Front-End Filtering and Quantisation Effects on GNSS Signal Processing

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***Abstract*—Traditionally, the effects of presampling filtering and of quantisation on the processing of GNSS signals have been dealt with in isolation. Analysis of the losses incurred during the quantisation process has almost invariably been based on the assumption that the signals at IF are distorted by additive white Gaussian noise. This paper, in contrast, considers the joint effect of filtering and quantisation, illustrates the need to consider these losses jointly and presents novel expressions for the total loss incurred.**

I. Introduction

The performance of the digital matched filter (DMF) has been of interest for many decades. In particular, the performance of low resolution receiver designs, which employ an integrate-and-dump (ID) DMF, has been examined [1]. Early work in this area has mainly focused on the performance of baseband non-return-to-zero (NRZ) signals which are distorted by additive white Gaussian noise (AWGN) [2]. Such analysis has presented analytical expressions for the loss due to pre-filtering, sampling and quantisation in a DMF. In the context of global navigation satellite signals (GNSS), recent simulation-based work has focused on these effects when applying matched-filtering to direct sequence spread spectrum (DSSS) signals [3]. Unfortunately, assumptions made in [2] regarding the receiver structure are violated in typical GNSS receivers and so analytical results for a GNSS-type receiver are still lacking.

The effects of nonlinear operations on correlated noise have, however, been extensively studied and results from [4], [5], [6] can be applied to this problem. This paper utilises these approaches to analyse the performance of the DMF when applied to wideband (DSSS) signals. Novel results describing the losses incurred when a DMF is used, as opposed to its analogue counterpart, are presented here and these expressions are validated by extensive simulation.

Section II introduces the signal model and defines the operation of the intermediate frequency (IF) filter and quantiser. The ID matched filter and the characteristics of the AWGN are also defined. A performance measure for the DMF is described in Section III. A joint analysis of the processing loss incurred by filtering and quantisation is derived in III-B. A simulation-based validation of this theory is provided in Section IV. Finally, a comparison of this approach to traditional loss estimates is provided in Section V.

II. Signal and System Model

The system model used in this analysis assumes that a single satellite signal, distorted by AWGN, is incident on the receiver's IF stage. This IF signal, $r(t)$, is given by:

$$\begin{aligned} r(t) &= s(t) + n(t) \\ s(t) &= \sqrt{2P}\sin(\omega_c t + \phi)\, c(t-\tau), \end{aligned} \tag{1}$$

where P is the received signal power, ω_c and ϕ represent the intermediate carrier frequency and phase respectively, $c(t-\tau)$ models the signal spreading code sequence, delayed by time τ, the additive noise, $n(t)$, is white and has a two-sided power spectral density (PSD) of $N_0/2$ W/Hz. It is postulated here that the bandlimiting effects of the signal can be modelled by an equivalent discrete time filter, represented by the transfer function, $H_f(z)$ [7]. The following expressions describe an equivalent signal processing procedure which will be used to evaluate the receiver performance. With this model in mind, it is assumed that a perfect anti-aliasing filter can be applied to $r(t)$ and that the result is sampled at a sample rate of F_S, to produce the discrete time sequence:

$$r_w[n] = s_w[n] + n_w[n]. \tag{2}$$

This signal is subsequently passed through the filter, $H_f(z)$, which yields the signal:

$$r_f[n] = s_f[n] + n_f[n]. \tag{3}$$

Finally, this filtered sequence is quantised by a B bit quantiser, $Q_B^{A_g}[x]$, producing the signal:

$$r[n] = Q_B^{A_g}[r_f[n]]. \tag{4}$$

These operations are depicted in Fig. 1. The quantiser is defined as [8]:

$$Q_B^{A_g}[x] = -\left(2^B - 1\right) + 2\sum_{i=-L}^{L} u(A_g x - i), \tag{5}$$

where A_g is a gain applied to the operand prior to quantisation, $u(t)$ is the unit step function and $L = 2^{(B-1)} - 1$. This representation, rather than adjusting threshold levels, scales the input signal with the gain A_g.

The quantised signal, $r[n]$, can now be represented as the sum of a useful signal term, $r_s[n]$, and a noise term, $r_n[n]$,

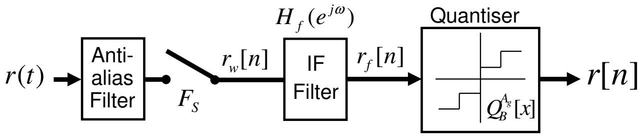

Fig. 1. Block Diagram of Front-End

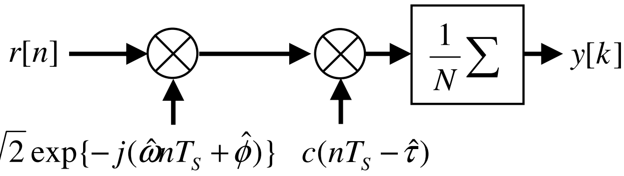

Fig. 2. Block Diagram of Correlator

where:

$$r_s[n] = \mathrm{E}[r[n]], \tag{6}$$
$$r_n[n] = r[n] - r_s[n] \tag{7}$$

and $\mathrm{E}[x]$ denotes the expectation of x.

Generally, a DMF is applied to this signal, which produces samples at a rate $1/T_I$, where T_I is an integer multiple of the spreading code period. The output of this filter, shown in Fig. 2, can be expressed as:

$$y[k] = \frac{1}{N}\sum_{n=0}^{N-1} c(nT_S - \hat{\tau})\sqrt{2}\,\mathrm{e}^{-j\left(\hat{\omega}(nT_S+kT_I)+\hat{\phi}\right)}\, r[n+kN]. \tag{8}$$

Receiver local replica signals are often represented with relatively high resolution when compared with the received signal [9], [10]. It is thus assumed here that the quantisation effects of the local replica signal are negligible and so, for the sake of simplicity, the complex exponential and code replica of (8) are not quantised. Receiver estimates of the signal variables, ω and ϕ, are respectively represented by $\hat{\omega}$ and $\hat{\phi}$.

III. Analysis

To perform a comparative loss analysis on a particular system, the performance of an ideal receiver is chosen as a reference. This ideal receiver is assumed to have an all pass front-end filter, $H(z) = 1$, and quantiser given by $Q[x] = x$. When the received signal and the local replica are perfectly synchronised, the signal to noise ratio SNR of the real channel, referred to as the coherent SNR and denoted SNR_i, can be computed as:

$$\mathrm{SNR}_i = \frac{\mu_i^2}{\sigma_i^2}, \tag{9}$$

where $\mu_i = \mathrm{E}[\Re\{y[k]\}]$ and $\sigma_i^2 = \mathrm{Var}[\Re\{y[k]\}]$. Var[ ] denotes variance and $\Re\{x\}$ is the real part of x. In the case of the ideal receiver, the coherent SNR, SNR_i^{Ideal}, is given by: $2PT_I/N_0$ [11] and, thus, the loss factor associated with a non-ideal receiver is expressed as:

$$L = \frac{\mathrm{SNR}_i}{\mathrm{SNR}_i^{Ideal}} = \mathrm{SNR}_i\,\frac{N_0}{2PT_I}. \tag{10}$$

This loss factor will be used herein to compare the performance of different receiver configurations.

A. Moment Evaluation

The filter described in (8) can be expressed as the convolution of $r[n]$ and the impulse response:

$$h_c[n] = \frac{w[n]}{N} c((N-n)T_S - \hat{\tau})\sqrt{2}e^{-j\left(\hat{\omega}((N-n)T_S+kT_I)+\hat{\phi}\right)}, \tag{11}$$

where:

$$w[n] = \begin{cases} 1 & \text{for } 0 \le n < N \\ 0 & \text{otherwise} \end{cases}, \tag{12}$$

where T_S, the sample period, is the reciprocal of F_S. From (11), the correlator output and its mean value are, respectively, given by:

$$y[n] = (h_c * r)[0] \tag{13}$$
$$\mathrm{E}[y[n]] = (h_c * \mathrm{E}[r])[0] = (h_c * r_s)[0] \tag{14}$$

where $(a*b)[m]$ is the convolution of $a[n]$ and $b[n-m]$. Synchronisation of the local replica with the received signal implies a maximisation of the correlation power in the in-phase channel. Thus, the mean correlator output can be expressed as:

$$\mu_i = \max_{\hat{\phi},\hat{\tau},\hat{\omega}} \Re\{(h_c * r_s)[0]\}, \tag{15}$$

where $\max_a f(a)$ is the maximum value of the function $f(a)$ over all values of a. The correlator output noise power can be evaluated in a similar manner:

$$\sigma_i^2 = \Re\left\{\mathrm{E}\left[((h_c * r_n)[0])((h_c * r_n)[0])^*\right]\right\} = \Re\left\{\sum_n\sum_m h_c[n]\,h_c^*[m]\,R_n[n-m]\right\}, \tag{16}$$

where $(\ )^*$ denotes complex conjugation. Letting $n \to n+m$,

$$\sigma_i^2 = \sum_n R_n[n]\sum_m \Re\{h_c[n+m]\,h_c^*[m]\} = \sum_n R_n[n]\,\Re\{R_c[n]\}, \tag{17}$$

where $R_n[n]$ and $R_c[n]$, respectively, denote the autocorrelation functions of the correlator impulse response and the received noise.

B. Joint Filtering and Quantisation Effects

The loss incurred by the filtering and quantisation of a signal can be evaluated by substituting (15) and (17) into (10). This, however, requires the evaluation of the signal mean, $r_s[n]$ and the noise autocorrelation function, $R_n[n]$, and these issues are the focus of this section.

The signal $s_f[n]$ is the result of passing the signal $s_w[n]$ through the filter $H_f(z)$. This filter is represented by impulse

$$\frac{\partial^k R(\tau)}{\partial\rho(\tau)^k} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f_1^{(k)}(x_1) f_2^{(k)}(x_2) \exp\left(-\frac{x_1^2+x_2^2-2\rho(\tau)x_1x_2}{2[1-\rho^2(\tau)]}\right)}{2\pi\sqrt{1-\rho^2(\tau)}} dx_1 dx_2 \tag{18}$$

response, $h_f[n]$, and frequency response, $H_f\left(e^{j\omega}\right)$. The filtered signal is thus given by:

$$r_f[n] = (r_w * h_f)[n], \tag{19}$$
$$s_f[n] = (s_w * h_f)[n], \tag{20}$$
$$n_f[n] = (n_w * h_f)[n]. \tag{21}$$

Assuming that the received SNR is low (< -25 dB), it has been shown [8] that the mean quantiser output, i.e. the expectation of (4), is given by:

$$r_s[n] = \frac{2s_f[n]}{\sqrt{2\pi\sigma_{IF}^2}}\left(1 + 2\sum_{i=-L}^{L}\exp\left(\frac{-i^2}{2A_g^2\sigma_{IF}^2}\right)\right) \tag{22}$$
$$= s_f[n]K_D, \tag{23}$$

where σ_{IF}^2 is the variance of $n_f[n]$, the noise incident on the quantiser. Substituting (23) into (15) will thus yield the mean correlator output, μ_i:

$$\begin{aligned}\mu_i &= \max_{\hat{\phi},\hat{\tau},\hat{\omega}} \Re\left\{K_Q (h_c * s_f)[0]\right\}\\ &= \sqrt{P}K_Q \max_{\hat{\phi},\hat{\tau},\hat{\omega}} \Re\left\{e^{j(\delta\phi)}\left(R_{Code}(\delta\tau + nT_S) * h_f[n]\right)\right\}\\ &= \sqrt{P}K_Q K_F,\end{aligned} \tag{24}$$

where $R_{Code}(\tau)$ is the autocorrelation function of $c(t)$, $\delta\phi = \hat{\phi} - \phi$, $\delta\tau = \hat{\tau} - \tau$ and the variables K_Q and K_F represent the attenuation of the signal mean due to quantisation and filtering, respectively.

From (20) and (19), the autocorrelation of the noise component of the signal $r_f[n]$ is given by:

$$\begin{aligned}R_{n_f}[m] &= \mathrm{E}\left[(n_w * h_f)[n](n_w * h_f)[n-m]\right]\\ &= \mathrm{E}\left[\sum_j n_w[j]h_f[n-j]\sum_k n_w[k]h_f[n-m-k]\right]\\ &= \sum_j\sum_k \mathrm{E}\left[n_w[j]n_w[k]\right]h_f[n-j]h_f[n-m-k].\end{aligned} \tag{25}$$

Exploiting the fact that $n_w[n]$ is delta correlated, with variance equal to $N_0F_S/2$, we find:

$$\begin{aligned}R_{n_f}[m] &= \frac{N_0F_S}{2}\sum_k h_f[n-k]h_f[n-m-k]\\ &= \frac{N_0F_S}{2}\sum_k h_f[k]h_f[k-m]\\ &= \frac{N_0F_S}{2}R_f[m].\end{aligned} \tag{26}$$

$R_f[m]$ represents the autocorrelation function of the impulse response of the front-end filter. The noise described by (26) is incident on the receiver's quantiser, and results in the noise $r_n[n]$. It has been shown [4] that quantisation may alter

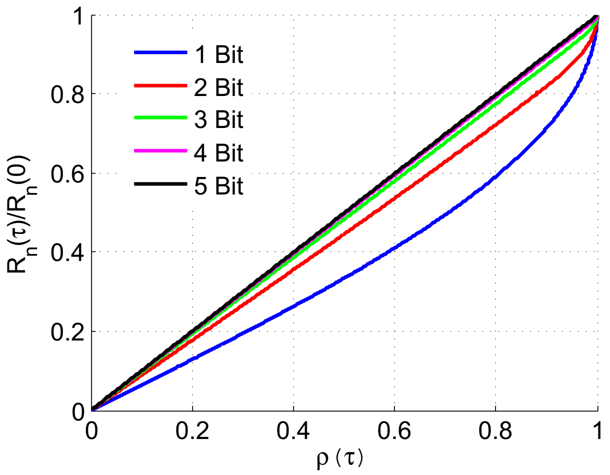

Fig. 3. Relationship between input and output correlation coefficients

the autocorrelation function of non-white noise. Thus, the autocorrelation function of $r_n[n]$, denoted $R_n[n]$, may differ to that of $n_f[n]$, and must be evaluated. Previous work on nonlinear operations on noise [5] yielded (18), which can be applied to this problem.

If $f_1(x)$ and $f_2(x)$ are two memoryless, nonlinear functions and $x_1 = x(t)$ and $x_2 = x(t+\tau)$ are samples of a Gaussian random process, then (18) defines the k^{th} partial derivative of the cross-correlation function of $f_1(x_1)$ and $f_2(x_2)$, denoted $R(\tau)$, with respect to the correlation coefficient of $x(t)$, denoted $\rho(\tau)$, and the k^{th} partial derivatives of $f_1(x)$ and $f_2(x)$ with respect to x, respectively denoted $f_1^{(k)}(x)$ and $f_2^{(k)}(x)$.

As (18) is defined in terms of a zero mean, unit variance noise, we define $f_1(x)$ and $f_2(x)$ in terms of (5) such that:

$$f_1(x) = f_2(x) = Q_B^{A_g}[\sigma_{IF}x], \tag{27}$$

which have first derivative given by

$$f_1^{(1)}(x) = \frac{\partial}{\partial x}Q_B^{A_g}[\sigma_{IF}x] = 2A_g\sigma_{IF}\sum_{i=-L}^{L}\delta(A_g\sigma_{IF}x - i), \tag{28}$$

where $\delta(x)$ is the Dirac delta function. The correlation coefficient of the incident noise is defined as:

$$\rho(\tau) = \frac{R_f[n]}{R_f[0]}. \tag{29}$$

Substituting (28) into (18), and performing the double integration, the first partial derivative of the autocorrelation function of the quantised process is found to be:

$$\frac{\partial R_n(\tau)}{\partial\rho(\tau)} = 4\sum_{i=-L}^{L}\sum_{k=-L}^{L}\frac{\exp\left(-\frac{1}{A_g^2\sigma_{IF}^2}\frac{i^2+k^2-2\rho(\tau)ik}{2[1-\rho^2(\tau)]}\right)}{2\pi\sqrt{1-\rho^2(\tau)}} \tag{30}$$

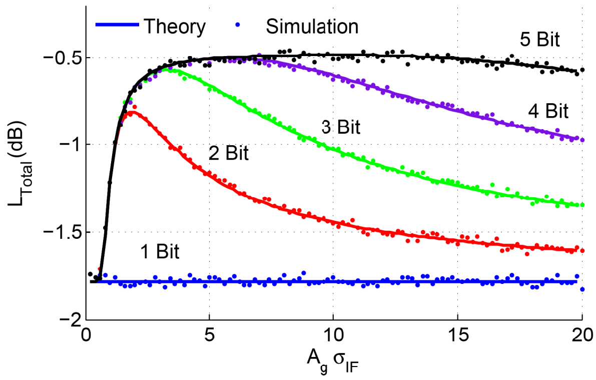

Fig. 4. Loss Simulation: Example 1

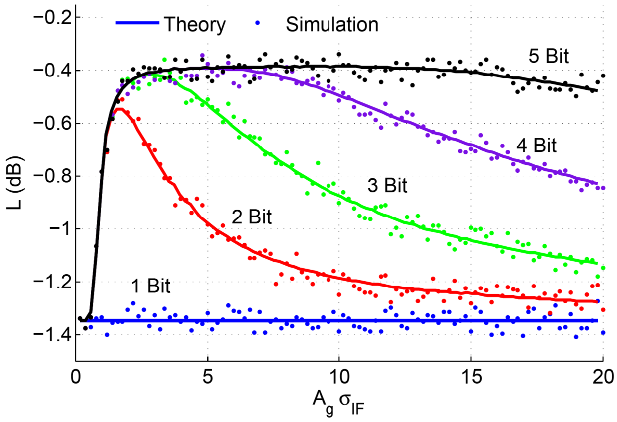

Fig. 5. Loss Simulation: Example 2

and, subsequently, integrating with respect to $\rho(\tau)$ yields:

$$R_n(\tau) = \frac{2}{\pi} \sum_{i=-L}^{L} \sum_{k=-L}^{L} \int_0^{\rho(\tau)} \frac{\exp\left(-\frac{1}{A_g^2 \sigma_{IF}^2} \frac{i^2+k^2-2rik}{2[1-r^2]}\right)}{\sqrt{1-r^2}} dr. \tag{31}$$

It has been shown [6] that, in the case of a one bit quantiser ($B = 1$), this admits a closed form solution:

$$R_n(\tau) = \frac{2}{\pi} \sin^{-1}\left(\rho(\tau)\right). \tag{32}$$

Figure 3 illustrates a typical relationship between quantiser input and output (defined by (31)) correlation functions for a variety of quantisation levels. Substituting (31) into (17), the variance of the output of the accumulator can now be calculated. In turn, (9) and (10) can be used to evaluate the joint loss due to filtering and quantisation.

IV. Simulation Results

This section details the experiments conducted to validate the theoretical analysis of Section III-B. A software simulation environment was developed to model the signal and system defined in Section II using the GPS L1 C/A signal [12] as a case study. Monte Carlo simulations were run in this environment using a variety of different front-end filters and quantisation levels. For each simulation, the value of A_g was swept over a suitable range and the resulting SNR loss was calculated. This was then compared to the theoretical loss, as defined by (10) and (31). The results, two of which are presented here, were found to be a very close match.

Fig. 4 shows the results using a sample rate of 9 MHz, a center frequency of 2.25 MHz and a carrier to noise ratio of 40 dBHz. An eighth order elliptic filter was used with passband edges of $f_c \pm 1$ MHz and a stopband attenuation of 20 dB.

Fig. 5 shows the results using a sample rate of 40 MHz, a center frequency of 10 MHz and a carrier to noise ratio of 30 dBHz. In this case, a sixth order elliptic filer was used, again with passband edges of $f_c \pm 1$ MHz and a stopband attenuation of 20 dB.

V. Significance of Joint Analysis

This sections examines the significance of the joint analysis of losses incurred through filtering and quantisation. This is done by comparing the joint analysis described in III-B with the loss coefficient obtained through a separate analysis.

A. Filtering Loss

Firstly, a filtering loss coefficient, L_{Filt}, is derived by assuming that the signal is not quantised, i.e. $Q_B^{A_g}[x] = x$. Applying this assumption to (24), the mean correlator output is given by:

$$\begin{aligned}\mu_i &= \sqrt{P} \max_{\hat{\phi},\hat{\tau},\hat{\omega}} \Re\left\{e^{j(\delta\phi)}\left(R_{Code}(\delta\tau + nT_S) * h_f\right)[0]\right\} \\ &= \sqrt{P} K_F.\end{aligned} \tag{33}$$

In the absence of quantisation the noise correlation functions, $R_{n_f}[n]$ and $R_n[n]$, are equal. Using (17), the correlator output variance is given by:

$$\sigma_i^2 = \sum_n R_{n_f}[n] \Re\{R_c[n]\} \tag{34}$$

$$= \frac{N_0 F_S}{2} \sum_n R_f[n] \Re\{R_c[n]\}. \tag{35}$$

The resultant loss is thus given by:

$$L_{Filt} = \frac{K_F^2}{N \sum_n R_f[n] \Re\{R_c[n]\}}, \tag{36}$$

where we note that $F_S T_I = N$.

B. Quantisation Loss

Secondly, a quantisation loss coefficient, L_{Quant}, can be calculated by assuming that the signal is not filtered, i.e. $H_f(z) = 1$. This assumption reduces (24) to the following:

$$\mu_i = \sqrt{P} K_Q, \tag{37}$$

where the variance of the noise incident on the quantiser, σ_{IF}^2, is given (from (26)) by:

$$\sigma_{IF}^2 = \frac{N_0}{2T_I}. \tag{38}$$

Since the noise incident on the quantiser is delta correlated, so too will be the quantised noise, incident on the correlator. The correlation function is thus given by:

$$R_n[n] = \sigma_Q^2 \delta[n], \tag{39}$$

$$\sigma_Q^2 = \frac{2}{\pi} \sum_{i=-L}^{L} \sum_{k=-L}^{L} \int_0^1 \frac{\exp\left(-\frac{1}{A_g^2 \sigma_{IF}^2} \frac{i^2+k^2-2rik}{2[1-r^2]}\right)}{\sqrt{1-r^2}} dr. \tag{40}$$

Substitution of this expression into (17) yields, after some simplification:

$$\sigma_i^2 = \frac{\sigma_Q^2}{N}, \tag{41}$$

from which the loss due to quantisation alone is found:

$$L_{Quant} = \frac{K_Q^2 N_0 F_S}{2\sigma_Q^2}. \tag{42}$$

C. Comparison of Loss Estimates

In order to illustrate the significance of the joint loss analysis (described in Section III-B) the loss estimate, which we will now denote L_{Joint}, is compared to a loss estimate derived from separate analyses of filtering and quantisation effects. This latter estimate, denoted L_{Prod}, is described in Sections V-A and V-B, respectively. L_{Prod} is defined in (43), as is traditionally done in the literature [3][13], as the product of individual loss coefficients:

$$L_{Prod} = L_{Filt} L_{Quant}. \tag{43}$$

As a case study, the receiver configuration described in Section IV (pertaining to the results depicted in Figure 5) is examined. Both loss estimates are evaluated for each quantiser configuration (1 to 5 bit) and across a range of $A_g \sigma_{IF}$ values. For clarity, a linear scale is used for the vertical axis. The result, depicted in Figure 6, illustrates that these estimates differ significantly. It is clear that L_{Prod} consistently underestimates the loss. In the case of one bit quantisation, the estimates differ by a factor of approximately 1.25.

This phenomenon can be explained by examining (31) and Figure 3. The noise sequence $r_n[n]$ decorrelates faster than does $n_f[n]$, essentially the noise is whitened as it is quantised. When the quantisation and filtering are considered separately, the whitening of the correlation function $R_{n_f}[n]$ to $R_n[n]$ is not reflected in the summation (17). As a result, (17) will overestimate the variance of the correlator output, resulting in an underestimation in the loss coefficient.

VI. Conclusions

This paper has presented a novel characterisation of combined filtering and quantisation processing loss for a generalised B-bit symmetric quantiser. Using the GPS C/A signal as a case study, Section IV has validated this theory through extensive simulation while Section V has illustrated that a joint analysis is necessary in order to account accurately for these losses.

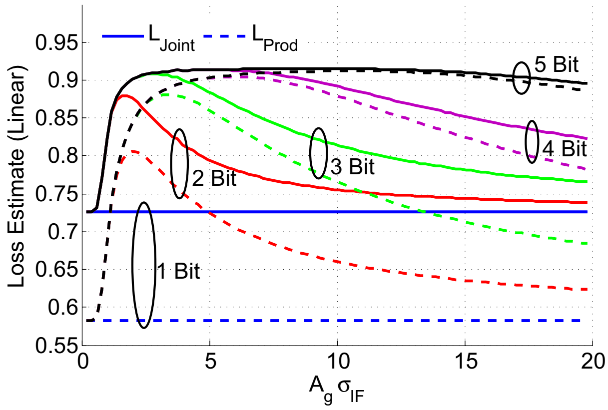

Fig. 6. Comparison of Loss Estimates

The characterisation presented here can be used to compute accurate estimates of the total processing gain of a CDMA receiver. Moreover, they can be applied to the joint optimisation of front-end filter and quantiser in order to reduce the overall processing loss in a GNSS receiver. Further research is being conducted to find a relationship between the front-end filter and the quantiser which reduces the overall processing loss.

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