Research statement

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1 Overview

My research interest is in Dynamical systems. I have recently (May 2022) finished my first postdoc at Loughborough University (UK), and now I am a postdoc at Pennsylvania State University (US), this is a one-year position. My PhD research (under supervision of Yulij Ilyashenko) was on discrete dynamical systems, mainly on Milnor attractors and skew products with one-dimensional fiber and hyperbolic base, and during my postdoc at Loughborough University I was working on averaging method for small perturbations of Hamiltonian systems exhibiting separatrix crossing (under supervision of Anatoly Neishtadt).

2 Milnor attractors and skew products

2.1 Milnor attractors

The major part of my PhD research was related to Milnor's definition of attractor [1]. For a continuous map $f : X \mapsto X$, Milnor defined an attractor of f to be a closed subset of X that attracts almost any point with respect to Lebesgue measure, and contains no smaller subset that does so. This definition looks natural, but it is quite hard to work with. One of the difficulties is that a Milnor attractor is not preserved after a conjugacy by a homeomorphism. In fact, even a transitive C^1-smooth Anosov diffeomorphism can have a nontrivial Milnor attractor, as we proved in [2] (a result by C. Bonatti, S. Minkov, I. Shilin and myself).

In our example the Milnor attractor is a horseshoe. Moreover, there is a physical measure supported on this horseshoe such that its basin of attraction has full Lebesgue measure. Note that a C^2-smooth transitive Anosov diffeomorphism has full Minor attractor due to classical results on physical measures of Anosov diffeomorphisms. Our construction is implicit, we construct a class of Anosov diffeomorphism having an invariant horseshoe and prove that the required properties are satisfied on a residual subset of this class.

2.2 Step skew products

A *step skew product* is a skew product over the shift on the space $\{1, \dots, k\}^{\mathbb{Z}}$ of two-sided sequences such that the fiber map depends only on the zeroth element of the sequence in the base. The k fiberwise maps form an iterated function system. Step skew products are often used as a model of partially hyperbolic dynamics. Robust properties found in the class of step skew products can often be recreated for smooth partially hyperbolic skew products and their small perturbations, yielding a property locally generic in the space of all diffeomorphisms (this is done, e.g., in [3]).

In [4] (joint work with V. Kleptsyn, Yu. Kudryashov) we study step skew products with the fiber a circle. One of our main results is that generically (on an open and dense set in $C^r, r \geq 1$) such skew product is either transitive or has an absorbing domain (a finite union of intervals). In the later case, the dynamics is similar to the dynamics of skew products with the fiber a segment. Both of these two classes are rather well studied (let us mention the papers [5] on the transitive case and [6] on skew products with segment fiber), so it is nice to know that they exhaust all possibilities. We also study conditions required for global synchronoization (for the transitive case) and present several examples illustrating possible degenerate dynamics. It would be interesting (and important for many applications, e.g. time-periodic perturbations of chaotic systems) to obtain a similar alternative for partially hyperbolic skew products.

A key part of our proof is to show that if there is no absorbing domain, one can create a map with Diophantine rotation number in the semigroup generated by the fiberwise maps after a small perturbation. To this end, we consider rotation number for random iterations and show that it changes after we add a small constant c to all fiberwise maps, then Diophantine rotation number is obtained by the intermediate value theorem.

2.3 Attractors of step and smooth skew products

Another part of my PhD research was about Milnor attractors of step skew products with the fiber a circle [7] (joint with I. Shilin), and also of partially hyperbolic skew products over Anosov diffeomorphism with the fiber a circle [8]. This research was motivated by the observation that in the class of *boundary-preserving* skew products with the fiber a segment the Milnor attractor may have very strange properties. In the example [9] the Milnor attractor is Lyapunov unstable, while in the example [3] it has positive, but not full Lebesgue measure. Both properties are locally generic for boundary preserving skew products.

In [7] and [8] we prove Lyapunov stability of Milnor attractor for a generic skew product with the fiber a circle together with an alternative that the attractor has either full or zero measure. It follows that the same also holds when the fiber is a segment that is mapped inside its interior by the fiberwise maps. Thus the unusual properties of Milnor attractor listed above are only possible in the (less natural) class of boundary-preserving skew products. The main ingredients of the proofs are a semicontinuity argument and the fact that the Milnor attractor is saturated by the unstable leaves. An interesting open question about skew products with one-dimensional fiber is whether generically the Milnor attractor is not just Lyapunov stable, but also asymptotically stable.

3 Separatrix crossing

3.1 Averaging method, separatrix crossing, resonances

The averaging method is a classical powerful tool in perturbation theory of dynamical systems. In particular, it is widely used for perturbations of integrable Hamiltonian systems. For perturbations of one-frequency systems far from separatrices averaging method works for all initial data. Two main obstructions to the use of averaging method that arise in more complicated settings are resonances and separatrix crossings. Starting with two-frequency systems, resonances between the angular velocities of angle variables are possible, and small measure of initial data can be captured in resonances (this means that the resonance is preserved along solutions of the perturbed system for a large time). Separatrix crossings are possible even for one-frequency systems: due to the perturbation solutions of the perturbed system can cross separatrices of the unperturbed systems. Most recently I have been working together with Anatoly Neishtadt on problems related to separatrix crossing for small perturbations of one- and two-frequency systems.

3.2 One-frequency systems

Our project on separatrix crossing started with estimating the accuracy of order 2 averaging method for small perturbations of one-frequency Hamiltonian systems near separatrices of the unperturbed system [10]. Order 2 averaging is important, because it allows to track the evolution of the phase (i.e., the angle from the pair of action-angle variables of the unperturbed system; it is fast variable for the perturbed system). Order 1 averaging can only be used to describe the evolution of slow variables (e.g., the action of the unperturbed system). Using these estimates, we obtained a formula for the phase change while approaching the separatrices for perturbations of one-frequency systems. Such formulas were already known when the perturbed system is also Hamiltonian (e.g., for slow-fast Hamiltonian systems with one fast phase [11]); our results are valid for arbitrary perturbations of one-frequency Hamiltonian systems (with the Hamiltonian depending on a parameter that slowly changes for the perturbed system).

Knowing the phase at the moment of separatrix crossing is important for the study of probabilistic phenomena associated with separatrix crossing as it determines the domain where the trajectory is captured after separatrix crossing (when capture into multiple domains is possible) and also the jump of slow variables caused by separatrix crossing. For the case of Hamiltonian perturbations there are formulas (e.g., [12]) for the jump of slow variables. It is still an open question to find an analogue of these formulas for the case of arbitrary perturbation; together with our formula for the phase change this will allow to study consequtive separatrix crossings for the general case, as done in [13, 14] for the case of Hamiltonian perturbations.

3.3 Two-frequency systems

The main subject of our project on separatrix crossing are perturbations of two-frequency systems near separatrices. This is the simplest setting where there are *both* resonances and separatrix crossing. Again, the perturbation may be arbitrary, but our results are also new for Hamiltonian perturbations. We establish [15] estimates for the measure of initial data such that the corresponding solution is approximately described by the solution of the averaged system and the accuracy of this approximation. This work is my main mathematical achievement. It extends Neishtadt's Theorem [16] (1975) on two-frequency systems far from separatrices to the general case with separatrix crossing and provides rigorous justification for the use of averaging method in this setting, which is frequently encountered in applications. This was a challenging problem with many difficulties to overcome. Unlike one frequency systems, for two-frequency systems there are resonances and thus capture into resonances is possible. For systems with separatrix crossing resonant zones accumulate to the separatrices, there is a small zone near the separatrices where different resonant zones overlap.

The main part of this problem was to describe the passage through resonant zones near the separatrices. There is a standard procedure that allows to describe passage through a resonant zone using an auxiliary equation, an equation of pendulum type with a small perturbation. In our case, if this procedure is applied straightforwardly, the amplitude of the perturbation (for the auxiliary system) grows near the separatrices. However, additional analysis allows to split this perturbation into a Hamiltonian part and a dissipative part that does not grow near the separatrices. One interesting corollary of our analysis is that there are limit auxiliary systems when resonances approach the separatrices and such limit systems can be written in terms of Melnikov function [17] describing the splitting of separatrices.

Two other parts of the problem were the study of passage through non-resonant zones and through the resonance overlap zone near the separatrices. In the resonance overlap zone the dynamics is complicated due to Chirikov criterion [18], but, fortunately, this zone is so small that volume arguments yield a good estimate for the time spent in this zone by most solutions.

References

[1] John Milnor. "On the concept of attractor". *The theory of chaotic attractors*. Springer, 1985, pp. 243–264.

[2] Christian Bonatti, Stanislav Minkov, Alexey Okunev, and Ivan Shilin. "Anosov diffeomorphism with a horseshoe that attracts almost any point". *Discrete & Continuous Dynamical Systems-A* 40.1 (2020), p. 441.

[3] Yulij Ilyashenko. "Thick attractors of boundary preserving diffeomorphisms". *Indagationes Mathematicae* 22.3-4 (2011), pp. 257–314.

[4] Victor Kleptsyn, Yury Kudryashov, and Alexey Okunev. “Classification of generic semigroup actions of circle diffeomorphisms”. *arXiv preprint arXiv:1804.00951* (2018).

[5] VA Antonov. “Modeling cyclic evolution processes: Synchronization by means of a random signal”. *Leningradskii Universitet Vestnik Matematika Mekhanika Astronomiia* (1984), pp. 67–76.

[6] V Kleptsyn and D Volk. “Physical Measures for Nonlinear Random Walks on Interval”. *Moscow Mathematical Journal* 14.2 (2014), pp. 339–365.

[7] Alexey Okunev and Ivan Shilin. “On the attractors of step skew products over the Bernoulli shift”. *Proceedings of the Steklov Institute of Mathematics* 297.1 (2017), pp. 235–253.

[8] Alexey Okunev. “Milnor attractors of skew products with the fiber a circle”. *Journal of Dynamical and Control Systems* 23.2 (2017), pp. 421–433.

[9] Ittai Kan. “Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin”. *Bulletin of the American Mathematical Society* 31.1 (1994), pp. 68–74.

[10] Anatoly Neishtadt and Alexey Okunev. “Phase change and order 2 averaging for one-frequency systems with separatrix crossing”. *Nonlinearity* 35.8 (2022), p. 4469.

[11] Anatoly Neishtadt and Alexei Vasiliev. “Phase change between separatrix crossings in slow–fast Hamiltonian systems”. *Nonlinearity* 18.3 (2005), p. 1393.

[12] John Cary, Dominique Escande, and Jonathan Tennyson. “Adiabatic-invariant change due to separatrix crossing”. *Physical Review A* 34.5 (1986), p. 4256.

[13] Analoly Neishtadt, Vladislav Sidorenko, and Dmitry Treschev. “Stable periodic motions in the problem on passage through a separatrix”. *Chaos* 7.1 (1997), p. 2.

[14] A.A. Vasiliev, A.I. Neishtadt, C Simó, and D.V. Treschev. “Stability islands in domains of separatrix crossings in slow-fast Hamiltonian systems”. *Proceedings of the Steklov Institute of Mathematics* 259.1 (2007), pp. 236–247.

[15] Anatoly Neishtadt and Alexey Okunev. “Averaging and passage through resonances in two-frequency systems near separatrices”. *arXiv preprint arXiv:2108.08540* (2021).

[16] A.I. Neishtadt. “Passage through resonances in a two-frequency problem”. *Akademiia Nauk SSSR Doklady*. Vol. 221. 1975, pp. 301–304.

[17] V.K. Melnikov. “On the stability of a center for time-periodic perturbations”. *Trudy moskovskogo matematicheskogo obshchestva* 12 (1963), pp. 3–52.

[18] Boris V Chirikov. "Resonance processes in magnetic traps". *The Soviet Journal of Atomic Energy* 6.6 (1960), pp. 464–470.