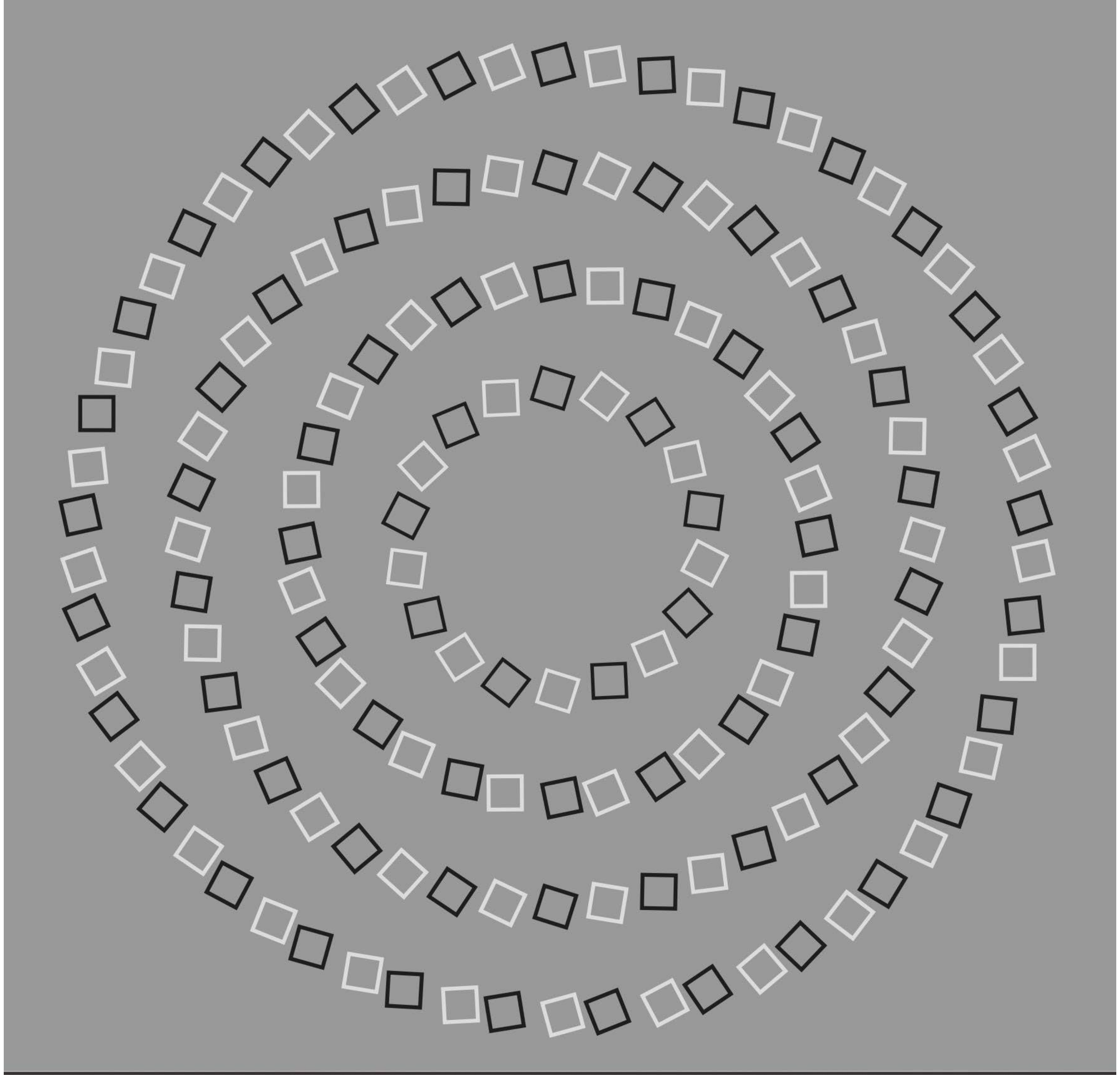
Networked Robotics Systems, Cooperative Control and Swarming (ROB-GY 6333 Section A)

- **Today's lecture:**
  Consensus Continued



Gershgorin Disc/Circle Theorem

- Eigenvalues (spectrum) of a square matrix (are important for control)
  - Obvious for diagonal matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
  - Less obvious for diagonally dominant $\begin{bmatrix} 1 & 0.1 & 0 \\ 0.1 & 2 & 0.1 \\ 0 & 0.1 & 3 \end{bmatrix}$
  - Really not obvious for any other matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$
  - Proof is neat, makes you appreciate how a matrix product is defined as a series of dot-products between A's rows and B's columns

- **Theorem:** Let A be an $n \times n$ matrix and $\mathcal{R}_i$ denote the circle in the complex plane $\mathbb{C}$ with center a_{ii} and radius $\sum_{j \neq i} |a_{ij}|$

$$\mathcal{R}_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}$$

Then all of the eigenvalues of A are contained in the union $\bigcup_i \mathcal{R}_i$

- **Short Proof:** Suppose λ is an eigenvalue with associated eigenvector v (normalized such that $|v|_\infty = 1$).

$$Av = \lambda v$$

$$\sum_{j=k}^{n} a_{kj} v_j = \lambda v_k$$

$$\sum_{j \neq k}^{n} a_{kj} v_j = (\lambda - a_{kk}) v_k$$

$$\sum_{j \neq k}^{n} |a_{kj}| |v_j| \geq |\lambda - a_{kk}| |v_k|$$

- **Theorem:** Let A be an $n \times n$ matrix and $\mathcal{R}_i$ denote the circle in the complex plane $\mathbb{C}$ with center a_{ii} and radius $\sum_{j \neq i} |a_{ij}|$

$$\mathcal{R}_i = \left\{ z \in \mathbb{C} : | z - a_{ii} | \leq \sum_{j \neq i} | a_{ij} | \right\}$$

Then all of the eigenvalues of A are contained in the union $\bigcup_i \mathcal{R}_i$

- **Short Proof:** Suppose λ is an eigenvalue with associated eigenvector v (normalized such that $|v|_\infty = 1$).

$$Av = \lambda v$$

$$\sum_{j=k}^{n} a_{kj} v_j = \lambda v_k$$

$$\sum_{j \neq k}^{n} a_{kj} v_j = (\lambda - a_{kk}) v_k$$

$$\sum_{j \neq k}^{n} | a_{kj} || v_j | \geq | \lambda - a_{kk} || v_k |$$

Exact Looser

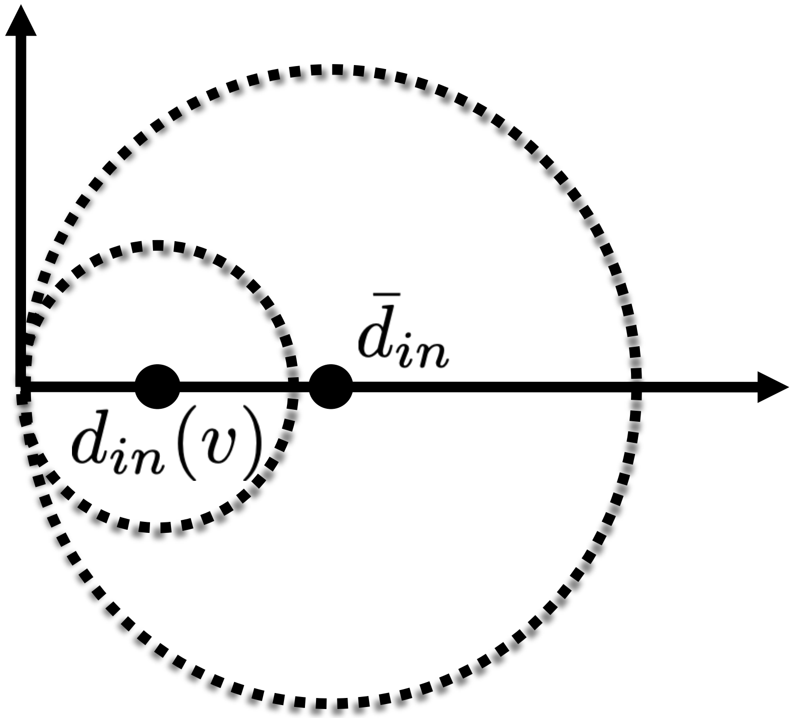
$$| \lambda - a_{kk} | \leq \sum_{j \neq k}^{n} | a_{kj} |$$

For each eigenvalue and eigenvector pair, there must be a choice of k where $v_k = 1$

Nifty reminder that matrix norms exist and tied to eigenvalues

Gershgorin Disc/Circle Theorem

- Because of how Laplacians are created, all its eigenvalues must be non-negative.
    - Silent on the key issue of number of zero eigenvalues



Algebraic Connectivity

- We are going to see, at the level of differential equations, why the number of zero eigenvalues matters
- Short proof for swarms represented by **undirected** graphs:
  - Design a swarm control law from scratch simply based on neighbor-distances
    - Eigenvector-Agreement Set ("all-the-same-values" vector) appears as a fixed point
  - Imagine the swarm is disconnected.
    - There are more eigenvectors! (smaller "all-the-same-values" vectors)
    - Working backwards from physical control problem back to linear algebra
      - Food for thought: Can there be multiple zero eigenvalues and only one/multiple linearly independent eigenvectors? **Hold that thought**

Control Theory Again

- For this equation: $\dot{x} = Ax$
  - Stable
    - Asymptotically stable
    - Marginally stable
  - Unstable
- For this equation for undirected graphs: $\dot{x} = -Lx$
  - **Stable**

Control Theory Again

- For this equation: $\dot{x} = Ax$
  - Stable
    - Asymptotically stable
    - Marginally stable
  - Unstable
- For this equation for undirected graphs: $\dot{x} = -Lx$
  - **Stable**
    - **Marginally stable**

Control Theory Again

- For this equation: $\dot{x} = Ax$
  - Stable (has NO eigenvalues with positive real part)
    - Asymptotically stable (has NO eigenvalues with zero real part)
    - Marginally stable (has at least one eigenvalue with zero real part)
  - Unstable (has at least one eigenvalue with positive real part)
- For this equation for undirected graphs: $\dot{x} = -Lx$
  - **Stable**
    - **Marginally stable (–*L* has a zero eigenvalue)**

Control Theory Again

- For this equation: $\dot{x} = Ax$
  - Stable (has NO eigenvalues with positive real part)
    - Asymptotically stable (has NO eigenvalues with zero real part)
    - Marginally stable (has at least one eigenvalue with zero real part)
  - Unstable (has at least one eigenvalue with positive real part)
- For this equation for undirected graphs: $\dot{x} = -Lx$
  - **Stable**
    - **Marginally stable (–*L* has a zero eigenvalue)**

- **These ideas about stability are only for linear time-invariant (LTI) systems**

Linear Algebra: Similarity Transforms

- Change of basis

$$H = I\omega$$

$$\Updownarrow$$

$$\tilde{H} = \tilde{I}\tilde{\omega} \longrightarrow (RH) = \tilde{I}(R\omega)$$

$$\tilde{\omega} = R\omega$$

$$\tilde{H} = RH$$

$$H = R^{-1}\tilde{I}R\omega$$

$$\boxed{\begin{aligned} I &= R^{-1}\tilde{I}R \\ RIR^{-1} &= \tilde{I} \end{aligned}}$$

Diagonalization as Similarity Transform

- Symmetric matrices are always diagonalizable by some orthogonal matrix *Q* composed of eigenvectors. Note: $Q^{-1} = Q^T$

$$\dot{x} = -Lx$$

$$\dot{x} = QDQ^{-1}x$$

$$Q^{-1}\dot{x} = DQ^{-1}x$$

$$Q^{-1}\dot{x} = \dot{\tilde{x}}$$

$$Q^{-1}x = \tilde{x}$$

- Given *D*, what is the first element of $\tilde{x}, \dot{\tilde{x}}$?

$$D \in \mathbb{R}^{N\times N} = \begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_p \end{bmatrix}$$

$$\dot{\tilde{x}}_1 = 0$$

$$\tilde{x}_1 = \frac{1}{N}\begin{bmatrix}1 & \cdots & 1\end{bmatrix}x(0) \longrightarrow \frac{\mathbf{1}\mathbf{1}^T x(0)}{N} = \begin{bmatrix}\bar{x}(0) \\ \vdots \\ \bar{x}(0)\end{bmatrix}$$

Reached Agreement

Symmetric Laplacian (Undirected Graph)

- **Desired behavior:**
  - Converge to average
- **Control:**
  - Marginally stable (is good here!)
- **Laplacian structure:**
  - **Single** zero eigenvalue with eigenvector $\frac{1}{N}\begin{bmatrix}1 & \cdots & 1\end{bmatrix}$ and $Q^{-1} = Q^T$
- Given *D*, what is the first element of $\tilde{x}, \dot{\tilde{x}}$?

$$D \in \mathbb{R}^{N \times N} = \begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_p \end{bmatrix}$$

$$\dot{\tilde{x}}_1 = 0$$

$$\tilde{x}_1 = \frac{1}{N}\begin{bmatrix}1 & \cdots & 1\end{bmatrix} x(0)$$

Invariant/constant of motion

$$\longrightarrow \frac{\mathbf{1}\mathbf{1}^T x(0)}{N} = \begin{bmatrix} \overline{x}(0) \\ \vdots \\ \overline{x}(0) \end{bmatrix}$$

Reached Agreement

Symmetric Laplacian (Undirected Graph)

- **Theorem:** Symmetric matrices have a full set of eigenvectors and eigenvectors can be chosen to be orthogonal
- Multiple zero eigenvalues → multiple LI eigenvectors → Disagreement

Jordan Normal Form (When Diagonalization Fails)

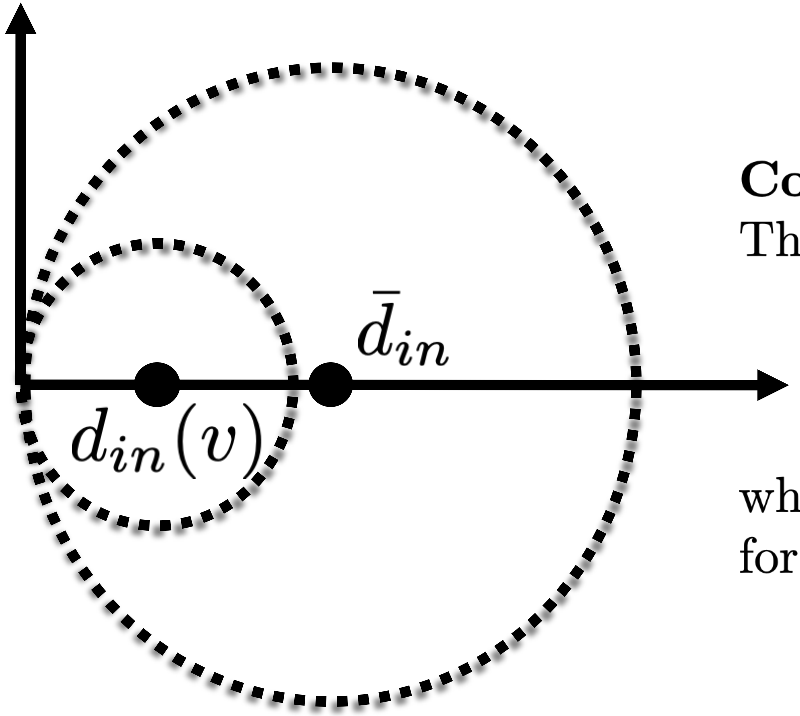
- P^{-1} is made of left eigenvectors (true for diagonalization)
  - "Proof" $A = PJP^{-1}$

$$P^{-1}A = JP^{-1}$$

  - Not that important but it is nice to make the connection
  - Left eigenvector [1 1 1 … 1 1] is related to column sums being zero

How about directed graphs?

- **Intuition:** A directed graph …
- **Math:** Gershgorin's Theorem tells us that the worst we can expect are zero eigenvalues, but it does not tell us the number of zero eigenvalues.



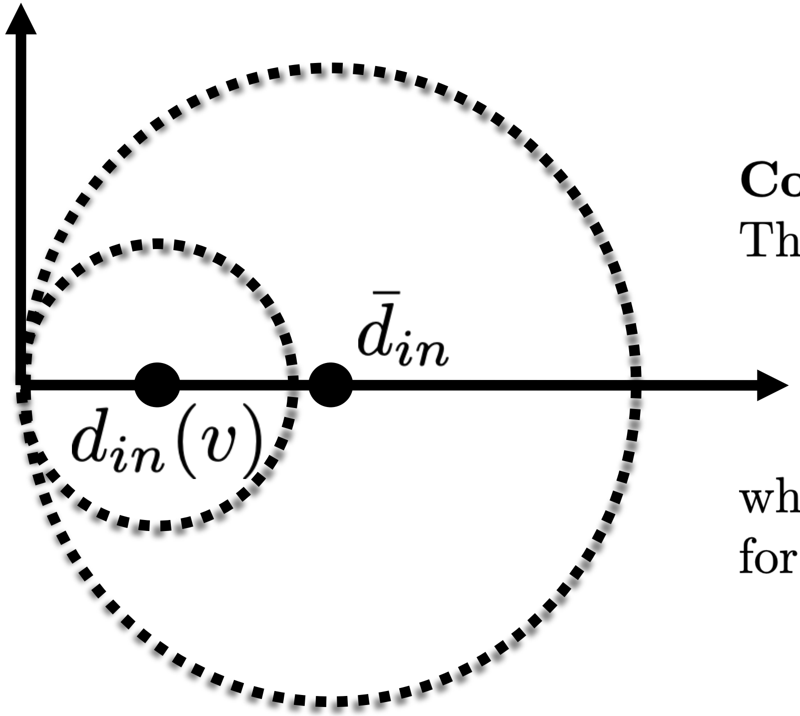
Corollary: Digraph Laplacian Let $\mathcal{D}$ be a weighted digraph on n vertices. Then the spectrum of $L(\mathcal{D})$ lies in the region

$$\left\{ z \in \mathbb{C} \;\middle|\; |z - \bar{d}_{in}(\mathcal{D})| \leq \bar{d}_{in}(\mathcal{D}) \right\}$$

where $\bar{d}_{in}(\mathcal{D})$ denotes the maximum (weighted) in-degree in $\mathcal{D}$. This means that for every digraph, the eigenvalues have non negative real parts $(Re(\lambda) \geq 0)$ $\square$

How about directed graphs?

- **Intuition:** A directed graph might not arrive at a consensus
- **Math:** Gershgorin's Theorem tells us that the worst we can expect are zero eigenvalues, but it does not tell us the number of zero eigenvalues.



Corollary: Digraph Laplacian Let $\mathcal{D}$ be a weighted digraph on n vertices. Then the spectrum of $L(\mathcal{D})$ lies in the region

$$\left\{ z \in \mathbb{C} \;\middle|\; |z - \bar{d}_{in}(\mathcal{D})| \leq \bar{d}_{in}(\mathcal{D}) \right\}$$

where $\bar{d}_{in}(\mathcal{D})$ denotes the maximum (weighted) in-degree in $\mathcal{D}$. This means that for every digraph, the eigenvalues have non negative real parts $(Re(\lambda) \geq 0)$ $\square$

Different Perspectives: Directed Graphs

- **Desired behavior:**
  - Converge to average
- **Control:**
  - Marginally stable (is good here!)
- **Laplacian structure:**
  - Not necessarily diagonalizable (asymmetric), but will still have **Jordan normal form (JNF)**

Quiz (Easy)

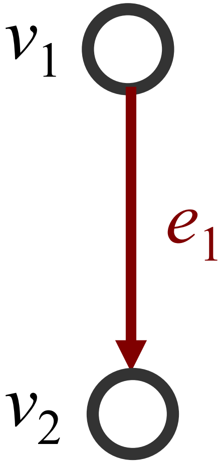
- Find adjacency matrix

Quiz (Easy)

- Find adjacency matrix
- We expect v_1 and v_2 to meet at the middle.
  - Think simple symmetry, how could the system be biased towards v_1 and v_2?

Quiz (Harder)

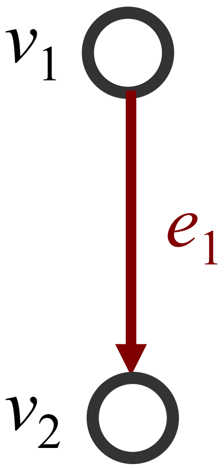
- Find adjacency matrix



$$[A]_{ij} = \begin{cases} w_{ij} & \text{if } (v_j, v_i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Quiz (Harder)

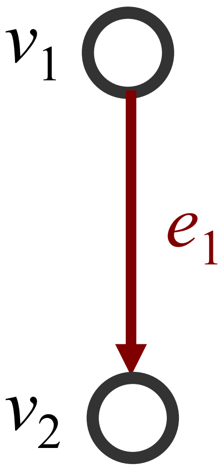
- Find adjacency matrix
- Find Laplacian
  - Is there consensus?
  - What kind of consensus?



$$[A]_{ij} = \begin{cases} w_{ij} & \text{if } (v_j, v_i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Quiz (Harder)

- Find adjacency matrix
- Find Laplacian
  - Is there consensus?
  - What kind of consensus?



$$[A]_{ij} = \begin{cases} w_{ij} & \text{if } (v_j, v_i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

- Diagonalize the Laplacian

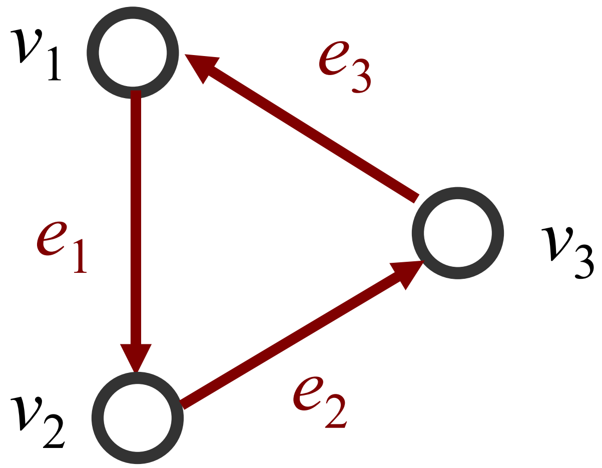
$$L = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

$$L = PDP^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

- Look at first row of P^{-1}, the left-eigenvector $q_1^{\mathrm{T}} = [1\ 0]$,
- That tells us the consensus is $[1\ 0] * [x_1\ x_2]^T = x_1$
- v_1 is a leader (doesn't move), v_2 is a follower

Quiz (Even Harder)

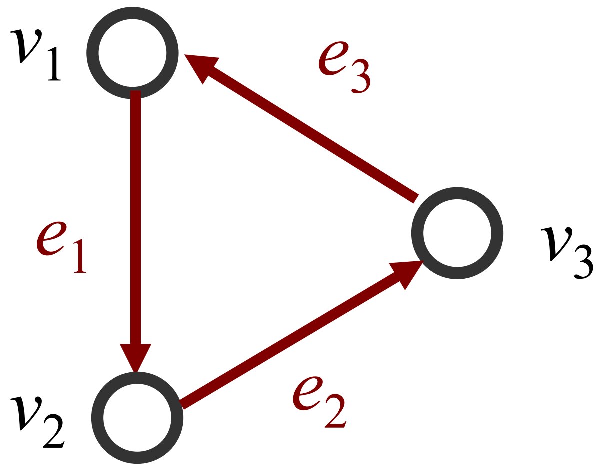
- Find adjacency matrix
- Find Laplacian
  - Is there consensus?
  - What kind of consensus?



$$[A]_{ij} = \begin{cases} w_{ij} & \text{if } (v_j, v_i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Quiz (Even Harder)

- Find adjacency matrix
- Find Laplacian
    - Is there consensus?
    - What kind of consensus?



$$L = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[A]_{ij} = \begin{cases} w_{ij} & \text{if } (v_j, v_i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Laplacian is a **circulant matrix**, even the matrix looks like a cycle

Quiz (Even Harder)

- Diagonalize matrix with help from WolframAlpha

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = S.J.S^{-1}$$

where

$$S = \begin{pmatrix} 1 & -\frac{1}{2} i\left(\sqrt{3} + -i\right) & \frac{1}{2} i\left(\sqrt{3} + i\right) \\ 1 & \frac{1}{2} i\left(\sqrt{3} + i\right) & -\frac{1}{2} i\left(\sqrt{3} + -i\right) \\ 1 & 1 & 1 \end{pmatrix}$$

- Only one zero eigenvalue

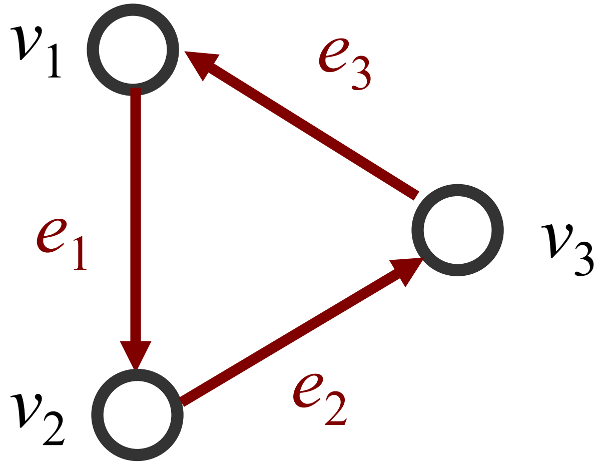
$$J = \begin{pmatrix} \boxed{0} & 0 & 0 \\ 0 & \frac{1}{2}\left(3 - i\sqrt{3}\right) & 0 \\ 0 & 0 & \frac{1}{2}\left(3 + i\sqrt{3}\right) \end{pmatrix}$$

- Consensus is average [1/3 1/3 1/3]

$$S^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} i\left(\sqrt{3} + i\right) & -\frac{1}{6} i\left(\sqrt{3} + -i\right) & \frac{1}{3} \\ -\frac{1}{6} i\left(\sqrt{3} + -i\right) & \frac{1}{6} i\left(\sqrt{3} + i\right) & \frac{1}{3} \end{pmatrix}$$

Quiz (Even Harder)

- Dog chasing its own tail, arrives at a consensus!



Observations?

Formal notions

- We are now ready to speak in the language of graph theory again
- **I am going to present things without proof, but they are fun to work out examples in Python/MATLAB, caveats are that in MATLAB for the digraph() command, the adjacency matrix is defined as the transpose of what we do in class.**
- **You can recreate all the videos in this lecture in code.**

Rooted out-branching

- A **rooted out-branching** is a graph that:
  - contains no cycles
  - has one vertex (**root**) connected by a **directed** path to any other vertex
- **Fact:** a directed graph $\mathcal{D}$ contains a rooted-out branching if and only if the rank of its Laplacian is $N-1$ (i.e., only one zero eigenvalue)
  - In that case the null space **N**(L(D)) is spanned by $[1\ 1\ \ldots\ 1\ 1]^T$ ("all-the-same-values" vector)

Consensus Protocol on Digraphs

- The **consensus protocol** $\dot{x} = -Lx$ converges to the **agreement set** for any initial conditions if the graph $\mathcal{D}$ contains a **rooted-out branching**
- The **reached agreement** is $\mathbf{1}{q_1}^T x(0)$ where q_1 is the left eigenvector of *L* with left eigenvalue 0 (i.e. ${q_1}^T L = 0$) and $q_1^T \mathbf{1} = 1$
  - For a symmetric Laplacian (undirected graph) the left and right eigenvectors are the same, the “all-the-same-values” vector.

Balanced Digraph

- A digraph is **balanced** if each vertex has equal in-degree and out-degree
- If a digraph is balanced, then $\mathbf{1}^T$ is a left eigenvector of L
- If a graph is balanced and contains a rooted-out branching then it converges to the agreement value $\mathbf{1}\mathbf{1}^T x(0)/N$
- Symmetric Laplacians are always balanced!

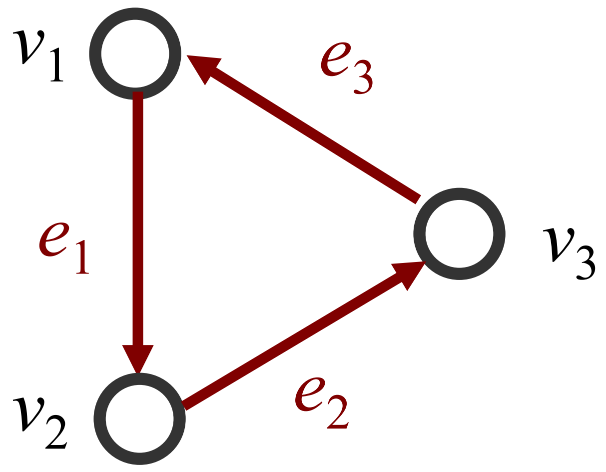
- **The agreement protocol over a digraph reaches the average consensus for every initial condition if and only if it contains a rooted out-branching and it is balanced.**
  - **Which is equivalent to the condition that the graph is strongly connected or that it is weakly connected and balanced**

Careful

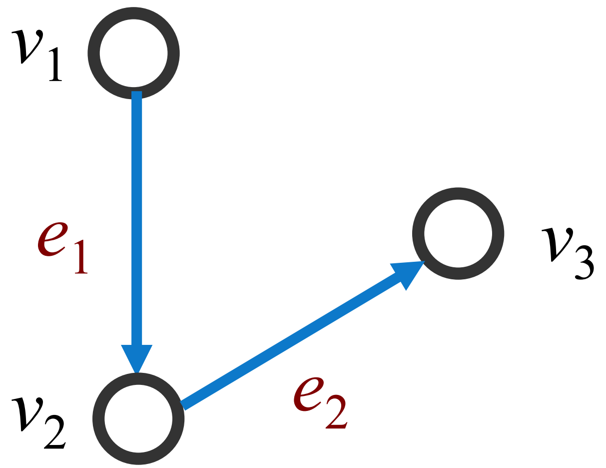
The agreement protocol over a digraph reaches the average consensus for every initial condition if and only if it contains a rooted out-branching and it is balanced.

Which is equivalent to the condition that the graph is strongly connected or that it is weakly connected and balanced

- All cycle graphs are strongly connected
  - All cycle graphs contain a rooted out-branching (simply cut one of the edges)

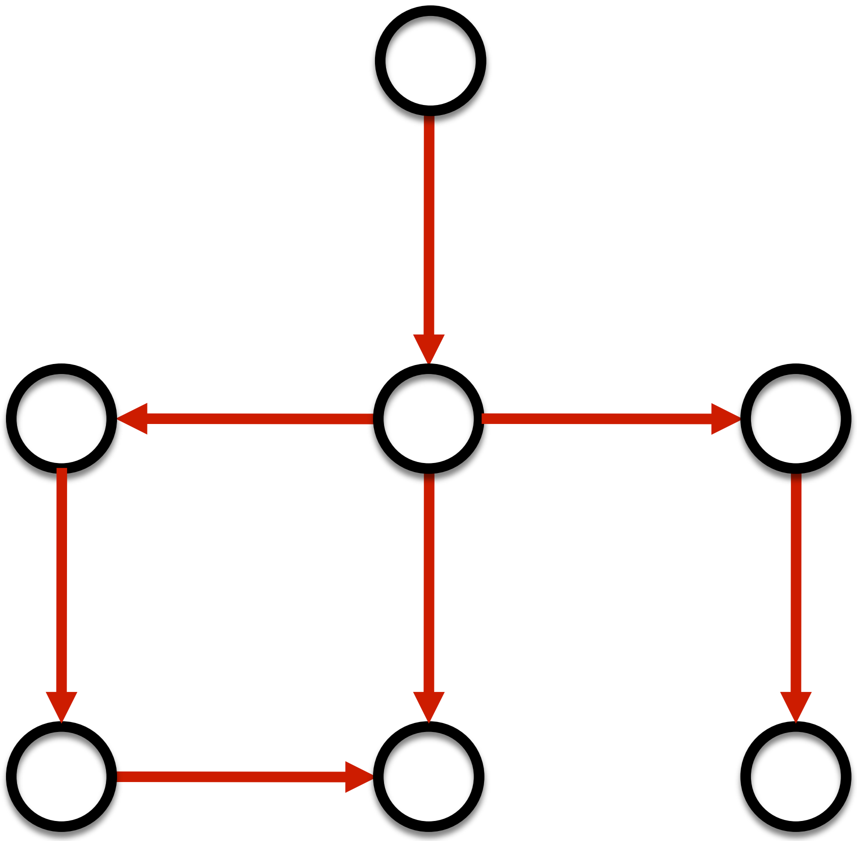


The above cycle graph **contains** the rooted out-branching below



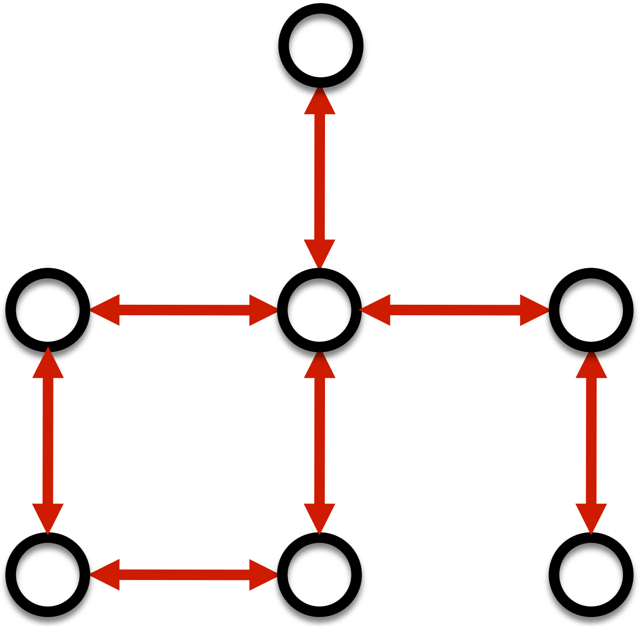
Rooted-out branching with leader

- What if a graph has a rooted-out branching but the root does not have any incoming edges?

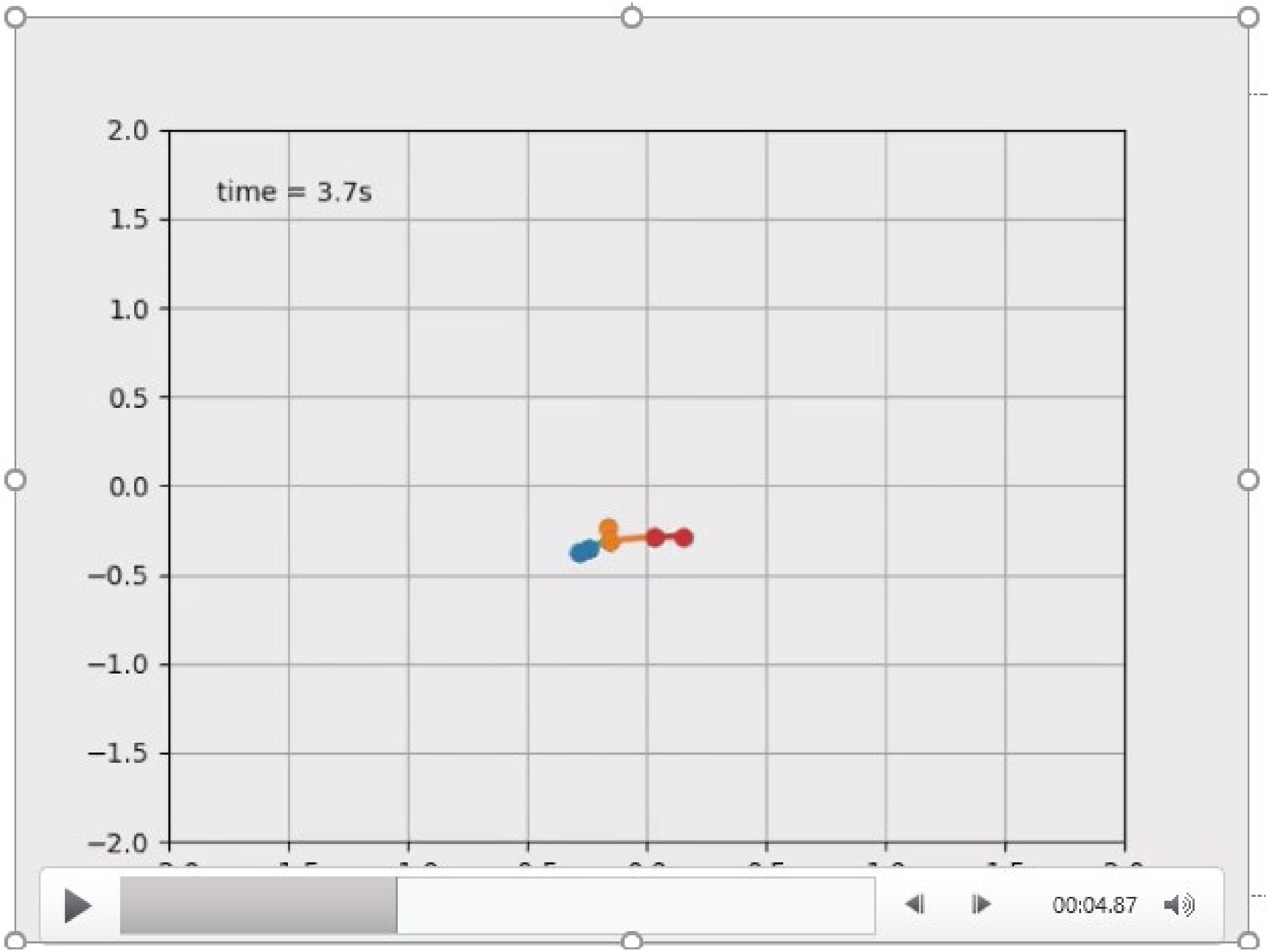


Rooted-out branching + Balanced

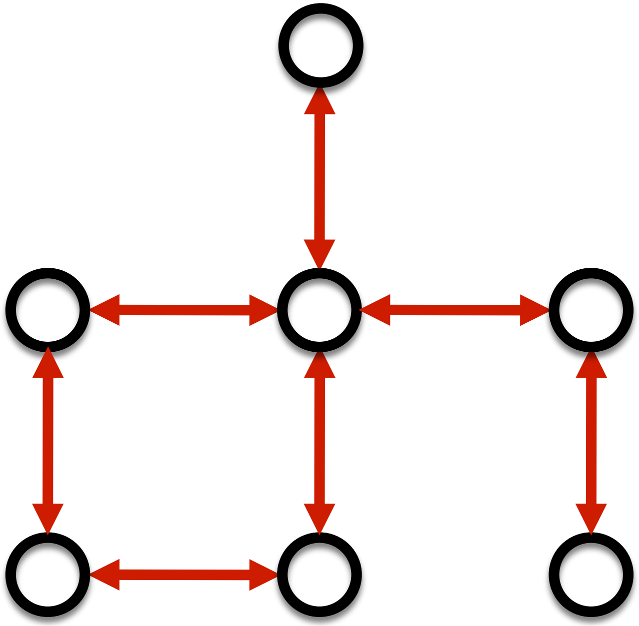
$$\lambda_2 = 1$$



Rooted-out branching + Balanced



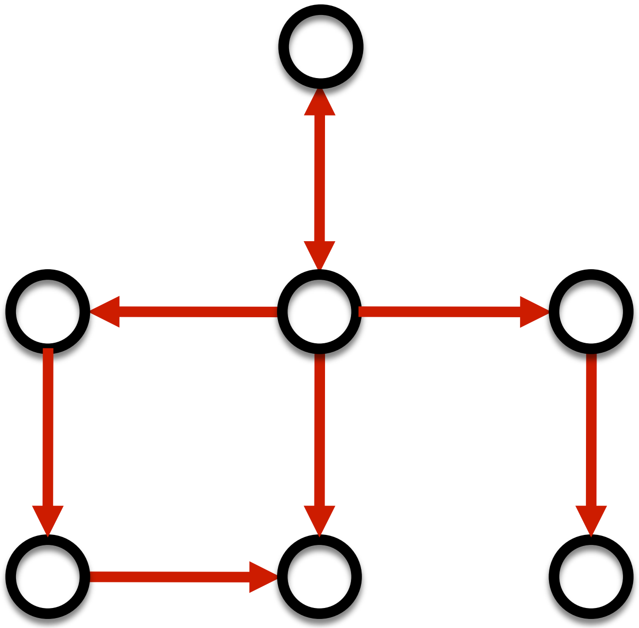
$$\lambda_2 = 1$$



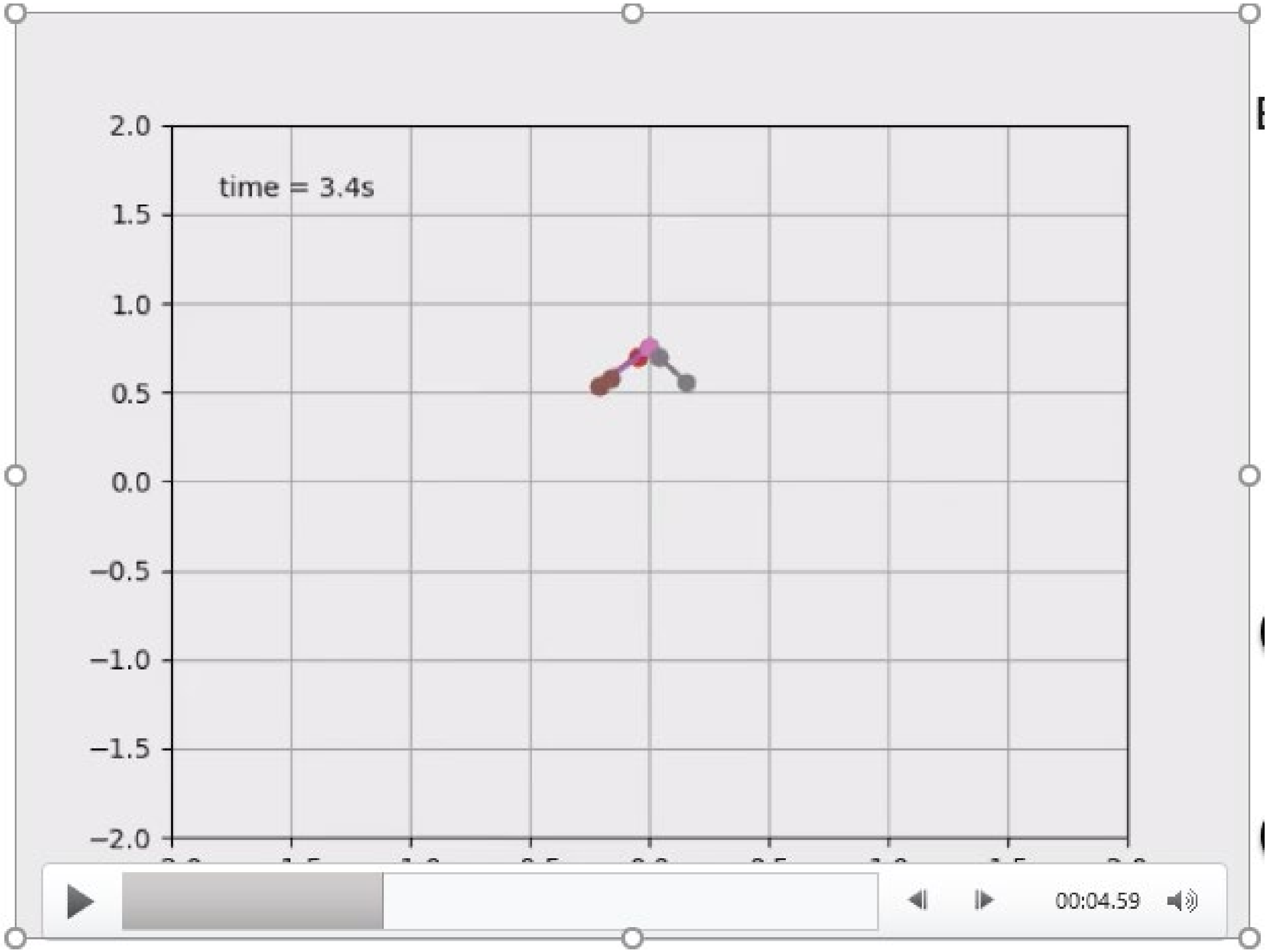
Rooted-out branching + Not Balanced

Edge incoming in root

$$\lambda_2 = 1$$

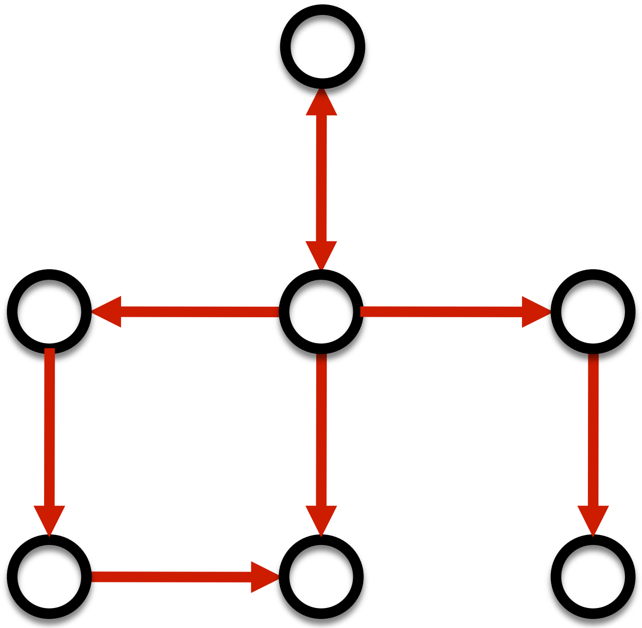


Rooted-out branching + Not Balanced



- Edge incoming in root

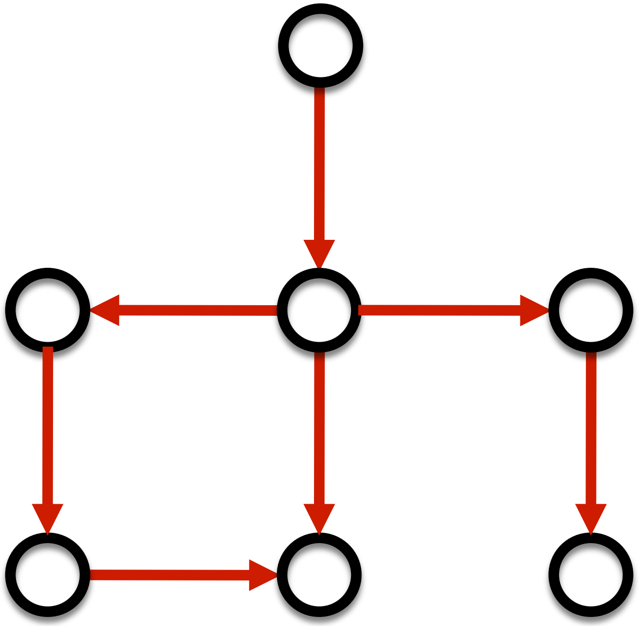
$$\lambda_2 = 1$$



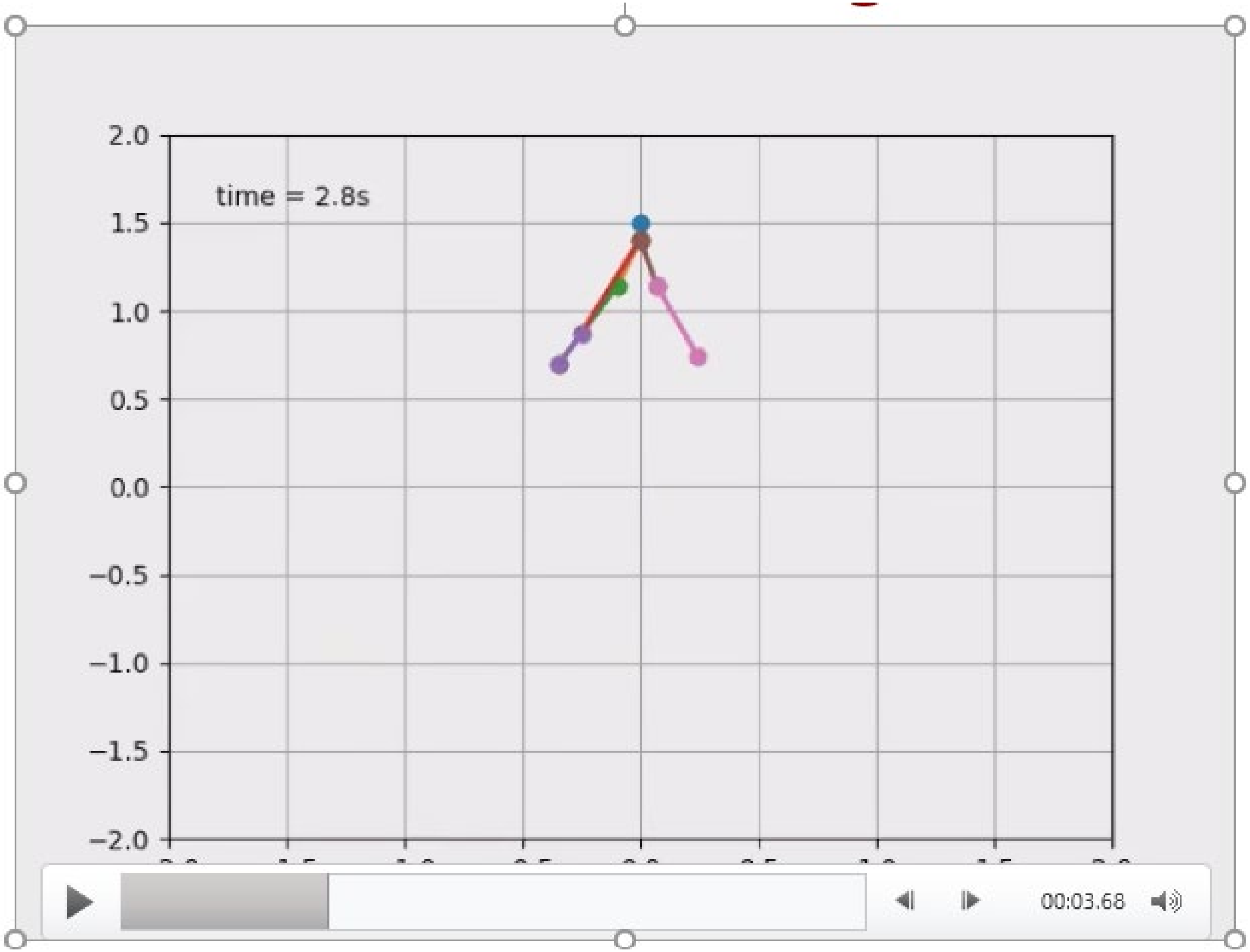
Rooted-out branching + Not Balanced

- No incoming edge to root

$$\lambda_2 = 1$$

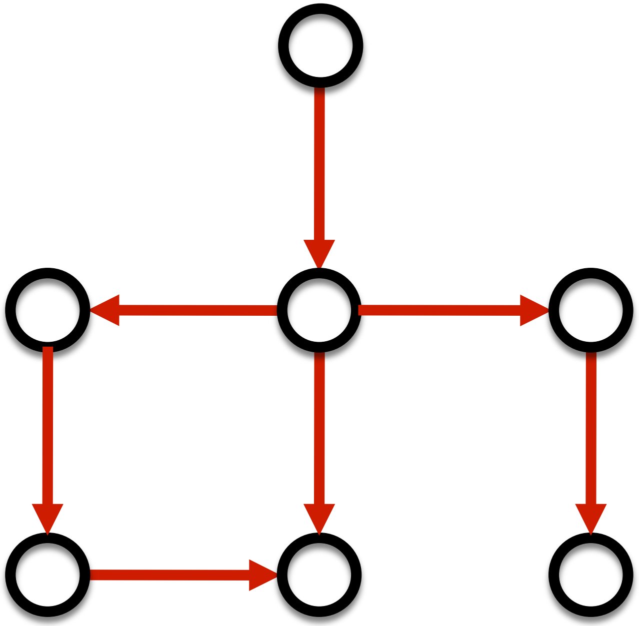


Rooted-out branching + Not Balanced



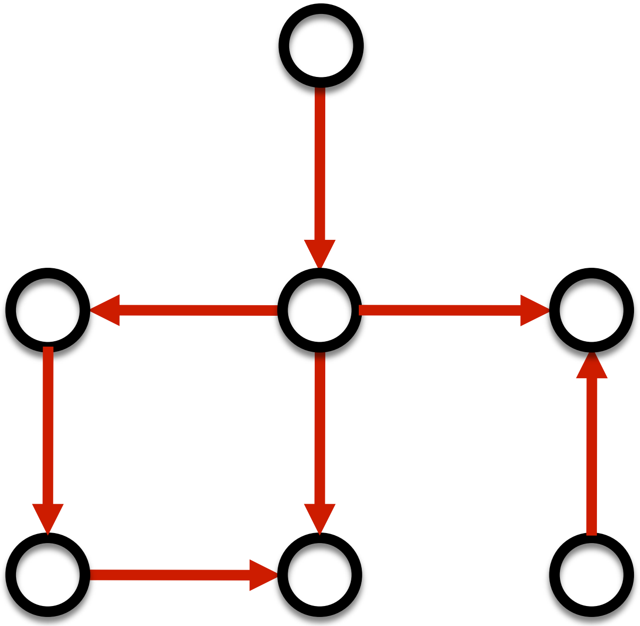
- No incoming edge to root

$$\lambda_2 = 1$$

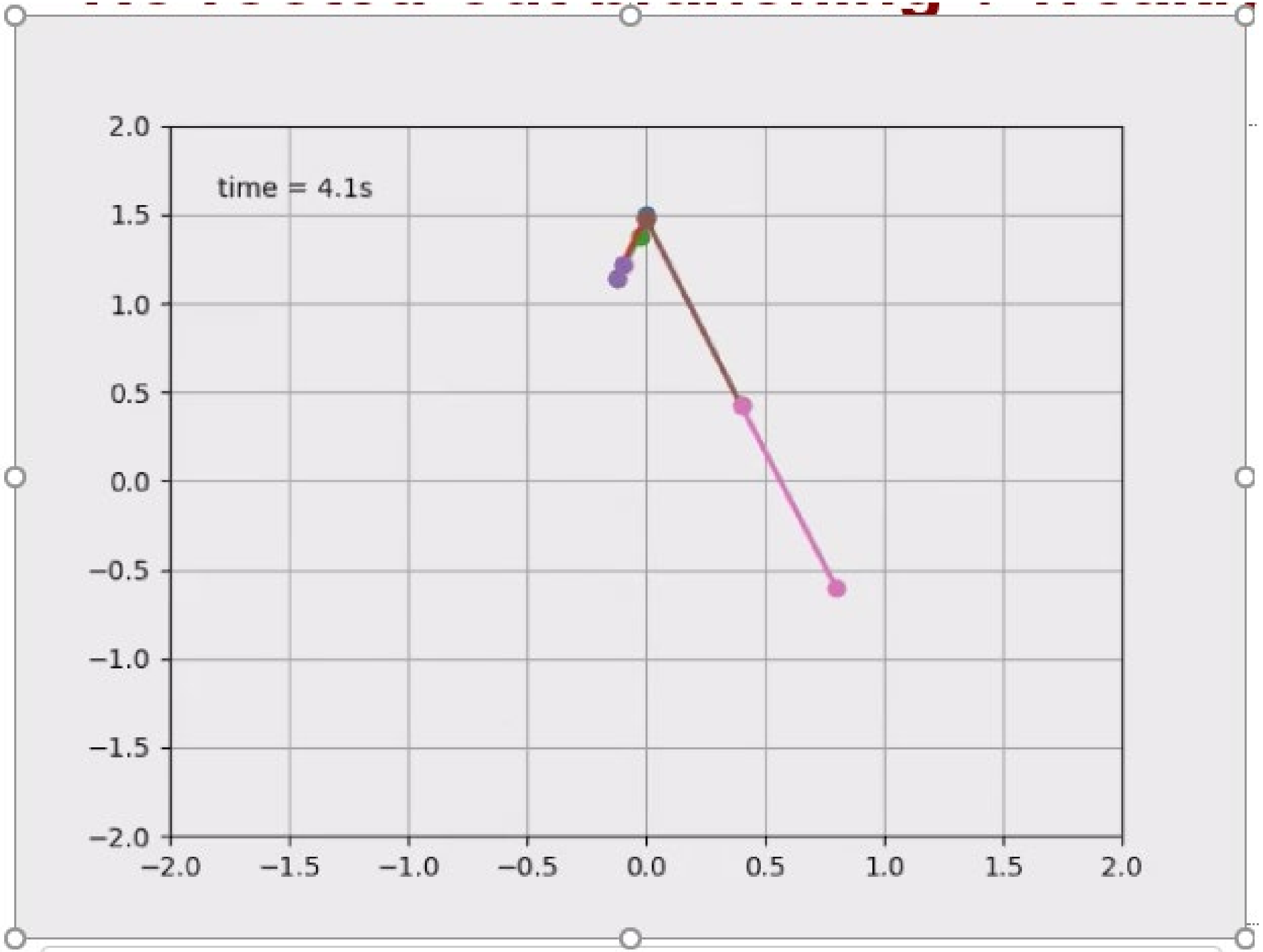


No rooted-out branching + Weakly-connected

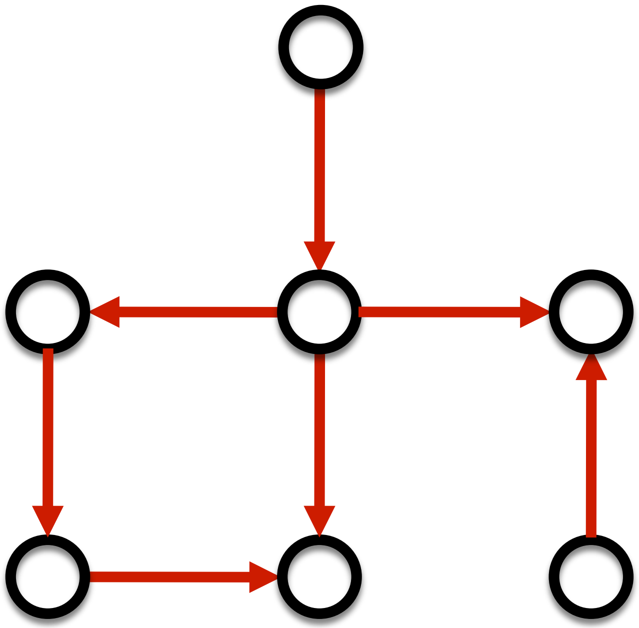
$$\lambda_2 = 0$$



No rooted-out branching + Weakly-connected



$$\lambda_2 = 0$$



Making Connections

- All graphs (undirected, directed, weighted directed) are marginally stable
  - The eigenvalues of L have positive or zero real part
    (i.e., the eigenvalues of $-L$ have negative or zero real part)
    - Gershgorin Theorem and structure of Laplacian Matrix
- Marginally stable alone does not imply agreement can be reached
- Marginally stable with a single zero eigenvalue does imply agreement!
- A connected graph must have only a single zero eigenvalue (rank = $n - 1$)
  - Argument from a single eigenvector $[1\ 1\ \dots\ 1\ 1]^T$
- **??????????**

Summary

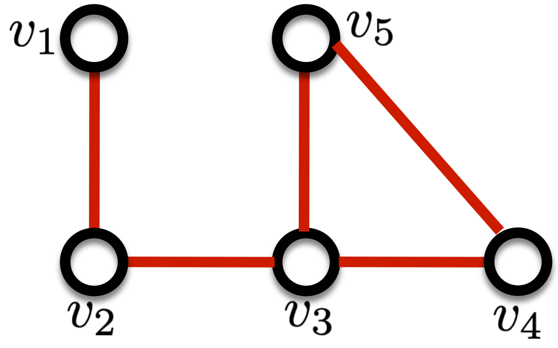
- Find local control rule to get agents to **agree on the same value** in a distributed manner

The agreement set $\mathcal{A} \subset \mathbb{R}^N$ is the subspace $\text{span}\{\mathbf{1}\}$ that is

$$\mathcal{A} = \{x \in \mathbb{R}^N \mid x_i = x_j, \forall i, j\}$$

- We found a consensus (or agreement) protocol

Graph Theory



Linear Algebra
Differential Equations

Graph Laplacian

$$\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x}$$

Vector of agents states

Control Theory

$$\begin{aligned}
u_1 &= (x_2 - x_1) \\
u_2 &= (x_3 - x_2) + (x_1 - x_2) \\
u_3 &= (x_2 - x_3) + (x_4 - x_3) + (x_5 - x_3) \\
u_4 &= (x_3 - x_4) + (x_5 - x_4) \\
u_5 &= (x_3 - x_5) + (x_4 - x_5)
\end{aligned}$$

$$\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x}$$

Undirected graphs

Convergence to agreement <=> connected graph

Agreement value:

$$\lim_{t\to\infty} \mathbf{x}(t) = \frac{\mathbf{1}\mathbf{1}^T\mathbf{x}_0}{N}$$

$\mathbf{1}^T\mathbf{x}(t)$ is a **constant** of motion

Weighted directed graphs

Convergence to agreement <=> graph contains rooted-out branching

Agreement value:

$$\mathbf{1}q_1^T x_0$$

where q_1 is the left eigenvector of L with eigenvalue 0 and $q_1^T\mathbf{1} = 1$

If graph is balanced => $\dfrac{\mathbf{1}\mathbf{1}^T\mathbf{x}_0}{N}$

$q_1^T\mathbf{x}(t)$ is a **constant** of motion

The rate of (exponential) convergence is defined by the second smaller (real part) eigenvalue of L, also known as algebraic connectivity or Fiedler eigenvalue

Next week: Preview

- So far we have an agreement protocol for **static** graphs with trivial dynamics
- What happens when the graph topology changes?
  - Time-varying systems
- How do we analyze networks with more complex dynamics?
  - Nonlinear systems

Invariant sets analysis

- How to analyze differential equations without solving them

Solving ODEs is not always useful

- Consider the following equation $$\dot{x} = \frac{dx}{dt} = \sin(x)$$

- We can separate variables and get $$dt = \frac{dx}{\sin(x)}$$

- Integrating each side gives

$$t = \int \csc x \, dx = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|$$

- While it is the solution to the equation, it does not give use much insights!

Solving ODEs is not always useful

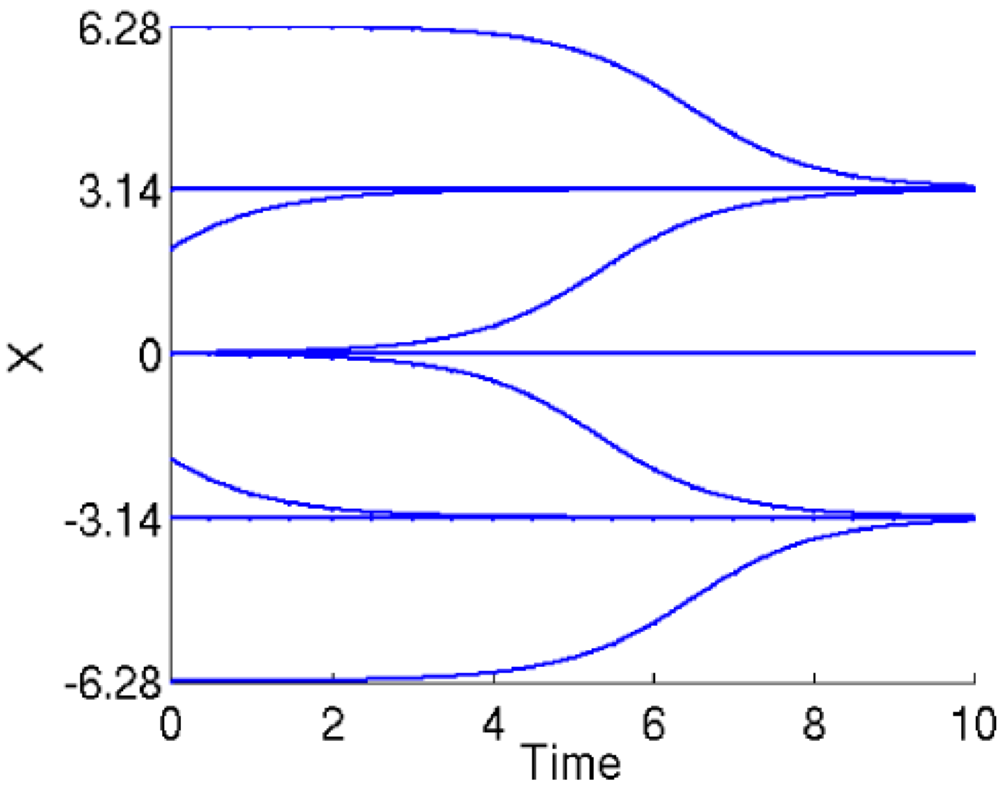
$$t = \int \csc x \, dx = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|$$

- Given some initial conditions $x_0(t = 0)$ what happens when $t \to \infty$?
- Is this behavior the same for every initial condition?
- What are the **qualitative** features of the solution?

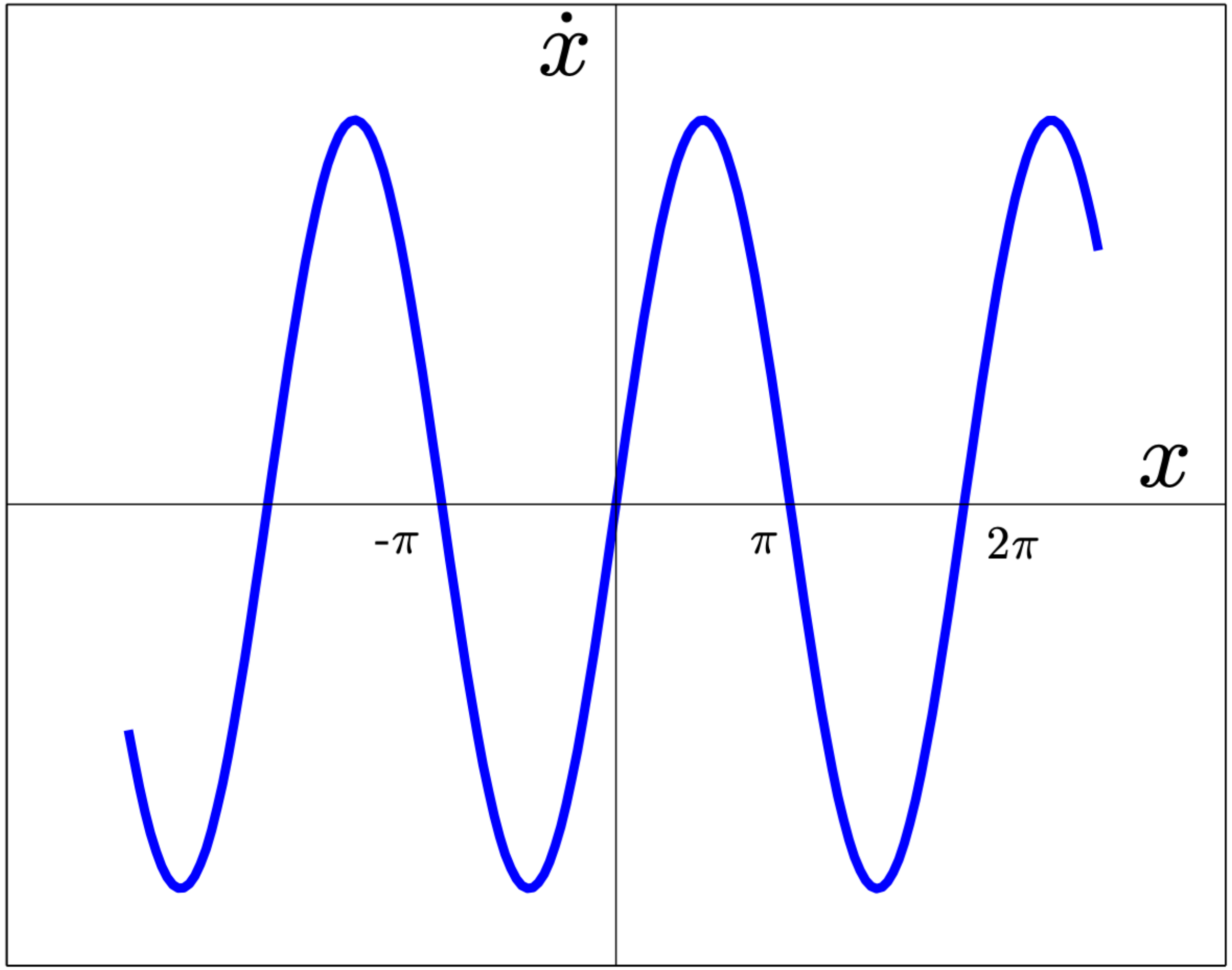
- The solution of the ODE does not help us answer these questions

Pictures Help

We can understand the behavior of the system by numerically integrating the equations for several initial conditions



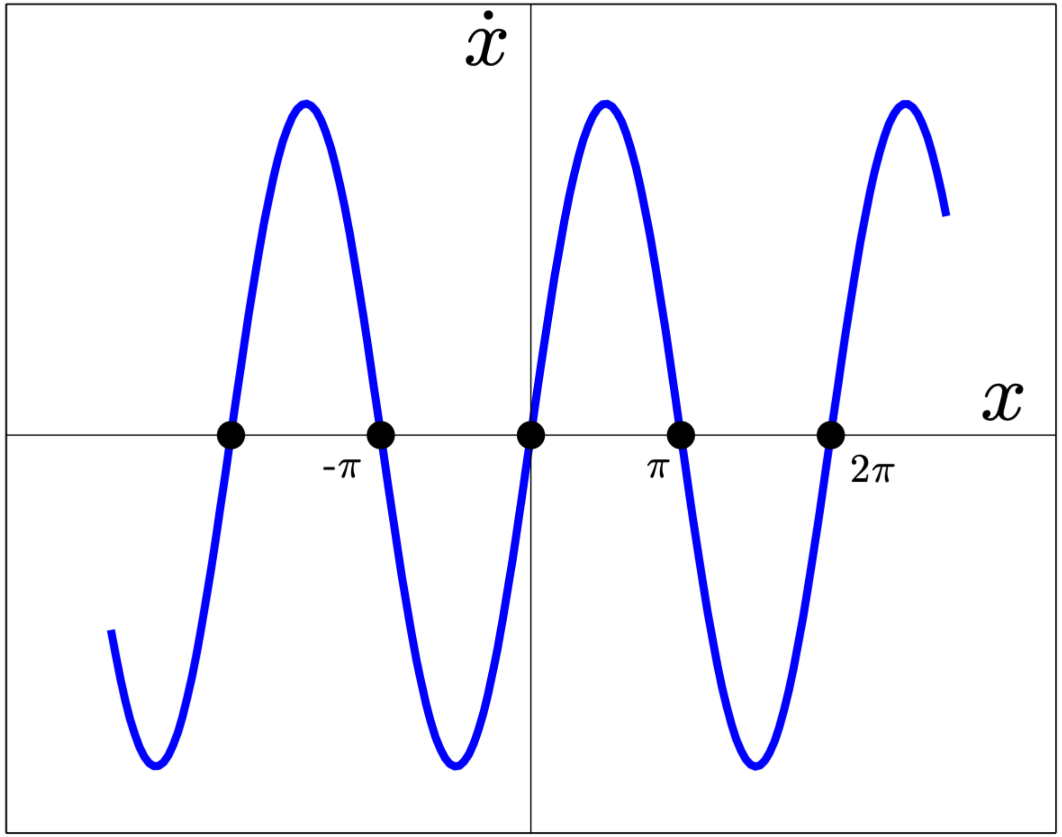
Pictures Help



We plot $\dot{x} = \sin(x)$ as a function of x

To better understand the flow of x

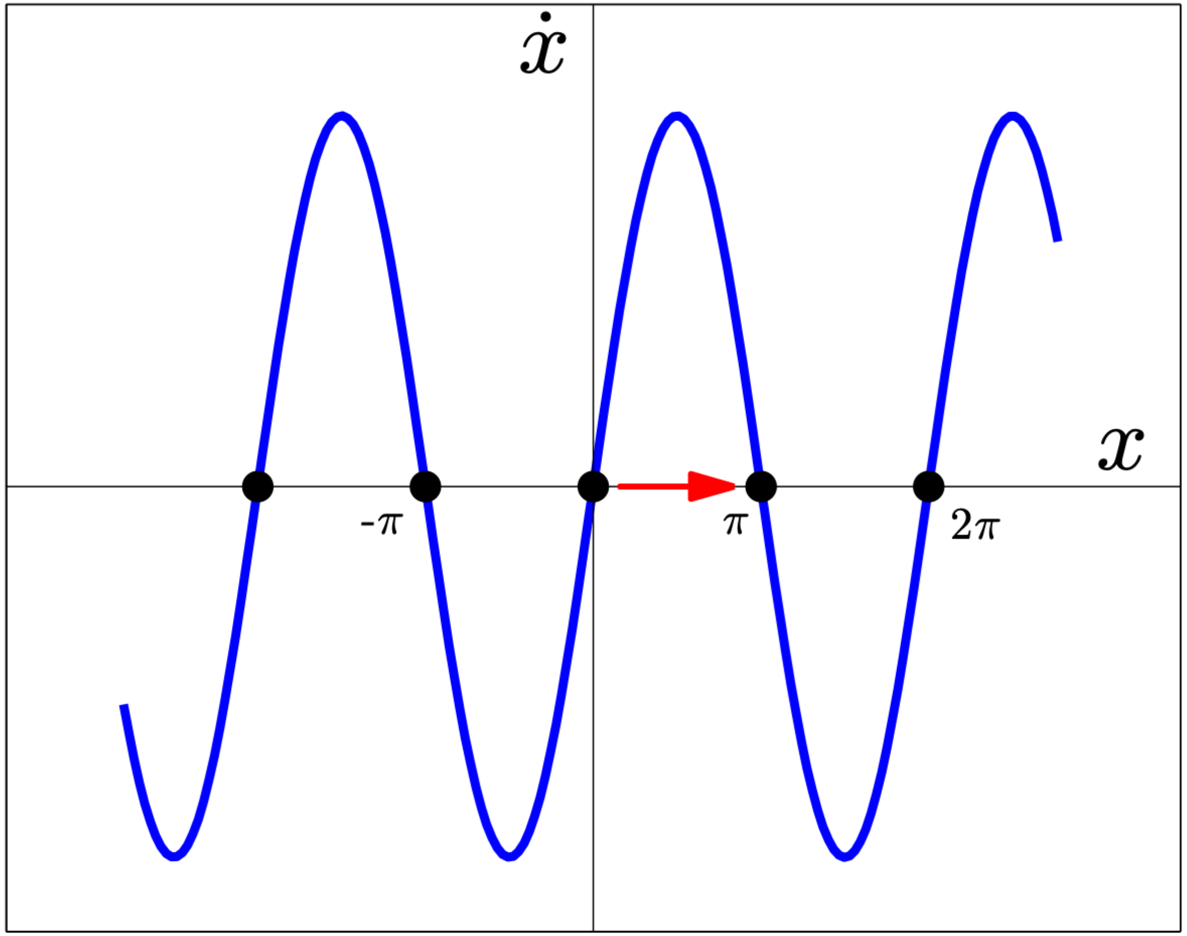
Pictures Help



If $x = [-2\pi, -\pi, 0, \pi, 2\pi]$ then $\dot{x} = 0$
And $x(t)$ does not change anymore

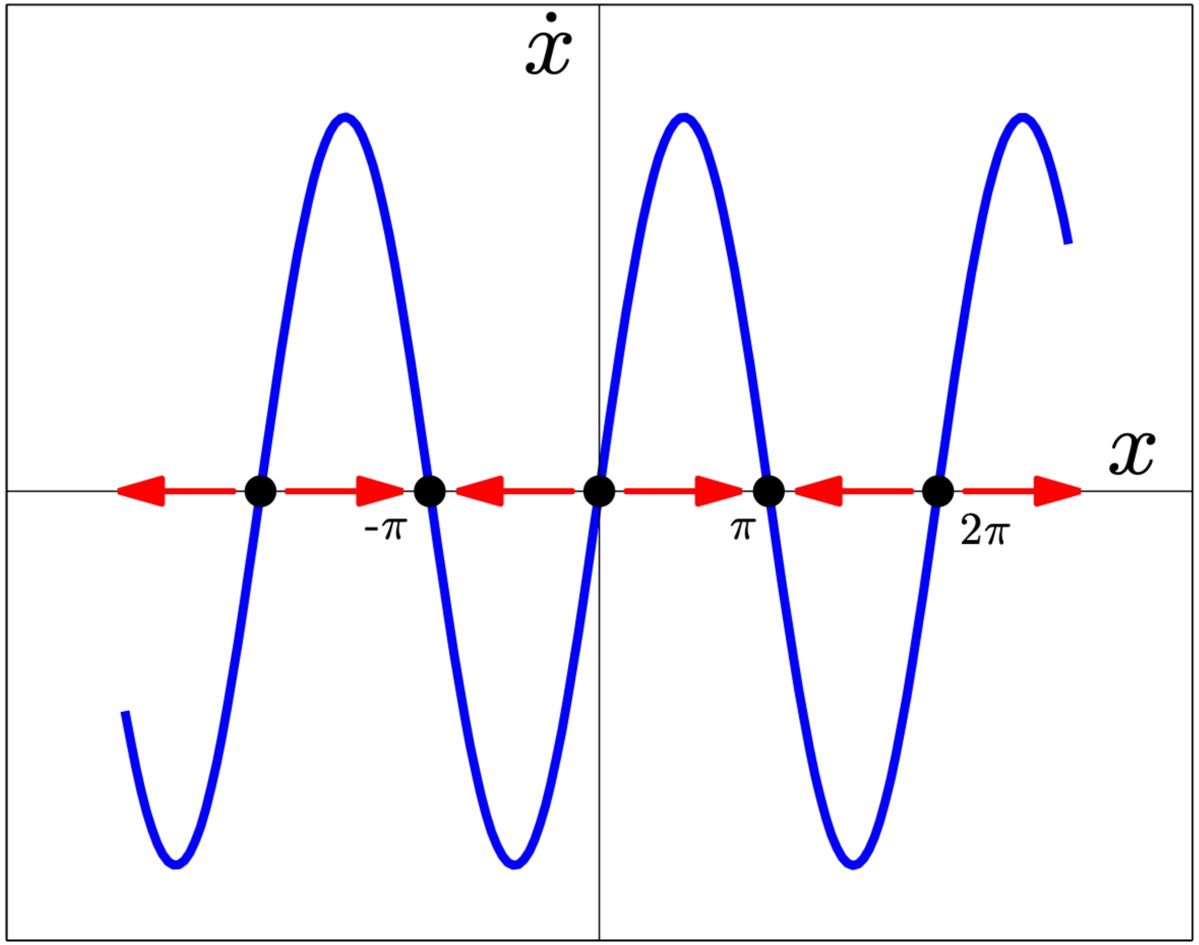
These points are called *Fixed Points*

Pictures Help



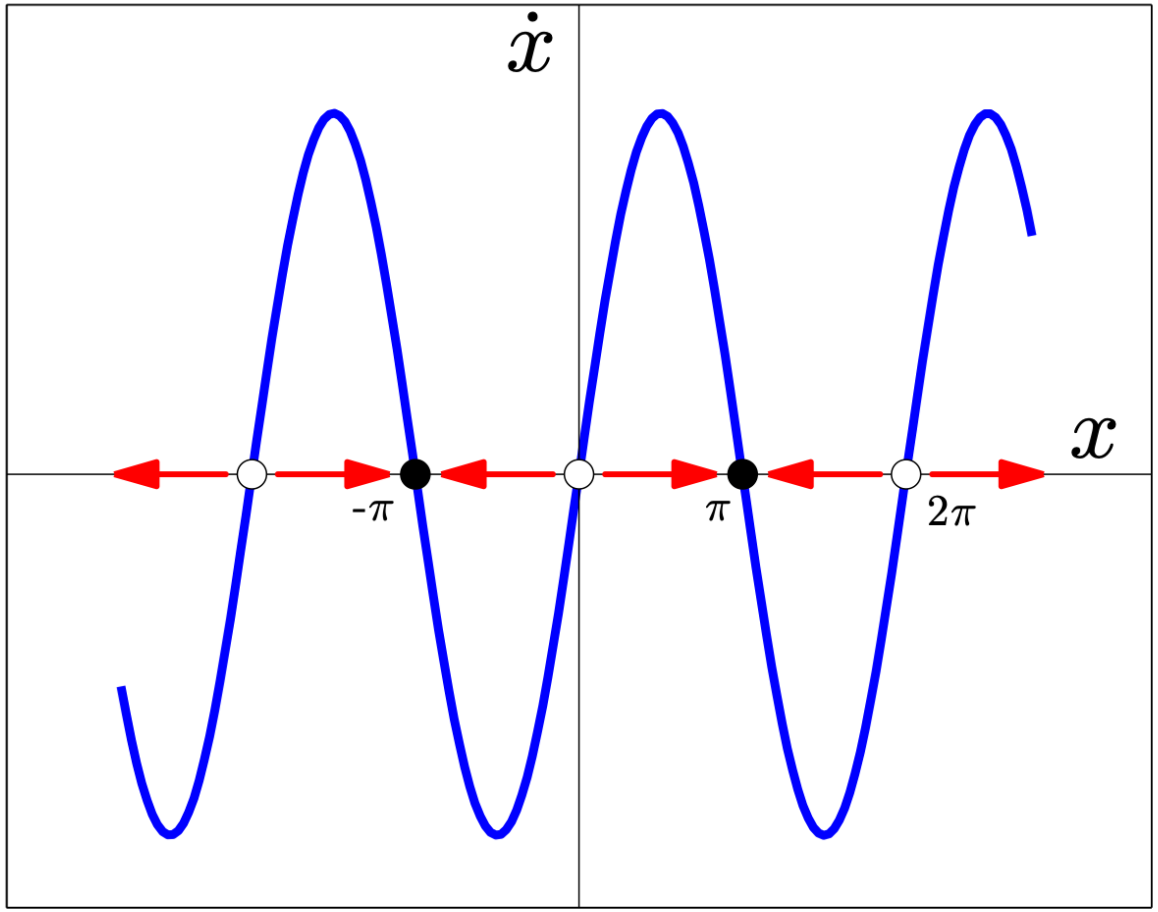
If $0 < x < \pi$ then $\dot{x} > 0$ and x converges to π

Pictures Help



For each region between fixed points we find the convergence properties

Pictures Help



We see 2 kinds of fixed points

Stable Fixed Points are $-\pi$ and π

Unstable Fixed Points are -2π, 0 and 2π

Lyapunov Stability

- Think about the stability of fixed points rather than system
  - Global stability vs local stability