Digital Signal Processing

Fourth Edition

John G. Proakis

Department of Electrical and Computer Engineering
Northeastern University
Boston, Massachusetts

Dimitris G. Manolakis

MIT Lincoln Laboratory
Lexington, Massachusetts

PEARSON
Prentice Hall

Upper Saddle River, New Jersey 07458

would suggest a sampling rate $F_s \geq 2F_H$; however, as we show in this section, there are sampling techniques that allow sampling rates consistent with the bandwidth B, rather than the highest frequency, F_H, of the signal spectrum. Sampling of bandpass signals is of great interest in the areas of digital communications, radar, and sonar systems.

6.4.1 Uniform or First-Order Sampling

Uniform or first-order sampling is the typical periodic sampling introduced in Section 6.1. Sampling the bandpass signal in Figure 6.4.1(a) at a rate $F_s = 1/T$ produces a sequence $x(n) = x_a(nT)$ with spectrum

$$X(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \tag{6.4.1}$$

The positioning of the shifted replicas $X(F - kF_s)$ is controlled by a single parameter, the sampling frequency F_s. Since bandpass signals have two spectral bands, in general, it is more complicated to control their positioning, in order to avoid aliasing, with the single parameter F_s.

Integer Band Positioning. We initially restrict the higher frequency of the band to be an integer multiple of the bandwidth, that is, $F_H = mB$ (*integer band positioning*). The number $m = F_H/B$, which is in general fractional, is known as the *band position*. Figures 6.4.1(a) and 6.4.1(d) show two bandpass signals with even ($m = 4$) and odd ($m = 3$) band positioning. It can be easily seen from Figure 6.4.1(b) that, for integer-positioned bandpass signals, choosing $F_s = 2B$ results in a sequence with a spectrum without aliasing. From Figure 6.4.1(c), we see that the original bandpass signal can be reconstructed using the reconstruction formula

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT) g_a(t - nT) \tag{6.4.2}$$

where

$$g_a(t) = \frac{\sin \pi Bt}{\pi Bt} \cos 2\pi F_c t \tag{6.4.3}$$

is the inverse Fourier transform of the bandpass frequency gating function shown in Figure 6.4.1(c). We note that $g_a(t)$ is equal to the ideal interpolation function for lowpass signals [see (6.1.21)], modulated by a carrier with frequency F_c.

It is worth noticing that, by properly choosing the center frequency F_c of $G_a(F)$, we can reconstruct a continuous-time bandpass signal with spectral bands centered at $F_c = \pm(kB + B/2)$, $k = 0, 1, \ldots$. For $k = 0$ we obtain the equivalent baseband signal, a process known as *down-conversion*. A simple inspection of Figure 6.4.1 demonstrates that the baseband spectrum for $m = 3$ has the same spectral structure as the original spectrum; however, the baseband spectrum for $m = 4$ has been "inverted." In general, when the band position is an *even* integer the baseband spectral images are inverted versions of the original ones. Distinguishing between these two cases is important in communications applications.

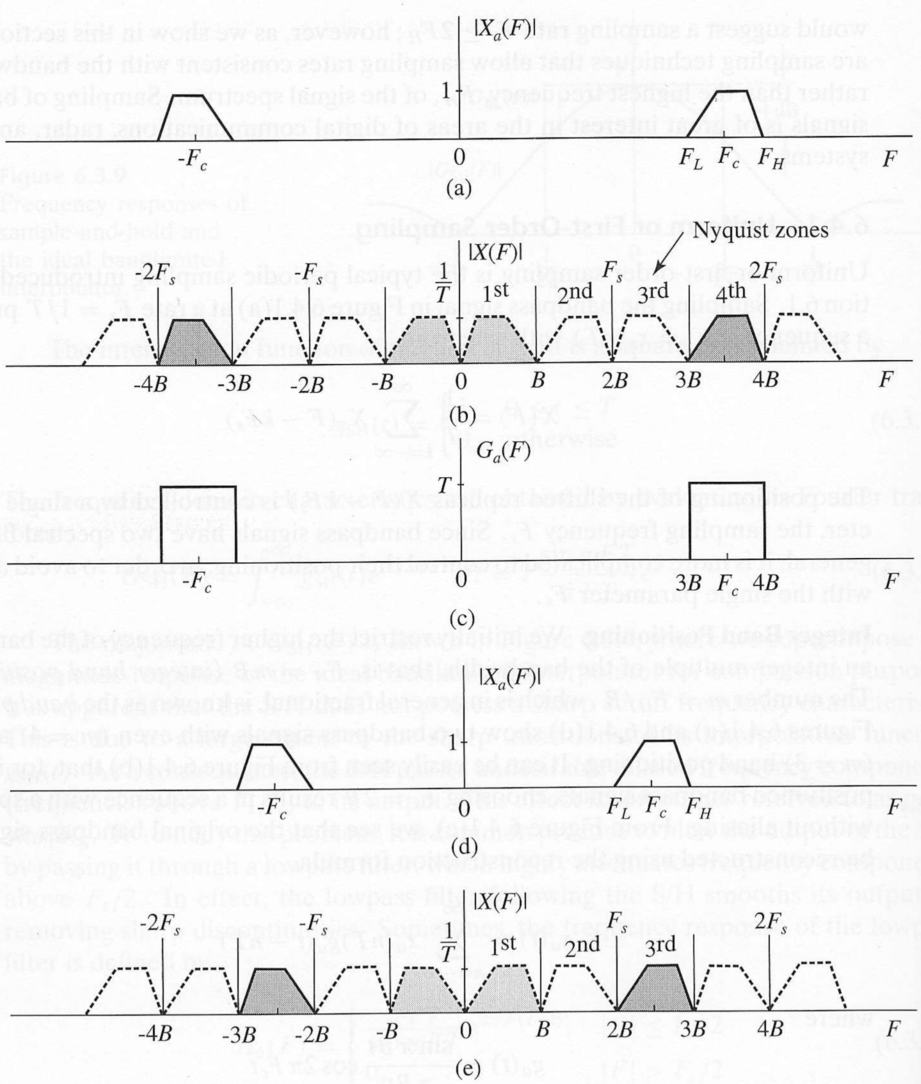

Figure 6.4.1 Illustration of bandpass signal sampling for integer band positioning.

Arbitrary Band Positioning. Consider now a bandpass signal with arbitrarily positioned spectral bands, as shown in Figure 6.4.2. To avoid aliasing, the sampling frequency should be such that the $(k-1)$th and kth shifted replicas of the "negative" spectral band do not overlap with the "positive" spectral band. From Figure 6.4.2(b) we see that this is possible if there is an integer k and a sampling frequency F_s that satisfy the following conditions:

$$2F_H \leq kF_s \tag{6.4.4}$$

$$(k-1)F_s \leq 2F_L \tag{6.4.5}$$

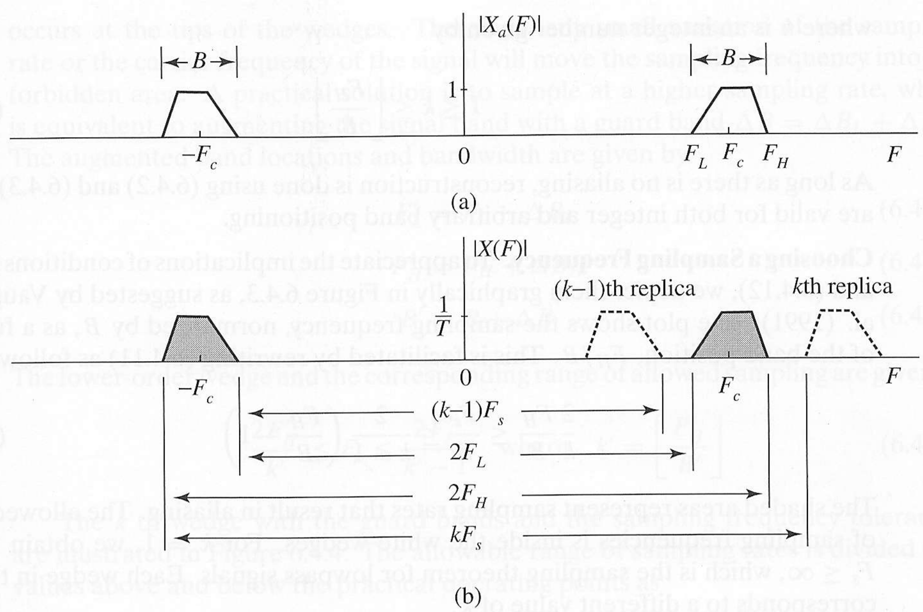

Figure 6.4.2 Illustration of bandpass signal sampling for arbitrary band positioning.

which is a system of two inequalities with two unknowns, k and F_s. From (6.4.4) and (6.4.5) we can easily see that F_s should be in the range

$$\frac{2F_H}{k} \leq F_s \leq \frac{2F_L}{k-1} \tag{6.4.6}$$

To determine the integer k we rewrite (6.4.4) and (6.4.5) as follows:

$$\frac{1}{F_s} \leq \frac{k}{2F_H} \tag{6.4.7}$$

$$(k-1)F_s \leq 2F_H - 2B \tag{6.4.8}$$

By multiplying (6.4.7) and (6.4.8) by sides and solving the resulting inequality for k we obtain

$$k_{\max} \leq \frac{F_H}{B} \tag{6.4.9}$$

The maximum value of integer k is the number of bands that we can fit in the range from 0 to F_H, that is

$$k_{\max} = \left\lfloor \frac{F_H}{B} \right\rfloor \tag{6.4.10}$$

where $\lfloor b \rfloor$ denotes the integer part of b. The minimum sampling rate required to avoid aliasing is $F_{s\max} = 2F_H/k_{\max}$. Therefore, the range of acceptable uniform sampling rates is determined by

$$\frac{2F_H}{k} \leq F_s \leq \frac{2F_L}{k-1} \tag{6.4.11}$$

where k is an integer number given by

$$1 \leq k \leq \left\lfloor \frac{F_H}{B} \right\rfloor \tag{6.4.12}$$

As long as there is no aliasing, reconstruction is done using (6.4.2) and (6.4.3), which are valid for both integer and arbitrary band positioning.

Choosing a Sampling Frequency. To appreciate the implications of conditions (6.4.11) and (6.4.12), we depict them graphically in Figure 6.4.3, as suggested by Vaughan et al. (1991). The plot shows the sampling frequency, normalized by B, as a function of the band position, F_H/B. This is facilitated by rewriting (6.4.11) as follows:

$$\frac{2}{k}\frac{F_H}{B} \leq \frac{F_s}{B} \leq \frac{2}{k-1}\left(\frac{F_H}{B} - 1\right) \tag{6.4.13}$$

The shaded areas represent sampling rates that result in aliasing. The allowed range of sampling frequencies is inside the white wedges. For $k = 1$, we obtain $2F_H \leq F_s \leq \infty$, which is the sampling theorem for lowpass signals. Each wedge in the plot corresponds to a different value of k.

To determine the allowed sampling frequencies, for a given F_H and B, we draw a vertical line at the point determined by F_H/B. The segments of the line within the allowed areas represent permissible sampling rates. We note that the theoretically minimum sampling frequency $F_s = 2B$, corresponding to integer band positioning,

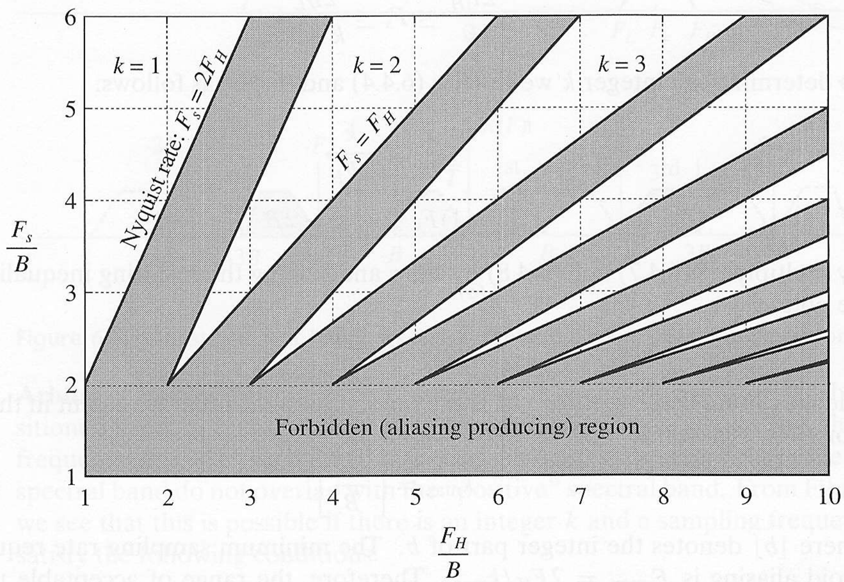

Figure 6.4.3 Allowed (white) and forbidden (shaded) sampling frequency regions for bandpass signals. The minimum sampling frequency $F_s = 2B$, which corresponds to the corners of the alias-free wedges, is possible for integer-positioned bands only.

occurs at the tips of the wedges. Therefore, any small variation of the sampling rate or the carrier frequency of the signal will move the sampling frequency into the forbidden area. A practical solution is to sample at a higher sampling rate, which is equivalent to augmenting the signal band with a guard band $\Delta B = \Delta B_L + \Delta B_H$. The augmented band locations and bandwidth are given by

$$F'_L = F_L - \Delta B_L \tag{6.4.14}$$

$$F'_H = F_H + \Delta B_H \tag{6.4.15}$$

$$B' = B + \Delta B \tag{6.4.16}$$

The lower-order wedge and the corresponding range of allowed sampling are given by

$$\frac{2F_H}{k'} \le F_s \le \frac{2F_L}{k'-1} \quad \text{where} \quad k' = \left\lfloor \frac{F_H}{B'} \right\rfloor \tag{6.4.17}$$

The k'th wedge with the guard bands and the sampling frequency tolerances are illustrated in Figure 6.4.4. The allowable range of sampling rates is divided into values above and below the practical operating points as

$$\Delta F_s = \frac{2F'_L}{k'-1} - \frac{2F'_H}{k'} = \Delta F_{\mathrm{sL}} + \Delta F_{\mathrm{sH}} \tag{6.4.18}$$

From the shaded orthogonal triangles in Figure 6.4.4, we obtain

$$\Delta B_L = \frac{k'-1}{2} \Delta F_{\mathrm{sH}} \tag{6.4.19}$$

$$\Delta B_H = \frac{k'}{2} \Delta F_{\mathrm{sL}} \tag{6.4.20}$$

which shows that symmetric guard bands lead to asymmetric sampling rate tolerance.

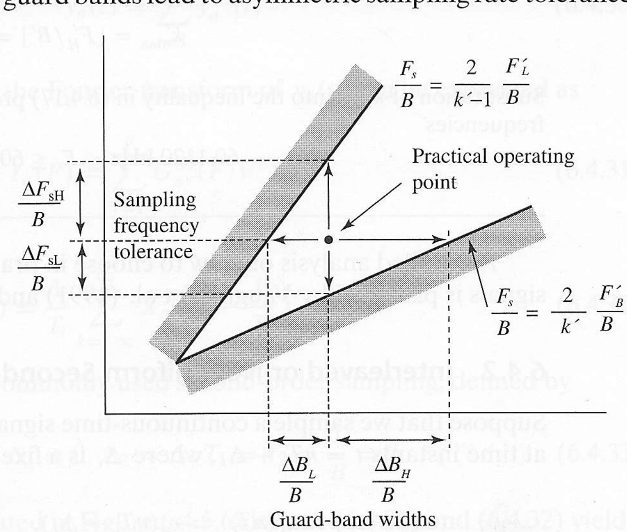

Figure 6.4.4
Illustration of the relationship between the size of guard bands and allowed sampling frequency deviations from its nominal value for the kth wedge.

If we choose the practical operating point at the vertical midpoint of the wedge, the sampling rate is

$$F_s = \frac{1}{2}\left(\frac{2F'_H}{k'} + \frac{2F'_L}{k'-1}\right) \tag{6.4.21}$$

Since, by construction, $\Delta F_{\text{sL}} = \Delta F_{\text{sH}} = \Delta F_s/2$, the guard bands become

$$\Delta B_L = \frac{k'-1}{4}\Delta F_s \tag{6.4.22}$$

$$\Delta B_H = \frac{k'}{4}\Delta F_s \tag{6.4.23}$$

We next provide an example that illustrates the use of this approach.

EXAMPLE 6.4.1

Suppose we are given a bandpass signal with $B = 25$ kHz and $F_L = 10{,}702.5$ kHz. From (6.4.10) the maximum wedge index is

$$k_{\max} = \lfloor F_H/B \rfloor = 429$$

This yields the theoretically minimum sampling frequency

$$F_s = \frac{2F_H}{k_{\max}} = 50.0117 \text{ kHz}$$

To avoid potential aliasing due to hardware imperfections, we wish to use two guard bands of $\Delta B_L = 2.5$ kHz and $\Delta B_H = 2.5$ kHz on each side of the signal band. The effective bandwidth of the signal becomes $B' = B + \Delta B_L + \Delta B_H = 30$ kHz. In addition, $F'_L = F_L - \Delta B_L = 10{,}700$ kHz and $F'_H = F_H + \Delta B_H = 10{,}730$ kHz. From (6.4.17), the maximum wedge index is

$$k'_{\max} = \lfloor F'_H/B' \rfloor = 357$$

Substitution of $k_{\max}$ into the inequality in (6.4.17) provides the range of acceptable sampling frequencies

$$60.1120 \text{ kHz} \le F_s \le 60.1124 \text{ kHz}$$

A detailed analysis on how to choose in practice the sampling rate for bandpass signals is provided by Vaughan et al. (1991) and Qi et al. (1996).

6.4.2 Interleaved or Nonuniform Second-Order Sampling

Suppose that we sample a continuous-time signal $x_a(t)$ with sampling rate $F_i = 1/T_i$ at time instants $t = nT_i + \Delta_i$, where Δ_i is a fixed time offset. Using the sequence

$$x_i(nT_i) = x_a(nT_i + \Delta_i), \qquad -\infty < n < \infty \tag{6.4.24}$$

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converters and digital signal processors it is more convenient and economic to sample directly the bandpass signal, as described in Section 6.4.1, and then obtain $x_I(n)$ and $x_Q(n)$ using the discrete-time approach developed in Section 6.5.

roadmap

6.5 Sampling of Discrete-Time Signals

(discrete downconversion)

In this section we use the techniques developed for the sampling and representation of continuous-time signals to discuss the sampling and reconstruction of lowpass and bandpass discrete-time signals. Our approach is to conceptually reconstruct the underlying continuous-time signal and then resample at the desired sampling rate. However, the final implementations involve only discrete-time operations. The more general area of sampling rate conversion is the subject of Chapter 11.

6.5.1 Sampling and Interpolation of Discrete-Time Signals

Suppose that a sequence $x(n)$ is sampled periodically by keeping every Dth sample of $x(n)$ and deleting the $(D-1)$ samples in between. This operation, which is also known as decimation or down-sampling, yields a new sequence defined by

$$x_d(n) = x(nD), \qquad -\infty < n < \infty \tag{6.5.1}$$

Without loss of generality we assume that $x(n)$ has been obtained by sampling a signal $x_a(t)$ with spectrum $X_a(F) = 0$, $|F| > B$ at a sampling rate $F_s = 1/T \geq 2B$, that is, $x(n) = x_a(nT)$. Therefore, the spectrum $X(F)$ of $x(n)$ is given by

$$X(F) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(F - kF_s) \tag{6.5.2}$$

We next sample $x_a(t)$ at time instants $t = nDT$, that is, with a sampling rate F_s/D. The spectrum of the sequence $x_d(n) = x_a(nDT)$ is provided by

$$X_d(F) = \frac{1}{DT}\sum_{k=-\infty}^{\infty} X_a\left(F - k\frac{F_s}{D}\right) \tag{6.5.3}$$

This process is illustrated in Figure 6.5.1 for $D = 2$ and $D = 4$. We can easily see from Figure 6.5.1(c) that the spectrum $X_d(F)$ can be expressed in terms of the periodic spectrum $X(F)$ as

$$X_d(F) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(F - k\frac{F_s}{D}\right) \tag{6.5.4}$$

To avoid aliasing, the sampling rate should satisfy the condition $F_s/D \geq 2B$. If the sampling frequency F_s is fixed, we can avoid aliasing by reducing the bandwidth of $x(n)$ to $(F_s/2)/D$. In terms of the normalized frequency variables, we can avoid aliasing if the highest frequency $f_{\max}$ or $\omega_{\max}$ in $x(n)$ satisfies the conditions

$$f_{\max} \leq \frac{1}{2D} = \frac{f_s}{2} \quad \text{or} \quad \omega_{\max} \leq \frac{\pi}{D} = \frac{\omega_s}{2} \tag{6.5.5}$$

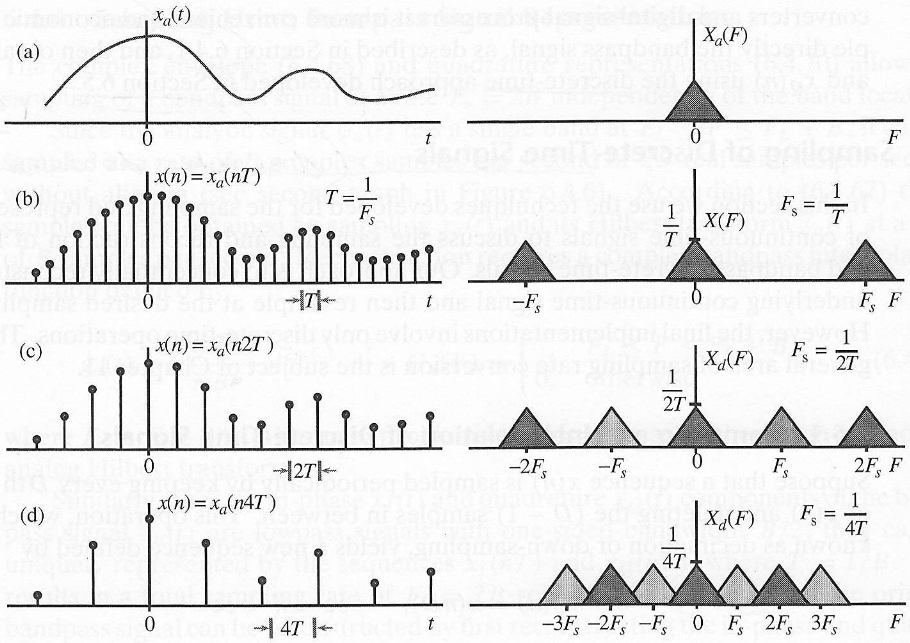

Figure 6.5.1 Illustration of discrete-time signal sampling in the frequency domain.

In continuous-time sampling the continuous-time spectrum $X_a(F)$ is repeated an infinite number of times to create a periodic spectrum covering the infinite frequency range. In discrete-time sampling the periodic spectrum $X(F)$ is repeated D times to cover one period of the periodic frequency domain.

To reconstruct the original sequence $x(n)$ from the sampled sequence $x_d(n)$, we start with the ideal interpolation formula

$$x_a(t) = \sum_{m=-\infty}^{\infty} x_d(m) \frac{\sin \frac{\pi}{DT}(t - mDT)}{\frac{\pi}{DT}(t - mDT)} \tag{6.5.6}$$

which reconstructs $x_a(t)$ assuming that $F_s/D \geq 2B$. Since $x(n) = x_a(nT)$, substitution into (6.5.6) yields

$$x(n) = \sum_{m=-\infty}^{\infty} x_d(m) \frac{\sin \frac{\pi}{D}(n - mD)}{\frac{\pi}{D}(n - mD)} \tag{6.5.7}$$

This is not a practical interpolator, since the $\sin(x)/x$ function is infinite in extent. In practice, we use a finite summation from $m = -L$ to $m = L$. The quality of this approximation improves with increasing L. The Fourier transform of the ideal bandlimited interpolation sequence in (6.5.7) is

$$g_{\mathrm{BL}}(n) = D \frac{\sin(\pi/D)n}{\pi n} \overset{\mathcal{F}}{\longleftrightarrow} G_{\mathrm{BL}}(\omega) = \begin{cases} D, & |\omega| \leq \pi/D \\ 0, & \pi/D < |\omega| \leq \pi \end{cases} \tag{6.5.8}$$

Therefore, the ideal discrete-time interpolator has an ideal lowpass frequency characteristic.

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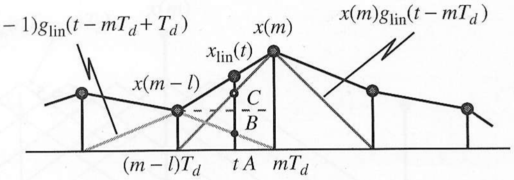

Figure 6.5.2 Illustration of continuous-time linear interpolation.

To understand the process of discrete-time interpolation, we will analyze the widely used linear interpolation. For simplicity we use the notation $T_d = DT$ for the sampling period of $x_d(m) = x_a(mT_d)$. The value of $x_a(t)$ at a time instant between mT_d and $(m+1)T_d$ is obtained by raising a vertical line from t to the line segment connecting the samples $x_d(mT_d)$ and $x_d(mT_d + T_d)$, as shown in Figure 6.5.2. The interpolated value is given by

$$x_{\text{lin}}(t) = x(m-1) + \frac{x(m) - x(m-1)}{T_d}[t - (m-1)T_d],\ (m-1)T_d \le t \le mT_d \quad (6.5.9)$$

which can be rearranged as follows:

$$x_{\text{lin}}(t) = \left[1 - \frac{t-(m-1)T_d}{T_d}\right] x(m-1) + \left[1 - \frac{mT_d - t)}{T_d}\right] x(m) \quad (6.5.10)$$

To put (6.5.10) in the form of the general reconstruction formula

$$x_{\text{lin}}(t) = \sum_{m=-\infty}^{\infty} x(m) g_{\text{lin}}(t - mT_d) \quad (6.5.11)$$

we note that we always have $t - (m-1)T_d = |t - (m-1)T_d|$ and $mT_d - t = |t - mT_d|$ because $(m-1)T_d \le t \le mT_d$. Therefore, we can express (6.5.10) in the form (6.5.11) if we define

$$g_{\text{lin}}(t) = \begin{cases} 1 - \dfrac{|t|}{T_d}, & |t| \le T_d \\ 0, & |t| > T_d \end{cases} \quad (6.5.12)$$

The discrete-time interpolation formulas are obtained by replacing t by nT in (6.5.11) and (6.5.12). Since $T_d = DT$, we obtain

$$x_{\text{lin}}(n) = \sum_{m=-\infty}^{\infty} x(m) g_{\text{lin}}(n - mD) \quad (6.5.13)$$

where

$$g_{\text{lin}}(n) = \begin{cases} 1 - \dfrac{|n|}{D} & |n| \le D \\ 0, & |n| > D \end{cases} \quad (6.5.14)$$

As expected from any interpolation function, $g_{\text{lin}}(0) = 1$ and $g_{\text{lin}}(n) = 0$ for $n = \pm D, \pm 2D, \ldots$. The performance of the linear interpolator can be assessed by comparing its Fourier transform

$$G_{\text{lin}}(\omega) = \frac{1}{D}\left[\frac{\sin(\omega D/2)}{\sin(\omega/2)}\right]^2 \quad (6.5.15)$$

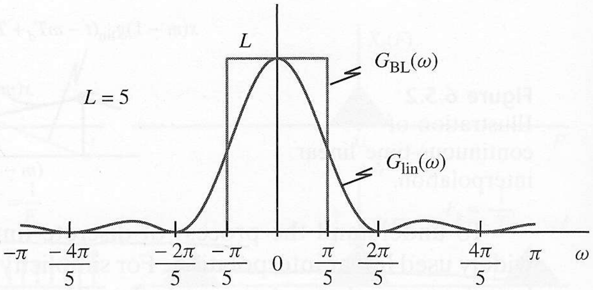

Figure 6.5.3
Frequency response of ideal and linear discrete-time interpolators.

to that of the ideal interpolator (6.5.8). This is illustrated in Figure 6.5.3 which shows that the linear interpolator has good performance only when the spectrum of the interpolated signal is negligible for $|\omega| > \pi/D$, that is, when the original continuous-time signal has been oversampled.

Main point

Equations (6.5.11) and (6.5.13) resemble a convolution summation; however, they are *not* convolutions. This is illustrated in Figure 6.5.4 which shows the computation of interpolated samples $x(nT)$ and $x((n+1)T)$ for $D = 5$. We note that only a subset of the coefficients of the linear interpolator is used in each case. Basically, we decompose $g_{\text{lin}}(n)$ into D components and we use one at a time periodically to compute the interpolated values. This is essentially the idea behind the polyphase filter structures discussed in Chapter 11. However, if we create a new sequence $\tilde{x}(n)$ by inserting $(D-1)$ zero samples between successive samples of $x_d(m)$, we can compute $x(n)$ using the convolution

$$x(n) = \sum_{k=-\infty}^{\infty} \tilde{x}(k) g_{\text{lin}}(n-k) \tag{6.5.16}$$

at the expense of unnecessary computations involving zero values. A more efficient implementation can be obtained using equation (6.5.13).

Sampling and interpolation of a discrete-time signal essentially corresponds to a change of its sampling rate by an integer factor. The subject of sampling rate conversion, which is very important in practical applications, it is extensively discussed in Chapter 11.

6.5.2 Representation and Sampling of Bandpass Discrete-Time Signals

The bandpass representations of continuous-time signals, discussed in Section 6.4.3, can be adapted for discrete-time signals with some simple modifications that take into consideration the periodic nature of discrete-time spectra. Since we cannot require that the discrete-time Fourier transform is zero for $\omega < 0$ without violating its periodicity, we define the analytic signal $\psi(n)$ of a bandpass sequence $x(n)$ by

$$\Psi(\omega) = \begin{cases} 2X(\omega), & 0 \le \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases} \tag{6.5.17}$$

where $X(\omega)$ and $\Psi(\omega)$ are the Fourier transforms of $x(n)$ and $\psi(n)$, respectively.

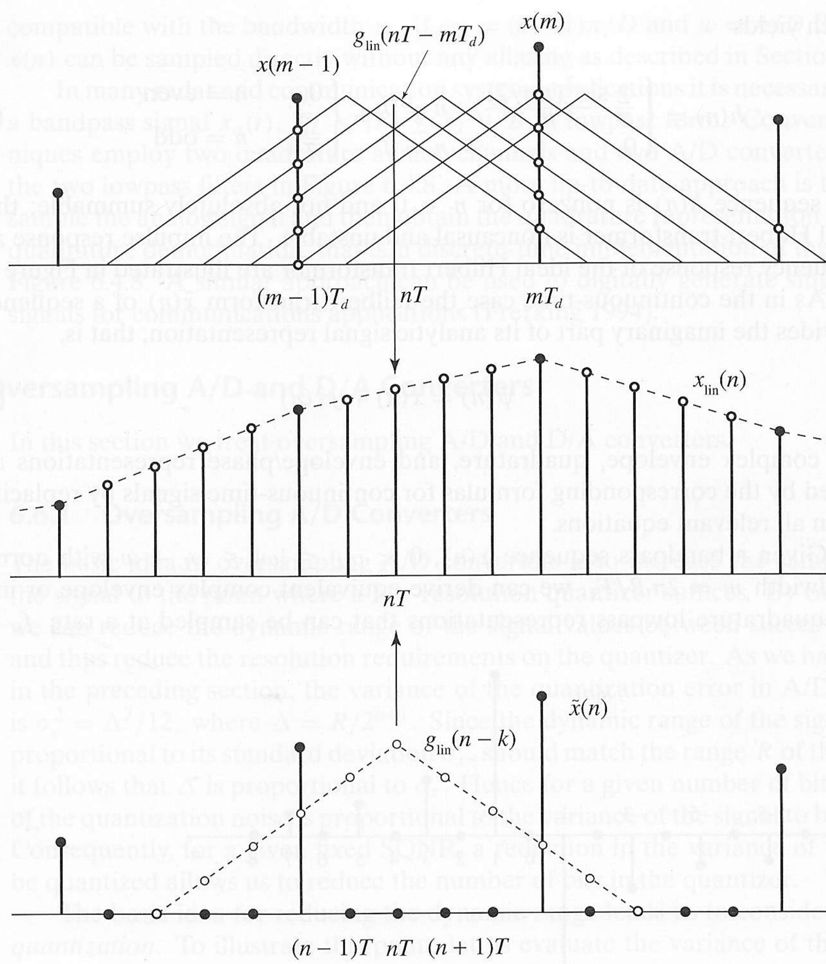

Figure 6.5.4 Illustration of linear interpolation as a linear filtering process.

The ideal discrete-time Hilbert transformer, defined by

$$H(\omega) = \begin{cases} -j, & 0 < \omega < \pi \\ j, & -\pi < \omega < 0 \end{cases} \tag{6.5.18}$$

is a 90-degree phase shifter as in the continuous-time case. We can easily show that

$$\Psi(\omega) = X(\omega) + j\hat{X}(\omega) \tag{6.5.19}$$

where

$$\hat{X}(\omega) = H(\omega)X(\omega) \tag{6.5.20}$$

To compute the analytic signal in the time domain, we need the impulse response of the Hilbert transformer. It is obtained by

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{0} je^{j\omega n}\,d\omega - \frac{1}{2\pi}\int_{0}^{\pi} je^{j\omega n}\,d\omega \tag{6.5.21}$$

which yields

$$h(n) = \begin{cases} \dfrac{2}{\pi}\dfrac{\sin^2(\pi n/2)}{n}, & n \neq 0 \\ 0, & n = 0 \end{cases} = \begin{cases} 0, & n = \text{even} \\ \dfrac{2}{\pi n}, & n = \text{odd} \end{cases} \tag{6.5.22}$$

The sequence $h(n)$ is nonzero for $n < 0$ and not absolutely summable; thus, the ideal Hilbert transformer is noncausal and unstable. The impulse response and the frequency response of the ideal Hilbert transformer are illustrated in Figure 6.5.5.

As in the continuous-time case the Hilbert transform $\hat{x}(n)$ of a sequence $x(n)$ provides the imaginary part of its analytic signal representation, that is,

$$\psi(n) = x(n) + j\hat{x}(n) \tag{6.5.23}$$

The complex envelope, quadrature, and envelope/phase representations are obtained by the corresponding formulas for continuous-time signals by replacing t by nT in all relevant equations.

Given a bandpass sequence $x(n)$, $0 < \omega_L \leq |\omega| \leq \omega_L + w$ with normalized bandwidth $w = 2\pi B/F_s$, we can derive equivalent complex envelope or in-phase and quadrature lowpass representations that can be sampled at a rate $f_s = 1/D$

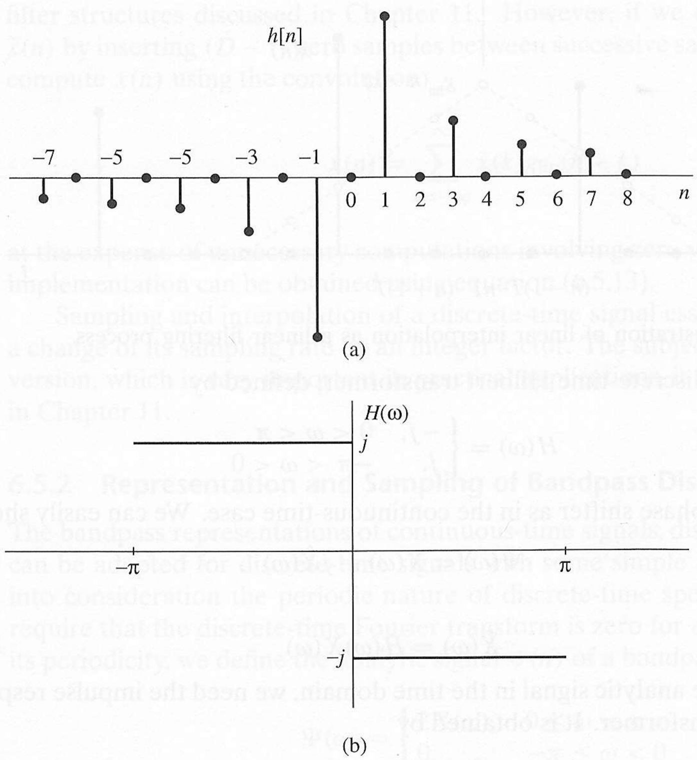

Figure 6.5.5 Impulse response (a) and frequency response (b) of the discrete-time Hilbert transformer.

compatible with the bandwidth w. If $\omega_L = (k-1)\pi/D$ and $w = \pi/D$ the sequence $x(n)$ can be sampled directly without any aliasing as described in Section 11.7.

In many radar and communication systems applications it is necessary to process a bandpass signal $x_a(t)$, $F_L \leq |F| \leq F_L + B$ in lowpass form. Conventional techniques employ two quadrature analog channels and two A/D converters following the two lowpass filters in Figure 6.4.8. A more up-to-date approach is to uniformly sample the analog signal and then obtain the quadrature representation using digital quadrature demodulation, that is, a discrete-time implementation of the first part of Figure 6.4.8. A similar approach can be used to digitally generate single sideband signals for communications applications (Frerking 1994).

6.6 Oversampling A/D and D/A Converters

In this section we treat oversampling A/D and D/A converters.

6.6.1 Oversampling A/D Converters

The basic idea in oversampling A/D converters is to increase the sampling rate of the signal to the point where a low-resolution quantizer suffices. By oversampling, we can reduce the dynamic range of the signal values between successive samples and thus reduce the resolution requirements on the quantizer. As we have observed in the preceding section, the variance of the quantization error in A/D conversion is $\sigma_e^2 = \Delta^2/12$, where $\Delta = R/2^{b+1}$. Since the dynamic range of the signal, which is proportional to its standard deviation σ_x, should match the range R of the quantizer, it follows that Δ is proportional to σ_x. Hence for a given number of bits, the power of the quantization noise is proportional to the variance of the signal to be quantized. Consequently, for a given fixed SQNR, a reduction in the variance of the signal to be quantized allows us to reduce the number of bits in the quantizer.

The basic idea for reducing the dynamic range leads us to consider *differential quantization*. To illustrate this point, let us evaluate the variance of the difference between two successive signal samples. Thus we have

$$d(n) = x(n) - x(n-1) \tag{6.6.1}$$

The variance of $d(n)$ is

$$\begin{aligned}
\sigma_d^2 &= E[d^2(n)] = E\{[x(n) - x(n-1)]^2\} \\
&= E[x^2(n)] - 2E[x(n)x(n-1)] + E[x^2(n-1)] \\
&= 2\sigma_x^2[1 - \gamma_{xx}(1)]
\end{aligned} \tag{6.6.2}$$

where $\gamma_{xx}(1)$ is the value of the autocorrelation sequence $\gamma_{xx}(m)$ of $x(n)$ evaluated at $m = 1$. If $\gamma_{xx}(1) > 0.5$, we observe that $\sigma_d^2 < \sigma_x^2$. Under this condition, it is better to quantize the difference $d(n)$ and to recover $x(n)$ from the quantized values $\{d_q(n)\}$. To obtain a high correlation between successive samples of the signal, we require that the sampling rate be significantly higher than the Nyquist rate.