>BUT</u>. <u>$\frac{1}{q}$-Dirichlet fun</u>, is <u>simple dicon</u>.

④ $f(x)=\begin{cases} x^2 & \text{if } x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q} \end{cases}$ is <u>cont'n at 0</u> & <u>discont'n at all the other</u> points

MONOTONIC FUNCTION $f : (a,b) \to \mathbb{R}$

- f mono incr if $x \le y \Rightarrow f(x) \le f(y)$

  .. decr .. $\Rightarrow f(x) \ge f(y)$

- Thm. f mono incr on $(a,b) \Rightarrow f(x^+), f(x^-)$ exist $\forall x \in (a,b)$

  In fact, $\sup_{t \in (a,x)} f(t) \le f(x) \le \inf_{t \in (x,b)} f(t)$

  sup & inf exists because $f(x)$ is bounded.

  call A. claim $A = f(x^-)$.

  Given $\varepsilon > 0$, consider $A - \varepsilon$.

  $\exists\, \delta$ s.t. $A - \varepsilon < f(x-\delta) \le A$ since A is sup.

  but then any $t \in (x-\delta, x)$ must satisfy $f(x-\delta) < f(t) < A$,

  So $f(t) \in (A-\varepsilon, A)$, as desired.

  Similarly arg. suits the other side.

- COR. Mono fun. has no 2^{nd} kind discont'n. (following above thm).

- Thm. Let f be monotonic on (a,b). Then the set of points of (a,b) at which f is discontinuous is at most countable.

proof. $\forall x$ where f is discont'n, pick $r(x) \in \mathbb{Q}$ s.t. $f(x^-) < r(x) < f(x^+)$.

If $x, y \in D$, $r(x) \neq r(y)$ because f mon.

Get 1-1 ~~cor~~ correspondence between D and subset of $\mathbb{Q}$. ■

(Since every discont'n has a gap, we don't want infinitely many gaps or intervals. Those are disjoint. There can't be uncountably many disjoint intervals in $\mathbb{R}$)

- Lemma: Every open set in $\mathbb{R}$ can be expressed as a countable number of open intervals (because $\mathbb{R}$ is second countable).

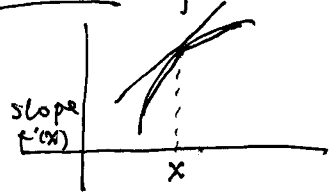
proof: In a disjoint union of open intervals $(I_j)_{j \in J}$ each interval I_j contains a rational number q_j which enables to define an injection $J \to \mathbb{Q}$. ■

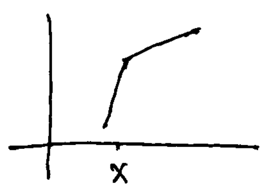
DIFFERENTIATION

- Def'n A function $f : [a,b] \to \mathbb{R}$ is differentiable at $x \in [a,b]$ if this limit exists:

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \quad (a < t < b,\ t \neq x)$$

derivative of x, a function of x, slope of secant line.





not differentiable at x.

Q If f cont'n on $[a,b]$, is f diff'able on $[a,b]$? NO.

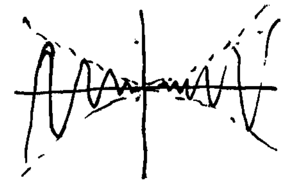
Q If f diff on $[a,b]$, could f be cont'n on $[a,b]$?

yes. why? If $t \to x$. $\lim_{t\to x} f(t)-f(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}\cdot(t-x) = f'(x)\cdot 0 = 0$. ✓

$\lim_{t\to x} f(t) = f(x)$: cont'n.

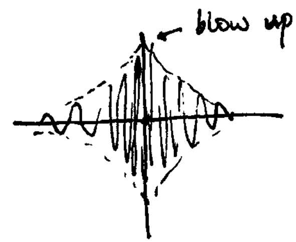
Q If f is diff'ion $[a,b]$, must f' be contin? NO.

$$f(x) = \begin{cases} x^{4/3}\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f 

- f' always satisfies IVP &

  f' has no simple discontinuity.

f' 

- Call a function f a C^1-function

  if f' exists and is contin.

C^k-function if k^{th} derivative $f^{(k)}$ exists & contin.

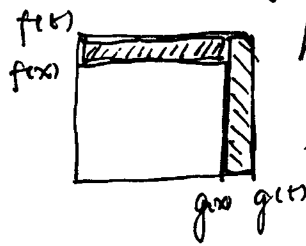
C^0- ··· cont'n.

C^∞- ··· all derivative exists (SMOOTH).

• If f' is limit, then sum, prod, quotient rules follow.

$(f+g)' = f' + g'$ $\qquad$ $(fg)' = f'g + fg'$

(sum(limits) = limit(sums)) $\qquad$ Let $h = fg$, then



$h(t) - h(x) = \cancel{h} f(t)(g(t)-g(x)) + g(x)(f(t)-f(x))$

• Thm. There exists functions $\mathbb{R} \to \mathbb{R}$ that are continuous everywhere, but differentiable nowhere!

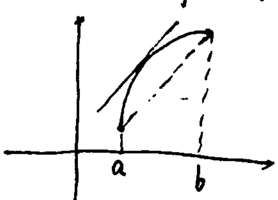
Here's one: $\quad f(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x) \quad 0<b<1,\ a \text{ odd} \in \mathbb{Z},\ ab > 1 + \frac{3}{2}\pi.$

• The Mean Value Thm.

• If f is cont'n on $[a,b]$, diff'l on (a,b), then

$\exists$ point $c \in (a,b)$ s.t. $\quad f(b) - f(a) = (b-a) f'(c)$.

(The slope of the secant line is equal to ~~some point~~ the derivative of some point.)



☆ It connects value of f to value of f' without using limits!!!

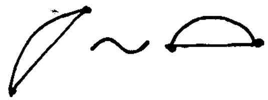
EX. app. If $f'(x)>0$ for all $x \in (a,b)$, then show $f(b)>f(a)$.

proof. $f(b)-f(a) \underset{\substack{\uparrow \\ \text{MVT}}}{=} (b-a)f'(c) > 0$

proof. ① If h on $[a,b]$ has local max at $c \in [a,b]$ & $h'(c)$ exists $\Rightarrow h'(c)=0$.

(Idea. $\dfrac{h(t)-h(c)}{t-c}$ $\Big\}$ ⊖ right $t>c$; ⊕ left $t<c$ $\Big\}$ right limit & left limit exist & equal $\Rightarrow$ limit must be 0!

② General MVT



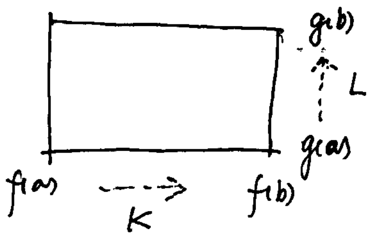
transformation : so we can use ①

If $f(x), g(x)$ cont'n on $[a,b]$

diff. on (a,b)

then $\exists\, c \in (a,b)$ s.t. $[f(b)-f(a)]\,g'(c) = [g(b)-g(a)]\,f'(c)$.

Idea.



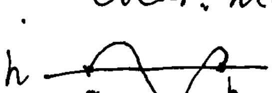
LHS. is the rate that ~~K~~ L sweeps the area.

RHS. is the rate that K sweeps the area.

Consider $h(x) = [f(b)-f(a)]\,g(x) - [g(b)-g(a)]\,f(x)$

distance of areas swept by time x.

clear: $h(a)=h(b)=0$ so by ①, $\exists\, c$ s.t. $h'(c)=0$



TAYLOR'S THEOREM

suppose know $f(a)$, what approximate $f(b)$

MVT says: $f(b) = f(a) + \underline{f'(c)(b-a)}$ some $c \in (a,b)$ "error" not precisely know.

- This suggests: $f(b) = f(a) + f'(a)(b-a) + \underline{\text{error}} \leftarrow$ In fact, $\underline{\frac{f''(c)}{2}}(b-a)^2$

  It gives how good the error is. $c \in (a,b)$

More generally, if $P_{n-1}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}$

(poly deg $n-1$)

~~Thm~~ • Taylor's thm. If $f^{(n-1)}$ $\underline{\text{cont'n}}$. on $[a,b]$, $f^{(n)}$ $\underline{\text{exists}}$ on (a,b)

then $P_{n-1}(x)$ $\underline{\text{approximate}}$ $f(x)$ and $f(x) - P_{n-1}(x) + \frac{f^{(n)}(c)}{n!}(x-a)^n$ $c \in (a,b)$

- when $n=1$, the MVT.

- $P_n(x)$ is "$\underline{\text{best}}$" poly approx of order n at a, i.e., same value of $\left.\begin{matrix} f, f', f'', \ldots, f^{(n)} \\ P_n, P_n', P_n'', \ldots, P_n^{(n)} \end{matrix}\right\}$ at x

$\underline{\text{Proof}}$. clearly, for some number M, $f(b) = P_{n-1}(b) + M(b-a)^n$

Let $g(x) = f(x) - P_{n-1}(x) - M(x-a)^n$,

$g^{(n)}(x) = f^{(n)}(x) - Mn!$

So it's $\underline{\text{enough to show}}$ $g^{(n)}(c) = 0$ for some $c \in (a,b)$

Check $g(a) = 0$ (see $f(a) = P_{n-1}(a)$)

$g'(a)=0,\quad g''(a)=0,\quad \cdots,\quad g^{(n-1)}(a)=0$.

Also, $g(b)=0$ by def'n.

$g(a)=0$ $\qquad\qquad\qquad\qquad g(b)=0$

$g'(a)=0 \qquad g'(x_1)=0$

$g''(a)=0 \quad g''(x_2)=0$

$\vdots$

$$f^{(n)}(x) - M\cdot n! = 0$$
$$M = \frac{f^{(n)}(x)}{n!}$$

$g^{(n)}(x_n)=0$ for some x_n between a and x_{n-1}, that is, between $\overset{a}{\alpha}$ and $\overset{b}{\beta}$. ■

SEQUENCES OF FUNCTIONS

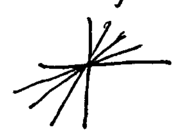
Q What does it mean for seq. of funs to converge?

$$f_1(x),\ f_2(x),\ f_3(x),\ \cdots$$

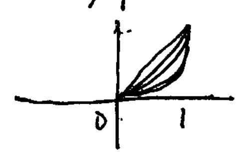
A. Pointwise converge

Fix x, does $\{f_n(x)\}$ converge? If so, ptwise limit $f(x) = \lim_{n\to\infty} f_n(x)$

EX1. $f_n(x) = \frac{x}{n} \xrightarrow[\text{ptwise}]{} f(x)=0$

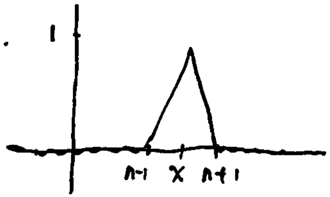


EX2. on $[0,1]$, $f_n(x) = x^n$



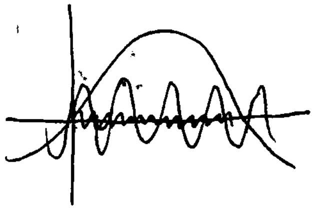
$f_n(x) = x^n \xrightarrow[\text{ptwise}]{} f(x) = \begin{cases} 1 & \text{at } x=1 \\ 0 & \text{others} \end{cases}$

EX3.



$f_n \xrightarrow[pt]{} f(x)=0$

EX4



$f_n(x)=\frac{1}{n}\sin(n^2x) \xrightarrow[pt]{} f(x)=0$

Q what properties preserved by pointwise limits?

CONTINUITY? NO ②

derivatives? No ④

Integral/area? NO ③

Need stronger notion, let $\|f\| = \sup\limits_{x\in E} |f(x)|$.

this is usual convergence in metric space $C_b(E)$ the cont'n, bounded, fns on E

$d(f,g) = \|f-g\|$

Def'n (Uniform convergence) say $f_n \xrightarrow{u} f$, "f_n converge uniformly to f" on E

if $\forall \varepsilon>0, \exists N$ s.t. $n \geq N \Rightarrow \|f_n - f\| < \varepsilon$.

(same N works for all $x \in E$)

can draw an ε-ribbon about f and f_n is eventually stays in the ribbon.

- FACT (Analysis II) $C_b(E)$ is complete. so we have Cauchy criterion.
- Thm $f_n \xrightarrow{u} f$ on $E \iff \forall \varepsilon>0\ \exists N$ s.t. $\forall n,m>N$ & $\forall x\in E$, $|f_n(x)-f_m(x)|<\varepsilon$

like a real sequence

EX. $f_n [0,1] \to \mathbb{R}^2$, it's Cauchy → It converges

 Like so, The limit? It's cont'n !

Thm. If $f_n \xrightarrow{u} f$, f_n cont'n, thus f cont'n.

proof. use bound:

$$|f(x)-f(y)| \leq |f(x)-f_n(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)|$$

FIX x. $\forall \varepsilon > 0$ STEP 1. choose f_n so $\|f_n - f\| < \frac{\varepsilon}{3}$.

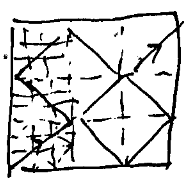
STEP 2. Then f_n cont'n, $\exists \delta > 0$ s.t. $|x-y| < \delta$

So $\forall \varepsilon > 0$, we found $\delta > 0$, s.t. $|f(x)-f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. ∎

Appl. Thm. $\exists f$ $[0,1] \to [0,1]^2$ box, that is space filling !

(The box is filled, every point will be hit by f).

CONSTRUCTION USING LIMIT

 in a finer and finer way !