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Homework-3
FEG904 - Fixed Income Securities
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Problem 1
Consider the three bonds quoted in Table 1 (settlement: 2/15/94). For each bond take the average of bid and ask prices. Calculate discount factors and spot rates at six-month intervals, and implied six-month forward rates (${}_1f_{1},{}_2f_1$).

Consider now the three STRIPS prices of Table 2 (settlement: 2/15/94). For each STRIPS take the average of bid and ask prices. Based on the average prices, calculate discount factors, spot rates, and forward rates.

Solution:

Table 1

Coupon rate	Maturity	Bid	Ask
6 7/8	8/15/94	101:19	101:21
5 1/2	2/15/95	101:17	101:19
4 5/8	8/15 /95	100:20	100:22

Table 2

Maturity	Bid	Ask
8/15/94	98:09	98:09
2/15/95	96:06	96:07
8/15/95	94:00	94:01

a. For the three bonds quoted in Table 1:

- First, we calculate the average of bid and ask prices and coupon payments for these three bonds:

$$
\begin{aligned}
P(1) &= 100 \times \frac{101\frac{19}{32}\% + 101\frac{21}{32}\%}{2} = 101.6250 \\
C(1) &= 100 \times \frac{6\frac{7}{8}\%}{2} = 3.4375 \\
P(2) &= 100 \times \frac{101\frac{17}{32}\% + 101\frac{19}{32}\%}{2} = 101.5625 \\
C(2) &= 100 \times \frac{5\frac{1}{2}\%}{2} = 2.75 \\
P(3) &= 100 \times \frac{100\frac{20}{32}\% + 100\frac{22}{32}\%}{2} = 100.6563 \\
C(3) &= 100 \times \frac{4\frac{5}{8}\%}{2} = 2.3125
\end{aligned}
$$

- Second, we use coupon bonds to derive the discount factors:

$$
\begin{aligned}
d(1) &= \frac{101.6250}{103.4375} = 0.9825 \\
d(2) &= \frac{101.5625 - (0.9825 \times 2.7500)}{102.7500} = 0.9621 \\
d(3) &= \frac{100.6563 - (0.9825 \times 2.3125) - (0.9621 \times 2.3125)}{102.3125} = 0.9399
\end{aligned}
$$

- Then, spot rates can be calculated based on the discount factors:

$$y_1 = (\frac{1}{0.9825}) - 1 = 1.7835\%$$
$$y_2 = (\frac{1}{0.9621})^{\frac{1}{2}} - 1 = 1.9481\%$$
$$y_3 = (\frac{1}{0.9399})^{\frac{1}{3}} - 1 = 2.0890\%$$

- Finally, the implied 6-month forward rates can be obtained using spot rates:

$$_1f_1 = \frac{(1+y_2)^2}{(1+y_1)} - 1 = \frac{d(1)}{d(2)} - 1 = 2.1129\%$$
$$_2f_1 = \frac{(1+y_3)^3}{(1+y_2)^2} - 1 = \frac{d(2)}{d(3)} - 1 = 2.3715\%$$

b. For the three STRIPS in Table 2:

- First, we calculate the average of bid and ask prices for these STRIPS:

$$P(1) = 100 \times \frac{98\frac{9}{32}\% + 98\frac{9}{32}\%}{2} = 98.2813$$
$$P(2) = 100 \times \frac{96\frac{6}{32}\% + 96\frac{7}{32}\%}{2} = 96.2031$$
$$P(3) = 100 \times \frac{94 + 94\frac{1}{32}\%}{2} = 94.0156$$

- Second, we use the prices of the STRIPS to derive the discount factors:

$$d(1) = \frac{98.2813}{100} = 0.9828$$
$$d(2) = \frac{96.2031}{100} = 0.9620$$
$$d(3) = \frac{94.0156}{100} = 0.9402$$

- Then, spot rates can be calculated based on the discount factors:

$$y_1 = (\frac{1}{0.9828}) - 1 = 1.7488\%$$
$$y_2 = (\frac{1}{0.9620})^{\frac{1}{2}} - 1 = 1.9543\%$$
$$y_3 = (\frac{1}{0.9402})^{\frac{1}{3}} - 1 = 2.0783\%$$

- Finally, the implied 6-month forward rates can be obtained using spot rates:

$$_1f_1 = \frac{(1+y_2)^2}{(1+y_1)} - 1 = \frac{d(1)}{d(2)} - 1 = 2.1601\%$$
$$_2f_1 = \frac{(1+y_3)^3}{(1+y_2)^2} - 1 = \frac{d(2)}{d(3)} - 1 = 2.3267\%$$

Problem 2
Consider the first two bonds quoted in Table 1 (for settlement on 2/15/94). Consider now a $7\frac{3}{4}$ with maturity 2/15/95 and bid and ask quotes 103:24 and 103:25. Construct a portfolio of the $6\frac{7}{8}$ and the $7\frac{3}{4}$ which replicates the cash flows of the $5\frac{1}{2}$. Also, verify that the bid price of the replicating portfolio is lower than the ask price of the $5\frac{1}{2}$. Consider now the first two STRIPS of Table 2. Again, construct a portfolio of the two STRIPS which replicates the cash

flows of the $5\frac{1}{2}$ and verify that there are no arbitrage opportunities.

Solution:

a. Let X be the number of the $6\frac{7}{8}$ in the replicating portfolio and let Y be the number of the $7\frac{3}{4}$ in the replicating portfolio.

   Then in order to replicate the cash flows from the third bond we need:

   - First cash flow

$$CF_1 = 100 \times \frac{5\frac{1}{2}\%}{2} = 2.750$$
$$CF_1 = X \times (100 + 100 \times \frac{6\frac{7}{8}}{2}\%) + Y \times \frac{7\frac{3}{4}\%}{2} = 103.4375X + 3.875Y$$

   - Second cash flow

$$CF_2 = 100 + 100 \times \frac{5\frac{1}{2}}{2} = 102.750$$
$$CF_2 = Y \times (100 + \frac{7\frac{3}{4}}{2}) = 103.875Y$$

   Two equations and two unknowns, so we can use some simple math to solve for X and Y:

$$X = -1.0470\%$$
$$Y = 98.9170\%$$

   Then we can price the replicating portfolio:

$$\begin{aligned} P(r)^b &= X \times P(1)^b + Y \times P(2)^b \\ &= X \times 101\frac{19}{32} + Y \times 103\frac{24}{32} \\ &= 101.5626 \end{aligned}$$

   While $P_a = 101\frac{19}{32} = 101.5938 > P(r)^b = 101.5626$, the bid price of the replicating portfolio is lower than the ask price of the $5\frac{1}{2}$.

b. Let X be the number of the first STRIPS in the replicating portfolio and let Y be the number of the second STRIPS in the replicating portfolio.

   Then in order to replicate the cash flows from the third bond we need

   - First cash flow

$$CF_1 = 100 \times \frac{5\frac{1}{2}\%}{2} = 2.750$$
$$CF_1 = 100 \times X$$

   - Second cash flow

$$CF_2 = 100 + 100 \times \frac{5\frac{1}{2}}{2} = 102.750$$
$$CF_2 = 100 \times Y$$

   Two equations and two unknowns, so we can use some simple math to solve for X and Y:

$$X = 0.0275$$
$$Y = 1.0275$$

Then we can price the replicating portfolio:

$$\begin{aligned}
P(r)^a &= X \times P(1)^a + Y \times P(2)^a \\
&= X \times 98\frac{9}{32} + Y \times 96\frac{7}{32} \\
&= 101.5675 \\
P(r)^b &= X \times P(1)^b + Y \times P(2)^b \\
&= X \times 98\frac{9}{32} + Y \times 96\frac{6}{32} \\
&= 101.5354
\end{aligned}$$

There are no arbitrage opportunities since:

$$\begin{aligned}
P(r)^a &= 101.5675 > P^b = 101\frac{17}{32} = 101.5313 \\
P(r)^b &= 101.5354 < P^a = 101\frac{19}{32} = 101.5938
\end{aligned}$$

Problem 3
Consider investment in three STRIPS, the first STRIPS matures in six months, the second STRIPS matures in one year, the third STRIPS matures in eighteen months. The price of the first STRIPS is 98:12, the price of the second STRIPS is 96:20, the price of the third STRIPS is 94:26. Based on the prices above, calculate the implied forward rates ${}_1f_1$ and ${}_1f_2$. Assume you expect the one-period and two-period spot rates in six months to equal the corresponding forward rates: $E(y_1') =_1 f1$ and $E(y_2') =_1 f_2$. Also, assume that realized future spot rates match your expectations: $y_1' =_1 f_1$ and $y_2' =_1 f_2$. Calculate the one-period realized rates of return on the three STRIPS.

Solution:

a. First, discount factors are calculated using the prices of the STRIPS:

$$\begin{aligned}
d(1) &= \frac{98\frac{12}{32}}{100} = 98.3750/100 = 0.9838 \\
d(2) &= \frac{96\frac{20}{32}}{100} = 96.6250/100 = 0.9663 \\
d(3) &= \frac{94\frac{26}{32}}{100} = 94.8125/100 = 0.9481
\end{aligned}$$

Then, we can use discount factors to calculate the forward rates:

$$\begin{aligned}
{}_1f_1 &= \frac{d(1)}{d(2)} - 1 = 1.8111\% \\
{}_1f_2 &= (\frac{d(1)}{d(3)})^{\frac{1}{2}} - 1 = 1.8614\%
\end{aligned}$$

b. We know the return from holding a t-period zero-coupon bond with face value of 1 for one period:

$$RET = \frac{P_{t-1}}{P_t} - 1$$

- For the first STRIPS

$$RET_1 = \frac{100}{98.3750} - 1 = \frac{100}{98.3750} - 1 = 1.6518\%$$

- For the second STRIPS

$$RET_2 = \frac{100/(1+y_1')}{96.6250} - 1 = \frac{100/(1+_1f_1,)}{96.6250} - 1 = \frac{100/(1+1.8111\%)}{96.6250} - 1 = 1.6518\%$$

- For the third STRIPS

$$RET_3 = \frac{100/(1+y_2')^2}{94.8125} - 1 = \frac{100/(1+_1f_2,)^2}{94.8125} - 1 = \frac{100/(1+1.8614\%)^2}{94.8125} - 1 = 1.6518\%$$

Problem 4

(Joseph Jett exercise) Consider two Treasury coupon bonds maturing in six months and one year, with (annualized) coupon rates $5\frac{7}{8}$ and $4\frac{1}{2}$, respectively. The corresponding prices are 101:16 and 100:10, respectively. Calculate the discount factors d1 and d2 , the spot rates y_1 and y_2 , and the forward rate 1f1 implied by the above prices. Now, consider the STRIPS that make up the $4\frac{1}{2}$ and assume they are fairly priced (no arbitrage opportunities). Assume the $4\frac{1}{2}$ coupon bond (FV =$100,000) is in your portfolio and you use it to create the two STRIPS. Your firm treats the reconstitution of Treasuries as a sale of STRIPS and the purchase of a coupon bond. Assume that you can record the sale at the forward price of the STRIPS for delivery immediately before the next coupon payment; while the purchase of the coupon bond is recorded at the current spot price. Calculate the profit or loss that you would show as a result of the transaction.

Solution:

- For the two Treasury coupon bonds:
  - First, we calculate the prices and coupon payments for these two bonds:

$$P(1) = 100 \times 101\frac{16}{32}\% = 101.500$$
$$C(1) = 100 \times \frac{5\frac{7}{8}\%}{2} = 2.9375$$
$$P(2) = 100 \times 100\frac{10}{32}\% = 100.3125$$
$$C(2) = 100 \times \frac{4\frac{1}{2}\%}{2} = 2.2500$$

  - Second, we use Treasury coupon bonds to derive the discount factors:

$$d(1) = \frac{101.500}{102.9375} = 0.9860$$
$$d(2) = \frac{100.3125 - (0.9860 \times 2.2500)}{102.2500} = 0.9594$$

  - Then, spot rates can be calculated based on the discount factors:

$$y_1 = (\frac{1}{0.9860}) - 1 = 1.4163\%$$
$$y_2 = (\frac{1}{0.9594})^{\frac{1}{2}} - 1 = 2.0964\%$$

  - Finally, the implied 6-month forward rates can be obtained using spot rates:

$$_1f_1 = \frac{(1+y_2)^2}{(1+y_1)} - 1 = \frac{d(1)}{d(2)} - 1 = 2.7812\%$$

- Sale of STRIPS: Record the sale at the forward price of the STRIPS for delivery immediately before the next coupon:

$$CF_1 = C(2) = 2.2500$$
$$CF_2 = \frac{C(2)+100}{1+_1f_1} = \frac{102.2500}{1+2.7812\%} = 99.4832$$
$$Gain = 1000 \times (CF_1 + CF_2) = 100 \times (2.2500 + 99.4832) = \$101,733.1820$$

- The result of the transaction:

$$\begin{aligned} Profit &= Gain - Cost \\ &= 101,733.1820 - 100.3125 \times 1000 \\ &= \$1,420.6820 \end{aligned}$$