Two ways to Cauchy-Schwarz inequality

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This document only serves as informal personal thoughts.

Cauchy-Schwarz inequality is a very important inequality in linear algebra, especially for the inner-product space (which can be thought of as a vector space with inner-product operation allowed). It states the connectivity between the inner product (which maps vector to scalar) and the norm (which is the length of the vector).

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$$

There are many ways to prove this inequality. Here I give two proofs I have seen for the proof.

Proof. 1. Construct two related inequalities.

$$0 \leq \| \; \|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u} \; \|^2 = 2||\vec{u}||^2||\vec{v}||^2 + 2||\vec{u}||||\vec{v}||\vec{u} \cdot \vec{v} \tag{1}$$

$$0 \leq \| \; \|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u} \; \|^2 = 2||\vec{u}||^2||\vec{v}||^2 - 2||\vec{u}||||\vec{v}||\vec{u} \cdot \vec{v} \tag{2}$$

Then move the terms properly to show Cauchy-Schwarz inequality. [1]

2. This one comes from orthogonality, which says $||proj_{\vec{v}}\vec{u}|| \leq ||\vec{u}||$.

$$proj_{\vec{v}}\vec{u} = c\vec{v} \tag{3}$$

$$proj_{\vec{v}}\vec{u} \cdot (\vec{u} - proj_{\vec{v}}\vec{u}) = 0 \tag{4}$$

$$c\vec{v} \cdot (\vec{u} - c\vec{v}) = 0 \tag{5}$$

$$c = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||||\vec{v}||} \tag{6}$$

$$||proj_{\vec{v}}\vec{u}|| = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||||\vec{v}||}||\vec{v}|| \leq ||\vec{u}|| \tag{7}$$

Then move the terms properly to show Cauchy-Schwarz inequality.[2]

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References

[1] Carlos Fernandez-Granda. *Probability and Statistics for Data Science.*

[2] David C Lay. *Linear algebra and its applications, 2012.* Addison Wesley, Boston.