Lecture 21

- Special cases of DFT

$$\begin{aligned} Ff[0] &= \sum_{n=0}^{N-1} f[n]w^{-n}[0] \\ &= \sum_{n=0}^{N-1} f[n] \end{aligned}$$

analogous to

$$Ff(0) = \int_{-\infty}^{\infty} f(t)dt$$

- Two special discrete signals:

$$\begin{gathered} 1 = [1, \ldots, 1] \\ \delta_0 = [1, 0, \ldots] \\ \delta_k = [0, . . , 1, 0...] \\ F\delta_0 = \sum_{n=0}^{N-1} \delta_0[n]w^{-n} = [1, \ldots, 1] = 1 \\ F\delta_0 = 1 \\ F\delta_k = \sum_{n=0}^{N-1} \delta_k[n]w^{-n} = w^{-k} \\ F\delta_k = w^{-k} \\ Fw^k = \sum_{n=0}^{N-1} w^k[n]w^{-n} \\ Fw^k[m] = \sum_{n=0}^{N-1} w^k[n]w^{-n}[m] = w^k \cdot w^m = \begin{cases} 0, & k \neq m, \\ N, & k = m. \end{cases} \\ Fw^k = N\delta_k \end{gathered}$$

- Different perspective on the definition of the DFT, at least on computation.
  - Can view DFT as $N \times N$ matrix.
  - Computing Ff is matrix multiplication.

$$\begin{gathered} Ff[m] = \sum_{n=0}^{N-1} f[n]w^{-n}[m] \\ w^{-n}[m] = e^{-2\pi inm/N} \end{gathered}$$

  Write $\hat{w} = e^{2\pi i/N}$ Nth root of unity, 1.

$$\begin{gathered} \hat{w}^{nm} = e^{2\pi inm/N} \\ Ff[m] = \sum_{n=0}^{N-1} f[n]\hat{w}^{-nm} \end{gathered}$$

Computing Ff is multiplying by matrix $[\hat{w}^{-nm}]$.

$$(F)_{nm} = (\hat{w}^{-nm})$$

- F is symmetric.
- F^* adjoint of F is conjugate transpose.
- $F^*F = FF^* = NI$.
- Another way of getting inverse Fourier transform

$$F^{-1} = \frac{1}{N}F^*$$

- To compute Ff requires N^2 operations.

- The FFT algorithm reduces the operations to $O(N \log N)$.
- Back to general properties of DFT.
- **Must consider inputs and outputs to DFT as periodic of period N.** that is why in practice, we need padding the signal.
- This is so because of periodicity of discrete complex exponential.

$$w[m] = e^{2\pi im/N}$$
$$w[m+N] = e^{2\pi i(m+N)/N} = e^{2\pi im/N}e^{2\pi i} = w[m]$$

- Likewise, w^n is also periodic of period N.
- $Ff = \sum_{n=0}^{N-1} f[n]w^{-n}$ is periodic of period N.
- So $F = Ff$ is periodic of period N.
- Same reasoning applies in taking F^{-1}. $F^{-1}f = \frac{1}{N}\sum_{n=0}^{N-1} f[n]w^n$ is periodic of period N. Also, produces periodic signal of period N.
- Thus also ought to consider inputs to the DFT as periodic signals of period N.
- *Here is the issue: it forces periodicity into a situation where the periodicity really might not be present.*
- Simple but helpful consequence is "**independence indexing**".
  - Because of periodicity, can use any consecutive N numbers as index.

$$Ff = \sum_{n=1}^{N} f[n]w^{-n}$$
$$f[N] = f[0]$$

  - Likewise can consider f[n] for n<0 $f[-1] = f[N-1]$.
  - If N is even, then

$$Ff = \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f[n]w^{-n}$$

- Can now introduce reversed signals and formulate duality.

- $f^-[n] = f[-n]$
- Find e.g.

$$F(f^-) = (Ff)^-$$
$$FFf = Nf^-$$

Lecture 22 FFT

- The FFT Algorithm reduces complexity from $O(N^2)$ to $O(N \log N)$.
- Exploit structure in DFT via algebraic properties of complex exponential.
- Our approach is to write DFT matrix as product of simpler matrices. The simpler matrices have lots of zeros, so fewer mult required.
- The purpose is to write order N DFT as combination of two DFTs of order $N/2$. Then iterate.
- To do this, need to assume N is a power of 2.

Lecture 23 Linear System

- linear system — An appreciation.
- Want to see how Fourier transform applies to linear systems.
    - impulse response and transfer function.
    - complex exponentials as eigenfunctions of linear time invariant systems.
- Basic definitions
    - A mapping from inputs to outputs that satisfies principle of superposition

$$L(v_1 + v_2) = Lv_1 + Lv_2$$
$$L(\alpha v) = \alpha Lv$$

    - Extension

$$L(\sum_{i=1}^{n} \alpha_i v_i) = \sum_{i=1}^{n} \alpha_i Lv_i$$

    - Can extend this to infinite sums (and to integrals) that generally requires additional assumptions on operators L.
    - Assume some kind of continuity of L.
    - Example
        - Direct proportionality $Lv = \alpha v$.

$$L(v_1 + v_2) = \alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2 = Lv_1 + Lv_2$$

        - **All linear systems can be understood as direct proportionality.**
        - **More generally, multiplication**
        - Linear (parameter can depend on t):

$$Lv(t) = \alpha(t)v(t)$$

        - e.g. switch on for a duration a,

$$Lv(t) = \pi_a(t)v(t)$$

        - Sampling $Lv(t) = \mathrm{III}_p(t)v(t)$ is linear.
    - Slight but important generalisation — direct proportion plus adding, i.e. matrix multiplication. A is $n \times m$ and v is an m-vector, Av is linear.
- Linear systems with special properties derive from special properties of A.
    - Ex. A is symmetric. $A^* = A$. Or unitary.
    - Often look for eigenvectors and eigenvalues of A. $Av = \lambda v$ if $v \neq 0$.
    - If have eigenvectors and eigenvalues. That form a bases for all inputs, then can analyse A easily. v any input can write

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

$$Av = \sum_{i=1}^{n} A(\alpha_i v_i) = \sum_{i=1}^{n} \alpha_i A v_i = \sum_{i=1}^{n} \alpha_i \lambda_i v_i$$

    - **Spectral theorem** says A is hermitian, then can find an orthonormal basis of **eigenvectors**.
- *Not that matrix multiplication is good example of finite dimensional linear systems; It is the only example.*
- *Any finite dimensional linear system can be realised as matrix multiplication.*
- Example
    - Inputs poly's of degree <= n
    - Take $L = \frac{d}{dx}$ linear system. L can be described by an $(n+1) \times (n+1)$ matrix. Find it.
- There is an analogous statement for infinite dimensional continuous case.
    - *Linear system that generalizes matrix multiplication is "integration against a kernel".*
    - Input is a function $v(x)$.
    - "Kernel" is a function $k(x, y)$
    - Linear operator:

$$Lv(x) = \int_{-\infty}^{\infty} k(x, y)v(y)dy$$

    - L is a linear system.
    - It is the infinite dimensional, continuous analog of matrix multiplication.
    - Think of v as infinite column vector. For the kernel, x is the index of the row, y is the index of the column, integral is the sum.
    - What else is here?
    - Special linear systems arise by extra assumptions on $k(x, y)$.
        - Symmetry: $k(x, y) = k(y, x)$.

        - Hermitian: $k(x, y) = \overline{k(y, x)}$.
    - Example

$$Ff(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t)dt$$
$$k(s, t) = e^{-2\pi ist}$$

    - Another example — convolution

$$Lv = h * v$$
$$Lv(x) = \int_{-\infty}^{\infty} h(x-y)v(y)dy$$

    For convolution, the kernel depends on $x - y$, not x and y separately.

    - In particular, if we shift x and y by same amount a, then kernel does not change.
    - This leads to convolution as a "linear time invariant system".
- Not just that "integration against a kernel" is a good example of linear systems. It is the *only* example.

Lecture 24

- For discrete, finite dimensional case, any linear system is given by multiplication by a matrix.
- An analogous result holds in continuous, infinite dimensional case.
- We'll see that linear system is given by integration against a kernel.
- Any linear system looks like

$$Lv(x) = \int_{-\infty}^{\infty} k(x, y)v(y)dy$$

- Have to produce $k(x, y)$ for a general linear system.
- Need a digression on cascading (composing) linear systems.
    - L and M are each linear, then $w = MLv$ is linear.

$$Lv(x) = \int_{-\infty}^{\infty} k(x, y)v(y)dy$$
$$MLv(v) = \int_{-\infty}^{\infty} M_x k(x, y)v(y)dy$$
$$= \int_{-\infty}^{\infty} M_x(k(x, y))v(y)dy$$

    - Back to main plot:

    Any linear system is given integration against a kernel.

    - $v(x) = \int \delta(x-y)v(y)dy$
    - L is linear system, find Lv by apply to the integral:

$$
\begin{aligned}
Lv(x) &= L(\int_{-\infty}^{\infty} \delta(x-y)v(y)dy) \\
&= \int_{-\infty}^{\infty} L_x(\delta(x-y))v(y)dy
\end{aligned}
$$

Set

$$k(x,y) = L_x(\delta(x-y))$$

```
Then we have
```

$$Lv(x) = \int_{-\infty}^{\infty} k(x,y)v(y)dy$$

- Call $k(x,y)$ the impulse response.
  - It is how system L response to an impluse input $\delta(x-y)$.
- **Schwartz kernel theorem**: L is a linear operator on distributions, then there is a unique kernel k, another distribution, so that

$$Lv = < k, v >$$

- Example, what is the impulse response for Fourier transform?

$$
\begin{aligned}
F(\delta(x-y)) = e^{-2\pi ixy} \\
Ff(x) = \int e^{-2\pi isx} f(y)dy \\
k(x,y) = e^{-2\pi ixy} \Rightarrow F(\delta(x-y)) = e^{-2\pi ixy}
\end{aligned}
$$

- Finite dimensional, discrete case.

  $Lv = Av$

  A is matrix, what is the response?

  The impulse response is A.
- Special case of convolution

$$Lv = h * v = \int_{-\infty}^{\infty} h(x-y)v(y)dy$$

- Conclude that

$$L\delta(x-y) = h(x-y)$$

  can also check directly that this is so!
- The relationship between convolution and delay (shift).
- Delay operator

$$\tau_a v(x) = v(x-a)$$

- You showed: convolution of a delayed signal is the delay of the convolution of the signal

$$h * \tau_a v = \tau_a(h * v)$$

- Reinterprate in linear systems

$$Lv = h * v$$

- Say that L is time invariant (shift invariant)

$$\begin{gathered} w = Lv \\ w(x-a) = L(v(x-a)) \end{gathered}$$

- We just saw: If system is given by convoluton, then we have invariance. *The converse is also true!*
- If L is time invariant, then L is given by convolution.
- We know that

$$\begin{gathered} Lv(x) = \int_{-\infty}^{\infty} L_x(\delta(x-y))f(y)dy \\ What\ is\ L_x(\delta(x-y))? \\ Let\ L\delta(x) = h(x) \\ Then\ L(\delta(x-y)) = L(\tau_y(x)) \\ L\tau_a = \tau_a L \end{gathered}$$

- The difference between a general linear system and a time invariant system is its impulse is not a function of x and y independently, but rather a function of x-y.
- Switch function is not a time invariant system

$$h(x,y) = \pi(y)\delta(x-y)$$

- Conclusion:
  - Any linear system is integration against a kernel (impulse response).
  - System is time invariant if and only if it is given by convolution.
- Every time convolution comes into the picture, Fourier transform cannot be far behind.

Lecture 25

- Structure of linear systems

$$\begin{gathered} h(x,y) = L(\delta(x-y)) \\ w(x) = Lv(x) = \int_{-\infty}^{\infty} h(x,y)v(y)dy \end{gathered}$$

- LTI system: say that L is time invariant or shift invariant, the following happens

$$\begin{gathered} w(x) = Lv(x) \\ w(x-y) = L(v(x-y)) \end{gathered}$$

- Impulse response for LTI system

$$
\begin{gathered}
h(x) = L(\delta(x)) \\
h(x-y) = L(\delta(x-y)) \\
w(x) = \int_{-\infty}^{\infty} h(x-y)v(y)dy = (h*v)(x)
\end{gathered}
$$

- System is time invariant if and only if it is given by convolution.
- Same considerations hold for discrete systems given by multipication of matrix

$$
w = Lv
$$

  - L is LTI iff

$$
\begin{gathered}
w = h*v \\
h = L\delta \\
h[n-m] = L\delta_m \\
w[n-m] = h*v[n-m]
\end{gathered}
$$

  - If we write system as matrix multiplication

$$
w = Av
$$

    Then A has a special form for time invariant.

  - e.g.

$$
\begin{gathered}
h = (1,2,3,4) \\
w = Av = h*v \\
What\ is\ A?
\end{gathered}
$$

    Columns of A

$$
\begin{gathered}
A\delta_0, \delta_0 = (1,0,0,0) \\
A\delta_1, \delta_1 = (0,1,0,0) \\
A\delta_2, \delta_2 = (0,0,1,0) \\
A\delta_3, \delta_3 = (0,0,0,1) \\
A\delta_0 = h*\delta_0 = h = (1,2,3,4) \\
A\delta_1 = h*\delta_1 \\
(h*\delta_1)[m] = h[m-1] \\
h*\delta_1 = [4,1,2,3]\ \text{(always assume that signal is periodic)} \\
A\delta_2 = h*\delta_2 = (3,4,1,2) \\
A\delta_3 = h*\delta_3 = (2,3,4,1)
\end{gathered}
$$

    System given by w = Av.

    A is circulent matrix.

- Bring in the Fourier Transform

$$
\begin{gathered}
w = h*v \\
Fw = (Fh)(Fv) \\
W(s) = H(s)V(s)
\end{gathered}
$$

- H(s) is called the transfer function.
- In the frequency domain, the system is given by *direct proportion*.
- **Last great fact on LTI systems: Complext exponentials are eigenfunctions.**

$$
\begin{gathered}
w = Lv = h * v \\
W(s) = H(s)V(s) \\
v(x) = e^{2\pi ivs} \\
What\ is\ Lv(x)? \\
F(e^{2\pi i\nu s}) = \delta(s-\nu) \\
W(s) = H(\nu)\delta(s-\nu) \\
w(x) = H(\nu)e^{2\pi ivs} \\
L(e^{2\pi ivs}) = H(\nu)e^{2\pi ivs}
\end{gathered}
$$

  This says: $e^{2\pi ivs}$ is eigenfunction of any LTI system.

  The corresponding eigenvalue is $H(\nu)$.

- Not true that $\sin$ and $\cos$ are themselves are eigenfunctions of an LTI system.
  - e.g.

$$
\begin{aligned}
v(x) &= \cos(2\pi\nu x) \\
Lv(x) = L\cos(2\pi\nu x) &= L(\frac{1}{2}(e^{2\pi i\nu x} + e^{-2\pi i\nu x})) \\
&= \frac{1}{2}(Le^{2\pi i\nu x} + Le^{-2\pi i\nu x}) \\
&= \frac{1}{2}(H(\nu)e^{2\pi i\nu x} + H(-\nu)e^{-2\pi i\nu x})
\end{aligned}
$$

  Now stuck without any further assumptions.

  - What if h is real, then $H(\nu) = \overline{H(\nu)}$

$$
\begin{aligned}
L(\cos(2\pi i\nu x)) &= \frac{1}{2}(H(\nu)e^{2\pi i\nu x} + \overline{H(\nu)e^{2\pi i\nu x}}) \\
&= Re(H(\nu)e^{2\pi i\nu x}) \\
&= |H(\nu)|\cos(2\pi\nu x + \phi)
\end{aligned}
$$

- Same considerations hold for discrete case

$$
\begin{gathered}
w = h * v \\
W[m] = H[m]V[m]
\end{gathered}
$$

  Discrete complext exponentials are eigenvectors

$$
\begin{gathered}
v = w^k \\
F(w^k) = N\delta_k \\
HN\delta_k = H[k]N\delta_k \\
w = H[k]w^k \\
L(w^k) = H[k]w^k
\end{gathered}
$$

  See that

$$1, w, w^2, \ldots, w^{N-1}$$

form a basis of eigenvectors for any LTI system.

- e.g.

$$\begin{gathered} w = h * v \\ h = (1, 2, 3, 4) \\ w = Ax \end{gathered}$$

Eigenvectors of the system are eigenvectos of A.

Let's do this via LTI system.

Eigenvalues are H[0], H[1], H[2], H[3]

$$\begin{aligned} H = Fh &= \sum_{k=0}^{3} h[k] w^{-k} \\ &= \sum_{k=0}^{3} (k+1) w^{-k} \end{aligned}$$

Eigenvalues are $-10, -2+2i, -2, -2-2i$.