Networked Robotics Systems, Cooperative Control and Swarming (ROB-GY 6333 Section A)

- **Today's lecture:**
  Consensus

Quick Review of Control Theory

$$\dot{x} = ax$$

- Control theory is playing with differential equations
- Is the above differential equation (the system dynamics) stable?
- Depends on a
  - $a < 0$ ($a = 0$ is marginally stable)
  - If a is complex, $\mathrm{Re}\{a\} < 0$ (left-half plane)
- Can we change a to stabilize the system?

Quick Review of Control Theory

$$\dot{x} = ax + u$$

- Can we change *a* to stabilize the system?
  - Not quite, but we can add a control input *u*.
- Many choices for *u*:
  - Explicit function of time:
    - $u(t) = e^t$, $u(t) = e^{-t}$, …
- Pick the simplest possible *u*:
  - Function of the state:
    - $u = kx$

Quick Review of Control Theory

$$\dot{x} = Ax$$

- Control (i.e., stabilizing the system) does not require solving the differential equation
  - Simulation does.
- How do we solve the above equation?

$$x(t) = e^{At}x(0)$$

Matrix Exponential

$$\mathrm{e}^{A} = \sum_{k=0}^{\infty} \frac{1}{k!} A^{k}$$

$$e^{\mathrm{diag}(a_i)} = \mathrm{diag}(e^{a_i})$$

- $e^{0} = I$
- $e^{aX} e^{bX} = e^{(a+b)X}$
- $e^{X} e^{-X} = I$
- If $XY = YX$ then $e^{X} e^{Y} = e^{Y} e^{X} = e^{X+Y}$
- If Y is invertible then $e^{YXY^{-1}} = Y e^{X} Y^{-1}$
- $e^{X^{T}} = (e^{X})^{T}$

Matrix Exponential

$$\mathrm{e}^{A} = \sum_{k=0}^{\infty} \frac{1}{k!} A^{k}$$

$$e^{\operatorname{diag}(a_i)} = \operatorname{diag}(e^{a_i})$$

- What matrices are easy to raise to powers?
- Diagonal matrices!
- Can all matrices be diagonalized?
- Almost

Real Symmetric Matrices

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

- All real symmetric matrices are diagonalizable
- Any real symmetric matrix A, with eigenvalues λ_i and unit eigenvectors q_i can be written as:

$$A = Q\mathrm{diag}(\lambda_i)Q^T$$

  where Q is made of the eigenvectors q_i.

- Note that Q is an orthonormal matrix
  - $QQ^T = Q^T Q = I$
  - $Q^{-1} = Q^T$

Jordan Normal Form

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

- Any square complex matrix has a Jordan normal form:

Any square matrix A, with eigenvalues λ_i can be written in block diagonal form as

$$A = P \begin{bmatrix} J_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & J_p \end{bmatrix} P^{-1}$$

where each block J_i is a square matrix of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

Exponentiating Jordan Blocks

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

$$\left[\begin{array}{ccc|cc} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]^n = \left[\begin{array}{ccc|cc} \lambda_1^n & \binom{n}{1}\lambda_1^{n-1} & \binom{n}{2}\lambda_1^{n-2} & 0 & 0 \\ 0 & \lambda_1^n & \binom{n}{1}\lambda_1^{n-1} & 0 & 0 \\ 0 & 0 & \lambda_1^n & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_2^n & \binom{n}{1}\lambda_2^{n-1} \\ 0 & 0 & 0 & 0 & \lambda_2^n \end{array}\right]^n$$

Matrix Exponential

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

- Straightforward to apply for diagonal systems, less so for Jordan normal form

$$e^{At} = e^{\left\{ P \left(\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} t \right) P^{-1} \right\}} = P e^{\lambda_1 t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} P^{-1}$$

Linear Systems 101 $\quad \dot{x} = Ax \quad \Rightarrow \quad x(t) = \mathrm{e}^{At} x_0$

- $x(t) = 0$ always a solution to the equation and is a **fixed point**
  - (i.e., $x(0) = 0$ implies that $x(t)=0$ for all $t > 0$)
  - If A is rank deficient (i.e. if A has a non-empty null space), any point in the **nullspace** of A is also a **fixed point**
- **Stability:**
  - If all eigenvalues of A have strictly negative real parts then the system is **stable** and $x(t)$ converges to 0 when $T \to \infty$
  - If one eigenvalue of A has a positive real part, then the system is **unstable** and $x(t)$ goes to infinity for any $x(0)$ not a **fixed point**
- **Controllability: "stability at whichever point you like"**

Back to Graph Theory

- What kind of graph is a social media graph?

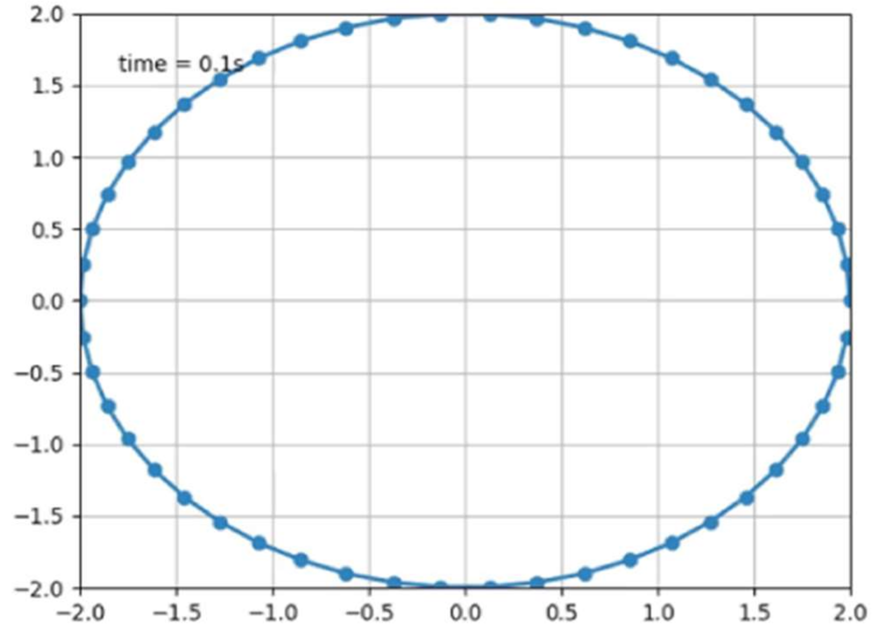
Back to Graph Theory

- Directed/Undirected Graph
- Degree
- Neighborhood
- Adjacency
- Connectedness
- Path, Cycle, Tree, Forest
- Adjacency Matrix
- Degree Matrix
- Incidence Matrix
- Laplacian Matrix

Is this stable?

- Consider an undirected graph

$$\dot{x} = -Lx$$

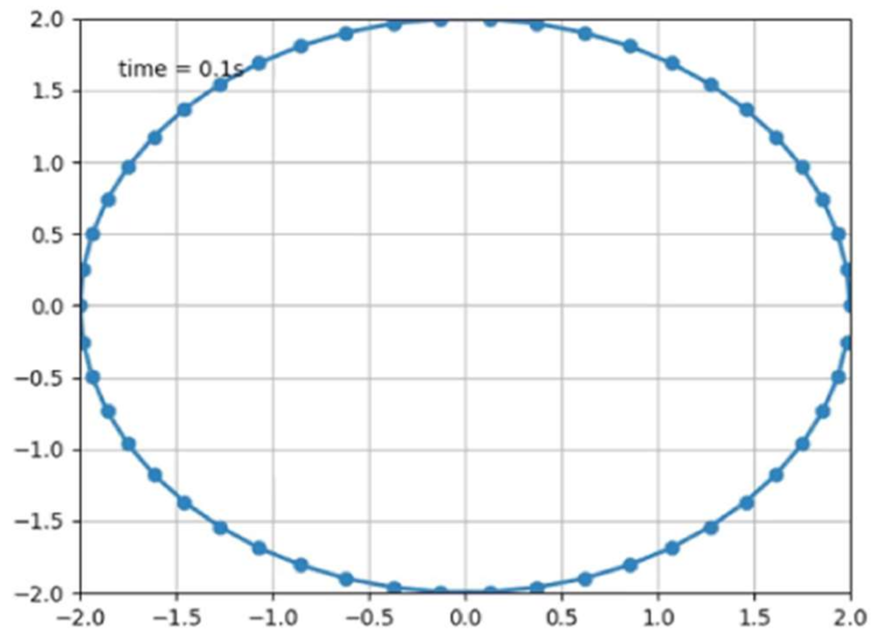


Is this stable?

- Consider an undirected graph

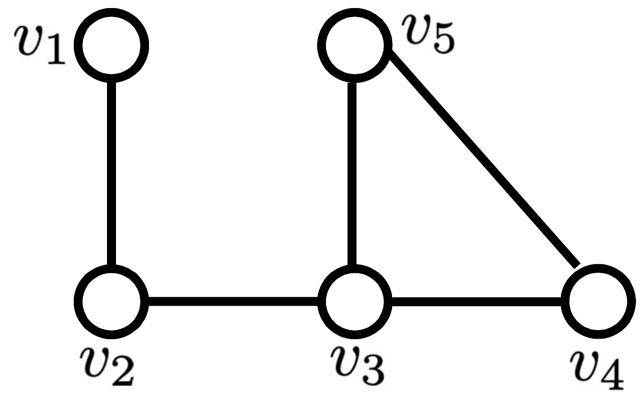
$$\dot{x} = -Lx$$

- Yes, because the Laplacian matrix is symmetric



Laplacian matrix: Example

$$L \in \mathbb{R}^{n \times n} = \Delta - A = EE^T$$



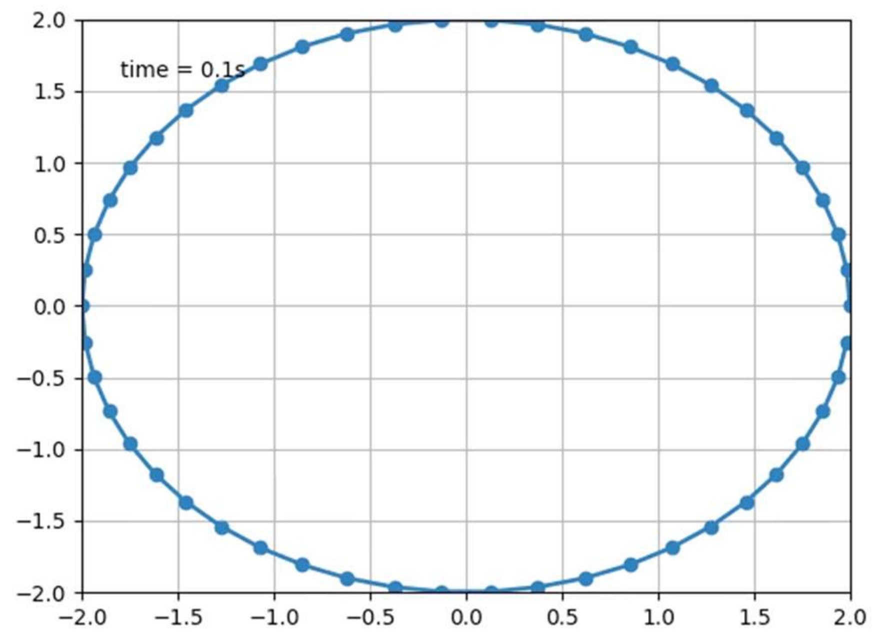
$$\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

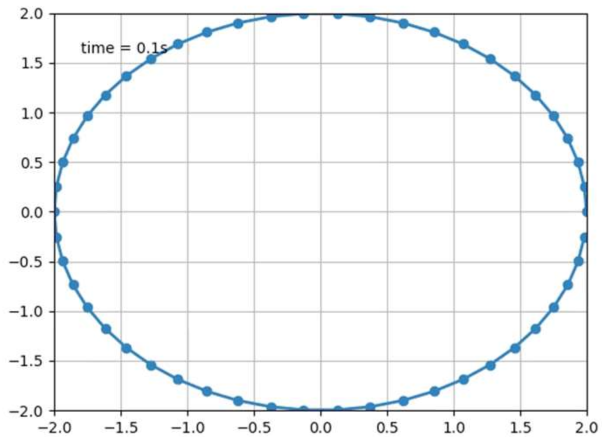
Is this stable?

- Consider an undirected graph

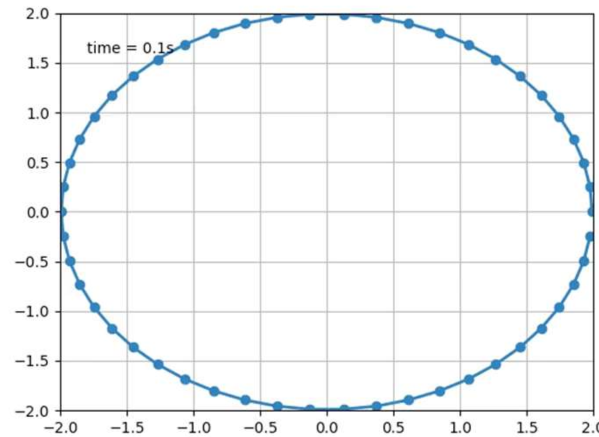
$$\dot{x} = -Lx$$

- Yes, because the Laplacian matrix is symmetric
- How to speed things up?

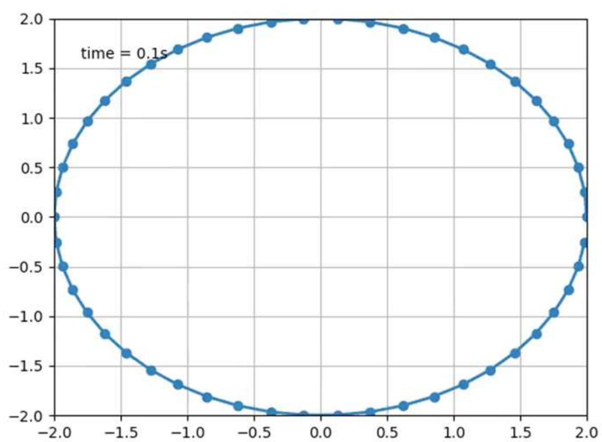




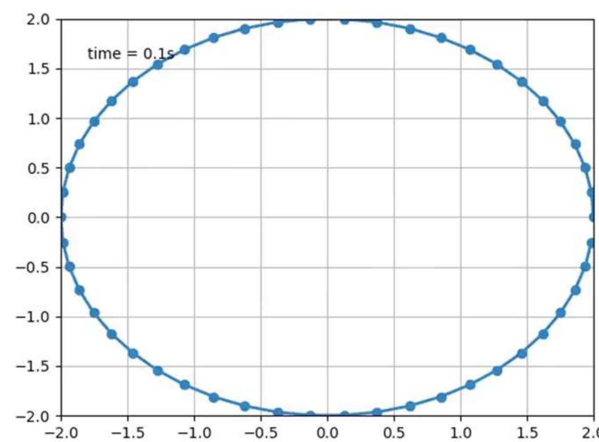
2-regular ring



4-regular ring



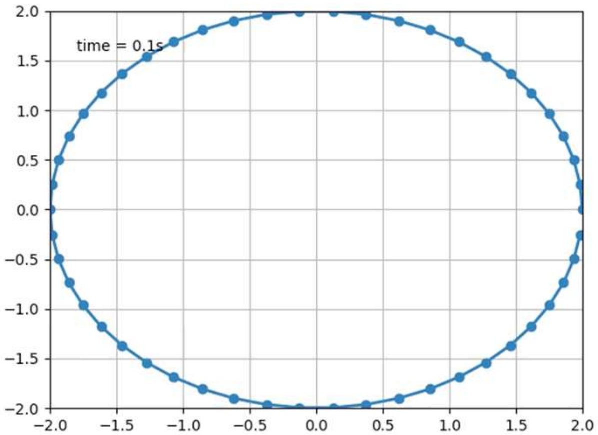
8-regular ring



10-regular ring

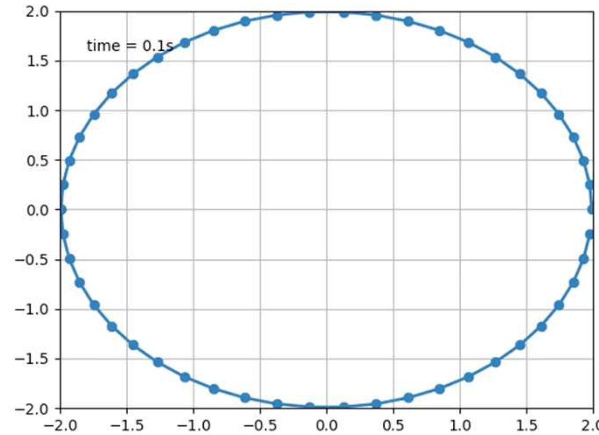
Algebraic Connectivity

- Recall: graph is connected if and only if Laplacian matrix has only one 0 eigenvalue
- The **algebraic connectivity** (Fiedler value or Fiedler eigenvalue) is the second-smallest eigenvalue of the Laplacian matrix.



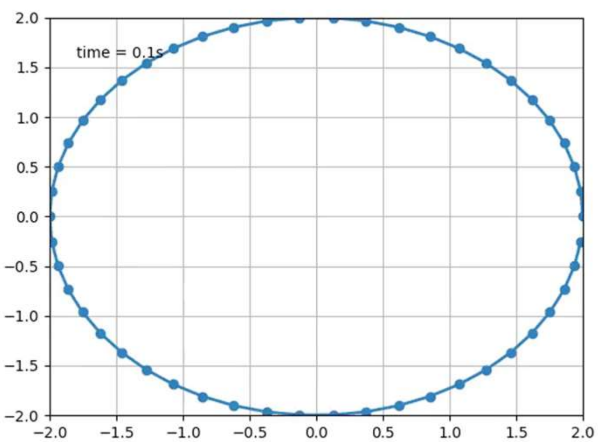
$\lambda_2 = 0.0158$

2-regular ring



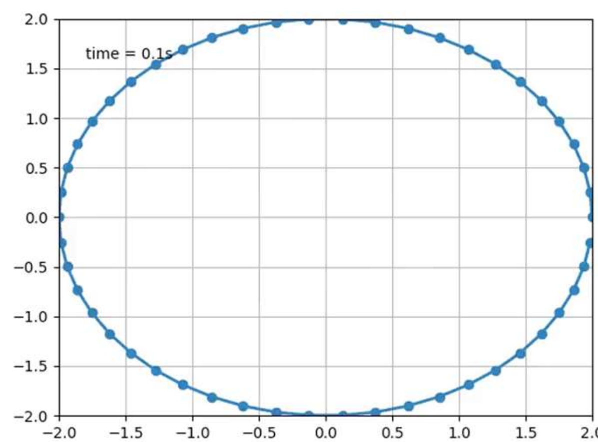
$\lambda_2 = 0.0786$

4-regular ring



$\lambda_2 = 0.4664$

8-regular ring



$\lambda_2 = 5.5743$

10-regular ring

Locality $\dot{x} = -Lx$

- This is a “local” control law
- The control on each vertex depends on its own degree (degree matrix) and its neighbors (adjacency matrix)
- Structure of the Laplacian is very useful!

$$\nabla \cdot \nabla f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\frac{1}{(\Delta X)^2} \begin{vmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{vmatrix} = \begin{vmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{vmatrix}$$

Are directed graphs stable?

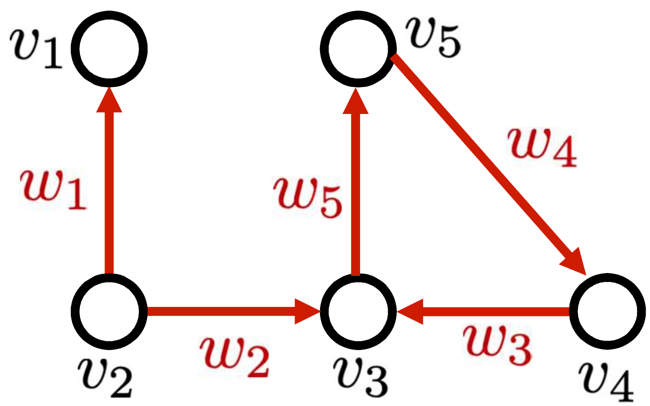
Directed Graph: Gershgorin Disc Theorem

- The eigenvalues of an *n* x *n* matrix *A* lie in the union of the *n* discs in the complex plane defined as follows:

$$D_i = \left\{ z \in \mathbb{C} : | x - a_{ii} | \leq \sum_{j \neq i} | a_{ij} | \right\}, i = 1 : n$$

- Guarantees that for a directed graph, the control law $\dot{x} = -Lx$ is always stable.

Weighted Directed Graphs



- Adjacency Matrix $$[A]_{ij} = \begin{cases} w_{ij} & \text{if } (v_j, v_i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

- Weighted in-degree $$d_{in}(i) = \sum_{j|(v_j,v_i)\in\mathcal{E}} w_{ij}$$

Weighted Directed Graphs: Laplacian

$$L = \Delta - A$$

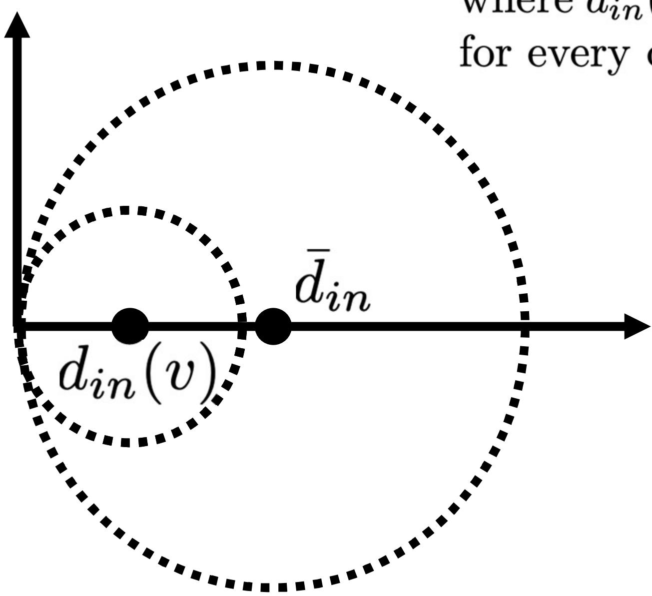
- The Laplacian matrix L is NOT necessarily symmetric
- The sum of the columns of L is 0 (not necessarily the sum of rows)
- **1** is an eigenvector of L with associated eigenvalue of 0 ($L\mathbf{1} = 0$)

Weighted Directed Graph: Gershgorin Disc Theorem

Corollary: Digraph Laplacian Let $\mathcal{D}$ be a weighted digraph on n vertices. Then the spectrum of $L(\mathcal{D})$ lies in the region

$$\left\{ z \in \mathbb{C} \;\middle|\; |z - \bar{d}_{in}(\mathcal{D})| \leq \bar{d}_{in}(\mathcal{D}) \right\}$$

where $\bar{d}_{in}(\mathcal{D})$ denotes the maximum (weighted) in-degree in $\mathcal{D}$. This means that for every digraph, the eigenvalues have non negative real parts ($Re(\lambda) \geq 0$) $\square$

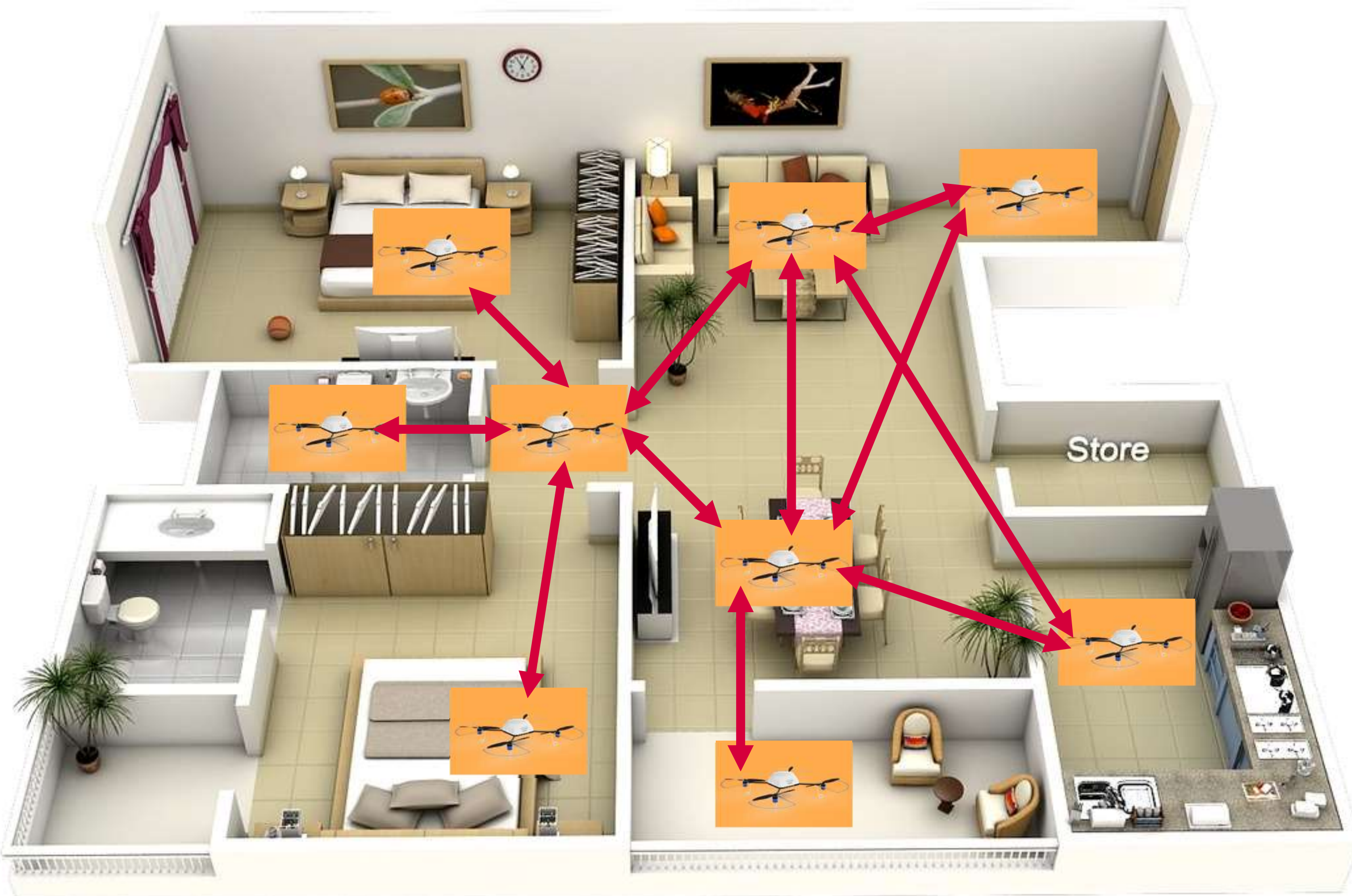


The Consensus Problem

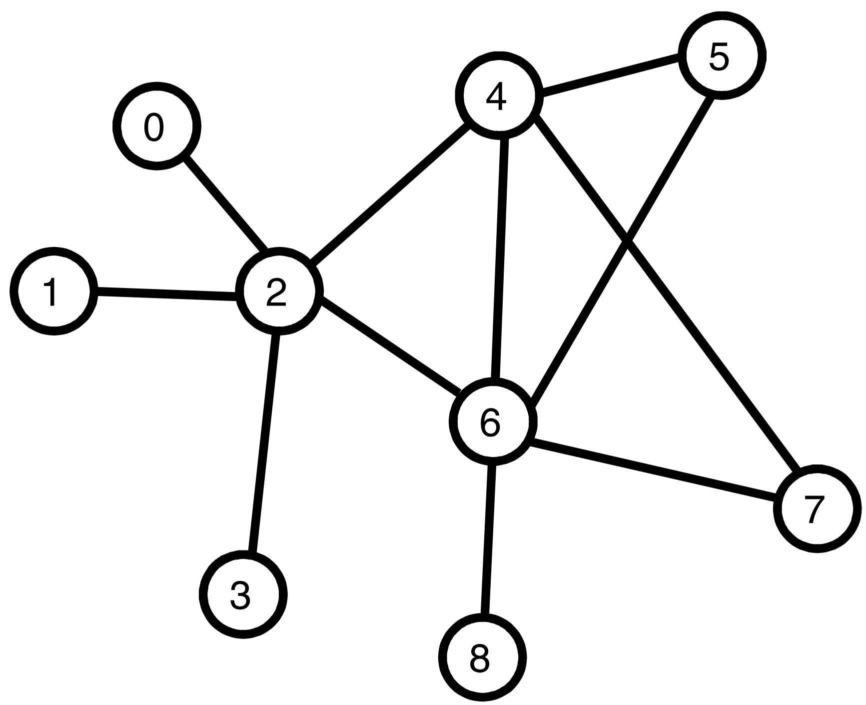
Consensus

- Meet at the same place: **Rendezvous**
- Look in the same direction: **Alignment**
- Agree on the estimation of a certain quantity (e.g. temperature): **Distributed estimation**
- Agree on the same time: **Synchronization**

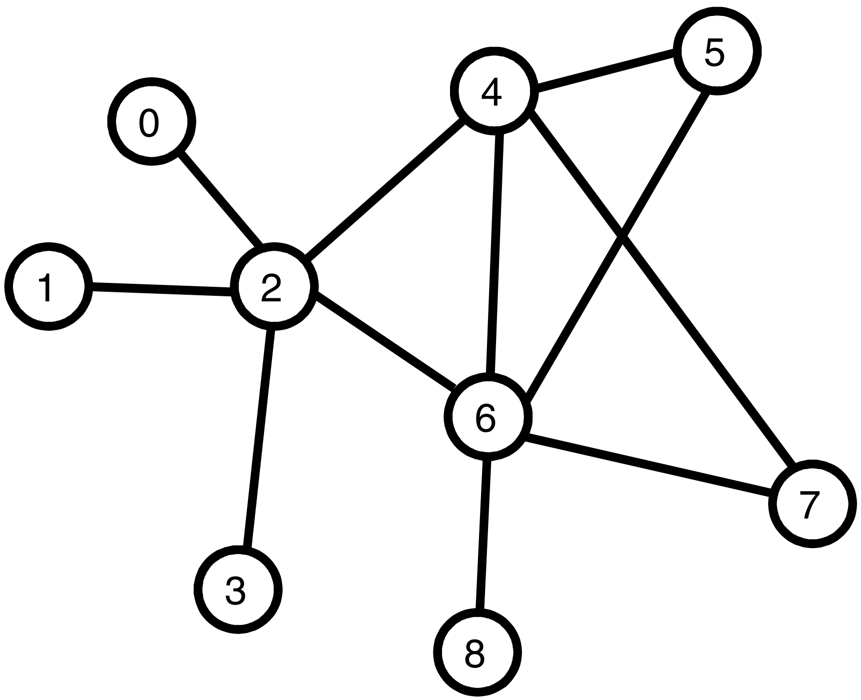
Temperature Estimation



Temperature Estimation



Temperature Estimation

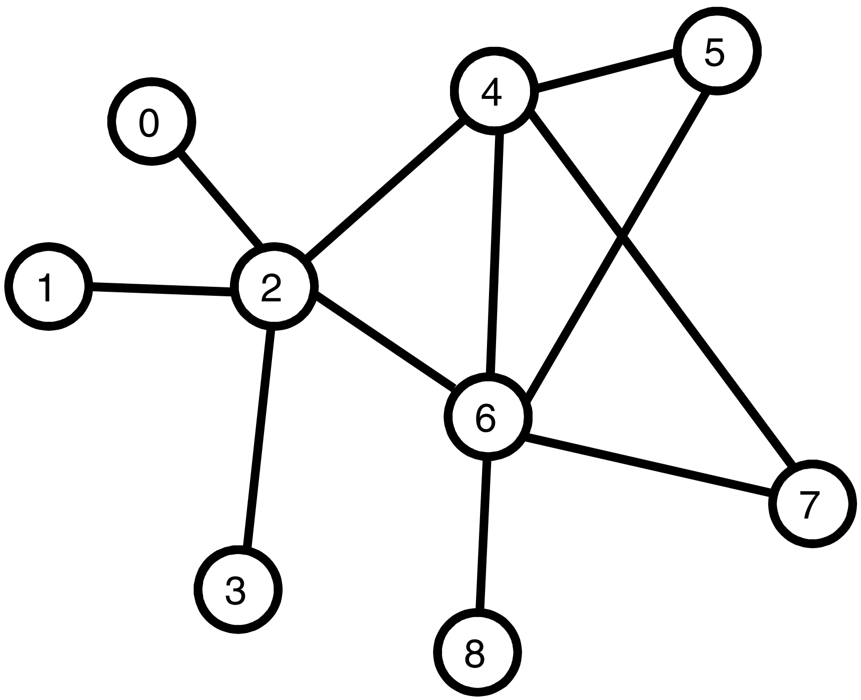


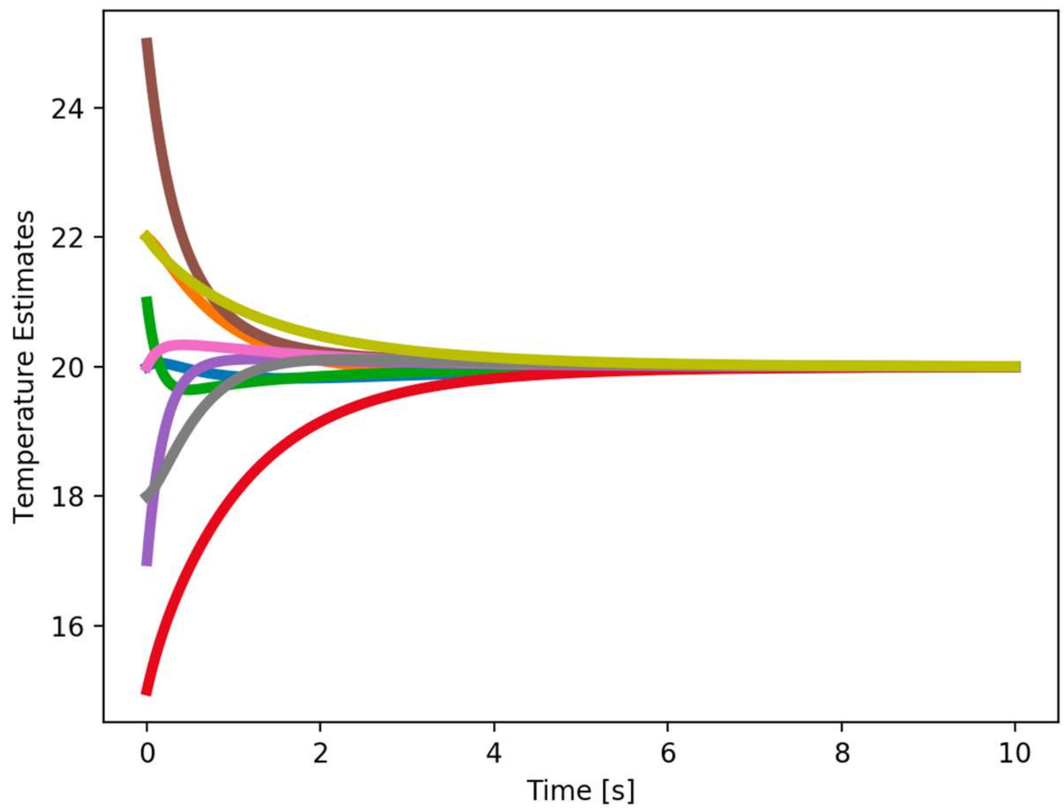
$$L = \begin{bmatrix} 1. & 0. & -1. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 1. & -1. & 0. & 0. & 0. & 0. & 0. & 0. \\ -1. & -1. & 5. & -1. & -1. & 0. & -1. & 0. & 0. \\ 0. & 0. & -1. & 1. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & -1. & 0. & 4. & -1. & -1. & -1. & 0. \\ 0. & 0. & 0. & 0. & -1. & 2. & -1. & 0. & 0. \\ 0. & 0. & -1. & 0. & -1. & -1. & 5. & -1. & -1. \\ 0. & 0. & 0. & 0. & -1. & 0. & -1. & 2. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & -1. & 0. & 1. \end{bmatrix}$$

$$\lambda_i = \{[0. \quad 0.54109706 \quad 1. \quad 1. \quad 1.08021725 \quad 2. \quad 4.14022933 \quad 5.61477866 \quad 6.62367771]\}$$

Temperature Estimation

Initial Temperature Measurements $= \begin{bmatrix} 20. & 22. & 21. & 15. & 17. & 25. & 20. & 18. & 22. \end{bmatrix}$





The Consensus Problem

- Find a local control rule to get agents to agree on the same value in a distributed manner
- **Definition:** *N* dynamical systems (agents) are said to be in agreement if they all have the same state
- Consider *N* agents with trivial 1D dynamics $\dot{x}_i = u_i$

The agreement set $\mathcal{A} \subset \mathbb{R}^N$ is the subspace $\text{span}\{\mathbf{1}\}$ that is

$$\mathcal{A} = \{x \in \mathbb{R}^N \mid x_i = x_j, \forall i, j\}$$

The Consensus Protocol

- The **consensus protocol** $\dot{x} = -Lx$

  converges to the **agreement set** for any initial conditions if and only if *L* is connected. In particular it converges to the **average**

$$\frac{\mathbf{1}\mathbf{1}^T x_0}{N}$$

- **Note:**
  - $\mathbf{1}^T x_0$ is a constant of motion
  - Rate of (exponential) convergence defined by the second smallest eigenvalue of *L* (**algebraic connectivity**)

“Homework”

- Play around with the MATLAB code “Week2.m”
- Try figuring it out in Python
    - good practice if you are new to Python.

Next Week

- Consensus for directed graphs