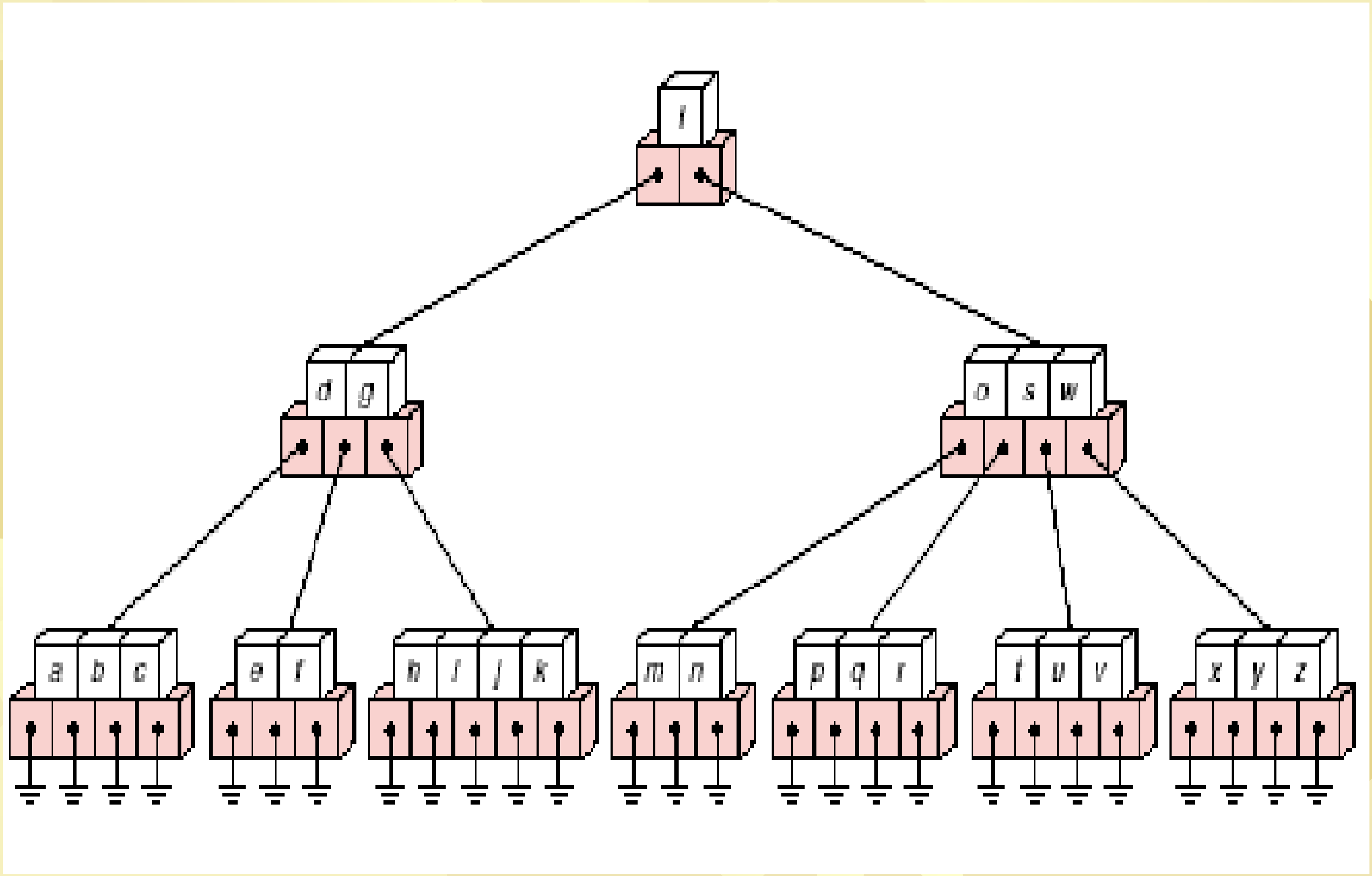
Balanced Multiway Trees (B-Trees)

- ***A B-tree of order m*** **is an m-way search tree in which**
  - **All leaves are on the same level.**
  - **All internal nodes except the root have at most m nonempty children, and at least $\lceil m/2 \rceil$ nonempty children.**
  - **The number of keys in each internal node is one less than the number of its nonempty children, and these keys partition the keys in the children in the fashion of a search tree.**
  - **The root has at most m children, but may have as few as 2 if it is not a leaf, or none if the tree consists of the root alone.**

Balanced Multiway Trees (B-Trees)



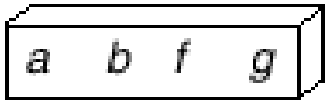
Insertion into a B-Tree

- In contrast to binary search trees, B-trees are not allowed to grow at their leaves; instead, they are forced to grow at the root. General insertion method:
  - Search the tree for the new key. This search (if the key is truly new) will terminate in failure at a leaf.
  - Insert the new key into the leaf node. If the node was not previously full, then the insertion is finished.

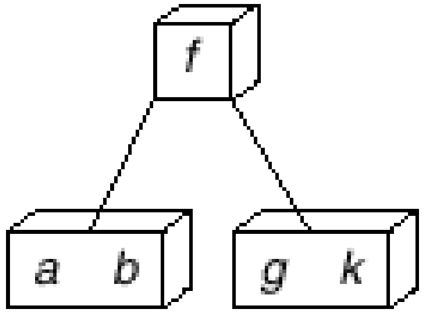
- When a key is added to a full node, then the node splits into two nodes, side by side on the same level, except that the median key is not put into either of the two new nodes.
- When a node splits, move up one level, insert the median key into this parent node, and repeat the splitting process if necessary.
- When a key is added to a full root, then the root splits in two and the median key sent upward becomes a new root. This is the only time when the B-tree grows in height.

Growth of a B-Tree

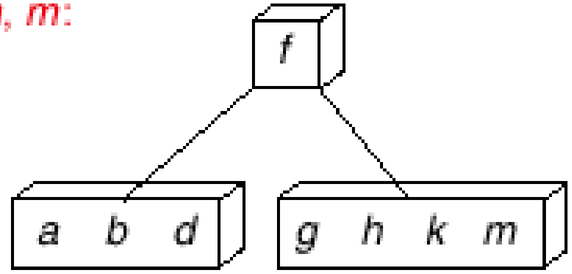
1. *a, g, f, b:*



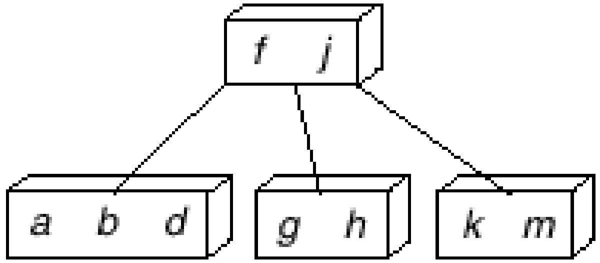
2. *k:*



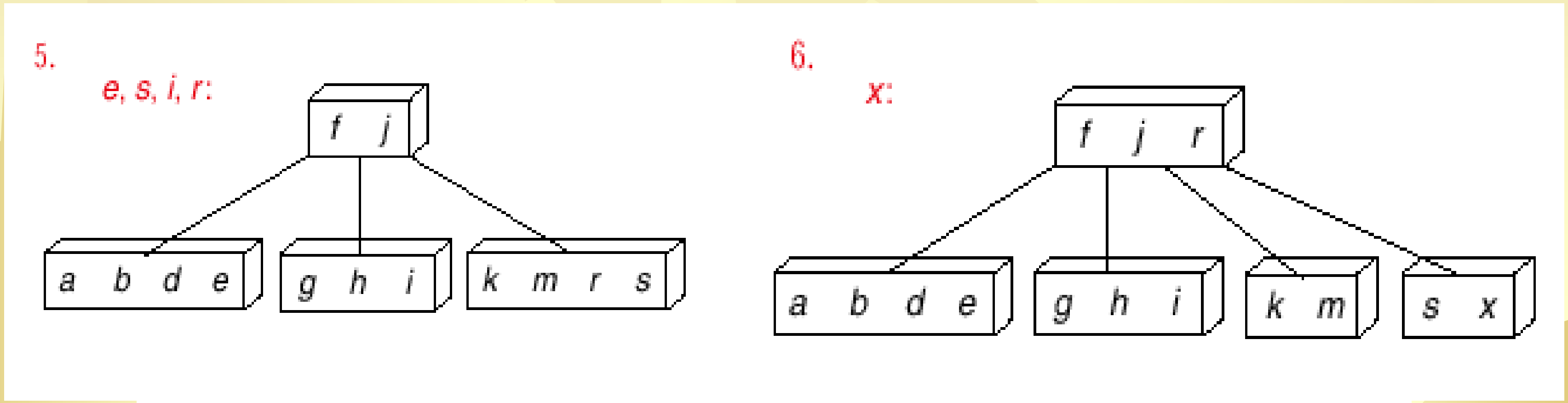
3. *d, h, m:*

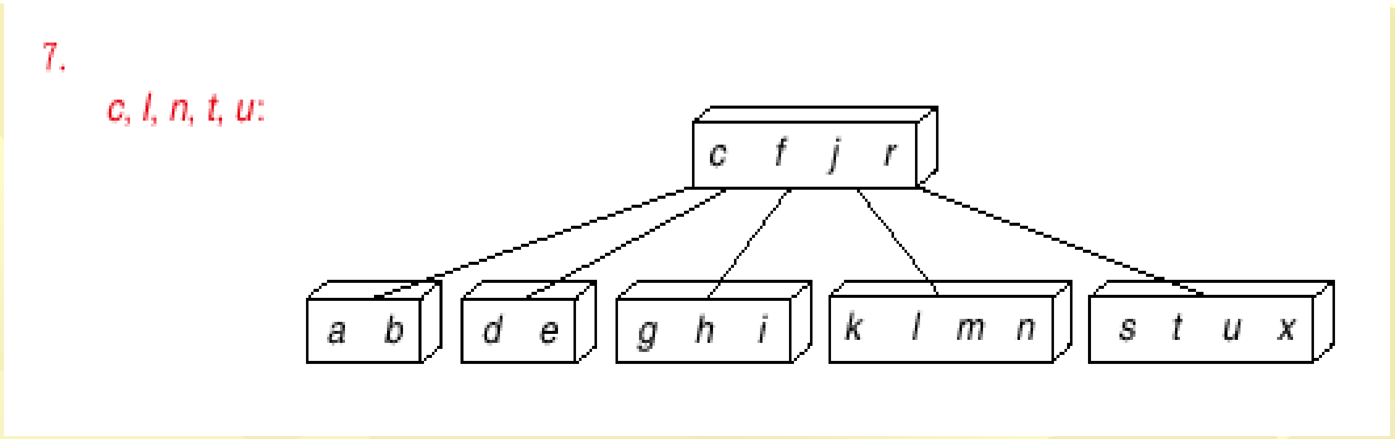


4. *j:*

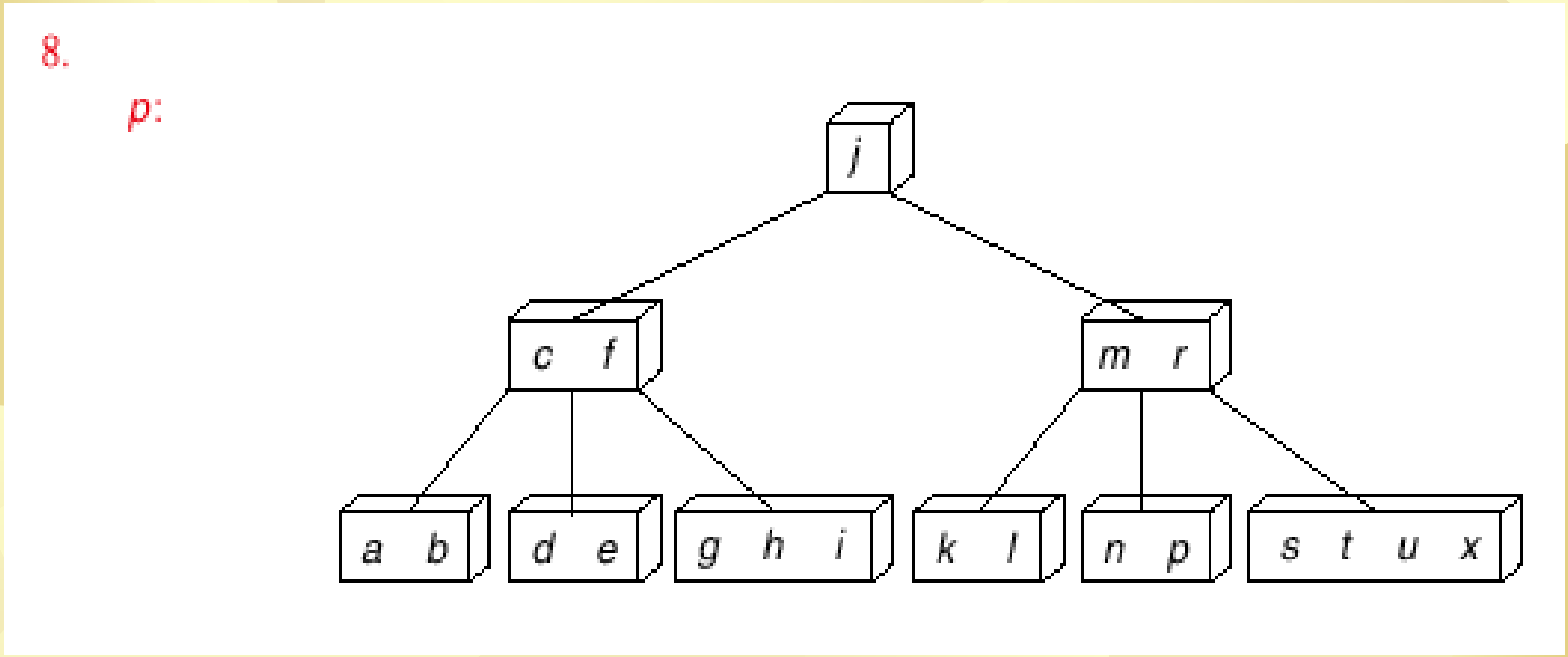


Growth of a B-Tree





Growth of a B-Tree



B-Tree Declarations in C++

- We add the order as a second template parameter.
- For example,

```
B_tree<int, 5> sample_tree;
```

declares sample_tree as a B_tree of order 5 that holds integer records.

B-tree class declaration

```
template <class Record, int order>
class B_tree {
   public:             // Add public methods
   private:                      // data members
      B_node<Record, order> *root;
                 // Add private auxiliary functions here.
};
```

Node declaration

```
template <class Record, int order>
struct B_node {
                // data members
   int count;
   Record data [ order - 1];
   B_node <Record, order> *branch[order];
                // constructor
   B_node( );
};
```

Conventions

- count gives the number of records in the B_node.
- If count is nonzero then the node has count + 1 children.
- branch[0] points to the subtree containing all records with keys less than that in data[0].

Conventions

- For 1≤position ≤ count - 1, branch[position] points to the subtree with keys strictly between those in the subtrees pointed to by data[position - 1] and data[position].
- branch[count] points to the subtree with keys greater than that of data[count - 1].

Searching in a B-Tree

```
template <class Record, int order>
Error_code B_tree<Record, order> ::
    search_tree(Record &target)
/* Post: If there is an entry in the B-tree whose key
    matches that in target , the parameter target is
    replaced by the corresponding Record from the
    B-tree and a code of success is returned.
    Otherwise a code of not present is returned.
    Uses: recursive search tree */
{
    return recursive_search_tree(root, target);
}
```

Searching in a B-Tree

```
template <class Record, int order>
Error_code B_tree<Record, order> ::
    recursive_search_tree(B_node<Record,
    order> *current, Record &target)
/* Pre: current is either NULL or points to a subtree
    of the B_tree .
  Post: If the Key of target is not in the subtree, a
    code of not_present is returned. Otherwise, a
    code of success is returned and target is set to
    the corresponding Record of the subtree.
 Uses: recursive_search_tree recursively and
    search_node */
```

```
{
    Error_ code result = not_present;
    int position;
    if (current != NULL) {
        result = search_node(current, target,
                                              position);
        if (result == not_present)
                result = Recursive_ search_ tree (
                current->branch[position], target);
        else
                target = current->data[position];
    }
    return result;
}
```

Searching in a B-Tree

- Recursive_ search_ tree
- This function has been written recursively to exhibit the similarity of its structure to that of the insertion function.
- The recursion is tail recursion and can easily be replaced by iteration.

Searching a Node

- search_node
- This function determines if the target is present in the current node, and, if not, finds which of the count+1 branches will contain the target key.

```
template <class Record, int order>
Error_code B_tree<Record,
    order> ::search_node(
B_node<Record, order> *current, const Record
    &target, int &position)
/* Pre: current points to a node of a B_tree .
    Post: If the Key of target is found in*current ,
    then a code of success is returned,the
    parameter position is set to the index of target ,
    and the corresponding Record is copied to
    target . Otherwise, a code of  not present is
    returned,And position is set to the branch index
    on which to continue the search.
    Uses: Methods of class Record . */
```

```
{
   position = 0;
   while (position < current->count &&
                target > current->data[position])
        position++;         //Perform a sequential
   search                           //through the keys.
   if (position < current->count &&
                target == current->data[position])
        return success;
   else
        return not_present;
}
```

Searching a Node

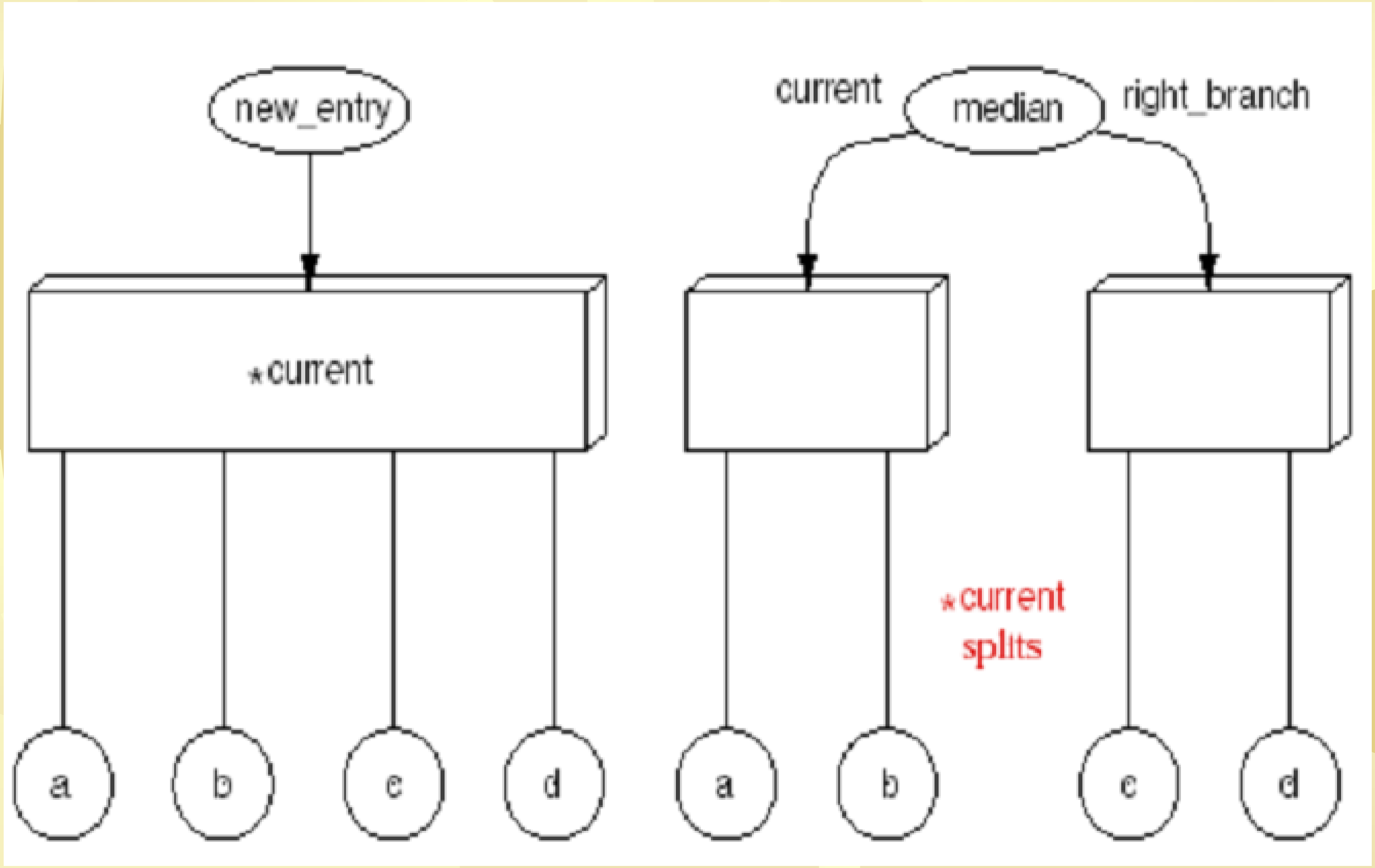
- For B-trees of large order, this function should be modified to use binary search instead of sequential search.
- Another possibility is to use a linked binary search tree instead of a sequential array of entries for each node.

Insertion: Parameters and push_down

- Insertion is done with recursion in a function called push_down.
- We require that the record new_entry being inserted is not already present in the tree.
- The recursive function push_down uses three more output parameters.
- current is the root of the current subtree under consideration.

- If *current splits to accommodate new_entry, push_ down returns a code of overflow, and the following come into use:
  - The old node *current contains the left half of the entries.
  - median gives the median record.
  - right_branch points to a new node containing the right half of the former *current.

Insertion: Parameters and push_down



Public Insertion Method

```
template <class Record, int order>
Error_code B_tree<Record, order> ::
    insert(const Record &new_entry)
/* Post: If the Key of new_entry is already in the
    B_tree , a code of duplicate_error is returned.
    Otherwise, a code of success is returned and the
    Record new_entry is inserted into the B-tree in
    such a way that the properties of a B-tree are
    preserved.
   Uses: Methods of struct B_node and the auxiliary
    function push_down . */
```

```
{
   Record median;
   B_node<Record, order> *right_branch,
   *new_root;
   Error_code result = push_down(root,
        new_entry, median, right_branch);
if (result == overflow) {
    // The whole tree grows in height.
    //Make a brand new_root for the whole B-tree.
    new_root = new B_node<Record, order>;
    new_root->count = 1;
    new_ root->data[0] = median;
    new_ root->branch[0] = root;
    new_ root->branch[1] = right_branch;
    root = new_ root;
    result = success;
}
return result;
}
```

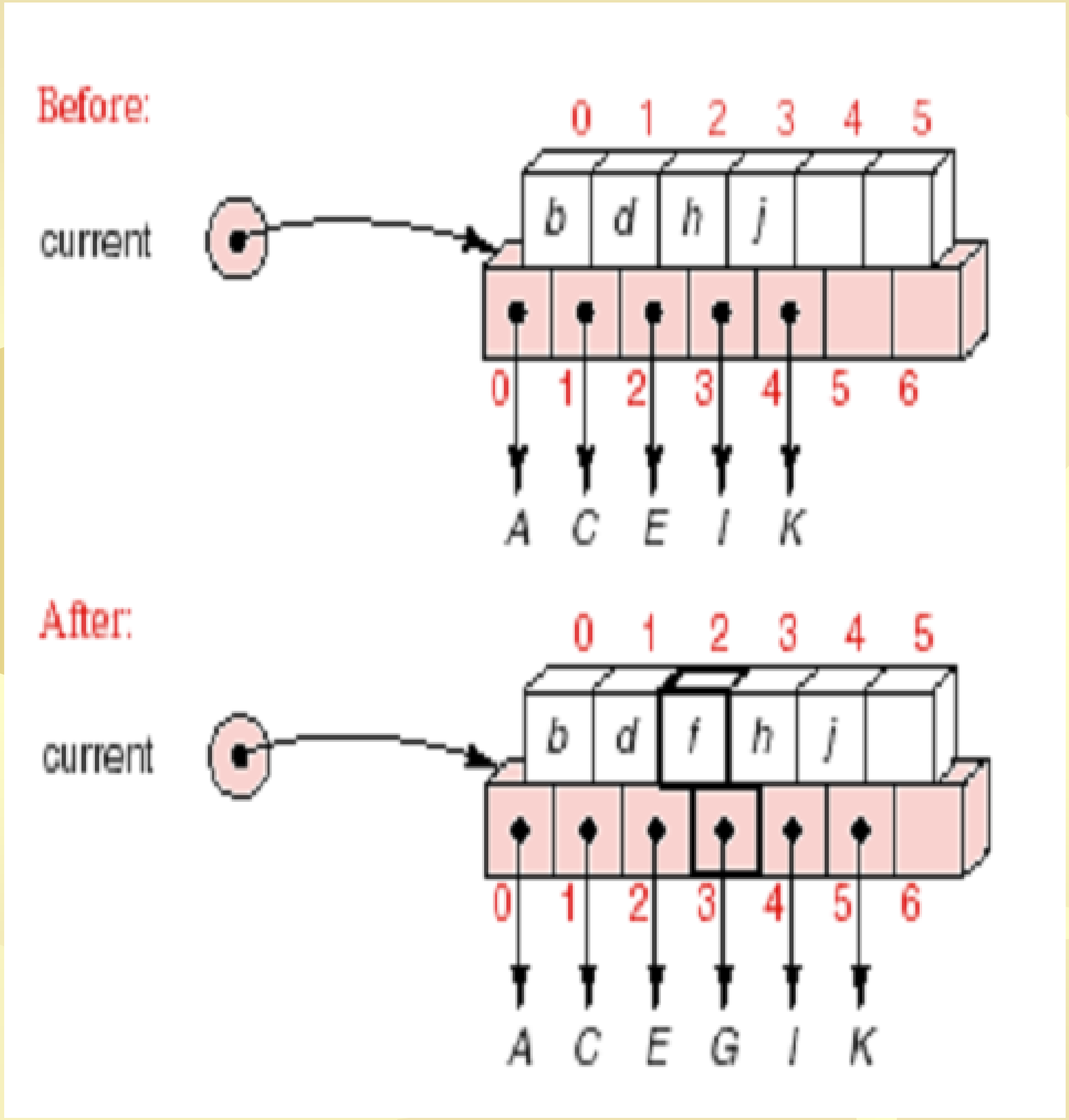
Recursive Insertion into a Subtree

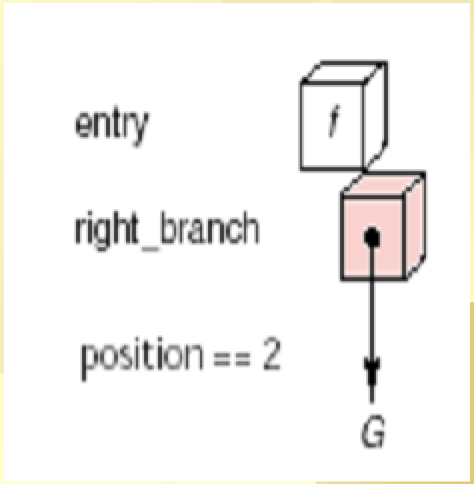
```
template <class Record, int order>
Error_code B_tree<Record, order> ::
   push_down(
         B_node<Record, order> *current,
         const Record &new_entry,
         Record &median,
         B_node<Record,
   order>*&right_branch)
```

```
/* Pre: current is either NULL or points to a node of
   a B_tree .
Post: If an entry with a Key matching that of
   new_entry is in the subtree to which current
   points, a code of duplicate_error is returned.
   Otherwise,new_entry is inserted into the subtree:
   If this causes the height of the subtree to grow,a
   code of overflow is returned, and the Record
   median is extracted to be reinserted higher in
   the B-tree, together with the subtree
   right_branch on its right. If the height does not
   grow, a code of success is returned.
Uses: Functions push_down (called recursively),
   search_node ,split_node , and push_in . */
```

```
{
   Error_code result;
   int position;
   if (current == NULL) {
         // Since we cannot insert in an empty tree,
         //the recursion terminates.
         median = new_entry;
         right_branch = NULL;
         result = overflow;
}
```

```
else {                    // Search the current node.
if (search_node(current, new_entry, position)
    == success)
    result = duplicate_error;
else {
    Record extra_entry;
    B_node<Record, order> *extra_branch;
    result = push_down(
                  current->branch [position],
                  new_entry,
                  extra_entry,
                  extra_branch);
    if (result == overflow) {
// Record extra_entry now must be added to current
        if (current->count < order - 1) {
                result = success;
                push_in(current, extra_entry,
                                extra_branch, position);}
        else split_node( current, extra_entry,
                        extra_branch, position,
                        right_branch, median);
// Record median and its right_ branch will go up to a higher
    node.
        }}}
    return result;
}
```


Before:
0 1 2 3 4 5
current
b d h j
0 1 2 3 4 5 6
A C E I K
After:
0 1 2 3 4 5
current
b d f h j
0 1 2 3 4 5 6
A C E G I K

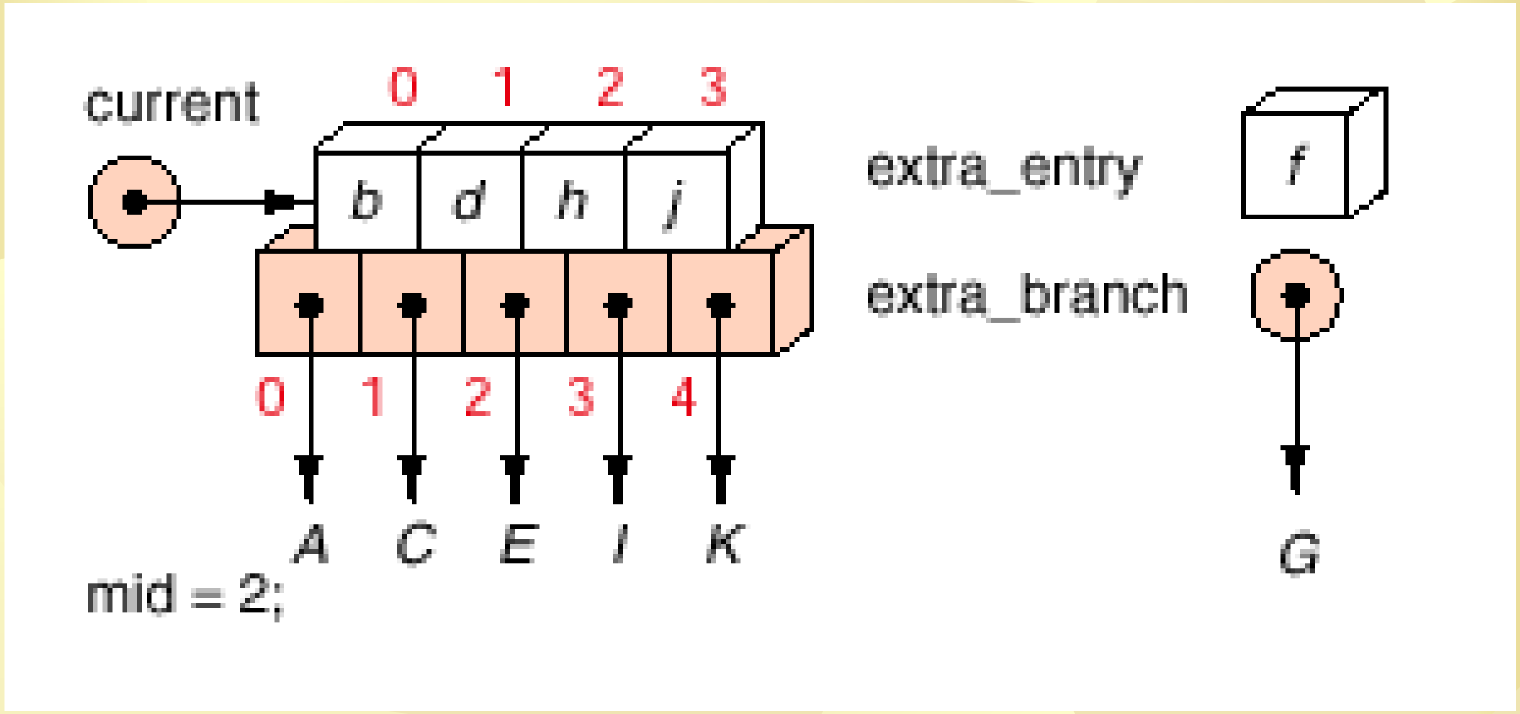

entry
f
right_branch
position == 2
G

```
template <class Record, int order>
void B_tree<Record, order> ::
push_in(   B_node<Record,  order> *current,
           const Record &entry,
           B_node<Record,        order>
   *right_branch,
           int position)
/*  Pre:  current points to a node of a B_tree . The
   node*current is not full and entry belongs
   in*current at index position .
Post: entry has been inserted along with its right-
   hand  branch right_branch into *current at
   index position . */
```

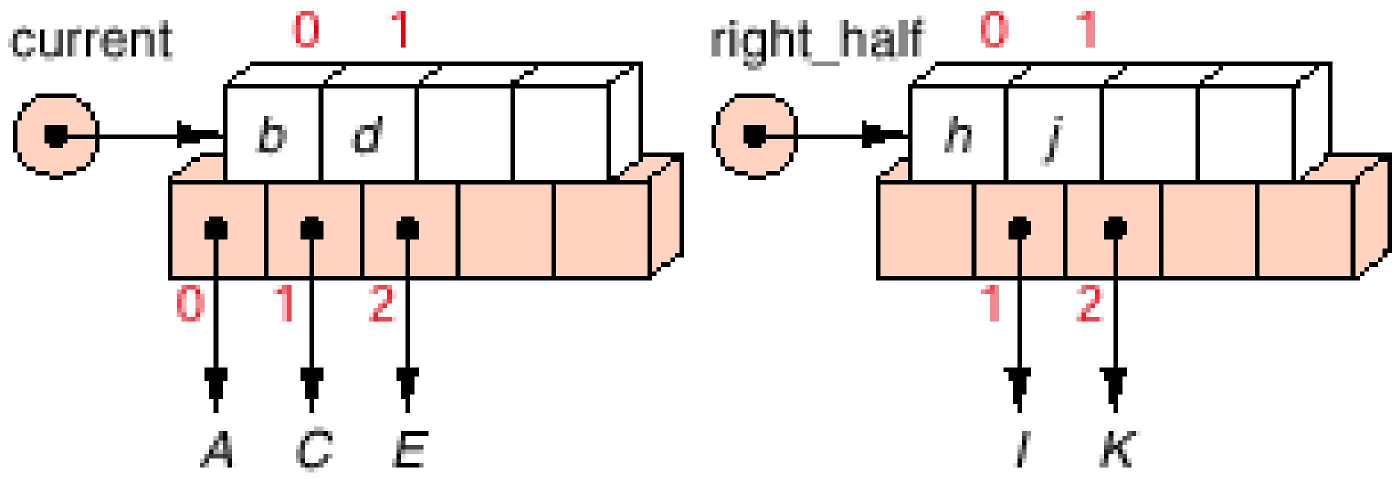
```
{
   for (int i = current->count; i > position; i--) {
                                                  // Shift all later data to the right.
        current->data[i] = current->data[i-1];
        current->branch[ i+1] = current->branch[i];
   }
   current->data[position] = entry;
   current->branch[position + 1] = right_branch;
   current->count ++;
}
```

Example of Splitting a Full Node

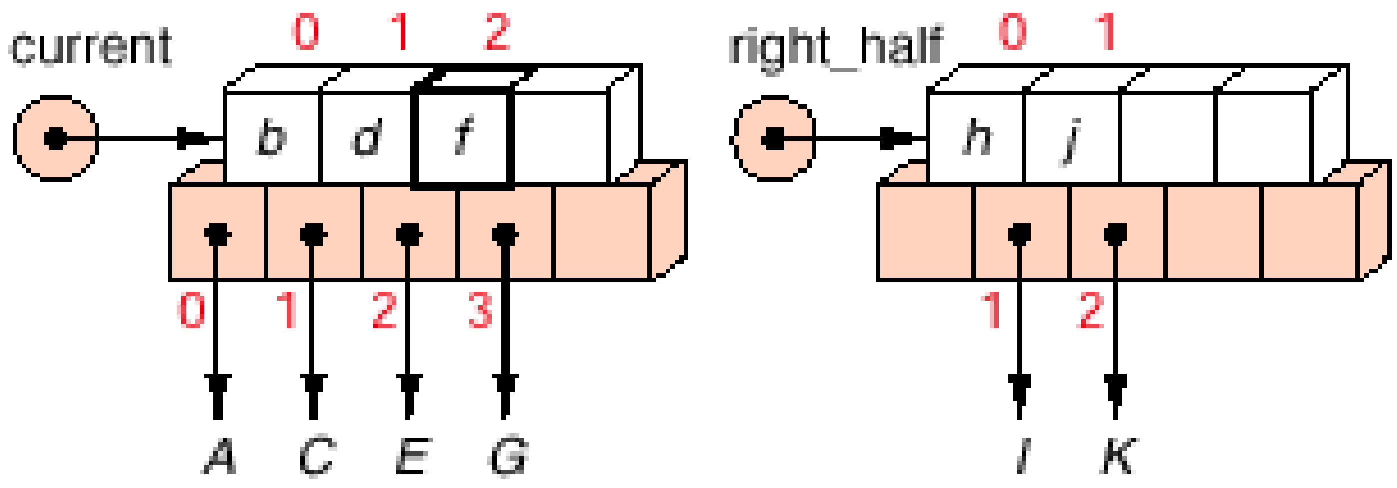
- **Case 1: position == 2; order == 5; (extra_entry belongs in left half.)**



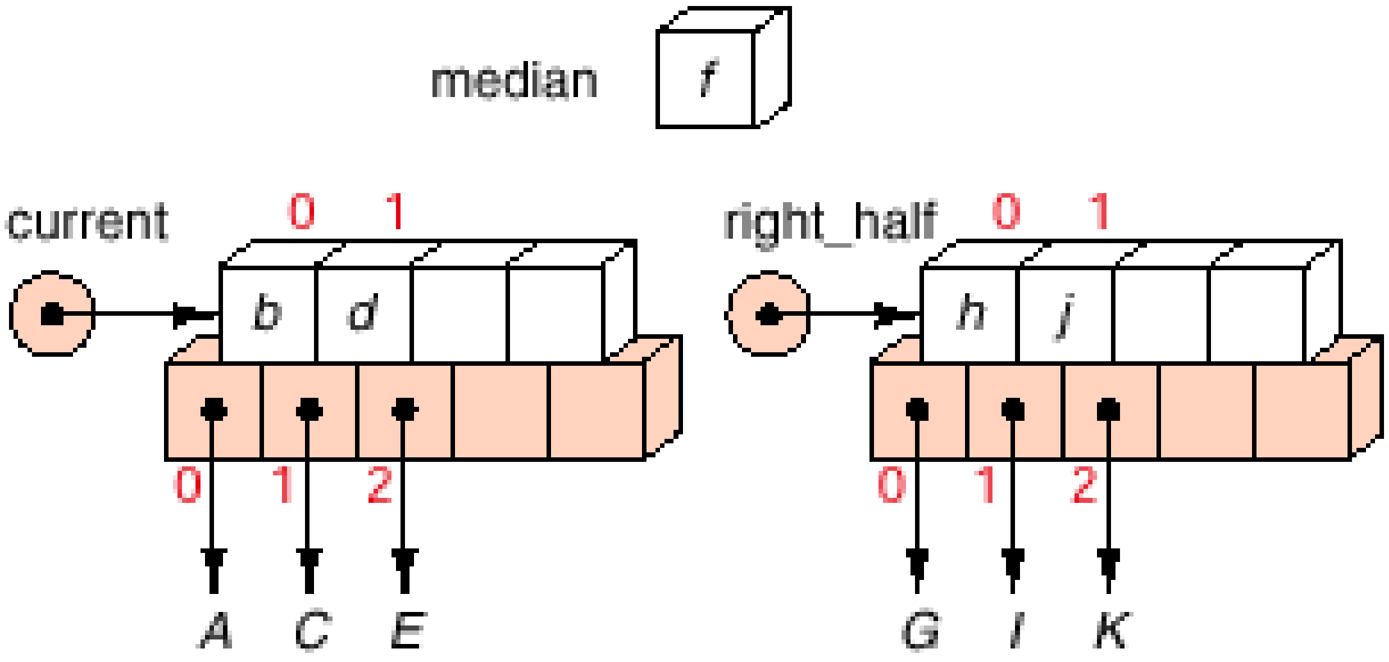
Shift entries right:



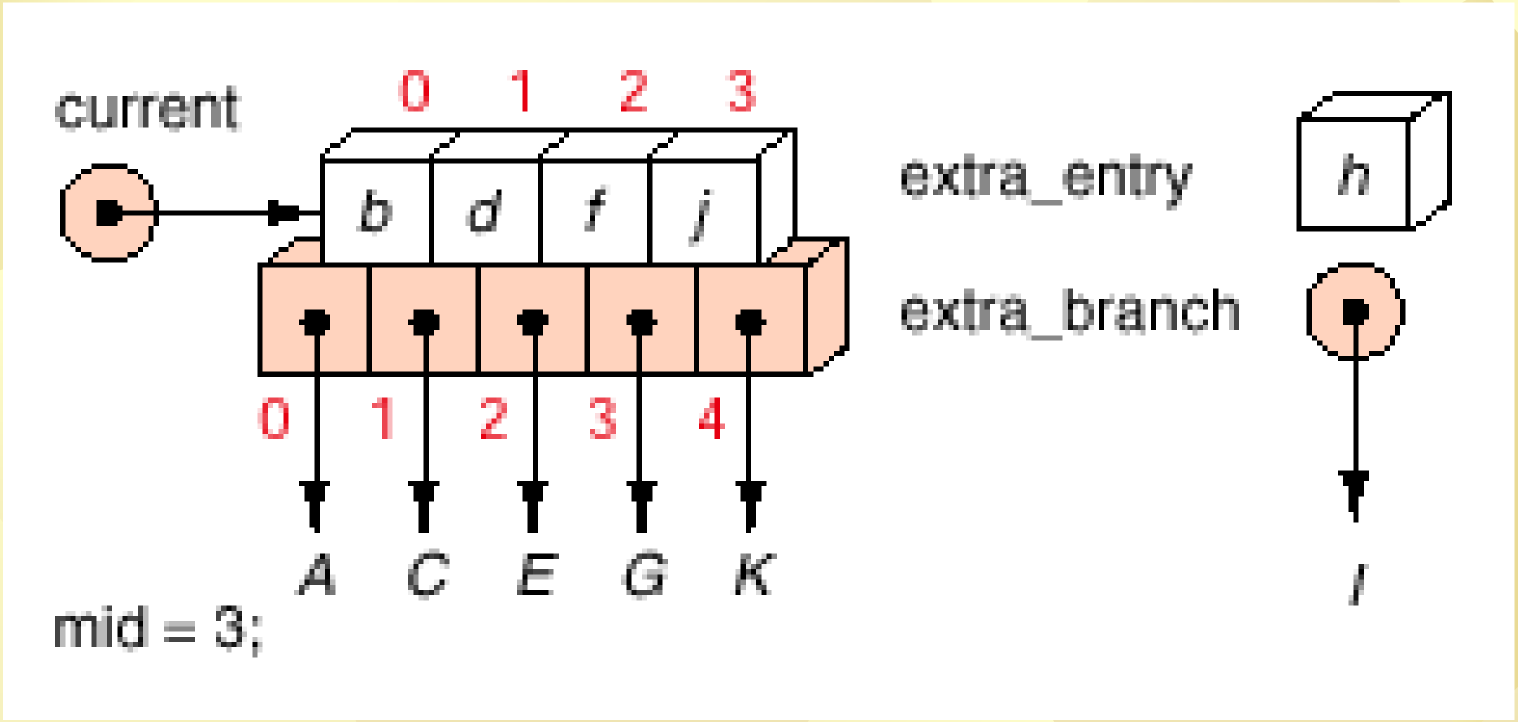
Insert extra_entry and extra_branch:



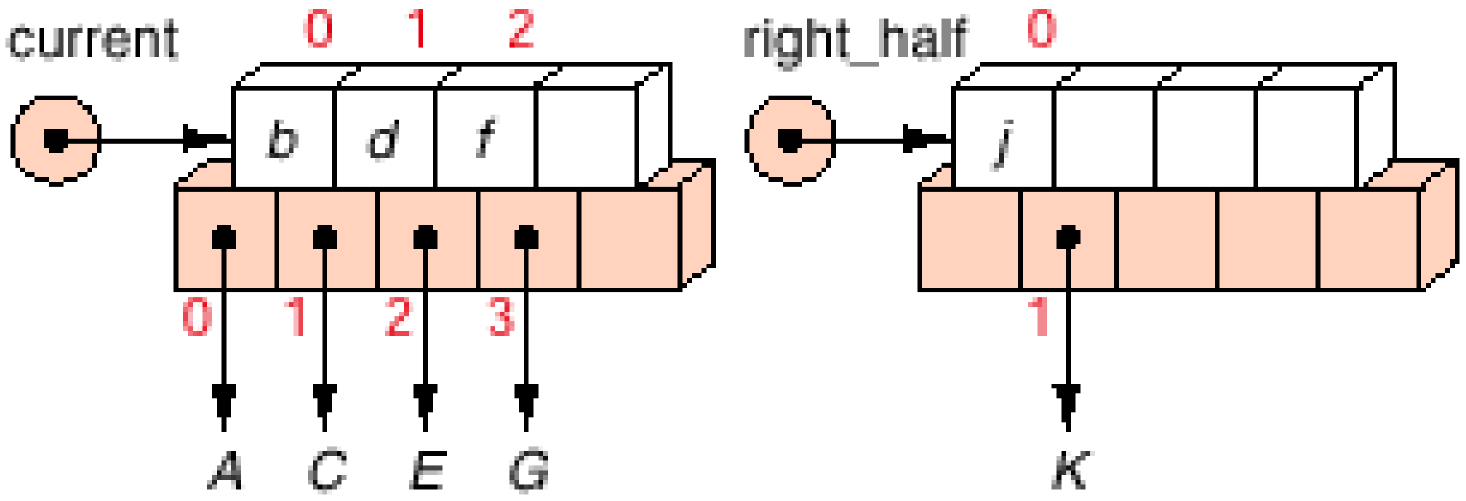
Remove median; move branch:



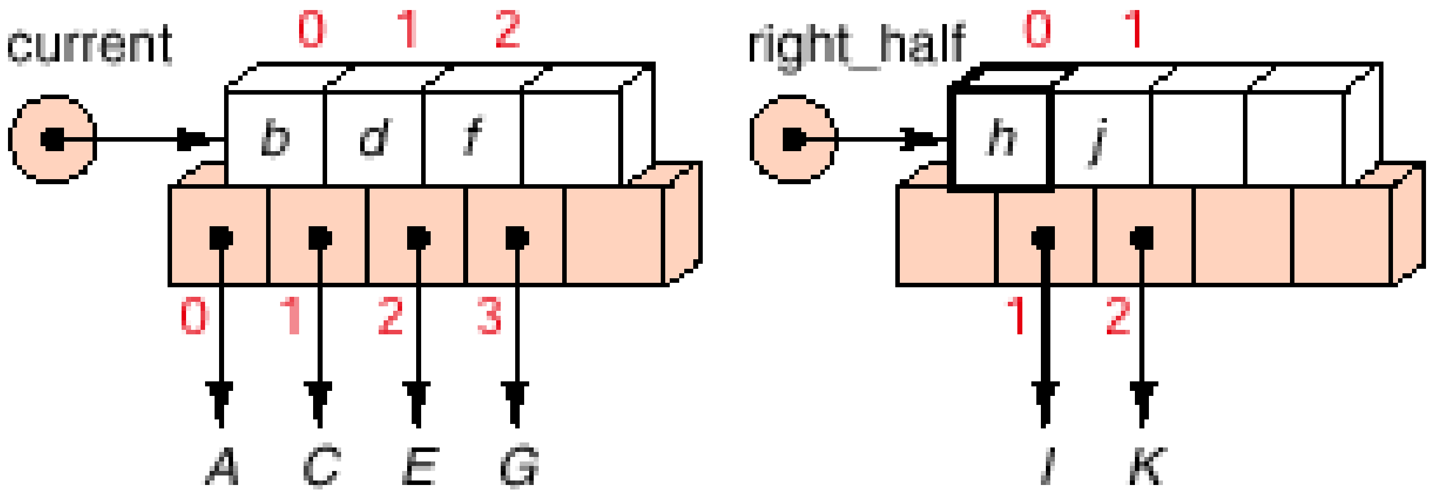
Case 2: position == 3; order == 5; (extra_entry belongs in right half.)



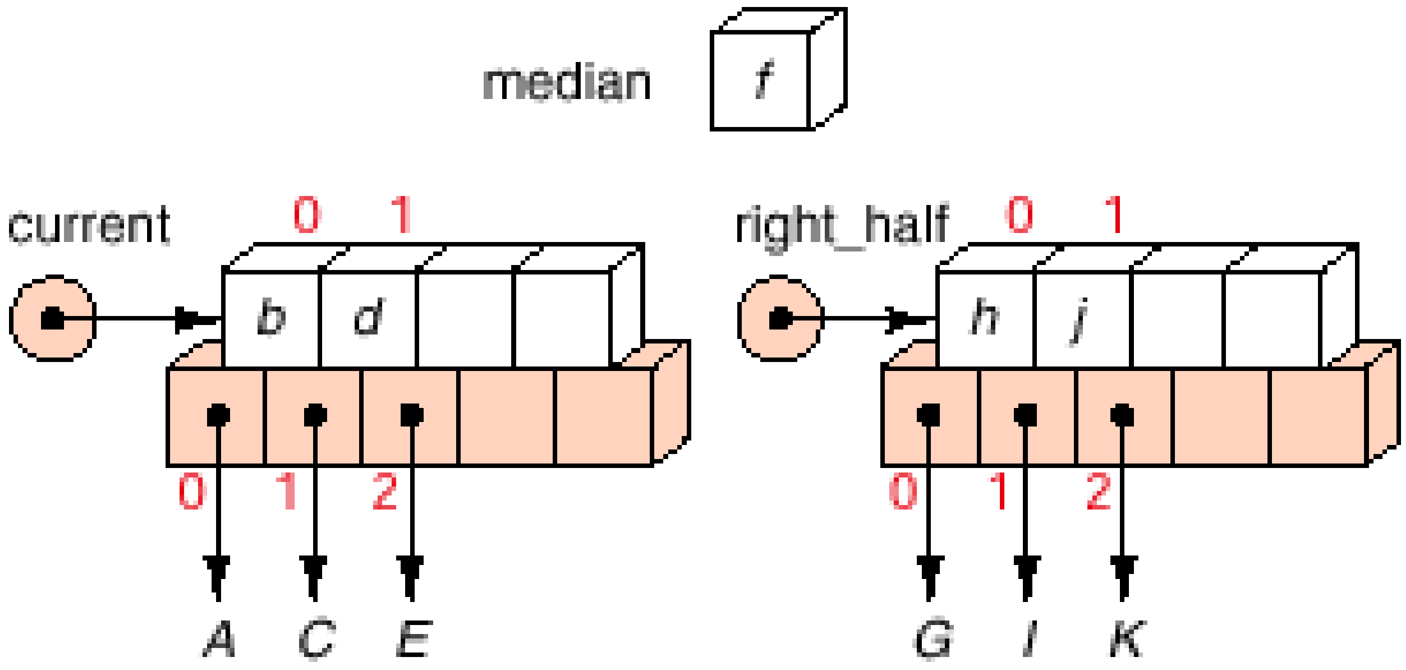
Shift entry right:



Insert extra_entry and extra_branch:



Remove median; move branch:



Function split node

```
template <class Record, int order>
void B_tree<Record, order> :: split_node(
       B_node<Record, order> *current, // node to be split
       const Record &extra_entry, // new_entry to insert
       B_node<Record, order> *extra_branch,
                        // subtree on right of extra_entry
       int position,// index in node where extra_entry goes
      B_node<Record, order> * &right_half, // new node
                                          // for right half of entries
      Record &median) // median entry (in neither half)
```

/* Pre: current points to a node of a B_tree . The node*current is full, but if there were room, the record extra_entry with its right-hand pointer extra_branch would belong in*current at position position ,0δposition < order .

Post: The node*current with extra_entry and pointer extra_branch at position position are divided into nodes *current and *right_half separated by a Record median .

Uses: Methods of struct B_node , function push_in . */

```
{
    right_half = new B_node<Record, order>;
    int mid = order/2; // The entries from mid on will go to right
    half .
    if (position <= mid) { // First case: extra_entry belongs in
    left half.
        for (int i = mid; i < order - 1; i++) { // Move entries to
    right_half .
                right_half->data[i - mid] = current->data[i];
                right_half->branch[i + 1 - mid] =
                            current->branch[i +1];
        }
        current->count = mid;
        right_half->count = order - 1 - mid;
        push_in(current, extra_entry,
                extra_branch, position);
}
else {  // Second case: extra_entry belongs in right_half.
    mid++;      // Temporarily leave the median in left half.
    for (int i = mid; i < order - 1; i++) {
                                            // Move entries to right_half .
          right_half->data[i - mid] = current->data[i];
          right_half->branch[i + 1 - mid] =
                              current->branch[i + 1];
    }
    current->count = mid;
    right_half->count = order - 1 - mid;
    push_in(right_half, extra_entry, extra branch,
    position - mid);
}
    median = current->data[current->count - 1];
                                  // Remove median from left half.
    right_half->branch[0] = current->branch[current-
    >count];
    current->count--;
}
```

Deletion from a B-Tree

- If the entry that is to be deleted is not in a leaf, then its immediate predecessor (or successor) under the natural order of keys is guaranteed to be in a leaf.
- We promote the immediate predecessor (or successor) into the position occupied by the deleted entry, and delete the entry from the leaf.
- If the leaf contains more than the minimum number of entries,then one of them can be deleted with no further action.

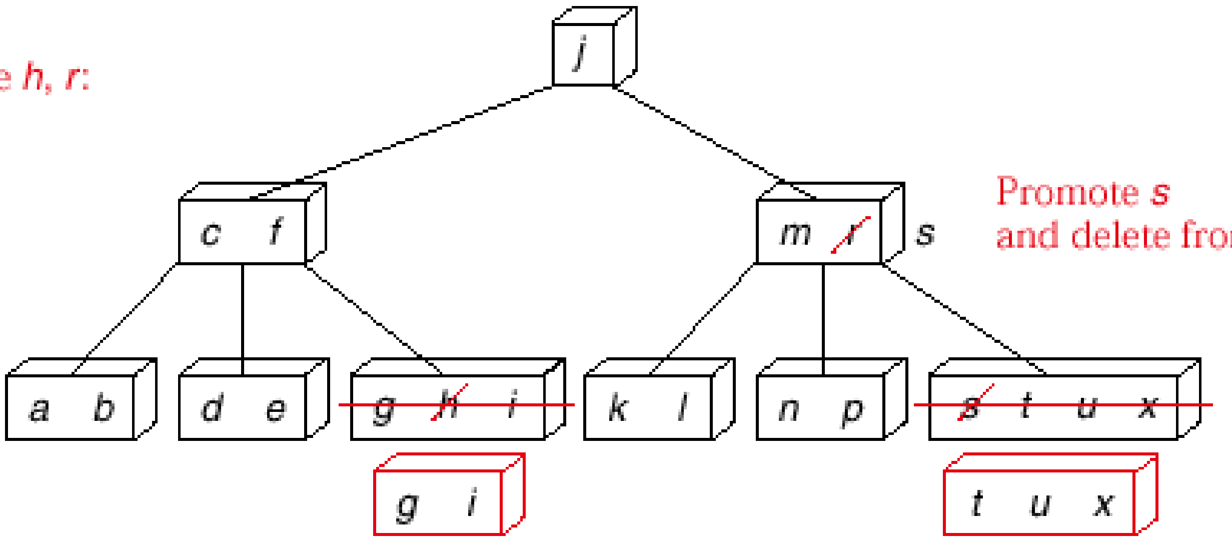
Deletion from a B-Tree

- If the leaf contains the minimum number, then we first look at the two leaves (or, in the case of a node on the outside,one leaf) that are immediately adjacent to each other and are children of the same node. If one of these has more than the minimum number of entries, then one of them can be moved into the parent node, and the entry from the parent moved into the leaf where the deletion is occurring.

Deletion from a B-Tree

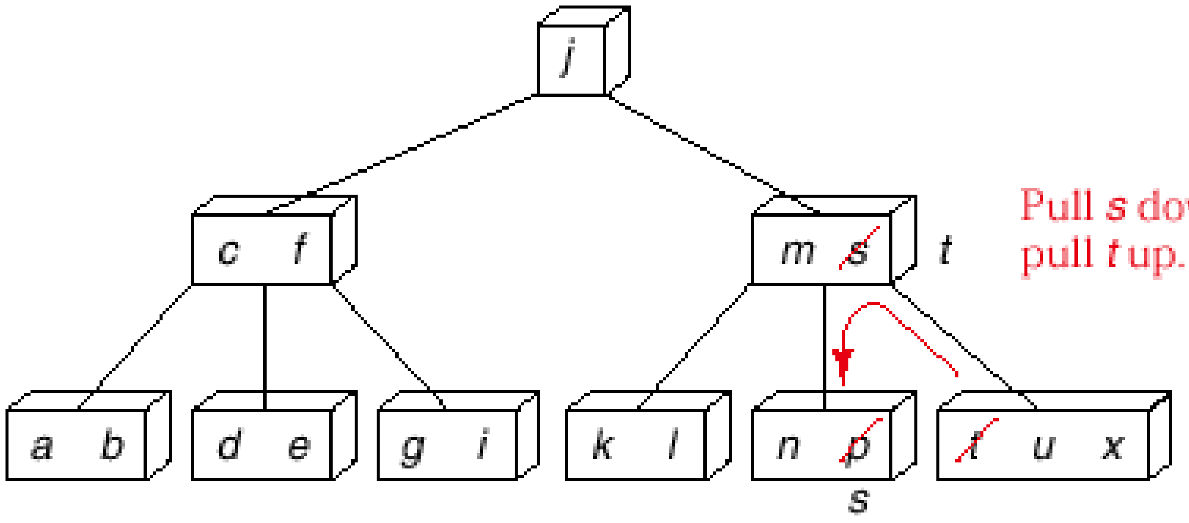
- If the adjacent leaf has only the minimum number of entries,then the two leaves and the median entry from the parent can all be combined as one new leaf, which will contain no more than the maximum number of entries allowed.
- If this step leaves the parent node with too few entries, then the process propagates upward. In the limiting case, the last entry is removed from the root, and then the height of the tree decreases.

1. Delete *h*, *r*:



Promote *s* and delete from leaf.

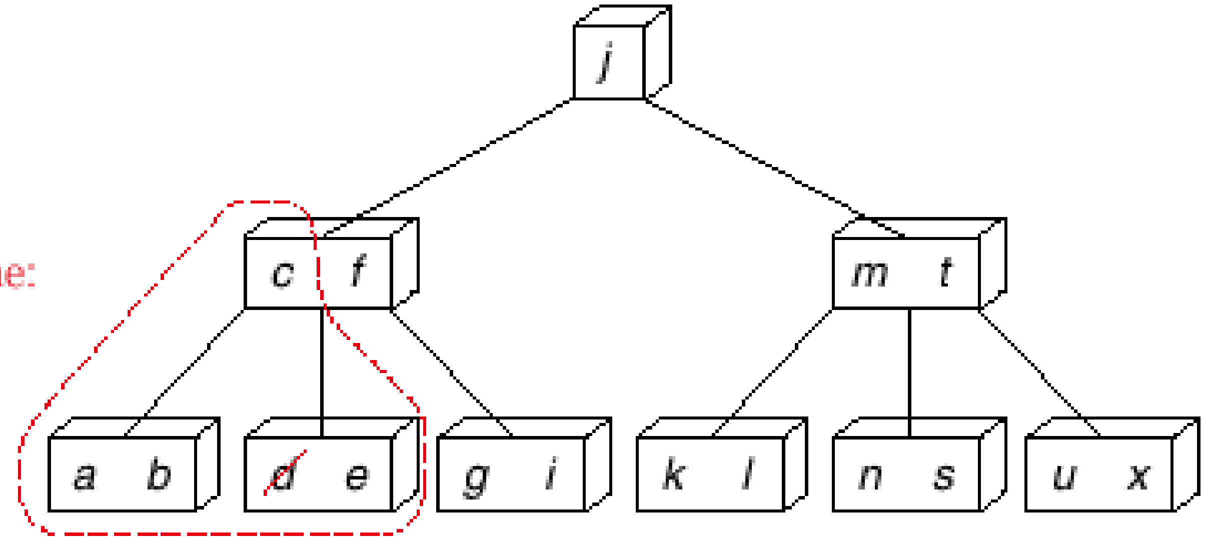
2. Delete *p*:



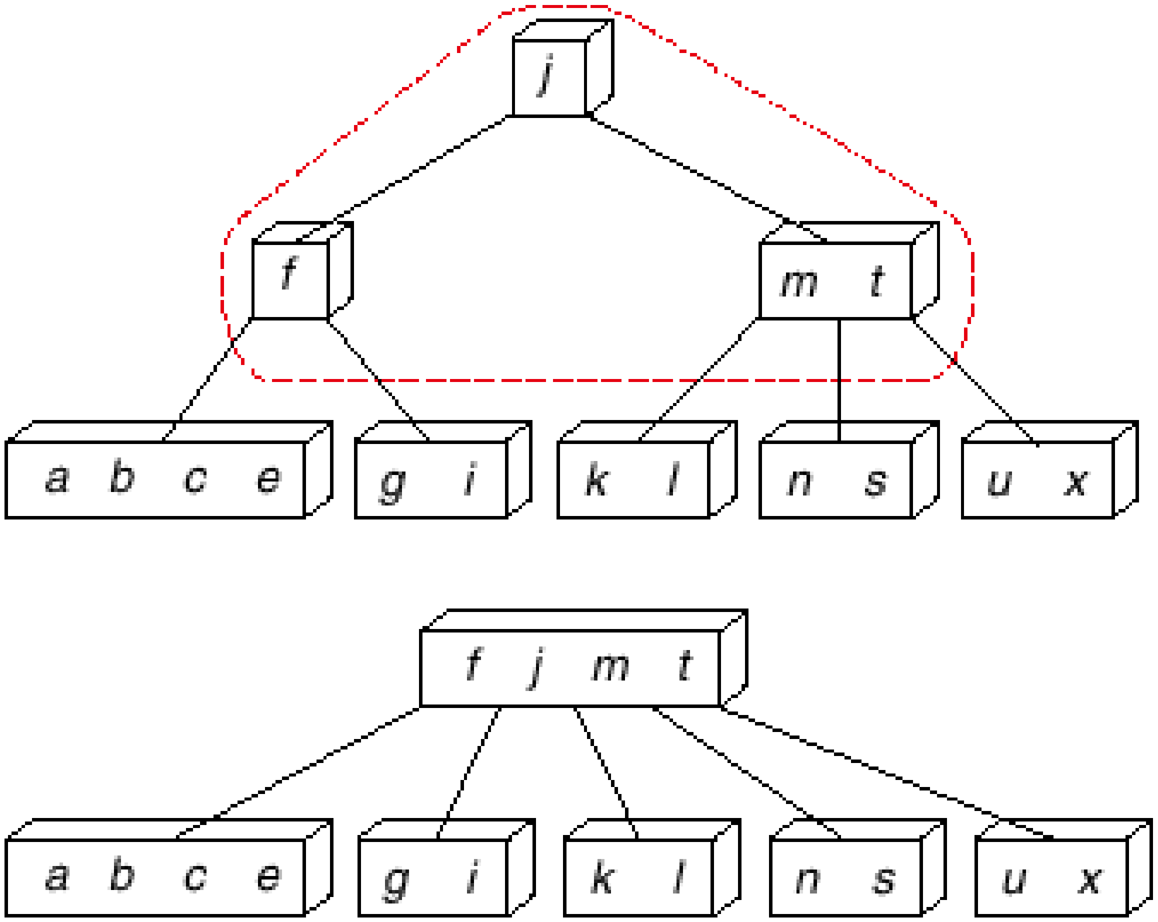
Pull *s* down; pull *t* up.

3. Delete *d*:

Combine:



Combine:



Public Deletion Method

```
template <class Record, int order>
Error_code B_tree<Record, order> ::
    remove(const Record &target)
/* Post: If a Record with Key matching that of
            target belongs to the B_tree , a code of
            success is returned and the corresponding
    node
            is removed from the B-tree.Otherwise, a code
            of not_present is returned.
    Uses: Function recursive_remove */
```

```
{
   Error_code result;
   result = recursive_remove(root, target);
   if (root != NULL && root->count == 0)
   {                                   // root is now empty.
        B_node<Record, order> *old_root = root;
        root = root->branch[0];
        delete old_root;
   }
   return result;
}
```

Recursive Deletion

template <class Record, int order>

Error_code B_tree<Record, order> :: recursive_remove(B_node<Record, order> *current, const Record &target)

/ Pre: current is either NULL or points to the root node of a subtree of a B_tree .*

Post: If a Record with Key matching that of target belongs to the subtree, a code of success is returned and the corresponding node is removed from the subtree so that the properties of a B-tree are maintained. Otherwise, a code of not _present is returned.

*Uses: Functions search_node , copy_in_predecessor , recursive_remove (recursively), Remove_data , and restore . */*

```
{  Error_code result;
   int position;
   if (current == NULL) result = not_present;
   else {
      if (search_node(current, target, position)
                          == success) {
                          // The target is in the current node.
         result = success;
         if (current->branch[position] != NULL) {
                                 // not at a leaf node
            copy_in_predecessor(current, position);
            recursive_remove(current-
   >branch[position],
            current->data[position]);
      }
```

```
    else remove_data(current, position);
                        // Remove from a leaf node.
    }
        else result = recursive_remove(
                current-branch[position], target);
        if (current->branch[position] != NULL)
        if (current->branch[position]->count <
                                (order - 1)/2)
                restore(current, position);
    }
    return result;
}
```

Auxiliary Functions

Remove data from a leaf

```
template <class Record, int order>
void B_tree<Record, order> ::remove_data(
B_node<Record, order> *current, int position)
/* Pre: current points to a leaf node in a B-tree with an
   entry at position .
   Post: This entry is removed from*current . */
{
   for (int i = position; i < current->count - 1; i++)
         current->data[i] = current->data[i + 1];
   current->count--;
}
```

Auxiliary Functions

Replace data by its immediate predecessor

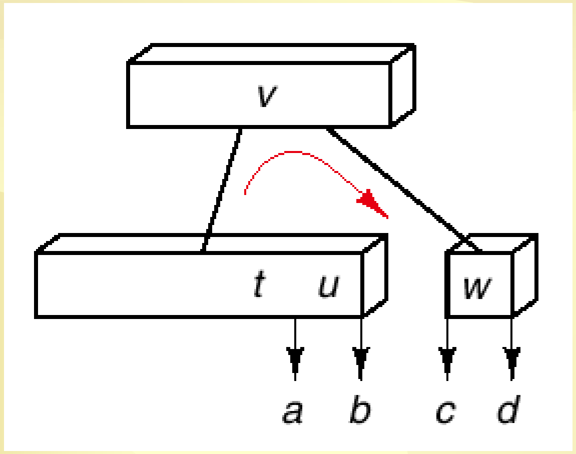
```
template <class Record, int order>
void B_tree < Record, order > ::
    copy_in_predecessor(
                B_node<Record, order> *current,
                int position)
```

/ Pre: current points to a non-leaf node in a B-tree with an entry at position .*

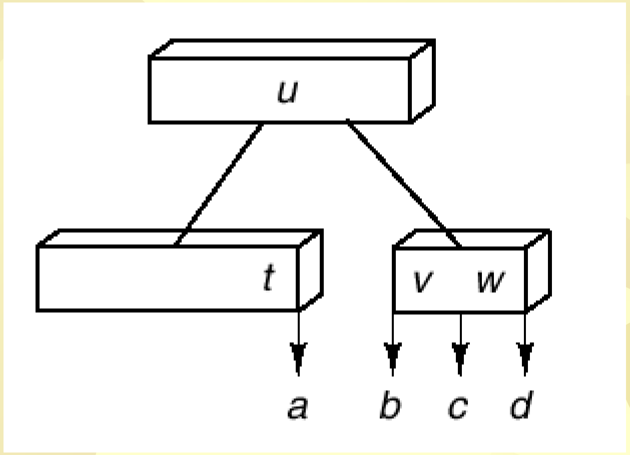
Post: This entry is replaced by its immediate predecessor under order of keys./*

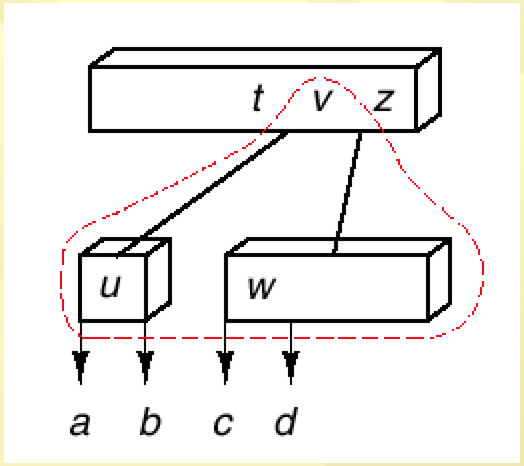
```
{
   B_node<Record, order>
        *leaf = current->branch[position];
                    // First go left from the current entry.
   while (leaf->branch[leaf->count] != NULL)
        leaf = leaf->branch[leaf->count];
                    // Move as far rightward as possible.
        current->data[position] =
                leaf->data[leaf->count - 1];
}
```

Restore Minimum Number of Entries

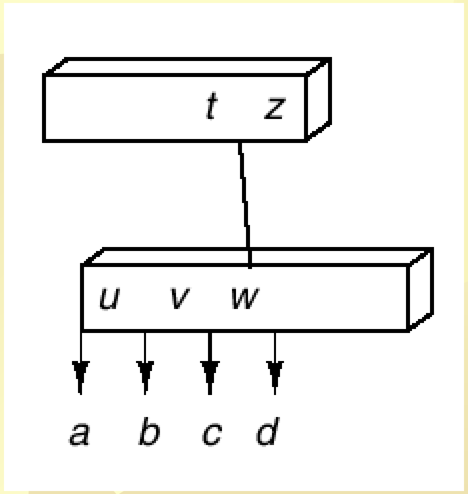


move_right $\longrightarrow$





combine $\longrightarrow$



Function to Restore Minimum Node Entries

```
template <class Record, int order>
void B_tree<Record, order> ::restore(
   B_node<Record, order> *current, int position)
/* Pre: current points to a non-leaf node in a B-tree; the
   node to which current->branch[position] points has
   one too few entries.
  Post: An entry is taken from elsewhere to restore the
   minimum number of entries in the node to which
   current->branch[position] points.
  Uses: move_left ,move_right ,combine . */
```

```
{
    if (position == current->count) // case: rightmost branch
        if (current->branch[position - 1]->count
                                > (order - 1)/2)
                move_right(current, position - 1);
        else
                combine(current, position);
    else if (position == 0)    // case: leftmost branch
        if (current->branch[1]->count > (order - 1)/2)
                move_left(current, 1);
        else
                combine(current, 1);
    else // remaining cases: intermediate branches
        if (current->branch[position - 1]-
    >count                    > (order - 1)/2)
                move_right(current, position -
    1);
        else if (current->branch[position + 1]
                ->count > (order - 1)/2)
                move_left(current, position + 1);
        else combine(current, position);
}
```

Function move_left

```
template <class Record, int order>
void B_tree<Record, order> ::
move_left(B_node<Record, order>
   *current,
        int position)
/* Pre: current points to a node in a B-tree with more
   than the minimum number of entries in branch
   position and one too few entries in branch position -
   1 .
  Post: The leftmost entry from branch position has
   moved into current , which has sent an entry into the
   branch position - 1 . */
```

```
{
B_node<Record, order>
    *left_branch = current->branch[position - 1],
    *right_branch = current->branch[position];
left_branch->data[left_branch->count] =
    current->data[position - 1]; // Take entry from the
    parent.
left_branch->branch[++left_branch->count] =
    right_branch->branch[0];
current->data[position - 1] = right_branch-
    >data[0];
// Add the right-hand entry to the parent.
right_branch->count--;
for (int i = 0; i < right_branch->count; i++) {
                    // Move right-hand entries to fill the hole.
    right_branch->data[i] = right_branch->data[i
    + 1];
    right_branch->branch[i] =
                    right_branch->branch[i + 1];
}
right_branch->branch[right_branch->count] =
    right_branch->branch[right_branch->count +
    1];
}
```

Function move_right

```
template <class Record, int order>
void B_tree<Record, order> ::
move_right(B_node<Record, order>
    *current, int position)
/* Pre: current points to a node in a B-tree with more
    than the minimum number of entries in branch
    position and one too few entries in branch position +
    1 .
Post: The rightmost entry from branch position has
    moved into current , which has sent an entry into the
    branch position + 1 . */
```

```
{
    B_node<Record, order>
            *right_branch = current->branch[position + 1],
            *left_branch = current->branch[position];
    right_branch->branch[right_branch->count + 1] =
            right_branch->branch[right_branch->count];
    for (int i = right_branch->count ; i > 0; i--) {
                                            // Make room for new entry.
            right_branch->data[i] =
                        right_branch->data[i - 1];
            right_branch->branch[i] =
                        right_branch->branch[i - 1];
    }
```

```
     right_branch->count++;
     right_branch->data[0] =
            current->data[position];
                              // Take entry from parent.
     right_branch->branch[0] =
            left_branch->branch[
                   left_branch->count--];
     current->data[position] =
            left_branch->data[
                   left_branch->count];
}
```

Function combine

```
template <class Record, int order>
void B_tree<Record, order> ::
combine(B_node<Record, order>
    *current, int position)
/* Pre: current points to a node in a B-tree with entries
    in the branches position and position - 1 , with too
    few to move entries.
 Post: The nodes at branches position - 1 and position
    have been combined into one node, which also
    includes the entry formerly in current at index
    position - 1 . */
```

```
{  int i;
   B_node<Record, order>
         *left_branch = current->branch[position -
   1],
        *right_branch = current->branch[position];
   left_branch->data[left_branch->count] =
         current->data[position - 1];
   left_branch->branch[++left_branch->count] =
         right_branch->branch[0];
```

```
    for (i = 0; i < right_branch->count; i++) {
            left_branch->data[left_branch->count] =
                        right_branch->data[i];
            left_branch->branch[++left_branch->count] =
                        right_branch->branch[i + 1];
    }
    current->count--;
    for (i = position - 1; i < current->count; i++) {
            current->data[i] = current->data[i+ 1];
            current->branch[i +1] = current->branch[i +2];
    }
    delete right_branch;
}
```