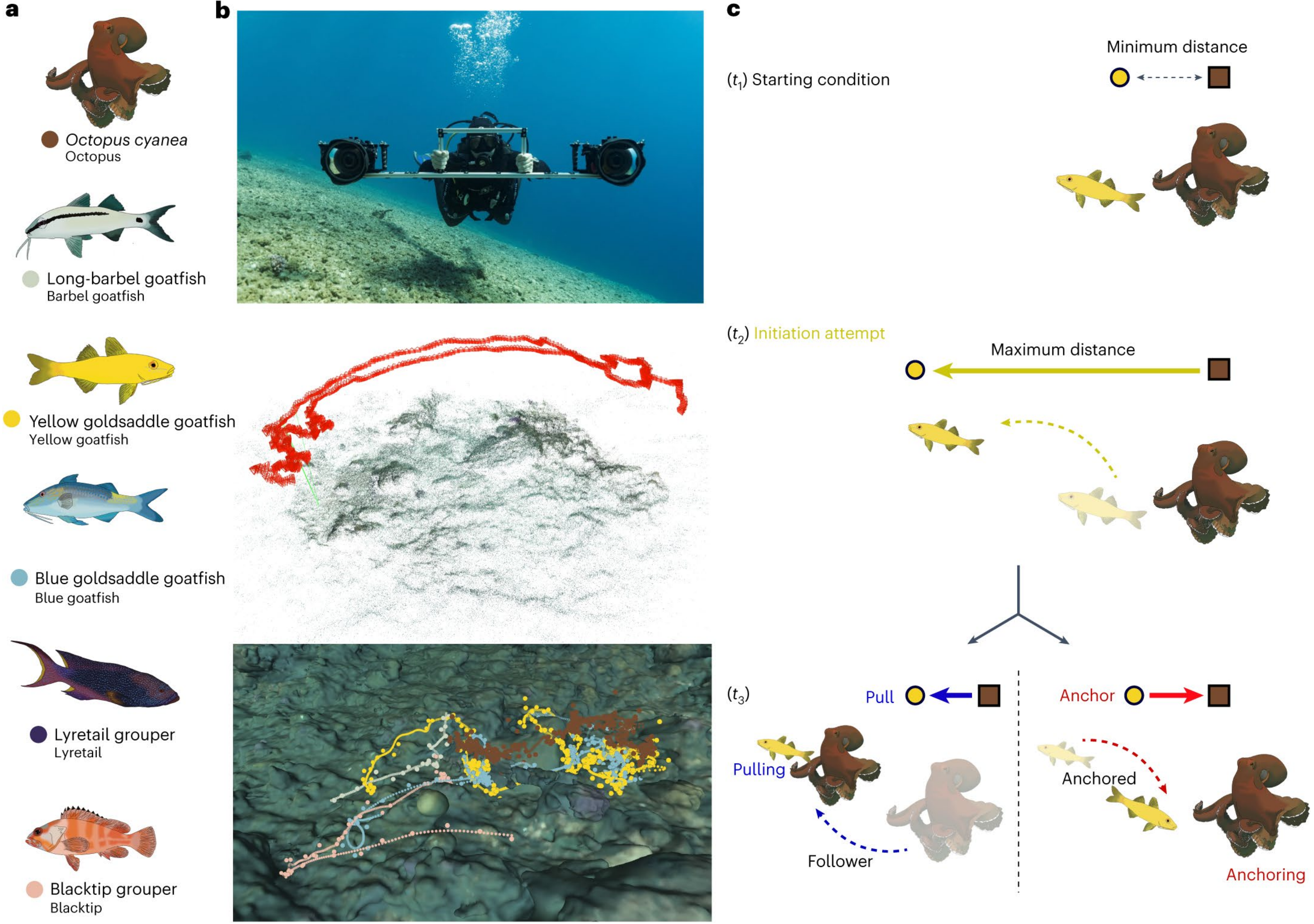
Networked Robotics Systems, Cooperative Control and Swarming (ROB-GY 6333 Section A)

- **Today's lecture:**
    - Consensus and Lyapunov Stability
    - Formation Control

What is leadership?

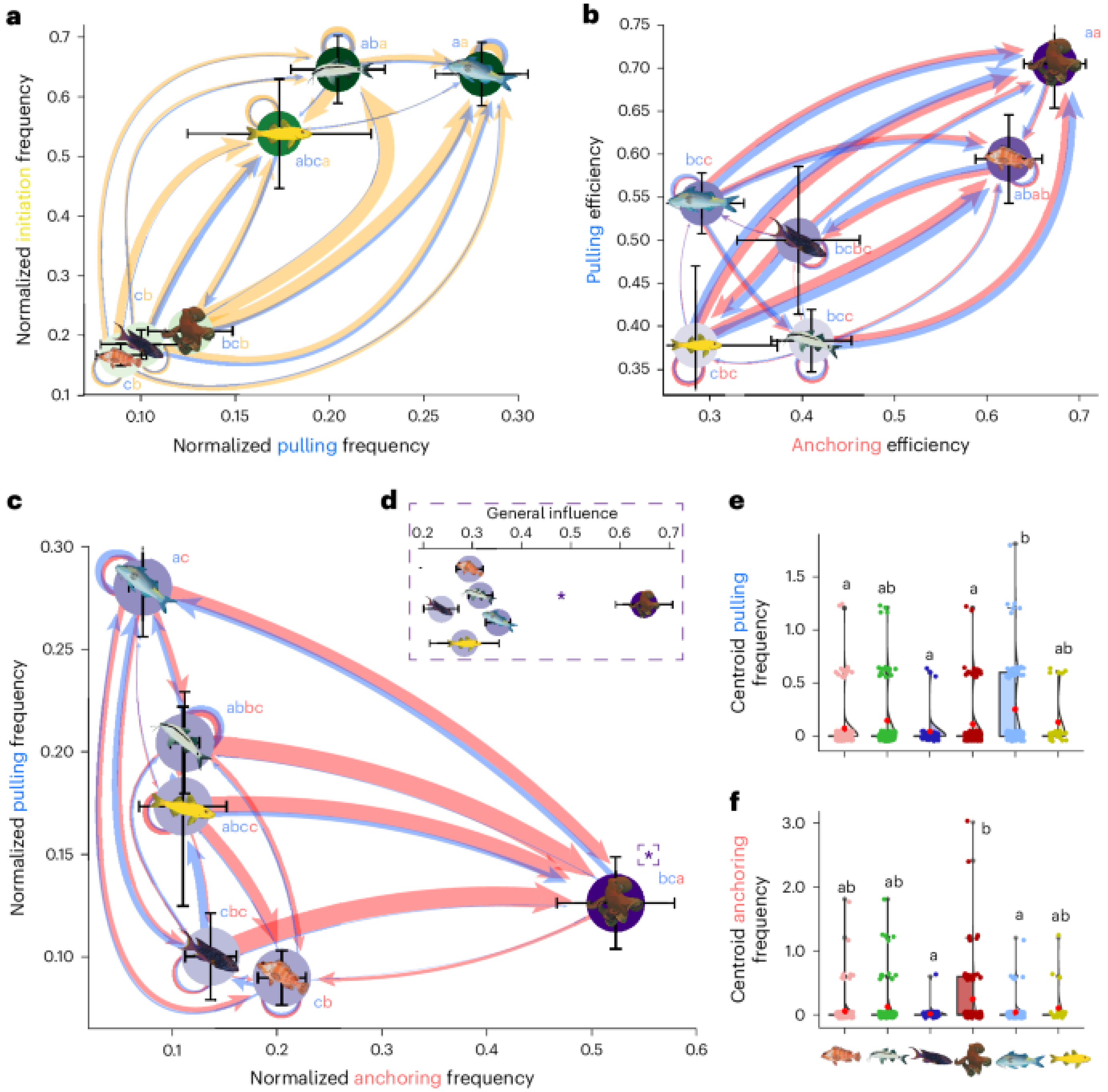
Can you identify leaders from behavior?



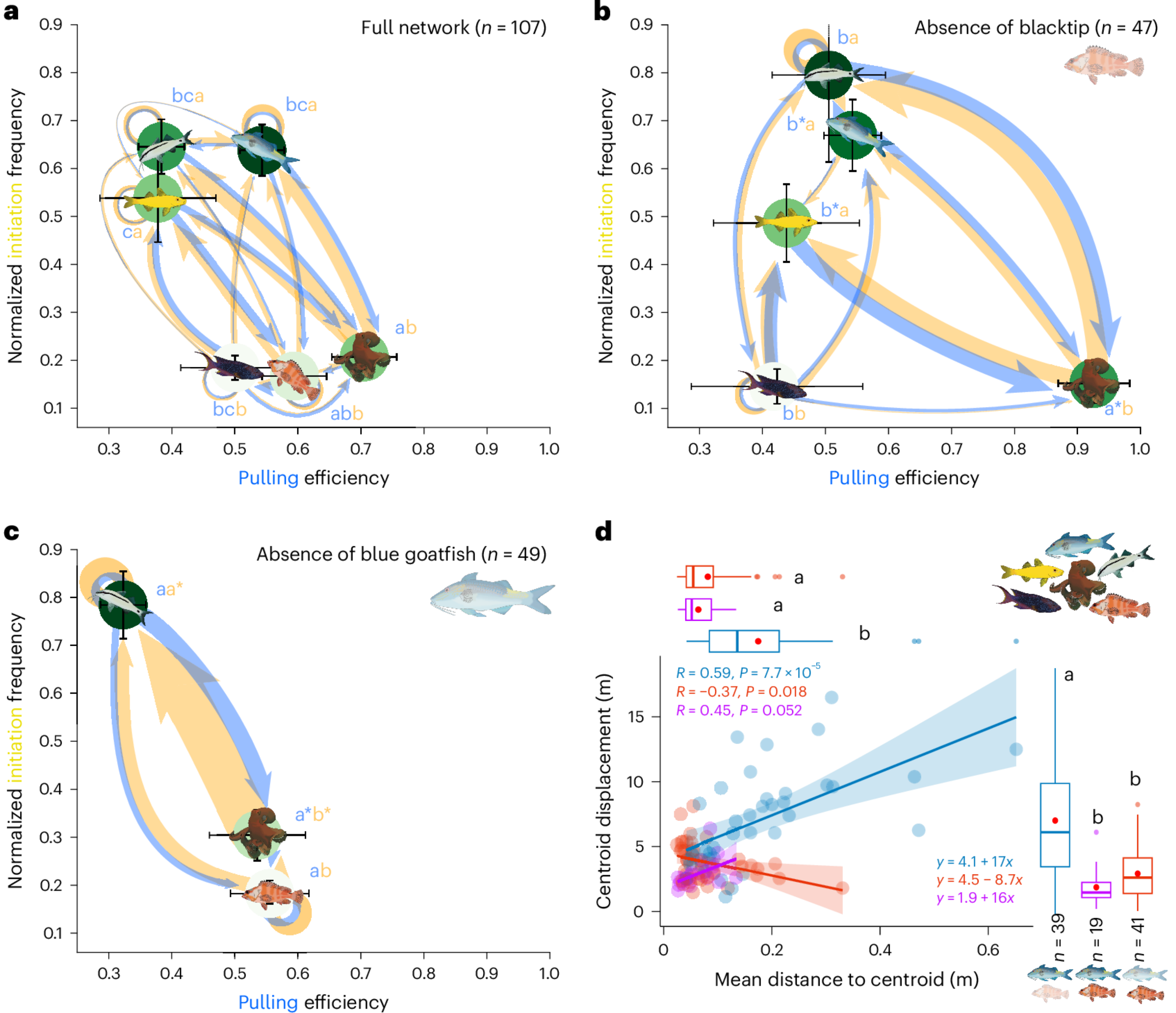
E. Sampaio, et al., (2024) Multidimensional social influence drives leadership and composition-dependent success in octopus–fish hunting groups, *Nature Ecology & Evolution*

Abstract

"Social influence is hierarchically distributed over multiscale dimensions representing role specializations: fish (particularly goatfish) drive environmental exploration, deciding where, while the octopus decides if, and when, the group moves. Thus, **'classical leadership' can be insufficient to describe complex heterogeneous systems**, in which leadership instead can be driven by both stimulating and inhibiting movement. Furthermore, group composition altered individual investment and collective action, **triggering partner control mechanisms (that is, punching)** and benefits for the de facto leader, the octopus. This seemingly non-social invertebrate flexibly adapts to heterospecific actions, showing hallmarks of social competence and cognition. These findings expand our current understanding of what leadership is and what sociality is."



Multidimensionality of leadership.



Group composition effects on hierarchical networks, individual investment and group properties.

Inverse Problem

- Multi-agent network (Octopus-Fish hunting group)
  - e.g., Types of behavior (pulling vs. anchoring)
- Quantitative and graphical description
  - e.g., Pulling/anchoring frequency
    - Infer findings about hierarchy and leadership
- Reminder that we do not have to use graph theory, and different fields have invented their own quantitative methods

Recap

- Choose system and behavior we want:
    - **System: Swarm of *n* drones**
    - **Behavior: Rendezvous at center of swarm**
- Translate behavior to desired mathematical properties:
    - **Linear time-invariant system with one zero eigenvalue corresponding to a special eigenvector (agreement set of $[1\ 1\ 1\ \ldots\ 1\ 1\ 1]^T$)**
- Find consensus protocol satisfying mathematical properties:
    - **Laplacian matrix**

Recap

- Choose system and behavior we want:
  - **System: Swarm of *n* drones**
  - **Behavior: Rendezvous at center of swarm**
- Translate behavior to desired mathematical properties:
  - **Classical Control Theory**
- Find consensus protocol satisfying mathematical properties:
  - **Graph Theory**

Review

- Main benefit of classical control theory (linear system)
  - Show that the consensus protocol $\dot{x} = -Lx$ “works”:
    - It is stable
    - It is globally stable
    - It is **not** asymptotically stable.

Linear Time-Varying System

- Tricky but doable because *L* is a matrix of constant values.
    - There is some added complexity depending on whether the graph is undirected or directed, balanced or unbalance, has or lacks rooted out-branching, but it is all rather straightforward
- What about the stability of a time-varying consensus protocol?

$$\dot{x} = A(t)x$$

- Connections can weaken or break as agents lose communication or move further apart from each other.

Overview

- In general, things are both **nonlinear** and **time-varying**.
- To be general, we consider **dynamical systems**, a broad enough category to everything including:
    - Hysteretic systems: where the system's behavior depends on both its current state, and the history or previous trajectory of the system up to the current state
    - Anticipatory systems: where the system's behavior depends on its expected/anticipated state in the future
    - And systems that can be described with this:

$$\dot{x} = f(x)$$

Overview

- So we will now borrow more advanced concepts from control theory to deal with time-varying and possible nonlinear consensus protocols
- As before, we will try to show if a multi-agent network is stable or not, what kind of stable it is, and make some connections with how the graph looks like.
- We will need to talk about Lyapunov functions, but first we need to redefine what stability means…

Review (For Undirected Graph)

- **Ending:**
  - Equilibrium/Fixed Point

$$x_1 = \cdots = x_n \Leftrightarrow \begin{matrix} \dot{x}_1 = 0 \\ \vdots \\ \dot{x}_n = 0 \end{matrix}$$

- Right eigenvector of Laplacian

$$\dot{x} = -Lx = 0x$$

- **Getting There:**
  - Invariant

$$\frac{d}{dt}(x_1 + \cdots + x_n) = 0$$

- Left eigenvector of Laplacian

$$-L = P\Lambda P^{-1}$$

$$P^{-1}\dot{x} = \Lambda P^{-1}x$$

Lyapunov Function

- We will soon see that the Lyapunov function combines the **ending**, and the process of **getting to the ending** into a single function
    - Can be used for both linear and non-linear systems

The real numbers are the completion of rational numbers

Real Analysis

- **The real numbers are the completion of rational numbers**
  - The rationals are missing limits (Euclid's proof of irrationality of $\sqrt{2}$)

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots = 1$$

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \ldots = 1$$

$$\sum_{i=0}^{\infty} \frac{-1^i}{(2k+1)^i} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots = ?$$

  - When we construct the real numbers, we are inventing a definition that is useful to us: (rationals + limits of all Cauchy sequences modulo equivalence classes)
    - With limits, you can have continuity and derivatives (smoothness)
    - And Taylor polynomials

Arbitrarily close

- This type of language/machinery shows up in delta-epsilon proofs
- Get used to the machinery
  - Demonstrating proximity is a process

$$\sum_{i=0}^{\infty} \frac{-1^i}{(2k+1)^i} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots = ?$$

$$x_n = 1, \frac{2}{3}, \frac{13}{15}, \frac{76}{105}, \frac{263}{315}, \ldots$$

$$|x_n - \pi/4| = 0.21460\ldots, -0.11873\ldots, 0.08127\ldots, -0.06159\ldots, 0.04952\ldots$$

$$\boxed{\forall \varepsilon > 0, \exists N \mid (|x_n - L| < \varepsilon \ \forall n > N)}$$

$$\boxed{\forall \varepsilon > 0, \exists N \mid (|x_m - x_n| < \varepsilon, \quad \forall m,n > N)}$$

Lyapunov Stability

- Fundamental tool to study the stability of dynamical systems $\dot{x} = f(x)$
- Fixed points and invariant sets
  - **Fixed point** $\dot{x} = 0$
  - **Invariant sets**
    - A set $\mathcal{A}$ is an invariant set if $x(\bar{t}) \in \mathcal{A} \rightarrow \forall t \geq \bar{t}, \quad x(t) \in \mathcal{A}$

  - **Think about being "stuck" at a specific point (fixed point), or "stuck" inside some region or set of regions/points (invariant sets)**

Lyapunov Stability

- What is stability (in the sense of Lyapunov)?
  - the origin ($x = 0$) is **stable** if a trajectory starting close to it does not go away as time increases

$$\forall \varepsilon > 0,\ \exists \delta > 0 \ \text{ s.t. } \ \| x(0) \| \leq \delta \rightarrow \| x(t) \| \leq \varepsilon \ \forall t \geq 0$$

  - the origin ($x = 0$) is **unstable** if it is not stable

Lyapunov Stability

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$$\forall \varepsilon > 0,\ \exists \delta > 0 \ \text{ s.t. } \ \| x(0) \| \leq \delta \rightarrow \| x(t) \| \leq \varepsilon \ \forall t \geq 0$$

    - the origin ($x = 0$) is **asymptotically stable** if a trajectory starting sufficiently close to it converges to it as time increases

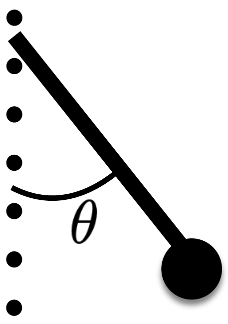
$x = 0$ is **stable** and $\exists \delta > 0 \ \text{ s.t. } \ \| x(0) \| \leq \delta \ \text{ s.t. } \ \| x(t) \| \rightarrow 0 \ \text{ as } \ t \rightarrow \infty$

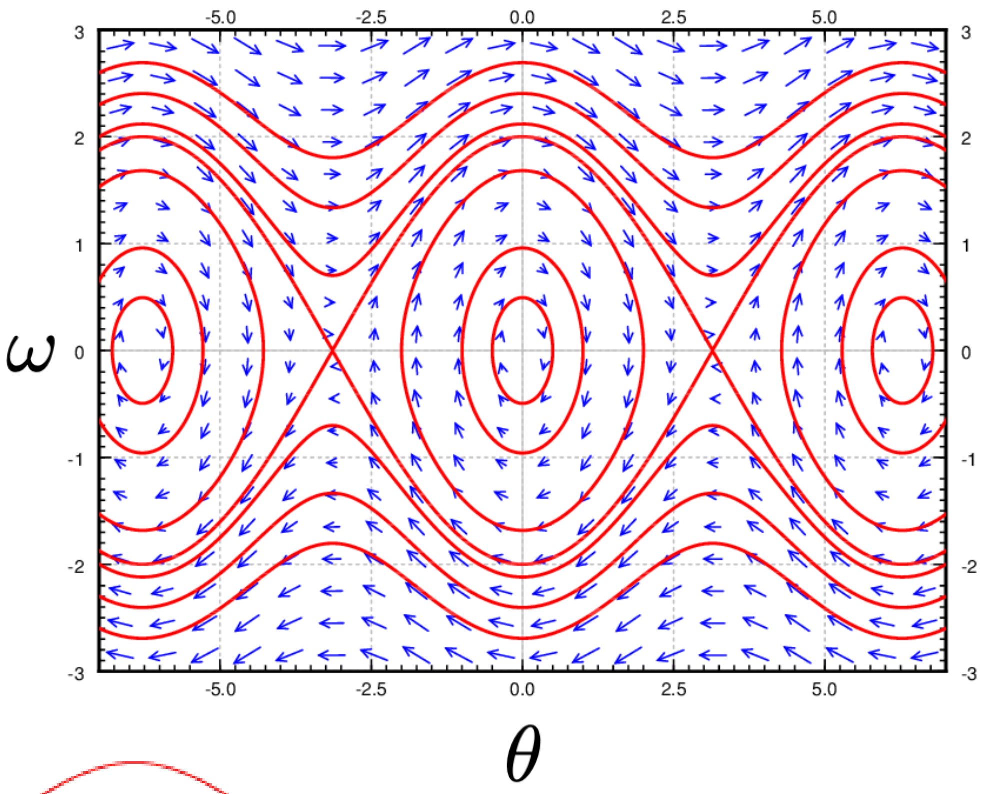
  - the origin ($x = 0$) is **unstable** if it is not stable

Undamped pendulum

$$\dot{\theta} = \omega$$

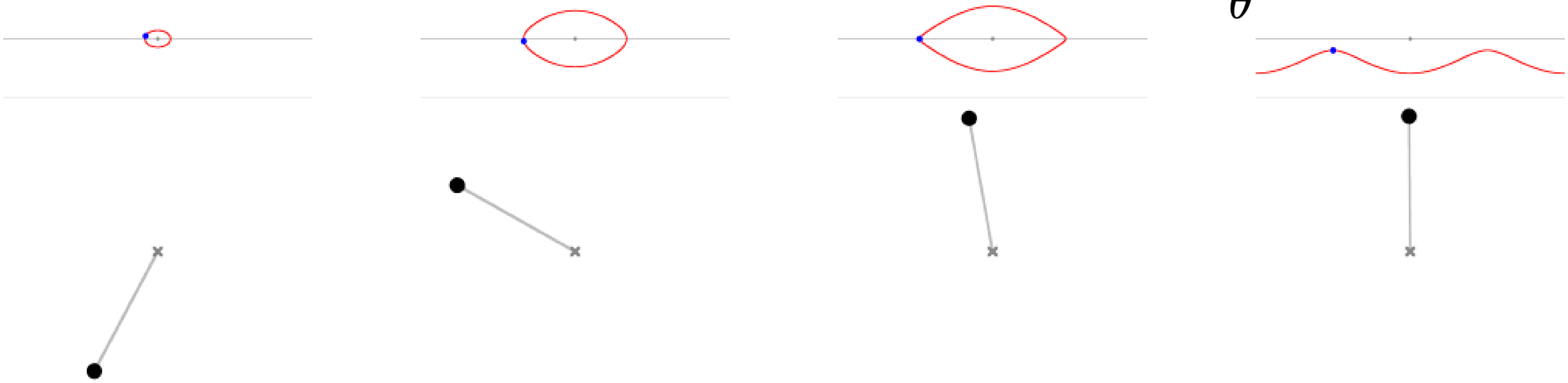
$$\dot{\omega} = -a \sin\theta$$





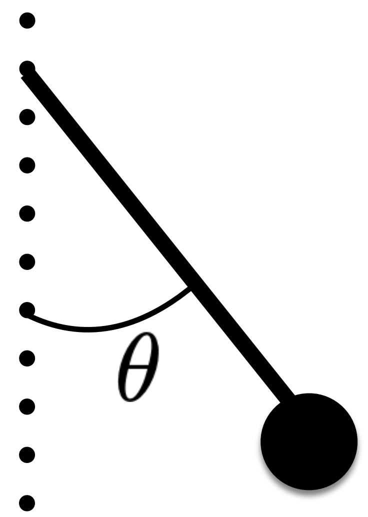
$\theta = \omega = 0$ **Stable fixed point**

$\theta = \pi \quad \omega = 0$ **Unstable fixed point**



Damped pendulum

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -a \sin\theta - b\omega$$



$$\theta = \omega = 0$$

Asymptotically stable fixed point when b>0

$$\theta = \pi \quad \omega = 0$$

Unstable fixed point

Lyapunov stability

- What is stability (in the sense of Lyapunov)?
  - the origin is **globally asymptotically stable** if it is asymptotically stable with any initial x(0)
  - the origin is **exponentially stable** if trajectories converge exponentially fast to it

x $= 0$ is *exponentially stable* if there exists $\delta > 0$, $c > 0$, $\lambda > 0$ such that $||x(0)|| \leq \delta$ implies that $||x(t)|| \leq c||x(0)||\mathrm{e}^{-\lambda t}$ for all $t \geq 0$

  - In other words, does it stabilize everywhere and how fast does it do so?

Lyapunov stability

A **Lyapunov function** for $\dot{x} = f(x)$, with respect to the origin, is a real-valued, positive definite $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$, such that $\dot{V} < 0,\ \forall x \neq 0$ along trajectories of $x(t)$.

A **weak Lyapunov function** for $\dot{x} = f(x)$, with respect to the origin, is a real-valued, positive definite $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$, such that $\dot{V} \leq 0,\ \forall x \neq 0$ along trajectories of $x(t)$.

If $V(x) \to \infty$ when $||x|| \to \infty$, then V is radially unbounded

Lyapunov stability

1. If the system admits a **weak** Lyapunov function, the origin is **stable**
2. If the system admits a Lyapunov function, the origin is **asymptotically stable**
3. If the system admits a **radially unbounded** Lyapunov function, the origin is **globally asymptotically stable**
4. If the system admits a Lyapunov function and $\dot{V} \leq -\alpha V$ for some $\alpha > 0$ the origin is **exponentially stable**

Lyapunov stability

- A Lyapunov function allows to check the stability of a system without solving the equations (very powerful tool)
- Finding a Lyapunov function for a system can be very difficult (often try to find an energy-like function)
- Conditions are only **sufficient** → failure to find a Lyapunov function does not allow to conclude anything about the system

LaSalle invariance principle

- **Recall:** Definition of invariant set $\mathcal{A}$ $\quad x(\bar{t}) \in \mathcal{A} \rightarrow \forall t \geq \bar{t}, \quad x(t) \in \mathcal{A}$

- Let $V(x)$ be a weak Lyapunov function for the system. Let $\mathcal{M}$ be the largest invariant set contained in

$$\{x \in \mathbb{R}^n \mid \dot{V} = 0\}$$

Then every solution $x(t)$ that remains bounded is such that

$$\inf_{y \in \mathcal{M}} \| x(t) - y \| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

Discrete Versions

$$x(n+1) = f(x(n))$$

Discrete dynamics

$$x_e = f(x_e)$$

Fixed points for discrete dynamics

Lyapunov:

If $V(x) > 0$ for all $x \neq 0$ and if $V(x(n+1)) - V(x(n)) < 0$ for $x(n) \neq 0$ then the origin is asymptotically stable.

LaSalle:

If $V(x) \geq 0$ and if $V(x(n+1)) - V(x(n)) \leq 0$ for $x(n) \neq 0$ then the x will converge to the largest invariant set contained in $\{x | V(f(x)) - V(x) = 0\}$.

The consensus protocol with Lyapunov

Undirected graphs

$$\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x}$$

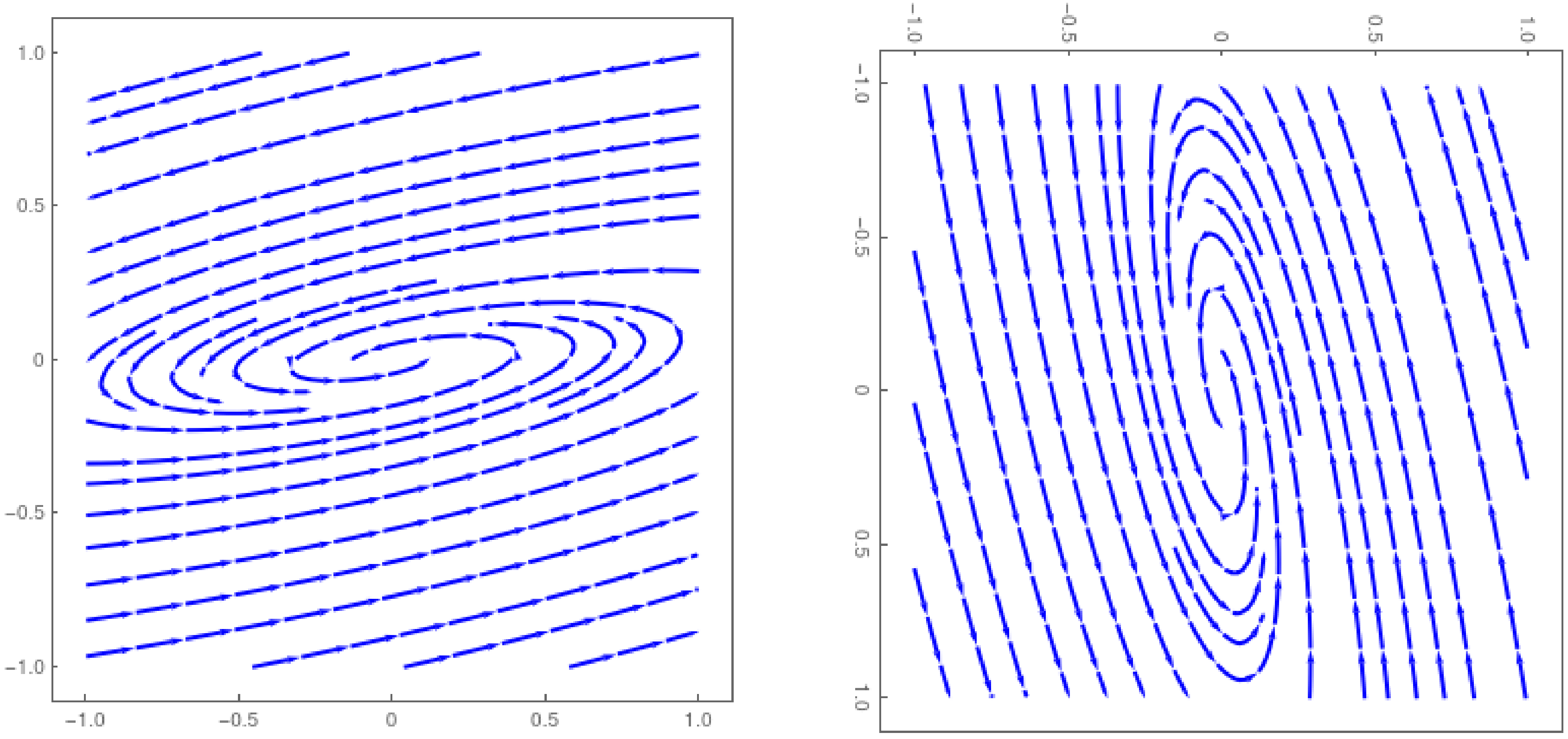
$$V(x) = \frac{1}{2}x^T x$$

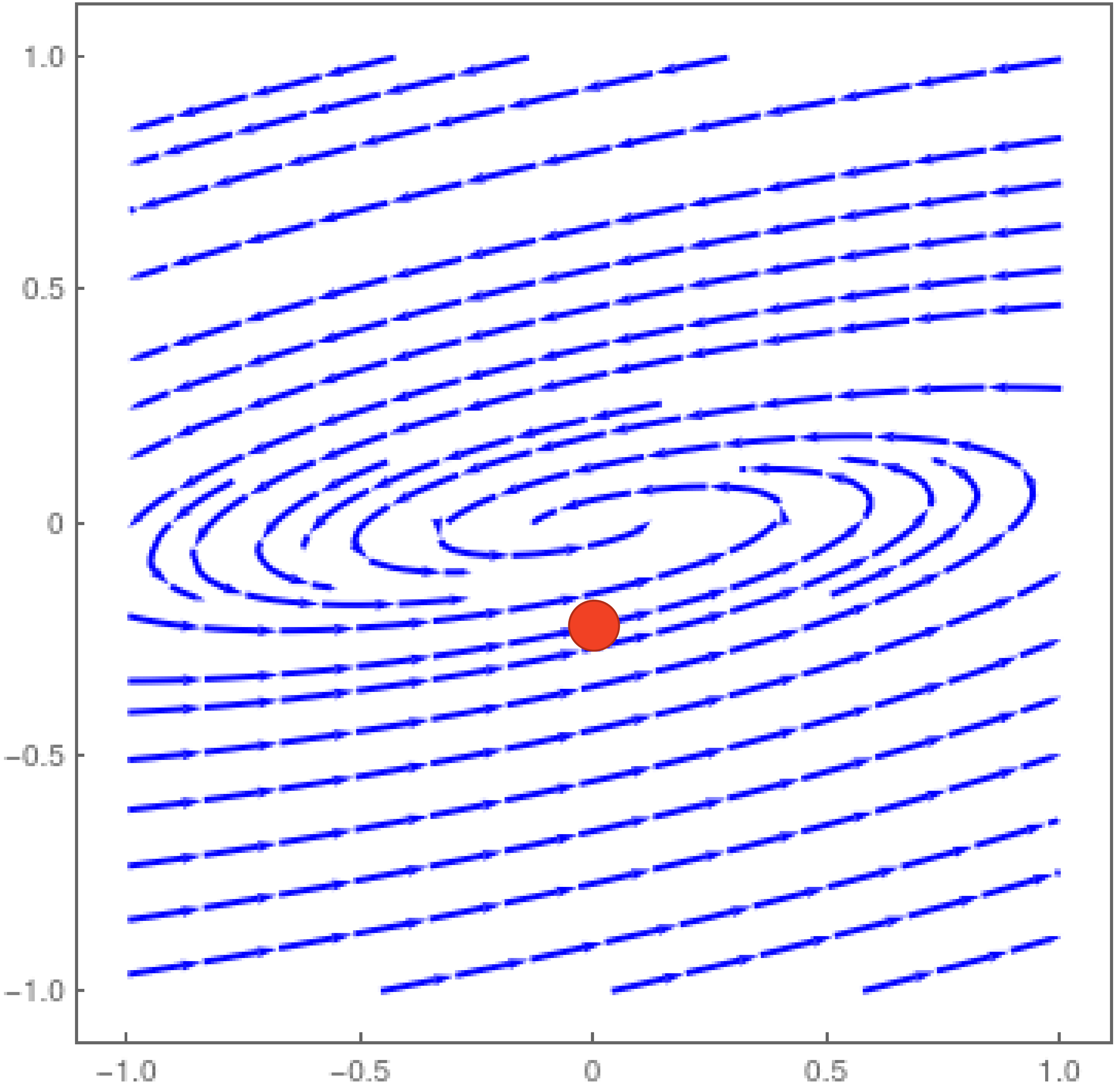
$$\dot{V}(x) = -x^T L x \leq 0$$

$$\{x \in \mathbb{R}^n | \dot{V} = 0\} = \mathbf{span}\{\mathbf{1}\}$$ when graph is connected

Switched Systems

- In general switching between stable systems may not result in a stable system!





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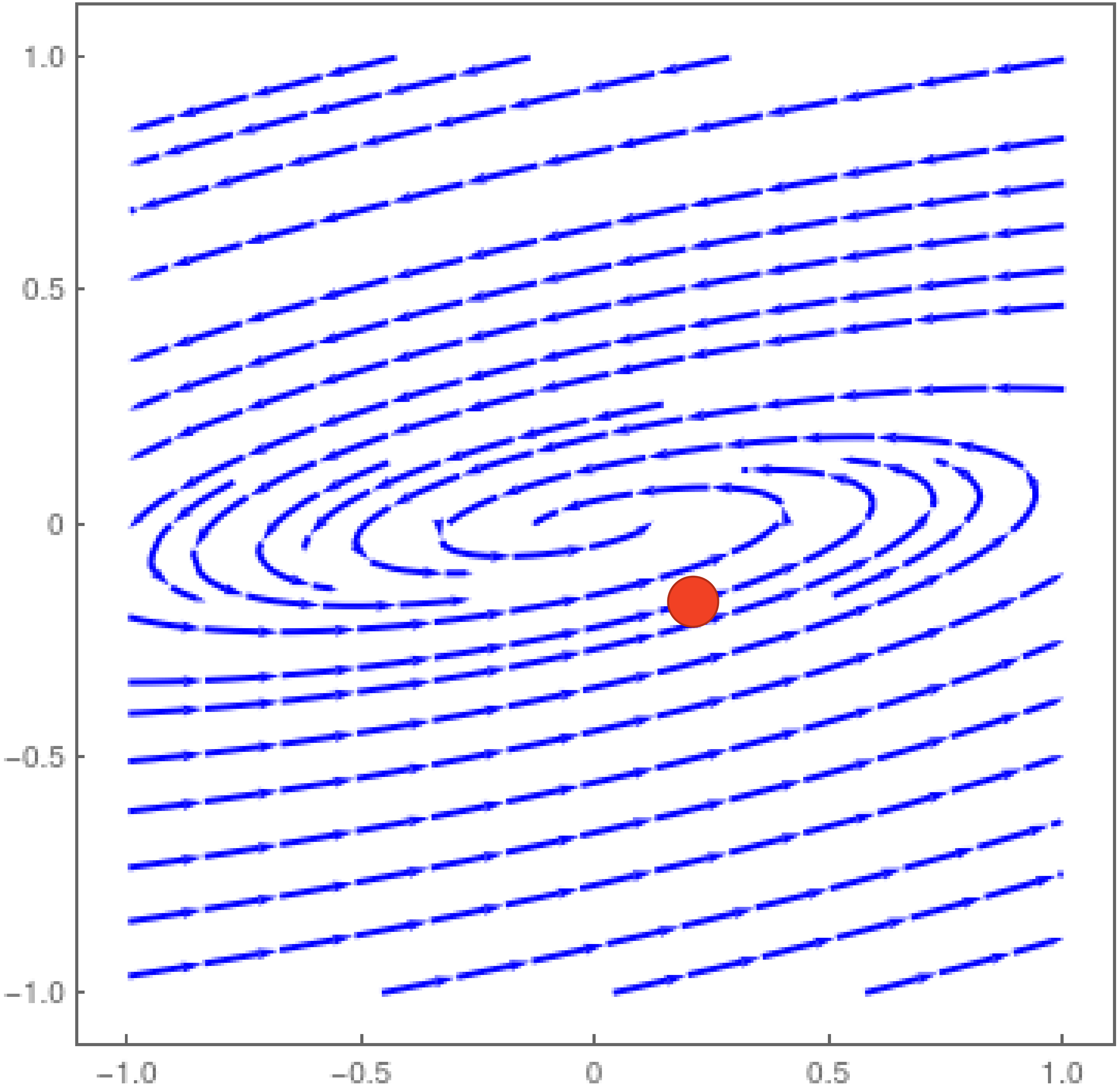

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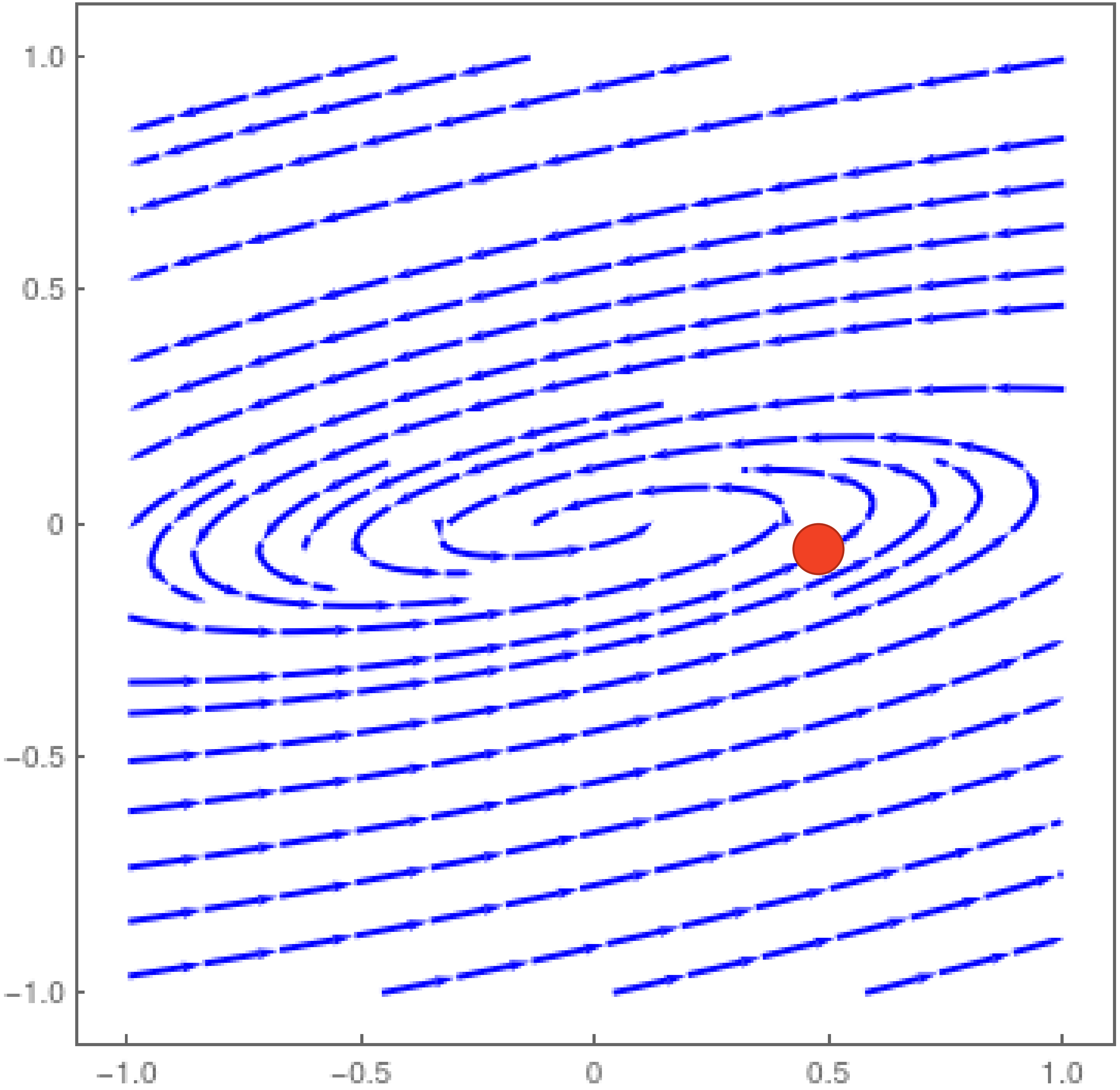
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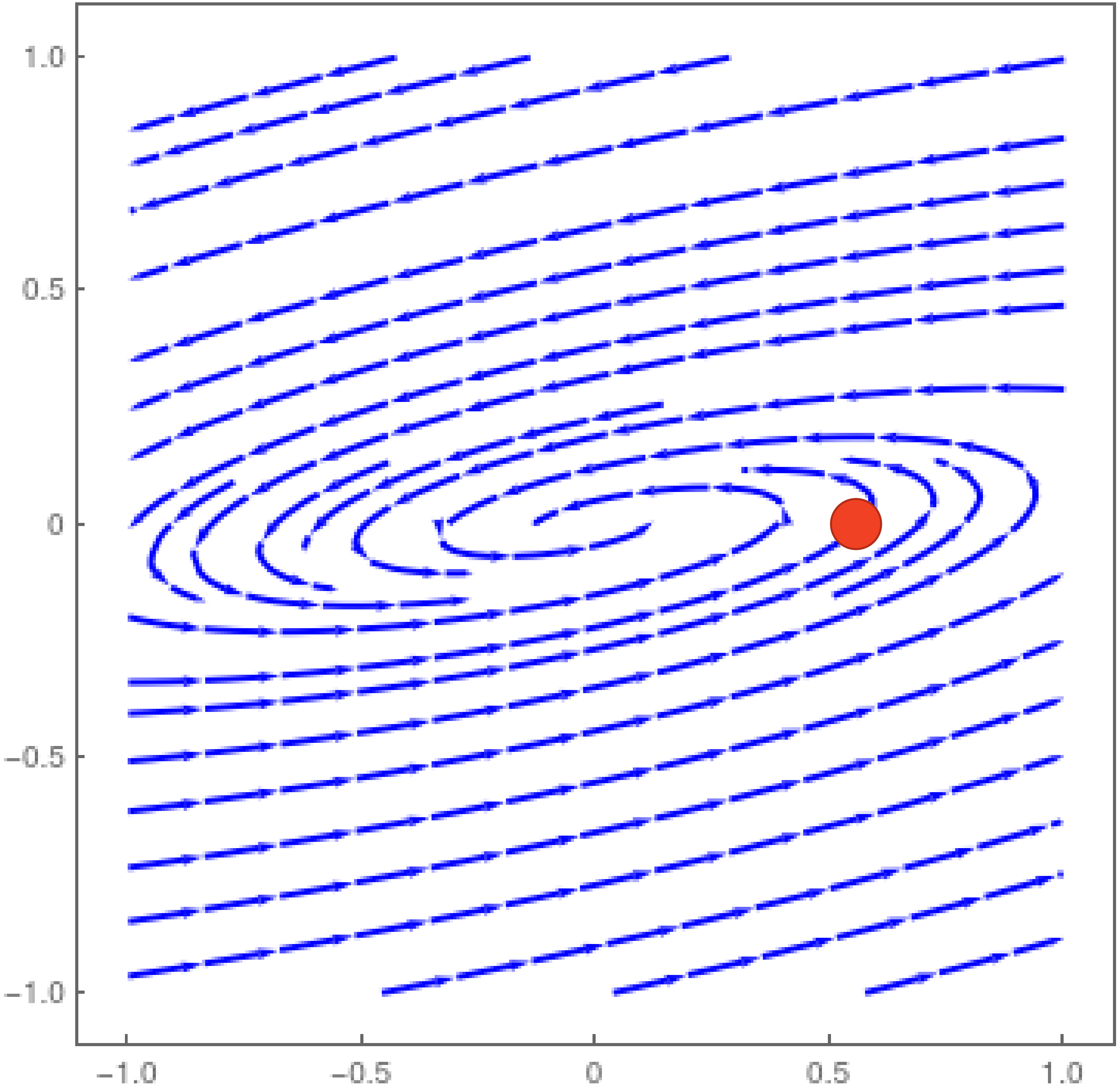

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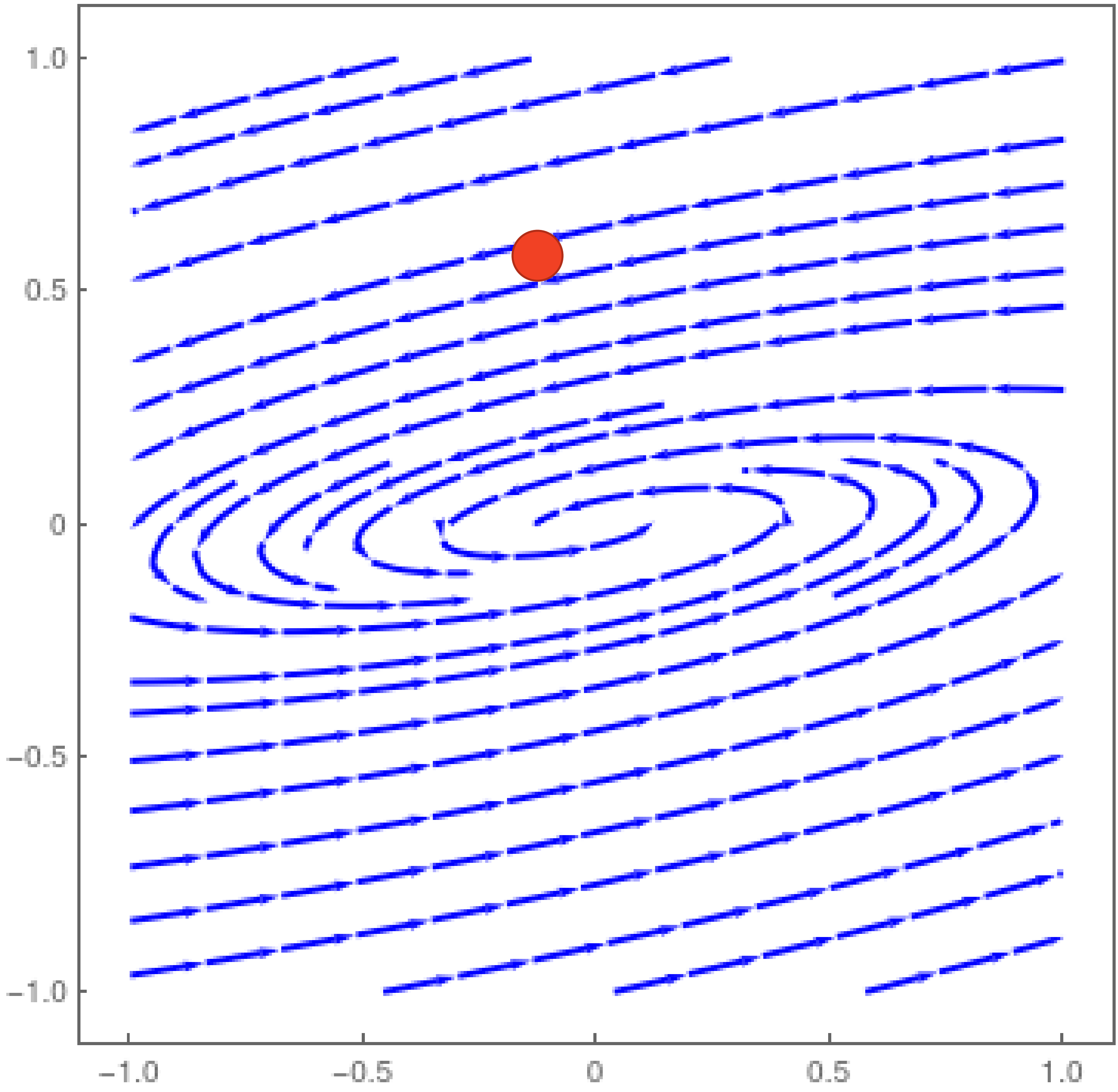

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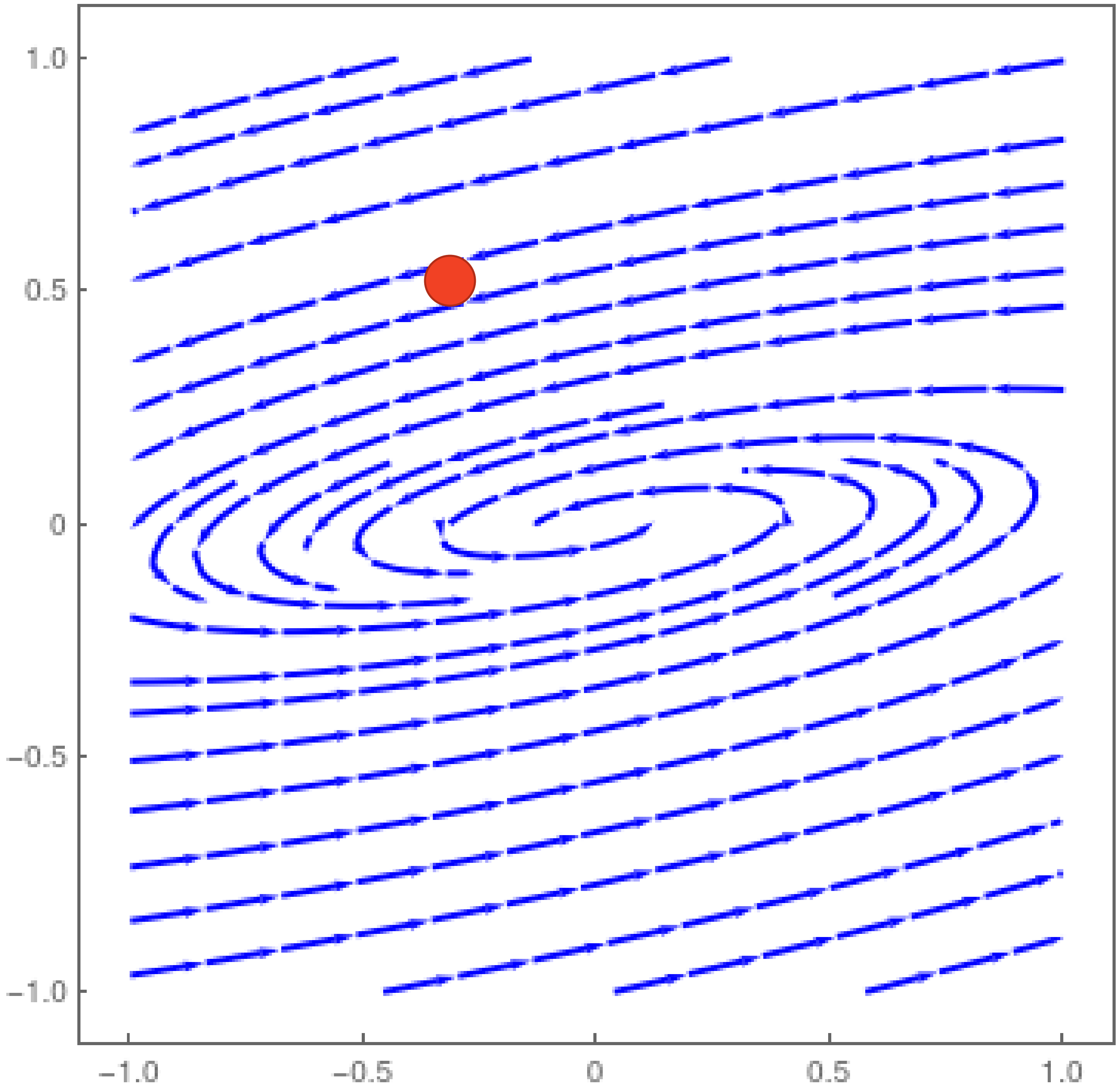

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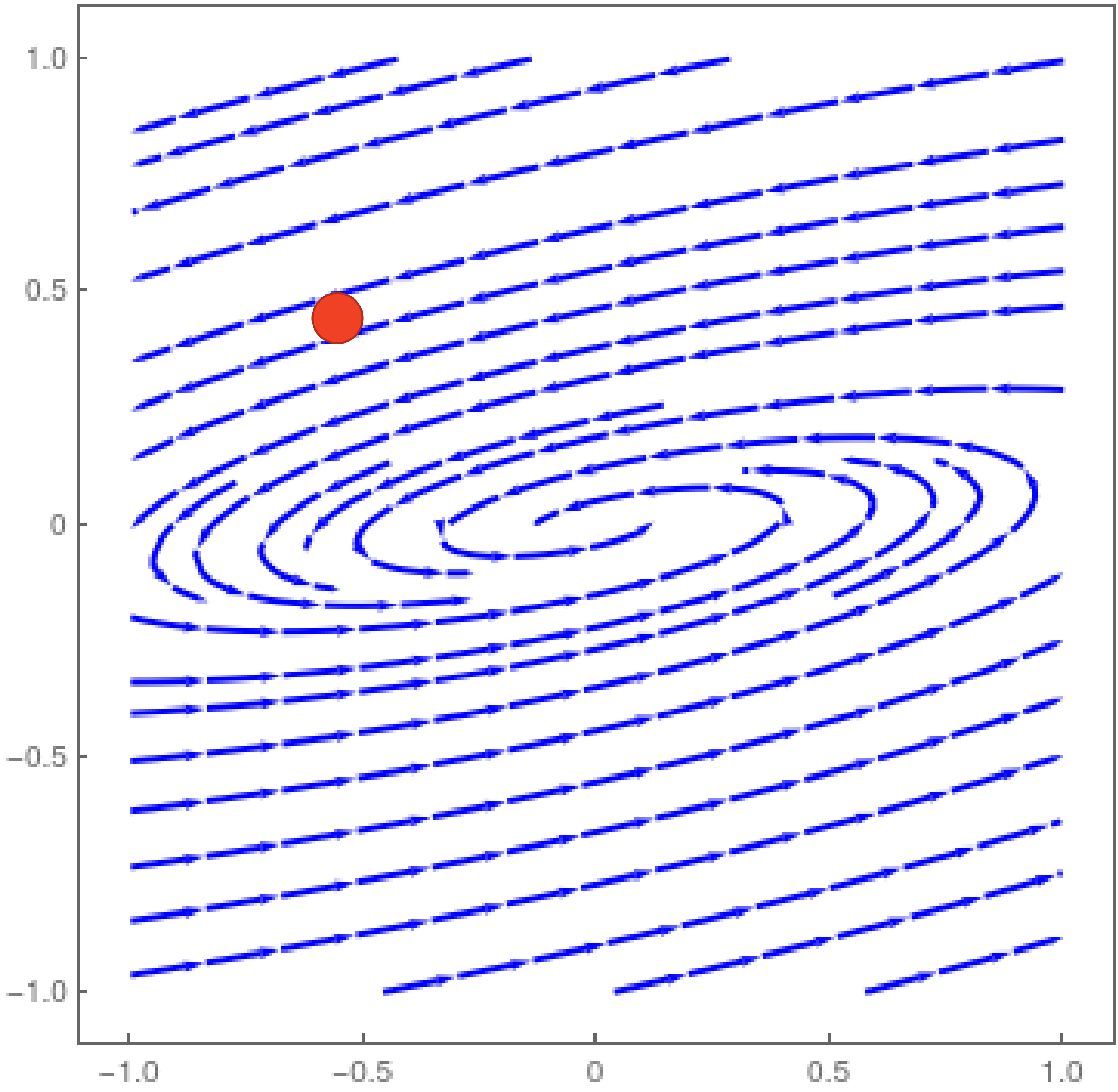

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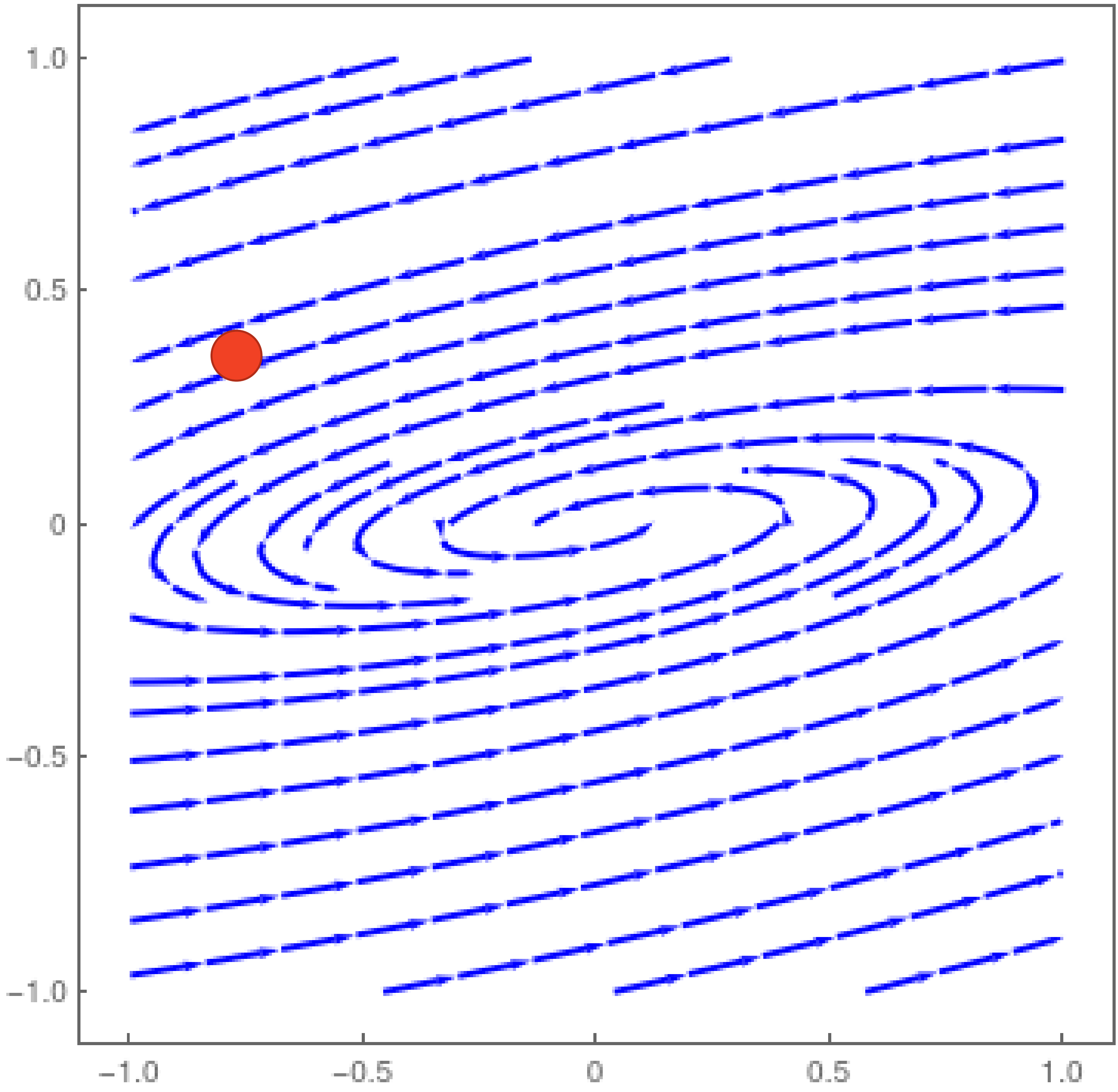

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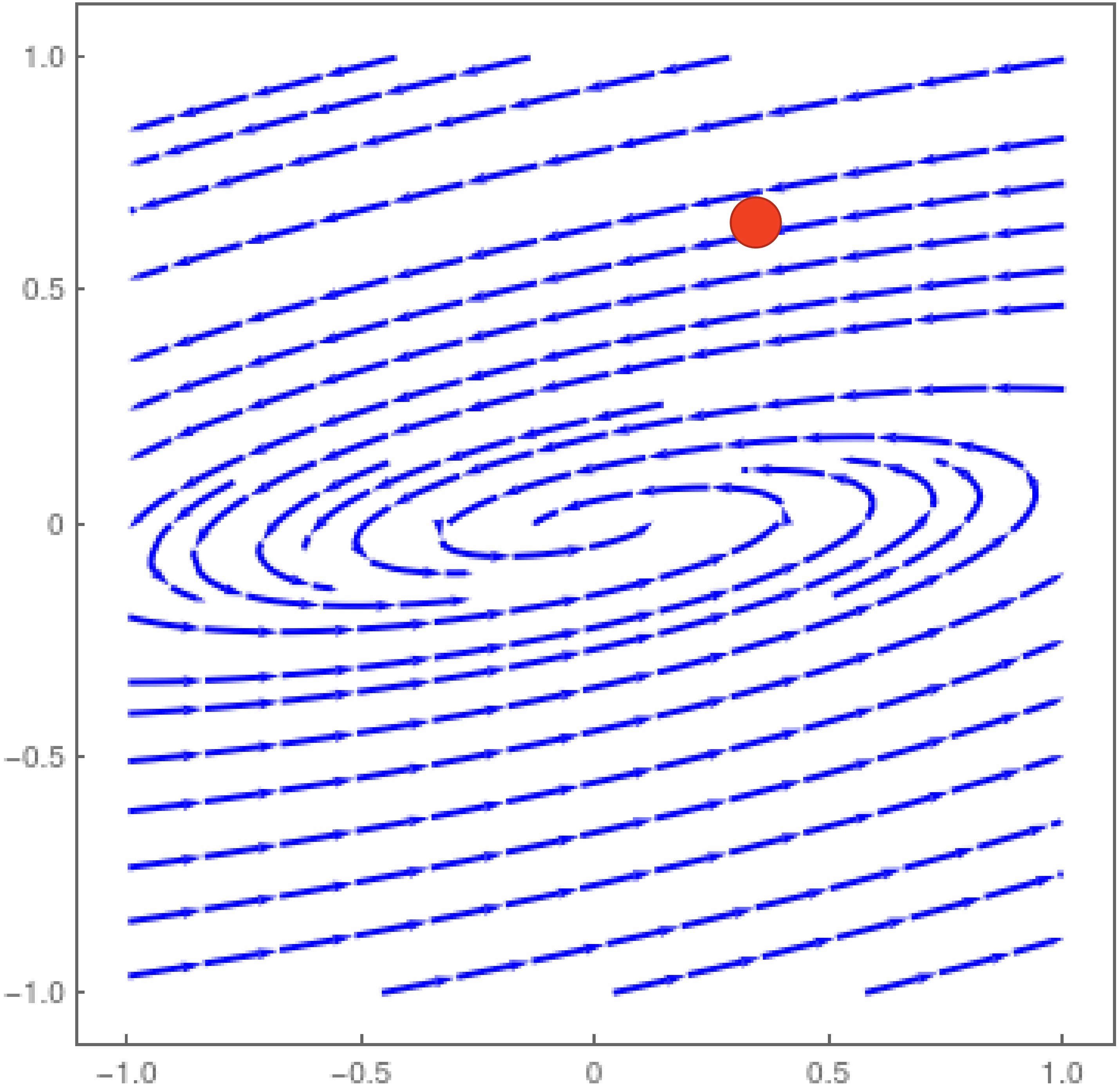

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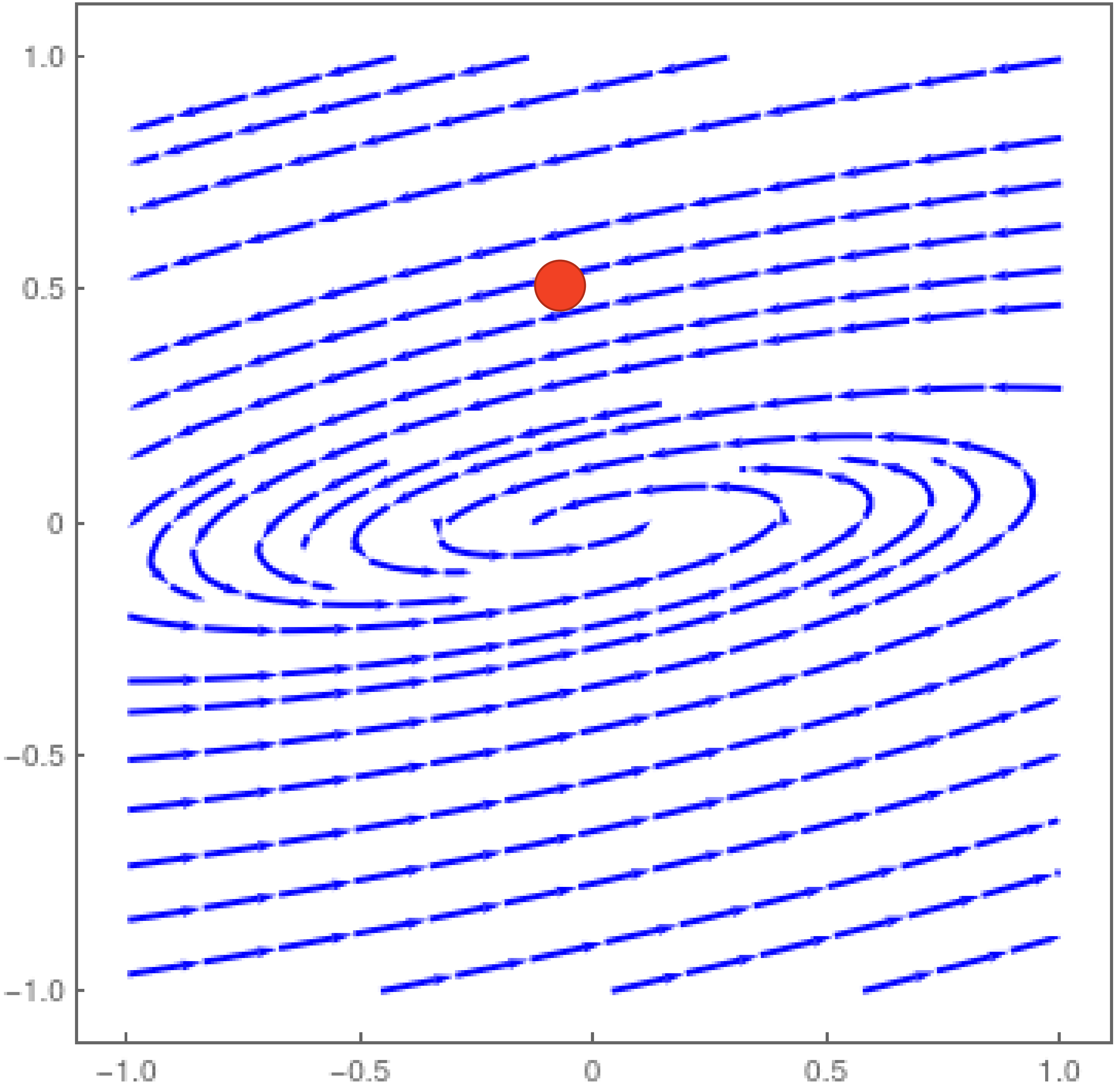
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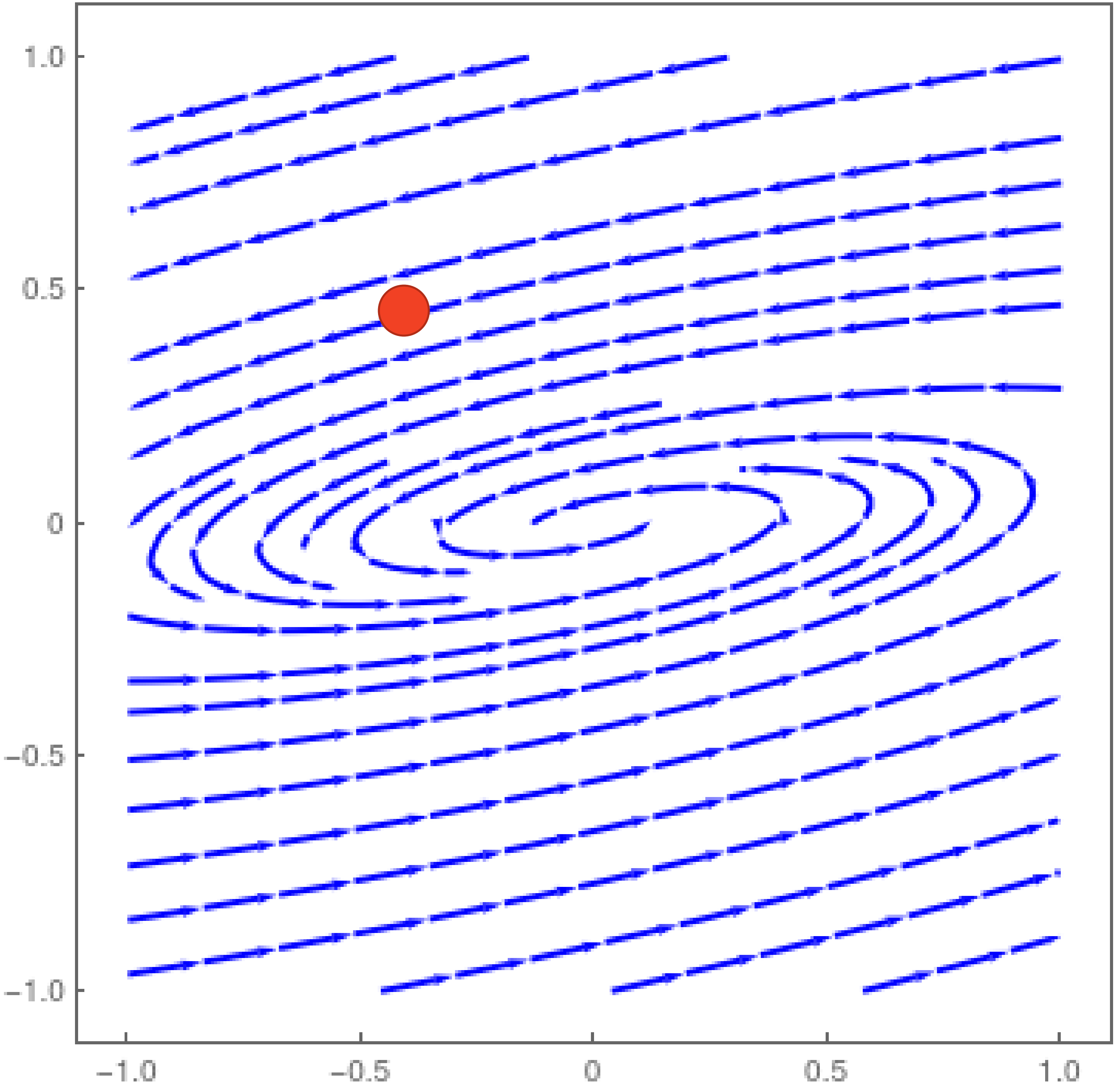
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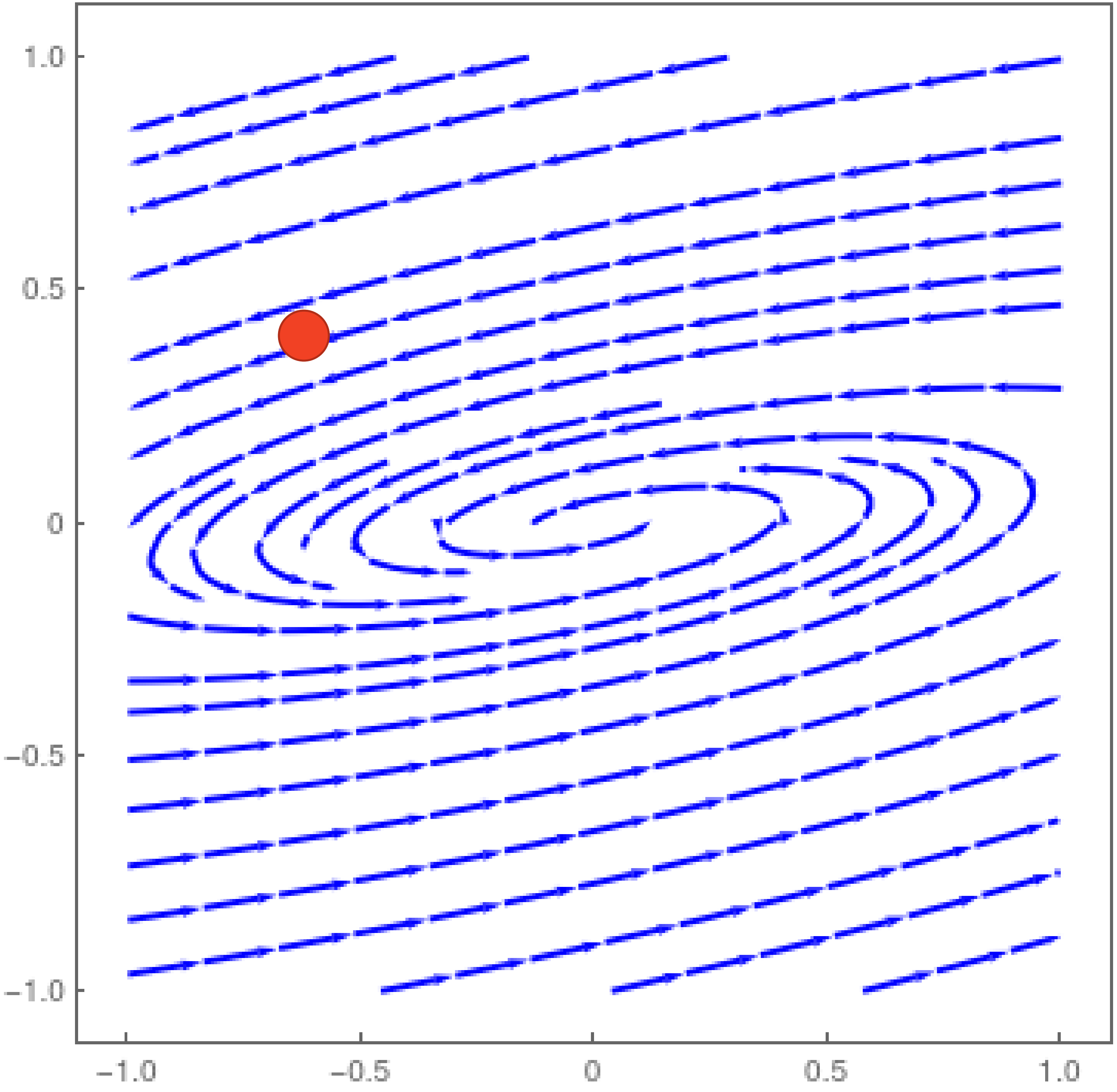

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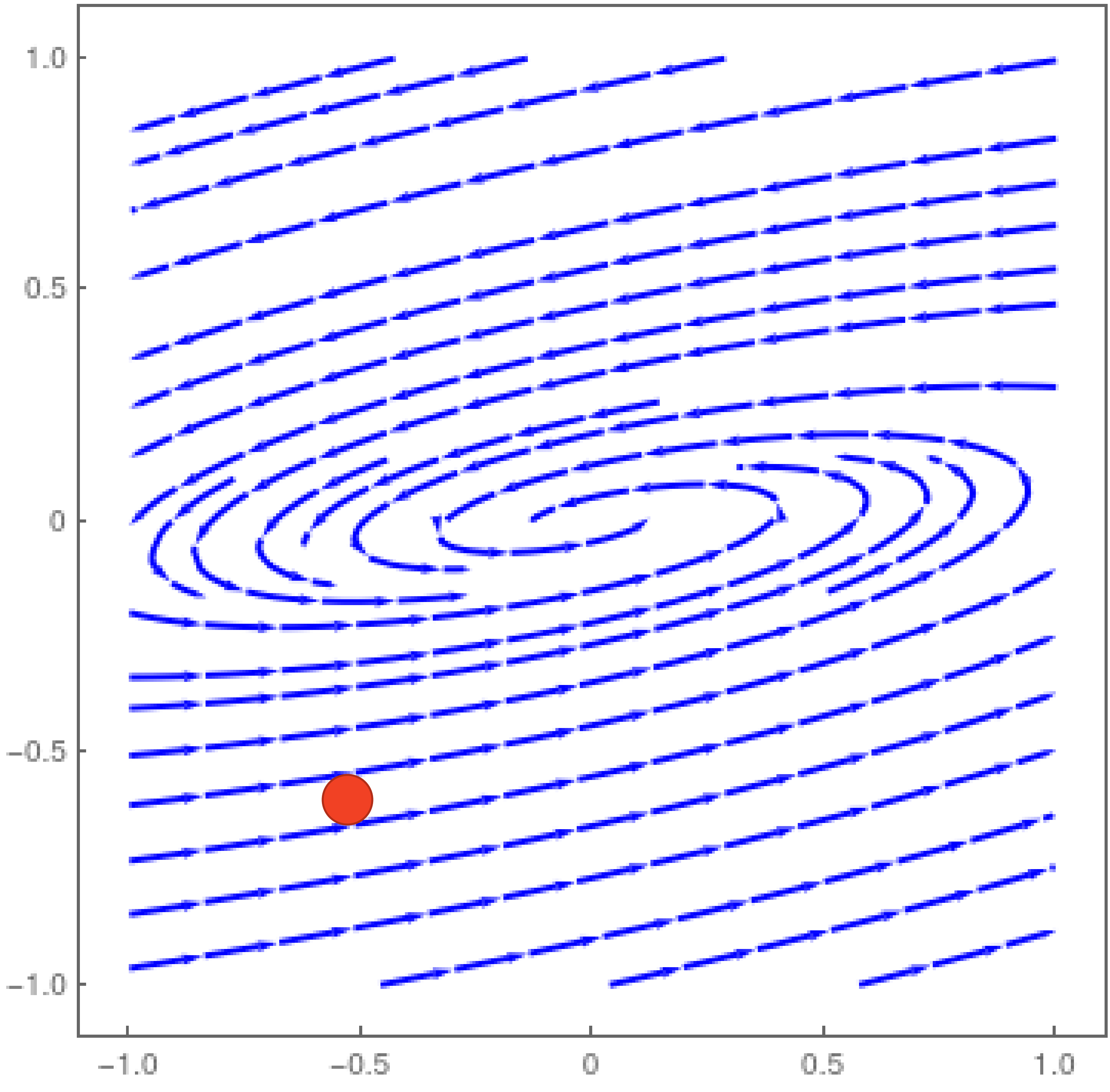
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Switching connected graphs

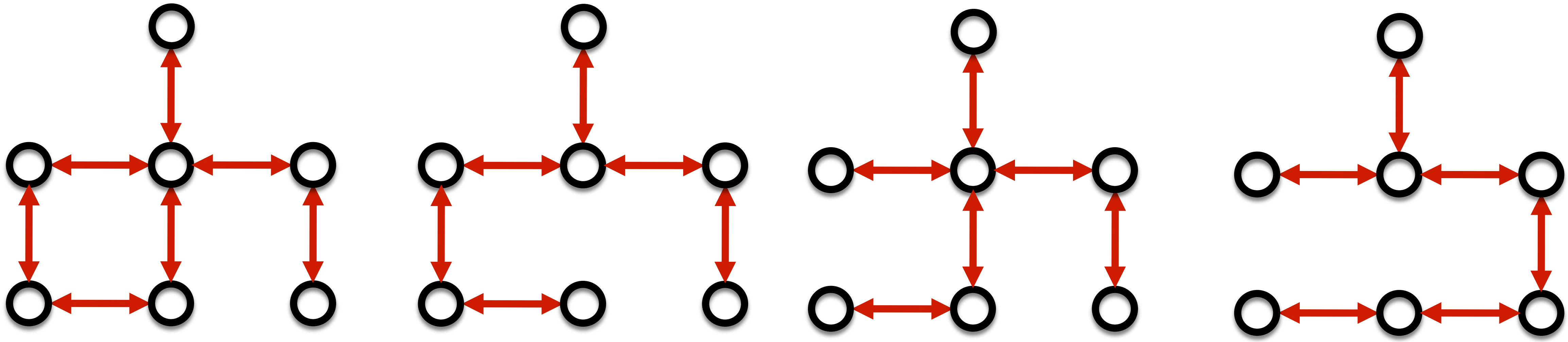
- Now that we know to be careful
- I have a network of robots that switches between different graphs, but each graph is stable. Is the resulting behavior of the robot over time also stable?
    - We will find a Lyapunov function
    - Check what kind of Lyapunov function it is
    - Try to say something about stability of the consensus protocol using properties/theorems like LaSalle's Invariance Principle

Switching connected graphs

- Note that each undirected graph is connected, so they all converge to the same spot, the average of their initial positions
- The Laplacian is now a time-varying function

$$\{D_1, D_2 \cdots D_p\}$$

$$\dot{x} = -L(D_j)x$$



D_1 D_2 D_3 D_4

Switching connected graphs

- Naively, this should work. Even though the Laplacian is changing over time, at every single snapshot in time, the system is trying to converge to the same final destination.
- Let’s check this by first trying to find a Lyapunov function
    - What is the easiest positive definite function you can think of?

A **Lyapunov function** for $\dot{x} = f(x)$, with respect to the origin, is a real-valued, positive definite $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$, such that $\dot{V} < 0, \ \forall x \neq 0$ along trajectories of $x(t)$.

Switching connected graphs

- Anything quadratic will work. Try this:

$$V(x) = \frac{1}{2} x^T x$$

  - We are lucky that this works, but it's not completely a coincidence.
  - Lyapunov function are energy-like. Think about the energy stored in a spring ($1/2kx^2$) or capacitor ($1/2CV^2$)
  - We will make use of this energy analogy in the future
  - Just like a group of marbles rolling down a hill to minimize their potential energy, a drone swarm converges to a rendezvous point

Switching connected graphs

$$\dot{x} = -L(D_j)x$$

- Check derivative of $V(x)$

$$\{D_1, D_2 \cdots D_p\}$$

$$V(x) = \frac{1}{2}x^T x \qquad \dot{V} = -x^T L(D_j)x \qquad \dot{V} \in \{x^T L(D_j)x | j \in \{1 \dots p\}\}$$

- Since each graph is connected:
  - The invariant set (i.e., the positions where the swarm ends up) is independent of the graph.
    - And the set where $\dot{V} = 0$ is also the same for each graph!
- We have checked that V(x) satisfies all the requirements to be a weak Lyapunov function

Switching connected graphs

- Because it is a weak Lyapunov function, LaSalle's invariance principle guarantees that the agreement protocol $\dot{x} = -L(D_j)x$ works!
  - It will converge to the agreement subspace as we expect!
- We only considered undirected graphs here, but it is **also valid for switching strongly-connected directed graphs**

Switching connected graphs (Discrete time)

Let a digraph $\mathcal{D}$ and $\delta > 0$. Then $[\mathrm{e}^{-\delta L(D)}]_{ij} > 0$ if and only if $i = j$ or there is a directed path from j to i in $\mathcal{D}$

The digraph $\mathcal{D}$ contains a rooted out branching if and only if for any $\delta > 0$ at least one of the columns of $\mathrm{e}^{-\delta L(\mathcal{D})}$ is positive

$$V(z) = \max_i z_i - \min_j z_j$$

$$V(z(k+1)) < V(z(k))$$

For discrete-time systems, pick this as your Lyapunov function. Easy to check that the function is positive definite

For digraphs with rooted-out branching, convergence to the agreement subset

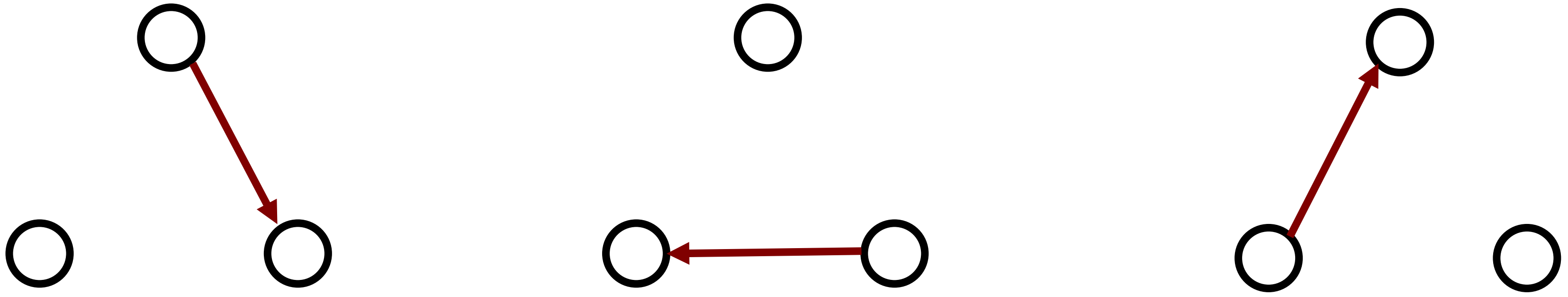
Details in page 76 of textbook (Chapter 4.1)

Switching between disconnected graphs

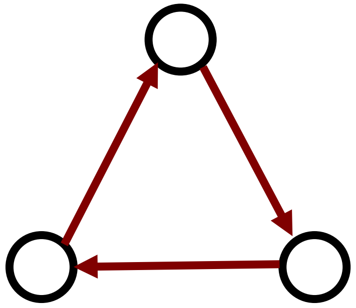
- Imagine a swarm with bad internet, so they often are disconnecting and reconnecting with each other.
- Or one group splits off and then rejoins the main group.
- Can they still converge?
- We hope so.

Switching between disconnected graphs

- We need to explain what union of graphs over time means:
- If I switch between these three graphs



- Their union over time (overlapping all three on top) is:



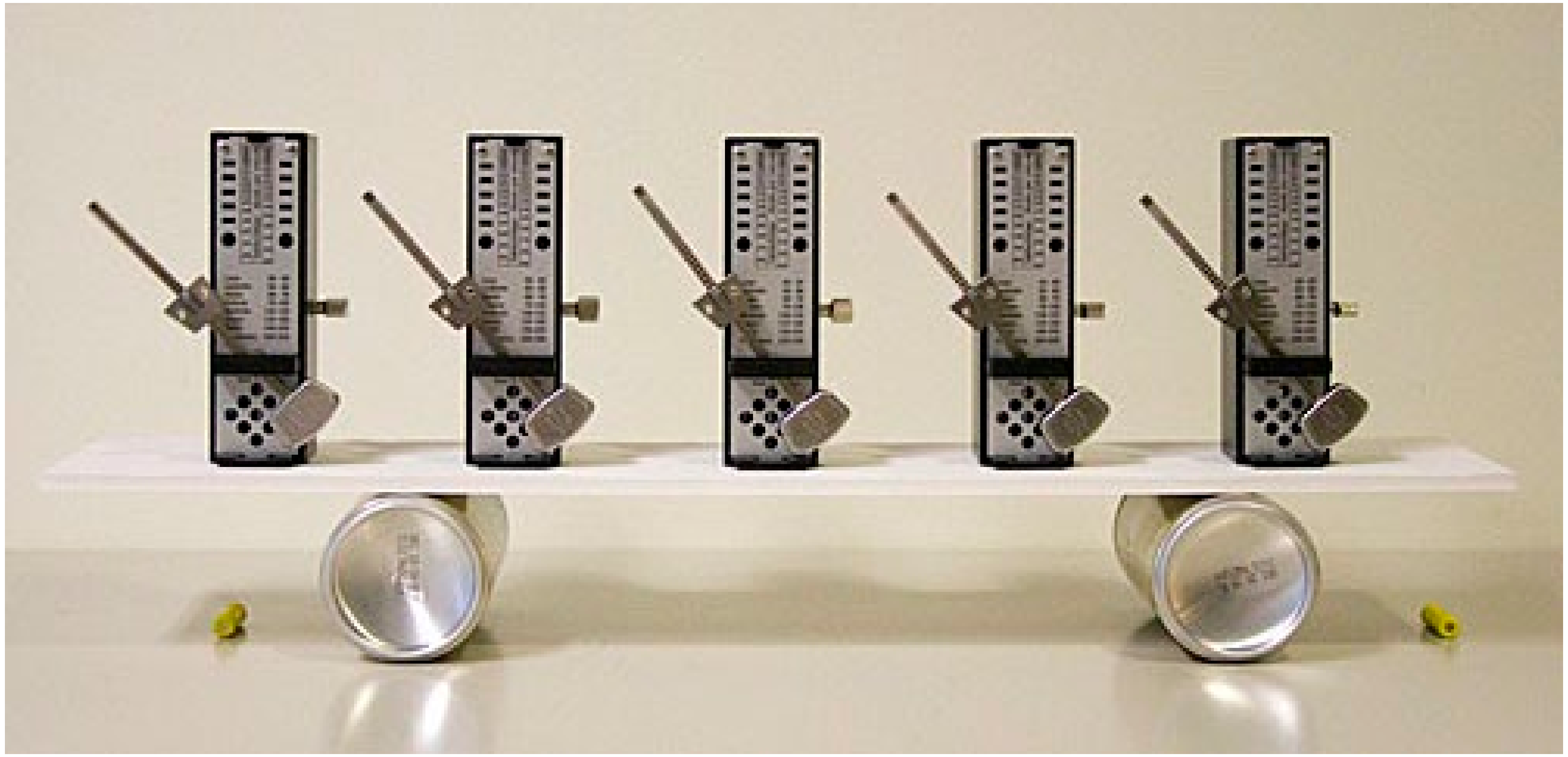
Switching between disconnected graphs

- The agreement protocol has very strong properties for graphs with rooted-out branching:
  - Convergence for switching digraphs as long as their **union over some time *T*** contains a rooted out-branching
  - Convergence for such complicated systems can be proved using Lyapunov and LaSalle stability theorems.

- More complicated agreement protocols exist (e.g. taking into account the dynamics of the agents, nonlinear rules, etc.)
- Lyapunov arguments can be used for many other systems (e.g. to prove synchronisation of the metronomes, prove stability of nonlinear control systems, etc.)

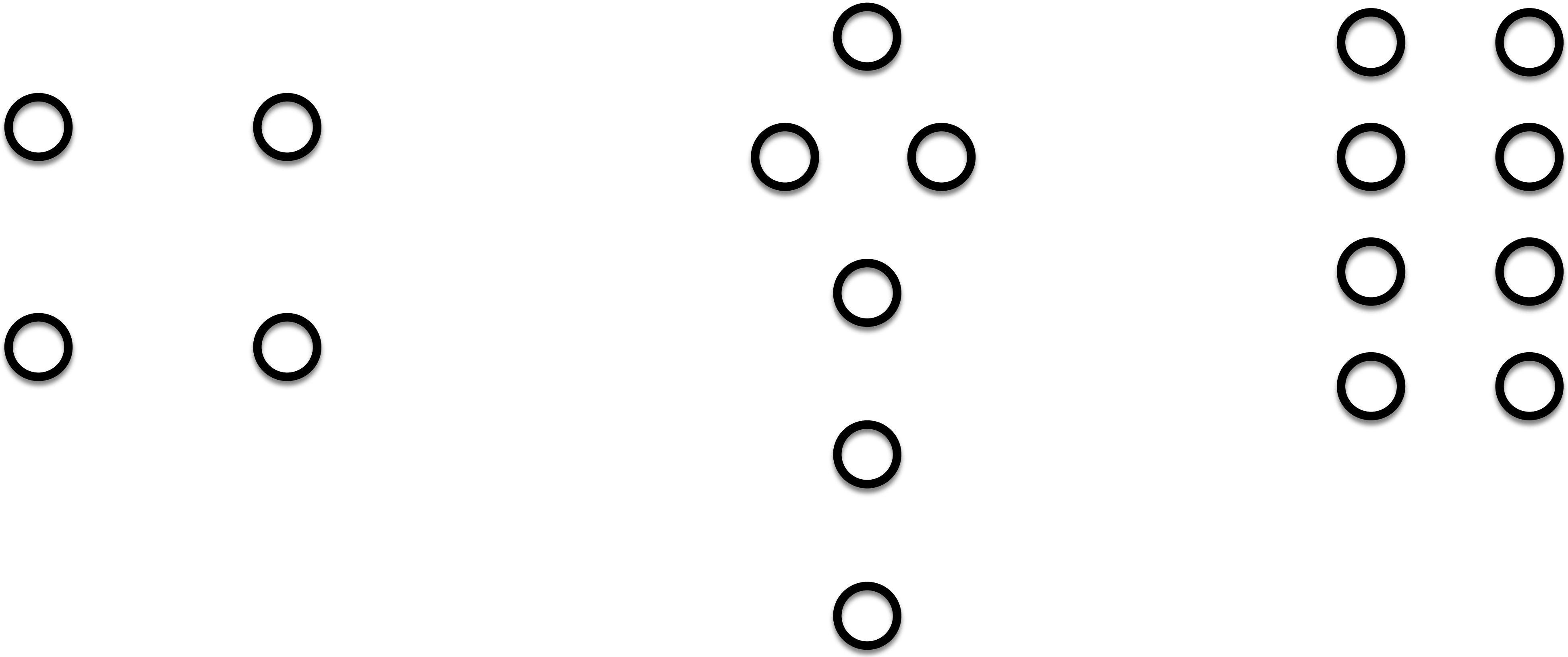
Spontaneous Synchronization

- Metronomes



Formation Control: Beyond Rendezvous

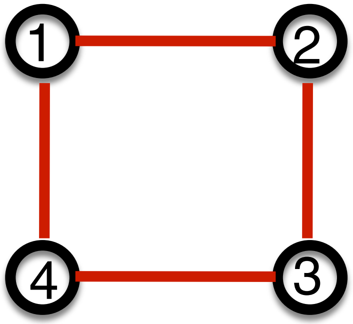
- Formation control → control of geometrical patterns formed by multi-agent systems



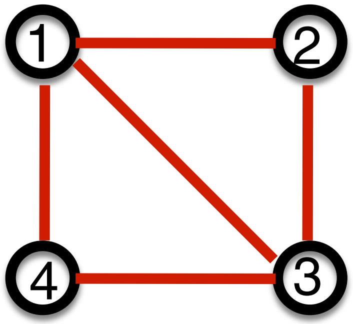
Formation Specifications: Pairwise Geometric Constraints

- **Specification:** set of desired relative distances between agents

$$D = \{d_{ij} \in \mathbb{R} | d_{ij} > 0, i, j = 1, \cdots, n \quad i \neq j\}$$



$$D = \{d_{12} = 1,\ d_{23} = 1,\ d_{34} = 1,\ d_{41} = 1\}$$



$$D = \left\{d_{12} = 1,\ d_{23} = 1,\ d_{34} = 1,\ d_{41} = 1,\ d_{13} = \sqrt{2}\right\}$$

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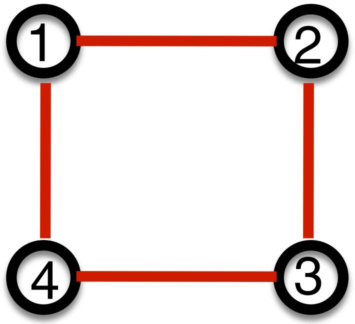

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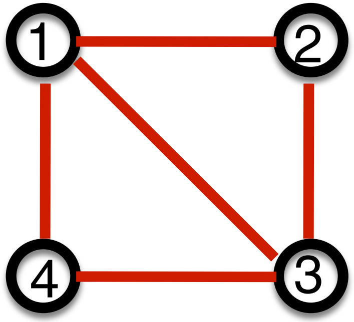
Set of pair-wise distance constraints → weighted edges of a graph

A formation can be modeled as a weighted graph!

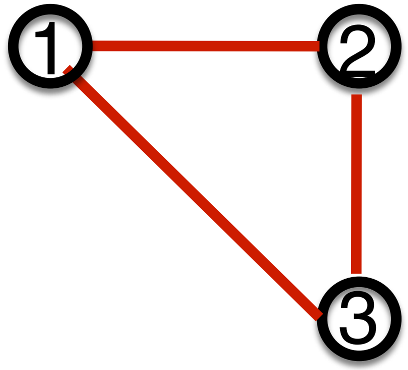


$$D = \left\{d_{12} = 1,\ d_{23} = 1,\ d_{34} = 1,\ d_{41} = 1,\ d_{13} = \sqrt{2}\right\}$$

Formation: Distance Constraints

- Formation specification needs to be feasible

$$D = \{d_{12} = 1,\ d_{23} = 1,\ d_{31} = \sqrt{2}\}$$

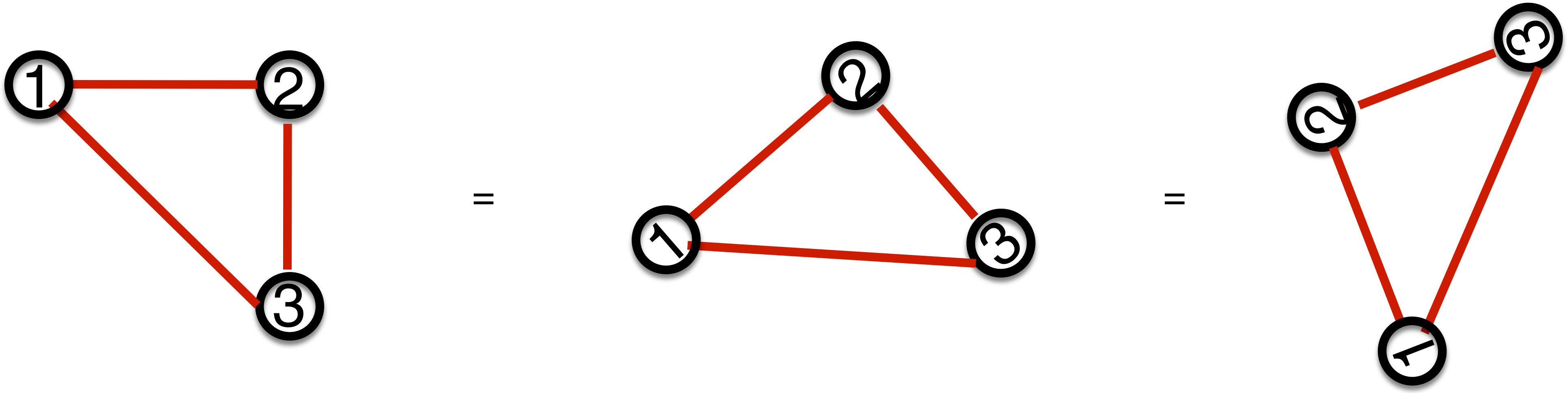


$$D = \{d_{12} = 1,\ d_{23} = 1,\ d_{31} = 3\}$$

Formation: Distance Constraints

- The formation can be translated or rotated as a whole

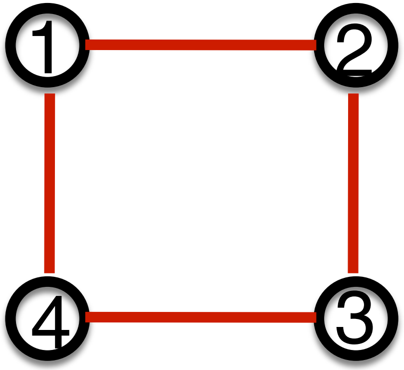
$$D = \{d_{12} = 1,\ d_{23} = 1,\ d_{31} = \sqrt{2}\}$$



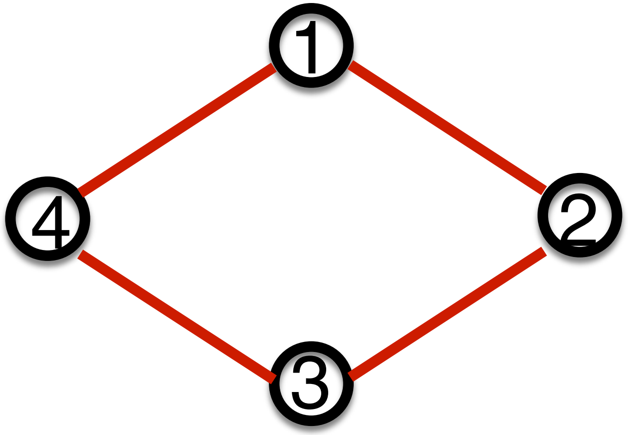
Formation: Distance Constraints

- Some specifications have multiple possibilities

$$D = \{d_{12} = 1,\ d_{23} = 1,\ d_{34} = 1,\ d_{41} = 1\}$$



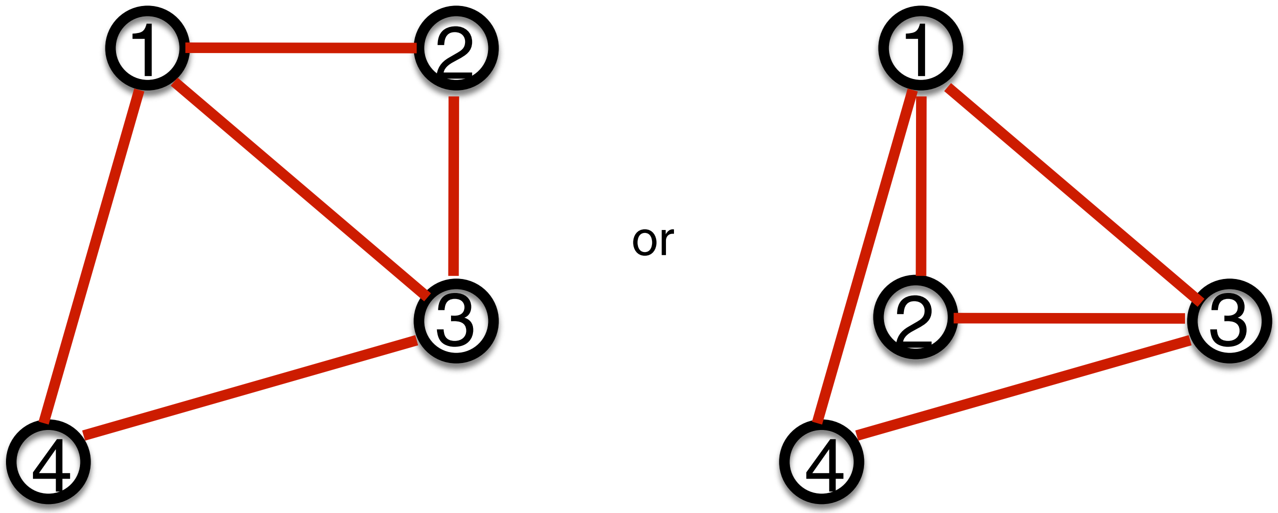
or



Formation: Distance Constraints

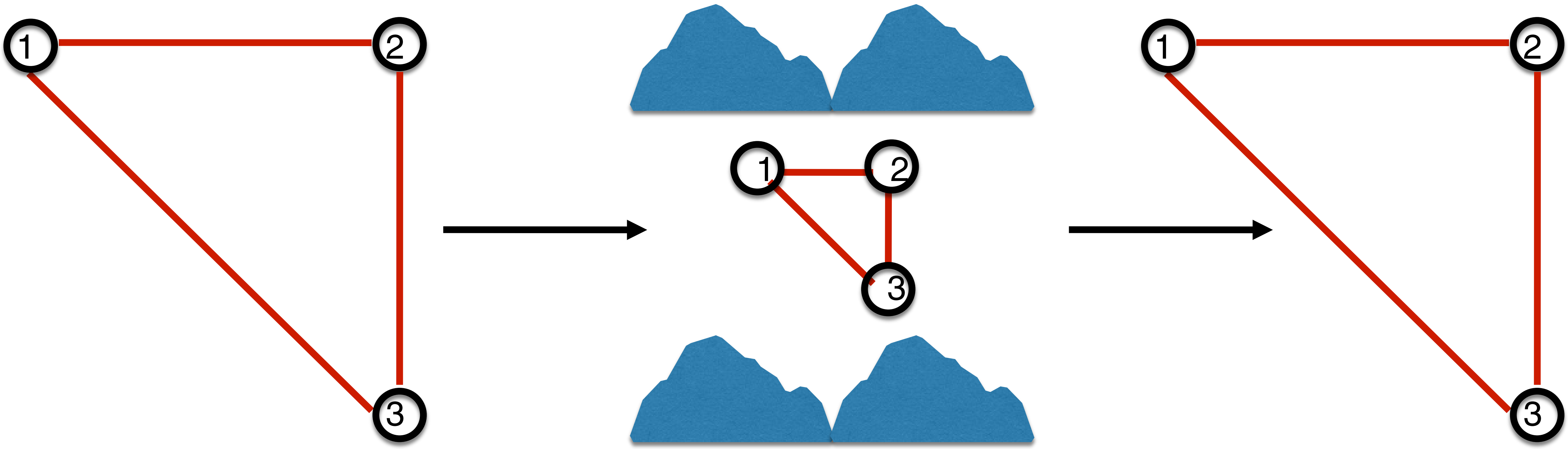
- Some specifications have multiple possibilities

$$D = \{d_{12} = 1,\ d_{23} = 1,\ d_{34} = 1.5,\ d_{41} = 1.5,\ d_{13} = \sqrt{2}\}$$



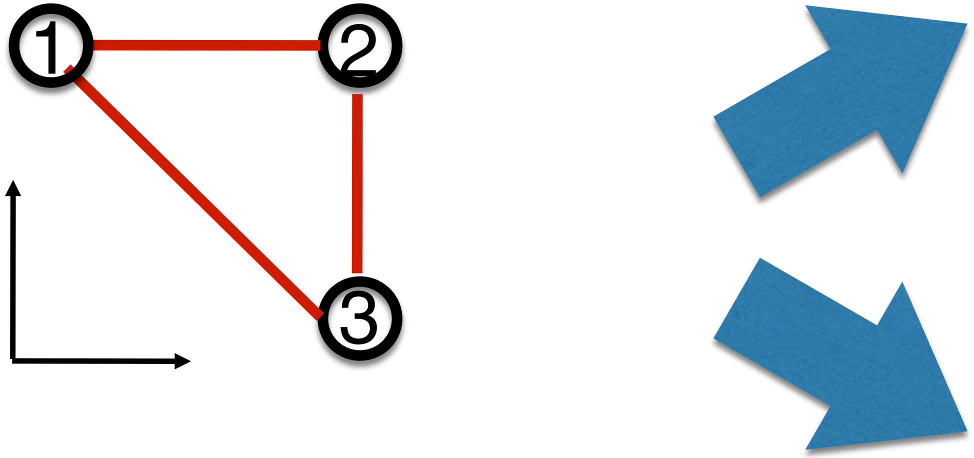
Formation: Bearing Constraints

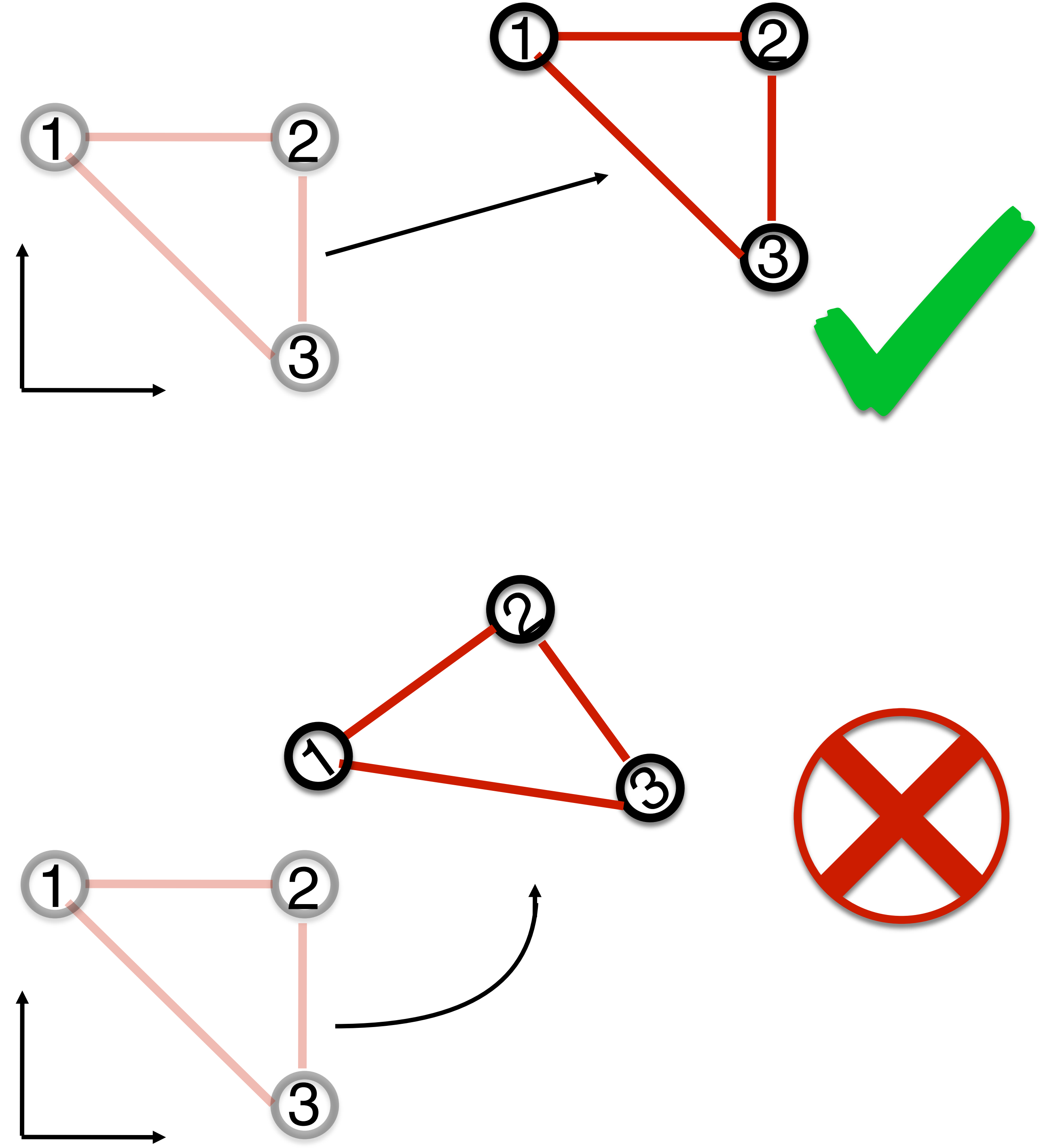
- Angles are constant (bearing constraints) ←→ Distances are constant up to a common factor
- Useful to negotiate obstacles



Formation

- Constant absolute orientation + relative distances





Next Week

- Rigidity and Frameworks