Some facts about matrix ranks

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This document only serves as informal personal thoughts.

Suppose A is a $m \times n$ matrix. Two facts are easily derived about the row operations on matrix. (Row operations include interchange, replacement, and scaling.)

Fact 1. Row operations on matrix A will not change the linear relationship between columns of A.

Fact 2. If row reduction by row operations on matrix A generates matrix B in echelon form, then any row in B is a linear combination of rows in A.

The following facts illustrate the intrinsic properties of a matrix.

Fact 3. $\dim ColA = \dim RowA$

Proof. Row reduce A into its echelon form B. Then we can find the x pivots and pivot columns of A (which are the non-zero columns in B). These pivot columns are independent due to Fact 1, so the basis for $ColA$ consists of the pivot columns, leading to

$$\dim ColA = x$$

Then consider the row space of A. By Fact 2, the row space of B is exactly the same row space of A. Obviously, the basis of the row space of B is its non-zero rows, since each row cannot be a linear combination of the rows below it. The non-zero rows must consist of all pivots, resulting in

$$\dim RowA = x$$

In all, we have

$$\dim ColA = \dim RowA$$

□

Fact 4. rank(A)=rank($A^\top$)=rank($A^\top A$)=rank($AA^\top$)

Proof. Because $\dim RowA = \dim ColA^\top$ and from Fact 3, we have

$$\dim ColA = \dim ColA^\top$$

which is the fact that

$$rank(A) = rank(A^\top)$$

Now prove that $rank(A^\top A) = rank(A)$.
Take $\vec{x} \in NulA$,

$$\begin{aligned} A\vec{x} &= \vec{0} \\ \Rightarrow A^\top A\vec{x} &= \vec{0} \end{aligned}$$

Thus $\vec{x} \in NulA^\top A$, showing that $NulA \subseteq NulA^\top A$.
Then show that $NulA^\top A \subseteq NulA$. Take $\vec{x} \in NulA^\top A$,

$$\begin{aligned} A^\top A\vec{x} &= \vec{0} \\ \Rightarrow \vec{x}^\top A^\top A\vec{x} &= 0 \\ \Rightarrow \vec{x}^\top A^\top A\vec{x} &= 0 \\ \Rightarrow (A\vec{x})^\top A\vec{x} &= 0 \\ \Rightarrow A\vec{x} &= \vec{0} \end{aligned}$$

Thus $\vec{x} \in A$, showing that $NulA^\top A \subseteq NulA$. Until now we have

$$\begin{gathered} NulA^\top A = NulA \\ \dim NulA^\top A = \dim NulA \\ rank(A^\top A) + \dim NulA^\top A = rank(A) + \dim NulA = n \\ rank(A^\top A) = rank(A) \end{gathered}$$

Just take the transpose of A, we have $rank(AA^\top) = rank(A^\top)$.

□

Fact 5. If A and B are two conformable matrices, $rank(AB) \le \min(rank(A), rank(B))$.

Fact 6. If A and B are two conformable matrices, $rank(A + B) \le rank(A) + rank(B)$.

Fact 7. If B is a square matrix of full rank, $rank(AB) = rank(A)$.