Some fundamentals of complex exponentials

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Complex exponential or complex sinusoids is a complex-valued function usually used in signal processing field. Its real part is cosine and its imaginary part is a sine.

$$\exp(ix) = \cos(x) + i\sin(x)$$

Remember the magnitude of the complex exponential is

$$|\exp(ix)|^2 = \exp(ix)\overline{\exp(ix)} = \cos(x)^2 + \sin(x)^2 = 1.$$

Frequency and period

The period of the original complex exponential is 2π and the corresponding frequency is $\frac{1}{2\pi}$ (since both $\cos(\cdot)$ and $\sin(\cdot)$ have period of 2π).

$$\exp(i(x + 2\pi)) = \exp(ix)\exp(i2\pi) = \exp(ix)$$

For convenience, we usually set the period and the frequency to 1. In order to do so, we scale the phase, which is x, to $2\pi t$. Note that we convert the variable x to t, usually denoting time. This illustrates that the signal is varying against the time.

Now we have the representation $\exp i2\pi t$ with unit period and frequency. To have an elegant representation in frequency-domain, we are interested in the integer-valued frequencies– 1,2,3,... So we incorporate another coefficient f to denote frequency explicitly.

$$\exp(i2\pi f(t + p)) = \exp(i2\pi ft)\exp(i2\pi fp) = \exp(i2\pi ft)$$
$$fp = 1$$

where we can see f governs the period as well as the frequency.

Facts and calculus

It is obvious that for a set of complex exponentials of any integer frequency $f \in \mathcal{Z}$, they all have period 1. Thus the family of such sinusoids form an orthonormal set of functions on the unit interval $[-\frac{1}{2}, \frac{1}{2}]$. Though we take this unit interval explicitly, it does not matter which interval we take as long as the length of the interval is 1.

Now we denote the complex exponential $\exp(i2\pi ft)$ by $h_f(t)$. To see the orthogonal relation between them, we need some facts of triangular relations.

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$
$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$
$$\cos(-x) = \cos(x)$$
$$\sin(-x) = -\sin(x)$$
$$\overline{h_k(t)} = h_k(-t) = h_{-k}(t)$$
$$h_k(t)\overline{h_j(t)} = h_{k-j}(t)$$

$$\begin{aligned}
< h_k(t), h_j(t) > &= \int_{-0.5}^{0.5} h_k(t)\overline{h_j(t)}dt \\
&= \int_{-0.5}^{0.5} h_{k-j}(t)dt \\
&= \frac{\cos(\pi(k-j)) - \cos(-\pi(k-j))}{i2\pi(k-j)} \\
&= 0
\end{aligned}$$