2-4

10. $\frac{\partial M}{dy} = 3y^2 = \frac{\partial N}{dx}$

$M = x^3 + y^3$

$N = 3xy^2$

$\int M \partial x = \frac{x^4}{4} + xy^3 + g(y) = f(x,y)$

$\frac{\partial f(x,y)}{\partial y} = 3xy^2 + g'(y) = 3xy^2 \Rightarrow g'(y) = 0$

$f(x,y) = \frac{x^4}{4} + xy^3 \Rightarrow \underline{\frac{x^4}{4} + xy^3 = c}$, Yes, exact

22. $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$

$M = e^x + y$

$N = 2 + x + ye^y$

$\int M \partial x = e^x + xy + g(y) = f(x,y)$

$\frac{\partial f(x,y)}{\partial y} = x + g'(y) = 2 + x + ye^y$

$g'(y) = 2 + ye^y \Rightarrow g(y) = 2y + ye^y\ e^y + c$

$f(x,y) = e^x + xy + 2y + ye^y - e^y$

$e^x + xy + 2y + ye^y - e^y = c \Rightarrow 0 + 1 + 2 + e - e = c \Rightarrow c = 3$

ans. $\underline{e^x + xy + 2y + ye^y - e^y = 3}$

2-5

18. $\frac{dy}{dx} - \frac{1+x}{x} y = y^2$

$u = y^{-1} \Rightarrow y = u^{-1} \Rightarrow \frac{dy}{dx} = -u^{-2} \frac{du}{dx}$

$-u^{-2}\frac{du}{dx} - \frac{1+x}{x} u^{-1} = u^{-2}$

$-\frac{du}{dx} - \frac{1+x}{x} u = 1$

$If = e^{\int \frac{1+x}{x} dx}$

$= e^{\int \frac{1}{x}+1 dx}$

$= e^{\ln|x|+x}$

$= |x| e^{x}$

$u = \frac{c}{xe^x} - \frac{1}{xe^x}\int xe^x\, dx$

$\Rightarrow u = \frac{c}{xe^x} - \frac{1}{xe^x}(xe^x - e^x)$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$xe^x - \int e^x dx$

$= xe^x - e^x$

$\Rightarrow y = \frac{xe^x}{c + e^x - xe^x}$

26.

let $u = x+y \Rightarrow \frac{du}{dx} = \frac{dy}{dx} + 1$

$\frac{dy}{dx} = \sin(x+y) \Rightarrow \frac{du}{dx} - 1 = \sin u$

$\frac{du}{dx} = (\sin u + 1) \Rightarrow \frac{du}{\sin u + 1} = dx$

$\frac{1 - \sin u\, du}{(1+\sin u)(1-\sin u)} = dx \Rightarrow \frac{1-\sin u}{\cos^2 u} du = dx$

$(\sec^2 u - \tan u \sec u)\, du = dx$

$\tan u - \sec u = x + C \Rightarrow \tan(x+y) - \sec(x+y) = x + C$

3-1

28.

$$\begin{vmatrix} \cos\ln x & \sin\ln x \\ \frac{-\sin\ln x}{x} & \frac{\cos\ln x}{x} \end{vmatrix} = \frac{\cos\ln x\cos\ln x + \sin\ln x\sin\ln x}{x}$$

$$= \frac{\cos(\ln x - \ln x)}{x} = \frac{\cos 0}{x} = \frac{1}{x} \Rightarrow (0, \infty)$$

linear independent

$$y = \sin\ln x,\ dy = \frac{\cos\ln x}{x},\ d^2y = \frac{\frac{\sin\ln x}{x}x - \cos\ln x}{x^2}$$

$$= \frac{\sin\ln x - \cos\ln x}{x^2}$$

$$y = \cos\ln x,\ dy = \frac{-\sin\ln x}{x},\ d^2y = \frac{-\cos\ln x - \sin\ln x}{x^2}$$

$$\sin\ln x - \cos\ln x - \sin\ln x + \cos\ln x = 0$$

$$-\cos\ln x - \sin\ln x + \cos\ln x + \sin\ln x = 0 \quad \text{Solution}$$

Solution and independent = General

3-2

4.

$$y = u\sin 3x$$

$$y' = u'\sin 3x + 3u\cos 3x$$

$$y'' = u''\sin 3x + 3u'\cos 3x + 3(u'\cos 3x - 3u\sin 3x)$$

$$= u''\sin 3x + 6u'\cos 3x - 9u\sin 3x$$

$$u''\sin 3x + 6u'\cos 3x - 9u\sin 3x + 9u\sin 3x = 0$$

$$u''\sin 3x + 6u'\cos 3x = 0$$

let $w = u'$

$$w'\sin 3x + 6w\cos 3x = 0$$

$$w' + \frac{6\cos 3x}{\sin 3x}w = 0$$

If: $e^{\int 6\cot 3x\,dx}$

$$= e^{2(\ln\sin 3x)}$$

$$W = \frac{c_1}{(\sin 3x)^2} + \frac{1}{(\sin 3x)}\int (\sin 3x)^2\, 0 = (\sin 3x)^2$$

$$W = \frac{c}{(\sin 3x)^2} = u' = c_1 (\csc 3x)^2$$

$$u = \int u'\,dx = \int c_1 (\csc 3x)^2 = \frac{-c_1 \cot 3x}{3} + C_2$$

$$(c_1, c_2) \Rightarrow (1, 0) \Rightarrow u = -\frac{\cot 3x}{3}$$

$$y_2(x) = -\frac{\cot 3x}{3} \sin 3x = -\frac{\cos 3x}{3}$$

3-3

2.

$$m^2 - 36 = 0$$

$$(m+6)(m-6) = 0$$

$$m = 6 \vee -6$$

$$y = c_1 e^{6x} + c_2 e^{-6x}$$

3-5

12. $m^2 - 4m + 3 = (m-3)(m-1)$

$$c_1 e^{3x} + c_2 e^{x}$$

$$W = \begin{vmatrix} e^{3x} & e^{x} \\ 3e^{3x} & e^{x} \end{vmatrix} = e^{4x} - 3e^{4x} = -2e^{4x}$$

$$W_1 \begin{vmatrix} 0 & e^{x} \\ e^{x} & e^{x} \end{vmatrix} = -e^{2x}$$

$$W_2 = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & e^{x} \end{vmatrix} = e^{4x}$$

$$u_1' = \frac{W_1}{W} = \frac{1}{2e^{2x}}, \quad u_2' = \frac{W_2}{W} = -\frac{e^{4x}}{2e^{4x}} = -\frac{1}{2}$$

$$u_1 = -\frac{e^{2x}}{4}, \quad u_2 = \frac{-1}{2}x$$

$$y = c_1e^{3x} + c_2e^{x} - e^{x} - \frac{1}{2}xe^{x}$$

24.

$$2m^2+m-1 = (2m-1)(m+1)$$

$$y_c = c_1e^{\frac{1}{2}x} + c_2e^{-x}$$

$$W = \begin{vmatrix} e^{\frac{1}{2}x} & e^{-x} \\ \frac{1}{2}e^{\frac{1}{2}x} & -e^{-x} \end{vmatrix} = -e^{-\frac{1}{2}x} - \frac{1}{2}e^{-\frac{1}{2}x} = -\frac{3}{2}e^{-\frac{1}{2}x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{x+1}{2} & -e^{-x} \end{vmatrix} = -\frac{(x+1)e^{-x}}{2}$$

$$W_2 = \begin{vmatrix} e^{\frac{1}{2}x} & 0 \\ \frac{1}{2}e^{\frac{1}{2}x} & \frac{x+1}{2} \end{vmatrix} = \frac{(x+1)}{2}e^{\frac{1}{2}x}$$

$$u_1' = \frac{W_1}{W} = \frac{-\frac{(x+1)}{2}e^{-x}}{-\frac{3}{2}e^{-\frac{1}{2}x}} = \frac{(x+1)}{3}e^{-\frac{1}{2}x}$$

$$u = (x+1), dv = e^{-\frac{1}{2}x} \Rightarrow du = dx, v = -2e^{-\frac{1}{2}x}$$

$$-2(x+1)e^{-\frac{1}{2}x} + 2\int e^{-\frac{1}{2}x}dx$$

$$-2(x+1)e^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x} \Rightarrow u_1 = -2e^{-\frac{1}{2}x} - \frac{2}{3}xe^{-\frac{1}{2}x}$$

$$u_2' = \frac{W_2}{W} = \frac{\frac{x+1}{2}e^{\frac{1}{2}x}}{-\frac{3}{2}e^{-\frac{1}{2}x}} = \frac{-(x+1)}{3}e^x$$

$$u = x+1,\ dv = e^x \Rightarrow du = dx,\ v = e^x$$

$$(x+1)e^x - \int e^x dx = (x+1)e^x - e^x = xe^x$$

$$u_2 = \frac{-1}{3}xe^x$$

$$y = c_1 e^{\frac{1}{2}x} + c_2 e^{-x} - \frac{2}{3}x - \frac{1}{3}x - 2$$

$$y = c_1 e^{\frac{1}{2}x} + c_2 e^{-x} - x - 2$$

$$y' = \frac{1}{2}c_1 e^{\frac{1}{2}x} + -c_2 e^{-x} - 1$$

$$y(0) = 1$$

$$y'(0) = 0$$

$$c_1 + c_2 - 2 = 1$$

$$\frac{1}{2}c_1 - c_2 = 1$$

$$c_1 + c_2 = 3$$

$$\frac{1}{2}c_1 - c_2 = 1$$

$$c_1 = \frac{8}{3}$$

$$c_2 = \frac{1}{3}$$

$$y = \frac{8}{3}e^{\frac{1}{2}x} + \frac{1}{3}e^{-x} - x - 2$$

3-6

8. $m(m-1)+3m-4 = m^2-m+3m-4 = m^2+2m-4$

$y = c_1 x^{-1+\sqrt{5}} + c_2 x^{-1-\sqrt{5}}$

14. $m(m-1)-7m+41=0$

$m^2-m-7m+41=0$

$m^2-8m+41=0$

$\frac{4 \pm \sqrt{64-4\times41}}{2} = \frac{4 \pm \sqrt{-100}}{2} = 2 \pm 5i$

$y = x^2[c_1\cos(5\ln x) + c_2\sin(5\ln x)]$

24. $m(m-1)+m-1 = m^2-1 = (m-1)(m+1)=0 \Rightarrow m=1 \vee -1$

$y_c = c_1 x + c_2 x^{-1}$

$y'' + \frac{y'}{x} - \frac{y}{x^2} = \frac{1}{x^2(x+1)}$

$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & \frac{-1}{x^2} \end{vmatrix} = -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x}$

$W_1 = \begin{vmatrix} 0 & \frac{1}{x} \\ \frac{1}{x^2(x+1)} & \frac{-1}{x^2} \end{vmatrix} = -\frac{1}{x^3(x+1)}$

$W_2 \begin{vmatrix} x & 0 \\ 1 & \frac{1}{x^2(x+1)} \end{vmatrix} = \frac{1}{x(x+1)}$

$$u_1' = \frac{W_1}{W} = \frac{1}{2x^2(x+1)} = \frac{1}{2}\left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right), \quad u_2' = \frac{W_2}{W} = \frac{-1}{2(x+1)}$$

$$u_1 = \frac{1}{2}\left(-\ln x - \frac{1}{x} + \ln(x+1)\right), \quad u_2 = \frac{-1}{2}\ln(x+1)$$

$$y = c_1 x + c_2 x^{-1} + \frac{1}{2}x\left(\ln\frac{x+1}{x} - \frac{1}{x}\right) - x^{-1}\frac{1}{2}\ln(x+1)$$

$$y = c_1 x + c_2 x^{-1} + \frac{1}{2}\left(x\ln\frac{x+1}{x} - \frac{\ln(x+1)}{x} - 1\right)$$

3-9.

14.

$$y'' + \alpha^2 y = 0$$

$$y = c_1\cos\alpha x + c_2\sin\alpha x$$

$$c_1 = 0$$

$$y = c_2\sin\alpha x$$

$$y' = -\alpha c_1\sin\alpha x + \alpha c_2\cos\alpha x$$

$$0 = -\alpha c_1\sin\frac{\pi}{2}\alpha + \alpha c_2\cos\frac{\pi}{2}\alpha$$

$$0 = \alpha c_2\cos\frac{\pi}{2}\alpha$$

$$\alpha = (2k+1) \quad k \in N$$

$$\lambda = \alpha^2 = (2k+1)^2, \quad c_2\sin(2k+1)x$$