CHAPTER 5

Optimal filtering with linear system models

5.1 INTRODUCTION

The previous chapters addressed the stochastic modeling problem: how do you develop system models that account for uncertainties in a proper, yet practical, fashion? Chapter four in particular developed practical dynamic system models in the form of linear stochastic differential or difference state equations, with associated linear output equations. Now we can exploit this background to solve a class of optimal estimation problems: equipped with such linear models and incomplete, noise-corrupted data from available sensors, how do you optimally estimate the quantities of interest to you?

Section 5.2 will formulate the problem in detail, and 5.3 will then derive the discrete-time (sampled data) optimal estimator for that problem formulation, the Kalman filter [35]. Section 5.4 investigates the characteristics of the processes within the filter structure, both to define algorithm behavior and to allow systematic event (failure) detection and adaptation. Section 5.5 delineates criteria other than that used in the derivation, with respect to which the algorithm is also optimal. Computational aspects and alternate but equivalent forms of the filter are considered in Sections 5.6 and 5.7, and stability in Section 5.8. In Sections 5.9 and 5.10 the problem formulation is extended and the resulting algorithms described. Finally, estimation for the case of continuously available measurement data and the relation of these results to Wiener filtering are explored in Sections 5.11 and 5.12.

5.2 PROBLEM FORMULATION

Assume that modeling techniques have produced an adequate system description in the form of a linear stochastic differential equation to describe the state propagation, with discrete-time noise-corrupted linear measurements available

as the system outputs. Let the *state* process $\mathbf{x}(\cdot,\cdot)$ of the system model satisfy the linear equation

$$\mathbf{dx}(t) = \mathbf{F}(t)\mathbf{x}(t)\,dt + \mathbf{B}(t)\mathbf{u}(t)\,dt + \mathbf{G}(t)\,\mathbf{d\beta}(t) \tag{5-1a}$$

or, in the less rigorous white noise notation,

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t) \tag{5-1b}$$

where $\mathbf{x}(\cdot,\cdot)$ is an n-vector state process, one sample of which would generate a state time history: $\mathbf{x}(t,\omega_i) = \mathbf{x}(t)$ would be the system state at time t; $\mathbf{u}(\cdot)$ is an r-vector of piecewise continuous deterministic control input functions (more general input functions are possible, but piecewise continuous is adequate for our purposes); $\mathbf{F}(\cdot)$ is an n-by-n system dynamics matrix (of piecewise continuous functions in its general form); $\mathbf{B}(\cdot)$ is an n-by-r deterministic input matrix; and $\mathbf{G}(\cdot)$ is an n-by-s noise input matrix. If (5-1a) is used, then $\boldsymbol{\beta}(\cdot,\cdot)$ is s-vector Brownian motion with statistics (for all $t, t' \in T$, $t \geq t'$):

$$\begin{aligned} E\{\boldsymbol{\beta}(t)\} &= \mathbf{0} \\ E\{[\boldsymbol{\beta}(t) - \boldsymbol{\beta}(t')][\boldsymbol{\beta}(t) - \boldsymbol{\beta}(t')]^{\mathrm{T}}\} &= \int_{t'}^{t} \mathbf{Q}(\tau)\,d\tau \end{aligned} \tag{5-2a}$$

with $\mathbf{Q}(\cdot)$ an s-by-s matrix of piecewise continuous functions (most generally) such that $\mathbf{Q}(t)$ is symmetric and positive semidefinite for all $t \in T$. On the other hand, if (5-1b) is used, then $\mathbf{w}(\cdot,\cdot)$ is s-vector white Gaussian noise with statistics

$$\begin{aligned} E\{\mathbf{w}(t)\} &= \mathbf{0} \\ E\{\mathbf{w}(t)\mathbf{w}(t')^{\mathrm{T}}\} &= \mathbf{Q}(t)\,\delta(t - t') \end{aligned} \tag{5-2b}$$

with the same description of $\mathbf{Q}(\cdot)$ as just given.

The state differential equation (5-1) is propagated forward from the *initial condition* $\mathbf{x}(t_0)$. For any particular operation of the real system, the initial state assumes a specific value $\mathbf{x}(t_0)$. However, because this value may not be known precisely a priori, it will be modeled as a random vector that is normally distributed. Thus, the description of $\mathbf{x}(t_0)$ is completely specified by the mean $\hat{\mathbf{x}}_0$ and covariance $\mathbf{P}_0$:

$$E\{\mathbf{x}(t_0)\} = \hat{\mathbf{x}}_0 \tag{5-3a}$$

$$E\{[\mathbf{x}(t_0) - \hat{\mathbf{x}}_0][\mathbf{x}(t_0) - \hat{\mathbf{x}}_0]^{\mathrm{T}}\} = \mathbf{P}_0 \tag{5-3b}$$

where $\mathbf{P}_0$ is an n-by-n matrix that is symmetric and positive semidefinite. Allowing $\mathbf{P}_0$ to be positive semidefinite, instead of requiring positive definiteness, admits the case of singular $\mathbf{P}_0$: the case in which some initial states or some linear combinations of initial states are known precisely.

Measurements are available at discrete time points, $t_1, t_2, \ldots t_i, \ldots$ (often, but not necessarily, equally spaced in time), and are modeled by the relation

(for all $t_i \in T$):

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \tag{5-4}$$

where $\mathbf{z}(\cdot,\cdot)$ is an m-vector discrete-time measurement process, one sample of which provides a particular measurement time history: $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$ would be the measurement numbers that become available at time t_i; $\mathbf{H}(\cdot)$ is an m-by-n measurement matrix; $\mathbf{x}(\cdot,\cdot)$ is the state vector process: $\mathbf{x}(t_i,\cdot)$ is a random vector corresponding to the state vector process at the particular time t_i; and $\mathbf{v}(\cdot,\cdot)$ is an m-vector discrete-time white Gaussian noise with statistics (for all t_i, $t_j \in T$):

$$E\{\mathbf{v}(t_i)\} = \mathbf{0} \tag{5-5a}$$

$$E\{\mathbf{v}(t_i)\mathbf{v}^{\mathrm{T}}(t_j)\} = \begin{cases} \mathbf{R}(t_i) & t_i = t_j \\ \mathbf{0} & t_i \neq t_j \end{cases} \tag{5-5b}$$

In this description, $\mathbf{R}(t_i)$ is an m-by-m, symmetric, positive definite matrix for all $t_i \in T$. Positive definiteness of $\mathbf{R}(t_i)$ implies that all components of the measurement vector are corrupted by noise, and there is no linear combination of these components that would be noise-free. The measurements modeled as in (5-4) are all that we have available from the real system under consideration.

It is further assumed that $\mathbf{x}(t_0)$, $\boldsymbol{\beta}(\cdot,\cdot)$ or $\mathbf{w}(\cdot,\cdot)$, and $\mathbf{v}(\cdot,\cdot)$ are independent of each other. Since all are assumed Gaussian, this is equivalent to assuming that they are uncorrelated with each other.

It is desired to combine the measurement data taken from the actual system with the information provided by the system model and statistical description of uncertainties, in order to obtain an "optimal" estimate of the system state. In general, the "optimality" of the estimate depends upon what performance criterion is chosen. We will adopt the Bayesian point of view and seek the means of propagating the conditional probability density of the state, conditioned on the entire history of measurements taken [50]. Once this is accomplished, then the "optimal estimate" can be defined, but attention will be focused on the *entire conditional density itself*, as it embodies considerably more information in general than any single estimate value. Under the assumptions of this problem formulation, the conditional density will be shown to remain Gaussian at all times, and therefore the mean, mode, median, and essentially any logical choice of estimate based upon the conditional density will all converge upon the same estimate value.

The problem formulation just described is not the most general possible. The case of $\mathbf{R}(t_i)$ being positive semidefinite instead of positive definite, or even null in the extreme case, can be considered and will warrant subsequent attention. Furthermore, the assumed independence (uncorrelatedness) of the dynamic driving noise and the measurement noise may not provide an adequate model in some applications. For instance, in applications of optimal estimators to aircraft

navigation systems, commonly used models embody INS (inertial navigation system) noise sources in the dynamic driving noise and onboard radar uncertainties in the measurement noise: since the aircraft's own vibration affects both of those systems, there is in fact some degree of correlation among the noise sources. Admitting such correlations into the problem formulation is also possible, though again the derivation is more complicated. The given problem statement will serve as the basis for the derivation in the next section. Subsequently, extensions to the problem formulation and resulting solutions will be considered.

Before the estimator is derived, it will be convenient to introduce a new notational representation. Define a composite vector which comprises the entire measurement history to the current time, and denote it as $\mathbf{Z}(t_i)$, where

$$\mathbf{Z}(t_i) = \begin{bmatrix} \mathbf{z}(t_1) \\ \mathbf{z}(t_2) \\ \vdots \\ \mathbf{z}(t_i) \end{bmatrix} \tag{5-6}$$

This is a vector of growing dimension: at time t_i, it is a vector random variable of dimension $(i \cdot m)$ that models the information of the entire measurement history. Its realized value, analogous to $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$, is $\mathbf{Z}_i$, where

$$\mathbf{Z}_i = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_i \end{bmatrix} \tag{5-7}$$

This is then the history of actual measurement values obtained in a single experiment (trial). Finally, the dummy variable associated with $\mathbf{Z}(t_i)$, corresponding to ζ_i being associated with $\mathbf{z}(t_i)$, will be denoted as $\mathscr{Z}_i$.

5.3 THE DISCRETE-TIME (SAMPLED DATA) OPTIMAL ESTIMATOR: THE KALMAN FILTER

We are going to consider two measurement times, t_{i-1} and t_i, and will propagate optimal estimates from the point just after the measurement at time t_{i-1} has been incorporated into the estimate, to the point just after the measurement at time t_i is incorporated. This is depicted in Fig. 5.1 as propagating from time t_{i-1}^+ to time t_i^+.

Suppose we are at time t_{i-1} and have just taken and processed the measurement $\mathbf{z}(t_{i-1}, \omega_j) = \mathbf{z}_{i-1}$. From a Bayesian point of view, we are really interested in the probability density for $\mathbf{x}(t_{i-1})$ conditioned on the entire measurement history to that time, $f_{\mathbf{x}(t_{i-1})|\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi} | \mathbf{Z}_{i-1})$, and how this density can be propa-

FIG. 5.1 Estimate propagation. (a) Measurement $\mathbf{z}(t_{i-1}, \omega_j) = \mathbf{z}_{i-1}$ becomes available. (b) Measurement $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$ becomes available.

gated forward through the next measurement time to generate $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\xi|\mathbf{Z}_i)$. Once the densities are described explicitly, the optimal estimate of the state at time t_i can be determined.

To start the derivation, we will *assume* that $f_{\mathbf{x}(t_{i-1})|\mathbf{Z}(t_{i-1})}(\xi|\mathbf{Z}_{i-1})$ is a *Gaussian* conditional density:

$$f_{\mathbf{x}(t_{i-1})|\mathbf{Z}(t_{i-1})}(\xi|\mathbf{Z}_{i-1}) = [(2\pi)^{n/2}|\mathbf{P}(t_{i-1}^+)|^{1/2}]^{-1}\exp\{\cdot\}$$
$$\{\cdot\} = \{-\tfrac{1}{2}[\xi - \hat{\mathbf{x}}(t_{i-1}^+)]^{\mathrm{T}}\mathbf{P}^{-1}(t_{i-1}^+)[\xi - \hat{\mathbf{x}}(t_{i-1}^+)]\} \tag{5-8}$$

where we *define* $\hat{\mathbf{x}}(t_{i-1}^+)$ and $\mathbf{P}(t_{i-1}^+)$ to be the conditional mean and conditional covariance, respectively:

$$\hat{\mathbf{x}}(t_{i-1}^+) \triangleq E\{\mathbf{x}(t_{i-1})|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \tag{5-9}$$

$$\mathbf{P}(t_{i-1}^+) \triangleq E\{[\mathbf{x}(t_{i-1}) - \hat{\mathbf{x}}(t_{i-1}^+)][\mathbf{x}(t_{i-1}) - \hat{\mathbf{x}}(t_{i-1}^+)]^{\mathrm{T}}|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \tag{5-10}$$

We will be able to verify this assumption, and in fact this can be visualized as an inductive proof type of derivation, since $f_{\mathbf{x}(t_0)|\mathbf{Z}(t_0)}(\xi)$ is actually $f_{\mathbf{x}(t_0)}(\xi)$ [because the first measurement is at time t_1, so $\mathbf{Z}(t_0)$ is no measurement information at all], and $f_{\mathbf{x}(t_0)}(\xi)$ is assumed to be a Gaussian density. The following derivation considers the case from time t_{i-1} to time t_i, and combining this with the (essentially duplicate) results from t_0 to t_1 would complete an inductive proof.

Furthermore, in the process of deriving the estimator algorithm, we will be able to verify that the conditional covariance defined in (5-10) equals the unconditional covariance. In other words, the covariance recursion is *not* dependent upon the actual values of the measurements taken, and thus can be computed *without* knowledge of the realized measurement values $\mathbf{Z}_i$. For this reason, we will be able to *precompute* the time history of the covariance of the errors committed by using $\hat{\mathbf{x}}(t_i^+)$ as the optimal estimate of the state at time t_i [recall the discussion in Chapter 3: $\hat{\mathbf{x}}(t_i^+, \cdot)$ would be defined as $E\{\mathbf{x}(t_i)|\mathbf{Z}(t_i) = \mathbf{Z}(t_i, \cdot)\}$]. This will be of considerable practical significance, and will be exploited in both this chapter and the next.

Recall Fig. 5.1: we want to propagate the conditional density and associated estimate from time t_{i-1}^+, just after incorporating the measurement $\mathbf{z}(t_{i-1}, \omega_j) = \mathbf{z}_{i-1}$, to time t_i^+, just after incorporating $\mathbf{z}_i$. Let us decompose this into two

steps: (1) a time propagation from t_{i-1}^{+} to t_i^{-}, at time t_i just before the measurement $\mathbf{z}_i$ is incorporated, and (2) a measurement update from t_i^{-} to t_i^{+}. To make this derivation algebraically simpler, we will at first neglect the deterministic control inputs in (5-1). Later these will be incorporated by modifying only the mean equations of the algorithm: under our assumptions, these known inputs have no effect on the spread of density functions, only on their location.

First consider the *time propagation* from t_{i-1}^{+} to t_i^{-}. From the Bayesian point of view, we want to establish the conditional density of the state at time t_i, conditioned on the measurement history up through the previous sample time t_{i-1}: $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}\,|\,\mathbf{Z}_{i-1})$. Conceptually, we will first prove that this density is Gaussian under the assumptions of Section 5.2, and then it will be characterized completely by explicitly evaluating its mean and covariance.

By our model, $\mathbf{x}(t_i)$ can be written as

$$\mathbf{x}(t_i) = \boldsymbol{\Phi}(t_i, t_{i-1})\mathbf{x}(t_{i-1}) + \mathbf{w}_{\mathrm{d}}(t_{i-1}) \tag{5-11}$$

where, in the context of equivalent discrete-time models,

$$\mathbf{w}_{\mathrm{d}}(t_{i-1}) = \int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i, \tau)\mathbf{G}(\tau)\,\mathbf{d}\boldsymbol{\beta}(\tau) \tag{5-12}$$

Since (5-11) expresses $\mathbf{x}(t_i)$ as a linear combination of $\mathbf{x}(t_{i-1})$ and $\mathbf{w}_{\mathrm{d}}(t_{i-1})$, $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}\,|\,\mathscr{Z}_{i-1})$ will be Gaussian if we can show that $f_{\mathbf{x}(t_{i-1}),\,\mathbf{w}_{\mathrm{d}}(t_{i-1})|\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}, \boldsymbol{\rho}\,|\,\mathscr{Z}_{i-1})$ is Gaussian. By Bayes' rule,

$$f_{\mathbf{x}(t_{i-1}),\,\mathbf{w}_{\mathrm{d}}(t_{i-1})|\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}, \boldsymbol{\rho}\,|\,\mathscr{Z}_{i-1}) = \frac{f_{\mathbf{x}(t_{i-1}),\,\mathbf{w}_{\mathrm{d}}(t_{i-1}),\,\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}, \boldsymbol{\rho}, \mathscr{Z}_{i-1})}{f_{\mathbf{Z}(t_{i-1})}(\mathscr{Z}_{i-1})}$$

Because $\mathbf{w}_{\mathrm{d}}(t_{i-1})$ is independent of $\mathbf{x}(t_{i-1})$ and $\mathbf{Z}(t_{i-1})$, the numerator in this expression can be decomposed and the result recombined by another application of Bayes' rule to obtain

$$\begin{aligned} f_{\mathbf{x}(t_{i-1}),\,\mathbf{w}_{\mathrm{d}}(t_{i-1})|\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}, \boldsymbol{\rho}\,|\,\mathscr{Z}_{i-1}) &= \frac{f_{\mathbf{x}(t_{i-1}),\,\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}, \mathscr{Z}_{i-1}) f_{\mathbf{w}_{\mathrm{d}}(t_{i-1})}(\boldsymbol{\rho})}{f_{\mathbf{Z}(t_{i-1})}(\mathscr{Z}_{i-1})} \\ &= f_{\mathbf{x}(t_{i-1})|\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}\,|\,\mathscr{Z}_{i-1}) f_{\mathbf{w}_{\mathrm{d}}(t_{i-1})}(\boldsymbol{\rho}) \end{aligned}$$

The density $f_{\mathbf{x}(t_{i-1})|\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}|\mathscr{Z}_{i-1})$ has been assumed to be Gaussian in this induction, and $f_{\mathbf{w}_{\mathrm{d}}(t_{i-1})}(\boldsymbol{\rho})$ is Gaussian according to our dynamics model, so their product is also Gaussian. Thus, conditioned on $\mathbf{Z}(t_{i-1})$, $\mathbf{x}(t_{i-1})$ and $\mathbf{w}_{\mathrm{d}}(t_{i-1})$ are jointly Gaussian, and so $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}\,|\,\mathscr{Z}_{i-1})$ is in fact a Gaussian conditional density.

To specify the density completely, its mean and covariance will now be computed. The conditional mean is found by invoking the linearity of the con-

ditional expectation operator and the nonrandomness of $\mathbf{\Phi}(t_i, t_{i-1})$ to write

$$\begin{aligned} E\{\mathbf{x}(t_i)\big|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} &= E\{\mathbf{\Phi}(t_i, t_{i-1})\mathbf{x}(t_{i-1}) + \mathbf{w}_d(t_{i-1})\big|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \\ &= \mathbf{\Phi}(t_i, t_{i-1})E\{\mathbf{x}(t_{i-1})\big|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \\ &\quad + E\{\mathbf{w}_d(t_{i-1})\big|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \end{aligned}$$

But $\mathbf{w}_d(t_{i-1})$ is independent of $\mathbf{Z}(t_{i-1})$, so its conditional mean equals its unconditional mean, which was assumed to be zero, so

$$E\{\mathbf{x}(t_i)\big|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} = \mathbf{\Phi}(t_i, t_{i-1})E\{\mathbf{x}(t_{i-1})\big|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \tag{5-13}$$

Now let $\hat{\mathbf{x}}(t_i^-)$ denote the conditional mean of $\mathbf{x}(t_i)$ before the measurement $\mathbf{z}(t_i) = \mathbf{z}_i$ is taken and processed, i.e.,

$$\hat{\mathbf{x}}(t_i^-) \triangleq E\{\mathbf{x}(t_i)\big|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \tag{5-14}$$

In terms of this notation and that of (5-9), the conditional mean time propagation relation can be written as

$$\hat{\mathbf{x}}(t_i^-) = \mathbf{\Phi}(t_i, t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+) \tag{5-15}$$

Similarly, if we define $\mathbf{P}(t_i^-)$ to be the conditional covariance of $\mathbf{x}(t_i)$ before the measurement $\mathbf{z}(t_i) = \mathbf{z}_i$ is taken and processed,

$$\mathbf{P}(t_i^-) \triangleq E\{[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^-)][\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^-)]^{\mathrm{T}}\big|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \tag{5-16}$$

then the conditional covariance propagates in time as

$$\begin{aligned} \mathbf{P}(t_i^-) &= \mathbf{\Phi}(t_i, t_{i-1})\mathbf{P}(t_{i-1}^+)\mathbf{\Phi}^{\mathrm{T}}(t_i, t_{i-1}) \\ &\quad + \int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\mathbf{\Phi}^{\mathrm{T}}(t_i, \tau)\,d\tau \end{aligned} \tag{5-17}$$

If $\hat{\mathbf{x}}(t_i^-)$ is used as the estimate of $\mathbf{x}(t_i)$ before $\mathbf{z}_i$ is processed, then $[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^-)]$ is the error committed by the estimator $\hat{\mathbf{x}}(t_i^-)$ for the particular measurement history realization $\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}$. Consequently $\mathbf{P}(t_i^-)$ is the conditional covariance not only of the state, but also of the error committed by using the conditional mean as an estimator of the state (this error will be shown to be zero-mean). The density function we have been seeking can now be written explicitly as

$$\begin{aligned} f_{\mathbf{x}(t_i)|\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}\big|\mathbf{Z}_{i-1}) &= [(2\pi)^{n/2}\big|\mathbf{P}(t_i^-)\big|^{1/2}]^{-1}\exp\{\cdot\} \\ \{\cdot\} &= \{-\tfrac{1}{2}[\boldsymbol{\xi} - \hat{\mathbf{x}}(t_i^-)]^{\mathrm{T}}\mathbf{P}^{-1}(t_i^-)[\boldsymbol{\xi} - \hat{\mathbf{x}}(t_i^-)]\} \end{aligned} \tag{5-18}$$

where $\hat{\mathbf{x}}(t_i^-)$ and $\mathbf{P}(t_i^-)$ are given by (5-15) and (5-17), respectively.

Now we want to consider incorporating the measurement that becomes available at time t_i, $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$, so as to generate the density $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}\big|\mathbf{Z}_i)$. Repeated application of Bayes' rule will allow us to write this density in terms of three other densities, each of which can be evaluated rather easily. It is such

a desirable result that motivates and guides the particular usage of Bayes' rule that follows. For convenience and compactness, the arguments of the density functions will be omitted. Bayes' rule and the definition of $\mathbf{Z}(t_i)$ as the composite of $\mathbf{Z}(t_{i-1})$ and $\mathbf{z}(t_i)$ yield

$$
\begin{aligned}
f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)} &= \frac{f_{\mathbf{x}(t_i),\,\mathbf{Z}(t_i)}}{f_{\mathbf{Z}(t_i)}} \\
&= \frac{f_{\mathbf{x}(t_i),\,\mathbf{z}(t_i),\,\mathbf{Z}(t_{i-1})}}{f_{\mathbf{z}(t_i),\,\mathbf{Z}(t_{i-1})}} \\
&= \frac{f_{\mathbf{z}(t_i)|\mathbf{x}(t_i),\,\mathbf{Z}(t_{i-1})} f_{\mathbf{x}(t_i),\,\mathbf{Z}(t_{i-1})}}{f_{\mathbf{z}(t_i)|\mathbf{Z}(t_{i-1})} f_{\mathbf{Z}(t_{i-1})}} \\
&= \frac{f_{\mathbf{z}(t_i)|\mathbf{x}(t_i),\,\mathbf{Z}(t_{i-1})} f_{\mathbf{x}(t_i)|\mathbf{Z}(t_{i-1})} f_{\mathbf{Z}(t_{i-1})}}{f_{\mathbf{z}(t_i)|\mathbf{Z}(t_{i-1})} f_{\mathbf{Z}(t_{i-1})}}
\end{aligned}
$$

Canceling like terms yields the final result,

$$
f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)} = \frac{f_{\mathbf{z}(t_i)|\mathbf{x}(t_i),\,\mathbf{Z}(t_{i-1})} f_{\mathbf{x}(t_i)|\mathbf{Z}(t_{i-1})}}{f_{\mathbf{z}(t_i)|\mathbf{Z}(t_{i-1})}} \tag{5-19}
$$

The task at hand is to evaluate each density on the right hand side of (5-19), seeking eventually to show that $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\xi|\mathscr{Z}_i)$ is Gaussian and to display it explicitly.

The second numerator term has already been established, and is given by (5-18), so let us consider the other numerator term,

$$
f_{\mathbf{z}(t_i)|\mathbf{x}(t_i),\,\mathbf{Z}(t_{i-1})}(\zeta_i|\xi, \mathscr{Z}_{i-1})
$$

According to our system model, the measurement $\mathbf{z}(t_i)$ is given by

$$
\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \tag{5-20}
$$

We desire the density function for the random variable $\mathbf{z}(t_i)$, conditioned not only upon knowledge of the previous measurement history but also upon the fact that we *know* $\mathbf{x}(t_i)$ has assumed the realization ξ. That fact fixes the random variable $\mathbf{H}(t_i)\mathbf{x}(t_i)$ at the single known value of $\mathbf{H}(t_i)\xi$, with no uncertainty. Moreover, $\mathbf{v}(t_i)$ is independent of $\mathbf{x}(t_i)$ and $\mathbf{Z}(t_{i-1})$, and is assumed Gaussian with mean zero and covariance matrix $\mathbf{R}(t_i)$. Conditioned on $\mathbf{x}(t_i) = \xi$ and $\mathbf{Z}(t_{i-1}) = \mathscr{Z}_{i-1}$, $\mathbf{z}(t_i)$ is a linear combination of a known vector and a Gaussian random vector, so $f_{\mathbf{z}(t_i)|\mathbf{x}(t_i),\,\mathbf{Z}(t_{i-1})}(\zeta_i|\xi, \mathscr{Z}_{i-1})$ is a Gaussian density, completely specified by its mean and covariance. The mean is given by

$$
\begin{aligned}
E\{\mathbf{z}(t_i)|\mathbf{x}(t_i) = \xi, \mathbf{Z}(t_{i-1}) = \mathscr{Z}_{i-1}\} &= \mathbf{H}(t_i)E\{\mathbf{x}(t_i)|\mathbf{x}(t_i) = \xi, \mathbf{Z}(t_{i-1}) = \mathscr{Z}_{i-1}\} \\
&\quad + E\{\mathbf{v}(t_i)|\mathbf{x}(t_i) = \xi, \mathbf{Z}(t_{i-1}) = \mathscr{Z}_{i-1}\} \\
&= \mathbf{H}(t_i)\xi
\end{aligned} \tag{5-21}
$$

The covariance matrix is:

$$E\{[\mathbf{z}(t_i) - \mathbf{H}(t_i)\xi][\mathbf{z}(t_i) - \mathbf{H}(t_i)\xi]^{\mathrm{T}} | \mathbf{x}(t_i) = \xi, \mathbf{Z}(t_{i-1}) = \mathscr{Z}_{i-1}\} = \mathbf{R}(t_i) \quad (5\text{-}22)$$

Thus, we can write

$$f_{\mathbf{z}(t_i)|\mathbf{x}(t_i),\,\mathbf{Z}(t_{i-1})}(\zeta_i | \xi, \mathscr{Z}_{i-1}) = [(2\pi)^{m/2}|\mathbf{R}(t_i)|^{1/2}]^{-1} \exp\{\cdot\}$$
$$\{\cdot\} = \{-\tfrac{1}{2}[\zeta_i - \mathbf{H}(t_i)\xi]^{\mathrm{T}}\mathbf{R}^{-1}(t_i)[\zeta_i - \mathbf{H}(t_i)\xi]\} \quad (5\text{-}23)$$

Having evaluated the numerator terms in (5-19), we now consider the denominator, $f_{\mathbf{z}(t_i)|\mathbf{Z}(t_{i-1})}(\zeta_i | \mathscr{Z}_{i-1})$. The measurement $\mathbf{z}(t_i)$ is again described as in (5-20), but now we are conditioning only on knowledge of the previous time history of measurements. First we want to show that $f_{\mathbf{z}(t_i)|\mathbf{Z}(t_{i-1})}(\zeta_i | \mathscr{Z}_{i-1})$ is Gaussian. Since $\mathbf{z}(t_i)$ is a linear combination of $\mathbf{x}(t_i)$ and $\mathbf{v}(t_i)$, we will be able to achieve this objective if we can show that, conditioned on $\mathbf{Z}(t_{i-1})$, $\mathbf{x}(t_i)$ and $\mathbf{v}(t_i)$ are jointly Gaussian. Bayes' rule yields:

$$f_{\mathbf{x}(t_i),\,\mathbf{v}(t_i)|\mathbf{Z}(t_{i-1})}(\xi, \eta | \mathscr{Z}_{i-1}) = f_{\mathbf{v}(t_i)|\mathbf{x}(t_i),\,\mathbf{Z}(t_{i-1})}(\eta | \xi, \mathscr{Z}_{i-1}) \cdot f_{\mathbf{x}(t_i)|\mathbf{Z}(t_{i-1})}(\xi | \mathscr{Z}_{i-1})$$

But $\mathbf{v}(t_i)$ is independent of $\mathbf{x}(t_i)$ and $\mathbf{Z}(t_{i-1})$, so this becomes

$$f_{\mathbf{x}(t_i),\,\mathbf{v}(t_i)|\mathbf{Z}(t_{i-1})}(\xi, \eta | \mathscr{Z}_{i-1}) = f_{\mathbf{v}(t_i)}(\eta) f_{\mathbf{x}(t_i)|\mathbf{Z}(t_{i-1})}(\xi | \mathscr{Z}_{i-1})$$

The two separate densities on the right hand side of this expression are each Gaussian, so their product is Gaussian, and thus $f_{\mathbf{z}(t_i)|\mathbf{Z}(t_{i-1})}(\zeta_i | \mathscr{Z}_{i-1})$ is itself Gaussian. The mean is calculated as:

$$\begin{aligned} E\{\mathbf{z}(t_i) | \mathbf{Z}(t_{i-1}) = \mathscr{Z}_{i-1}\} &= \mathbf{H}(t_i) E\{\mathbf{x}(t_i) | \mathbf{Z}(t_{i-1}) = \mathscr{Z}_{i-1}\} \\ &\quad + E\{\mathbf{v}(t_i) | \mathbf{Z}(t_{i-1}) = \mathscr{Z}_{i-1}\} \\ &= \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-) \end{aligned} \quad (5\text{-}24)$$

The covariance is computed as

$$\begin{aligned} &E\{[\mathbf{z}(t_i) - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)][\mathbf{z}(t_i) - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]^{\mathrm{T}} | \mathbf{Z}(t_{i-1}) = \mathscr{Z}_{i-1}\} \\ &\quad = E\{[\mathbf{H}(t_i)\mathbf{x}(t_i) - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-) + \mathbf{v}(t_i)] \\ &\qquad \cdot [\mathbf{H}(t_i)\mathbf{x}(t_i) - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-) + \mathbf{v}(t_i)]^{\mathrm{T}} | \mathbf{Z}(t_{i-1}) = \mathscr{Z}_{i-1}\} \\ &\quad = \mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i) \end{aligned} \quad (5\text{-}25)$$

Since $f_{\mathbf{z}(t_i)|\mathbf{Z}(t_{i-1})}(\zeta_i | \mathscr{Z}_{i-1})$ is a Gaussian density, we can now write:

$$f_{\mathbf{z}(t_i)|\mathbf{Z}(t_{i-1})}(\zeta_i | \mathscr{Z}_{i-1}) = [(2\pi)^{m/2}|\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)|^{1/2}]^{-1} \exp\{\cdot\}$$
$$\{\cdot\} = \{-\tfrac{1}{2}[\zeta_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]^{\mathrm{T}}[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)]^{-1}[\zeta_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]\} \quad (5\text{-}26)$$

At this point, we have written (5-19) to generate a measurement update expression for evaluating $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\xi | \mathscr{Z}_i)$, and we have depicted each of the

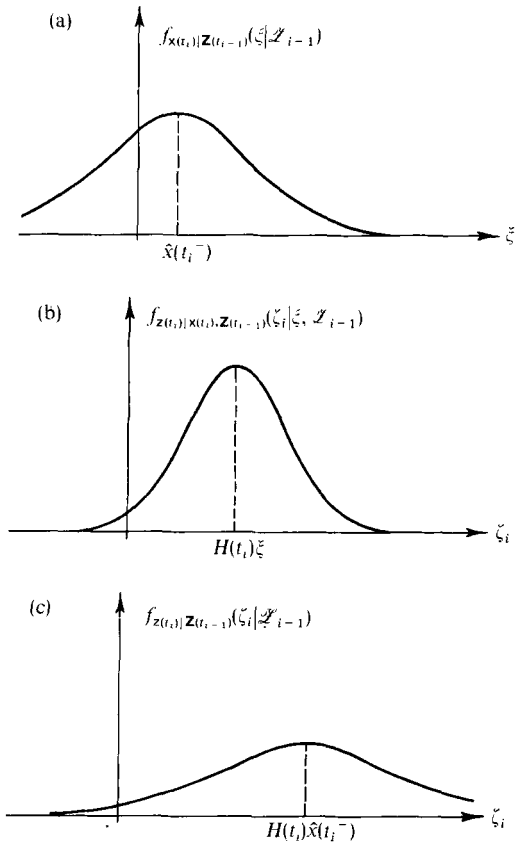

FIG. 5.2 Evaluation of densities for use in Bayes' rule. (a) Variance = $P(t_i^-)$. (b) Variance = $R(t_i)$. (c) Variance = $H(t_i)P(t_i^-)H(t_i) + R(t_i)$.

separate Gaussian densities explicitly. This is portrayed graphically in Fig. 5.2. Substituting (5-18), (5-23), and (5-26) into (5-19) yields

$$f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathscr{Z}_i) = \frac{|\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)|^{1/2}}{(2\pi)^{n/2}|\mathbf{P}(t_i^-)|^{1/2}|\mathbf{R}(t_i)|^{1/2}} \exp\{-\tfrac{1}{2}(\cdot)\}$$

$$\begin{aligned}(\cdot) = {} & [\boldsymbol{\zeta}_i - \mathbf{H}(t_i)\boldsymbol{\xi}]^{\mathrm{T}}\mathbf{R}(t_i)^{-1}[\boldsymbol{\zeta}_i - \mathbf{H}(t_i)\boldsymbol{\xi}] \\ & + [\boldsymbol{\xi} - \hat{\mathbf{x}}(t_i^-)]^{\mathrm{T}}\mathbf{P}(t_i^-)^{-1}[\boldsymbol{\xi} - \hat{\mathbf{x}}(t_i^-)] \\ & - [\boldsymbol{\zeta}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]^{\mathrm{T}}[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)]^{-1}[\boldsymbol{\zeta}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]\end{aligned} \tag{5-27}$$

It is not immediately evident that (5-27) is in fact of Gaussian form: the three separate determinant terms would have to be equivalent to a single determinant square root in the denominator of the leading coefficient, and the sum of the

three quadratics in the exponential would have to be equivalent to a single quadratic form. We will demonstrate the quadratic form equivalency, and the manipulation of determinants is left to Problem 5.4 at the end of the chapter.

To achieve the desired result, we will require use of "the matrix inversion lemma," valid for positive definite $\mathbf{P}$ and $\mathbf{R}$:

$$[\mathbf{P}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1} = \mathbf{P} - \mathbf{P}\mathbf{H}^T[\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}]^{-1}\mathbf{H}\mathbf{P} \tag{5-28}$$

This lemma is important enough to warrant proving; one such proof is outlined in Problem 5.2. It is of special interest to us because the left hand side involves inversion of n-by-n matrices, whereas the right hand side requires m-by-m matrix inversion: in most problems of interest, m will be significantly less than n. A more general lemma, admitting positive semidefinite $\mathbf{P}$, can be proven to yield

$$[\mathbf{I} + \mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1}\mathbf{P} = \mathbf{P} - \mathbf{P}\mathbf{H}^T[\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}]^{-1}\mathbf{H}\mathbf{P} \tag{5-28'}$$

but we will not need this generality here. In fact, in the ensuing proof, we will *assume* $\mathbf{P}(t_i^-)$ to be positive definite and at the end of the development we will return to establish under what conditions the assumption is valid. Once (5-28) is proven, it is straightforward to generate two other useful matrix identities (see Problem 5.3) as well:

$$[\mathbf{P}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1}\mathbf{H}^T\mathbf{R}^{-1} = \mathbf{P}\mathbf{H}^T[\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}]^{-1} \tag{5-29}$$

$$\mathbf{H}[\mathbf{P}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1}\mathbf{H}^T = \mathbf{R} - \mathbf{R}[\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}]^{-1}\mathbf{R} \tag{5-30}$$

What follows is somewhat difficult to motivate, in much the same way as scalar completion of squares is, except by keeping the overall objective firmly in mind: to generate a single quadratic form from the sum of three quadratics in (5-27). Through expanding, exploiting algebraic identities, and regrouping, we seek to combine all terms into a single quadratic. To make the algebra more tractable, we will omit the time notation, and denote $\hat{\mathbf{x}}(t_i^-)$ and $\mathbf{P}(t_i^-)$ by $\hat{\mathbf{x}}^-$ and $\mathbf{P}^-$, respectively.

REDUCTION TO A SINGLE QUADRATIC FORM First expand the terms denoted by $(\cdot)$ in (5-27) to get the sum of 12 terms, which can be combined conveniently as (recalling that the transpose of a scalar is just the scalar itself)

$$\begin{aligned}(\cdot) = {} & \xi^T[\mathbf{P}^{-\,-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]\xi - 2\xi^T[\mathbf{P}^{-\,-1}\hat{\mathbf{x}}^- + \mathbf{H}^T\mathbf{R}^{-1}\zeta_i] \\ & + \zeta_i^T[\mathbf{R}^{-1} - (\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R})^{-1}]\zeta_i + 2\hat{\mathbf{x}}^{-T}\mathbf{H}^T[\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R}]^{-1}\zeta_i \\ & + \hat{\mathbf{x}}^{-T}[\mathbf{P}^{-\,-1} - \mathbf{H}^T(\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R})^{-1}\mathbf{H}]\hat{\mathbf{x}}^-\end{aligned}$$

Consider the third term. The matrix which appears in it can be obtained by premultiplying and postmultiplying (5-30) by $\mathbf{R}^{-1}$ (which exists since $\mathbf{R}$ is positive definite):

$$\begin{aligned}\mathbf{R}^{-1}\mathbf{H}[\mathbf{P}^{-\,-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1}\mathbf{R}^{-1} &= \mathbf{R}^{-1}\mathbf{R}\mathbf{R}^{-1} - \mathbf{R}^{-1}\mathbf{R}[\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R}]^{-1}\mathbf{R}\mathbf{R}^{-1} \\ &= \mathbf{R}^{-1} - [\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R}]^{-1}\end{aligned}$$

so that the third term can be rewritten as

$$\{+\zeta_i^T\mathbf{R}^{-1}\mathbf{H}[\mathbf{P}^{-\,-1}+\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1}\mathbf{H}^T\mathbf{R}^{-1}\zeta_i\}$$

Now the objective is to operate on the fourth and fifth terms so as to express them in terms of a quadratic form as $\mathbf{x}_1^T[\mathbf{P}^{-\,-1}+\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1}\mathbf{x}_2$, with $\mathbf{x}_1$ and $\mathbf{x}_2$ some n-vectors, so that subsequent combination of terms will be possible.

Now consider the fourth term. If (5-29) is premultiplied by $\mathbf{P}^{-\,-1}$ (which exists since $\mathbf{P}^-$ is assumed positive definite), we get

$$\begin{aligned}\mathbf{P}^{-\,-1}[\mathbf{P}^{-\,-1}+\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1}\mathbf{H}^T\mathbf{R}^{-1} &= \mathbf{P}^{-\,-1}\mathbf{P}^-\mathbf{H}^T(\mathbf{H}\mathbf{P}^-\mathbf{H}^T+\mathbf{R})^{-1}\\ &= \mathbf{H}^T[\mathbf{H}\mathbf{P}^-\mathbf{H}^T+\mathbf{R}]^{-1}\end{aligned}$$

Thus, the fourth term can be rewritten as

$$\{+2\hat{\mathbf{x}}^{-T}\mathbf{P}^{-\,-1}[\mathbf{P}^{-\,-1}+\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1}\mathbf{H}^T\mathbf{R}^{-1}\zeta_i\}$$

Finally, look at the fifth term. If (5-28) is premultiplied and postmultiplied by $\mathbf{P}^{-\,-1}$, we get

$$\begin{aligned}\mathbf{P}^{-\,-1}[\mathbf{P}^{-\,-1}+\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1}\mathbf{P}^{-\,-1} &= \mathbf{P}^{-\,-1}\mathbf{P}^-\mathbf{P}^{-\,-1}-\mathbf{P}^{-\,-1}\mathbf{P}^-\mathbf{H}^T[\mathbf{H}\mathbf{P}^-\mathbf{H}^T+\mathbf{R}]^{-1}\mathbf{H}\mathbf{P}^-\mathbf{P}^{-\,-1}\\ &= \mathbf{P}^{-\,-1}-\mathbf{H}^T[\mathbf{H}\mathbf{P}^-\mathbf{H}^T+\mathbf{R}]^{-1}\mathbf{H}\end{aligned}$$

Thus the fifth term can be put into the form

$$\{+\hat{\mathbf{x}}^{-T}\mathbf{P}^{-\,-1}[\mathbf{P}^{-\,-1}+\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1}\mathbf{P}^{-\,-1}\hat{\mathbf{x}}^-\}$$

Now the third, fourth, and fifth terms in the original expansion can be combined so as to write the expansion equivalently as

$$\begin{aligned}(\cdot) = {}&\xi^T[\mathbf{P}^{-\,-1}+\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]\xi-2\xi^T[\mathbf{P}^{-\,-1}\hat{\mathbf{x}}^-+\mathbf{H}^T\mathbf{R}^{-1}\zeta_i]\\ &+[\zeta_i^T\mathbf{R}^{-1}\mathbf{H}+\hat{\mathbf{x}}^{-T}\mathbf{P}^{-1}][\mathbf{P}^{-1}+\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]^{-1}[\mathbf{H}^T\mathbf{R}^{-1}\zeta_i+\mathbf{P}^{-1}\hat{\mathbf{x}}^-]\end{aligned}$$

To simplify the remaining algebra, define the n-vector $\mathbf{a}$ and the n-by-n matrix $\mathbf{A}$ as

$$\mathbf{a}\triangleq[\mathbf{P}^{-\,-1}\hat{\mathbf{x}}^-+\mathbf{H}^T\mathbf{R}^{-1}\zeta_i]\qquad\mathbf{A}\triangleq[\mathbf{P}^{-\,-1}+\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}]$$

In terms of this notation, the expansion becomes

$$(\cdot)=\xi^T\mathbf{A}\xi-2\xi^T\mathbf{a}+\mathbf{a}^T\mathbf{A}^{-1}\mathbf{a}$$

Again motivated by the desired to achieve a single quadratic form, we can write this equivalently as

$$\begin{aligned}(\cdot) &= \xi^T\mathbf{A}\xi-2\xi^T\mathbf{A}\mathbf{A}^{-1}\mathbf{a}+\mathbf{a}^T\mathbf{A}^{-1}\mathbf{A}\mathbf{A}^{-1}\mathbf{a}\\ &= (\xi-\mathbf{A}^{-1}\mathbf{a})^T\mathbf{A}(\xi-\mathbf{A}^{-1}\mathbf{a})\end{aligned}$$

This is the single quadratic form we have been seeking. ■

Combined with a similar development for the determinant terms, the preceding reduction has shown that the conditional probability density $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\xi\,|\,\mathbf{Z}_i)$ is indeed a Gaussian density, with mean $(\mathbf{A}^{-1}\mathbf{a})$ and covariance $\mathbf{A}^{-1}$. Consistent with the previous definition of $\hat{\mathbf{x}}(t_{i-1}^+)$, the mean of this conditional density is denoted as $\hat{\mathbf{x}}(t_i^+)$:

$$\begin{aligned}\hat{\mathbf{x}}(t_i^+) &\triangleq E\{\mathbf{x}(t_i)\,|\,\mathbf{Z}(t_i)=\mathbf{Z}_i\}=\mathbf{A}^{-1}\mathbf{a}\\ &=[\mathbf{P}(t_i^-)^{-1}+\mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i)]^{-1}[\mathbf{P}(t_i^-)^{-1}\hat{\mathbf{x}}(t_i^-)+\mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)\mathbf{z}_i]\end{aligned}\tag{5-31}$$

Similarly, the covariance is denoted as $\mathbf{P}(t_i^+)$:

$$\mathbf{P}(t_i^+) \triangleq E\{[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^+)][\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^+)]^T | \mathbf{Z}(t_i) = \mathbf{Z}_i\} = \mathbf{A}^{-1}$$
$$= [\mathbf{P}(t_i^-)^{-1} + \mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i)]^{-1} \tag{5-32}$$

In terms of these statistics, the density $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i)$ can be written explicitly as:

$$f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i) = [(2\pi)^{n/2}|\mathbf{P}(t_i^+)|^{1/2}]^{-1} \exp\{\cdot\}$$
$$\{\cdot\} = \{-\tfrac{1}{2}[\boldsymbol{\xi} - \hat{\mathbf{x}}(t_i^+)]^T\mathbf{P}(t_i^+)^{-1}[\boldsymbol{\xi} - \hat{\mathbf{x}}(t_i^+)]\} \tag{5-33}$$

This is portrayed in Fig. 5.3. As indicated by the notation, the conditional mean $\hat{\mathbf{x}}(t_i^+)$ is chosen as the optimal estimate. Not only is it the conditional mean, but also the conditional mode: it maximizes the conditional density of $\mathbf{x}(t_i)$ conditioned on the entire measurement history (i.e., it is more probable to be in the interval between $[\hat{\mathbf{x}}(t_i^+) - \varepsilon]$ and $[\hat{\mathbf{x}}(t_i^+) + \varepsilon]$ than any equivalent-sized region of $\boldsymbol{\xi}$). It is also the conditional median and satisfies essentially any criterion of optimality once $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i)$ has been established. As mentioned previously, if we do in fact use $\hat{\mathbf{x}}(t_i^+)$ as the optimal estimate, then $\mathbf{P}(t_i^+)$ is not only the state covariance but also the covariance of the error committed by that estimate of the state value.

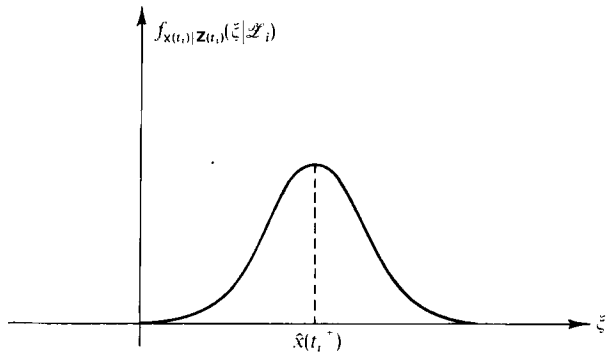

FIG. 5.3 Conditional probability density after measurement incorporation. Variance = $P(t_i^+)$.

Although (5-31) and (5-32) are valid expressions, they involve inversions of n-by-n matrices, where n is the dimension of the state vector. If m, the dimension of the measurement vector, is significantly smaller than n (as is often the case), then the matrix inversion lemma can yield equivalent but more efficient expressions that require only m-by-m inversions. Substituting (5-28) and (5-29) into (5-31) yields:

$$\hat{\mathbf{x}}(t_i^+) = [\mathbf{P}^- - \mathbf{P}^-\mathbf{H}^T(\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R})^{-1}\mathbf{H}\mathbf{P}]\mathbf{P}^{-\ -1}\hat{\mathbf{x}}^- + [\mathbf{P}^-\mathbf{H}^T(\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R})^{-1}]\mathbf{z}_i$$

Regrouping these terms and applying (5-28) directly to (5-32) yields the measurement update equations as:

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{P}(t_i^-)\mathbf{H}^T(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{R}(t_i)]^{-1}[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)] \tag{5-34}$$

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{P}(t_i^-)\mathbf{H}^T(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{R}(t_i)]^{-1}\mathbf{H}(t_i)\mathbf{P}(t_i^-) \tag{5-35}$$

Recall that this derivation was based upon the assumption that $\mathbf{P}(t_i^-)$ was positive definite. Let us investigate the conditions under which this is valid. First, we will determine if $\mathbf{P}(t_{i-1}^+)$ being positive definite is sufficient to make $\mathbf{P}(t_i^+)$ positive definite. In (5-17), if $\mathbf{P}(t_{i-1}^+)$ is assumed positive definite, then $\mathbf{\Phi}(t_i, t_{i-1})\mathbf{P}(t_{i-1}^+)\mathbf{\Phi}^T(t_i, t_{i-1})$ is also positive definite by the properties of state transition matrices; the integral term is at worst positive semidefinite, so $\mathbf{P}(t_i^-)$ is positive definite if $\mathbf{P}(t_{i-1}^+)$ is. In generating $\mathbf{P}(t_i^+)$ from $\mathbf{P}(t_i^-)$ as in (5-35), there would seem to be some question about preserving positive definiteness because a term is subtracted from $\mathbf{P}(t_i^-)$ to obtain $\mathbf{P}(t_i^+)$. However, if the equivalent expression (5-31) is considered, this preservation becomes evident. Since $\mathbf{P}(t_i^-)$ is assumed positive definite, $\mathbf{P}(t_i^-)^{-1}$ is also positive definite. Added to this is the term $\mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i)$, which is positive semidefinite [since $\mathbf{R}(t_i)$ is assumed positive definite, $\mathbf{R}(t_i)^{-1}$ is positive definite, and so $\mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i)$ is an n-by-n matrix of rank at most m], so their sum $\mathbf{P}(t_i^+)^{-1}$ is positive definite, and so is its inverse, $\mathbf{P}(t_i^+)$.

Thus, we can conclude that once $\mathbf{P}(t_i^-)$ or $\mathbf{P}(t_i^+)$ becomes positive definite, the covariances will remain positive definite from that time forward (although they may asymptotically approach singularity). For that reason, look at the initial time interval and determine under what conditions

$$\mathbf{P}(t_1^-) = \mathbf{\Phi}(t_1, t_0)\mathbf{P}_0\mathbf{\Phi}^T(t_1, t_0) + \int_{t_0}^{t_1} \mathbf{\Phi}(t_1, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^T(\tau)\mathbf{\Phi}^T(t_1, \tau)\, d\tau$$

is positive definite. Two sufficient (not necessary) conditions would be if $\mathbf{P}_0$ were positive definite or if the integral term were separately positive definite [i.e., $\mathbf{Q}(t)$ is positive definite for all $t \in [t_0, t_1)$ and the system description is completely controllable from the points of entry of the dynamic driving noise]. Neither of these are very restrictive assumptions, yet we really only require $\mathbf{P}(t_1^-)$ itself to be positive definite.

The derivation just presented was not the most general possible. For instance, $\mathbf{R}(t_i)$ need not be positive definite for the algorithm to operate properly (as long as $[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{R}(t_i)]$ is always invertible), even though this was required in our derivation. Nevertheless, the assumption made does encompass the vast majority of applications of practical interest.

To complete the derivation, let us add the effects of deterministic control inputs. As described previously, the *only* change in the estimator algorithm is that the state estimate (conditional mean) time propagation relation (5-15)

becomes

$$\hat{\mathbf{x}}(t_i^-) = \mathbf{\Phi}(t_i, t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+) + \int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau$$

Note that the discrete-time (sampled data) *Kalman filter algorithm* just derived entails the time propagation and measurement updating of conditional mean and covariance equations. However, because all probability densities of interest have been shown to be Gaussian, this algorithm does in fact portray the entire conditional density of the state conditioned on the measurements taken: the Bayesian objective of propagating *all* probability information has been fulfilled.

To summarize the algorithm, the optimal state estimate is *propagated* from measurement time t_{i-1} to measurement time t_i by the relations

$$\hat{\mathbf{x}}(t_i^-) = \mathbf{\Phi}(t_i, t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+) + \int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau \tag{5-36}$$

$$\mathbf{P}(t_i^-) = \mathbf{\Phi}(t_i, t_{i-1})\mathbf{P}(t_{i-1}^+)\mathbf{\Phi}^{\mathrm{T}}(t_i, t_{i-1}) + \int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\mathbf{\Phi}^{\mathrm{T}}(t_i, \tau)\,d\tau \tag{5-37}$$

At measurement time t_i, the measurement $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$ becomes available. The estimate is *updated* by defining the Kalman filter gain $\mathbf{K}(t_i)$ and employing it in both the mean and covariance relations:

$$\mathbf{K}(t_i) = \mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)]^{-1} \tag{5-38}$$

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i)[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)] \tag{5-39}$$

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{K}(t_i)\mathbf{H}(t_i)\mathbf{P}(t_i^-) \tag{5-40}$$

The initial conditions for the recursion are given by

$$\hat{\mathbf{x}}(t_0) = E\{\mathbf{x}(t_0)\} = \hat{\mathbf{x}}_0 \tag{5-41}$$

$$\mathbf{P}(t_0) = E\{[\mathbf{x}(t_0) - \hat{\mathbf{x}}_0][\mathbf{x}(t_0) - \hat{\mathbf{x}}_0]^{\mathrm{T}}\} = \mathbf{P}_0 \tag{5-42}$$

Figure 5.4 is a block diagram portrayal of the algorithm. The mathematical system model inherently in the filter structure generates $\hat{\mathbf{x}}(t_i^-)$, the best prediction of the state at time t_i before the measurement at time t_i, $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$, is

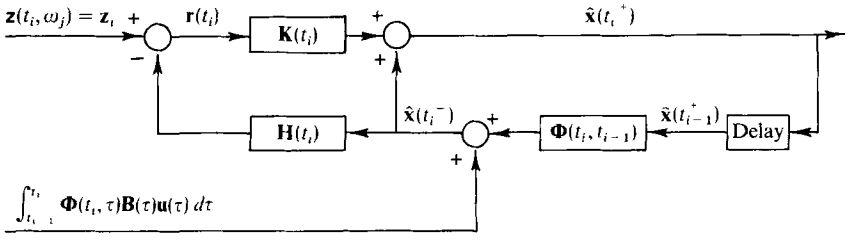

FIG. 5.4 Sampled-data Kalman filter block diagram.

processed. Moreover, this same system model allows generation of $[\mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]$, which is the best prediction of what the measurement at time t_i will be before it is actually taken [recall (5-24)]. The input to the algorithm is $\mathbf{z}_i$, the realized value of the measurement $\mathbf{z}(t_i)$. The measurement *residual* $\mathbf{r}(t_i)$ is then generated as the difference between the true measurement value $\mathbf{z}_i$ and the best prediction of it before it is actually taken:

$$\mathbf{r}(t_i) = \mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-) \tag{5-43}$$

(Many term this quantity the *innovations* and reserve the name "residual" for the quantity $[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^+)]$, which does not appear explicitly in the algorithm; we choose to retain the name residual because of the important procedure known as "residual monitoring," in which the sequence of $\mathbf{r}(t_i)$ values are monitored for adaptive purposes.) Then the residual is passed through an optimal weighting matrix $\mathbf{K}(t_i)$ to generate a correction term to be added to $\hat{\mathbf{x}}(t_i^-)$ to obtain $\hat{\mathbf{x}}(t_i^+)$: the algorithm has a predictor–corrector structure.

To specify a Kalman filter algorithm completely, we need to define both the *structure* of the system model and a statistical description of the *uncertainties* in the model. The structure is established by $\mathbf{F}(t)$ or $\mathbf{\Phi}(t_i, \tau)$, $\mathbf{B}(t)$, $\mathbf{G}(t)$, and $\mathbf{H}(t_i)$ for all times of interest, and the uncertainties are specified by $\hat{\mathbf{x}}_0$, $\mathbf{P}_0$, and the time histories of $\mathbf{Q}(t)$ and $\mathbf{R}(t_i)$.

EXAMPLE 5.1 Recall the example of being lost at sea in Section 1.5 of the first chapter. Just after time t_2, after the trained navigator's measurement was incorporated, the state estimate and variance were established as $\hat{x}(t_2^+)$ and $P(t_2^+) = \sigma_x^2(t_2^+)$. The dynamics model was given by (1-10) as

$$\dot{\mathsf{x}}(t) = u + \mathsf{w}(t)$$

with u constant and $\mathsf{w}(\cdot, \cdot)$ described as a zero-mean white Gaussian noise of strength

$$E\{\mathsf{w}(t)\mathsf{w}(t+\tau)\} = \sigma_w^2\delta(\tau)$$

Thus, we can identify $\mathbf{F} = 0$ so $\mathbf{\Phi} = 1$, $\mathbf{B} = 1$, $\mathbf{G} = 1$, and $\mathbf{Q} = \sigma_w^2$.

The time propagation equation for the state estimate would be

$$\begin{aligned}\hat{x}(t_3^-) &= \Phi(t_3, t_2)\hat{x}(t_2^+) + \int_{t_2}^{t_3} \Phi(t_3, \tau)B(\tau)u(\tau)\,d\tau \\ &= 1 \cdot \hat{x}(t_2^+) + \int_{t_2}^{t_3} 1 \cdot 1 \cdot u\,d\tau \\ &= \hat{x}(t_2^+) + u[t_3 - t_2]\end{aligned}$$

This is the result quoted in (1-11). Similarly, the variance time propagation is as delineated in (1-12),

$$\begin{aligned}P(t_3^-) &= \Phi(t_3, t_2)P(t_2^+)\Phi^{\mathrm{T}}(t_3, t_2) + \int_{t_2}^{t_3} \Phi(t_3, \tau)G(\tau)Q(\tau)G^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t_3, \tau)\,d\tau \\ &= 1 \cdot P(t_2^+) \cdot 1 + \int_{t_2}^{t_3} 1 \cdot 1 \cdot \sigma_w^2 \cdot 1 \cdot 1\,d\tau \\ &= P(t_2^+) + \sigma_w^2[t_3 - t_2]\end{aligned}$$

At time t_3, a measurement becomes available, modeled as

$$\mathsf{z}(t_3) = \mathsf{x}(t_3) + \mathsf{v}(t_3)$$

where $v(\cdot,\cdot)$ is a white Gaussian discrete-time noise process of mean zero and variance $\sigma_{z_3}^2$: so $\mathbf{H} = 1$, $\mathbf{R} = \sigma_{z_3}^2$. Based on this model, the measurement update relations become

$$\begin{aligned} K(t_3) &= P(t_3^-)H^{\mathrm{T}}(t_3)[H(t_3)P(t_3^-)H^{\mathrm{T}}(t_3) + R(t_3)]^{-1} \\ &= P(t_3^-)/[P(t_3^-) + \sigma_{z_3}^2] \\ \hat{x}(t_3^+) &= \hat{x}(t_3^-) + K(t_3)[z_3 - H(t_3)\hat{x}(t_3^-)] \\ &= \hat{x}(t_3^-) + K(t_3)[z_3 - \hat{x}(t_3^-)] \\ P(t_3^+) &= P(t_3^-) - K(t_3)H(t_3)P(t_3^-) \\ &= P(t_3^-) - K(t_3)P(t_3^-) \end{aligned}$$

These are identical to (1-13)–(1-15). ■

The Kalman filter time propagation equations given by (5-36) and (5-37) are in a convenient form if the problem under consideration is modeled by time-invariant dynamics and dynamic driving noise with stationary statistics ($\mathbf{F}$, $\mathbf{B}$, $\mathbf{G}$, and $\mathbf{Q}$ all constants) with a fixed measurement sample period. In this case, $\boldsymbol{\Phi}(t_i, t_{i-1})$ and $\int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i,\tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\boldsymbol{\Phi}^{\mathrm{T}}(t_i,\tau)\,d\tau$ are the same for every sample period and need only be computed once. Moreover, if the deterministic inputs are held constant over each sample period (as provided by a digital controller operating at the same iteration rate), then $\mathbf{u}(\tau) = \mathbf{u}(t_{i-1})$ for all $\tau \in [t_{i-1}, t_i)$, and

$$\int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i,\tau)\mathbf{B}(\tau)\mathbf{u}(t_{i-1})\,d\tau = \left[\int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i,\tau)\mathbf{B}(\tau)\,d\tau\right]\mathbf{u}(t_{i-1})$$

and the bracketed term is also constant from sample to sample.

However, if the system or statistics are time varying, the time propagation equations would be more efficiently expressed in differential equation form. To generate this form, define $\hat{\mathbf{x}}(t/t_{i-1})$ and $\mathbf{P}(t/t_{i-1})$ for any $t \in [t_{i-1}, t_i)$ as the conditional mean and covariance conditioned on the measurements taken up to that time, i.e., up through $\mathbf{z}(t_{i-1}, \omega_j) = \mathbf{z}_{i-1}$:

$$\hat{\mathbf{x}}(t/t_{i-1}) = \boldsymbol{\Phi}(t, t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+) + \int_{t_{i-1}}^{t} \boldsymbol{\Phi}(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau \tag{5-44}$$

$$\begin{aligned} \mathbf{P}(t/t_{i-1}) &= \boldsymbol{\Phi}(t, t_{i-1})\mathbf{P}(t_{i-1}^+)\boldsymbol{\Phi}^{\mathrm{T}}(t, t_{i-1}) \\ &\quad + \int_{t_{i-1}}^{t} \boldsymbol{\Phi}(t,\tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\boldsymbol{\Phi}^{\mathrm{T}}(t,\tau)\,d\tau \end{aligned} \tag{5-45}$$

Differentiating these yields

$$\dot{\hat{\mathbf{x}}}(t/t_{i-1}) = \mathbf{F}(t)\hat{\mathbf{x}}(t/t_{i-1}) + \mathbf{B}(t)\mathbf{u}(t) \tag{5-46}$$

$$\dot{\mathbf{P}}(t/t_{i-1}) = \mathbf{F}(t)\mathbf{P}(t/t_{i-1}) + \mathbf{P}(t/t_{i-1})\mathbf{F}^{\mathrm{T}}(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t) \tag{5-47}$$

which would then be integrated over the interval from t_{i-1} to t_i, starting from the initial conditions

$$\hat{\mathbf{x}}(t_{i-1}/t_{i-1}) = \hat{\mathbf{x}}(t_{i-1}^+) \tag{5-48a}$$

$$\mathbf{P}(t_{i-1}/t_{i-1}) = \mathbf{P}(t_{i-1}^+) \tag{5-48b}$$

Integrating these to time t_i (as by fourth order Runge–Kutta technique or by sequentially integrating over partitions of this interval by a first order method) would yield $\hat{\mathbf{x}}(t_i^-)$ and $\mathbf{P}(t_i^-)$.

In some cases the dynamics model is itself a discrete-time model, for instance an equivalent discrete-time model as discussed in the previous chapter. Let us replace the dynamics model of (5-1) and (5-2) with the linear stochastic difference equation

$$\mathbf{x}(t_i) = \mathbf{\Phi}(t_i, t_{i-1})\mathbf{x}(t_{i-1}) + \mathbf{B}_d(t_{i-1})\mathbf{u}(t_{i-1}) + \mathbf{G}_d(t_{i-1})\mathbf{w}_d(t_{i-1}) \tag{5-49}$$

where $\mathbf{w}_d(\cdot,\cdot)$ is a discrete-time zero-mean white Gaussian noise sequence with covariance kernel

$$E\{\mathbf{w}_d(t_i)\mathbf{w}_d^{\mathrm{T}}(t_j)\} = \begin{cases} \mathbf{Q}_d(t_i) & t_i = t_j \\ \mathbf{0} & t_i \neq t_j \end{cases} \tag{5-50}$$

(Note that in the equivalent discrete model formulation, $\mathbf{G}_d$ was assumed to be the identity matrix.) The only change to the Kalman filter algorithm is that the time propagation equations (5-36) and (5-37) are replaced by

$$\hat{\mathbf{x}}(t_i^-) = \mathbf{\Phi}(t_i, t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+) + \mathbf{B}_d(t_{i-1})\mathbf{u}(t_{i-1}) \tag{5-51}$$

$$\mathbf{P}(t_i^-) = \mathbf{\Phi}(t_i, t_{i-1})\mathbf{P}(t_{i-1}^+)\mathbf{\Phi}^{\mathrm{T}}(t_i, t_{i-1}) + \mathbf{G}_d(t_{i-1})\mathbf{Q}_d(t_{i-1})\mathbf{G}_d^{\mathrm{T}}(t_{i-1}) \tag{5-52}$$

To specify a Kalman filter of this form completely, we again must depict both the structure [$\mathbf{\Phi}(t_i, t_{i-1})$, $\mathbf{B}_d(t_{i-1})$, $\mathbf{G}_d(t_{i-1})$, $\mathbf{H}(t_i)$ for all times of interest] and uncertainties [$\hat{\mathbf{x}}_0$, $\mathbf{P}_0$, and $\mathbf{Q}_d(t_{i-1})$ and $\mathbf{R}(t_i)$ for all times] of the model.

EXAMPLE 5.2 We now consider an example based on a scalar dynamics model: a simple representation of a gyro on test. (This example will be reexamined throughout Chapter 5.) Gyros are subject to long term drifts, and we would like to estimate the drift rate from laboratory data. Assume that gyro drift rate can be adequately modeled as a stationary exponentially time-correlated Gaussian process (in fact, this is a rather good model for the dominant drift effects). To keep the problem restricted to one state variable, we will further assume that we can measure instantaneous drift rate. Thus, though not totally realistic, this problem can be viewed as a portion of a more realistic problem, and we seek to exploit its simple form to illustrate the use of the estimator algorithm.

Figure 5.5 depicts the system model. Gyro drift rate is the state process $x(\cdot,\cdot)$, and since it is an exponentially time-correlated Gaussian process, it is shown as the output of a first order shaping filter (first order lag) driven by white Gaussian noise $w(\cdot,\cdot)$. The shaping filter break fre-

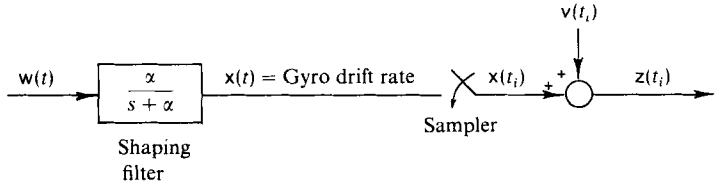

FIG. 5.5 System model for gyro on test example.

quency α is set at 1 rad/hr, i.e., the correlation time of the x process is one hour (reasonable for state-of-the-art gyros), and the statistics of $\mathsf{w}(\cdot,\cdot)$ are

$$E\{\mathsf{w}(t)\} = 0$$
$$E\{\mathsf{w}(t)\mathsf{w}(t+\tau)\} = Q\,\delta(\tau), \qquad Q = 2 \quad \text{deg}^2/\text{hr}$$

The units of Q seem strange at first since w is in units of deg/hr, but are valid because $\delta(\tau)$ carries units of $(\text{time})^{-1}$. It is assumed that sampled data measurements are taken every 0.25 hr, modeled as

$$\mathsf{z}(t_i) = \mathsf{x}(t_i) + \mathsf{v}(t_i)$$

where $\mathsf{v}(\cdot,\cdot)$ is a discrete-time zero-mean white Gaussian noise with

$$E\{\mathsf{v}(t_i)\mathsf{v}(t_j)\} = R\,\delta_{ij}, \qquad R = 0.5 \quad \text{deg}^2/\text{hr}^2$$

Note that declaring $\mathsf{v}(\cdot,\cdot)$ to be a white sequence is really assuming that the correlation time of any noise corrupting the analog measuring device output is short compared to the sample period of the sampler. It is desired to process these measurements to obtain an optimal estimate of the gyro drift rate $\mathsf{x}(t)$.

First generate the state differential equation. Since the Laplace domain transfer function of the shaping filter is:

$$\mathsf{x}(s)/\mathsf{w}(s) = \alpha/(s+\alpha)$$

we get, by cross multiplying

$$s\mathsf{x}(s) + \alpha\mathsf{x}(s) = \alpha\mathsf{w}(s)$$

or, taking the inverse Laplace transform

$$\dot{\mathsf{x}}(t) = -\alpha\mathsf{x}(t) + \alpha\mathsf{w}(t)$$

from which we can identify $\mathbf{F} = -\alpha = -1$, $\mathbf{G} = \alpha = 1$.

If we want to use (5-36) and (5-37), we will need the state transition matrix, which in this case is

$$\Phi(t_i, t_{i-1}) = \exp[-\alpha(t_i - t_{i-1})] = \exp[-1(0.25)] \cong 0.78$$

This could also be obtained through the inverse Laplace transform of $(s\mathbf{I} - \mathbf{F})^{-1}$.

It was assumed that $\mathsf{x}(\cdot,\cdot)$ is a stationary process, so we must determine its steady state variance to serve as P_0. The general stochastic process covariance relation

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^{\mathrm{T}}(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t)$$

becomes

$$\dot{P}(t) = -2\alpha P(t) + \alpha^2 Q$$

so that the steady state value, evaluated by solving $\dot{P}(t) = 0$, is

$$P = \alpha Q/2 = 1 \quad \text{deg}^2/\text{hr}^2$$

Thus, before any measurements are taken, we have the initial conditions of

$$\hat{x}(t_0) = \hat{x}_0 = 0 \qquad \text{(assumed)}$$
$$P(t_0) = P_0 = 1 \quad \text{deg}^2/\text{hr}^2$$

At this point, the filter can be completely delineated. To propagate the estimate from sample time t_{i-1} to the next time t_i, (5-36) and (5-37) yield

$$\hat{x}(t_i^-) = \Phi(t_i, t_{i-1})\hat{x}(t_{i-1}^+) = 0.78\hat{x}(t_{i-1}^+)$$

$$P(t_i^-) = \Phi^2(t_i, t_{i-1})P(t_{i-1}^+) + \int_{t_{i-1}}^{t_i} \Phi^2(t_i, \tau)G^2Q\, d\tau$$

$$= 0.78^2 P(t_{i-1}^+) + 2\int_{t_{i-1}}^{t_i} \exp[-2(t_i - \tau)]\, d\tau$$

$$= 0.61P(t_{i-1}^+) + 0.39$$

Note that these can also be interpreted as the Kalman filter using (5-51) and (5-52), based on the equivalent discrete-time model with $\mathbf{B}_d = 0$, $\mathbf{G}_d = 1$, and $\mathbf{Q}_d = 0.39$. This propagation can also be represented in differential equation form by using (5-46) and (5-47) to write

$$\dot{\hat{x}}(t/t_{i-1}) = F(t)\hat{x}(t/t_{i-1}) = -\hat{x}(t/t_{i-1})$$

$$\dot{P}(t/t_{i-1}) = 2F(t)P(t/t_{i-1}) + G^2(t)Q(t)$$

$$= -2P(t/t_{i-1}) + 2$$

which would be integrated forward from $\hat{x}(t_{i-1}/t_{i-1}) = \hat{x}(t_{i-1}^+)$, $P(t_{i-1}/t_{i-1}) = P(t_{i-1}^+)$ to time t_i. To update the estimate with the measurement z_i at time t_i, (5-38)–(5-40) yield

$$K(t_i) = \frac{P(t_i^-)H(t_i)}{H(t_i)P(t_i^-)H(t_i) + R(t_i)} = \frac{P(t_i^-)}{P(t_i^-) + 0.5}$$

$$\hat{x}(t_i^+) = \hat{x}(t_i^-) + K(t_i)[z_i - H(t_i)\hat{x}(t_i^-)]$$

$$= \hat{x}(t_i^-) + \frac{P(t_i^-)}{P(t_i^-) + 0.5}[z_i - \hat{x}(t_i^-)]$$

$$P(t_i^+) = P(t_i^-) - K(t_i)H(t_i)P(t_i^-)$$

$$= P(t_i^-) - \frac{P(t_i^-)^2}{P(t_i^-) + 0.5} = \frac{0.5P(t_i^-)}{P(t_i^-) + 0.5} \quad \blacksquare$$

In the derivation of the Kalman filter algorithm, $\mathbf{P}(t_i^-)$ and $\mathbf{P}(t_i^+)$ were *defined* in (5-10) and (5-16) as conditional covariances of the state at time t_i, conditioned on $\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}$ and $\mathbf{Z}(t_i) = \mathbf{Z}_i$, respectively. However, the recursion relations (5-37), (5-38), and (5-40) for these matrices do *not* depend upon the particular sequence of realized measurements. The covariances are statistically related to the measurement history through the $\mathbf{H}(t_i)$ and $\mathbf{R}(t_i)$ sequences, but they are not functions of the specific measurement values. Whereas the state estimates (conditional means) are a function of measurement realizations, the error committed by such an estimate will be shown to be independent of $\mathbf{Z}(t_i)$. Consequently, one can precompute the time history of $\mathbf{P}(t_i^-)$, $\mathbf{P}(t_i^+)$, and $\mathbf{K}(t_i)$ before actual measurement numbers $\mathbf{z}_i$ are available: in fact, even before the measuring devices themselves are available. Since $\mathbf{P}(t_i^-)$ and $\mathbf{P}(t_i^+)$ are both conditional state covariances and state estimation error covariances, this precomputability allows early design tradeoffs of estimation accuracy versus measuring device precision [i.e., $\mathbf{R}(t_i)$ time history] to ensure cost-effective systems that meet specifications.

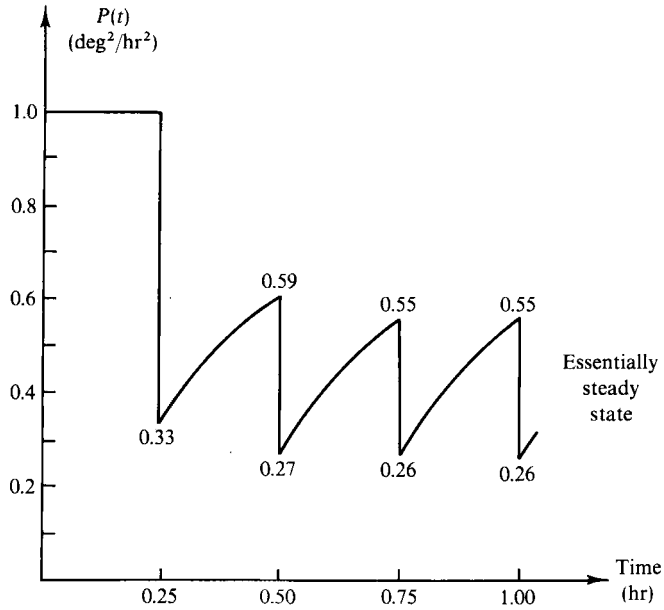

FIG. 5.6 Error variance time history for Example 5.3.

EXAMPLE 5.3 We can precompute the conditional state variance time history as generated by the Kalman filter for the problem of Example 5.2. Figure 5.6 portrays this variance time history through the fourth measurement sample time. [The values between sample times can be found from integrating (5-47), whereas (5-37) would generate only the $P(t_i^-)$ values.]

Note that the variance is constant over the first interval, since we started off in steady state conditions. Even on the first measurement, the estimate error variance is less than the measurement error variance of 0.5. Further, it decreases in time from 0.33 to 0.25: the algorithm uses the past history of measurement information, as propagated through its internal model, to yield this reduction.

Figure 5.6 indicates that steady state filter operation has been essentially achieved by the fourth measurement update time. To substantiate this claim, the steady state conditions can be calculated by equating $P(t_i^+)$ to $P(t_{i-1}^+)$:

$$P(t_i^+) = \frac{0.5P(t_i^-)}{P(t_i^-) + 0.5} = \frac{0.5[0.61P(t_{i-1}^+) + 0.39]}{[0.61P(t_{i-1}^+) + 0.39] + 0.5}$$

$$= \frac{0.30P(t_{i-1}^+) + 0.19}{0.61P(t_{i-1}^+) + 0.89}$$

Equating this to $P(t_{i-1}^+)$ yields

$$0.61P^{+2} + 0.59P^+ - 0.19 = 0$$

for which the positive solution is

$$P^+ = 0.255$$

Then, the steady state P^- is found from

$$P^- = 0.61P^+ + 0.39 = 0.546$$

These values do confirm the claim. ■

The filter performance exhibited in the previous example is typical for problems with time-invariant system models and stationary statistics: an initial transient in $\mathbf{P}$ and $\mathbf{K}$ followed by an essentially steady state filter operation. In many applications, the transient is short compared to the total time of interest (suggesting a possible approximation of using the steady state filter for all time if the resulting performance degradation is not prohibitive; this will be discussed further in the next chapter). A different $\mathbf{P}_0$ matrix will yield a different magnitude transient characteristic, but its duration will be the same and the steady state conditions are unaffected.

On the other hand, changing $\mathbf{Q}$ or $\mathbf{R}$ in the filter structure does affect the transient duration and the steady state operation. Increasing $\mathbf{Q}$ would indicate either stronger noises driving the dynamics or increased uncertainty in the adequacy of the model itself to depict the true dynamics accurately. This will increase both the rate of growth of the $\mathbf{P}(t)$ elements (or eigenvalues) between measurement times and their steady state values. As a reult, the filter gains will generally increase, thereby weighting the measurements more heavily: this is reasonable since increased $\mathbf{Q}$ dictates that we should put less confidence in the output of the filter's own dynamics model. By similar reasoning, increased $\mathbf{R}$ would indicate that the measurements are subjected to a stronger corruptive noise, and so should be weighted less by the filter. In fact, this will decrease the gain values, the eigenvalues of the $[\mathbf{K}(t_i)\mathbf{H}(t_i)\mathbf{P}(t_i^-)]$ term are smaller so the error variances going from t_i^- to t_i^+ decrease to a lesser extent, and the steady state covariance eigenvalues are larger. If the eigenvalues of $\mathbf{Q}$ are large compared to the eigenvalues of $\mathbf{R}$ (in the scalar case, if the Q/R ratio is large), steady state is quickly reached because the uncertainty involved in the state propagation is large compared to the accuracy of the measurements, so the new state estimate is heavily dependent upon the new measurement and not closely related to prior estimates.

EXAMPLE 5.4 To see the effects of variations in Q and R, Example 5.3 will be repeated for the following cases (case 1 is Example 5.3 itself).

Case	Q (deg^2/hr)	R (deg^2/hr^2)
1	2	0.5
2	4	0.5
3	2	1.0
4	4	1.0

P_0 was set to the stationary value, which is 1 deg^2/hr^2 for $Q = 2$ deg^2/hr, and 2 deg^2/hr^2 for $Q = 4$ deg^2/hr; the same P_0 could have been used for all cases, but the differing transient would quickly decay and the same steady state filter operation would be achieved. Table 5.1 displays the time history of $P(t_i^-)$, $P(t_i^+)$, and $K(t_i)$ values for the four cases.

In this scalar example, the "tracking" properties of the filter will be very evident: if $K(t_i)$ is approximately one, $\hat{x}(t_i^+)$ is approximately equal to z_i. Conversely, if $K(t_i)$ is very small, then $\hat{x}(t_i)$ is not "tracking" the measurements closely, but rather is heavily weighting the output of its own internal system model.

TABLE 5.1

Effects of Q and R Variation

	Time (hr)	0.25	0.50	0.75	1.00
Case 1	$Q = 2, R = 0.5$				
	$P(t_i^-)$	1.00	0.59	0.55	0.55
	$P(t_i^+)$	0.33	0.27	0.26	0.26
	$K(t_i)$	0.67	0.54	0.52	0.52
Case 2	$Q = 4, R = 0.5$				
	$P(t_i^-)$	2.00	1.02	0.99	0.98
	$P(t_i^+)$	0.40	0.34	0.33	0.33
	$K(t_i)$	0.80	0.67	0.66	0.66
Case 3	$Q = 2, R = 1.0$				
	$P(t_i^-)$	1.00	0.69	0.64	0.63
	$P(t_i^+)$	0.50	0.41	0.39	0.39
	$K(t_i)$	0.50	0.41	0.39	0.39
Case 4	$Q = 4, R = 1.0$				
	$P(t_i^-)$	2.00	1.19	1.11	1.09
	$P(t_i^+)$	0.67	0.54	0.52	0.52
	$K(t_i)$	0.67	0.54	0.52	0.52

Doubling Q and retaining the original R (case 2) causes the difference between $P(t_{i-1}^+)$ and $P(t_i^-)$ to increase over that of case 1: in steady state, this is $(0.98 - 0.33) = 0.65$ versus $(0.54 - 0.26) = 0.28$. Not only are the oscillations larger due to more rapid growth (more rapid input of uncertainty) between sample times, but the steady state $P(t_i^-)$ and $P(t_i^+)$ are larger as well. With the same measurement precision but more uncertainty in the system model, there is less certainty in the estimate. The gains $K(t_i)$ are larger, and the filter "tracks" the measurements to a greater degree.

Doubling R and keeping the same Q as in case 1 (case 3) causes the difference between $P(t_i^-)$ and $P(t_i^+)$ to decrease: in steady state, $(0.63 - 0.39) = 0.24$ as opposed to 0.28. Steady state $P(t_i^-)$ and $P(t_i^+)$ are larger, since there is now more noise corruption in the measurements. Furthermore, there is now greater uncertainty in the measurements relative to the model uncertainties, so the $K(t_i)$'s are smaller, and the filter no longer "tracks" the measurements as closely.

Doubling both Q and R yields case 4. Here the gain time history is identical to that of case 1: because we have a scalar linear system description, the ratio Q/R is the determining factor of steady state gain. In fact, for this problem, steady state filter gain can be shown to be

$$K = 0.16\{\sqrt{[(Q/R) + 2]^2 + 12.5(Q/R)} - [(Q/R) + 2]\}$$

Thus, proportionately increasing Q and R had no effect on the $K(t_i)$ time history, while $P(t_i^-)$ and $P(t_i^+)$ were doubled (a linear system with doubling of all input magnitudes).

"Tuning" a general Kalman filter involves achieving "good" values of $\mathbf{P}_0$, $\mathbf{Q}$, and $\mathbf{R}$, good in the sense that the best estimation accuracy is obtained from a specified Kalman filter structure. The task of determining the best set of matrices is considerably more difficult than the scalar case, but this example does provide some basic insights. ■

5.4 STATISTICS OF PROCESSES WITHIN THE FILTER STRUCTURE

The previous section derived the Kalman filter, through which the conditional probability density $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i)$ could be generated explicitly for all time. This density function, or the mean and covariance which define it, provides all the possible information obtainable about the system *state*. Now we investigate the statistical description of some of the processes within the filter structure itself.

The *error* committed by using $\hat{\mathbf{x}}(t_i^+)$ as an estimator of $\mathbf{x}(t_i)$ would be defined as

$$\mathbf{e}(t_i^+) \triangleq \mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^+) \tag{5-53}$$

where $\hat{\mathbf{x}}(t_i^+)$ is the random variable $E\{\mathbf{x}(t_i)|\mathbf{Z}(t_i) = \mathbf{Z}(t_i, \cdot)\}$, one realization of which would be the numerical output of the filter algorithm. Since $\mathbf{x}(t_i)$ and $\hat{\mathbf{x}}(t_i^+)$ can be shown to be jointly Gaussian, conditioned on $\mathbf{Z}(t_i)$, $\mathbf{e}(t_i^+)$ is a Gaussian random variable, and its density function is then completely specified by its mean and covariance. The mean is

$$\begin{aligned} E\{\mathbf{e}(t_i^+)|\mathbf{Z}(t_i) = \mathbf{Z}_i\} &= E\{\mathbf{x}(t_i)|\mathbf{Z}(t_i) = \mathbf{Z}_i\} - E\{\hat{\mathbf{x}}(t_i^+)|\mathbf{Z}(t_i) = \mathbf{Z}_i\} \\ &= \hat{\mathbf{x}}(t_i^+) - \hat{\mathbf{x}}(t_i^+) = \mathbf{0} \end{aligned} \tag{5-54}$$

i.e., the estimator is unbiased since the error is zero mean. The covariance is:

$$\begin{aligned} &E\{\mathbf{e}(t_i^+)\mathbf{e}^{\mathrm{T}}(t_i^+)|\mathbf{Z}(t_i) = \mathbf{Z}_i\} \\ &\quad = E\{[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^+)][\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^+)]^{\mathrm{T}}|\mathbf{Z}(t_i) = \mathbf{Z}_i\} \\ &\quad = E\{\mathbf{x}(t_i)\mathbf{x}^{\mathrm{T}}(t_i)|\mathbf{Z}(t_i) = \mathbf{Z}_i\} - E\{\mathbf{x}(t_i)|\mathbf{Z}(t_i) = \mathbf{Z}_i\}\hat{\mathbf{x}}^{\mathrm{T}}(t_i^+) \\ &\qquad - \hat{\mathbf{x}}(t_i^+)E\{\mathbf{x}^{\mathrm{T}}(t_i)|\mathbf{Z}(t_i) = \mathbf{Z}_i\} + \hat{\mathbf{x}}(t_i^+)\hat{\mathbf{x}}^{\mathrm{T}}(t_i^+) \\ &\quad = [\mathbf{P}(t_i^+) + \hat{\mathbf{x}}(t_i^+)\hat{\mathbf{x}}^{\mathrm{T}}(t_i^+)] - \hat{\mathbf{x}}(t_i^+)\hat{\mathbf{x}}^{\mathrm{T}}(t_i^+) - \hat{\mathbf{x}}(t_i^+)\hat{\mathbf{x}}^{\mathrm{T}}(t_i^+) + \hat{\mathbf{x}}(t_i^+)\hat{\mathbf{x}}^{\mathrm{T}}(t_i^+) \\ &\quad = \mathbf{P}(t_i^+) \end{aligned} \tag{5-55}$$

This proves the previous statement that the conditional covariance of the error committed by using $\hat{\mathbf{x}}(t_i^+)$ as an estimator is equal to the conditional covariance of $\mathbf{x}(t_i)$ itself. Consequently, we can write

$$f_{\mathbf{e}(t_i^+)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i) = [(2\pi)^{n/2}|\mathbf{P}(t_i^+)|^{1/2}]^{-1} \exp\{-\tfrac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}\mathbf{P}(t_i^+)^{-1}\boldsymbol{\xi}\} \tag{5-56}$$

This is functionally independent of the particular realized measurement values, $\mathbf{Z}_i$, since $\mathbf{P}(t_i^+)$ does not depend on $\mathbf{Z}_i$; for this reason, it is often written as $f_{\mathbf{e}(t_i^+)|\mathbf{Z}(t_i)}(\boldsymbol{\xi})$. By exploiting this and the concepts of marginal densities and Bayes' rule, the unconditional density $f_{\mathbf{e}(t_i^+)}(\boldsymbol{\xi})$ can be written as

$$f_{\mathbf{e}(t_i^+)}(\boldsymbol{\xi}) = \int_{-\infty}^{\infty} f_{\mathbf{e}(t_i^+),\mathbf{Z}(t_i)}(\boldsymbol{\xi}, \mathscr{Z}_i)\, d\mathscr{Z}_i = \int_{-\infty}^{\infty} f_{\mathbf{e}(t_i^+)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}) f_{\mathbf{Z}(t_i)}(\mathscr{Z}_i)\, d\mathscr{Z}_i$$

$$= f_{\mathbf{e}(t_i^+)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}) \int_{-\infty}^{\infty} f_{\mathbf{Z}(t_i)}(\mathscr{Z}_i)\, d\mathscr{Z}_i = f_{\mathbf{e}(t_i^+)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}) \cdot 1 \tag{5-57}$$

Thus, if we use $\hat{\mathbf{x}}(t_i^+)$ as our state estimator, the error committed, $\mathbf{e}(t_i^+)$, is *independent* of $\mathbf{Z}(t_i)$, the entire measurement history random variable.

Figure 5.7 portrays this graphically. If the time history of measurements $\mathbf{Z}_i$ changes, then the conditional mean $\hat{\mathbf{x}}(t_i^+)$ changes, but the shape of the density function $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i)$ [which is just $f_{\mathbf{e}(t_i^+)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i)$ shifted by the conditional mean] remains unchanged. This is a unique characteristic, not true of nonlinear estimation problems in general.

This independence says conceptually that the estimator gleans out as much information from the measurements as possible, and there is nothing left in the measurements that could tell you anything about the error. Geometrically, the

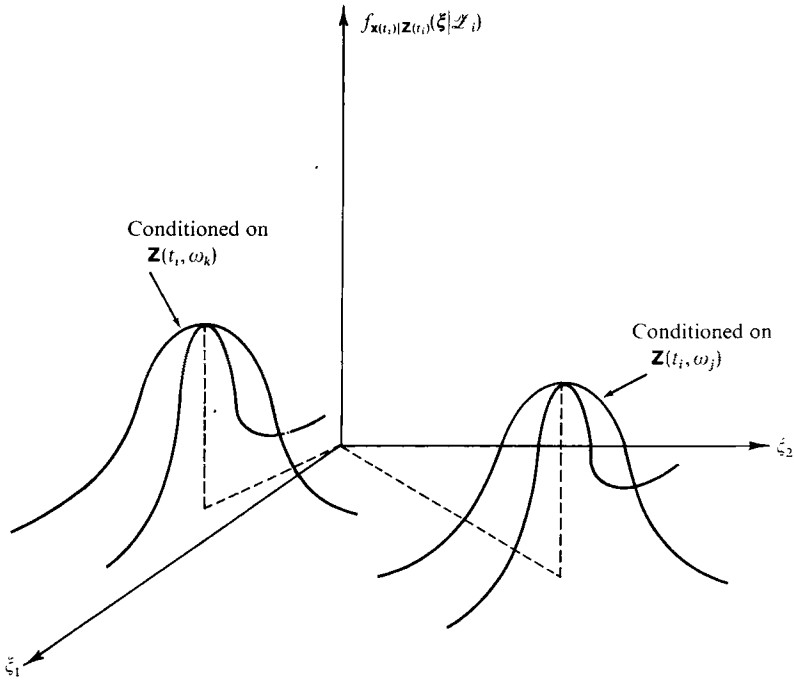

FIG. 5.7 Estimation error independence of measurement history.

error is orthogonal to the projection of the real $\mathbf{x}(t_i)$ onto the measurement subspace (in a Hilbert space of random variables): Kalman originally developed the filter recursion relations from this geometrical insight.

Similarly, if we define

$$\mathbf{e}(t_i^-) \triangleq \mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^-) \tag{5-58}$$

then, $f_{\mathbf{e}(t_i^-)|\mathbf{Z}(t_{i-1})}(\boldsymbol{\xi}|\mathbf{Z}_{i-1})$ can be shown to be Gaussian, with

$$E\{\mathbf{e}(t_i^-)|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} = \mathbf{0} \tag{5-59}$$

$$E\{\mathbf{e}(t_i^-)\mathbf{e}^{\mathrm{T}}(t_i^-)|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} = \mathbf{P}(t_i^-) \tag{5-60}$$

Two other closely related processes in the filter are the *residual* (*innovations*) and *new information* processes, denoted as $\mathbf{r}(\cdot,\cdot)$ and $\mathbf{s}(\cdot,\cdot)$, respectively, and defined for all $t_i \in T$ by

$$\mathbf{r}(t_i) \triangleq \mathbf{z}(t_i) - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-) \tag{5-61}$$

$$\mathbf{s}(t_i) \triangleq \mathbf{K}(t_i)\mathbf{r}(t_i) \tag{5-62}$$

The term $[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]$ in the filter measurement update equation (5-39) is a realization of $\mathbf{r}(t_i)$: the difference between the current measurement value $\mathbf{z}_i$ and the best prediction of its value before the measurement is actually taken. It is then a particular realization of $\mathbf{s}(t_i)$ that is added to $\hat{\mathbf{x}}(t_i^-)$ to obtain $\hat{\mathbf{x}}(t_i^+)$. We can rewrite $\mathbf{r}(t_i)$ as

$$\begin{aligned}
\mathbf{r}(t_i) &= \mathbf{z}(t_i) - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-) \\
&= \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-) \\
&= \mathbf{H}(t_i)\Bigg[\boldsymbol{\Phi}(t_i, t_{i-1})\mathbf{x}(t_{i-1}) + \int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau \\
&\quad + \int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i,\tau)\mathbf{G}(\tau)\,d\boldsymbol{\beta}(\tau)\Bigg] + \mathbf{v}(t_i) \\
&\quad - \mathbf{H}(t_i)\Bigg[\boldsymbol{\Phi}(t_i, t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+) + \int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau\Bigg] \\
&= \mathbf{H}(t_i)\boldsymbol{\Phi}(t_i, t_{i-1})\mathbf{e}(t_{i-1}^+) + \mathbf{H}(t_i)\int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i,\tau)\mathbf{G}(\tau)\,d\boldsymbol{\beta}(\tau) + \mathbf{v}(t_i)
\end{aligned} \tag{5-63}$$

But $\mathbf{e}(t_{i-1}^+)$, $\int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i,\tau)\mathbf{G}(\tau)\,d\boldsymbol{\beta}(\tau)$, and $\mathbf{v}(t_i)$ are all random variables that are independent of $\mathbf{Z}(t_{i-1})$, so $\mathbf{r}(t_i)$ *is independent of* $\mathbf{Z}(t_{i-1})$. But, by their definition, $\mathbf{r}(t_1)$, $\mathbf{r}(t_2)$, ..., $\mathbf{r}(t_{i-1})$ are linear functions of $\mathbf{Z}(t_{i-1})$, so $\mathbf{r}(t_i)$ is independent of all previous $\mathbf{r}(t_j)$'s. In other words, *the* $\mathbf{r}(t_i)$ *sequence is a white sequence.* In view of (5-61), this also demonstrates that $\mathbf{s}(t_i)$ *is independent of* $\mathbf{Z}(t_{i-1})$ and that the $\mathbf{s}(t_i)$ *sequence is white*: each new piece of information is independent of the information gained in the past or, geometrically, $\mathbf{s}(t_i)$ is orthogonal to $\hat{\mathbf{x}}(t_i^-)$.

Moreover, based on arguments of linear combinations of jointly Gaussian random variables being Gaussian, $\mathbf{r}(t_i)$ and $\mathbf{s}(t_i)$ are both *Gaussian* for all t_i. To describe these processes completely, we only need to specify means and covariance kernels (conditioned on measurements or not does not matter, but quantities *defined* as conditional entities will be identified, so we choose to condition on $\mathbf{Z}_{i-1}$):

$$E\{\mathbf{r}(t_i)\} = E\{\mathbf{r}(t_i)|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} = \mathbf{0} \tag{5-64a}$$

$$\begin{aligned} E\{\mathbf{r}(t_i)\mathbf{r}^{\mathrm{T}}(t_i)\} &= E\{\mathbf{r}(t_i)\mathbf{r}^{\mathrm{T}}(t_i)|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \\ &= E\{(\mathbf{H}(t_i)[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^-)] + \mathbf{v}(t_i)) \\ &\quad \cdot (\mathbf{H}(t_i)[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^-)] + \mathbf{v}(t_i))^{\mathrm{T}}|\mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \\ &= \mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i) \end{aligned} \tag{5-64b}$$

$$E\{\mathbf{s}(t_i)\} = \mathbf{K}(t_i)E\{\mathbf{r}(t_i)\} = \mathbf{0} \tag{5-65a}$$

$$\begin{aligned} E\{\mathbf{s}(t_i)\mathbf{s}^{\mathrm{T}}(t_i)\} &= \mathbf{K}(t_i)E\{\mathbf{r}(t_i)\mathbf{r}^{\mathrm{T}}(t_i)\}\mathbf{K}^{\mathrm{T}}(t_i) \\ &= \mathbf{K}(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)]\mathbf{K}^{\mathrm{T}}(t_i) \\ &= \mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)]^{-1}\mathbf{H}(t_i)\mathbf{P}(t_i^-) \end{aligned} \tag{5-65b}$$

Obtaining (5-65b) exploited the symmetry of $[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)]$; the result is singular in general, being n-by-n and of rank at most m. As expected, we recognize (5-65b) as the term that is subtracted from $\mathbf{P}(t_i^-)$ to obtain $\mathbf{P}(t_i^+)$: the decrease in estimation error covariance due to incorporating the information of the measurement at time t_i.

The residual sequence has been shown to be a white Gaussian sequence of mean zero and covariance $[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)]$. This can be exploited for the practical purposes of either sensor failure detection or reasonableness checking of measurement data. The preceding is a proper characterization of the residual process provided that the mathematical model upon which the filter was based accurately depicts the real system behavior. During operation of the filter, the actual residual sequence realization can be monitored and compared to this description. If the description appears valid up to a point in time, thereafter being violated consistently, one can deduce that something occurred in the real system to invalidate the model within the filter. If the violation occurs in only one component of a vector residual process, one can further deduce that the measuring device generating that particular residual component is the source of difficulty: a failure (soft or hard) can be declared in that sensor.

Optimal "likelihood function" methods or ad hoc techniques of event detection (hypothesis testing) can be used to perform a test for the occurrence of a sensor failure [51, 52]. Essentially, the N most recent residual signals are examined to determine whether they differ significantly from the statistical description of their values that assumes no failures [9, 55]. The number N is a design parameter. It is kept greater than one to prevent failure declarations due

to a single residual sample of large magnitude: consistently large residuals indicate abnormalities, whereas individual realizations of large magnitude are to be expected. On the other hand, it is inappropriate to use all the residual samples from the initial time to current time, since this would decrease the sensitivity to true failures as time progressed. Thus, a "moving window" of the N most recent samples, with N on the order of 5 to 20, would be used.

Statistical hypothesis testing theory indicates that a good choice of likelihood function [67] for event (failure) detection would be in the form of sum of natural logs of conditional densities for components of residuals: for the kth component,

$$L_{N_k}(t_i) = \sum_{j=i-N+1}^{i} \ln f_{r_k(t_j)|r_k(t_{j-1}),\ldots,r_k(t_1)}(\rho_j|\rho_{j-1},\ldots,\rho_1) \tag{5-66}$$

If the residual sequence can be assumed to be a set of independent zero-mean Gaussian random variables, then this can be rewritten as

$$L_{N_k}(t_i) = c_k(t_i) - \frac{1}{2}\sum_{j=i-N+1}^{i} \frac{r_k^2(t_j)}{\sigma_k^2(t_j)} \tag{5-67}$$

where $c_k(t_i)$ is a (slowly varying) negative term independent of the observed residual values (thus containing no information of direct use for failure detection), and $\sigma_k^2(t_j)$ is the estimate of the variance of possible kth residual values based on the assumption that no failures have occurred. The value of $1/\sigma_k^2(t_j)$ can be evaluated as the kth diagonal term of $[\mathbf{H}(t_j)\mathbf{P}(t_j^-)\mathbf{H}^T(t_j) + \mathbf{R}(t_j)]^{-1}$, a matrix that has *already* been computed in the filter algorithm. If $r_k^2(t_j)$ becomes consistently larger than that predicted by $\sigma_k^2(t_j)$ over the most recent N samples, $L_{N_k}(t_i)$ will become more and more negative; if its value goes beyond a predetermined threshold, a failure can be declared.

EXAMPLE 5.5 Figure 5.8a portrays a possible residual process realization in the filter described in Example 5.2. Until time t_F (time of sensor failure), the residual sequence is well described as a zero-mean white Gaussian sequence of variance $\sigma^2 = [\mathbf{HPH}^T + R]$: about 68% of the samples lie within the 1σ bounds, 95% within the 2σ bounds, etc. At t_F, the bias shifts markedly from zero. In Fig. 5.8b, the residual process strength increases markedly at time t_F. For either case, the likelihood function $L_N(t_i)$ would be as depicted in Fig. 5.8c: from time t_F on, the $r^2(t_j)/\sigma^2(t_j)$ terms are larger, so $L_N(t_i)$ grows more negative. Upon passing the threshold, a failure is declared. ■

Residual monitoring is also used for reasonableness checking of measurements before they are processed by the filter. If a spurious data point is received, we would want to reject it rather than let it corrupt the filter computations. If a measurement residual is greater than (for instance) the 3σ value computed by the filter, it can be declared as unacceptable. However, if this happens on a frequent basis, the cause may be either a sensor failure or a filter divergence problem (the filter's estimates do not correspond well to the real system behavior). In this latter case, it is critical not to reject large residuals, since they are the only means of correcting the divergence! Filter divergence [25] and more sophisticated use of residual monitoring (adaptive estimation algorithms,

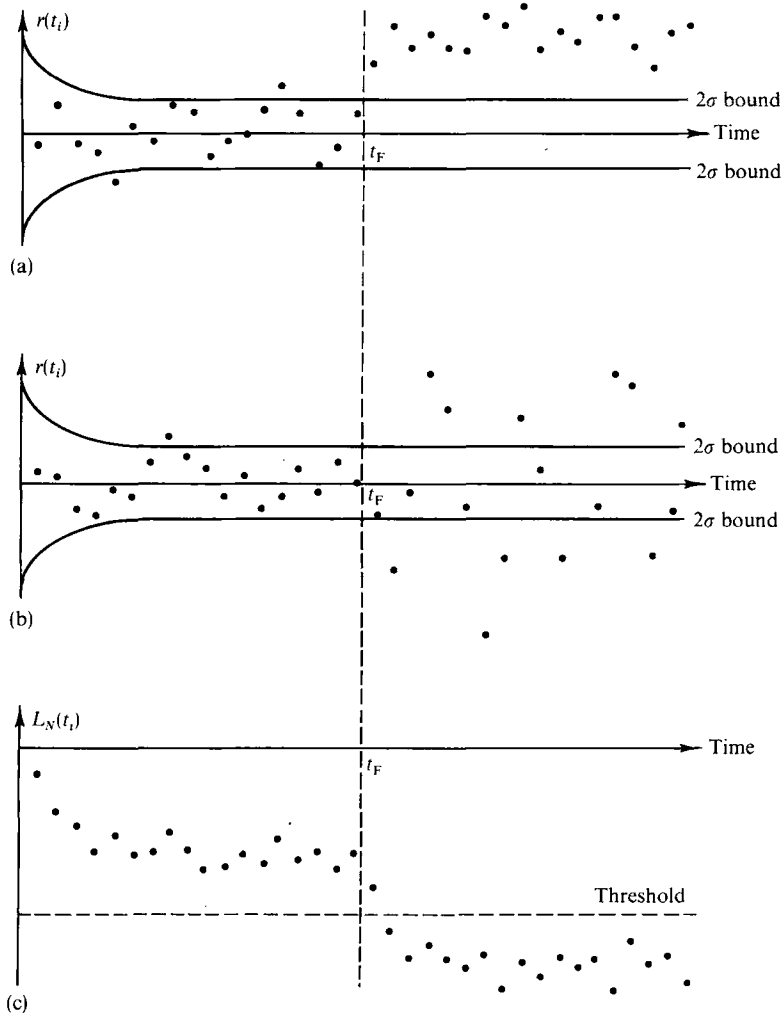

FIG. 5.8 Residual monitoring and sensor failure detection. (a) Residual bias shift. (b) Residual strength increase. (c) Likelihood function $L_N(t_i)$ for either case.

which modify their internal models and/or gains online by exploiting the observed residuals) will be discussed in detail in subsequent chapters.

5.5 OTHER CRITERIA OF OPTIMALITY

In Section 5.3 we derived the Kalman filter algorithm in a Bayesian manner by generating explicit recursions for the Gaussian conditional probability density for the states, conditioned on the entire measurement history, $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i)$. Then $\hat{\mathbf{x}}(t_i^+)$ was chosen as the "optimal estimate" because it was

the mean, mode, and median of this density function. There are other criteria of optimality that are logical for an estimation problem [16, 18, 29, 31, 64, 68, 82], and thus there are other means of deriving the filter relations. Some of these aspects will now be discussed.

By virtue of being the conditional mean, $\hat{\mathbf{x}}(t_i^+)$ is also the minimum mean square error (MMSE) *estimate* [54, 82]. In a general estimation problem, if $\hat{\mathbf{x}}_{\text{EST}}(t_i)$ is some estimator of $\mathbf{x}(t_i)$ and $\mathbf{e}_{\text{EST}}(t_i)$ is the error committed by this estimator,

$$\mathbf{e}_{\text{EST}}(t_i) = \mathbf{x}(t_i) - \hat{\mathbf{x}}_{\text{EST}}(t_i) \tag{5-68}$$

then the estimator that minimizes the cost function

$$J[\hat{\mathbf{x}}_{\text{EST}}(t_i)] = E\{\mathbf{e}_{\text{EST}}(t_i)^{\text{T}}\mathbf{e}_{\text{EST}}(t_i)\} \tag{5-69}$$

is the conditional mean. This is true for any form of probability distribution function $F_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i)$. Moreover, if $F_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i)$ is Gaussian, as is true for the particular problem addressed in this chapter, the conditional mean also minimizes *any* cost function of the general quadratic form

$$J[\hat{\mathbf{x}}_{\text{EST}}(t_i)] = E\{\mathbf{e}_{\text{EST}}(t_i)^{\text{T}}\mathbf{M}(t_i)\mathbf{e}_{\text{EST}}(t_i)\} \tag{5-70}$$

where $\mathbf{M}(t_i)$ for all $t_i \in T$ form a set of arbitrary symmetric, positive semi-definite matrices. For further discussion of *least squares estimation*, see [1, 5, 24, 27, 28, 32, 33, 57, 64–66, 73, 75–77].

EXAMPLE 5.6 Consider an estimation problem with a two dimensional state vector. Consider $\mathbf{M}(t_i)$ constant in time, set equal to any of the following:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$$

According to the above claim, the *same* estimator (the conditional mean) would optimize (5-70) for all five choices. Whether you are interested in estimating only x_1, only x_2, or both with any relative importance in estimation accuracy, the same estimator would be used. ■

Also due to its being the conditional mean, $\hat{\mathbf{x}}(t_i^+)$ minimizes the symmetric cost function criterion [14, 54, 69, 70, 82]. Define a general estimation error cost function as

$$J[\hat{\mathbf{x}}_{\text{EST}}(t_i)] = E\{C[\mathbf{e}_{\text{EST}}(t_i)]\} \tag{5-71}$$

If $C(\cdot)$ is symmetric and nondecreasing,

$$\begin{aligned} &C(\mathbf{0}) = 0 \\ &C(\mathbf{e}) = C(-\mathbf{e}) \\ &C(\mathbf{e}_2) \geq C(\mathbf{e}_1) \qquad \text{if} \quad \|\mathbf{e}_2\| > \|\mathbf{e}_1\| \end{aligned} \tag{5-72}$$

and if $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i)$ is unimodal (one-peaked), symmetric about the conditional mean, and satisfies

$$\lim_{\|\boldsymbol{\xi}\| \to \infty} C(\boldsymbol{\xi}) f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i) = 0 \tag{5-73}$$

then the cost function (5-71) is minimized by the conditional mean. For the problem at hand, the assumptions on $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i)$ are met, and we need only assume (5-72) to be true. Note particularly that there was no need to assume $C(\cdot)$ to be convex, so that all the cost functions depicted in Fig. 5.9 are admissible. Comparing this claim to the fact that the conditional mean is the MMSE estimate, optimality with respect to a more general cost criterion has

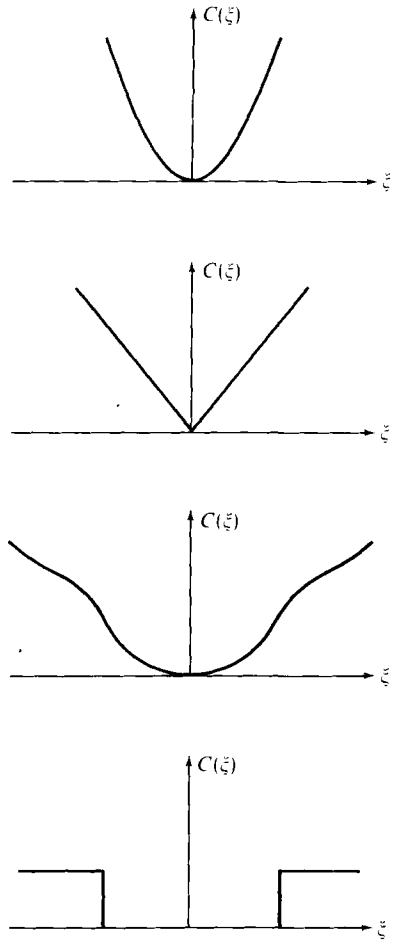

FIG. 5.9 Cost functions admissible according to Eq. (5-72).

been achieved, but at the expense of additional restrictions on the allowable class of stochastic processes [the assumptions on $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}|\mathbf{Z}_i)$].

To obtain a *maximum likelihood estimate* of the system state, an appropriate "likelihood function" [17, 67, 84] must be defined as a scalar function relating the available measurements (whose values are known), the state variables (the unknowns to be estimated), and any other pertinent parameters. One choice of a likelihood function (though not the "classical" choice) would be $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}\,|\,\mathbf{Z}_i)$ itself. By maximizing this likelihood function, we are actually finding the mode (location of the peak) of the conditional density, and the resulting estimator is often called the maximum a posteriori, or MAP, estimate.

To show that $\hat{\mathbf{x}}(t_i^+)$ *is the* MAP *estimate of the state* [30, 62, 66, 82] involves an algebraically simpler derivation than the reduction that followed Eq. (5-27) originally. Under the assumptions of our problem formulation, $f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}\,|\,\mathscr{Z}_i)$ is Gaussian, so it is more convenient to define the (log-) likelihood function as the natural logarithm of the conditional density,

$$L(\boldsymbol{\xi}, \mathscr{Z}_i) = \ln f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\boldsymbol{\xi}\,|\,\mathscr{Z}_i) \tag{5-74}$$

Maximizing this function yields the same estimate as maximizing the density itself, since for any function f, f and $\ln f$ attain their maxima at the same point. Substitute (5-27) into (5-74) to obtain

$$\begin{aligned} L(\boldsymbol{\xi}, \mathscr{Z}_i) = {} & \ln\{(2\pi)^{-n/2}|\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)|^{1/2} \\ & \times |\mathbf{P}(t_i^-)|^{-1/2}|\mathbf{R}(t_i)|^{-1/2}\} \\ & - \tfrac{1}{2}\{[\boldsymbol{\zeta}_i - \mathbf{H}(t_i)\boldsymbol{\xi}]^{\mathrm{T}}\mathbf{R}(t_i)^{-1}[\boldsymbol{\zeta}_i - \mathbf{H}(t_i)\boldsymbol{\xi}]\} \\ & - \tfrac{1}{2}\{[\boldsymbol{\xi} - \hat{\mathbf{x}}(t_i^-)]^{\mathrm{T}}\mathbf{P}(t_i^-)^{-1}[\boldsymbol{\xi} - \hat{\mathbf{x}}(t_i^-)]\} \\ & + \tfrac{1}{2}\{[\boldsymbol{\zeta}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]^{\mathrm{T}}[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)]^{-1} \\ & \times [\boldsymbol{\zeta}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]\} \end{aligned} \tag{5-75}$$

To generate the MAP estimate of $\mathbf{x}(t_i)$, denoted as $\hat{\mathbf{x}}_{\mathrm{MAP}}(t_i)$, we must solve

$$\cdot\ \partial L[\boldsymbol{\xi}, \mathscr{Z}_i]/\partial\boldsymbol{\xi}\Big|_{\xi\to\hat{\mathbf{x}}_{\mathrm{MAP}}(t_i)} = \mathbf{0}^{\mathrm{T}} \tag{5-76}$$

Performing this differentiation on (5-75) yields

$$\{\mathbf{H}^{\mathrm{T}}(t_i)\mathbf{R}^{-1}(t_i)[\boldsymbol{\zeta}_i - \mathbf{H}(t_i)\boldsymbol{\xi}] - \mathbf{P}(t_i^-)^{-1}[\boldsymbol{\xi} - \hat{\mathbf{x}}(t_i^-)]\}\Big|_{\xi\to\hat{\mathbf{x}}_{\mathrm{MAP}}(t_i)} = \mathbf{0}$$

or

$$\{\mathbf{H}^{\mathrm{T}}(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i)\boldsymbol{\xi} + \mathbf{P}(t_i^-)^{-1}\boldsymbol{\xi}\}\Big|_{\xi\to\hat{\mathbf{x}}_{\mathrm{MAP}}(t_i)} = \mathbf{H}^{\mathrm{T}}(t_i)\mathbf{R}^{-1}(t_i)\boldsymbol{\zeta}_i + \mathbf{P}(t_i^-)^{-1}\hat{\mathbf{x}}(t_i^-)$$

for which the solution is

$$\hat{\mathbf{x}}_{\text{MAP}}(t_i) = [\mathbf{P}(t_i^-)^{-1} + \mathbf{H}^{\text{T}}(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i)]^{-1}[\mathbf{H}^{\text{T}}(t_i)\mathbf{R}^{-1}(t_i)\boldsymbol{\zeta}_i + \mathbf{P}(t_i^-)^{-1}\hat{\mathbf{x}}(t_i^-)] \tag{5-77}$$

Comparing this to (5-31) reveals that $\hat{\mathbf{x}}_{\text{MAP}}(t_i) = \hat{\mathbf{x}}(t_i^+)$.

The classical maximum likelihood estimate (often denoted as MLE) is found by maximizing the likelihood function chosen to be the conditional density $f_{\mathbf{Z}(t_i)|\mathbf{x}(t_i)}(\mathscr{Z}_i|\boldsymbol{\xi})$, or its natural logarithm. Conceptually, maximizing this maximizes the probability of the event that *did* in fact occur, i.e., $\mathbf{Z}(t_i, \omega_j) = \mathscr{Z}_i$, expressed as a function of $\boldsymbol{\xi}$. It can be shown that, under the assumptions of the original derivation, the value of $\boldsymbol{\xi}$ which maximizes this likelihood function is again $\hat{\mathbf{x}}(t_i^+)$ if there is no a priori state information (an MLE does not incorporate such data): $\hat{\mathbf{x}}(t_i^+)$ *is the maximum likelihood estimate* (MLE) *of* $\mathbf{x}(t_i)$ *if* $\mathbf{P}_0 = \infty\mathbf{I}$, i.e. *if* $\mathbf{P}_0^{-1} = \mathbf{0}$, *and converges asymptotically to the* MLE *if* $\mathbf{P}_0^{-1} \neq \mathbf{0}$ [60].

As a maximum likelihood estimator, $\hat{\mathbf{x}}(t_i^+)$ possesses certain desirable characteristics [17, 23, 49, 82, 84]. Under rather general regularity conditions, a general maximum likelihood parameter estimator can be shown to be consistent (it converges to the true value as the number of measurement samples grows without bound), asymptotically unbiased, asymptotically normally distributed, and asymptotically efficient (as the number of samples grows without bound, it is unbiased, has finite covariance, and there is no other unbiased estimate whose covariance is smaller). For this particular problem formulation, these asymptotic properties are also true for a finite number of measurement samples.

Furthermore, $\hat{\mathbf{x}}(t_i^+)$ *is the minimum variance unbiased linear estimate* [2–5, 26, 66, 72] of the state. Consider the same problem formulation as in Section 5.2, except that the noises need not be Gaussian. Then $\hat{\mathbf{x}}(t_i^+)$ is the estimator out of the class of *linear* unbiased (zero-mean error) estimators that yields a minimum error variance (minimum trace of the error covariance matrix). It is the best linear estimator in this sense, but there are nonlinear estimators which may outperform it. In the original derivation, we imposed the Gaussian assumption and did *not* have to seek the best *linear* filter: under this additional assumption, the linear filter *is* the best filter of any kind.

The original derivation of the Kalman filter was based on the fact that $\hat{\mathbf{x}}(t_i^+)$ *is the orthogonal projection of the true state* $\mathbf{x}(t_i)$ *onto the subspace spanned by* $\mathbf{Z}(t_i)$ [35, 47, 64]. The orthogonal projection lemma of functional analysis is applied to the estimation problem as posed in the infinite-dimensional Hilbert space of random variables with finite second moments. We will not delve into the rigor of the proof, but the geometric insight is that if the estimate of $\mathbf{x}(t_i) \in \mathscr{X}$ is desired in the form of some $\hat{\mathbf{x}}_{\text{EST}}(t_i)$ confined to a subspace of $\mathscr{X}$, then the

best estimate (approximation) is the orthogonal projection of $\mathbf{x}(t_i)$ onto that subspace: such that the error defined in (5-68) is orthogonal to that subspace.

Thus, the Kalman filter algorithm is optimal with respect to many different criteria. This reveals the power and importance of this algorithm, the practical design and implementation of which will be detailed subsequently.

5.6 COVARIANCE MEASUREMENT UPDATE COMPUTATIONS

The most troublesome numerical aspect of the Kalman filter is the measurement update of the covariance matrix, so the properties of alternate computational forms are of substantial interest.

As originally obtained in the derivation of the estimator, this update can be written as

$$\mathbf{P}(t_i^+) = [\mathbf{P}(t_i^-)^{-1} + \mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i)]^{-1} \tag{5-78}$$

Although this form adds the measurement information in a simple manner that preserves symmetry well, it requires two n-by-n inversions each sample time (unless an inverse covariance is used in place of the covariance, as discussed in the next section), where n is the number of state variables.

Applying the matrix inversion lemma to (5-78) provided an alternate form of

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{P}(t_i^-)\mathbf{H}^T(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{R}(t_i)]^{-1}\mathbf{H}(t_i)\mathbf{P}(t_i) \tag{5-79a}$$

$$= \mathbf{P}(t_i^-) - \mathbf{K}(t_i)\mathbf{H}(t_i)\mathbf{P}(t_i^-) \tag{5-79b}$$

This involves m-by-m inversions, where m is the dimension of the measurement vector. For m significantly less than n, as is the case in many practical applications, (5-79) is therefore much more efficient than (5-78). However, (5-79) can involve the small difference of large numbers, especially if the measurements are very accurate. On a finite word length computer, this can cause serious numerical precision problems, even to the extent of not assuring positive definiteness of the result. (Once any computed covariance obtains a negative eigenvalue, all subsequent computations are erroneous since they are based on a theoretical impossibility. In practice, filters can be "tuned" or modified so as to be able to avoid or recover from such numerical errors, as will be seen in subsequent chapters.)

EXAMPLE 5.7 Consider the first measurement time in the gyro on test introduced in Example 5.2, but let R be changed from 5×10^{-1} deg^2/hr^2 to 5×10^{-4} deg^2/hr^2. Then $P(t_i^-) = 1$, $K(t_i)H(t_i)P(t_i^-) = 0.99950025$, and $P(t_i^+) = 0.00049975$ to eight significant figures. To three significant figures, $K(t_i)H(t_i)P(t_i^-)$ would be rounded up to one and $P(t_i^+)$ would be zero. If truncation were used rather than rounding, then $K(t_i)H(t_i)P(t_i^-)$ would be 0.999 and $P(t_i^+)$ would be 0.001. Both cases are seen to be very erroneous on a percentage basis. ■

The update (5-79b) is readily seen to be equivalent algebraically to

$$\mathbf{P}(t_i^+) = [\mathbf{I} - \mathbf{K}(t_i)\mathbf{H}(t_i)]\mathbf{P}(t_i^-) \tag{5-80}$$

Not only does this form fail to assure positive definiteness as true of (5-79b), but it suffers additionally from the fact that symmetry is not well preserved either. Whereas (5-79b) entailed subtracting one symmetric form from another, (5-80) is in the form of a product of a nonsymmetric matrix and a symmetric one, and thus it is a less desirable form.

If the state estimate update equation is rewritten as

$$\hat{\mathbf{x}}(t_i^+) = [\mathbf{I} - \mathbf{K}(t_i)\mathbf{H}(t_i)]\hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i)\mathbf{z}_i \tag{5-81}$$

it can be readily shown that an equivalent expression for $\mathbf{P}(t_i^+)$ is the "Joseph form" (after the man who first developed it):

$$\mathbf{P}(t_i^+) = [\mathbf{I} - \mathbf{K}(t_i)\mathbf{H}(t_i)]\mathbf{P}(t_i^-)[\mathbf{I} - \mathbf{K}(t_i)\mathbf{H}(t_i)]^{\mathrm{T}} + \mathbf{K}(t_i)\mathbf{R}(t_i)\mathbf{K}^{\mathrm{T}}(t_i) \tag{5-82}$$

This is in the form of the *sum* of two *symmetric* matrices, the first being positive definite and the second being positive semidefinite (n-by-n, and of rank at most m). Consequently, numerical computations based upon this form will be better conditioned, better assuring both the symmetry and positive definiteness of $\mathbf{P}(t_i^+)$ than previous forms. Furthermore, it is insensitive, to first order, to small errors $\delta\mathbf{K}(t_i)$ in the computed filter gain: for a first order error $\delta\mathbf{K}(t_i)$, the error in the $\mathbf{P}(t_i^+)$ computed by (5-82) is of second order, while the error in the previous forms is of first order,

$$\delta\mathbf{P}(t_i^+) = -\delta\mathbf{K}(t_i)\mathbf{H}(t_i)\mathbf{P}(t_i^-) \tag{5-83}$$

Similarly, it is less sensitive to arithmetic truncation than the other forms: especially in the cases in which the measurement noise is small, (5-79) and (5-80) will be subject to first order truncation error effects, while (5-82) will only be affected to second order. This becomes a crucial consideration for online applications in which the minimum computer wordlength that achieves adequate performance is sought.

Although the Joseph form has some desirable characteristics, it requires a considerably greater number of computations (multiplications and additions), so more computer time is required. (Section 7.8 will tabulate required operations for various forms; Problem 5.17 develops a more efficient implementation of the Joseph form.) A tradeoff must be analyzed to determine if the benefits warrant the additional loading. In fact, there are some cases, especially those characterized by long periods of essentially steady state behavior, in which the inherently greater number of adds and multiplies (each individual operation with truncation or roundoff effects of its own) causes larger numerical errors in the Joseph form than in the others.

To date, it is typical to perform the majority of the algorithm computations in single precision. Regardless of the form used, the covariance measurement update calculations are done in double precision to maintain numerical accuracy.

For online applications in which time constraints are critical, symmetry can be exploited by propagating and updating only lower triangular forms of the covariance matrix. This requires only $\frac{1}{2}n(n+1)$ scalar terms instead of n^2, which can be substantial for large n. However, due to symmetry preservation problems, some operational filters maintain all n^2 terms and periodically resymmetrize the covariance matrix by averaging the appropriate elements.

Symmetry can be exploited further by using a square root covariance formulation: express and compute the algorithm results in terms of $\mathbf{P}^{1/2}$ instead of $\mathbf{P}$ (a matrix $\mathbf{P}^{1/2}$ such that $\mathbf{P}^{1/2}\mathbf{P}^{1/2\,\mathrm{T}} = \mathbf{P}$ can always be defined for a symmetric positive definite $\mathbf{P}$ matrix). This attains equivalent numerical accuracy with approximately half the wordlength, while requiring a number of computations comparable to that of the Joseph form update for $\mathbf{P}$. A related technique, known as U–D covariance factorization, in which $\mathbf{P}$ is factored as $\mathbf{P} = \mathbf{U}\mathbf{D}\mathbf{U}^{\mathrm{T}}$, with $\mathbf{U}$ being upper triangular and unitary and $\mathbf{D}$ being diagonal, provides the same numerical benefits but with considerably less computational loading. "Square root filtering" is the subject of Chapter 7.

An alternate expression for the filter gain $\mathbf{K}(t_i)$, given originally by (5-38), is

$$\mathbf{K}(t_i) = \mathbf{P}(t_i^{+})\mathbf{H}^{\mathrm{T}}(t_i)\mathbf{R}^{-1}(t_i^{-}) \tag{5-84}$$

Equivalence can be demonstrated by substituting (5-79a) into (5-84) to obtain (5-38). Although (5-84) is a simpler expression, it is not very useful for the discrete-time update (sampled-data) filter formulation, since it requires $\mathbf{P}(t_i^{+})$ to be known before $\mathbf{K}(t_i)$ can be obtained! However, this equivalent expression will be of use for the continuous-measurement case and in fact is the computational form of the Kalman gain for that case.

5.7 INVERSE COVARIANCE FORM

The previous section mentioned the idea of expressing the optimal estimation algorithm in terms of the inverse of the covariance matrix, instead of the covariance itself. Though an algebraically equivalent result, this form will possess some unique characteristics, as allowing a startup procedure for the case of $\mathbf{P}_0^{-1}$ being singular. This form will be directly exploited in the optimal smoothers of Chapter 8 (Volume 2). Moreover, the inverse covariance matrix is directly related to the Fisher information matrix, allowing an interpretation of filter performance in terms of information theoretic concepts. Finally, the relationship of the optimal estimator to the classical Gauss–Markov theorem for a special class of problems becomes apparent from this form.

The usual recursion relations for the covariance matrix in the estimation algorithm can be written as

$$\mathbf{P}^{-1}(t_i^+) = \mathbf{P}^{-1}(t_i^-) + \mathbf{H}^{\mathrm{T}}(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i) \tag{5-85}$$

$$\mathbf{P}(t_{i+1}^-) = \boldsymbol{\Phi}(t_{i+1}, t_i)\mathbf{P}(t_i^+)\boldsymbol{\Phi}^{\mathrm{T}}(t_{i+1}, t_i) + \mathbf{G}_{\mathrm{d}}(t_i)\mathbf{Q}_{\mathrm{d}}(t_i)\mathbf{G}_{\mathrm{d}}{}^{\mathrm{T}}(t_i) \tag{5-86}$$

Applying the matrix inversion lemma to (5-85) yields the familiar Kalman filter equations. Instead, apply the lemma, one form of which states that for $\mathbf{X}$ and $\mathbf{Y}$ both n-by-k matrices,

$$(\mathbf{A} + \mathbf{X}^{\mathrm{T}}\mathbf{Y})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{X}^{\mathrm{T}}(\mathbf{I} + \mathbf{Y}\mathbf{A}^{-1}\mathbf{X}^{\mathrm{T}})^{-1}\mathbf{Y}\mathbf{A}^{-1} \tag{5-87}$$

to (5-86) by identifying

$$\mathbf{A} = \boldsymbol{\Phi}(t_{i+1}, t_i)\mathbf{P}(t_i^+)\boldsymbol{\Phi}^{\mathrm{T}}(t_{i+1}, t_i); \qquad \mathbf{X}^{\mathrm{T}} = \mathbf{G}_{\mathrm{d}}(t_i)\mathbf{Q}_{\mathrm{d}}(t_i); \qquad \mathbf{Y} = \mathbf{G}_{\mathrm{d}}{}^{\mathrm{T}}(t_i)$$

to yield, for $\mathbf{Q}_{\mathrm{d}}(t_i)$ nonsingular (not very restrictive),

$$\begin{aligned}\mathbf{P}^{-1}(t_{i+1}^-) = \mathbf{M}(t_{i+1}) - \mathbf{M}(t_{i+1})\mathbf{G}_{\mathrm{d}}(t_i)[\mathbf{G}_{\mathrm{d}}{}^{\mathrm{T}}(t_i)\mathbf{M}(t_{i+1})\mathbf{G}_{\mathrm{d}}(t_i) \\ + \mathbf{Q}_{\mathrm{d}}^{-1}(t_i)]^{-1}\mathbf{G}_{\mathrm{d}}{}^{\mathrm{T}}(t_i)\mathbf{M}(t_{i+1})\end{aligned} \tag{5-88}$$

where

$$\mathbf{M}(t_{i+1}) = \boldsymbol{\Phi}^{\mathrm{T}}(t_i, t_{i+1})\mathbf{P}^{-1}(t_i^+)\boldsymbol{\Phi}(t_i, t_{i+1}) \tag{5-89}$$

In (5-89), note the order of time indices in the state transition matrices: appropriate for backward time propagation of the system relations (forward propagation of adjoint relations). If $\mathbf{Q}_{\mathrm{d}}(t_i) \equiv \mathbf{0}$, then (5-88) is not applicable, and $\mathbf{P}^{-1}(t_{i+1}) = \mathbf{M}(t_{i+1})$. The inverse covariance time propagation equation (5-88) is analogous in form to the covariance update equation (5-79). The analog of the Joseph form (5-82) has also been derived, with superior numerical characteristics similar to that of (5-82). Define the gain matrix $\mathscr{K}(t_i)$ as

$$\mathscr{K}(t_i) = \mathbf{M}(t_{i+1})\mathbf{G}_{\mathrm{d}}(t_i)[\mathbf{G}_{\mathrm{d}}{}^{\mathrm{T}}(t_i)\mathbf{M}(t_{i+1})\mathbf{G}_{\mathrm{d}}(t_i) + \mathbf{Q}_{\mathrm{d}}^{-1}(t_i)]^{-1} \tag{5-90}$$

In terms of this gain, (5-88) becomes

$$\mathbf{P}^{-1}(t_{i+1}^-) = \mathbf{M}(t_{i+1}) - \mathscr{K}(t_i)\mathbf{G}_{\mathrm{d}}{}^{\mathrm{T}}(t_i)\mathbf{M}(t_{i+1}) \tag{5-91}$$

or, in the analog of the Joseph form [49],

$$\mathbf{P}^{-1}(t_{i+1}^-) = [\mathbf{I} - \mathscr{K}(t_i)\mathbf{G}_{\mathrm{d}}{}^{\mathrm{T}}(t_i)]\mathbf{M}(t_{i+1})[\mathbf{I} - \mathscr{K}(t_i)\mathbf{G}_{\mathrm{d}}{}^{\mathrm{T}}(t_i)]^{\mathrm{T}} + \mathscr{K}(t_i)\mathbf{Q}_{\mathrm{d}}^{-1}(t_i)\mathscr{K}^{\mathrm{T}}(t_i) \tag{5-92}$$

In certain circumstances, the a priori statistical information about the state may not be complete: there is no information about the state initial conditions in some or all directions of state space. This can be modeled as the limiting case of certain eigenvalues of $\mathbf{P}_0$ going to infinity, or those of $\mathbf{P}_0^{-1}$ going to zero. Because it remains finite, the inverse covariance would be more desirable to employ.

If $\mathbf{P}^{-1}(t_0) = \mathbf{P}_0^{-1}$ is singular, then until $\mathbf{P}^{-1}(t_i)$ attains full rank, a unique estimate of the full state cannot be made. To allow a viable startup procedure, the state estimates $\hat{\mathbf{x}}(t_i^-)$ and $\hat{\mathbf{x}}(t_i^+)$ are replaced by

$$\hat{\mathbf{y}}(t_i^-) \triangleq \mathbf{P}^{-1}(t_i^-)\hat{\mathbf{x}}(t_i^-) \tag{5-93a}$$

$$\hat{\mathbf{y}}(t_i^+) \triangleq \mathbf{P}^{-1}(t_i^+)\hat{\mathbf{x}}(t_i^+) \tag{5-93b}$$

The recursions for $\hat{\mathbf{y}}$ are then

$$\hat{\mathbf{y}}(t_i^+) = \hat{\mathbf{y}}(t_i^-) + \mathbf{H}^{\mathrm{T}}(t_i)\mathbf{R}^{-1}(t_i)\mathbf{z}_i \tag{5-94}$$

$$\hat{\mathbf{y}}(t_{i+1}^-) = [\mathbf{I} - \mathscr{K}(t_i)\mathbf{G}_{\mathrm{d}}{}^{\mathrm{T}}(t_i)]\boldsymbol{\Phi}^{\mathrm{T}}(t_i, t_{i+1})[\hat{\mathbf{y}}(t_i^+) + \mathbf{P}^{-1}(t_i^+)\boldsymbol{\Phi}(t_i, t_{i+1})\mathbf{B}_{\mathrm{d}}(t_i)\mathbf{u}(t_i)] \tag{5-95}$$

starting from the initial condition

$$\hat{\mathbf{y}}(t_0) = \mathbf{P}_0^{-1}\hat{\mathbf{x}}_0 \tag{5-96}$$

Once $\mathbf{P}^{-1}(t_i^+)$ becomes nonsingular, then its inverse can be computed to obtain $\mathbf{P}(t_i^+)$, and the optimal state estimate can be expressed as

$$\hat{\mathbf{x}}(t_i^+) = \mathbf{P}(t_i^+)\hat{\mathbf{y}}(t_i^+) \tag{5-97}$$

From that time forward, it is possible to revert to the more familiar covariance form or to continue in the inverse covariance form. The latter is more convenient in certain situations, as in smoothing.

Thus, the inverse covariance form of the optimal estimator has been described. Measurement updating is accomplished through (5-85) and (5-94), and time propagation by (5-89), (5-90), (5-91) or (5-92), and (5-95). Initial conditions are $\mathbf{P}^{-1}(t_0) = \mathbf{P}_0^{-1}$ and $\hat{\mathbf{y}}(t_0) = \mathbf{P}_0^{-1}\hat{\mathbf{x}}_0$. These relations are valid for $\mathbf{Q}_{\mathrm{d}}(t_i)$ positive definite; unless there is no driving noise, a positive definite s-by-s $\mathbf{Q}_{\mathrm{d}}(t_i)$ can always be generated for an n-by-n positive semidefinite $[\mathbf{G}_{\mathrm{d}}(t_i)\mathbf{Q}_{\mathrm{d}}(t_i)\mathbf{G}_{\mathrm{d}}{}^{\mathrm{T}}(t_i)]$ form of rank s. If there is no driving noise [$\mathbf{Q}_{\mathrm{d}}(t_i) = \mathbf{0}$ for all t_i], then the time propagation relations become

$$\mathbf{P}^{-1}(t_{i+1}^-) = \mathbf{M}(t_{i+1}) \tag{5-98}$$

$$\hat{\mathbf{y}}(t_{i+1}^-) = \boldsymbol{\Phi}^{\mathrm{T}}(t_i, t_{i+1})[\hat{\mathbf{y}}(t_i^+) + \mathbf{P}^{-1}(t_i^+)\boldsymbol{\Phi}(t_i, t_{i+1})\mathbf{B}_{\mathrm{d}}(t_i)\mathbf{u}(t_i)] \tag{5-99}$$

A concept related to the inverse covariance is the *Fisher information matrix* which is a measure of the certainty of the state estimate due to measurement data *alone*; i.e., the a priori information of $\mathbf{x}(t_0)$ being modeled as Gaussian with mean $\hat{\mathbf{x}}_0$ and covariance $\mathbf{P}_0$ is disregarded. The information matrix $\mathscr{J}(t_i, t_1)$ is given by

$$\mathscr{J}(t_i, t_1) = \sum_{j=1}^{i} \boldsymbol{\Phi}^{\mathrm{T}}(t_j, t_i)\mathbf{H}^{\mathrm{T}}(t_j)\mathbf{R}^{-1}(t_j)\mathbf{H}(t_j)\boldsymbol{\Phi}(t_j, t_i) \tag{5-100}$$

where, as noted previously, $\Phi(t_j, t_i)$ for $j < i$ is the transition matrix for propagating the system state backward in time. To relate this concept directly to the previous algorithm, ignore the dynamics driving noise: assume that there is no $\mathbf{w}_d(t_i)$ sequence, or equivalently, that $\mathbf{Q}_d(t_i) = \mathbf{0}$ for all t_i. Under this assumption, (5-85), (5-89), and (5-98) yield

$$\mathcal{J}(t_i, t_1) = \mathbf{P}^{-1}(t_i^+) - \Phi^T(t_0, t_i)\mathbf{P}_0^{-1}\Phi(t_0, t_i) \tag{5-101}$$

If there were no a priori information about the state, or formally if $\mathbf{P}_0^{-1} = \mathbf{0}$, then the information matrix is the inverse of the corresponding estimator error covariance. The larger the eigenvalues of $\mathcal{J}(t_i, t_1)$, the smaller the eigenvalues of $\mathbf{P}(t_i^+)$, and the more precise our estimate is. If any eigenvalues of $\mathcal{J}(t_i, t_1)$ are zero, there are directions in state space along which our measurements give us no information. Not surprisingly, this information matrix is directly related to the observability matrix studied in Chapter 2 [see Eq. (2-74)].

Expressing the definition of the information matrix, (5-100), for times t_i and t_{i-1}, and equating like terms, yields the following recursion:

$$\mathcal{J}(t_i, t_1) = \Phi^T(t_{i-1}, t_i)\mathcal{J}(t_{i-1}, t_1)\Phi(t_{i-1}, t_i) + \mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i) \tag{5-102}$$

From this relation, it can be seen that the "information" contained in a single measurement at time t_i is $[\mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i)]$: the term added to $\mathbf{P}^{-1}(t_i^-)$ to generate $\mathbf{P}^{-1}(t_i^+)$.

EXAMPLE 5.8 Reconsider the estimator for the gyro on test, Example 5.2, in inverse variance form. Initial conditions are

$$P^{-1}(t_0) = P_0^{-1} = 1 \quad \text{hr}^2/\text{deg}^2$$
$$\hat{y}(t_0) = P_0^{-1}\hat{x}_0 = 0$$

Equations (5-85) and (5-94) yield the measurement updates as

$$P^{-1}(t_i^+) = P^{-1}(t_i^-) + H^T R^{-1} H = P^{-1}(t_i^-) + 2$$
$$\hat{y}(t_i^+) = \hat{y}(t_i^-) + H^T R^{-1} z_i = \hat{y}(t_i^-) + 2z_i$$

The time propagations are

$$M(t_{i+1}) = (\Phi^{-1})^T P^{-1}(t_i^+)\Phi^{-1} = (0.78)^{-2} P^{-1}(t_i^+) = 1.64 P^{-1}(t_i^+)$$

$$\mathcal{K}(t_i) = \frac{M(t_{i+1})G_d}{G_d^T M(t_{i+1})G_d + Q_d^{-1}} = \frac{M(t_{i+1})}{M(t_{i+1}) + (0.39)^{-1}} = \frac{M(t_{i+1})}{M(t_{i+1}) + 2.56}$$

$$P^{-1}(t_{i+1}^-) = M(t_{i+1}) - \mathcal{K}(t_i)G_d^T M(t_{i+1})$$

$$\hat{y}(t_{i+1}^-) = [1 - \mathcal{K}(t_i)G_d^T](\Phi^{-1})^T \hat{y}(t_i^+) = 1.28[1 - \mathcal{K}(t_i)]\hat{y}(t_i^+)$$

The time history of the inverse variance (and gains) can be precomputed, and is displayed in Fig. 5.10. These results are directly comparable to Fig. 5.6, the plot of the error variance for the same problem. Note that the "information" added at each measurement time is $H^T R^{-1} H = 2\ \text{hr}^2/\text{deg}^2$. ■

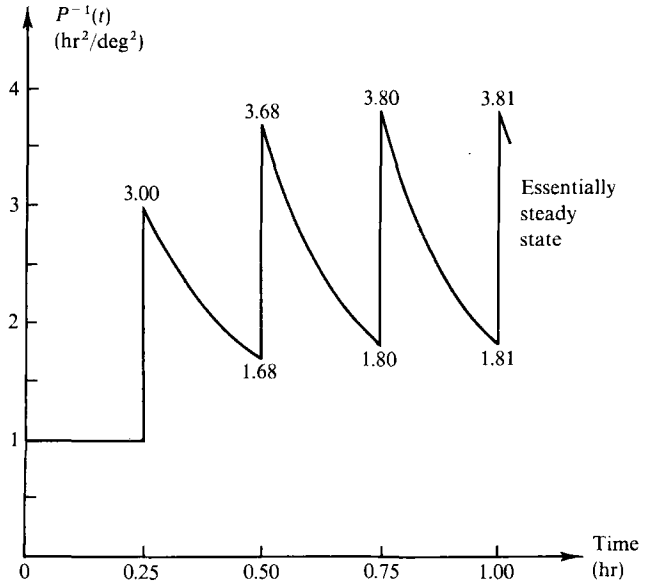

FIG. 5.10 Inverse variance time history for Example 5.8.

5.8 STABILITY

This section specifies the rather nonrestrictive conditions under which the filter algorithm is stable [22, 53]. Issues involved in the stability of stochastically driven systems are not completely resolved, but Lyapunov stability theory has been applied to the homogeneous portion of the filter to establish *zero-input stability* criteria. In fact, conditions will be established under which the filter is a uniformly, asymptotically stable linear system, which then implies *bounded input–bounded output* (BIBO) *stability*: for bounded inputs into the filter, the output (state estimate) is bounded. We note that zero-input and BIBO stability are significantly more distinct issues for nonlinear systems.

The state equations (5-36) and (5-39) of the Kalman filter algorithm can be rewritten as

$$\begin{aligned}\hat{\mathbf{x}}(t_i^+) &= [\mathbf{I} - \mathbf{K}(t_i)\mathbf{H}(t_i)]\hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i)\mathbf{z}_i \\ &= [\mathbf{I} - \mathbf{K}(t_i)\mathbf{H}(t_i)]\boldsymbol{\Phi}(t_i, t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+) \\ &\quad + [\mathbf{I} - \mathbf{K}(t_i)\mathbf{H}(t_i)] \int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau + \mathbf{K}(t_i)\mathbf{z}_i \end{aligned} \quad (5\text{-}103)$$

We want to consider the stability of the homogeneous part of the filter,

$$\hat{\mathbf{x}}_{\mathrm{h}}(t_i^+) = [\mathbf{I} - \mathbf{K}(t_i)\mathbf{H}(t_i)]\boldsymbol{\Phi}(t_i, t_{i-1})\hat{\mathbf{x}}_{\mathrm{h}}(t_{i-1}^+) \quad (5\text{-}104)$$

a linear discrete system model with state transition matrix equal to

$$[\mathbf{I} - \mathbf{K}(t_i)\mathbf{H}(t_i)]\mathbf{\Phi}(t_i, t_{i-1}).$$

To specify the sufficient conditions for stability succinctly, we must introduce some system theory concepts and terminology. We assume a system model as described in Section 5.2. Such a system representation is said to be *stochastically controllable* if there exist positive numbers α and β, $0 < \alpha < \beta < \infty$, and a time interval Δt such that, for all $t \geq t_0 + \Delta t$,

$$\alpha\mathbf{I} \leq \int_{t-\Delta t}^{t} \mathbf{\Phi}(t, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\mathbf{\Phi}^{\mathrm{T}}(t, \tau)\, d\tau \leq \beta\mathbf{I} \tag{5-105}$$

where $\mathbf{M}_1 \geq \mathbf{M}_2$ means $(\mathbf{M}_1 - \mathbf{M}_2) \geq \mathbf{0}$, i.e., $(\mathbf{M}_1 - \mathbf{M}_2)$ is positive semidefinite. This implies that the system is completely controllable with respect to the points of entry of the dynamic driving noise (see Section 2.5), which further implies that the driving noise affects all of the states. However, (5-105) is a stricter requirement than complete controllability: not only must the integral be positive definite, but must be bounded both above and below.

Analogously, the discrete-time system representation (5-49), which may have arisen as an equivalent discrete-time system model, is stochastically controllable if there exist α and β, $0 < \alpha < \beta < \infty$, and a positive integer N such that, for all $i \geq N$,

$$\alpha\mathbf{I} \leq \sum_{j=i-N+1}^{i} \mathbf{\Phi}(t_i, t_j)\mathbf{G}_{\mathrm{d}}(t_{j-1})\mathbf{Q}_{\mathrm{d}}(t_{j-1})\mathbf{G}_{\mathrm{d}}^{\mathrm{T}}(t_{j-1})\mathbf{\Phi}^{\mathrm{T}}(t_i, t_j) \leq \beta\mathbf{I} \tag{5-106}$$

As in (5-105), this is stricter than, and implies, complete controllability with respect to the points of entry of the dynamic driving noise $\mathbf{w}_{\mathrm{d}}(\cdot, \cdot)$.

The sampled-data system representation of Section 5.2 is said to be *stochastically observable* if there exist positive numbers α and β, $0 < \alpha < \beta < \infty$, and a positive integer N such that, for all $i \geq N$,

$$\alpha\mathbf{I} \leq \sum_{j=i-N+1}^{i} \mathbf{\Phi}^{\mathrm{T}}(t_j, t_i)\mathbf{H}^{\mathrm{T}}(t_j)\mathbf{R}^{-1}(t_j)\mathbf{H}(t_j)\mathbf{\Phi}(t_j, t_i) \leq \beta\mathbf{I} \tag{5-107}$$

Due to the requirement of being bounded both above and below, this is a stronger condition than, and implies, complete observability with respect to the points of exit of the measurements from the system model. Thus, it also implies that the effects of changes of any states can be observed in the outputs. Note that the summation term that appears in (5-107) is in fact the information matrix $\mathcal{J}(t_i, t_{i-N+1})$.

In Section 5.10, optimal estimation for the case of continuously available measurements will be discussed. The measurement model will be

$$\mathbf{z}(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}(t) \tag{5-108}$$

with $\mathbf{v}(\cdot,\cdot)$ a zero-mean white Gaussian noise with $E\{\mathbf{v}(t)\mathbf{v}^{\mathrm{T}}(t+\tau)\} = \mathbf{R}_{\mathrm{c}}(t)\delta(\tau)$. Such a system representation is similarly said to be stochastically observable if there exist positive numbers α and β, $0 < \alpha < \beta < \infty$, and a time interval Δt such that, for all $t \geq t_0 + \Delta t$,

$$\alpha\mathbf{I} \leq \int_{t-\Delta t}^{t} \mathbf{\Phi}^{\mathrm{T}}(\tau, t)\mathbf{H}^{\mathrm{T}}(\tau)\mathbf{R}_{\mathrm{c}}^{-1}(\tau)\mathbf{H}(\tau)\mathbf{\Phi}(\tau, t)\, d\tau \leq \beta\mathbf{I} \tag{5-109}$$

The integral in this expression is the information matrix, $\mathscr{J}(t, t-\Delta t)$, appropriate to this continuous-time model.

Note that the condition of stochastic controllability is not met if $\mathbf{Q}(\tau) \equiv \mathbf{0}$ over the entire interval in (5-105) or $\mathbf{Q}_{\mathrm{d}}(t_{j-1}) \equiv \mathbf{0}$ over the summation range in (5-106): the case of no dynamic driving noise. Neither is it met if these strengths have infinite eigenvalues over a finite length of time. Similarly, stochastic observability is violated if $\mathbf{R}^{-1}(t_j)$ or $\mathbf{R}_{\mathrm{c}}^{-1}(\tau)$ are zero over the entire time of interest (infinite noise corruption) or if they are infinite over any finite time (the case of perfect measurements).

If the system model upon which the Kalman filter is based is stochastically observable and stochastically controllable, then the filter is uniformly asymptotically globally stable. This means that if we consider the homogeneous equation (5-104), then

$$\lim_{i \to \infty} \left\| \hat{\mathbf{x}}_{\mathrm{h}}(t_i^{+}) \right\| = 0 \tag{5-110}$$

i.e., in the limit as the number of data samples grows without bound, the norm ("length" or magnitude) of $\hat{\mathbf{x}}_{\mathrm{h}}(t_i^{+})$ goes to zero (asymptotic), no matter what the initial conditions (global), and the rate of convergence is not a function of absolute time (uniform). Mathematically, the system model is uniformly asymptotically stable if there exist positive constants α and β such that, for *all* $t_i \geq t_0$, the state transition matrix to transition $\hat{\mathbf{x}}_{\mathrm{h}}(t_0)$ to time t_i has a norm bounded above by $\alpha \exp[-\beta(t_i - t_0)]$. The proof of this claim through explicit generation of an appropriate Lyapunov function is omitted, but can be found in the work of Kalman [36], Deyst and Price [22], Sorenson [72], Jazwinski [31], and McGarty [53] (the last reference correcting errors made in previous derivations).

It is important that the sufficient conditions for filter stability do not include stability of the original system model itself. The system model (and the actual system itself) can be unstable, and the filter equations may simultaneously be stable. Even if the system states are in fact growing without bound, as for example in the onset of nuclear reactor runaway, the errors committed by the filter in estimating those states will remain bounded. Thus, if the "true" state time history were as in Fig. 5.11, the state estimate could track this behavior as in plot (a) rather than exhibit an error growing unbounded so as to indicate no system instability, as in plot (b). This is a very desirable filter characteristic.

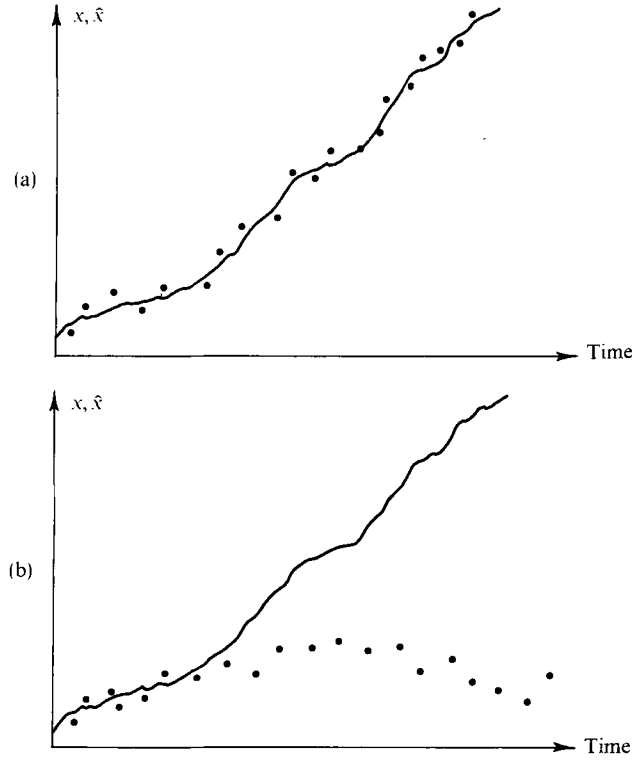

FIG. 5.11 Estimation performance for unstable systems. (a) Unstable system and stable filter. (b) Unstable system and unstable filter. The solid lines indicate "true state" and the dots indicate state estimates.

As might be suggested by the filter derivation of Section 5.3, if the system model is stochastically controllable, then $\mathbf{P}(t_i^+)$ is positive definite for all $i \geq N$. If it is both stochastically controllable and stochastically observable, $\mathbf{P}(t_i^+)$ is also uniformly bounded from above for all $i \geq N$. Furthermore, under these conditions, if $\mathbf{P}_1(t_i^+)$ and $\mathbf{P}_2(t_i^+)$ are two solutions to the filter recursions for different initial conditions $\mathbf{P}_{10} \geq \mathbf{0}$ and $\mathbf{P}_{20} \geq \mathbf{0}$, respectively, then $[\mathbf{P}_1(t_i^+) - \mathbf{P}_2(t_i^+)]$ converges uniformly asymptotically globally to $\mathbf{0}$. (For proofs, see Jazwinski [31].) This last claim indicates that as more measurement information is incorporated, the effect of $\mathbf{P}_0$ (which is often subject to uncertainty itself) is "forgotten." Since the same would be true of the $[\mathbf{P}_1(t_i^+) - \mathbf{P}_2(t_i^+)]$ sequence when started from some time later than t_0, this also indicates that the effect of numerical errors in computing $\mathbf{P}(t_i^+)$ are similarly "forgotten." However, numerical errors associated with finite computer wordlength warrant particular attention, in that the stability claims just made are based upon an assumption of unbounded wordlength. Chapter 7 will discuss this problem and its solution in greater detail.

5.9 CORRELATION OF DYNAMIC DRIVING NOISE AND MEASUREMENT NOISE

Assume that a sampled-data problem is adequately described by the discrete-time (possibly equivalent discrete-time) system model

$$\mathbf{x}(t_i) = \mathbf{\Phi}(t_i, t_{i-1})\mathbf{x}(t_{i-1}) + \mathbf{B}_d(t_{i-1})\mathbf{u}(t_{i-1}) + \mathbf{G}_d(t_{i-1})\mathbf{w}_d(t_{i-1}) \quad (5\text{-}49)$$

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \quad (5\text{-}4)$$

with the usual Gaussian description of the initial conditions and zero-mean white noise processes $\mathbf{w}_d(\cdot,\cdot)$ and $\mathbf{v}(\cdot,\cdot)$:

$$E\{\mathbf{x}(t_0)\} = \hat{\mathbf{x}}_0, \qquad E\{[\mathbf{x}(t_0) - \hat{\mathbf{x}}_0][\mathbf{x}(t_0) - \hat{\mathbf{x}}_0]^T\} = \mathbf{P}_0 \quad (5\text{-}3)$$

$$E\{\mathbf{w}_d(t_i)\mathbf{w}_d^T(t_j)\} = \begin{cases} \mathbf{Q}_d(t_i) & t_i = t_j \\ \mathbf{0} & t_i \neq t_j \end{cases} \quad (5\text{-}50)$$

$$E\{\mathbf{v}(t_i)\mathbf{v}^T(t_j)\} = \begin{cases} \mathbf{R}(t_i) & t_i = t_j \\ \mathbf{0} & t_i \neq t_j \end{cases} \quad (5\text{-}5)$$

The Kalman filter for this problem formulation was previously investigated under the assumption that the dynamic driving noise $\mathbf{w}_d(\cdot,\cdot)$ and measurement noise $\mathbf{v}(\cdot,\cdot)$ were uncorrelated; the algorithm is specified by (5-38)–(5-42) and (5-51)–(5-52).

Now we want to consider an extension of these results, allowing correlation between the two noise processes (assumed jointly Gaussian), the need for which was motivated in Section 5.2. Let this correlation be described by

$$E\{\mathbf{w}_d(t_i)\mathbf{v}^T(t_j)\} = \begin{cases} \mathbf{C}(t_i) & t_i = t_j \\ \mathbf{0} & t_i \neq t_j \end{cases} \quad (5\text{-}111)$$

A generalized derivation [36] yields the optimal estimation algorithm with the same initial conditions and measurement update relations, i.e., leaving (5-38)–(5-42) unaltered, but with time propagation equations modified from (5-51) and (5-52) to (note the time index has been changed for convenience):

$$\begin{aligned}\hat{\mathbf{x}}(t_{i+1}^-) = {} & \mathbf{\Phi}(t_{i+1}, t_i)\hat{\mathbf{x}}(t_i^+) + \mathbf{B}_d(t_i)\mathbf{u}(t_i) \\ & + \mathbf{G}_d(t_i)\mathbf{C}(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{R}(t_i)]^{-1}[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)] \end{aligned} \quad (5\text{-}112)$$

$$\begin{aligned}\mathbf{P}(t_{i+1}^-) = {} & \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{P}(t_i^+)\mathbf{\Phi}^T(t_{i+1}, t_i) + \mathbf{G}_d(t_i)\mathbf{Q}_d(t_i)\mathbf{G}_d^T(t_i) \\ & - \mathbf{G}_d(t_i)\mathbf{C}(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{R}(t_i)]^{-1}\mathbf{C}^T(t_i)\mathbf{G}_d^T(t_i) \\ & - \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{K}(t_i)\mathbf{C}^T(t_i)\mathbf{G}_d^T(t_i) - \mathbf{G}_d(t_i)\mathbf{C}(t_i)\mathbf{K}^T(t_i)\mathbf{\Phi}^T(t_{i+1}, t_i) \end{aligned} \quad (5\text{-}113)$$

Note that if t_1 is assumed to be the first measurement time, (5-51) and (5-52) are used for the first sample period.

EXAMPLE 5.9 Return to the gyro on test example, but now let

$$E\{w_d(t_i)v(t_i)\} = C = 0.2$$

Then the time propagation relations become (note $G_d = 1$)

$$\hat{x}(t_{i+1}^-) = 0.78\hat{x}(t_i^+) + 0.2[P(t_i^-) + 0.5]^{-1}[z_i - \hat{x}(t_i^-)]$$
$$P(t_{i+1}^-) = (0.78)^2 P(t_i^+) + 0.39 - (0.2)^2[P(t_i^-) + 0.5]^{-1} - 2[0.78]K(t_i)[0.2]$$

or, since $K(t_i) = P(t_i^+)H(t_i)R(t_i)^{-1} = 2P(t_i^+)$,

$$P(t_{i+1}^-) = 0.78(0.78 - 0.80)P(t_i^+) + 0.39 - 0.04[P(t_i^-) + 0.5]^{-1}$$
$$= -0.016P(t_i^+) + 0.39 - 0.04[P(t_i^-) + 0.5]^{-1}$$

Calculating the error variance and gain time history yields

Time	0	0.25	0.50	0.75	1.00
$P(t_i^-)$	-	1.00	0.35	0.34	0.34
$P(t_i^+)$	1	0.33	0.21	0.20	0.20
$K(t_i)$	-	0.67	0.42	0.40	0.40

This can be compared to Fig. 5.6 for the case of no correlation between $w_d(\cdot,\cdot)$ and $v(\cdot,\cdot)$. The decrease in steady state values from $P(t_i^-) = 0.55$ to 0.34 and $P(t_i^+)$ from 0.26 to 0.20 is due to the exploitation of the correlation between the dynamic noise and the noise that corrupts the observable outputs: the z_i realizations reveal more about the noise process $w_d(\cdot,\cdot)$. ■

If (5-49) is an equivalent discrete-time system model, the previous problem formulation corresponds to correlation between the noise corrupting the measurement at time t_i and the dynamic driving noise over the ensuing sample period. It is also useful to consider a model involving correlation between the dynamic noise over a sample period and the noise corrupting the measurement at the end of that interval. Thus, consider the same problem formulation, but replacing (5-11) with

$$E\{\mathbf{w}_d(t_{i-1})\mathbf{v}^T(t_j)\} = \begin{cases} \mathbf{C}(t_j) & t_j = t_i \\ \mathbf{0} & t_j \neq t_i \end{cases} \tag{5-114}$$

For this formulation, the generalized filter algorithm entails the original initial conditions and time propagation equations, i.e., (5-41), (5-42), (5-51), and (5-52). However, the measurement update relations (5-38)–(5-40) are replaced by (note the similarity to the result of Problem 3.19)

$$\mathbf{K}_c(t_i) = [\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{G}_d(t_{i-1})\mathbf{C}(t_i)][\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{R}(t_i) + \mathbf{H}(t_i)\mathbf{G}_d(t_{i-1})\mathbf{C}(t_i) + \mathbf{C}^T(t_i)\mathbf{G}_d{}^T(t_{i-1})\mathbf{H}^T(t_i)]^{-1} \tag{5-115}$$

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}_c(t_i)[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)] \tag{5-116}$$

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{K}_c(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-) + \mathbf{C}^T(t_i)\mathbf{G}_d{}^T(t_{i-1})] \tag{5-117}$$

EXAMPLE 5.10 Again consider the gyro on test example with correlation between $w_d(\cdot,\cdot)$ and $v(\cdot,\cdot)$, but now in the form of

$$E\{w_d(t_{i-1})v(t_i)\} = C = 0.2$$

Then the measurement update equations become (with $G_d = 1$):

$$K_c(t_i) = [P(t_i^-) + 0.2][P(t_i^-) + 0.5 + 0.2 + 0.2]^{-1} = [P(t_i^-) + 0.2]/[P(t_i^-) + 0.9]$$
$$\hat{x}(t_i^+) = \hat{x}(t_i^-) + K_c(t_i)[z_i - x(t_i^-)]$$
$$P(t_i^+) = P(t_i^-) - K_c(t_i)[P(t_i^-) + 0.2]$$

The error variance and gain time histories can be computed as

Time	0	0.25	0.50	0.75	1.00
$P(t_i^-)$	-	1.00	0.54	0.49	0.48
$P(t_i^+)$	1	0.24	0.16	0.15	0.15
$K_c(t_i)$	-	0.63	0.51	0.50	0.49

Here there is no correlation between the $v(t_i)$ corrupting the current measurement and $w_d(t_i)$ driving the dynamics during the next sample period, so there is a greater spreading of errors over the next interval than in Example 5.9: in steady state, $[P(t_{i+1}^-) - P(t_i^+)]$ here is $(0.48 - 0.15) = 0.33$ versus $(0.34 - 0.20) = 0.14$. With less confidence in the dynamics model here, the filter gain is higher (0.49 versus 0.40 in steady state) to weight the measurement data more heavily. By being correlated with $w_d(t_{i-1})$, $v(t_i)$ is also correlated with $x(t_i)$, so the measurement $z(t_i)$ is more strongly correlated with $x(t_i)$, and thus the lower $P(t_i^+)$ value here (0.15 versus 0.20). ■

These examples reveal the improved estimation precision due to exploiting the noise correlation: observing a particular realization of $\mathbf{v}(\cdot,\cdot)$ as the corruption on the measurements yields probabilistic information about the realizations of $\mathbf{w}_d(\cdot,\cdot)$ that have driven the system dynamics. As expected, either of these formulations reduce to the original Kalman filter algorithm if $\mathbf{C}(t_i) \equiv \mathbf{0}$ for all time. Moreover, if $\mathbf{v}(t_i)$ is correlated with both $\mathbf{w}_d(t_{i-1})$ and $\mathbf{w}_d(t_i)$, as

$$E\left\{\begin{bmatrix}\mathbf{w}_d(t_{i-1})\\ \mathbf{w}_d(t_i)\\ \mathbf{v}(t_i)\end{bmatrix}[\mathbf{w}_d^T(t_{i-1})\mathbf{w}_d^T(t_i)\mathbf{v}^T(t_i)]\right\} = \begin{bmatrix}\mathbf{Q}_d(t_{i-1}) & \mathbf{0} & \mathbf{C}_1(t_i)\\ \mathbf{0} & \mathbf{Q}_d(t_i) & \mathbf{C}_2(t_i)\\ \mathbf{C}_1^T(t_i) & \mathbf{C}_2^T(t_i) & \mathbf{R}(t_i)\end{bmatrix}$$

the above results can be combined. In practical applications, a tradeoff analysis would be conducted to determine whether the performance improvement afforded by this algorithm warrants the increased computer burden beyond that of the more conventional Kalman filter formulation.

5.10 TIME-CORRELATED MEASUREMENT NOISE; PERFECT MEASUREMENTS

In the optimal estimator derivation of Section 5.3, the covariance of the measurement corruption noise, $\mathbf{R}(t_i)$, was assumed positive definite because $\mathbf{R}^{-1}(t_i)$ was required in certain steps of that derivation. However, $\mathbf{R}^{-1}(t_i)$ does not ap-

pear in the final recursions, so positive definiteness may not in fact be necessary. In fact, the algorithm will operate as long as $[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{R}(t_i)]$ is invertible. The particular case of $\mathbf{R}(t_i) \equiv \mathbf{0}$ for all time is of interest because of its applicability to problems in which measurements are corrupted only by time-correlated noise as well as to formulations involving "perfect" measurements. For these cases in which $\mathbf{R}(t_i)$ is singular, stochastic observability may well be violated, but recall that this is part of a sufficient rather than necessary condition for filter stability. Numerical problems will be accentuated, however (see Chapter 7).

First let us demonstrate that time-correlated measurement noise does lead to a filter formulation in which $\mathbf{R}(t_i)$ is zero for all time. Let the system model be described (in white noise notation) as in Fig. 5.12. The system dynamics model is

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t) \tag{5-118}$$

with $\mathbf{w}(\cdot,\cdot)$ a zero-mean white Gaussian noise of strength $\mathbf{Q}(t)$ for all $t \in T$. The available discrete-time measurements are modeled as

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{n}_f(t_i) \tag{5-119}$$

where $\mathbf{n}_f(\cdot,\cdot)$ is a zero-mean time-correlated Gaussian process: the output of a time-correlated noise generator, i.e., a shaping filter. This shaping filter is described by (the subscript f denotes filter)

$$\dot{\mathbf{x}}_f(t) = \mathbf{F}_f(t)\mathbf{x}_f(t) + \mathbf{G}_f(t)\mathbf{w}_f(t) \tag{5-120}$$

$$\mathbf{n}_f(t) = \mathbf{H}_f(t)\mathbf{x}_f(t) \tag{5-121}$$

where $\mathbf{w}_f(\cdot,\cdot)$ is a zero-mean white Gaussian noise of strength $\mathbf{Q}_f(t)$ for all $t \in T$.

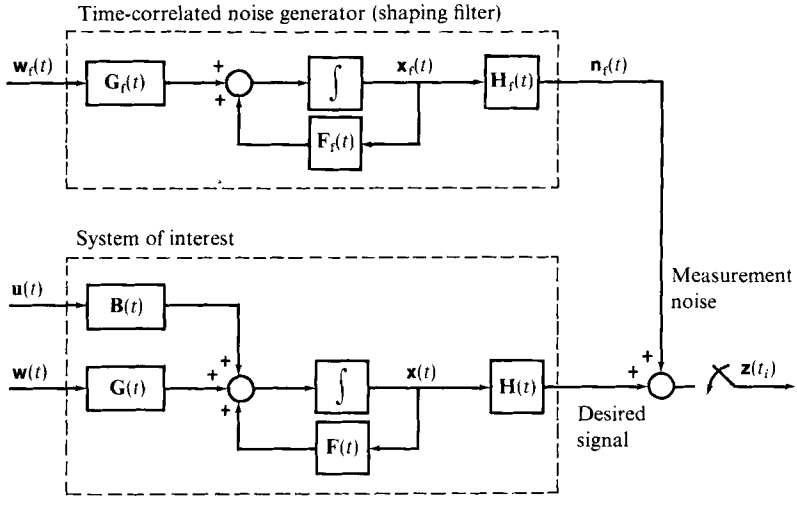

FIG. 5.12 Time-correlated measurement noise model.

The overall system model can be expressed in terms of the augmented state $\mathbf{x}_a(\cdot,\cdot)$ as described in Section 4.11,

$$\mathbf{x}_a(\cdot,\cdot) = \left[\begin{array}{c} \mathbf{x}(\cdot,\cdot) \\ \hline \mathbf{x}_f(\cdot,\cdot) \end{array}\right] \tag{5-122}$$

as

$$\left[\begin{array}{c} \dot{\mathbf{x}}(t) \\ \hline \dot{\mathbf{x}}_f(t) \end{array}\right] = \left[\begin{array}{c|c} \mathbf{F}(t) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{F}_f(t) \end{array}\right]\left[\begin{array}{c} \mathbf{x}(t) \\ \hline \mathbf{x}_f(t) \end{array}\right] + \left[\begin{array}{c} \mathbf{B}(t) \\ \hline \mathbf{0} \end{array}\right]\mathbf{u}(t) + \left[\begin{array}{c|c} \mathbf{G}(t) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{G}_f(t) \end{array}\right]\left[\begin{array}{c} \mathbf{w}(t) \\ \hline \mathbf{w}_f(t) \end{array}\right] \tag{5-123}$$

$$\mathbf{Q}_a(t) = \left[\begin{array}{c|c} \mathbf{Q}(t) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{Q}_f(t) \end{array}\right] \tag{5-124}$$

$$\mathbf{z}(t_i) = \left[\mathbf{H}(t_i) \;\vdots\; \mathbf{H}_f(t_i)\right]\left[\begin{array}{c} \mathbf{x}(t_i) \\ \hline \mathbf{x}_f(t_i) \end{array}\right] \tag{5-125}$$

This is in the form of a linear system driven only by white Gaussian noise $\mathbf{w}_a(\cdot,\cdot)$, but from which is available perfect measurements of certain linear combinations of states at discrete times.

Now let us consider the problem involving perfect measurements in general. The optimal estimator will be generated by a limiting process on previous results, and the difference in the characteristics of the resulting algorithm will be described. Let the available measurements be modeled as

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i)$$

with the zero-mean white Gaussian noise $\mathbf{v}(\cdot,\cdot)$ being of strength $\mathbf{R}(t_i) = \varepsilon\mathbf{I}$, $\varepsilon > 0$ (so that $\mathbf{R}(t_i)$ is positive definite). The filter update equations are

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{P}(t_i^-)\mathbf{H}^T(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \varepsilon\mathbf{I}]^{-1}[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]$$
$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{P}(t_i^-)\mathbf{H}^T(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \varepsilon\mathbf{I}]^{-1}\mathbf{H}(t_i)\mathbf{P}(t_i^-)$$

Now observe the result of letting $\varepsilon \to 0$, i.e., the limit of perfect measurements. Although ε must be nonzero to be assured of the existence of density functions used in the original derivation, characteristic functions can be used to maintain validity of this operation. As $\varepsilon \to 0$, $\mathbf{P}(t_i^+)$ will become singular, unlike the previous characterization of the algorithm. The result of the limiting process is

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{P}(t_i^-)\mathbf{H}^T(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i)]^{-1}[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)] \tag{5-126}$$

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{P}(t_i^-)\mathbf{H}^T(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i)]^{-1}\mathbf{H}(t_i)\mathbf{P}(t_i^-) \tag{5-127}$$

Let us investigate the invertibility of $[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i)]$ and the singularity of $\mathbf{P}(t_i^+)$ further. Let $\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i)$ be written out as

$$\begin{bmatrix} z_1(t_i) \\ \vdots \\ z_m(t_i) \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1^T(t_i) \\ \vdots \\ \mathbf{h}_m^T(t_i) \end{bmatrix}\begin{bmatrix} x_1(t_i) \\ \vdots \\ x_n(t_i) \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1^T(t_i)\mathbf{x}(t_i) \\ \vdots \\ \mathbf{h}_m^T(t_i)\mathbf{x}(t_i) \end{bmatrix}$$

Thus, if $\mathbf{H}(t_i)$ is of full rank [i.e., if $\mathbf{h}_1(t_i), \ldots, \mathbf{h}_m(t_i)$ are a linearly independent set of m vectors], then there are m directions in state space along which we have perfect measurements (if it is not of full rank, then there are rank $[\mathbf{H}(t_i)]$ such directions). With respect to some coordinate frame in state space, then, we can get perfect measurements of m (rank $[\mathbf{H}(t_i)]$) out of the n states. Figure 5.13 depicts the case of a two-dimensional measurement and a three-dimensional state: by knowing z_1 and z_2, we know the state in two of three directions perfectly, i.e., in a two-dimensional subspace of R^3.

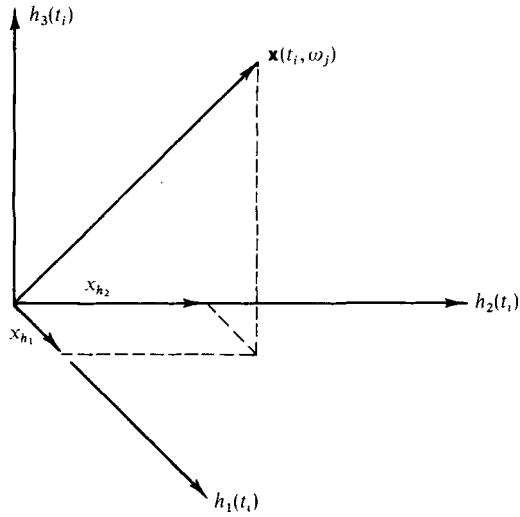

FIG. 5.13 Perfect two-dimensional measurement in three-dimensional state space. $x_{h_1} = (1/|\mathbf{h}_1(t_i)|)z_1$; $x_{h_2} = (1/|\mathbf{h}_2(t_i)|)z_2$.

If $\mathbf{P}(t_i^-)$ is positive definite (of rank n) and $\mathbf{H}(t_i)$ is of full rank m, then $[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i)]$ is a positive definite m-by-m matrix, and so has an inverse. $\mathbf{P}(t_i^+)$ will then be singular, of rank $(n - m)$; m eigenvalues are zero. This singularity is readily seen by premultiplying (5-127) by $\mathbf{H}(t_i)$ to obtain

$$\begin{aligned}\mathbf{H}(t_i)\mathbf{P}(t_i^+) &= \mathbf{H}(t_i)\mathbf{P}(t_i^-) - \mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i)]^{-1}\mathbf{H}(t_i)\mathbf{P}(t_i^-) \\ &= \mathbf{H}(t_i)\mathbf{P}(t_i^-) - \mathbf{H}(t_i)\mathbf{P}(t_i^-) = \mathbf{0} \qquad (5\text{-}128)\end{aligned}$$

Heuristically, when you take a perfect m-vector measurement, the error probability density collapses in the directions along which you can determine the values of the state components exactly.

Thus, we want $\mathbf{P}(t_i^-)$ to be positive definite (of rank n), whereas $\mathbf{P}(t_{i-1}^+)$ is singular (positive semidefinite, of rank $n - m$), with

$$\mathbf{P}(t_i^-) = \mathbf{\Phi}(t_i, t_{i-1})\mathbf{P}(t_{i-1}^+)\mathbf{\Phi}^T(t_i, t_{i-1}) + \int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^T(\tau)\mathbf{\Phi}^T(t_i, \tau)\, d\tau$$

One *sufficient* condition for this to be true would be for the integral term itself to be of rank n: for the system representation to be stochastically controllable over each sample period.

EXAMPLE 5.11 Consider the gyro example again, but now let the measurement be corrupted by a bias only, with no additional white noise corruption, as depicted in Fig. 5.14. The bias is modeled as the output of an undriven integrator: the shaping filter is simply $\dot{x}_f(t) = 0$. In augmented state vector form, this becomes

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_f(t) \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix} + \begin{bmatrix} \alpha \\ 0 \end{bmatrix} w(t)$$

$$z(t_i) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x(t_i) \\ x_f(t_i) \end{bmatrix}$$

The initial conditions are assumed to be

$$E\{x(t_0)\} = E\{x_f(t_0)\} = 0$$
$$E\{x^2(t_0)\} = E\{x_f^2(t_0)\} = 1 \qquad E\{x(t_0)x_f(t_0)\} = 0$$

so the appropriate estimator initial conditions are

$$\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{P}(t_0) = \mathbf{P}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the given augmented $\mathbf{F}_a$, the state transition matrix is

$$\mathbf{\Phi}_a(t_i, t_{i-1}) = \begin{bmatrix} e^{-\alpha(t_i - t_{i-1})} & 0 \\ 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 0.78 & 0 \\ 0 & 1 \end{bmatrix}$$

The time propagation relations are, by (5-36) and (5-37),

$$\begin{bmatrix} \hat{x}(t_i^-) \\ \hat{x}_f(t_i^-) \end{bmatrix} = \begin{bmatrix} 0.78 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}(t_{i-1}^+) \\ \hat{x}_f(t_{i-1}^+) \end{bmatrix} = \begin{bmatrix} 0.78\hat{x}(t_{i-1}^+) \\ \hat{x}_f(t_{i-1}^+) \end{bmatrix}$$

$$\mathbf{P}(t_i^-) = \mathbf{\Phi}(t_i, t_{i-1})\mathbf{P}(t_{i-1}^+)\mathbf{\Phi}^T(t_i, t_{i-1}) + \int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i, \tau)\mathbf{G}\mathbf{Q}\mathbf{G}^T\mathbf{\Phi}^T(t_i, \tau)\, d\tau$$

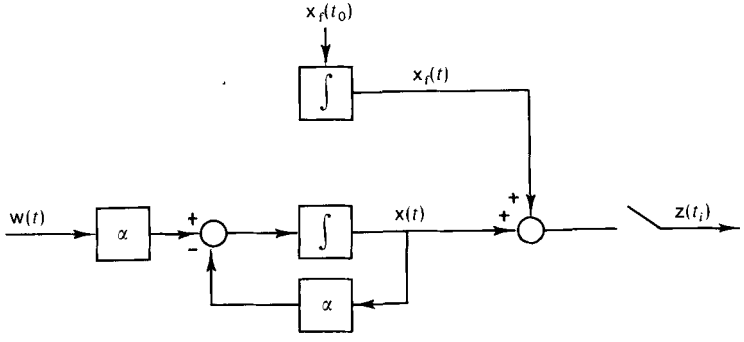

FIG. 5.14 Gyro corrupted by a bias.

The measurement update relations are given by (5-126) and (5-127) as

$$\begin{bmatrix} K_1(t_i) \\ K_2(t_i) \end{bmatrix} = \begin{bmatrix} [P_{11}(t_i^-) + P_{12}(t_i^-)]/[P_{11}(t_i^-) + 2P_{12}(t_i^-) + P_{22}(t_i^-)] \\ [P_{12}(t_i^-) + P_{22}(t_i^-)]/[P_{11}(t_i^-) + 2P_{12}(t_i^-) + P_{22}(t_i^-)] \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}(t_i^+) \\ \hat{x}_f(t_i^+) \end{bmatrix} = \begin{bmatrix} \hat{x}(t_i^-) \\ \hat{x}_f(t_i^-) \end{bmatrix} + \begin{bmatrix} K_1(t_i) \\ K_2(t_i) \end{bmatrix} [z_i - \hat{x}(t_i^-) - \hat{x}_f(t_i^-)]$$

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{K}(t_i)\mathbf{H}\mathbf{P}(t_i^-)$$

By direct addition of the above expressions, it can be seen that

$$\hat{x}(t_i^+) + \hat{x}_f(t_i^+) = z_i$$

Such a phenomenon invariably occurs due to the perfect measurement of $[\mathbf{x}(t_i) + \mathbf{x}_f(t_i)]$. As shown in Fig. 5.15, when a measurement $\mathbf{z}(t_i, \omega_j) = z_i$ becomes available, we know the value of $[\mathbf{x}(t_i, \omega_j) + \mathbf{x}_f(t_i, \omega_j)]$ exactly: the probability density describing the possible values of $\mathbf{x}(t_i)$ and $\mathbf{x}_f(t_i)$ has collapsed down to being nonzero only above the line $[\mathbf{x}(t_i, \omega_j) + \mathbf{x}_f(t_i, \omega_j)] = z_i$. (As time goes on between measurements, the probability density spreads out again.) Thus, $\mathbf{P}(t_i^+)$ is singular, of rank one. If we were to rotate coordinates (by similarity transformation) to the $\xi_1^* - \xi_2^*$ coordinates in the figure, there would be no uncertainty in the ξ_1^* direction just after a measurement, and so

$$\mathbf{P}^*(t_i^+) = \begin{bmatrix} P_{11}^*(t_i^+) & P_{12}^*(t_i^+) \\ P_{12}^*(t_i^+) & P_{22}^*(t_i^+) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & P_{22}^*(t_i^+) \end{bmatrix} \quad \blacksquare$$

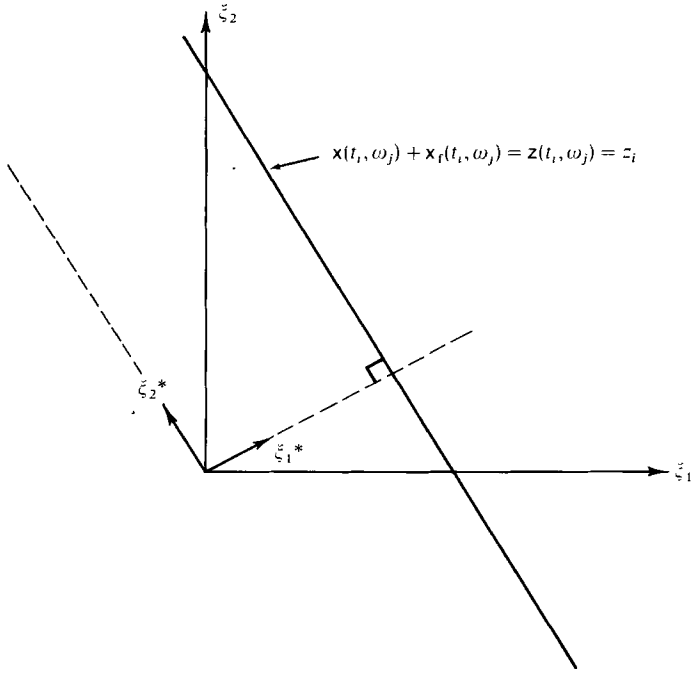

FIG. 5.15 Perfect measurement for Example 5.11.

As suggested in the example, it is convenient to define a coordinate transformation such that the perfect measurements are in fact direct measurements of individual state variables. Actually, a more general transformation than a simple rotation can be exploited to advantage. Assume $\mathbf{H}(t_i)$ to be full rank: this is not restrictive, since if it were not, we would have redundant perfect information. An n-by-n nonsingular transformation matrix $\mathbf{T}(t_i)$ can be defined for each $t_i \in T$ as

$$\mathbf{T}(t_i) = \begin{bmatrix} \mathbf{H}(t_i) \\ \hline \mathbf{J}(t_i) \end{bmatrix} \tag{5-129}$$

where $\mathbf{H}(t_i)$ is the measurement matrix and $\mathbf{J}(t_i)$ is any convenient $(n - m)$-by-n matrix that yields a nonsingular (and thus invertible) $\mathbf{T}(t_i)$. Since this procedure is computationally attractive for the time-invariant $\mathbf{H}$ case, we confine our attention to

$$\mathbf{T}(t_i) = \mathbf{T} = \begin{bmatrix} \mathbf{H} \\ \hline \mathbf{J} \end{bmatrix} \tag{5-129'}$$

Through this transformation, the problem can be expressed in terms of $\mathbf{x}^*(t)$, where

$$\mathbf{x}^*(t) = \mathbf{T}\mathbf{x}(t), \qquad \mathbf{x}(t) = \mathbf{T}^{-1}\mathbf{x}^*(t) \tag{5-130}$$

as the equivalent

$$\dot{\mathbf{x}}^*(t) = \mathbf{F}^*(t)\mathbf{x}^*(t) + \mathbf{B}^*(t)\mathbf{u}(t) + \mathbf{G}^*(t)\mathbf{w}(t), \qquad \mathbf{z}(t) = \mathbf{H}^*\mathbf{x}^*(t) \tag{5-131}$$

with $\mathbf{F}^*(t) = \mathbf{T}\mathbf{F}(t)\mathbf{T}^{-1}$, $\mathbf{B}^*(t) \doteq \mathbf{T}\mathbf{B}(t)$, and $\mathbf{G}^*(t) = \mathbf{T}\mathbf{G}(t)$. In view of (5-130),

$$\mathbf{x}^*(t_i) = \mathbf{T}\mathbf{x}(t_i) = \begin{bmatrix} \mathbf{H} \\ \hline \mathbf{J} \end{bmatrix} \mathbf{x}(t_i) = \begin{bmatrix} \mathbf{z}(t_i) \\ \hline \mathbf{y}(t_i) \end{bmatrix} \tag{5-132}$$

In other words, the first m components of $\mathbf{x}^*(t_i)$ are just $\mathbf{z}(t_i)$, so $\mathbf{H}^* = [\mathbf{I} \mid \mathbf{0}]$, where $\mathbf{I}$ is m-by-m and the zero matrix is m-by-$(n - m)$. The remaining $(n - m)$ components of $\mathbf{x}^*(t_i)$ are identified as $\mathbf{y}(t_i)$: by choice of $\mathbf{J}$, this vector can often be a vector of variables to be estimated.

The optimal estimate of $\mathbf{x}^*(t_i)$ is obtained by operating on the measurement vector $\mathbf{z}_i$. The advantage in using the transformed variables is that the first m rows and columns of $\mathbf{P}(t_i^+)$ are identically zero and need not be computed. If desired, the optimal estimate of the original state vector $\mathbf{x}(t_i)$ can then be computed as $\hat{\mathbf{x}}(t_i^+) = \mathbf{T}^{-1}\hat{\mathbf{x}}^*(t_i^+)$.

EXAMPLE 5.12 In Example 5.11, the suggested coordinate rotation can be accomplished by

$$\mathbf{T} = \begin{bmatrix} 1 & 1 \\ \hline -1 & 1 \end{bmatrix}$$

But, since we are interested specifically in obtaining an estimate of the first component of $\mathbf{x}_a(t_i)$, a more convenient choice of $\mathbf{J}$ would be $[1\ 0]$, yielding

$$\mathbf{T} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus we get

$$\begin{bmatrix} x_1^*(t_i) \\ x_2^*(t_i) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t_i) \\ x_f(t_i) \end{bmatrix} = \begin{bmatrix} x(t_i) + x_f(t_i) \\ x(t_i) \end{bmatrix} = \begin{bmatrix} z(t_i) \\ x(t_i) \end{bmatrix}$$

Thus, the $\mathbf{y}(t_i)$ of (5-132) is in fact $x(t_i)$, the variable to be estimated. The defining matrices are

$$\mathbf{F}^* = \mathbf{TFT}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\alpha & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ 0 & -\alpha \end{bmatrix}$$

$$\mathbf{G}^* = \mathbf{TG} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$$

$$\mathbf{H}^* = [1\ \ 0]$$

The filter update relations are

$$\mathbf{K}^*(t_i) = \mathbf{P}^*(t_i^-)\mathbf{H}^{*\mathrm{T}}[\mathbf{H}^*\mathbf{P}^*(t_i^-)\mathbf{H}^{*\mathrm{T}}]^{-1} = \begin{bmatrix} P_{11}^*(t_i^-)/P_{11}^*(t_i^-) \\ P_{12}^*(t_i^-)/P_{11}^*(t_i^-) \end{bmatrix} = \begin{bmatrix} 1 \\ P_{12}^*(t_i^-)/P_{11}^*(t_i^-) \end{bmatrix}$$

$$\mathbf{P}^*(t_i^+) = \begin{bmatrix} 0 & 0 \\ 0 & P_{22}^*(t_i^-) - [P_{12}^*(t_i^-)^2/P_{11}^*(t_i^-)] \end{bmatrix}$$

In higher-dimensioned problems, there would be greater computational advantage to using transformed variables. ■

The construction of a state estimate given perfect measurements from a system that has no dynamic driving noise (but initial conditions are not known perfectly) can be developed by means of *Luenberger observers* [45, 46, 48]. Coordinate transformations similar to that of (5-129) are utilized in the design of observers, in which the state estimate is generated as $\hat{\mathbf{x}}(t_i^+) = \mathbf{T}^{-1}\hat{\mathbf{x}}^*(t_i)$, where $\hat{\mathbf{x}}^*(t_i)$ is given by (5-132), with $\mathbf{z}(t_i)$ provided by the measuring devices and $\mathbf{y}(t_i)$ the output of an $(n - m)$-dimensional linear system (the "minimal order observer"). The inherent freedom of choice of $\mathbf{J}$ is exploited to achieve desirable dynamic performance of the algorithm. *Observer-estimators* [78, 79] can be developed for the problem of state reconstruction in which noises drive *some* states and corrupt *some* measurements. This is of practical significance in cases involving large differences in precision of sensors so that, with respect to the finite wordlength of the digital computer being used, some measurements "look perfect" compared to others. Observer theory has also been extended to the stochastic case [42, 80, 81]. For details, see the cited references.

The problem of time-correlated measurements can be handled by an alternate approach [10] that avoids state augmentation (and thus increased state dimension). Instead, consecutive measurements are differenced to generate pseudo-measurements in which the corrupting noise is white. Consider the

discrete-time system and shaping filter representations

$$\mathbf{x}(t_{i+1}) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{x}(t_i) + \mathbf{G}_{\mathrm{d}}(t_i)\mathbf{w}_{\mathrm{d}}(t_i) \tag{5-133a}$$

$$\mathbf{x}_{\mathrm{f}}(t_{i+1}) = \mathbf{\Phi}_{\mathrm{f}}(t_{i+1}, t_i)\mathbf{x}_{\mathrm{f}}(t_i) + \mathbf{G}_{\mathrm{df}}(t_i)\mathbf{w}_{\mathrm{df}}(t_i) \tag{5-133b}$$

with $\mathbf{w}_{\mathrm{d}}(\cdot,\cdot)$ and $\mathbf{w}_{\mathrm{df}}(\cdot,\cdot)$ zero-mean independent white Gaussian noises of strengths $\mathbf{Q}_{\mathrm{d}}(t_i)$ and $\mathbf{Q}_{\mathrm{df}}(t_i)$, respectively, for all $t_i \in T$, and assume that $\mathbf{x}(t_0)$ and $\mathbf{x}_{\mathrm{f}}(t_0)$ are uncorrelated. Further assume the measurement to be of the form

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{x}_{\mathrm{f}}(t_i) \tag{5-133c}$$

Note that this implies $\mathbf{x}_{\mathrm{f}}(t_i)$ is an m-vector process and that $\mathbf{H}_{\mathrm{f}}(t_i) \triangleq \mathbf{I}$ for all t_i: a definite restriction, except that the useful case of exponentially time-correlated noise on each scalar measurement fits this description. Now define the pseudo-measurement process $\mathbf{z}_{\mathrm{d}}(\cdot,\cdot)$, with d denoting difference, as

$$\mathbf{z}_{\mathrm{d}}(t_i) = \mathbf{z}(t_{i+1}) - \mathbf{\Phi}_{\mathrm{f}}(t_{i+1}, t_i)\mathbf{z}(t_i) \tag{5-134}$$

Although $\mathbf{z}_{\mathrm{d}}(t_i, \omega_j)$ will yield information about $\mathbf{x}(t_i)$, it does not become available until one sample period later, since it requires knowledge of $\mathbf{z}(t_{i+1}, \omega_j)$. Writing $\mathbf{z}_{\mathrm{d}}(t_i)$ in terms of (5-133) yields

$$\begin{aligned}
\mathbf{z}_{\mathrm{d}}(t_i) &= \mathbf{z}(t_{i+1}) - \mathbf{\Phi}_{\mathrm{f}}(t_{i+1}, t_i)\mathbf{z}(t_i) \\
&= [\mathbf{H}(t_{i+1})\mathbf{x}(t_{i+1}) + \mathbf{x}_{\mathrm{f}}(t_{i+1})] \\
&\quad - [\mathbf{\Phi}_{\mathrm{f}}(t_{i+1}, t_i)\mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{\Phi}_{\mathrm{f}}(t_{i+1}, t_i)\mathbf{x}_{\mathrm{f}}(t_i)] \\
&= \mathbf{H}(t_{i+1})\mathbf{\Phi}(t_{i+1}, t_i)\mathbf{x}(t_i) + \mathbf{H}(t_{i+1})\mathbf{G}_{\mathrm{d}}(t_i)\mathbf{w}_{\mathrm{d}}(t_i) \\
&\quad + \mathbf{\Phi}_{\mathrm{f}}(t_{i+1}, t_i)\mathbf{x}_{\mathrm{f}}(t_i) + \mathbf{G}_{\mathrm{df}}(t_i)\mathbf{w}_{\mathrm{df}}(t_i) \\
&\quad - \mathbf{\Phi}_{\mathrm{f}}(t_{i+1}, t_i)\mathbf{H}(t_i)\mathbf{x}(t_i) - \mathbf{\Phi}_{\mathrm{f}}(t_{i+1}, t_i)\mathbf{x}_{\mathrm{f}}(t_i) \\
&= [\mathbf{H}(t_{i+1})\mathbf{\Phi}(t_{i+1}, t_i) - \mathbf{\Phi}_{\mathrm{f}}(t_{i+1}, t_i)\mathbf{H}(t_i)]\mathbf{x}(t_i) \\
&\quad + [\mathbf{H}(t_{i+1})\mathbf{G}_{\mathrm{d}}(t_i)\mathbf{w}_{\mathrm{d}}(t_i) + \mathbf{G}_{\mathrm{df}}(t_i)\mathbf{w}_{\mathrm{df}}(t_i)]
\end{aligned} \tag{5-135}$$

This is in the form of a linear combination of the original system states, corrupted by a zero-mean white Gaussian noise of strength $\mathbf{R}_{\mathrm{d}}(t_i)$,

$$\begin{aligned}
\mathbf{R}_{\mathrm{d}}(t_i) &= \mathbf{H}(t_{i+1})\mathbf{G}_{\mathrm{d}}(t_i)\mathbf{Q}_{\mathrm{d}}(t_i)\mathbf{G}_{\mathrm{d}}{}^{\mathrm{T}}(t_i)\mathbf{H}^{\mathrm{T}}(t_{i+1}) \\
&\quad + \mathbf{G}_{\mathrm{df}}(t_i)\mathbf{Q}_{\mathrm{df}}(t_i)\mathbf{G}_{\mathrm{df}}^{\mathrm{T}}(t_i)
\end{aligned} \tag{5-136}$$

Since $\mathbf{z}(t_{i+1}, \omega_j)$ is required before $\mathbf{z}_{\mathrm{d}}(t_i)$ can be processed, this actually yields an optimal smoothing problem formulation rather than an optimal filtering problem, and the algorithm can be developed using the results of Chapter 8 (Volume 2). In application, the measurement $\mathbf{z}(t_{i+1}, \omega_j)$ is taken at time t_{i+1}, $\mathbf{z}_{\mathrm{d}}(t_i, \omega_j)$ is thereby computed, $\hat{\mathbf{x}}(t_i)$ is then calculated, and the estimate propagated to the current real time t_{i+1} as $\hat{\mathbf{x}}(t_{i+1}^-)$.

5.11 CONTINUOUS-TIME FILTER

As discussed previously, practical application of optimal estimation almost invariably involves implementation on a digital computer, which inherently dictates sampled-data format for measurements. Consequently, attention has been concentrated on this formulation. This section provides a formal derivation of the continuous-data Kalman filter [15, 37]; it can be made rigorous, and the concepts involved are of considerable theoretical significance, but the additional difficulty and effort is not warranted in view of our objective to attain efficient, practical algorithms. One might consider generating the continuous filter and then discretizing it for eventual implementation, but there is a serious drawback to this procedure, discussed subsequently.

Consider the same continuous-time dynamics model as used before

$$\mathbf{dx}(t) = \mathbf{F}(t)\mathbf{x}(t)\,dt + \mathbf{B}(t)\mathbf{u}(t)\,dt + \mathbf{G}(t)\,\mathbf{d\beta}(t) \tag{5-137}$$

or, in white noise notation

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t) \tag{5-137'}$$

where $\boldsymbol{\beta}(\cdot,\cdot)$ is Brownian motion of diffusion $\mathbf{Q}(t)$ for all $t \in T$, or $\mathbf{w}(\cdot,\cdot)$ is zero-mean white Gaussian noise of strength $\mathbf{Q}(t)$ for all $t \in T$. Let $\mathbf{x}(t_0)$ be modeled as a Gaussian random variable with mean $\hat{\mathbf{x}}_0$ and covariance $\mathbf{P}_0$.

We want to consider continuously available measurements, modeled by the process $\mathbf{z}_c(\cdot,\cdot)$ defined by

$$\mathbf{z}_c(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}_c(t) \tag{5-138}$$

where $\mathbf{v}_c(\cdot,\cdot)$ is a zero-mean white Gaussian noise with

$$E\{\mathbf{v}_c(t)\mathbf{v}_c^{\mathrm{T}}(t+\tau)\} = \mathbf{R}_c(t)\,\delta(\tau) \tag{5-139}$$

The subscript c is meant to distinguish these continuous time processes from the analogous discrete-time processes considered previously.

To derive the desired result, we will consider a discrete-time measurement process and examine the result of letting the time between sample times decrease in the limit to zero. Thus, the measurements are described by

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \tag{5-140}$$

where $\mathbf{v}(\cdot,\cdot)$ is a zero-mean white Gaussian sequence with

$$E\{\mathbf{v}(t_i)\mathbf{v}^{\mathrm{T}}(t_i)\} = \mathbf{R}(t_i) = \mathbf{R}_c(t_i)/\Delta t_i \tag{5-141a}$$

$$E\{\mathbf{v}(t_i)\mathbf{v}^{\mathrm{T}}(t_j)\} = \mathbf{0}, \qquad i \neq j \tag{5-141b}$$

where $\mathbf{R}(t_i)$ and $\mathbf{R}_c(t_i)$ are positive definite and symmetric for all $t_i \in T$ and Δt_i is the time interval $[t_{i+1} - t_i]$. Without any real loss of generality, we let all Δt_i's be the same, Δt, in the derivation. The covariance $\mathbf{R}_c(t_i)$ in (5-141a) will eventually become the strength of the continuous-time white Gaussian noise

process $\mathbf{v}_c(\cdot,\cdot)$ that corrupts the continuous-time measurements. This description results in an autocorrelation function as depicted in Fig. 5.16, defined for discrete values of τ. Note that as $\Delta t \to 0$, a Dirac delta function (an infinite impulse at $\tau = 0$) is achieved. We further assume that $\mathbf{x}(t_0)$, $\boldsymbol{\beta}(\cdot,\cdot)$ or $\mathbf{w}(\cdot,\cdot)$, and $\mathbf{v}(\cdot,\cdot)$ are independent, and that $\mathbf{F}(\cdot)$, $\mathbf{B}(\cdot)$, $\mathbf{G}(\cdot)$, $\mathbf{H}(\cdot)$, $\mathbf{Q}(\cdot)$, and $\mathbf{R}_c(\cdot)$ are at least piecewise continuous.

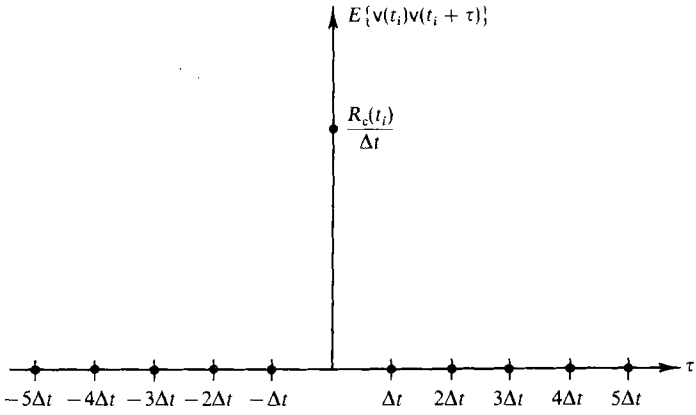

FIG. 5.16 Autocorrelation function for discrete-time $\mathbf{v}(\cdot,\cdot)$.

The discrete-time Kalman filter for this problem formulation is given by Eqs. (5-36)–(5-42) with $\mathbf{R}(t_i)$ repeated by $\mathbf{R}_c(t_i)/\Delta t$. In the time propagation relations, the state transition matrix can be expanded as

$$\boldsymbol{\Phi}(t_i, t_{i-1}) = \mathbf{I} + \mathbf{F}(t_{i-1})\Delta t + \mathcal{O}(\Delta t^2) \tag{5-142}$$

where $\mathcal{O}(\Delta t^2)$ is composed of terms involving powers of Δt greater than or equal to two, such that

$$\lim_{\Delta t \to 0} \frac{\mathcal{O}(\Delta t^2)}{\Delta t} = \mathbf{0}$$

First consider the state estimate equations: substituting (5-142) into the $\hat{\mathbf{x}}(t_i^-)$ equation, (5-36), yields

$$\begin{aligned}\hat{\mathbf{x}}(t_i^-) &= [\mathbf{I} + \mathbf{F}(t_{i-1})\Delta t]\hat{\mathbf{x}}(t_{i-1}^+) + \int_{t_{i-1}}^{t_i} [\mathbf{I} + \mathbf{F}(\tau)\{t_i - \tau\}]\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau + \mathcal{O}(\Delta t^2) \\ &= \hat{\mathbf{x}}(t_{i-1}^+) + \mathbf{F}(t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+)\Delta t + \mathbf{B}(\sigma)\mathbf{u}(\sigma)\Delta t + \mathcal{O}(\Delta t^2) \end{aligned} \tag{5-143}$$

where σ is somewhere in the interval $[t_{i-1}, t_i)$, by the mean value theorem. Substituting (5-143) into the $\hat{\mathbf{x}}(t_i^+)$ equation, (5-39), yields

$$\begin{aligned}\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_{i-1}^+) &+ \mathbf{F}(t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+)\Delta t + \mathbf{B}(\sigma)\mathbf{u}(\sigma)\Delta t + \mathcal{O}(\Delta t^2) \\ &+ \mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i)\Delta t + \mathbf{R}_c(t_i)]^{-1}\Delta t[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]\end{aligned}$$

Now when $\hat{\mathbf{x}}(t_{i-1}^{+})$ is brought to the left hand side of this equation, the entire result divided by Δt, and the limit taken as $\Delta t \to 0$, we get

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{P}(t)\mathbf{H}^{\mathrm{T}}(t)\mathbf{R}_{\mathrm{c}}^{-1}(t)[\mathbf{z}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)] \quad (5\text{-}144)$$

where, in the limit, $\hat{\mathbf{x}}(t_{i-1}^{+}) \to \hat{\mathbf{x}}(t_i^{-}) \to \hat{\mathbf{x}}(t_i^{+}) \triangleq \hat{\mathbf{x}}(t)$ and $\mathbf{P}(t_{i-1}^{+}) \to \mathbf{P}(t_i^{-}) \to \mathbf{P}(t_i^{+}) \triangleq \mathbf{P}(t)$. Note that $\mathbf{z}(\cdot)$ is a sample from the continuous-time measurement process $\mathbf{z}_{\mathrm{c}}(\cdot,\cdot)$. Performing similar operations on the covariance equations yields

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^{\mathrm{T}}(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t) - \mathbf{P}(t)\mathbf{H}^{\mathrm{T}}(t)\mathbf{R}_{\mathrm{c}}^{-1}(t)\mathbf{H}(t)\mathbf{P}(t) \quad (5\text{-}145)$$

This has not been very rigorous, but existence of the desired limits can be proven by means of probability one arguments and the concept of martingales, to prove, among other things, that

$$\lim_{k\to\infty} E\{\mathbf{x}(t) \mid \mathbf{z}(t_1) = \mathbf{z}_1, \ldots, \mathbf{z}(t_k) = z_k; t_1, \ldots, t_k \le t\}$$
$$= E\{\mathbf{x}(t) \mid \mathbf{z}_{\mathrm{c}}(\tau) = \mathbf{z}(\tau); t_0 \le \tau \le t\}$$

We will gloss over these aspects here. Thus, the continuous-time Kalman filter is specified by the differential equations (5-144) and (5-145), which are integrated forward from the initial conditions

$$\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0 \quad (5\text{-}146a)$$

$$\mathbf{P}(t_0) = \mathbf{P}_0 \quad (5\text{-}146b)$$

Figure 5.17 portrays the basic system model and the continuous-time Kalman filter based upon this model. From this figure it is evident that within the structure of the filter is a mathematical model of the real system that provides the measurement data input to the filter, just as in the discrete-time measurement case. The filter incorporates such a model, driven by an optimal gain times the difference between the actual measurements received, $\mathbf{z}(t)$, and the optimal estimates of what these should be based on the mathematical model output, $\mathbf{H}(t)\hat{\mathbf{x}}(t)$, the residual $[\mathbf{z}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)]$.

In the continuous-time case, the Kalman filter gain is seen to be

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^{\mathrm{T}}(t)\mathbf{R}_{\mathrm{c}}^{-1}(t) \quad (5\text{-}147)$$

If $\mathbf{P}(t)$ is "large" (having large eigenvalues), then the residual is heavily weighted: if we are very uncertain of the current estimate $\hat{\mathbf{x}}(t)$, then the new information from the measurements is emphasized. Similarly, if $\mathbf{R}_{\mathrm{c}}(t)$ is "small," i.e., if the measurements are very accurate, then the measurement information is weighted heavily. In fact, $\mathbf{P}(t)\mathbf{H}^{\mathrm{T}}(t)\mathbf{R}_{\mathrm{c}}^{-1}(t)$ can be interpreted heuristically as a signal to noise ratio.

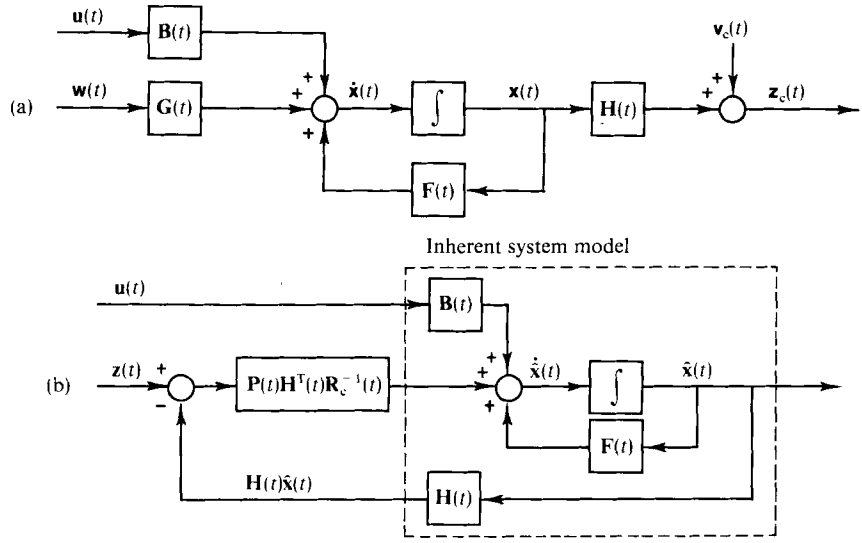

FIG. 5.17 (a) Continuous-time system model and (b) Kalman filter.

An algebraically equivalent form for the error covariance equation, (5-145), in terms of the Kalman gain $\mathbf{K}(t)$ of (5-147), is

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^{\mathrm{T}}(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t) - \mathbf{K}(t)\mathbf{R}_{\mathrm{c}}(t)\mathbf{K}^{\mathrm{T}}(t) \quad (5\text{-}148)$$

The first two terms of either of these expressions indicate the homogeneous system effects (usually stabilizing), the third term is the covariance increasing effect due to the dynamic noise $\mathbf{w}(\cdot, \cdot)$, and the fourth term is the error covariance decreasing effect of incorporating the measurement information. Thus it is reasonable that removing the last term, i.e., removing the continuously available measurements, yields the relation for propagating the error covariance between sample times in the sampled-data Kalman filter.

Equation (5-145) or (5-148) is a continuous-time matrix Riccati differential equation. As a regular differential equation, it has the usual properties of existence, uniqueness, and continuity of solutions. However, Riccati equations are very difficult to integrate numerically, being very sensitive to integration step size and often exhibiting unstable computed solutions despite theoretical solution stability.

Most practical estimation problems are characterized by system dynamics most naturally modeled by differential, rather than difference, equations. However, the designer knows from the outset that a digital computer will be employed in the eventual estimator implementation, thus dictating sampled-data rather than continuous-time measurements. Consequently, there are two means of systematic estimator design. First, we could take the continuous-time system model, design the continuous-time filter, and then discretize the result. Second, we could determine an equivalent discrete-time model and generate

the discrete-time filter from it. However, the first approach is fraught with the difficulties of integrating Riccati differential equations, while the second involves significantly better behaved recursions. Moreover, the discretization of a continuous filter is an approximation to an optimal discrete-time filter, whereas a discrete-time filter based on an *equivalent* discrete-time model involves no approximations. Thus, the preferable design approach is to discretize the model first and then generate the filter.

EXAMPLE 5.13 Once again we examine the gyro on test, but now assuming that measurement data are available continuously, modeled as

$$\mathbf{z}_c(t) = \mathbf{x}(t) + \mathbf{v}_c(t)$$

where $\mathbf{v}_c(\cdot,\cdot)$ is a zero-mean white Gaussian noise with $E\{\mathbf{v}_c(t)\mathbf{v}_c(t+\tau)\} = R_c(t)\delta(\tau)$. In view of (5-141a), if we want the continuous-time estimator performance to approximate that of the discrete-time filter, the appropriate noise strength would be

$$R_c(t) = R_c = [R(t_i)\Delta t] = (0.5 \quad \text{deg}^2/\text{hr}^2)(0.25 \quad \text{hr}) = 0.125 \quad \text{deg}^2/\text{hr}$$

The estimator is given by

$$\dot{\hat{x}}(t) = -\alpha\hat{x}(t) + [P(t)/R_c][z(t) - \hat{x}(t)] = -\hat{x}(t) + [8P(t)][z(t) - \hat{x}(t)]$$

integrated forward from the initial condition $\hat{x}(t_0) = 0$, where $P(t)$ satisfies the Riccati equation

$$\dot{P}(t) = -2\alpha P(t) + Q - [P^2(t)/R_c] = -2P(t) + 2 - 8P^2(t)$$

with an initial condition of $P(t_0) = 1$. For this problem, a steady state value of $P(t)$ is achieved, and it can be found by setting $\dot{P}(t) = 0$:

$$2 - 2P - 8P^2 = 0$$

for which the positive solution is

$$P = 0.392$$

The total solution for $P(t)$ is, letting $a = \sqrt{\alpha^2 + (Q/R_c)} = \sqrt{17}$

$$P(t) = \frac{a + [Q - \alpha P_0]\tanh(a[t - t_0])}{a + [\alpha - R_c^{-1}P_0]\tanh(a[t - t_0])} = \frac{\sqrt{17} + \tanh(\sqrt{17}[t - t_0])}{\sqrt{17} + 9\tanh(\sqrt{17}[t - t_0])}$$

Figure 5.18 plots this result, superimposed upon the result of Example 5.3. The continuous-time solution passes through the region bounded by the oscillations of the discrete-time filter. As $\Delta t \to 0$, these variations converge to the continuous-time result. For instance, consider halving the sample period. As a result, the increase in $P(t)$ is not as great before the next measurement becomes available. However, to maintain constant R_c, the discrete-time $R(t_i)$ would have to be doubled according to (5-141a) [we want to let $\Delta t \to 0$ while letting $\mathbf{v}(t_i) \to \mathbf{v}_c(t_i)$ of strength R_c]. Each decrease in $P(t)$ due to measurement updating is thus less, since the measurements are now corrupted by stronger noise.

Note that if the product $[R(t_i)\Delta t]$ were not held constant, the discrete-time solutions would not converge to the continuous-time solution. If $R(t_i)$ were kept constant instead, more of equally accurate data would be available by halving the sample period. In the limit, perfect continuous measurements would be achieved, and $P(t)$ would instantaneously go to zero at t_0 and stay there for all time. ■

The numerical characteristics of the Riccati differential equation solution, especially sensitivity to integration step size, motivate either avoiding methods

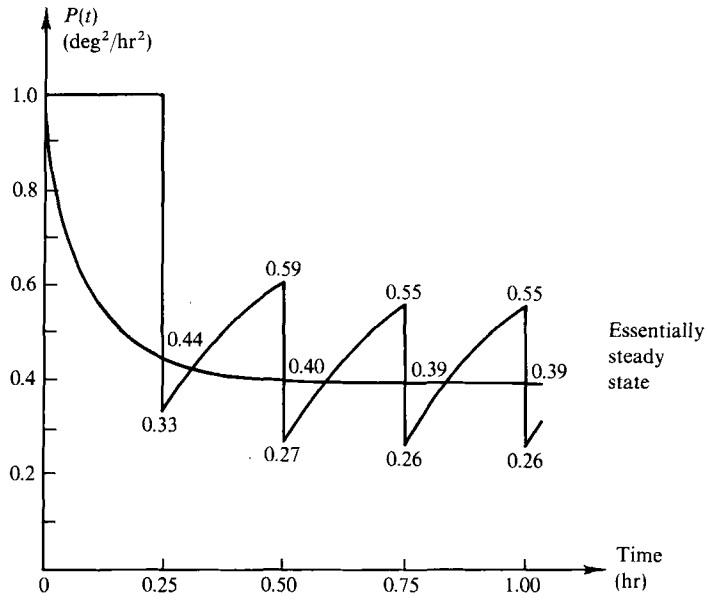

FIG. 5.18 Error variance of continuous-time filter for gyro example.

dependent on its solution or seeking solution techniques other than straightforward integration. To generate a solution to an n-by-n Riccati matrix differential equation, it is possible to exploit an associated $2n$-by-n linear matrix differential equation. Specifically, a solution to (5-145) can be expressed as

$$\dot{\mathbf{P}}(t) = \mathbf{U}(t)\mathbf{V}^{-1}(t) \tag{5-149}$$

where $\mathbf{U}(t)$ and $\mathbf{V}(t)$ are n-by-n matrices satisfying the homogeneous linear differential equation and initial condition

$$\begin{bmatrix} \dot{\mathbf{U}}(t) \\ \hline \dot{\mathbf{V}}(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{F}(t) & \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t) \\ \hline \mathbf{H}^{\mathrm{T}}(t)\mathbf{R}_c^{-1}(t)\mathbf{H}(t) & -\mathbf{F}^{\mathrm{T}}(t) \end{array}\right] \begin{bmatrix} \mathbf{U}(t) \\ \hline \mathbf{V}(t) \end{bmatrix} \tag{5-150a}$$

$$\begin{bmatrix} \mathbf{U}(t_0) \\ \hline \mathbf{V}(t_0) \end{bmatrix} = \begin{bmatrix} \mathbf{P}_0 \\ \hline \mathbf{I} \end{bmatrix} \tag{5-150b}$$

where $\mathbf{V}(t)$ is always invertible.

Proof of Equivalency First, the assumed form (5-149) satisfies the initial condition

$$\mathbf{P}(t_0) = \mathbf{U}(t_0)\mathbf{V}^{-1}(t_0) = \mathbf{P}_0\mathbf{I}^{-1} = \mathbf{P}_0$$

Differentiating the assumed form yields (dropping the time notation for convenience)

$$\dot{\mathbf{P}} = \dot{\mathbf{U}}\mathbf{V}^{-1} + \mathbf{U}\, d\{\mathbf{V}^{-1}\}/dt$$

But, since $\mathbf{V}\mathbf{V}^{-1} = \mathbf{I}, d\{\mathbf{V}\mathbf{V}^{-1}\}/dt = \mathbf{0} = \dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\, d\{\mathbf{V}^{-1}\}/dt$, so

$$d\mathbf{V}^{-1}/dt = -\mathbf{V}^{-1}\dot{\mathbf{V}}\mathbf{V}^{-1}$$

if $\mathbf{V}(t)$ is in fact invertible, giving

$$\dot{\mathbf{P}} = \dot{\mathbf{U}}\mathbf{V}^{-1} - \mathbf{U}\mathbf{V}^{-1}\dot{\mathbf{V}}\mathbf{V}^{-1}$$

Substituting the partitions of (5-150a) into this yields

$$\begin{aligned}\dot{\mathbf{P}} &= (\mathbf{F}\mathbf{U} + \mathbf{G}\mathbf{Q}\mathbf{G}^{\mathrm{T}}\mathbf{V})\mathbf{V}^{-1} - \mathbf{U}\mathbf{V}^{-1}(\mathbf{H}^{\mathrm{T}}\mathbf{R}_c^{-1}\mathbf{H}\mathbf{U} - \mathbf{F}^{\mathrm{T}}\mathbf{V})\mathbf{V}^{-1} \\ &= \mathbf{F}\mathbf{U}\mathbf{V}^{-1} + \mathbf{G}\mathbf{Q}\mathbf{G}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{-1} - \mathbf{U}\mathbf{V}^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{R}_c^{-1}\mathbf{H}\mathbf{U}\mathbf{V}^{-1} + \mathbf{U}\mathbf{V}^{-1}\mathbf{F}^{\mathrm{T}}\mathbf{V}\mathbf{V}^{-1} \\ &= \mathbf{F}\mathbf{P} + \mathbf{G}\mathbf{Q}\mathbf{G}^{\mathrm{T}} - \mathbf{P}\mathbf{H}^{\mathrm{T}}\mathbf{R}_c^{-1}\mathbf{H}\mathbf{P} + \mathbf{P}\mathbf{F}^{\mathrm{T}}\end{aligned}$$

which is in fact the original Riccati equation to be solved, (5-145). The matrix $\mathbf{V}(t)$ can be shown to be of full rank, and thus invertible, for all time. ■

Although (5-150) is a homogeneous linear differential equation, it embodies unstable modes; straightforward integration is not generally practicable, but eigenvalue techniques provide a useful means of attaining the steady state $\mathbf{P}$ satisfying (5-145) for the case of time-invariant systems with stationary noises [12, 58, 59]. Other means of solving (5-145) include iterative procedures [38–41], perturbation methods [61], the partitioned algorithm approach [43, 44], matrix factorization methods [63], the matrix sign function method [6, 19], and other means of enhancement in numerical integration [85].

The stability characteristics of the continuous-time filter are analogous to those of the discrete-time filter, described in Section 5.8. Here we consider the homogeneous portion of the filter state equations,

$$\dot{\hat{\mathbf{x}}}(t) = [\mathbf{F}(t) - \mathbf{P}(t)\mathbf{H}^{\mathrm{T}}(t)\mathbf{R}_c^{-1}(t)\mathbf{H}(t)]\hat{\mathbf{x}}(t) + \mathbf{P}(t)\mathbf{H}^{\mathrm{T}}(t)\mathbf{R}_c^{-1}(t)\mathbf{z}(t) \quad (5\text{-}151)$$

Then, if the system model upon which the continuous-time Kalman filter is based is stochastically observable (5-109) and stochastically controllable (5-105), then the filter is uniformly asymptotically globally stable. As in the sampled-data case, it is possible to generate a stable filter from an unstable system model.

Unlike the discrete-time case, $\mathbf{R}_c^{-1}(t)$ appears explicitly in the continuous-time Kalman filter gain, (5-147), so that a singular $\mathbf{R}_c(t)$ will require a substantial modification in the optimal estimator algorithm. Care must be taken when performing a limiting procedure to derive the optimal continuous-time estimator for problems characterized by time-correlated or no measurement noise, as considered in Section 5.10. (As in that section, filtering in the presence of only time-correlated measurement noise is a special case of filtering with perfect measurements, once state vector augmentation is exploited.) In the limit, the problem formulation and the optimal estimator, or *Deyst filter* [11, 13, 20, 21, 74], are as follows. Let the system dynamics be modeled as in (5-137) and let the continuous-time measurements be free of white noise corruption:

$$\mathbf{z}_c(t) = \mathbf{H}(t)\mathbf{x}(t) \quad (5\text{-}152)$$

Assume that $\mathbf{F}(\cdot)$, $\mathbf{B}(\cdot)$, $\mathbf{G}(\cdot)$, and $\mathbf{Q}(\cdot)$ are continuous with continuous first derivatives and $\mathbf{H}(\cdot)$ is continuous with continuous first and second derivatives.

Then the optimal estimator is specified in terms of an n-dimensional state $\mathbf{y}(t)$ as

$$\hat{\mathbf{x}}(t) = \mathbf{y}(t) + \mathbf{W}(t)\mathbf{z}(t) \tag{5-153a}$$

$$\dot{\mathbf{y}}(t) = \{[\mathbf{I} - \mathbf{W}(t)\mathbf{H}(t)]\mathbf{F}(t) - \dot{\mathbf{W}}(t)\mathbf{H}(t) - \mathbf{W}(t)\dot{\mathbf{H}}(t)\}\hat{\mathbf{x}}(t) + [\mathbf{I} - \mathbf{W}(t)\mathbf{H}(t)]\mathbf{B}(t)\mathbf{u}(t) \tag{5-153b}$$

$$\mathbf{A}(t) = \mathbf{P}(t)\dot{\mathbf{H}}^{\mathrm{T}}(t) + [\mathbf{P}(t)\mathbf{F}^{\mathrm{T}}(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t)]\mathbf{H}^{\mathrm{T}}(t) \tag{5-153c}$$

$$\mathbf{W}(t) = \mathbf{A}(t)[\mathbf{H}(t)\mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t)\mathbf{H}^{\mathrm{T}}(t)]^{-1} \tag{5-153d}$$

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^{\mathrm{T}}(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t) - \mathbf{W}(t)\mathbf{A}^{\mathrm{T}}(t) \tag{5-153e}$$

where (5-153a) and (5-153b) determine the basic filter structure, $\mathbf{W}(t)$ in (5-153d) is the filter weighting (gain) matrix, and (5-153e) is a matrix Riccati differential equation for the estimate error covariance. Figure 5.19 portrays the estimator structure. Note that, unlike the Kalman filter, the measurement $\mathbf{z}(t)$ can appear in the filter output *directly* with no integration. Because of this direct feedthrough, in stationary-noise, time-invariant-system cases this filter can be described by a matrix of transfer functions, each of which are able to have a numerator of degree equal to that of the denominator. On the other hand, transfer functions corresponding to the steady state Kalman filter must have numerators of degree less than the corresponding denominators.

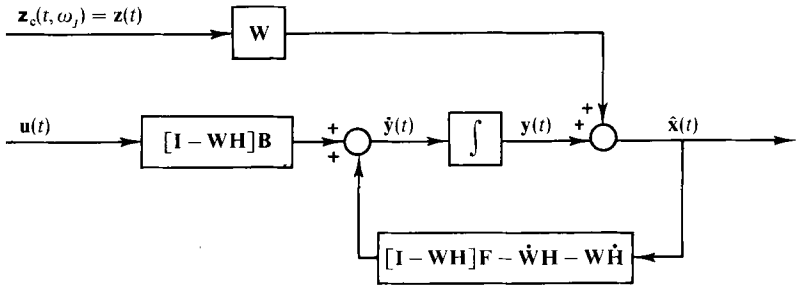

FIG. 5.19 Deyst filter structure.

Direct feedthrough of $\mathbf{z}(t)$ also affects the initialization of the filter. Before any measurement data are taken, the a priori statistics of $\mathbf{x}(t_0)$ are $\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$ and $\mathbf{P}(t_0) = \mathbf{P}_0$. However, at time t_0 there is a discontinuity [the form of which for $\hat{\mathbf{x}}(t_0^+)$ and $\mathbf{P}(t_0^+)$ is evident from a discrete-time Kalman filter update]:

$$\hat{\mathbf{x}}(t_0^+) = \hat{\mathbf{x}}_0 + \mathbf{P}_0\mathbf{H}^{\mathrm{T}}(t_0)[\mathbf{H}(t_0)\mathbf{P}_0\mathbf{H}^{\mathrm{T}}(t_0)]^{-1}[\mathbf{z}(t_0) - \mathbf{H}(t_0)\hat{\mathbf{x}}_0] \tag{5-154a}$$

$$\mathbf{P}(t_0^+) = \mathbf{P}_0 - \mathbf{P}_0\mathbf{H}^{\mathrm{T}}(t_0)[\mathbf{H}(t_0)\mathbf{P}_0\mathbf{H}^{\mathrm{T}}(t_0)]^{-1}\mathbf{H}(t_0)\mathbf{P}_0 \tag{5-154b}$$

$$\mathbf{A}(t_0^+) = \mathbf{P}(t_0^+)\dot{\mathbf{H}}^{\mathrm{T}}(t_0) + [\mathbf{P}(t_0^+)\mathbf{F}^{\mathrm{T}}(t_0) + \mathbf{G}(t_0)\mathbf{Q}(t_0)\mathbf{G}^{\mathrm{T}}(t_0)]\mathbf{H}^{\mathrm{T}}(t_0) \tag{5-154c}$$

$$\mathbf{W}(t_0^+) = \mathbf{A}(t_0^+)[\mathbf{H}(t_0)\mathbf{G}(t_0)\mathbf{Q}(t_0)\mathbf{G}^{\mathrm{T}}(t_0)\mathbf{H}^{\mathrm{T}}(t_0)]^{-1} \tag{5-154d}$$

$$\mathbf{y}(t_0^+) = \hat{\mathbf{x}}(t_0^+) - \mathbf{W}(t_0^+)\mathbf{z}(t_0) \tag{5-154e}$$

Thus, the filter cannot be initialized until the measurement value at time t_0 becomes available.

The m-by-m matrix $[\mathbf{H}(t)\mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t)\mathbf{H}^{\mathrm{T}}(t)]$ must be nonsingular to be assured of the existence of its inverse, required by the algorithm in (5-153d). Unless $\mathbf{H}(t)$ is of full rank for all $t \in T$, it will be singular; but linearly dependent rows of $\mathbf{H}$ can always be ignored since this is just redundant perfect information. Heuristically, this matrix represents the strength of first integrals of white noise in the measurements; its singularity implies that there exist one or more measurements, or linear combinations thereof, that contain no first integrals of white noise. If there is no white noise entering the system model just one integration before the measurement, one or more differentiators can be inserted into the filter input channel to generate derivatives of the measurements as "new" measurements (since we are more than one integration away from a white noise source, the differentiation process is not troublesome).

EXAMPLE 5.14 Consider the gyro with a bias as in Examples 5.11 and 5.12, with model depicted as in Fig. 5.14, but now assuming that measurements are available continuously. In terms of the transformed coordinates of Example 5.12, we have [$\mathsf{x}_1(t) = \mathsf{z}(t)$, $\mathsf{x}_2(t)$ = gyro drift rate to be estimated]

$$\begin{bmatrix} \dot{\mathsf{x}}_1(t) \\ \dot{\mathsf{x}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} \mathsf{x}_1(t) \\ \mathsf{x}_2(t) \end{bmatrix} + \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \mathsf{w}(t)$$

$$\mathsf{z}(t) = [1 \quad 0] \begin{bmatrix} \mathsf{x}_1(t) \\ \mathsf{x}_2(t) \end{bmatrix}$$

To apply the Deyst filter equations, $[\mathbf{HGQG}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}]$ must be evaluated:

$$[\mathbf{HGQG}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}] = [1 \quad 0] \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} Q [\alpha \quad \alpha] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha^2 Q$$

This is invertible, as expected, since the measurement is separated from the white noise $\mathsf{w}(\cdot,\cdot)$ by only one integration.

Consider the covariance. The initial condition is given in terms of the components of $\mathbf{P}_0$ by (5-154b)

$$\mathbf{P}(t_0{}^+) = \begin{bmatrix} 0 & 0 \\ 0 & P_{0_{22}} - [P_{0_{12}}^2/P_{0_{11}}] \end{bmatrix}$$

where the first row and column are zero because we know $\mathsf{z}(t_0)$ exactly. The Riccati equation, (5-153e), becomes

$$\begin{bmatrix} \dot{P}_{11}(t) & \dot{P}_{12}(t) \\ \dot{P}_{12}(t) & \dot{P}_{22}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -P_{22}(t)^2/Q \end{bmatrix}$$

again because $\mathsf{z}(t)$ is known exactly. Thus, the only Riccati equation to solve is the scalar equation for $\dot{P}_{22}(t)$, for which the solution is

$$P_{22}(t) = [QP_{22}(t_0{}^+)]/[Q + P_{22}(t_0{}^+)\{t - t_0\}]$$

This reduction of the dimension of the Riccati matrix equation by m, the number of measurements, always occurs in the transformed coordinates. Note that as $t \to \infty$, $P_{22}(t) \to 0$.

Substituting into the rest of the filter equations yields

$$\hat{x}_1(t) = y_1(t) + z(t)$$
$$\hat{x}_2(t) = y_2(t) + [1 - P_{22}(t)/(\alpha Q)]z(t)$$

where

$$\dot{y}_1(t) = 0$$
$$\dot{y}_2(t) = -[P_{22}(t)/Q]\hat{x}_2(t) + [\dot{P}_{22}(t)/(\alpha Q)]z(t)$$

But it can be shown that $y_1(t_0^+) = 0$, so $y_1(t) = 0$ for all time t, and then

$$\hat{x}_1(t) = z(t)$$

as anticipated. A block diagram of the filter is given in Fig. 5.20. Note the direct feedthrough of $z(t)$ through C_1. In this filter, $\hat{x}_2(t)$ is in fact the estimate of the gyro drift rate. ■

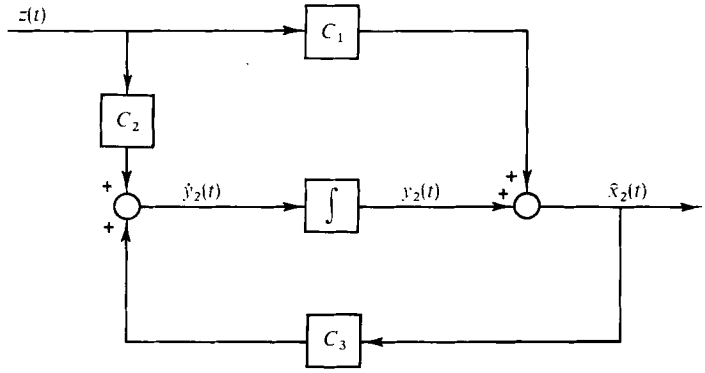

FIG. 5.20 Deyst filter for Example 5.14. $C_1 = 1 - [P_{22}(t)/\alpha Q]$, $C_2 = \dot{P}_{22}(t)/\alpha Q = -P_{22}(t)^2/\alpha Q^2$, $C_3 = -P_{22}(t)/Q$.

EXAMPLE 5.15 To see the effect of singular $[\mathbf{HGQG}^T\mathbf{H}^T]$, consider the same problem as the preceding, but let the gyro drift rate plus bias be put through a first order lag $[b/(s + b)]$ before becoming available as measured output. Figure 5.21 presents a state model block diagram in untransformed coordinates. The augmented system equations are:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & 0 & 0 \\ b & b & -b \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} w(t)$$

$$z(t) = [0 \quad 0 \quad 1]\mathbf{x}(t)$$

Since the white noise enters the model more than one integration back from the measurement, $[\mathbf{HGQG}^T\mathbf{H}^T]$ is singular:

$$\mathbf{HGQG}^T\mathbf{H}^T = [0 \quad 0 \quad 1] \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} Q [\alpha \quad 0 \quad 0] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

To circumvent this singularity problem, define a "new measurement"

$$z'(t) = (1/b)\dot{z}(t) + z(t)$$

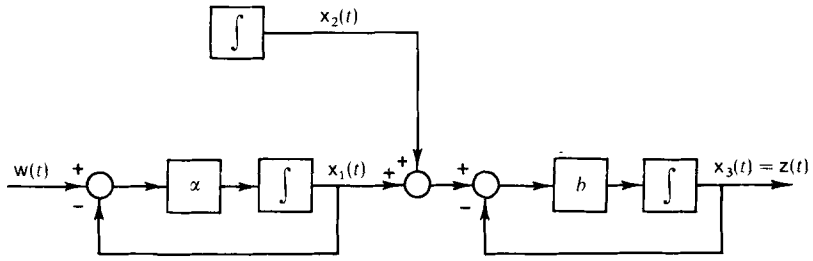

FIG. 5.21 System model for Example 5.15.

Because of the differentiation of $z(t)$ in this definition, the new measurement is separated by only one integration from the white noise. In fact, in the Laplace domain, $z'(s) = [(s + b)/b]z(s)$, or

$$z'(t) = x_1(t) + x_2(t)$$

We have already processed this measurement in the previous example.

Note that if a white noise entered the model at the same location where $x_2(t)$ enters, the Deyst filter could be employed with no difficulty. Moreover, if the output of the lag $[b/(s + b)]$ were corrupted by white noise, a Kalman filter could be used directly. ■

5.12 WIENER FILTERING AND FREQUENCY DOMAIN TECHNIQUES

The purpose of this section is to relate the optimal filtering results obtained by time-domain (state space) methods to frequency domain techniques. Furthermore, the pioneering work of Wiener [34, 36, 83] will be presented, culminating in the Wiener–Hopf equation. His original problem formulation and the Wiener–Hopf equation itself are in the time domain, but under restrictions (namely, system time-invariance, noise stationarity, and infinite data length) that allow frequency domain interpretation. Although these results have been extended to less restrictive assumptions, the most useful means of solving the Wiener–Hopf equation to yield systematic design capability, the Bode–Shannon technique, is a frequency domain procedure. In the case of time-invariant system models and stationary noises, the Wiener filter will be shown to be equivalent to the steady state Kalman or Deyst filter appropriate to the given problem. Throughout this section, the power spectrum and frequency domain system concepts developed in Sections 4.3 and 4.12 will be exploited.

First the optimal linear estimation problem will be discussed in the communication theory context and notation appropriate to Wiener filter development. Assume that an input signal $i(\cdot,\cdot)$ is the sum of a wanted signal $s(\cdot,\cdot)$ and an unwanted noise $n(\cdot,\cdot)$

$$i(t) = s(t) + n(t) \tag{5-155}$$

for all $t \in T$. The signal may in fact be deterministic, but can in general be a stochastic process. It is desired to generate the device which will accept $i(t)$

as an input and yield $\mathsf{s}(t + \Delta t)$ or some function of it as an output. (If $\Delta t = 0$, the device is called a *filter*; if $\Delta t > 0$ the device is a *predictor*; if $\Delta t < 0$, the device is a *smoother*. We will pursue the concept of a filter.)

Despite this desired function, the device that will actually be generated will accept $\mathsf{i}(t)$ and produce an output $\mathsf{y}(t)$ for all $t \in T$. The desired output of the device is $\mathsf{d}(t)$, the output of some specified linear system described by impulse response function $T_d(\cdot)$ in response to be being driven by $\mathsf{s}(t)$. For each sample $\mathsf{s}(t, \omega_j) = s(t)$ for all $t \in T$, $d(t)$ is

$$d(t) = \int_{-\infty}^{t} T_d(t - \tau)s(\tau)\,d\tau \tag{5-156}$$

(Note that time-invariance is assumed by writing $T_d(\cdot)$ instead of $T_d(\cdot, \cdot)$, but this can be generalized.) In most cases of interest, $T_d(\tau)$ is $\delta(\tau)$ for all τ: in other words, $\mathsf{d}(t) = \mathsf{s}(t)$, which is to say that we *desire* filter output $\mathsf{y}(t)$ to be the signal $\mathsf{s}(t)$ itself!

Thus, a block diagram of the optimal filtering problem would be as depicted in Fig. 5.22. Note that we cannot really separate out $\mathsf{s}(t)$ to put through the "desired operation," or else there would be no filtering problem at all. The filter error, $\mathsf{e}(t)$, is the difference between the actual filter output $\mathsf{y}(t)$ and the desired output $\mathsf{d}(t)$

$$\mathsf{e}(t) = \mathsf{y}(t) - \mathsf{d}(t) \tag{5-157}$$

or, in most cases, $[\mathsf{y}(t) - \mathsf{s}(t)]$. From the many possible criteria for optimality, that of least mean square error is particularly tractable if one can specify process autocorrelation functions and the system impulse response function or transfer function of the desired operation.

In the general problem formulation, the characterization of the input and noise is important; if these are stochastic processes, it is particularly important to know whether they are stationary or not. Also important is the amount of design freedom: do we assume that the filter structure is to be linear or that it is time-invariant? To ensure practical value of the design, it is usually required that the filter be realizable (nonanticipative): that the optimal filter impulse response function satisfy $T_{FO}(\tau) = 0$ for all $\tau < 0$. (This was inherent in the

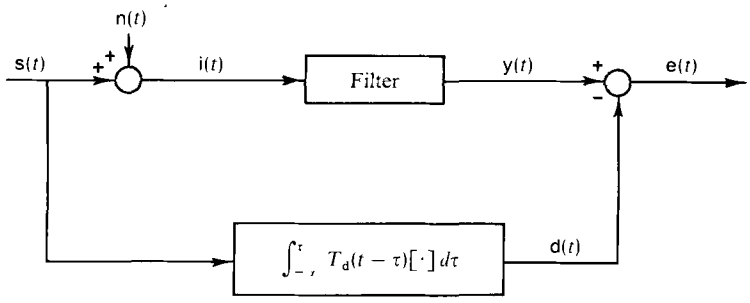

FIG. 5.22 Optimal filtering problem in communication theory context.

recursive state nature of the Kalman filter.) A final design parameter of the filter is its operating time: whether it operates on a finite measurement sample length or if it represents a steady state solution for an infinite length of measurement data.

In 1942, Wiener solved the filtering problem under the following assumptions: (1) the signal $s(t)$ and noise $n(t)$ are each samples from stationary random processes which have some distinguishing statistical characteristics, (2) the filter is a (time-invariant) linear device that operates on an infinite record of data, and (3) the optimality criterion is minimum mean square error. Note that only time-invariant devices need be considered since we seek a filter that produces an output with stationary statistics in response to a stationary input. Denoting a given filter impulse response function by $T_F(\cdot)$, realizations of (5-157) can be written as

$$e(t) = y(t) - d(t) = \int_{-\infty}^{\infty} T_F(\tau) i(t-\tau)\, d\tau - d(t)$$

where the limits of integration are valid because of infinite data record length and realizability. Thus, the mean square error for any filter is

$$\begin{aligned} E\{\mathsf{e}^2\} &= E\{\mathsf{y}^2\} - 2E\{\mathsf{yd}\} + E\{\mathsf{d}^2\} = \Psi_{yy}(0) - 2\Psi_{yd}(0) + E\{\mathsf{d}^2\} \\ &= \int_{-\infty}^{\infty} T_F(\tau_1)\left[\int_{-\infty}^{\infty} T_F(\tau_2)\Psi_{ii}(\tau_1 - \tau_2)\, d\tau_2\right] d\tau_1 \\ &\quad - 2\int_{-\infty}^{\infty} T_F(\tau)\Psi_{id}(\tau)\, d\tau + E\{\mathsf{d}^2\} \end{aligned} \tag{5-158}$$

where $\Psi_{xz}(\tau)$ denotes the correlation function of $\mathsf{x}(t)$ and $\mathsf{z}(t+\tau)$, $E\{\mathsf{x}(t)\mathsf{z}(t+\tau)\}$. By employing variational techniques, the filter may be written as the sum of the optimal filter described by impulse response function $T_{FO}(\cdot)$ plus a perturbation described by $\Delta T_F(\cdot)$:

$$T_F(t) = T_{FO}(t) + \varepsilon \Delta T_F(t)$$

for all time t of interest. This is substituted into (5-158), and then the necessary condition for a minimum is

$$0 = \left.\frac{\partial E\{\mathsf{e}^2\}}{\partial \varepsilon}\right|_{\varepsilon=0} = 2\int_{-\infty}^{\infty} \Delta T_F(t)\left[\int_{-\infty}^{\infty} T_{FO}(\tau)\Psi_{ii}(t-\tau)\, d\tau - \Psi_{id}(t)\right] dt$$

Since $\Delta T_F(t)$ is arbitrary for all $t \geq 0$ and zero for $t < 0$, for the above expression to be valid, the term within the brackets must itself equal zero. Thus is obtained the now famous *Wiener–Hopf equation*

$$\int_{-\infty}^{\infty} T_{FO}(\tau)\Psi_{ii}(t-\tau)\, d\tau - \Psi_{id}(t) = 0 \tag{5-159}$$

The solution to this integral equation is not a trivial task. To find $T_{FO}(\cdot)$, given $\Psi_{ii}(\cdot)$ and $\Psi_{id}(\cdot)$, it is usually necessary to employ integral transform techniques and solve for the filter transfer function in the frequency domain.

Perhaps the most useful solution method is the *Bode–Shannon technique* [7], which states that, if the desired transfer function $T_d(s)$ and the signal and noise spectra are all rational, then the solution for the optimal realizable filter is

$$T_{FO}(s) = \frac{1}{\bar{\Psi}_{ii}(s)_L}\left[\frac{\bar{\Psi}_{id}(s)}{\bar{\Psi}_{ii}(s)_R}\right]_{\mathscr{L}} \tag{5-160}$$

where the subscripts L and R denote *spectral factorization* and $\mathscr{L}$ denotes *separation by partial fraction expansion* (to be explained further as the entire procedure is specified).

The power spectral density of the input is given by

$$\bar{\Psi}_{ii}(s) = \bar{\Psi}_{ss}(s) + \bar{\Psi}_{sn}(s) + \bar{\Psi}_{ns}(s) + \bar{\Psi}_{nn}(s) \tag{5-161}$$

where, as before, the subscript i denotes input, s refers to signal, and n pertains to noise. If the noise and signal are uncorrelated, as if often the case, then (5-161) becomes

$$\bar{\Psi}_{ii}(s) = \bar{\Psi}_{ss}(s) + \bar{\Psi}_{nn}(s) \tag{5-161'}$$

This expression can be partitioned by spectral factorization (see Section 4.12) as

$$\bar{\Psi}_{ii}(s) = \bar{\Psi}_{ii}(s)_L \bar{\Psi}_{ii}(s)_R \tag{5-162}$$

where $\bar{\Psi}_{ii}(s)_L$ has all of its poles and zeros confined to the left half s plane (including half the pole and zero doubles on the imaginary axis), and $\bar{\Psi}_{ii}(s)_R$ similarly has all of its poles and zeros in the right half plane including the other half of the doubles on the $j\omega$ axis). These factors are used directly in (5-160).

The cross power spectral density between the input and the desired output is given by

$$\bar{\Psi}_{id}(s) = T_d(s)[\bar{\Psi}_{ss}(s) + \bar{\Psi}_{ns}(s)] \tag{5-163}$$

If the noise and signal are uncorrelated, this becomes:

$$\bar{\Psi}_{id}(s) = T_d(s)\bar{\Psi}_{ss}(s) \tag{5-163a}$$

We are especially interested in the case of $T_d(s) = 1$, for which

$$\bar{\Psi}_{id}(s) = \bar{\Psi}_{is}(s) = \bar{\Psi}_{ss}(s) + \bar{\Psi}_{ns}(s) \tag{5-163b}$$

and if the noise and signal are also uncorrelated, then this simplifies to

$$\bar{\Psi}_{id}(s) = \bar{\Psi}_{ss}(s) \tag{5-163c}$$

To use the Bode-Shannon technique, first the expression $[\bar{\Psi}_{id}(s)/\bar{\Psi}_{ii}(s)_R]$ is generated, and then it is separated by partial fraction expansion:

$$\left[\frac{\bar{\Psi}_{id}(s)}{\bar{\Psi}_{ii}(s)_R}\right] = \sum_{j=1}^{N}\sum_{k=1}\frac{A_{jk}}{(s+a_j)^k} \tag{5-164}$$

where N is the number of distinct poles and μ_j is the multiplicity of the jth pole.

This sum can then be separated into the sum of terms corresponding to left half plane poles, denoted as $[\bar{\Psi}_{id}(s)/\bar{\Psi}_{ii}(s)_R]_{\mathscr{L}}$, and a sum of terms corresponding to right half plane poles, denoted as $[\bar{\Psi}_{id}(s)/\bar{\Psi}_{ii}(s)_R]_{\mathscr{R}}$:

$$[\bar{\Psi}_{id}(s)/\bar{\Psi}_{ii}(s)_R] = [\bar{\Psi}_{id}(s)/\bar{\Psi}_{ii}(s)_R]_{\mathscr{L}} + [\bar{\Psi}_{id}(s)/\bar{\Psi}_{ii}(s)_R]_{\mathscr{R}} \qquad (5\text{-}165)$$

The term $[\bar{\Psi}_{id}(s)/\bar{\Psi}_{ii}(s)_R]_{\mathscr{L}}$ is then multiplied by $[1/\bar{\Psi}_{ii}(s)_L]$ to obtain $T_{FO}(s)$, the optimal filter transfer function.

This method is probably the easiest means of solving the Wiener–Hopf equation, but it is valid only for cases involving stationary signal and noise and infinite length data records. Thus, a systematic design technique is available for generating steady state optimal filters in the frequency domain, but its power and applicability is more restricted than that of the Kalman filter synthesis capability.

EXAMPLE 5.16 Assume that we have a signal and noise which are uncorrelated, with power spectral densities given by

$$\bar{\Psi}_{ss}(\omega) = A/(a^2 + \omega^2), \qquad \bar{\Psi}_{nn}(\omega) = \bar{\Psi}_0$$

i.e., an exponentially time-correlated signal corrupted by white noise. We want to design the optimal filter to accept the sum of these two and output the best representation of the signal alone. The desired output is the signal itself, so $T_d(s) = 1$ and

$$\bar{\Psi}_{id}(s) = \bar{\Psi}_{ss}(s) = \frac{A}{a^2 - s^2}$$

The input power spectral density is

$$\begin{aligned}\bar{\Psi}_{ii}(s) &= \bar{\Psi}_{ss}(s) + \bar{\Psi}_{nn}(s) \\ &= \frac{A}{a^2 - s^2} + \bar{\Psi}_0 = \bar{\Psi}_0 \frac{(a^2 + A/\bar{\Psi}_0) - s^2}{a^2 - s^2}\end{aligned}$$

Letting $c^2 = a^2 + (A/\bar{\Psi}_0)$, spectral factorization of $\bar{\Psi}_{ii}(s)$ yields

$$\bar{\Psi}_{ii}(s) = \underbrace{\left[\sqrt{\bar{\Psi}_0}\frac{c+s}{a+s}\right]}_{\bar{\Psi}_{ii}(s)_L}\underbrace{\left[\sqrt{\bar{\Psi}_0}\frac{c-s}{a-s}\right]}_{\bar{\Psi}_{ii}(s)_R}.$$

Thus, the term $[\bar{\Psi}_{id}(s)/\bar{\Psi}_{ii}(s)_R]$ becomes

$$\frac{\bar{\Psi}_{id}(s)}{\bar{\Psi}_{ii}(s)_R} = \left[\frac{A}{(a+s)(a-s)}\right]\left[\frac{1}{\sqrt{\bar{\Psi}_0}}\frac{a-s}{c-s}\right] = \frac{A}{\sqrt{\bar{\Psi}_0}}\left[\frac{1}{(a+s)(c-s)}\right]$$

This can be factored into the partial fraction expansion:

$$\left[\frac{\bar{\Psi}_{id}(s)}{\bar{\Psi}_{ii}(s)_R}\right] = \frac{A}{\sqrt{\bar{\Psi}_0}}\left[\frac{1/(a+c)}{a+s} + \frac{1/(a+c)}{c-s}\right]$$

so that

$$\left[\frac{\bar{\Psi}_{id}(s)}{\bar{\Psi}_{ii}(s)_R}\right]_{\mathscr{L}} = \frac{A}{\sqrt{\bar{\Psi}_0}(a+c)}\frac{1}{a+s}$$

Multiplying this by $1/\bar{\Psi}_{ii}(s)_L$ yields $T_{FO}(s)$ as

$$T_{FO}(s) = \left[\frac{1}{\sqrt{\bar{\Psi}_0}}\frac{a+s}{c+s}\right]\left[\frac{A}{\sqrt{\bar{\Psi}_0}(a+c)}\frac{1}{a+s}\right]$$

$$= \frac{A}{\bar{\Psi}_0(a+c)}\frac{1}{c+s} = \frac{A(c-a)}{\bar{\Psi}_0(c^2-a^2)}\frac{1}{c+s}$$

$$= \frac{c-a}{s+c}$$

where $c = \sqrt{a^2 + (A/\bar{\Psi}_0)}$.

Let us investigate the reasonableness of this first order lag as the optimal filter form. The break frequency of the filter is $c = \sqrt{a^2 + (A/\bar{\Psi}_0)}$. If the low frequency signal-to-noise ratio is much greater than one:

$$A/(\bar{\Psi}_0 a^2) \gg 1$$

then the break frequency is approximately

$$c \cong \sqrt{A/\bar{\Psi}_0} \gg a$$

and the magnitude of the filter transfer function is approximately one at low frequencies. Thus for the case in which the input contains a large proportion of valid information, the filter pays considerable attention to the input and does not attenuate it significantly until a frequency considerably beyond the signal break frequency, a.

When the signal-to-noise ratio is small,

$$A/(\bar{\Psi}_0 a^2) \ll 1$$

then $c \cong a$ and the low frequency gain is approximately $\frac{1}{2}[A/(\bar{\Psi}_0 a^2)] \ll 1$. Thus, the filter attenuates the input and breaks where the signal breaks. ■

Numerous attempts were made to extend Wiener's work by relaxing some of his assumptions [8, 71, 86]. By 1953 the necessary condition for a time-varying optimal filter operating on a finite length data record (of length Δt) generated by samples from nonstationary stochastic process inputs was shown to be

$$\int_0^{\Delta t} T_{FO}(\tau, \sigma)\Psi_{ii}(\sigma - t, \sigma - \tau)\,d\tau - \Psi_{id}(\sigma - t, \sigma) = 0 \qquad (5\text{-}166)$$

Although very general problems had been formulated and necessary conditions for optimality obtained in the form of integral equations, the solution to these equations is extremely difficult (if at all tractable) for all but the original, more restricted, Wiener–Hopf equation, (5-159). It would not be until 1960, with the advent of state space time–domain methods, the Kalman filter, and digital computers, that a practical design procedure would be available to generate optimal filters capable of operating on finite data samples of nonstationary stochastic process inputs.

The Kalman filter can readily be applied in this communication theory context. The wanted signal $\mathbf{s}(t)$ is considered to be $\mathbf{H}(t)\mathbf{x}(t)$, the output of a linear system driven by white noise: this is justifiable if the signal spectrum is assumed to be (well approximated as) *rational*. The corruptive noise which is

added to the signal is also considered to be *white* noise, $\mathbf{v}_c(\cdot,\cdot)$. Actually, by means of shaping filters, this formulation can be generalized to allow the corruptive noise to be the *sum* of time-correlated and white noises (if the corruption is composed only of time-correlated noise, then a Deyst filter must be used instead).

The continuous-time Kalman filter is given by (5-144)–(5-146) without the deterministic noise term, and in general is a time-varying system. If the system model is time invariant (**F**, **G**, and **H** constant) and the noises are stationary (**Q** and **R** constant), the filter may reach steady state performance in which the covariance **P** is a constant (sufficient conditions for stability being given in Section 5.11). For this condition of $\dot{\mathbf{P}}(t) = \mathbf{0}$, the Riccati equation becomes an algebraic relation

$$\dot{\mathbf{P}} = \mathbf{FP} + \mathbf{PF}^{\mathrm{T}} + \mathbf{GQG}^{\mathrm{T}} - \mathbf{PH}^{\mathrm{T}}\mathbf{R}_c^{-1}\mathbf{HP} = \mathbf{0} \tag{5-167}$$

In the steady state condition, the rate at which uncertainty increases (given by $\mathbf{GQG}^{\mathrm{T}}$) is just balanced by the rate at which new information enters ($\mathbf{PH}^{\mathrm{T}}\mathbf{R}_c^{-1}\mathbf{HP}$) and the dissipative effects of the system ($\mathbf{FP} + \mathbf{PF}^{\mathrm{T}}$). For a steady state covariance matrix, the optimal filter is also time invariant, given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{PH}^{\mathrm{T}}\mathbf{R}_c^{-1}[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)] \tag{5-168a}$$

$$= [\mathbf{F} - \mathbf{KH}]\hat{\mathbf{x}}(t) + \mathbf{Kz}(t) \tag{5-168b}$$

Taking the Laplace transform of this (neglecting initial conditions) yields

$$(s\mathbf{I} - \mathbf{F} + \mathbf{KH})\hat{\mathbf{x}}(s) = \mathbf{Kz}(s)$$

so that

$$\hat{\mathbf{x}}(s) = [(s\mathbf{I} - \mathbf{F} + \mathbf{KH})^{-1}\mathbf{K}]\mathbf{z}(s) \tag{5-169}$$

where the term in brackets is the transfer function representation of the steady state Kalman filter [56]. It is identical to the Wiener filter found for the case of white corruptive noise by solving the Wiener–Hopf equation (5-159) by the Bode–Shannon technique. In order words, *the steady state Kalman filter is equivalent to the Wiener filter for the same problem formulation.*

EXAMPLE 5.17 Consider the same problem as previously solved by Wiener filter design in Example 5.16, be now determine the Kalman filter to generate an optimal estimate of the wanted signal. First it is necessary to determine the shaping filter which will generate a stochastic process with the given power spectral density $\Phi_{ss}(\omega) = A/(a^2 + \omega^2)$. This is found to be a first order lag described by

$$G(s) = 1/(s + a)$$

driven by white noise with mean zero and strength A.

Thus, the system model upon which the filter is based is

$$\dot{x}(t) = -ax(t) + w(t), \qquad z_c(t) = x(t) + v_c(t)$$

with

$$E\{\mathsf{w}(t)\mathsf{w}(t+\tau)\} = A\,\delta(\tau), \qquad E\{\mathsf{v}_c(t)\mathsf{v}_c(t+\tau)\} = \Psi_0\,\delta(\tau) \qquad E\{\mathsf{w}(t)\mathsf{v}_c(t+\tau)\} = 0$$

The Kalman filter is specified by

$$\dot{\hat{x}}(t) = -a\hat{x}(t) + [P(t)/\Psi_0][z(t) - \hat{x}(t)]$$

where $P(t)$ satisfies the Riccati equation

$$\dot{P}(t) = -2aP(t) + A - [P^2(t)/\Psi_0]$$

In steady state, the variance is a constant, P, given by

$$P = \Psi_0[\sqrt{a^2 + (A/\Psi_0)} - a] = \Psi_0[c - a]$$

Thus, the steady state filter becomes a time-invariant system described by the differential equation:

$$\begin{aligned}\dot{\hat{x}}(t) &= -[a + (P/\Psi_0)]\hat{x}(t) + [P/\Psi_0]z(t)\\ &= -[a + c - a]\hat{x}(t) + [c - a]z(t)\\ &= -[c]\hat{x}(t) + [c - a]z(t)\end{aligned}$$

or equivalently by the Laplace transfer function

$$\frac{\hat{x}(s)}{z(s)} = \frac{c - a}{s + c}$$

This is identical to the result of Example 5.16.

Note again that this is a *steady state* Kalman filter. $P(t)$ is a complicated time function:

$$\begin{aligned}P(t) &= \Psi_0\sqrt{a^2 + (A/\Psi_0)}\tanh(t + k) - \Psi_0 A\\ k &= \tanh^{-1}\{[a + (P_0/\Psi_0)]/\sqrt{a^2 + (A/\Psi_0)}\}\end{aligned}$$

and the filter is generally a time-varying linear system. ■

Furthermore, *the steady state Deyst filter is equivalent to the Wiener filter for the same problem formulation.* For a system model given by (5-137) and (5-152), the Deyst filter was given by (5-153) and (5-154). For the particular case of a time-invariant system driven by stationary noise $\mathbf{w}(\cdot,\cdot)$, a steady state solution can be reached:

$$\hat{\mathbf{x}}(t) = \mathbf{y}(t) + \mathbf{W}\mathbf{z}(t) \tag{5-170a}$$

$$\dot{\mathbf{y}}(t) = [\mathbf{I} - \mathbf{W}\mathbf{H}]\mathbf{F}\hat{\mathbf{x}}(t) \tag{5-170b}$$

$$\mathbf{A} = [\mathbf{P}\mathbf{F}^{\mathrm{T}} + \mathbf{G}\mathbf{Q}\mathbf{G}^{\mathrm{T}}]\mathbf{H}^{\mathrm{T}} \tag{5-170c}$$

$$\mathbf{W} = \mathbf{A}[\mathbf{H}\mathbf{G}\mathbf{Q}\mathbf{G}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}]^{-1} \tag{5-170d}$$

$$\mathbf{0} = \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^{\mathrm{T}} + \mathbf{G}\mathbf{Q}\mathbf{G}^{\mathrm{T}} - \mathbf{W}\mathbf{A}^{\mathrm{T}} \tag{5-170e}$$

The Laplace transform transfer function of this filter is readily shown to be

$$\hat{\mathbf{x}}(s) = \{[s\mathbf{I} - (\mathbf{I} - \mathbf{W}\mathbf{H})\mathbf{F}]^{-1}[\mathbf{I} - \mathbf{W}\mathbf{H}]\mathbf{F}\mathbf{W} + \mathbf{W}\}\mathbf{z}(s) \tag{5-171}$$

This can be shown to be identical to the Wiener filter for the case of time-correlated measurement noise only. Note the direct feedthrough term $\mathbf{W}$ in

(5-171): unlike the Kalman filter, the steady state Deyst filter transfer function matrix can have numerators of order equal to the corresponding denominators, as previously claimed. The equivalency of the two forms is demonstrated for a particular application in Problems 5.26 and 5.27.

5.13 SUMMARY

This chapter formulated and solved the optimal estimation problem for the case in which a linear system model driven by white Gaussian noises and deterministic inputs adequately describes true system behavior. Because of its practical applicability through digital computer implementation, the sampled data formulation was emphasized throughout. Table 5.2 summarizes the Kalman filter algorithm for discrete-time measurements, comprised of relations for propagating the state estimate and error covariance from one measurement time to the next, and update equations for incorporating the next measurement

TABLE 5.2

Kalman Filter for Discrete-Time Measurements

State Dynamics Model	Time Propagation Relations
(1) Stochastic differential equation	
$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t)$	$\dot{\hat{\mathbf{x}}}(t/t_{i-1}) = \mathbf{F}(t)\hat{\mathbf{x}}(t/t_{i-1}) + \mathbf{B}(t)\mathbf{u}(t)$
$\mathbf{dx}(t) = \mathbf{F}(t)\mathbf{x}(t)\,dt + \mathbf{B}(t)\mathbf{u}(t)\,dt + \mathbf{G}(t)\,\mathbf{d}\boldsymbol{\beta}(t)$	$\dot{\mathbf{P}}(t/t_{i-1}) = \mathbf{F}(t)\mathbf{P}(t/t_{i-1}) + \mathbf{P}(t/t_{i-1})\mathbf{F}^{\mathrm{T}}(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t)$
(2) Solution to stochastic differential equation	
$\mathbf{x}(t) = \boldsymbol{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \tau)\mathbf{G}(\tau)\,\mathbf{d}\boldsymbol{\beta}(\tau)$	$\hat{\mathbf{x}}(t_i^-) = \boldsymbol{\Phi}(t_i, t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+) + \int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau$ $\mathbf{P}(t_i^-) = \boldsymbol{\Phi}(t_i, t_{i-1})\mathbf{P}(t_{i-1}^+)\boldsymbol{\Phi}^{\mathrm{T}}(t_i, t_{i-1}) + \int_{t_{i-1}}^{t_i} \boldsymbol{\Phi}(t_i, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\boldsymbol{\Phi}^{\mathrm{T}}(t_i, \tau)\,d\tau$
(3) Stochastic difference equation	
$\mathbf{x}(t_i) = \boldsymbol{\Phi}(t_i, t_{i-1})\mathbf{x}(t_{i-1}) + \mathbf{B}_{\mathrm{d}}(t_{i-1})\mathbf{u}(t_{i-1}) + \mathbf{G}_{\mathrm{d}}(t_{i-1})\mathbf{w}_{\mathrm{d}}(t_{i-1})$	$\hat{\mathbf{x}}(t_i^-) = \boldsymbol{\Phi}(t_i, t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+) + \mathbf{B}_{\mathrm{d}}(t_{i-1})\mathbf{u}(t_{i-1})$ $\mathbf{P}(t_i^-) = \boldsymbol{\Phi}(t_i, t_{i-1})\mathbf{P}(t_{i-1}^+)\boldsymbol{\Phi}^{\mathrm{T}}(t_i, t_{i-1}) + \mathbf{G}_{\mathrm{d}}(t_{i-1})\mathbf{Q}_{\mathrm{d}}(t_{i-1})\mathbf{G}_{\mathrm{d}}^{\mathrm{T}}(t_{i-1})$
Measurement Model	**Measurement Update Equations**
$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i)$	$\mathbf{K}(t_i) = \mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i)[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i)]^{-1}$ $\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i)[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]$ $\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{K}(t_i)\mathbf{H}(t_i)\mathbf{P}(t_i^-)$

into the estimate. Three forms of time propagation relations are enumerated in the table: (1) differential equations to be solved numerically, based on the stochastic differential equation description of state dynamics, (2) the discrete-time algorithm based upon the solution to the original stochastic differential equation, and (3) the discrete-time algorithm based upon a stochastic difference equation (especially viewed as an equivalent discrete-time model to the original continuous-time dynamics). By recursively generating the conditional mean and error covariance, the Kalman filter actually maintains an explicit description of the entire conditional density for the states conditioned on the entire measurement history, thereby fulfilling the objective of Bayesian estimation. The analytical and computational characteristics of the Kalman filter were analyzed, alternate and extended forms explored, continuous-time measurements considered and equivalency to Wiener filtering in the steady state investigated. The next chapter will further explore the design and implementation of practical Kalman filters.

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PROBLEMS

5.1 The two-dimensional random vector $\mathbf{x} = [\mathrm{x}_1 \mathrm{x}_2]^{\mathrm{T}}$ has the probability density:

$$f_{\mathbf{x}}(\xi) = \frac{1}{2\pi|\mathbf{P}|^{1/2}} \exp\{-\tfrac{1}{2}\xi^{\mathrm{T}}\mathbf{P}^{-1}\xi\}; \qquad \mathbf{P} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

A perfect measurement of x_1 is obtained as $\mathrm{z}(\omega_j) = \mathrm{x}_1(\omega_j) = z = 1$. What is the probability density of the vector $\mathbf{x}$, conditioned on the measurement $\mathrm{z}(\omega_j) = z$?

5.2 Prove the matrix inversion lemma, (5-28): for $\mathbf{P}$ and $\mathbf{R}$ positive definite,

$$(\mathbf{P}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1} = \mathbf{P} - \mathbf{P}\mathbf{H}^T(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R})^{-1}\mathbf{H}\mathbf{P}$$

Show this by looking at the partitioned matrix

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{P}^{-1} & \mathbf{H}^T \\ \hline \mathbf{H} & -\mathbf{R} \end{array}\right]$$

and letting $\mathbf{A}^{-1}$ be given by the partitioned matrix

$$\mathbf{A}^{-1} = \left[\begin{array}{c|c} \mathbf{D} & \mathbf{F} \\ \hline \mathbf{G}^T & \mathbf{E} \end{array}\right]$$

and solve the equations that result by setting $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ for the value of $\mathbf{D}$ (the upper left partition of $\mathbf{A}^{-1}$).

5.3 Having proven (5-28), use it to establish (5-29) and (5-30).

5.4 In reducing (5-27) to (5-33), it is necessary to show that

$$\frac{|\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R}|^{1/2}}{|\mathbf{P}^-|^{1/2}|\mathbf{R}|^{1/2}} = \frac{1}{|\mathbf{P}^+|^{1/2}}$$

To show this, we exploit three basic properties of determinants:

(1) if $\mathbf{A}$ and $\mathbf{B}$ are n-by-n, then $|\mathbf{A}\mathbf{B}| = |\mathbf{A}||\mathbf{B}|$,

(2) $|\mathbf{A}| = |\mathbf{A}^T|$,

(3)

$$\left|\left[\begin{array}{c|c} \mathbf{A}_1 & \mathbf{A}_2 \\ \hline \mathbf{0} & \mathbf{A}_3 \end{array}\right]\right| = |\mathbf{A}_1||\mathbf{A}_3|$$

(a) Show that

$$\mathbf{P}^* = \left[\begin{array}{c|c} \mathbf{P}^- & \mathbf{P}^-\mathbf{H}^T \\ \hline \mathbf{H}\mathbf{P}^- & \mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{P}^+ & \mathbf{P}^-\mathbf{H}^T \\ \hline \mathbf{0} & \mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R} \end{array}\right]\left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{K}^T & \mathbf{I} \end{array}\right]$$

so that $|\mathbf{P}^*| = |\mathbf{P}^+||\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R}|$.

(b) Show that the following matrix inversion of a partitioned, symmetric, positive definite matrix is valid

$$\left[\begin{array}{c|c} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \hline \mathbf{X}_{12}^T & \mathbf{X}_{22} \end{array}\right]^{-1} = \left[\begin{array}{c|c} \mathbf{I} & -\mathbf{X}_{11}^{-1}\mathbf{X}_{12} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right]\left[\begin{array}{c|c} \mathbf{X}_{11}^{-1} & \mathbf{0} \\ \hline \mathbf{0} & (\mathbf{X}_{22} - \mathbf{X}_{12}^T\mathbf{X}_{11}^{-1}\mathbf{X}_{12})^{-1} \end{array}\right]\left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline -\mathbf{X}_{12}^T\mathbf{X}_{11}^{-1} & \mathbf{I} \end{array}\right]$$

and use this to demonstrate that $|\mathbf{P}^*| = |\mathbf{P}^-||\mathbf{R}|$.

(c) Combine (a) and (b) to establish the result.

5.5 Verify the numerical results in Table 5.1 of Example 5.4.

5.6 Show that $\mathbf{x}(t_i)$ and $\hat{\mathbf{x}}(t_i^+)$ are jointly Gaussian, as claimed in Section 5.4. Also show that

$$E\{\mathbf{x}(t_i)\hat{\mathbf{x}}^T(t_i^+)|\mathbf{Z}(t_i) = \mathbf{Z}_i\} = E\{\mathbf{x}(t_i)|\mathbf{Z}(t_i) = \mathbf{Z}_i\}\hat{\mathbf{x}}^T(t_i^+) = \hat{\mathbf{x}}(t_i^+)\hat{\mathbf{x}}^T(t_i^+)$$

as claimed in Eq. (5-55).

5.7 Section 5.5 discussed logical definitions of an "optimal" state estimate other than that used to derive the filter results in this chapter. Under what conditions do some of these alternative approaches yield the same value for the "optimal" estimate for a general estimation problem (i.e., no linear model or Gaussian noise assumptions)?

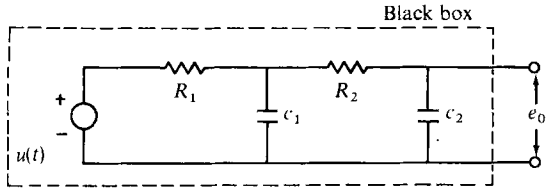

FIG. 5.P1 Circuit for Problem 5.8. $R_1 = R_2 = 1\ \Omega$; $c_1 = c_2 = 1$ F.

5.8 Consider the circuit shown in Fig. 5.P1. It has been constructed and sealed into the proverbial black box. Capacitor c_1 has a very low voltage rating and it is desired to monitor the voltage across c_1 to determine when it exceeds the capacitor limit. The only measurement that can be made on this system is the output voltage, e_0. However, thanks to an exceedingly good voltmeter, essentially perfect measurements can be made of this voltage at discrete times. In order to estimate the voltage across c_1, assume that $\mathsf{u}(t)$ can be described as

$$E\{\mathsf{u}(t)\} = 0, \qquad E\{\mathsf{u}(t_1)\mathsf{u}(t_2)\} = Q\,\delta(t_2 - t_1), \qquad Q = 2\ \mathrm{V}^2\,\mathrm{sec}$$

Determine an expression for the optimal estimate of the voltage across c_1. Assume that the system starts up with no charge in the capacitors. Plot the variance of the error in the estimate as a function of time, taking measurements every half second for two seconds.

Repeat the solution, assuming the voltmeter output to be the true voltage e_0 plus a zero-mean white Gaussian noise $\mathsf{v}(\cdot,\cdot)$ with $E\{\mathsf{v}(t_i)\mathsf{v}(t_j)\} = R\,\delta_{ij}$, $R = 0.2\ \mathrm{V}^2$.

5.9 Suppose the scalar process $\mathsf{y}(\cdot,\cdot)$ satisfied the differential equation

$$\ddot{\mathsf{y}}(t) + \mathsf{y}(t) = 0$$

where $\mathsf{y}(0)$ and $\dot{\mathsf{y}}(0)$ are modeled as jointly Gaussian random variables with

$$E[\mathsf{y}(0)] = 0, \qquad E[\dot{\mathsf{y}}(0)] = 0$$

$$E[\mathsf{y}(0)^2] = 4, \qquad E[\dot{\mathsf{y}}(0)^2] = 2, \qquad E[\mathsf{y}(0)\dot{\mathsf{y}}(0)] = 1$$

A discrete-time measurement process $\mathsf{z}(\cdot,\cdot)$ is available as the output from the system, with

$$\mathsf{z}(t_i) = \mathsf{y}(t_i) + \mathsf{v}(t_i)$$

where $\mathsf{v}(\cdot,\cdot)$ is a white Gaussian sequence, independent of $\mathsf{y}(0)$ and $\dot{\mathsf{y}}(0)$, with

$$E[\mathsf{v}(t_i)] = 0, \qquad E[\mathsf{v}(t_i)^2] = 1$$

Completely determine the optimal discrete-time estimator for $\dot{y}(t_i)$. What does "optimal" mean here?

Use the difference equation for the error covariance matrix (or the inverse of that matrix) to show that 2π sec is a poor choice of sample period.

5.10 Consider the scalar system model

$$\mathsf{x}(t_{i+1}) = \mathsf{x}(t_i) + \mathsf{w}_d(t_i)$$

where $\mathsf{x}(\cdot,\cdot)$ is the state and $\mathsf{w}_d(\cdot,\cdot)$ is a discrete-time Gaussian noise with

$$E[\mathsf{w}_d(t_i)] = 0, \qquad E[\mathsf{w}_d{}^2(t_i)] = \tfrac{1}{2}, \qquad E[\mathsf{w}_d(t_i)\mathsf{w}_d(t_j)] = 0 \qquad (i \neq j)$$

The initial state is modeled as Gaussian with statistics

$$E[\mathsf{x}(t_1)] = 1, \qquad E[\mathsf{x}^2(t_1)] = 2$$

Scalar measurements are available at times t_1 and t_2 as

$$\mathsf{z}(t_i) = \mathsf{x}(t_i) + \mathsf{v}(t_i) \qquad (i = 1, 2)$$

where $\mathsf{v}(\cdot, \cdot)$ is a Gaussian sequence with

$$E[\mathsf{v}(t_i)] = 0, \qquad E[\mathsf{v}^2(t_i)] = \tfrac{1}{4}, \qquad E[\mathsf{v}(t_i)\mathsf{v}(t_j)] = 0 \qquad (i \neq j)$$

Determine the explicit equations for the optimal estimate of x at time t_1 and t_2: let $\mathsf{z}(t_1, \omega_j) = z_1$, and $\mathsf{z}(t_2, \omega_j) = z_2$, and obtain explicit equations for the estimates $\hat{x}(t_i^-)$ and $\hat{x}(t_i^+)$, the estimate error variances $P(t_i^-)$ and $P(t_i^+)$, and optimal gain $K(t_i)$ for times t_1 and t_2. What is the value of the "information" added to $P^{-1}(t_i^-)$ by the measurements at times t_1 and t_2?

5.11 Suppose you have a system described by the relation

$$\mathsf{x}(t_i) = 0.7\mathsf{x}(t_{i-1}) + \mathsf{w}_d(t_{i-1}), \qquad t_i = 1, 2, 3 \ldots$$

starting from some known value $x(t_0 = 0) = x_0$, where the $\mathsf{w}_d(\cdot, \cdot)$ is a white Gaussian sequence described by statistics

$$E\{\mathsf{w}_d(t_i)\} = b = 0.2, \qquad E\{[\mathsf{w}_d(t_i) - b]^2\} = 0.01$$

Further suppose that at each sample time t_i, two separate measurements are available:

$$\mathsf{z}_1(t_i) = 2\mathsf{x}(t_i) + \mathsf{v}_1(t_i), \qquad \mathsf{z}_2(t_i) = \mathsf{x}(t_i)\sin t_i + \mathsf{v}_2(t_i)$$

where the sequences of $\mathsf{v}_1(\cdot, \cdot)$ and $\mathsf{v}_2(\cdot, \cdot)$ are independent white Gaussian sequences, each independent of $\mathsf{w}_d(\cdot, \cdot)$ with statistics

$$E\{\mathsf{v}_1(t_i)\} = 0, \qquad E\{\mathsf{v}_1^2(t_i)\} = 1$$
$$E\{\mathsf{v}_2(t_i)\} = 0, \qquad E\{\mathsf{v}_2^2(t_i)\} = \cos^2(t_i)$$

(a) Suppose you have an optimal estimate of the state at some time t_{i-1}, based upon the measurement history up through time t_{i-1}, and this estimate is $\hat{x}(t_{i-1}^+)$ with associated error variance $P(t_{i-1}^+)$. Write the equations for propagating the optimal estimate and error variance to the next sample time before the next measurement is taken, i.e., to obtain $\hat{x}(t_i^-)$ and $P(t_i^-)$. Explain your logic fully. If $\hat{x}(t_{i-1}^+) = 4$ and $P(t_{i-1}^+) = 1$, what are $\hat{x}(t_i^-)$ and $P(t_i^-)$?

(b) Is there redundancy in the phrase "independent white" Gaussian sequences in the problem statement?

(c) Let the measured values at t_i be

$$\mathsf{z}_1(t_i, \omega_j) = z_{i1} = 3, \qquad \mathsf{z}_2(t_i, \omega_j) = z_{i2} = 1$$

Show explicitly that the same $\hat{x}(t_i^+)$ and $P(t_i^+)$ are obtained by recursively updating the estimate with $\mathsf{z}_1(t_i)$ and then $\mathsf{z}_2(t_i)$ and by "batch processing" $\mathsf{z}_1(t_i)$ and $\mathsf{z}_2(t_i)$ simultaneously by defining a vector

$$\mathsf{z}(t_i) = \begin{bmatrix} \mathsf{z}_1(t_i) \\ \mathsf{z}_2(t_i) \end{bmatrix}$$

and updating. Think before brute force evaluations—it can save significant time and algebra in obtaining the "batch process" result.

(d) In part (c), a covariance matrix $\mathbf{R}(t_i)$ associated with the measurement noise was generated. What property of this matrix was critical to the equality of the two processing methods?

5.12 The performance function for the altitude hold mode of an airplane autopilot is given by the transfer function

$$\frac{h(s)}{h_c(s)} = \frac{0.3(s + 0.01)}{(s^2 + 0.006s + 0.003)}$$

where h represents altitude and h_c is commanded altitude. The altitude command h_c is modeled as a constant h_{c_0} plus white Gaussian noise $\delta h_c(t)$ in the command channel

$$h_c(t) = h_{c_0} + \delta h_c(t)$$

The constant h_{c_0} is modeled as a normal random variable with statistics

$$\text{Mean} = 10{,}000 \text{ ft}, \qquad \text{Variance} = 250{,}000 \text{ ft}^2$$

Noise in the command channel has the following statistics:

$$E[\delta h_c(t)] = 0, \qquad E[\delta h_c(t)\delta h_c(t+\tau)] = N_c\delta(\tau), \qquad N_c = 400 \text{ ft}^2 \text{ sec}$$

and $\delta h_c(\cdot,\cdot)$ is independent of all other processes.

Continuous measurements of altitude are available and we wish to process them to obtain the minimum variance estimate of altitude. The altitude measurements contain white noise, so the model is

$$h_m(t) = h(t) + \delta_m(t)$$

where $h_m(t)$ is measured altitude and $\delta_m(t)$ is independent white noise:

$$E[\delta_m(t)] = 0, \qquad E[\delta_m(t)\,\delta_m(t+\tau)] = N_m\,\delta(\tau). \qquad N_m = 900 \text{ ft}^2 \text{ sec}$$

(a) Determine the differential equations defining the minimum variance estimator of $h(t)$. Write these equations out in scalar form. Explain how you would determine the coefficients of these equations.

(b) Repeat, but use discrete-time measurements every second, with zero-mean white Gaussian noise of strength 900 ft^2 corrupting the measurements.

5.13 A radiometric area correlation guidance (RACG) system establishes a "position measurement" by taking a radiometric "picture" of the terrain directly below a vehicle and comparing (correlating) this with a prestored coordinatized map of the terrain. For simplicity, assume that one-dimensional position time propagation can be described by

$$\dot{x}(t_{i+1}) = x(t_i) + u\Delta t + w_d(t_i)$$

with u being a nominal vehicle velocity, $\Delta t = (t_{i+1} - t_i) = \text{constant}$, $w_d(\cdot,\cdot)$ being zero-mean white Gaussian noise with $E\{w_d^2(t_i)\} = Q_d = \text{constant}$, and $x(t_0)$ Gaussian with mean $\hat{x}_0$ and variance P_0. Assume that the "position measurement" at time t_i is well modeled as

$$z(t_i) = x(t_i) + v(t_i)$$

with $v(\cdot,\cdot)$ zero-mean white Gaussian noise with $E\{v^2(t_i)\} = R = \text{constant}$. Assume $x(t_0)$, $w_d(\cdot,\cdot)$, and $v(\cdot,\cdot)$ are independent of each other.

Let $\Delta t = 1$ min, and assume you want to minimize the rms error in position estimate at the end of a 10-min flight. Measurements are admissible at $t_i = 0, 1, \ldots, 9$ (not at $t_i = 10$), but because prestored maps consume significant computer memory, only two maps and thus only two measurements can actually be taken. The question is, where in the flight should they be scheduled?

(a) Let $P_0 = Q_d = R = (100 \text{ ft})^2$ and solve for the optimal rms terminal position error.

(b) Assume that if the rms position error at time of measurement update should exceed 250 ft, there is an unacceptably large probability that the true vehicle position is beyond the boundaries of the prestored map, precluding a valid position measurement at all. Solve the problem again under this additional constraint.

(c) Solve part (b) for $P_0 = (300 \text{ ft})^2$, $Q_d = R = (100 \text{ ft})^2$.

(d) Solve part (b) for $Q_d = (300 \text{ ft})^2$, $P_0 = R = (100 \text{ ft})^2$.

(e) Solve part (b) for $R = (300 \text{ ft})^2$, $P_0 = Q_d = (100 \text{ ft})^2$.

5.14 Show that the inverse covariance form estimator for the case of $\mathbf{Q}_d(t_i) \equiv \mathbf{0}$ for all t_i and $\mathbf{P}_0^{-1} = \mathbf{0}$ reduces to the classical solution for the linear, unbiased, minimum variance estimate of $x(t_i)$ [assuming $\mathbf{v}(\cdot,\cdot)$ is white but not necessarily Gaussian] given by the Gauss–Markov theorem:

$$\hat{\mathbf{x}}_{G-M}(t_i) = \mathscr{I}^{-1}(t_i, t_1) \sum_{j=1}^{i} \mathbf{\Phi}^T(t_j, t_i)\mathbf{H}^T(t_j)\mathbf{R}^{-1}(t_j)$$
$$\cdot \left\{ \mathbf{z}_j + \mathbf{H}(t_j) \sum_{k=j+1}^{i} \mathbf{\Phi}(t_j, t_k)\mathbf{B}(t_{k-1})\mathbf{u}(t_{k-1}) \right\}$$

5.15 Show that the optimal *prediction* of $\mathbf{x}(t_j)$ based on measurements through time $t_i < t_j$, $E\{\mathbf{x}(t_j)|\mathbf{Z}(t_i) = \mathbf{Z}_i\}$, can be evaluated by means of

$$\hat{\mathbf{x}}(t_j|t_i) = \mathbf{\Phi}(t_j, t_i)\hat{\mathbf{x}}(t_i^+) + \int_{t_i}^{t_j} \mathbf{\Phi}(t_j, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau$$

$$\mathbf{P}(t_j|t_i) = \mathbf{\Phi}(t_j, t_i)\mathbf{P}(t_i^+)\mathbf{\Phi}^T(t_j, t_i) + \int_{t_i}^{t_j} \mathbf{\Phi}(t_j, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^T(\tau)\mathbf{\Phi}^T(t_j, \tau)\, d\tau$$

Show that this can be generated recursively by iterating only the time propagation relations of a Kalman filter, without measurement updates, $(j - i)$ times from the initial conditions $\hat{\mathbf{x}}(t_i^+)$ and $\mathbf{P}(t_i^+)$.

One practical use of this idea is to partition a sample period Δt into N subintervals to maintain accuracy in numerical integration. Then the filter iteration period is $(\Delta t/N)$, and a measurement update is computed only every N propagations, generating optimal predictions at each intermediate point. Similarly, if one measurement becomes available every $N\,\Delta t$ sec and another every $M\,\Delta t$ sec, with M and N unequal integers, the filter iteration period could be set at a constant Δt sec, providing optimal state predictions at points where neither measurement becomes available.

5.16 Show that the Joseph form covariance measurement update for scalar measurements can be expressed equivalently in the computationally efficient manner:

$$\mathbf{a} = \mathbf{P}(t_i^-)\mathbf{H}^T(t_i), \qquad \mathbf{P}_1 = \mathbf{P}(t_i^-) - \mathbf{K}(t_i)\mathbf{a}^T$$

$$\mathbf{b} = \mathbf{P}_1\mathbf{H}^T(t_i) - \mathbf{K}(t_i)R(t_i), \qquad \mathbf{P}(t_i^+) = \mathbf{P}_1 - \mathbf{b}\mathbf{K}^T(t_i)$$

5.17 Let a signal of interest be the output of a first order lag driven by white Gaussian noise $\mathsf{w}_1(\cdot,\cdot)$, and let that signal be corrupted by exponentially time-correlated noise $\mathsf{n}(\cdot,\cdot)$, modeled as the output of a first order shaping filter driven by white Gaussian noise $\mathsf{w}_2(\cdot,\cdot)$, as depicted in Fig. 5.P2. The system (plant) is described by the transfer function $F_p(s) = 1/(s + \omega_p)$ and the noise shaping filter is described by $F_n(s) = 1/(s + \omega_n)$. Relevant statistics are

$$E\{\mathsf{w}_1(t)\} = 0, \qquad E\{\mathsf{w}_1(t)\mathsf{w}_1(t+\tau)\} = 2\,\delta(\tau)$$
$$E\{\mathsf{w}_2(t)\} = 0, \qquad E\{\mathsf{w}_2(t)\mathsf{w}_2(t+\tau)\} = 1\,\delta(\tau)$$

Assume $\mathsf{w}_1(\cdot,\cdot)$ and $\mathsf{w}_2(\cdot,\cdot)$ are independent and that the appropriate initial conditions are

$$E\{\mathsf{s}(t_0)\} = 0, \qquad E\{\mathsf{n}(t_0)\} = 0 \qquad E\{\mathsf{s}^2(t_0)\} = 1 \qquad E\{\mathsf{n}^2(t_0)\} = \tfrac{1}{2}$$

Determine a recursion equation for the variance of the error in the estimate of the signal using discrete-time measurements $\mathsf{z}(t_i)$. By performing a coordinate transformation, you should be able

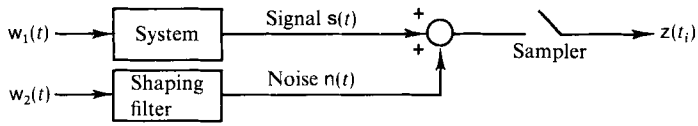

FIG. 5.P2 System schematic for Problem 5.17.

to express the recursion for the appropriate element of $\mathbf{P}^*(t_i^+)$ solely in terms of the value of that same element at time t_{i-1}.

5.18 Consider the first order system modeled by

$$\dot{\mathsf{x}}(t) = \mathsf{w}(t)$$

where $\mathsf{w}(\cdot,\cdot)$ is white Gaussian noise with statistics

$$E[\mathsf{w}(t)] = 0, \qquad E[\mathsf{w}(t)\mathsf{w}(t+\tau)] = 4\,\delta(\tau)$$

Assume that at time $t = 0$, the initial state $\mathsf{x}(0)$ is modeled as a Gaussian random variable with statistics

$$E[\mathsf{x}(0)] = 10, \qquad E[\{\mathsf{x}(0) - 10\}^2] = 25$$

The observed signal is

$$\mathsf{z}(t) = \mathsf{x}(t) + \mathsf{v}(t)$$

where $\mathsf{v}(\cdot,\cdot)$ is white Gaussian noise independent of $\mathsf{w}(\cdot,\cdot)$ with

$$E[\mathsf{v}(t)] = 0, \qquad E[\mathsf{v}(t)\mathsf{v}(t+\tau)] = 16\,\delta(\tau)$$

First determine the exact equations of the Kalman filter to estimate $x(t)$ for all t. Then investigate the steady state behavior of the filter as $t \to \infty$. Show that this steady state behavior is in fact time invariant; determine its transfer function.

5.19 Given the linear system model depicted in Fig. 5.P3, where $\mathsf{w}_1(\cdot,\cdot)$, $\mathsf{w}_2(\cdot,\cdot)$, $\mathsf{v}(\cdot,\cdot)$, $\mathsf{x}_1(0)$, and $\mathsf{x}_2(0)$ are mutually independent, zero mean, and Gaussian, with

$$E\{\mathsf{w}_1(t)\mathsf{w}_1(t+\tau)\} = \delta(\tau), \qquad E\{\mathsf{w}_2(t)\mathsf{w}_2(t+\tau) = \delta(\tau)$$
$$E\{\mathsf{x}_1(0)^2\} = 1, \qquad E\{\mathsf{x}_2(0)^2\} = 2, \qquad E\{\mathsf{v}(t)\mathsf{v}(t+\tau) = 2\delta(\tau)$$

Determine the optimal estimator for $x_3(t)$ using continuous measurements $z(t)$. (Question to answer first: How many state variables are required to describe the system?)

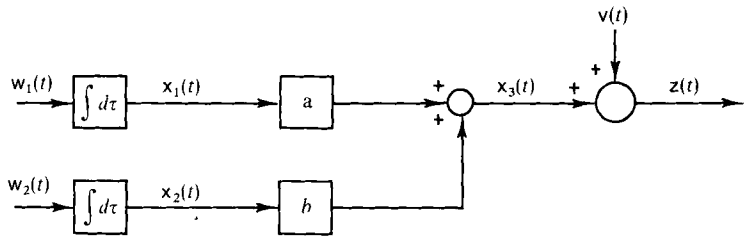

FIG. 5.P3 System model for Problem 5.19.

5.20 Recall the satellite orbit problem discussed in Examples 2.7 and 2.8. The perturbation equations for describing small deviations from an assumed circular orbit were given in Example 2.8. Note that there is no process noise included in these state equations. It is desired to measure these small orbital deviations from observations on the ground. Two proposals are presented:

(1) In an effort to keep the measurement stations rather simple and inexpensive, only angle (x_3) measurements will be made. However, the designer realizes the very likely possibility of measurement errors and includes an optimal filter in his proposal for estimating the states. The measurement may be represented as

$$\mathsf{z}_1(t) = \mathsf{x}_3(t) + \mathsf{v}_1(t)$$

where

$$E[\mathsf{v}_1(t)] = 0, \qquad E[\mathsf{v}_1(t)\mathsf{v}_1(t+\tau)] = R_1\,\delta(\tau)$$

(2) The second design proposes to use measurements of range (x_1) only. In this case,

$$\mathsf{z}_2(t) = \mathsf{x}_1(t) + \mathsf{v}_2(t)$$

where

$$E[\mathsf{v}_2(t)] = 0 \qquad E[\mathsf{v}_2(t)\mathsf{v}_2(t+\tau)] = R_2\,\delta(\tau)$$

It is your task to determine which of these proposals is superior. Are both system models observable? What does that indicate? Is there any benefit to incorporating both z_1 and z_2?

5.21 A linear system has the input/output transfer function

$$\frac{x(s)}{w(s)} = \frac{s+\alpha}{s+\beta}$$

The input $w(t)$ is known to be identically zero. The initial condition on the system, $\mathsf{x}(0)$, is modeled as a Gaussian random variable with

$$E\{\mathsf{x}(0)\} = 0, \qquad E\{\mathsf{x}(0)^2\} = 1$$

Continuous measurements of the form

$$\mathsf{z}(t) = \mathsf{x}(t) + \mathsf{v}(t)$$

are available with $\mathsf{v}(\cdot,\cdot)$ a white Gaussian noise with

$$E\{\mathsf{v}(t)\} = 0, \qquad E\{\mathsf{v}(t)\mathsf{v}(t+\tau)\} = \delta(\tau)$$

Determine the optimal estimator of $x(t)$ and the associated error variance explicitly for all $t \geq 0$.

To obtain an explicit evaluation of the variance as a function of time, the following fact can be useful. If the positive definite matrix $\mathbf{M}$ satisfies the differential equation

$$\dot{\mathbf{M}} = \mathbf{AM} + \mathbf{MA}^{\mathrm{T}} - \mathbf{MBM}$$

then $\mathbf{M}^{-1}$ satisfies the linear equation

$$\dot{\mathbf{M}}^{-1} = -\mathbf{A}^{\mathrm{T}}\mathbf{M}^{-1} - \mathbf{M}^{-1}\mathbf{A} + \mathbf{B}$$

5.22 Consider the unstable first order system

$$\dot{\mathsf{x}}(t) = \mathsf{x}(t) + \mathsf{w}(t)$$

with measurements

$$\mathsf{z}(t) = \mathsf{x}(t) + \mathsf{v}(t)$$

where

$$E\{\mathsf{w}(t)\} = 0, \qquad E\{\mathsf{w}(t)\mathsf{w}(t+\tau)\} = Q\,\delta(\tau)$$
$$E\{\mathsf{v}(t)\} = 0, \qquad E\{\mathsf{v}(t)\mathsf{v}(t+\tau)\} = R\,\delta(\tau)$$

Let $Q = 1$.

(a) Solve for the error variance, $P(t)$, as a function of time, assuming $P(t=0) = P_0$. Using your expression for $P(t)$, evaluate $\lim_{t\to\infty} P(t)$. For values of $R = 1, 2$, and 4, sketch the curve $P(t)$ as a function of time. (Use $P_0 = 1$.)

(b) Consider the homogeneous part of the filter which is given by the differential equation

$$\dot{y} = [1 - P/R]y$$

Using the Lyapunov function $V(\cdot,\cdot)$ defined by $V(y,t) = y^{\mathrm{T}}P^{-1}(t)y$, the system can be shown to be asymptotically stable. Why is this significant?

5.23 A continuous measurement process, $\mathsf{z}(\cdot,\cdot)$ is given as

$$\mathsf{z}(t) = at + \mathsf{n}(t)$$

where a is modeled as a Gaussian random variable with

$$E[\mathsf{a}] = 0, \qquad E[\mathsf{a}^2] = 1$$

and $\mathsf{n}(\cdot,\cdot)$ is a white Gaussian noise process with

$$E[\mathsf{n}(t)] = 0, \qquad E[\mathsf{n}(t)\mathsf{n}(t+\tau)] = 2\,\delta(\tau)$$

Obtain the optimal filter for estimating a. Is the filter a stable system?

5.24 In Example 5.16 of Section 5.12, it was stated that for small signal-to-noise ratio, $c \cong a$ and the low frequency gain is about $\frac{1}{2}[A/(\Phi_0 a^2)]$. Show this.

5.25 Show that Eq. (5-171) is valid.

5.26 Consider a system described by $F_p(s) = 1/s$ driven by white Gaussian noise $\mathsf{w}_1(\cdot,\cdot)$ whose output is corrupted by exponentially time-correlated noise described as the output of the noise shaping filter $F_n(s) = 1/(s + a)$ driven by white Gaussian noise $\mathsf{w}_2(\cdot,\cdot)$. See Fig. 5.P4. Noise statistics of the uncorrelated w_1 and w_2:

$$E[\mathsf{w}_1(t)] = 0, \qquad E[\mathsf{w}_1(t)\mathsf{w}_1(t+\tau)] = Q_1\,\delta(\tau)$$
$$E[\mathsf{w}_2(t)] = 0, \qquad E[\mathsf{w}_2(t)\mathsf{w}_2(t+\tau)] = Q_2\,\delta(\tau)$$

nitial condition statistics:

$$E\{\mathsf{x}_1(t_0)\} = 0, \qquad E\{\mathsf{x}_2(t_0)\} = 0$$
$$E[\mathsf{x}_1(t_0)^2] = \sigma_1^2, \qquad E[\mathsf{x}_2(t_0)^2] = \sigma_2^2, \qquad E[\mathsf{x}_1(t_0)\mathsf{x}_2(t_0)] = 0$$
$$E[\mathsf{z}(t_0)^2] = \sigma_z^2, \qquad E[\mathsf{z}(t_0)\mathsf{x}_1(t_0)] = \sigma_1^2$$

Determine the optimal continuous-time estimate for the state variable x_1. Observe the behavior as $t \to \infty$. Show that it can be described by a time-invariant system with transfer function

$$\frac{\hat{x}_1(s)}{z(s)} = \frac{c_1(s+a)}{s + ac_1} \qquad c_1 = \sqrt{\frac{Q_1}{Q_1 + Q_2}}$$

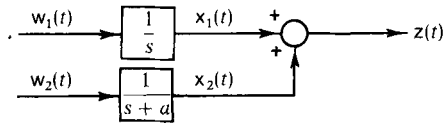

FIG. 5.P4 System configuration for Problem 5.26.

5.27 Show that the result of the previous problem is identical to the result of a Wiener filter design for the steady state problem.

5.28 Design an optimum linear filter (Wiener filter) to separate noise $\mathsf{n}(\cdot,\cdot)$ from a signal $\mathsf{s}(\cdot,\cdot)$ when these processes are uncorrelated with each other and

$$\Phi_{ss}(\omega) = 1/(\omega^2 + 1), \qquad \Phi_{nn}(\omega) = 2\omega^2/(\omega^4 + 1)$$

5.29 (a) An engineer presents you with a single input/single output black box that he says contains a steady state Kalman filter. To test the performance of the filter you subject it to a time-correlated noise with autocorrelation function

$$\Psi_{nn}(\tau) = E\{n(t)n(t+\tau)\} = Ne^{-5|\tau|}$$

and obtain an output whose power spectral density is

$$\Psi_{yy}(\omega) = \frac{\frac{5}{2}N(\omega^2 + 16)}{(\omega^2 + 4)(\omega^2 + 25)}$$

What do you think of the engineer's competence?

(b) Somewhat perplexed, you go back and ask him how he designed the filter. He tells you that he was faced with design of a filter to separate a signal from a signal-plus-noise input, where the signal power spectral density $\Psi_{ss}(\omega)$ and noise power spectral density $\Psi_{nn}(\omega)$ could be approximated as

$$\Psi_{ss}(\omega) = \frac{5/12}{\omega^2 + 4}, \qquad \Psi_{nn}(\omega) = \frac{7/12}{\omega^2 + 16}$$

and the signal and noise are not correlated with each other. Calculate the Wiener optimum filter, and compare your answer to his, and offer him any appropriate constructive criticism.

5.30 Show that the discrete-time Kalman filter algorithm of Table 5.2 can also be expressed in the following form ("innovations form") [32, 33]:

$$\hat{\mathbf{x}}(t_{i+1}^-) = \mathbf{\Phi}(t_{i+1}, t_i)\hat{\mathbf{x}}(t_i^-) + \mathbf{B}_d(t_i)\mathbf{u}(t_i) + \mathscr{K}(t_i)\boldsymbol{\nu}(t_i)$$
$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-) + \mathbf{\Sigma}(t_i)\boldsymbol{\nu}(t_i)$$

where

$$\mathbf{\Sigma}(t_i) = [\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{R}(t_i)]^{1/2}$$
$$\boldsymbol{\nu}(t_i) = \mathbf{\Sigma}^{-1}(t_i)[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]$$
$$\mathscr{K}(t_i) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i)\mathbf{\Sigma}^{-T}(t_i)$$
$$\mathbf{P}(t_{i+1}^-) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{P}(t_i^-)\mathbf{\Phi}^T(t_{i+1}, t_i) + \mathbf{G}_d(t_i)\mathbf{Q}_d(t_i)\mathbf{G}_d^T(t_i) - \mathscr{K}(t_i)\mathscr{K}^T(t_i)$$

Note that the square root matrix used in the preceding is defined such that $\mathbf{A}^{1/2}(\mathbf{A}^{1/2})^T = \mathbf{A}$ for a given matrix $\mathbf{A}$. Also show that $\boldsymbol{\nu}(\cdot, \cdot)$ is a zero-mean white Gaussian sequence with $E\{\boldsymbol{\nu}(t_i)\boldsymbol{\nu}^T(t_j)\} = \mathbf{I}\delta_{ij}$.