CHAPTER 2

Deterministic system models

2.1 INTRODUCTION

This chapter reviews the basics of deterministic system modeling, emphasizing time-domain methods. A strong foundation in deterministic models provides invaluable insights into, and motivation for, stochastic models to be developed subsequently. Especially because the ability to generate adequate models for a given application will typically be the critical factor in designing a practical estimation or control algorithm, considerably more detail will be developed herein than might be expected of a review.

Section 2.2 develops continuous-time dynamic models, perhaps the most natural description of most problems of practical interest. Attention progresses from linear, time-invariant, single input–single output systems models through nonlinear state models, exploiting as many analytical tools as practical to gain insights. Solutions to the state differential equations in these models are then discussed in Section 2.3. Because estimators and controllers will eventually be implemented digitally in most cases, discrete-time measurement outputs from continuous-time systems are investigated in Section 2.4. Finally, the properties of controllability and observability are discussed in Section 2.5.

2.2 CONTINUOUS-TIME DYNAMIC MODELS

Our basic models of dynamic systems will be in the form of state space representations [1–19]. These time-domain models will be emphasized because they can address a more general class of problems than frequency domain model formulations, but the useful interrelationships between the two forms will also be exploited for those problems in which both are valid. This section will begin with the simplest model forms and generalize to the most general model of a dynamic system.

Let us first restrict our attention to *linear, time-invariant, single input–single output system models*, those which can be adequately described by means of a linear constant-coefficient ordinary nth order differential equation of the form

$$\frac{d^n z(t)}{dt^n} + a_{n-1}\frac{d^{n-1}z(t)}{dt^{n-1}} + \cdots + a_0 z(t) = c_p \frac{d^p u(t)}{dt^p} + \cdots + c_0 u(t) \qquad (2\text{-}1)$$

where $u(t)$ is the system input at time t and $z(t)$ is the corresponding system output. Because of the linear, time-invariant structure, we can take the Laplace transform of Eq. (2-1), and rearrange (letting initial conditions be zero, but this can be generalized) to generate the system *transfer function* $G(s)$ to describe the output:

$$z(s) = G(s)u(s) \qquad (2\text{-}2)$$

$$G(s) = \frac{c_p s^p + c_{p-1}s^{p-1} + \cdots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \qquad (2\text{-}3)$$

The denominator of $G(s)$ reveals that we have an nth order system model, i.e., the homogeneous differential equation is of order n. The dynamic behavior of the system can be described by the *poles* of $G(s)$—the roots of this denominator.

A corresponding *state space representation* of the same system (for $n > p$) would be a first order vector differential equation with associated output relation:

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{b}u(t) \qquad (2\text{-}4)$$

$$z(t) = \mathbf{h}^{\mathrm{T}}\mathbf{x}(t) \qquad (2\text{-}5)$$

Here $\mathbf{x}$ is an n-dimensional state vector (the n dimensions corresponding to the fact that the system is described by nth order dynamics), the overdot denotes time derivative, $\mathbf{F}$ is a constant n-by-n matrix, $\mathbf{b}$ and $\mathbf{h}$ are constant n-dimensional vectors, and $^{\mathrm{T}}$ denotes transpose ($\mathbf{h}^{\mathrm{T}}$ is thus a 1-by-n matrix).

The state vector is a set of n variables, the values of which are sufficient to describe the system behavior completely. To be more precise, the *state* of a system at any time t is a minimum set of values $x_1(t), \ldots, x_n(t)$, which, along with the input to the system for all time τ, $\tau \geq t$, is sufficient to determine the behavior of the system for all $\tau \geq t$. In order to specify the solution to an nth order differential equation completely, we must prescribe n initial conditions at the initial time t_0 and the forcing function for all $t \geq t_0$ of interest: there are n quantities required to establish the system "state" at t_0. But t_0 can be any time of interest, so we can see that n variables are required to establish the state at any given time.

EXAMPLE 2.1 Consider the one-dimensional motion of a point mass on an ideal spring. If we specify the forcing function and the initial conditions on the position and velocity of the mass, we can completely describe the future behavior of the system through the solution of a second order differential equation. In this case, the state vector is two dimensional. ■

It can be shown that the relation between the state space representation given by (2-4) and (2-5) and the transfer function given by (2-3) is

$$G(s) = \mathbf{h}^{\mathrm{T}}[s\mathbf{I} - \mathbf{F}]^{-1}\mathbf{b} \tag{2-6}$$

The matrix $[s\mathbf{I} - \mathbf{F}]^{-1}$ is often given the symbol $\mathbf{\Phi}(s)$ and called the resolvent matrix; it is the Laplace transform of the system state transition matrix, to be discussed in the next section. Equation (2-6) reveals the fact that the poles of the transfer function, the defining parameters of the homogeneous system, are equal to the eigenvalues of the matrix $\mathbf{F}$. Given an nth order transfer function model with no pole-zero cancellations, it is possible to generate an n-dimensional state representation that duplicates its input–output characteristics. It is also possible to generate "nonminimal" state representations, of order greater than n, by adding extraneous or redundant variables to the state vector. However, such representations cannot be both observable and controllable, concepts to be discussed in Section 2.4. Figure 2.1 depicts these equivalent system models.

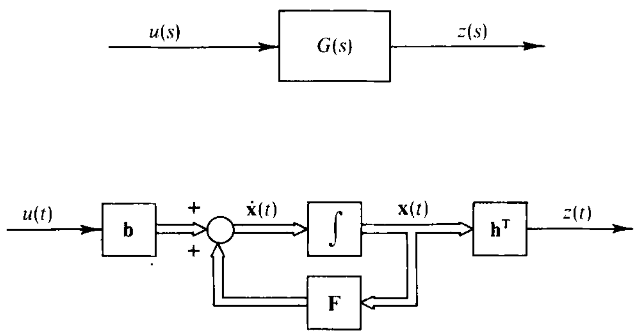

FIG. 2.1 Equivalent system representations. (⇨ denotes vector quantities in this figure.)

However, the state space representation is not unique. A certain set of state variables uniquely determines the system behavior, but there is an infinite number of such sets. There are four major types of state space representations: physical, standard controllable, standard observable, and canonical variables [1–3, 6, 15–17, 19]. Of these, the first two are the most readily generated: in the first, the states are physical quantities and Eqs. (2-4) and (2-5) result from combining the relationships (physical "laws," etc.) among these physical quantities; the second is particularly simple to derive from a previously determined transfer function model. The canonical form decouples the modes of the system dynamics and thereby facilitates both system analysis and numerical solutions to the state differential equation.

The various equivalent state representations can be related through *similarity transformations* (geometrically defining new basis vectors in n-dimensional state space). Given a system described by (2-4) and (2-5):

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{b}u(t); \qquad \mathbf{x}(t_0) = \mathbf{x}_0; \qquad z(t) = \mathbf{h}^{\mathrm{T}}\mathbf{x}(t)$$

we can define a new state vector $\mathbf{x}^*(t)$ through an invertible n-by-n matrix $\mathbf{T}$ as

$$\mathbf{x}(t) = \mathbf{T}\mathbf{x}^*(t) \tag{2-7a}$$

$$\mathbf{x}^*(t) = \mathbf{T}^{-1}\mathbf{x}(t) \tag{2-7b}$$

Substituting (2-7) into (2-4) yields

$$\mathbf{T}\dot{\mathbf{x}}^*(t) = \mathbf{F}\mathbf{T}\mathbf{x}^*(t) + \mathbf{b}u(t)$$

Premultiplying this by $\mathbf{T}^{-1}$, and substituting (2-7a) into (2-5) generates the result as

$$\dot{\mathbf{x}}^*(t) = \mathbf{F}^*\mathbf{x}^*(t) + \mathbf{b}^*u(t); \qquad \mathbf{x}^*(t_0) = \mathbf{x}_0{}^* \tag{2-8}$$

$$z(t) = \mathbf{h}^{*\mathrm{T}}\mathbf{x}^*(t) \tag{2-9}$$

where

$$\mathbf{F}^* = \mathbf{T}^{-1}\mathbf{F}\mathbf{T} \tag{2-10a}$$

$$\mathbf{b}^* = \mathbf{T}^{-1}\mathbf{b} \tag{2-10b}$$

$$\mathbf{h}^{*\mathrm{T}} = \mathbf{h}^{\mathrm{T}}\mathbf{T} \tag{2-10c}$$

Under a similarity transformation such as (2-10a), the eigenvalues, determinant, trace, and characteristic polynomial are all invariant. Thus, since the eigenvalues, i.e., system poles, remain unchanged, the transformation has not altered the dynamics of the system representation.

Physical variables are desirable for applications where feedback is needed, since they are often directly measurable. However, there is no uniformity in the structures of $\mathbf{F}$, $\mathbf{b}$, and $\mathbf{h}$, and little more can be said about this form without more knowledge of the specific dynamic system.

The *standard controllable form* can be generated directly from a transfer function or differential equation. Given either (2-1) or (2-3), one can write

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0\cdots 0 & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t); \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2-11}$$

$$z(t) = [c_0 \quad c_1 \quad \cdots \quad c_p \quad 0 \quad \cdots \quad 0]\mathbf{x}(t) \tag{2-12}$$

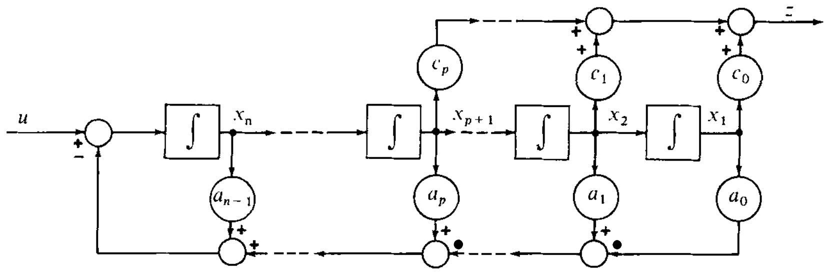

FIG. 2.2 Standard controllable form.

Figure 2.2 presents a block diagram of the standard controllable form state model.

EXAMPLE 2.2 Let a system be described by the transfer function

$$G(s) = \frac{\tau s + 1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Here we identify $a_0 = \omega_n^2$, $a_1 = 2\zeta\omega_n$, $c_0 = 1$, $c_1 = \tau$, to yield

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$z(t) = \begin{bmatrix} 1 & \tau \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

which can be portrayed by the block diagram in Fig. 2.3. Note that the state variables are the outputs of integrators, so that the corresponding inputs directly represent the differential equations. For instance, the output of the left integrator is $x_2(t)$, so the input is

$$\dot{x}_2(t) = -\omega_n^2 x_1(t) - 2\zeta\omega_n x_2(t) + u(t) \qquad \blacksquare$$

The standard controllable form derives its name from the fact that if a non-minimal representation is put into a form of this type, it will be a controllable, but not observable, representation (see Section 2.4).

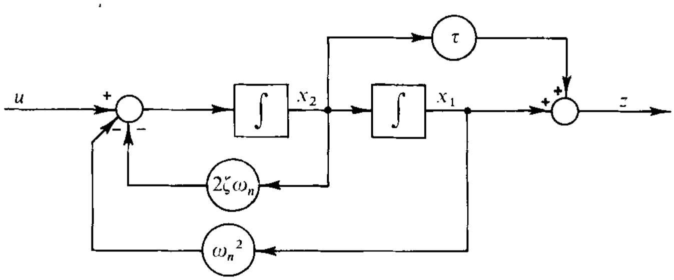

FIG. 2.3 Standard controllable form for Example 2.2.

The *standard observable form* derives its name analogously, and for minimal representations it is described by the same **F** matrix as in the standard controllable form, but has different forms for **b** and **h**:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u(t); \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2-13}$$

$$z(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{x}(t) \tag{2-14}$$

Again **F** is derived by inspection from either the differential equation or transfer function, and the b_i's are obtained by long division to generate the Laurent series for $G(s)$:

$$G(s) = b_1 s^{-1} + b_2 s^{-2} + \cdots + b_n s^{-n} + \cdots \tag{2-15}$$

Figure 2.4 depicts the block diagram for this form.

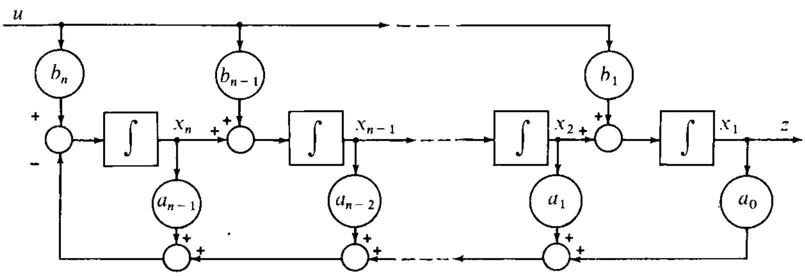

FIG. 2.4 Standard observable form.

EXAMPLE 2.3 Consider the same transfer function as in Example 2.2:

$$G(s) = \frac{\tau s + 1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Again $a_0 = \omega_n^2$ and $a_1 = 2\zeta\omega_n$. The b_i values are derived from

$$\begin{array}{r|l} & \tau s^{-1} + (1 - 2\zeta\omega_n\tau)s^{-2} + \cdots \\ \hline s^2 + 2\zeta\omega_n s + \omega_n^2 & \tau s + 1 \\ & \tau s + 2\zeta\omega_n\tau + \omega_n^2\tau s^{-1} \end{array}$$

so that $b_1 = \tau$, $b_2 = (1 - 2\zeta\omega_n\tau)$, and the desired result is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n 2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \tau \\ (1 - 2\zeta\omega_n\tau) \end{bmatrix} u(t)$$

$$z(t) = x_1(t)$$

The associated block diagram is given in Fig. 2.5. ■

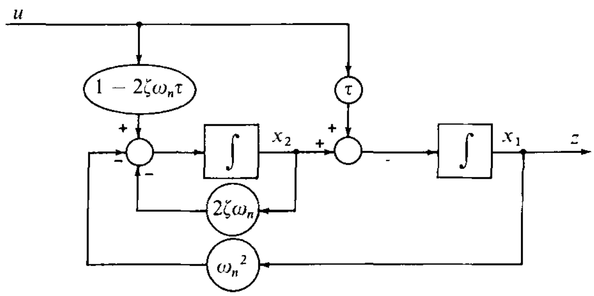

FIG. 2.5 Standard observable form for Example 2.3.

Canonical form provides decoupled system modes; the $\mathbf{F}$ matrix in this representation is a diagonal matrix whose entries are the eigenvalues of the system, if these eigenvalues are distinct:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t); \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2-16}$$

$$z(t) = [c_1 \quad c_2 \quad \cdots \quad c_n]\mathbf{x}(t) \tag{2-17}$$

The block diagram is portrayed in Fig. 2.6, from which the separation of system modes is evident.

To obtain this form from a transfer function, the n roots (eigenvalues) of the characteristic polynomial are determined, and $G(s)$ written in terms of the

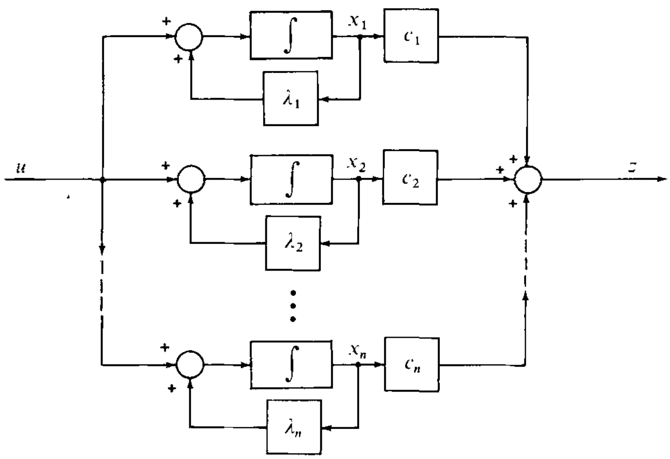

FIG. 2.6 Canonical form state model.

partial fraction expansion

$$G(s) = \frac{r_1}{s - \lambda_1} + \frac{r_2}{s - \lambda_2} + \cdots + \frac{r_n}{s - \lambda_n} \tag{2-18}$$

where λ_i is the ith root (eigenvalue), and r_i is the corresponding residue, given by

$$r_i = (s - \lambda_i)G(s)\Big|_{s=\lambda_i} \tag{2-19}$$

If we let $c_i = r_i$ for $i = 1, 2, \ldots, n$, then the canonical form in (2-16) and (2-17) is completely determined.

EXAMPLE 2.4 Consider the transfer function

$$G(s) = \frac{s+8}{s^2 + 8s + 12} = \frac{s+8}{(s+2)(s+6)}$$

This can be written as

$$G(s) = \frac{r_1}{s+2} + \frac{r_2}{s+6}$$

where

$$r_1 = \frac{s+8}{s+6}\Big|_{s=-2} = \frac{3}{2}, \qquad r_2 = \frac{s+8}{s+2}\Big|_{s=-6} = -\frac{1}{2}$$

yielding a canonical form representation of

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$z(t) = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \blacksquare$$

If it is desired to transform any state space representation into canonical variables, first the eigenvalues of the original **F** matrix are determined as solutions to

$$|\lambda\mathbf{I} - \mathbf{F}| = 0 \tag{2-20}$$

where $|\cdot|$ denotes determinant. To evaluate the c_i coefficients in (2-17), one can explicitly evaluate the transformation matrix **T** [explicit knowledge of **T** is also required to transform the initial conditions by $\mathbf{x}^*(t_0) = \mathbf{T}^{-1}\mathbf{x}(t_0)$]. The **F**, **b**, and **h** in the original representation are known; $\mathbf{F}^*$ for the canonical form is the diagonal matrix of eigenvalues, and $\mathbf{b}^*$ is an n-vector of ones. With such knowledge, the similarity transformation relations (2-10a) and (2-10b) can be written as

$$\mathbf{TF}^* = \mathbf{FT} \tag{2-21a}$$

$$\mathbf{Tb}^* = \mathbf{b} \tag{2-21b}$$

and solved simultaneously for $\mathbf{T}$. (These are a set of $n^2 + n$ equations, of which n^2 are independent.) Once the transformation matrix $\mathbf{T}$ is obtained, the desired $\mathbf{h}^{*\mathrm{T}}$ is found from

$$\mathbf{h}^{*\mathrm{T}} = \mathbf{h}^{\mathrm{T}}\mathbf{T} \tag{2-22}$$

Other means are possible, such as $\mathbf{T}$ being generated by arraying n eigenvectors in an n-by-n matrix, or by letting $\mathbf{T}$ be the Vandermonde matrix and not insist $\mathbf{b}$ be composed of all ones for the case of transforming from standard observable or standard controllable form [5].

EXAMPLE 2.5 Consider the system described in Example 2.4, equivalently modeled by the standard controllable form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$z(t) = [8 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The eigenvalues of $\mathbf{F}$ are the solutions to

$$|\lambda\mathbf{I} - \mathbf{F}| = \begin{vmatrix} \lambda & -1 \\ 12 & \lambda + 8 \end{vmatrix} = \lambda^2 + 8\lambda + 12 = 0$$

[i.e., the poles of $G(s)$, roots of the characteristic polynomial] from which we obtain $\lambda_1 = -2$, $\lambda_2 = -6$. Then (2-21) becomes:

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -8 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving these yields

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Thus, (2-22) gives $\mathbf{h}^{*\mathrm{T}}$ as

$$\mathbf{h}^{*\mathrm{T}} = [8 \quad 1] \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = [\tfrac{3}{2} \quad -\tfrac{1}{2}]$$

Note that these results agree with those determined in Example 2.4. ■

If there are repeated roots, then the canonical representation changes form somewhat. For instance, if the root λ_1 has a multiplicity of 3, then (2-16) becomes

$$\dot{\mathbf{x}}(t) = \left[\begin{array}{ccc|cc} \lambda_1 & 1 & 0 & 0 & \cdots \\ 0 & \lambda_1 & 1 & 0 & \cdots \\ 0 & 0 & \lambda_1 & 0 & \cdots \\ \hline 0 & 0 & 0 & \lambda_2 & \\ \vdots & \vdots & \vdots & & \ddots \end{array}\right] \mathbf{x}(t) + \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 1 \\ \vdots \end{array}\right] u(t); \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2-23}$$

i.e., a *Jordan canonical* **F** with all ones along the superdiagonal of the block with λ_1 as diagonal terms. Those superdiagonal terms must be ones in a minimally dimensioned single input–single output system model; they need not be all ones with multiple inputs or outputs.

If some of the eigenvalues are complex conjugate pairs, then the canonical **F** matrix will have complex entries along its diagonal, and $\mathbf{x}(t)$ will have some complex components. The *modified canonical form* [2, 5] maintains the desirable characteristic of mode separation, while regaining a totally real-valued system description. Given a canonical system model with $j = \sqrt{-1}$ and

$$\mathbf{F} = \left[\begin{array}{cc|cc} (\sigma + j\omega) & 0 & 0 & \cdots \\ 0 & (\sigma - j\omega) & 0 & \cdots \\ \hline 0 & 0 & \lambda_3 & \\ \vdots & \vdots & & \ddots \end{array}\right] \tag{2-24}$$

A similarity transformation described by

$$\mathbf{T} = \left[\begin{array}{cc|cc} 1/2 & -j/2 & 0 & \cdots \\ 1/2 & j/2 & 0 & \cdots \\ \hline 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{array}\right], \qquad \mathbf{T}^{-1} = \left[\begin{array}{cc|cc} 1 & 1 & 0 & \cdots \\ j & -j & 0 & \cdots \\ \hline 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{array}\right] \tag{2-25}$$

yields an equivalent real-valued system description as in (2-8) to (2-10), with

$$\mathbf{F}^* = \left[\begin{array}{cc|cc} \sigma & \omega & 0 & \cdots \\ -\omega & \sigma & 0 & \cdots \\ \hline 0 & 0 & \lambda_3 & \\ \vdots & \vdots & & \ddots \end{array}\right] \tag{2-26}$$

This idea can be generalized to the *modified Jordan canonical form* as well (see Brockett [2]).

Note that the entire previous discussion assumed that $n > p$ in the differential equation (2-1) or transfer function (2-3), i.e., that the order of the denominator of $G(s)$ is greater than the order of the numerator. If $n = p$, then the equivalent state space model is of a generalized form:

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{b}u(t); \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2-27}$$

$$z(t) = \mathbf{h}^{\mathrm{T}}\mathbf{x}(t) + du(t) \tag{2-28}$$

where now the output involves a direct feedthrough of the input u. The corresponding generalization of Eq. (2-6) is

$$G(s) = \mathbf{h}^{\mathrm{T}}[s\mathbf{I} - \mathbf{F}]^{-1}\mathbf{b} + d \tag{2-29}$$

EXAMPLE 2.6 Consider a system described by the "lead-lag" network, $y(s) = G(s)u(s)$, with

$$G(s) = \frac{s+a}{s+b}$$

This is equivalent to

$$G(s) = \frac{s+b+a-b}{s+b} = 1 + \frac{a-b}{s+b}$$

The term $(a-b)/(s+b)$ can be represented equivalently in standard controllable form

$$\dot{x}(t) = -bx(t) + u(t)$$
$$z'(t) = [a-b]x(t)$$

and then

$$z(t) = z'(t) + u(t) = [a-b]x(t) + u(t)$$

Figure 2.7 presents two equivalent block diagrams for this representation. It is obvious that (a) represents the equations just written. Diagram (b) obeys the same state differential equation, and then

$$z(t) = ax(t) + \dot{x}(t) = ax(t) - bx(t) + u(t) = [a-b]x(t) + u(t) \quad \blacksquare$$

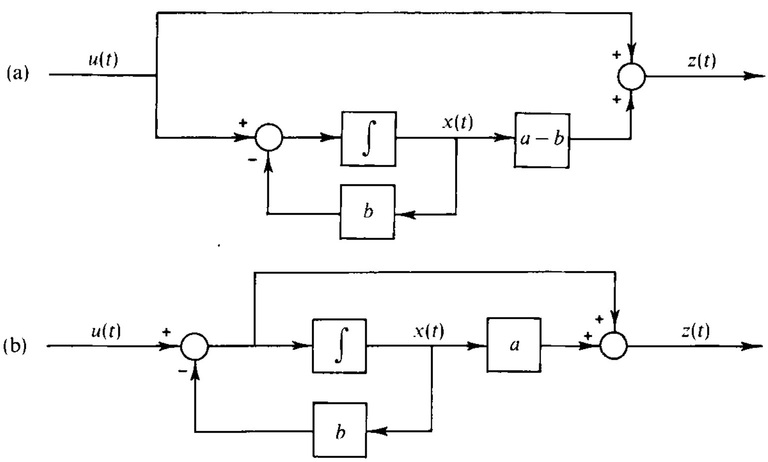

FIG. 2.7 State space models of $G(s) = (s+a)/(s+b)$.

State space representations are readily extended to *multiple input–multiple output* systems. Assume that there are r inputs into and m outputs from a system, described by r-dimensional $\mathbf{u}(t)$ and m-dimensional $\mathbf{z}(t)$, respectively. Then the state space model of such a time-invariant system would be

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t); \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2-30}$$

$$\mathbf{z}(t) = \mathbf{H}\mathbf{x}(t) \tag{2-31}$$

where $\mathbf{F}$ is the unaltered n-by-n matrix describing the homogeneous system dynamics, $\mathbf{B}$ is an n-by-r input matrix, and $\mathbf{H}$ is an m-by-n output matrix. Note the convention on $\mathbf{H}$: if the output is scalar, then $\mathbf{H}$ is a 1-by-n matrix, but this can be viewed as the transpose of an n-dimensional column vector, hence the previous notation $\mathbf{h}^{\mathrm{T}}$. Analogous to the previous discussion, the output equation can be generalized to

$$\mathbf{z}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{2-32}$$

but this direct feedthrough structure is often not required.

An equivalent time-invariant multiple input–multiple output model can be developed in the form of a matrix transfer function, whose entries would be the transfer functions of the individual components of the input and output vectors:

$$\mathbf{z}(s) = \mathbf{G}(s)\mathbf{u}(s) \tag{2-33}$$

$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1r}(s) \\ \vdots & & \vdots \\ G_{m1}(s) & \cdots & G_{mr}(s) \end{bmatrix} = \mathbf{H}[s\mathbf{I} - \mathbf{F}]^{-1}\mathbf{B} + \mathbf{D} \tag{2-34}$$

Again, $\mathbf{D}$ is often $\mathbf{0}$. However, this model is usually rather cumbersome, and the state space model is preferable.

Time-varying system models are generated readily through state space methods—the matrices defining the system structure simply vary with time:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t); \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2-35}$$

$$\mathbf{z}(t) = \mathbf{H}(t)\mathbf{x}(t) \tag{2-36}$$

or, generalized to

$$\mathbf{z}(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \tag{2-37}$$

Laplace transform methods are not readily extended to these cases. Such time-varying linear models arise most naturally from perturbations of a nonlinear set of relations about a nominal solution to the original nonlinear equations. This will be discussed further once we establish conditions under which the existence of such a nominal solution can be assumed.

The relations just given serve to define the most general deterministic linear system model. In Chapter 4, these will be extended to the stochastic linear system model of

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t) \tag{2-38}$$

$$\mathbf{z}(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}(t) \tag{2-39}$$

where $\mathbf{w}(t)$ is a dynamic driving noise and $\mathbf{v}(t)$ is a measurement corruption noise.

A *nonlinear state model* of a system can be described through a state differential equation and output relation of

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]; \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2-40}$$

$$\mathbf{z}(t) = \mathbf{h}[\mathbf{x}(t), \mathbf{u}(t), t] \tag{2-41}$$

where $\mathbf{f}[\cdot,\cdot,\cdot]$ is a mapping from $R^n \times R^r \times R^1$ into R^n [given any $\mathbf{x}(t) \in R^n$, $\mathbf{u}(t) \in R^r$, and $t \in R^1$ (= the real line), $\mathbf{f}$ can be evaluated to yield a vector $\dot{\mathbf{x}}(t) \in R^n$] and $\mathbf{h}[\cdot,\cdot,\cdot]$ is a mapping from $R^n \times R^r \times R^1$ into R^m. For time-invariant nonlinear models, $\mathbf{f}$ and $\mathbf{h}$ are not explicit functions of time, and analogous to the previous discussion for linear models, $\mathbf{h}$ may not be an explicit function of $\mathbf{u}(t)$.

EXAMPLE 2.7 A model of a satellite in planar orbit can be established through the approximation of a unit point mass in an inverse square law force field. Let r be the range from the force field center (earth center) to the satellite and θ be the angle between a reference coordinate axis through the field center and the line from the center to the satellite. Assume the satellite can thrust radially with thrust u_r and tangentially with thrust u_t. The motion of the satellite is then governed by a pair of coupled second order equations:

$$\ddot{r}(t) = r(t)\dot{\theta}^2(t) - \frac{G}{r^2(t)} + u_r(t)$$

$$\ddot{\theta}(t) = -\frac{2}{r(t)}\dot{\theta}(t)\dot{r}(t) + \frac{1}{r(t)}u_t(t)$$

These relations can be put into the form of (2-40) by using the states $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \theta$, and $x_4 = \dot{\theta}$ to write

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} f_1[\mathbf{x}(t), \mathbf{u}(t), t] \\ f_2[\mathbf{x}(t), \mathbf{u}(t), t] \\ f_3[\mathbf{x}(t), \mathbf{u}(t), t] \\ f_4[\mathbf{x}(t), \mathbf{u}(t), t] \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_1(t)x_4{}^2(t) - \dfrac{G}{x_1{}^2(t)} + u_r(t) \\ x_4(t) \\ -\dfrac{2}{x_1(t)}x_4(t)x_2(t) + \dfrac{1}{x_1(t)}u_t(t) \end{bmatrix} \blacksquare$$

Equations (2-40) and (2-41) are the form of a general deterministic nonlinear state-described system model, and Chapter 11 (Volume 2) will extend them to the stochastic model case. A more general class of system models called *dynamic system models* can be defined (see Desoer [7], etc.), but this level of generality will be adequate for our purposes.

2.3 SOLUTIONS TO STATE DIFFERENTIAL EQUATIONS

In this section, the solution to the state differential equations just described will be presented, starting with the most general case and then considering the simplifications made possible by progressively more restrictive models [4, 13].

To describe some underlying assumptions simply, consider the homogeneous nonlinear differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), t]; \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2-42}$$

First, assume that $\mathbf{f}(\cdot,\cdot)$ is piecewise continuous in its second argument. In other words, for each $\mathbf{x}_i \in R^n$, the mapping $\mathbf{f}(\mathbf{x}_i, \cdot)$ is continuous except possibly at a finite number of points, where left and right limits are well defined. Next, assume that $\mathbf{f}(\cdot,\cdot)$ is Lipschitz in its first argument, which is to say that there exists a piecewise continuous function $k(\cdot)$ such that, for all $t \in [0, \infty)$ and all $\mathbf{x}_1, \mathbf{x}_2 \in R^n$,

$$\|\mathbf{f}(\mathbf{x}_1, t) - \mathbf{f}(\mathbf{x}_2, t)\| < k(t)\|\mathbf{x}_1 - \mathbf{x}_2\| \tag{2-43}$$

where $\|\mathbf{v}\| = \max_i |v_i|$ for $\mathbf{v} \in R^n$. These two assumptions together imply, for *any* function $\boldsymbol{\psi}(\cdot)$ mapping $[t_0, \infty)$ into R^n, that the function that maps t into $\mathbf{f}[\boldsymbol{\psi}(t), t]$ is a piecewise continuous function. Therefore, for any such $\boldsymbol{\psi}(\cdot)$ we can integrate $\mathbf{f}[\boldsymbol{\psi}(t), t]$ with respect to time, and then the function that maps t into $\int_0^t \mathbf{f}[\boldsymbol{\psi}(\tau), \tau]\, d\tau$ is continuous. Moreover, by the fundamental theorem of calculus, its derivative is equal to $\mathbf{f}[\boldsymbol{\psi}(t), t]$ for all $t \in [t_0, \infty)$ except at the possible points of discontinuity.

With this introduction, it is possible to state one form of the *fundamental theorem of differential equations*: consider the differential equation and initial condition

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]; \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2-44}$$

where the function $\mathbf{f}(\cdot,\cdot,\cdot)$ that maps $R^n \times R^r \times [t_0, \infty)$ into R^n is assumed to be

(1) Lipschitz in its first argument,
(2) continuous in its second argument, and
(3) piecewise continuous in its third argument.

Then, for each $\mathbf{x}_0 \in R^n$ and each $t_0 \in [0, \infty)$ and any piecewise continuous r-vector-valued function $\mathbf{u}(\cdot)$, there *exists* a *unique continuous* mapping $\boldsymbol{\phi}(\cdot)$ from $[0, \infty)$ into R^n such that

$$\boldsymbol{\phi}(t_0) = \mathbf{x}_0 \tag{2-45}$$

and

$$\dot{\boldsymbol{\phi}}(t) = \mathbf{f}[\boldsymbol{\phi}(t), \mathbf{u}(t), t] \tag{2-46}$$

for all $t \in [0, \infty)$ except at the possible points of discontinuity. The function $\boldsymbol{\phi}(\cdot)$ is called the *solution* to the differential equation (2-44), and its value depends only on t, t_0, $\mathbf{x}_0$, and the values that $\mathbf{u}$ assumes in the interval $[t_0, t]$.

The proof will not be presented here (see Desoer [7], for example), but is based upon establishing local existence, using successive approximations to demonstrate constructively this existence globally, and then proving uniqueness. This procedure motivates the proof of existence of solutions to nonlinear stochastic differential equations as well, to be discussed in Chapter 11 (Volume 2).

Once a nominal solution to a nonlinear differential equation can be found, perturbations about this nominal solution can be considered. For a given $\mathbf{x}_0$, t_0, and input function $\mathbf{u}_0(\cdot)$, let (2-44) have a solution denoted as $\mathbf{x}_0(\cdot)$ for $t \in [t_0, \infty)$. What happens if the initial condition were perturbed to $(\mathbf{x}_0 + \Delta\mathbf{x}_0)$ and/or the input were perturbed to $[\mathbf{u}_0(\cdot) + \Delta\mathbf{u}(\cdot)]$? If these are "small" perturbations, we would expect the perturbed solution to be $[\mathbf{x}_0(\cdot) + \Delta\mathbf{x}(\cdot)]$ with $\Delta\mathbf{x}(t)$ "small" for all $t \in [t_0, \infty)$. (More precise conditions can be stated; see Desoer and Wong [8]), so that Taylor series about the nominal could be exploited:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] \\ &= \mathbf{f}[\mathbf{x}_0(t), \mathbf{u}_0(t), t] + \mathbf{F}(t)\{\mathbf{x}(t) - \mathbf{x}_0(t)\} + \mathbf{B}(t)\{\mathbf{u}(t) - \mathbf{u}_0(t)\} + \cdots \end{aligned} \tag{2-47}$$

where

$$\mathbf{F}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0(t),\, \mathbf{u}_0(t),\, t} = \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\mathbf{x}_0(t),\, \mathbf{u}_0(t),\, t} \tag{2-48}$$

$$\mathbf{B}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\mathbf{x}_0(t),\, \mathbf{u}_0(t),\, t} = \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial u_1} & \cdots & \dfrac{\partial f_1}{\partial u_r} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial u_1} & \cdots & \dfrac{\partial f_n}{\partial u_r} \end{bmatrix}\right|_{\mathbf{x}_0(t),\, \mathbf{u}_0(t),\, t} \tag{2-49}$$

Since $\dot{\mathbf{x}}_0(t) = \mathbf{f}[\mathbf{x}_0(t), \mathbf{u}_0(t), t]$, (2-47) yields

$$d\{\mathbf{x}(t) - \mathbf{x}_0(t)\}/dt = \mathbf{F}(t)\{\mathbf{x}(t) - \mathbf{x}_0(t)\} + \mathbf{B}(t)\{\mathbf{u}(t) - \mathbf{u}_0(t)\} + \cdots$$

By neglecting the higher order terms, one obtains an approximation to the true differential equation satisfied by $\{\mathbf{x}(t) - \mathbf{x}_0(t)\}$, called the *linearized perturbation equation* or equation of the first variation [2, 7], in the general form of a time-varying linear differential equation:

$$\delta\dot{\mathbf{x}}(t) = \mathbf{F}(t)\,\delta\mathbf{x}(t) + \mathbf{B}(t)\,\delta\mathbf{u}(t) \tag{2-50}$$

where $\delta\mathbf{x}(t) \cong \{\mathbf{x}(t) - \mathbf{x}_0(t)\}$ and $\delta\mathbf{u}(t) = \{\mathbf{u}(t) - \mathbf{u}_0(t)\}$.

EXAMPLE 2.8 Return to the model of satellite motion discussed in Example 2.7. Perturbations about a nominal trajectory can be described approximately by the model

$$\dot{\delta\mathbf{x}}(t) = \mathbf{F}(t)\,\delta\mathbf{x}(t) + \mathbf{B}(t)\,\delta\mathbf{u}(t)$$

where

$$\mathbf{F}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0(t),\,\mathbf{u}_0(t),\,t} = \left.\begin{bmatrix} 0 & 1 & 0 & 0 \\ \left[x_4^2 + \dfrac{2G}{x_1^3}\right] & 0 & 0 & 2x_1x_4 \\ 0 & 0 & 0 & 1 \\ \left[\dfrac{2x_2x_4}{x_1^2} - \dfrac{u_t}{x_1^2}\right] & -\dfrac{2x_4}{x_1} & 0 & -\dfrac{2x_2}{x_1} \end{bmatrix}\right|_{\mathbf{x}_0(t),\,\mathbf{u}_0(t),\,t}$$

$$\mathbf{B}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\mathbf{x}_0(t),\,\mathbf{u}_0(t),\,t} = \left.\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/x_1 \end{bmatrix}\right|_{\mathbf{x}_0(t),\,\mathbf{u}_0(t),\,t}$$

One particular solution admitted by the original nonlinear equation is that of a circular orbit: $r(t) = r_0$, $\dot{r}(t) = 0$, $\theta(t) = \omega t$, $\dot{\theta}(t) = \omega$, $u_t(t) = u_r(t) = 0$, $G = r_0{}^3\omega^2$ for all time. For this nominal, **F** and **B** are in fact time invariant:

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2r_0\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega/r_0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/r_0 \end{bmatrix} \qquad \blacksquare$$

The solution to linear differential equations can be written explicitly. If a proposed solution form satisfies the differential equation and the initial conditions, then it is *the unique solution* because the assumptions of the previous theorem will be met whenever $[\mathbf{B}(\cdot)\mathbf{u}(\cdot)]$ is piecewise continuous.

The solution to the linear time-varying differential equation and initial condition

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t); \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2-51}$$

for $\mathbf{F}(\cdot)$ and $[\mathbf{B}(\cdot)\mathbf{u}(\cdot)]$ piecewise continuous is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau \tag{2-52}$$

where $\mathbf{\Phi}(\cdot,\cdot)$ is the *state transition matrix* defined as the n-by-n matrix that satisfies the differential equation and initial condition

$$d[\mathbf{\Phi}(t, t_0)]/dt = \mathbf{F}(t)\mathbf{\Phi}(t, t_0) \tag{2-53a}$$

$$\mathbf{\Phi}(t_0, t_0) = \mathbf{I} \tag{2-53b}$$

Significant properties of the state transition matrix include:

(1) $\boldsymbol{\Phi}(t, t_0)$ is *uniquely* defined for all t and t_0 in $[0, \infty)$.

(2) The state transition matrix to propagate from any t_1 to t_3 equals the product of the separate transition matrices from t_1 to t_2 and t_2 to t_3 (called the semigroup property):

$$\boldsymbol{\Phi}(t_3, t_1) = \boldsymbol{\Phi}(t_3, t_2)\boldsymbol{\Phi}(t_2, t_1) \tag{2-54}$$

(3) $\boldsymbol{\Phi}(t, t_0)$ is nonsingular (invertible) and

$$\boldsymbol{\Phi}(t, t_0)\boldsymbol{\Phi}(t_0, t) = \boldsymbol{\Phi}(t, t) = \mathbf{I}$$

so that

$$\boldsymbol{\Phi}^{-1}(t, t_0) = \boldsymbol{\Phi}(t_0, t) \tag{2-55}$$

Let us show that the proposed solution, (2-52), does in fact satisfy both the differential equation and initial condition in (2-51):

$$\begin{aligned}\mathbf{x}(t_0) &= \boldsymbol{\Phi}(t_0, t_0)\mathbf{x}_0 + \int_{t_0}^{t_0} \boldsymbol{\Phi}(t_0, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau \\ &= \mathbf{I}\mathbf{x}_0 + \mathbf{0} = \mathbf{x}_0\end{aligned}$$

To demonstrate similar satisfaction of the differential equation will require use of Leibnitz' rule:

$$\frac{d}{dt}\int_{A(t)}^{B(t)} \mathbf{f}(t, \tau)\,d\tau = \int_{A(t)}^{B(t)} \frac{\partial \mathbf{f}(t, \tau)}{\partial t}\,d\tau + \mathbf{f}[t, B(t)]\frac{dB}{dt} - \mathbf{f}[t, A(t)]\frac{dA}{dt} \tag{2-56}$$

Differentiating (2-52) thus yields

$$\begin{aligned}\frac{d}{dt}\mathbf{x}(t) &= \dot{\boldsymbol{\Phi}}(t, t_0)\mathbf{x}_0 + \boldsymbol{\Phi}(t, t)\mathbf{B}(t)\mathbf{u}(t) + \int_{t_0}^{t} \dot{\boldsymbol{\Phi}}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau \\ &= \mathbf{F}(t)\boldsymbol{\Phi}(t, t_0)\mathbf{x}_0 + \mathbf{B}(t)\mathbf{u}(t) + \int_{t_0}^{t} \mathbf{F}(t)\boldsymbol{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau \\ &= \mathbf{F}(t)\left[\boldsymbol{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau\right] + \mathbf{B}(t)\mathbf{u}(t)\end{aligned}$$

But the terms in brackets are just the assumed form for $\mathbf{x}(t)$, so this is the desired solution.

Equation (2-52) is the appropriate solution form for linear time-invariant differential equations as well, but the state transition matrix in this case can be characterized further. If $\mathbf{F}$ is a constant matrix, then the associated $\boldsymbol{\Phi}(t, t_0)$ is not a function of the separate arguments t and t_0 but is a function only of the single parameter $(t - t_0)$. Thus, the general defining relationship for the state transition matrix reduces to

$$d[\boldsymbol{\Phi}(t - t_0)]/dt = \mathbf{F}\boldsymbol{\Phi}(t - t_0); \qquad \boldsymbol{\Phi}(0) = \mathbf{I} \tag{2-57}$$

to which the solution can be expressed as the matrix exponential

$$\mathbf{\Phi}(t, t_0) = \mathbf{\Phi}(t - t_0) = e^{\mathbf{F}(t-t_0)} \tag{2-58a}$$

Another expression for $\mathbf{\Phi}(t - t_0)$ for time-invariant systems can be obtained by taking the Laplace transform of (2-57), letting $t_0 = 0$:

$$\begin{aligned} s\mathbf{\Phi}(s) - \mathbf{\Phi}(t - t_0 = 0) &= \mathbf{F}\mathbf{\Phi}(s) \\ s\mathbf{\Phi}(s) - \mathbf{F}\mathbf{\Phi}(s) &= \mathbf{\Phi}(t - t_0 = 0) \\ [s\mathbf{I} - \mathbf{F}]\mathbf{\Phi}(s) &= \mathbf{I} \\ \mathbf{\Phi}(s) &= [s\mathbf{I} - \mathbf{F}]^{-1} \end{aligned} \tag{2-58b}$$

Thus $\mathbf{\Phi}(t - t_0)$ is the inverse Laplace transform of $[s\mathbf{I} - \mathbf{F}]^{-1}$, the resolvent matrix mentioned previously.

2.4 DISCRETE-TIME MEASUREMENTS

In many applications of estimation or control theory to actual problems, a digital computer performs online computations using data samples from a continuous-time dynamic process, generally (but not necessarily) taken at a fixed sample rate. Consequently, a discrete-time measurement equation will often be more pertinent than the continuous-time output equations already described. If t_i is a measurement sample time, then the measurement data can be represented as the sampled versions of (2-41) in the nonlinear case:

$$\mathbf{z}(t_i) = \mathbf{h}[\mathbf{x}(t_i), \mathbf{u}(t_i), t_i] \tag{2-59}$$

or of (2-36) or (2-37) for the linear case

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) \tag{2-60a}$$

or

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{D}(t_i)\mathbf{u}(t_i) \tag{2-60b}$$

It will be seen later that the computer software for implementing an optimal estimator or controller will embody a discrete-time model that is "equivalent" to the continuous-time model, in the sense that the discrete-time model's values of $\mathbf{x}(t_1), \mathbf{x}(t_2), \ldots$, are identical to those of the continuous-time model at these particular times. From Eq. (2-52), we can write for the linear model case:

$$\mathbf{x}(t_{i+1}) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{x}(t_i) + \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \tag{2-61}$$

Since a digital computer is assumed to apply the control, a very typical form of $\mathbf{u}(\cdot)$ would be a piecewise constant function: a measurement sample would be taken, the information processed, and a control input created and held constant until the following sample time. If we assume

$$\mathbf{u}(t) = \mathbf{u}(t_i) \qquad \text{for all} \quad t \in [t_i, t_{i+1}) \tag{2-62}$$

then (2-61) can be written as

$$\mathbf{x}(t_{i+1}) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{x}(t_i) + \left[\int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{B}(\tau)\, d\tau\right]\mathbf{u}(t_i) \tag{2-63}$$

This and (2-60) are then in the form of a discrete-time difference equation system model

$$\mathbf{x}(i+1) = \mathbf{\Phi}(i+1, i)\mathbf{x}(i) + \mathbf{B}(i)\mathbf{u}(i) \tag{2-64a}$$

$$\mathbf{z}(i) = \mathbf{H}(i)\mathbf{x}(i) + \mathbf{D}(i)\mathbf{u}(i) \tag{2-64b}$$

where i denotes instant, associated here with time t_i. System models of the form of (2-64) sometimes arise naturally from a basic problem application, as well as from "discretizing" a continuous-time model. However, in this case, one is not assured of such properties as nonsingularity of $\mathbf{\Phi}(i + 1, i)$. We will usually be concerned with problems described most fundamentally through differential, as opposed to difference, equations. For these problems, (2-60) and (2-63) will represent the "equivalent discrete-time system model." In the nonlinear case, the discrete-time dynamic model is

$$\mathbf{x}(i+1) = \boldsymbol{\phi}[\mathbf{x}(i), \mathbf{u}(i), i] \tag{2-65}$$

but an explicit "equivalency" relationship as in (2-63) cannot be written in a general context.

2.5 CONTROLLABILITY AND OBSERVABILITY

Observability and controllability [2, 9–12, 14, 16, 18] are properties of a specific state space representation for a system, rather than of the system itself. Thus, certain state space models will be more suitable for estimation or control purposes than others, even though both might accurately portray the input–output characteristics of a system.

Controllability is concerned with the effect of inputs upon states of a system model. A continuous-time system representation is said to be *completely controllable* if, for any vectors $\mathbf{x}_0, \mathbf{x}_1 \in R^n$ and any time t_0, there exists a piecewise continuous control function $\mathbf{u}(\cdot)$ such that the solution of the describing differential equation with $\mathbf{x}(t_0) = \mathbf{x}_0$ satisfies $\mathbf{x}(t_1) = \mathbf{x}_1$ for some finite t_1. In other words, a system model is completely controllable if any and all initial state variables $x_i(t_0) = x_{0i}$ can be transferred to any final state x_{1i} in finite time by applying a control $\mathbf{u}(t)$, $t_0 \le t \le t_1$. Consequently, in order to be completely controllable, the system model structure must at least be such that $\mathbf{u}$ can affect all of the state variables. It is possible to talk of a system being "controllable at a given t_0" (rather than any t_0), "controllable from $(\mathbf{x}_0, t_0)$ to $(\mathbf{x}_1, t_1)$" (further specifying particular initial and final conditions), and the like, but complete controllability will be a more significant concept for our purposes.

Consider the linear time-varying model given by Eqs. (2-35)–(2-37). Since the solution of the differential equation (2-35) is

$$\mathbf{x}_1 = \mathbf{x}(t_1) = \mathbf{\Phi}(t_1, t_0)\mathbf{x}_0 + \int_{t_0}^{t_1} \mathbf{\Phi}(t_1, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \tag{2-66}$$

we can premultiply both sides by $\mathbf{\Phi}(t_0, t_1)$ to obtain

$$\mathbf{\Phi}(t_0, t_1)\mathbf{x}_1 = \mathbf{x}_0 + \int_{t_0}^{t_1} \mathbf{\Phi}(t_0, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \tag{2-67}$$

But $\mathbf{\Phi}(t_0, t_1)\mathbf{x}_1$ is the result of propagating $\mathbf{x}_1$ backwards in time to time t_0 if no controls are applied. Therefore, complete controllability, being able to reach any $\mathbf{x}_1$ at t_1 from $\mathbf{x}_0$, is equivalent to saying $[\mathbf{\Phi}(t_0, t_1)\mathbf{x}_1 - \mathbf{x}_0]$ can be *any* vector in R^n. This is equivalent to saying the range space of $\int_{t_0}^{t_1} \mathbf{\Phi}(t_0, \tau)\mathbf{B}(\tau) \cdot d\tau$ (the space of possible mapping of piecewise continuous $\mathbf{u}$ by this operator) is all of R^n. It can be shown that this range space is equivalent to the range space of the n-by-n matrix (called the controllability Gramian):

$$\mathbf{W}(t_0, t_1) \triangleq \int_{t_0}^{t_1} \mathbf{\Phi}(t_0, \tau)\mathbf{B}(\tau)\mathbf{B}^{\mathrm{T}}(\tau)\mathbf{\Phi}^{\mathrm{T}}(t_0, \tau)\, d\tau \tag{2-68}$$

where the form of $\mathbf{W}(t_0, t_1)$ is motivated by a study of adjoints of linear operators. Thus, the system model (2-35)–(2-37) can be shown to be completely controllable if and only if any of the equivalent criteria are met for some t_1: (1) the range of $\mathbf{W}(t_0, t_1)$ is R^n, (2) the rank of $\mathbf{W}(t_0, t_1)$ is n, (3) $\mathbf{W}(t_0, t_1)$ is nonsingular and thus invertible, (4) $\mathbf{W}(t_0, t_1)$ is positive definite (it is always positive semidefinite), (5) the determinant of $\mathbf{W}(t_0, t_1)$ is nonzero. If the model is completely controllable, then one control which actually transfers $\mathbf{x}(t_0) = \mathbf{x}_0$ to $\mathbf{x}(t_1) = \mathbf{x}_1$ is given by:

$$\mathbf{u}(t) = -\mathbf{B}^{\mathrm{T}}(t)\mathbf{\Phi}^{\mathrm{T}}(t_0, t)\mathbf{W}(t_0, t_1)^{-1}[\mathbf{x}_0 - \mathbf{\Phi}(t_0, t_1)\mathbf{x}_1] \qquad \text{for all} \quad t \in [t_0, t_1] \tag{2-69}$$

If $\mathbf{W}(t_0, t_1)$ is singular and of rank $k < n$, then there are $(n - k)$ "uncontrollable states" in the representation. No matter what control is applied, no influence can be exerted on the system response in those directions of n-space. If it is possible to derive an equivalent model with no uncontrollable states, this would be a preferable basis for controller design. When noise is introduced into the dynamics model, it will be appropriate to investigate controllability with respect to both control inputs and noises, and this will be discussed subsequently.

Although the preceding criteria are precise mathematically, they are difficult to apply in practice because they need only be satisfied for some t_1. This is alleviated somewhat by the fact that the rank of $\mathbf{W}(t_0, t_1)$ is a monotonically increasing function of t_1, for fixed t_0. However, in the case of a time-invariant linear model, as (2-30)–(2-31), a more practical criterion can be achieved. Here, the range space of the n-by-nr matrix

$$\mathbf{W}_{\text{TI}} \triangleq [\mathbf{B} \;\vdots\; \mathbf{FB} \;\vdots\; \cdots \;\vdots\; \mathbf{F}^{n-1}\mathbf{B}] \tag{2-70}$$

can be shown equivalent to the range space of $\mathbf{W}(t_0, t_1)$. Thus, the system model (2-30)–(2-31) is completely controllable if and only if the range space of $\mathbf{W}_{\text{TI}}$ is R^n, or its rank is n, or equivalently, if there are n linearly independent columns in $\mathbf{W}_{\text{TI}}$. Each column of $\mathbf{W}_{\text{TI}}$ represents a vector in state space along which control is possible. If it is possible to control along n linearly independent directions in R^n, i.e., a basis of R^n, it is possible to control in all R^n.

For single input system models, $\mathbf{W}_{\text{TI}}$ becomes an n-by-n matrix, so in that case the system model is completely controllable if and only if $\mathbf{W}_{\text{TI}}$ is nonsingular, i.e., its determinant is nonzero. This can also be shown equivalent to the condition that $[s\mathbf{I} - \mathbf{F}]^{-1}\mathbf{b}$ has no pole-zero cancellations.

Analogously, a discrete-time system model is *completely controllable*, if, for any vectors $\mathbf{x}_0$, $\mathbf{x}_N \in R^n$, there exists a sequence $\mathbf{u}(0), \ldots, \mathbf{u}(N-1)$ such that the solution of the describing difference equation with $\mathbf{x}(0) = \mathbf{x}_0$ satisfies $\mathbf{x}(N) = \mathbf{x}_N$ for some finite N. The linear system representation of (2-64) is completely controllable if and only if the range space of the n-by-n matrix

$$\mathbf{W}_{\text{D}}(0, N) \triangleq \sum_{i=1}^{N} \mathbf{\Phi}(0, i)\mathbf{B}(i-1)\mathbf{B}^{\text{T}}(i-1)\mathbf{\Phi}^{\text{T}}(0, i) \tag{2-71}$$

is all of R^n, or any equivalent statements as after (2-68). The corresponding time-invariant linear system model is completely controllable if and only if the rank of the n-by-nr matrix $\mathbf{W}_{\text{DTI}}$ is n, where

$$\mathbf{W}_{\text{DTI}} \triangleq [\mathbf{B} \;\vdots\; \mathbf{\Phi B} \;\vdots\; \cdots \;\vdots\; \mathbf{\Phi}^{n-1}\mathbf{B}] \tag{2-72}$$

On the other hand, observability is concerned with the effect of states of a model upon the outputs. A continuous-time system representation is *completely observable* if, given $\mathbf{z}(t)$ and $\mathbf{u}(t)$ for all $t \in [t_0, t_1]$, it is possible to deduce $\mathbf{x}(t)$ for $t \in [t_0, t_1]$. Thus, a system model is completely observable if *any* state $x_i(t)$ can be determined exactly for $t \in [t_0, t_1]$ from knowledge of only the input and output over the interval $[t_0, t_1]$. To be completely observable, the representation structure must be such that the output $\mathbf{z}(t)$ is affected in some manner by the change of any single state variable. Moreover, the effect of any one state variable on the output must be distinguishable from the effect of any other state variable.

EXAMPLE 2.9 Consider a homogeneous system model as in Fig. 2.8a. Here, both x_1 and x_2 affect the output z, but there would be no way to obtain separate information about x_1 and x_2 just by observing the output z. From observations of z this model would appear identical to that depicted in Fig. 2.8b. ■

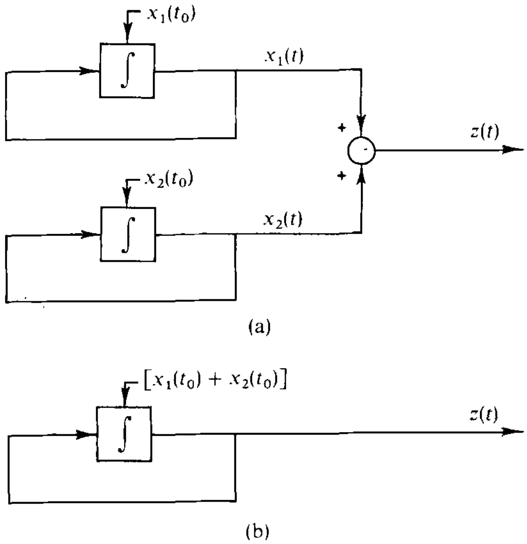

FIG. 2.8 Observability of system models. (a) Original model. (b) Equivalent model.

When a complete system model is generated by combining separate component models, it is not uncommon for the result to involve unobservable states. For instance, in a redundant system, a state to model a bias in one instrument might be indistinguishable from a similar bias state in a redundant instrument, considering only the effects on the output. If an estimator were based upon such a system model, the estimation errors along certain directions of state space would not decrease, regardless of how long measurements were taken. In such cases, it would be appropriate to combine such different physical quantities into a single state variable, such as to let a state be the sum of the biases in the redundant system example, to achieve an observable system model.

Now let us ask whether or not the linear time-varying system model described by (2-35) and (2-37) is completely observable. By the form of the solution to the differential equation, it is necessary and sufficient to be able to deduce $\mathbf{x}(t_0)$ from knowledge of $\mathbf{u}(t)$ and $\mathbf{z}(t)$ for all $t \in [t_0, t_1]$. Using the solution we can write

$$\begin{aligned}\mathbf{z}(t) &= \mathbf{H}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \\ &= \mathbf{H}(t)\left[\mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau\right] + \mathbf{D}(t)\mathbf{u}(t)\end{aligned}$$

so that

$$\mathbf{H}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}_0 = \mathbf{z}(t) - \mathbf{D}(t)\mathbf{u}(t) - \int_{t_0}^{t} \mathbf{H}(t)\mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \qquad (2\text{-}73)$$

Thus we are asking, under what conditions can this equation be solved for $\mathbf{x}_0$ uniquely? This is equivalent to asking, is the null space of $\mathbf{H}(t)\mathbf{\Phi}(t, t_0)\,\cdot$, the set of all vectors $\mathbf{x} \in R^n$ that are mapped into the zero function over the interval $[t_0, t_1]$, restricted to $\mathbf{0} \in R^n$? This is a difficult question to answer, but it can be shown that the null space of $\mathbf{H}(t)\mathbf{\Phi}(t, t_0)\,\cdot$ is the same as the null space of the n-by-n matrix (the observability Gramian)

$$\mathbf{M}(t_0, t_1) \triangleq \int_{t_0}^{t_1} \mathbf{\Phi}^{\mathrm{T}}(\tau, t_0)\mathbf{H}^{\mathrm{T}}(\tau)\mathbf{H}(\tau)\mathbf{\Phi}(\tau, t_0)\, d\tau \tag{2-74}$$

As in the case of controllability, the form of $\mathbf{M}(t_0, t_1)$ is motivated by adjoints of linear operators. Consequently, the model (2-35)–(2-37) is completely observable if and only if any of the following criteria are met for some t_1: (1) the null space of $\mathbf{M}(t_0, t_1)$ is $\mathbf{0} \in R^n$, (2) $\mathbf{M}(t_0, t_1)$ is nonsingular, i.e., invertible, (3) $\mathbf{M}(t_0, t_1)$ is positive definite, (4) the determinant of $\mathbf{M}(t_0, t_1)$ is nonzero. If $\mathbf{M}(t_0, t_1)$ is of rank $k < n$, then it is said that there are $(n - k)$ "unobservable states" in the model. If the model is completely observable, then $\mathbf{x}_0$ can be determined uniquely from $\mathbf{u}(t)$ and $\mathbf{z}(t)$, $t_0 \le t \le t_1$, by

$$\mathbf{x}_0 = \mathbf{M}(t_0, t_1)^{-1} \int_{t_0}^{t_1} \mathbf{\Phi}^{\mathrm{T}}(\tau, t_0)\mathbf{H}^{\mathrm{T}}(\tau)\mathbf{v}(\tau)\, d\tau \tag{2-75}$$

where

$$\mathbf{v}(\tau) = \mathbf{z}(\tau) - \mathbf{D}(\tau)\mathbf{u}(\tau) - \int_{t_0}^{\tau} \mathbf{H}(\tau)\mathbf{\Phi}(\tau, \sigma)\mathbf{B}(\sigma)\mathbf{u}(\sigma)\, d\sigma \tag{2-76}$$

Using that value of $\mathbf{x}_0$, the entire state history over $[t_0, t_1]$ can be determined from

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{\tau} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \tag{2-77}$$

As before, a more practical test is possible for time-invariant linear models. The system representation given by (2-30)–(2-31) is completely observable if and only if the range space of $\mathbf{M}_{\mathrm{TI}}$ is R^n, where $\mathbf{M}_{\mathrm{TI}}$ is the n-by-nm matrix

$$\mathbf{M}_{\mathrm{TI}} \triangleq [\mathbf{H}^{\mathrm{T}} \;\vdots\; \mathbf{F}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}} \;\vdots\; \cdots \;\vdots\; (\mathbf{F}^{\mathrm{T}})^{n-1}\mathbf{H}^{\mathrm{T}}] \tag{2-78}$$

or equivalently, if its rank is n or if there are n linearly independent columns in $\mathbf{M}_{\mathrm{TI}}$ (again, if we can observe along a basis of R^n, then we can observe any vector in R^n).

For single output systems, the preceding criterion becomes the condition that the n-by-n $\mathbf{M}_{\mathrm{TI}}$ is nonsingular, with nonzero determinant. This is equivalent to the condition that $\mathbf{h}^{\mathrm{T}}[s\mathbf{I} - \mathbf{F}]^{-1}$ has no pole-zero cancellations.

The corresponding discrete-time result is that the model described by (2-64) is completely observable if and only if the null space of the n-by-n matrix

$$\mathbf{M}_{\mathrm{D}}(0, N) \triangleq \sum_{i=1}^{N} \mathbf{\Phi}^{\mathrm{T}}(i, 0)\mathbf{H}^{\mathrm{T}}(i)\mathbf{H}(i)\mathbf{\Phi}(i, 0) \tag{2-79}$$

is $\mathbf{0} \in R^n$ for some finite N, or any equivalent statements as after (2-74). Note that the rank of each term in the sum is at most m, so there is in general some minimal number $N \geq n/m$ of measurements that must be taken before the model can be completely observable. A corresponding time-invariant linear model is completely observable if and only if the rank of the n-by-nm matrix $\mathbf{M}_{\mathrm{DTI}}$ is n, where

$$\mathbf{M}_{\mathrm{DTI}} \triangleq [\mathbf{H}^{\mathrm{T}} \;\vdots\; \mathbf{\Phi}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}} \;\vdots\; \cdots \;\vdots\; (\mathbf{\Phi}^{\mathrm{T}})^{n-1}\mathbf{H}^{\mathrm{T}}] \tag{2-80}$$

There is a noteworthy resemblance between the observability and controllability results just discussed. The interrelationship is a *duality* relationship that can be exploited substantially in linear system theory, but we will not pursue this matter further at this point.

2.6 SUMMARY

The state differential equation (2-35) and discrete-time output equation (2-60) will provide the basic linear deterministic system model, the structure of which will be extended in ensuing chapters by adding uncertainty to both the dynamics and measurement relations. The dynamic system response can be characterized by solving the state differential equation, facilitated by means of the concept of the state transition matrix. For such analysis, the interrelationships among inputs, states, and outputs are properly specified through the concepts of controllability and observability. Given a particular system model, an infinite variety of related time-domain models can represent the same input–output characteristics. Some particular forms are especially useful, in that they separate system modes or yield minimum required computations to depict the system behavior, and these can be exploited in practical applications.

In a similar manner, Eqs. (2-40) and (2-59) define a general nonlinear deterministic system model. The existence of solutions to nonlinear differential equations can be established under rather nonrestrictive assumptions, but their form cannot be characterized as fully as in the linear case. This model structure will be extended in Chapter 11 (Volume 2) to the stochastic case, motivated by the insights gained from the simpler models discussed in this chapter.

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PROBLEMS

2.1 Suppose we have a single input–single output system described by the following transfer function:

$$G(s) = (s^3 + 3s^2 + 5s + 8)/(s^4 + 7s^3 + 14s^2 + 8s)$$

(a) Derive the standard observable and standard controllable phase variable forms. Note that the $\mathbf{F}$ matrix of $\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{b}u(t)$ is singular—how is this interpreted physically? Draw the block diagrams for both and note the pattern of the feedback and feedforward loops.

(b) Represent the system in terms of canonical variables; draw the block diagram.

(c) Are the system representations just obtained observable and controllable?

2.2 For a system modeled by transfer function of

$$G(s) = [10(s + 4)]/(s^3 + 3s^2 + 2s)$$

(a) evaluate the standard controllable form of state variable representation,

(b) obtain the canonical form state variable representation from the phase variables, with and without the Vandermonde matrix, and

(c) obtain the canonical representation directly from the transfer function.

2.3 For a system modeled by $G(s) = 1/[(s^2 + 6s + 25)(s + 1)]$ generate the canonical form state variable model involving complex $\mathbf{F}$ and $\mathbf{x}(t)$. Transform the result appropriately to make $\mathbf{F}$ and $\mathbf{x}(t)$ be composed of only real-valued terms. From the transfer function, write the standard controllable form, and then obtain the same form as just stated without going through the intermediate step of complex $\mathbf{F}$ and $\mathbf{x}(t)$.

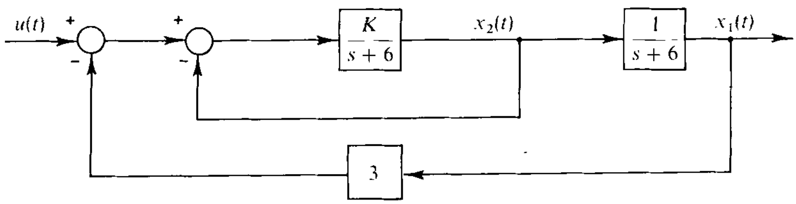

FIG. 2.P1 Blocks diagram for Problem 2.4.

2.4 A system can be represented by the block diagram of Fig. 2.P1.

(a) Determine the state equations in terms of $x_1(t)$ and $x_2(t)$.

(b) Determine the transfer function $\{x_1(s)/u(s)\}$.

(c) As a designer, you can control the gain K in the block diagram ($K > 0$). Describe how the system dynamics are affected by letting K be changed.

(d) Determine the canonical form state space description of the system. If and where appropriate, express in modified canonical form. Describe appropriate $\mathbf{F}$, $\mathbf{b}$, and $\mathbf{h}^{\mathrm{T}}$ without excessive algebra.

2.5 For a state equation of the form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

determine the resolvent matrix $\mathbf{\Phi}(s)$. Now let the output be expressed as

$$y = [c_1 \quad c_2 \quad c_3]\mathbf{x}$$

Determine the transfer function relating y to u. What is the differential equation that corresponds to this transfer function or state space description?

What is the steady state value of y to a unit impulse? To a unit step? Under what conditions are these evaluations valid? Show that if $c_1 = 1$ and $c_2 = c_3 = 0$, then the preceding system model is both observable and controllable for all values of a_1, a_2, and a_3.

2.6 A state space model of the system given in Fig. 2.P2 can be expressed in the form of the minimal realization

$$\frac{d}{dt}\begin{bmatrix} V(t) \\ f(t) \end{bmatrix} = \begin{bmatrix} -B/m & -1/m \\ 2K & 0 \end{bmatrix}\begin{bmatrix} V(t) \\ f(t) \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} F_s(t)$$

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} V(t) \\ f(t) \end{bmatrix}$$

where V = velocity of the mass and f = force through an equivalent spring of stiffness $2K$. Suppose you want to generate another minimal realization in terms of the variables f and df/dt. How would

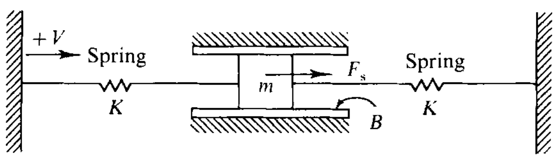

FIG. 2.P2 Diagram for system of Problem 2.6.

you do it? Write this form explicitly. Generate the similarity transformation matrix that relates these two realizations explicitly. Write the solution to these equations for df/dt, in both the time domain and the frequency domain.

2.7 Figure 2.P3 depicts a "stable platform," a system that is composed of a motor-driven gimbal, and which attempts to keep the platform rotating at a commanded angular velocity with respect to inertial space (often zero) despite interfering torques due to base motions or forces applied to the platform. Sensed angular velocity from a gyro on the platform is used by the controller to generate appropriate commands to the gimbal motor to achieve this purpose. Consider a single axis stable platform with direct drive and no tachometer feedback. Assume that its components can be described by the following relationships, where D indicates the time differentiation operation.

Controlled member: $I_{cm}D\omega_{im} = M_m + M_{intf}$.
Gyro equation: $De_{sig} = S_{sg}[\omega_{cmd} - \omega_{im} + \omega_d]$.
Control equation: $M_m = [S_c/S_{sg}]F_c(D)e_{sig}$.

The variables are defined as I_{cm}, moment of inertia of controlled member; ω_{im}, angular velocity of controlled member with respect to the inertial frame of reference; M_m, servo motor torque; M_{intf}, interfering torque; e_{sig}, signal voltage (gyro output); ω_{cmd}, commanded angular velocity; ω_d, drift rate of gyro; S_{sg}, signal generator gain; S_c, control gain; and $F_c(D)$, transfer function of control system.

(a) Show that the general performance equation for the stable platform is

$$[I_{cm}D^2 + S_cF_c(D)]\omega_{im} = S_cF_c(D)[\omega_{cmd} + \omega_d] + DM_{intf}$$

(b) Draw a system block diagram corresponding to the three component equations, and thereby to the general performance equation.

(c) Using the outputs of the integrators in this diagram as state variables, write out a state space description of the system for the case of $F_c(D) = 1$.

(d) To attain a controlled member "stable" with respect to inertial space, ω_{cmd} is set equal to zero. Using the result of part (c), evaluate the effect of a constant drift rate on the angular rate ω_{im} of the controlled member. Also evaluate the effect of a constant interfering torque on ω_{im}.

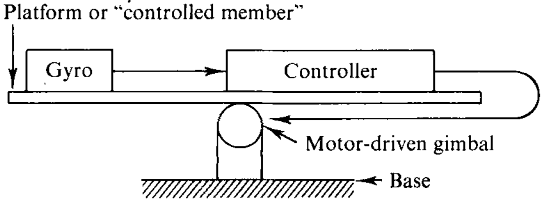

FIG. 2.P3 One-axis stable platform configuration.

2.8 This problem deals with a pitch attitude control system (PACS) for a rigid rocket vehicle employing a gimbaled rocket motor to obtain pitching torques. A pitching moment is obtained by tilting the rocket motor to give a transverse component of thrust, applied behind the center of gravity. The rocket motor is tilted in its gimbals by means of a hydraulic positioning system described by a transfer function of

$$0.1\Bigg/\left[\frac{s}{250} + 1\right]\left[\frac{s^2}{(365)^2} + \frac{2(0.616)}{365}s + 1\right] \quad \text{deg/V}$$

Two gyroscopes are used to produce measurements for feedback. One is an attitude gyro which produces an electrical signal proportional to pitch angle (with proportionality constant K_{ag} V/deg), and the other is a rate gyro which develops an electrical signal proportional to pitch rate (proportionality constant K_{rg} V-sec/deg). Attenuation controls enable you to vary the pitch rate signal level relative to the pitch angle signal level. The objective of a control system design would be to find the best ratio of these signals. These two signals are added, and their sum is then subtracted from a pitch angle command voltage, all by means of a summing amplifier. The pitch angle command voltage is produced by a guidance computer. The output of the summing amplifier is used as the actuating signal for the rocket motor positioning system.

(a) Prepare a block diagram for the PACS.

(b) Generate a time-domain model for this system in the form of a state differential equation and output relation.

(c) Generate a frequency-domain model in the form of a Laplace transform transfer function.

2.9 Consider the following system model (note that it is not in standard observable or standard controllable form):

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$z(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \mathbf{x}(t)$$

(a) Draw a block diagram of this state space model, explicitly labeling the states, input, and output.

(b) Is the system model completely controllable?

(c) Is the system model completely observable?

(d) Find the transfer function of the system, $G(s)$, that relates the output, $z(t)$, to the input, $u(t)$, through the observable and controllable portion of the system model.

(e) One system pole is located at $s = -1$. Plot the poles and zeros of $G(s)$. Is this a "nonminimum phase" and stable system—i.e., are there singularities in the right half plane?

(f) Develop the canonical form state equations to describe this system (using the transfer function directly is probably the most straightforward means).

(g) Can you determine a transformation matrix, $\mathbf{T}$, that will transform the given state equations into the canonical form equations?

(h) Assuming the system to be initially at rest (all appropriate initial conditions zero), find the system response, $z(t)$, to a unit impulse input, $u(t)$.

(i) Under the same assumptions of initial rest, compute $z(t)$ for a unit step input, $u(t)$.

There are a number of valid methods of attaining the answers to these questions. One matrix form that can be useful in some parts (though not necessarily required in any part) would be $\operatorname{adj}(f\mathbf{I} - \mathbf{F})$, where f is some appropriate quantity and $\mathbf{F}$ is the given $\mathbf{F}$ matrix in the problem; the evaluation of this form is

$$\operatorname{adj}(f\mathbf{I} - \mathbf{F}) = \begin{bmatrix} f^2 + 6f + 11 & f + 6 & 1 \\ -6 & f^2 + 6f & f \\ -6f & -11f - 6 & f^2 \end{bmatrix}$$

2.10 Explain why the following system realizations are or are not both observable and controllable.

(a)

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t)$$

$$z(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

(b)

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -3 & 0 \\ 0 & -5 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\mathbf{u}(t)$$

$$\mathbf{z}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\mathbf{x}(t)$$

(c)

$$\dot{\mathbf{x}}(y) = \begin{bmatrix} -3 & 0 \\ 0 & -5 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\mathbf{u}(t)$$

$$\mathbf{z}(t) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\mathbf{x}(t)$$

2.11 Calculate the controllability Gramian $\mathbf{W}(0, T)$ for the driven oscillator:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Is this system model completely controllable? Compare this to the answer achieved using $\mathbf{W}_{\mathrm{TI}}$.

2.12 Consider the single-axis error model for an inertial navigation system (INS) shown in Fig. 2.P4. In this simplified model, gyro drift rate errors are ignored. It is desired to estimate the vertical deflection process, e_ξ. Consider the system state vector consisting of the four states:

$$\mathbf{x}^{\mathrm{T}} = [e_{\mathrm{pb}} \quad \delta_{\mathrm{p}} \quad \delta_{\mathrm{v}} \quad e_{\xi\mathrm{b}}]$$

(a) For position measurements only (z_{v} not available), set up the observability matrix and determine whether this system model is observable.

(b) Now assume we have both position and velocity measurements. Is this system model observable?

(c) Now assume $e_{\mathrm{pb}} = 0$ and can be eliminated from the state error model. Is this system model observable with position measurements only?

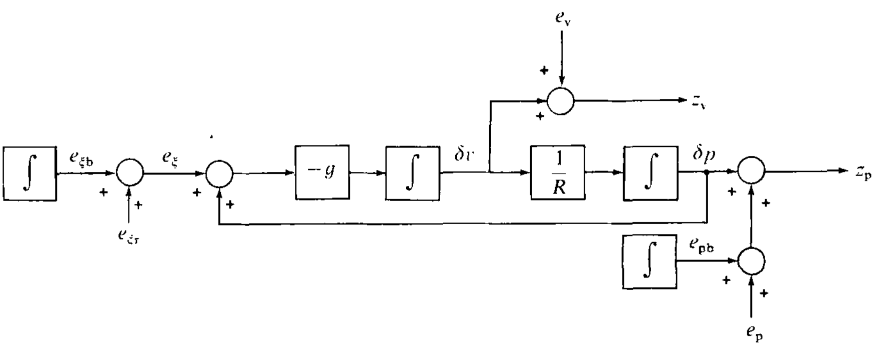

FIG. 2.P4 Single-axis error model for INS. $e_{\xi\mathrm{r}}$ = unbiased random vertical deflection, $e_{\xi\mathrm{b}}$ = vertical deflection bias, e_{v} = uncorrelated velocity measurement error, e_{pb} = bias position measurement error, e_{p} = uncorrelated position measurement error. From *Applied Optimal Estimation* by A. Gelb (ed.). © 1974. Used with permission of the M.I.T. Press.

2.13 Show that the state transition matrix corresponding to the **F** in Example 2.8 for a circular nominal trajectory with $r_0 = 1$ is given by:

$$\boldsymbol{\Phi}(t,0) = \begin{bmatrix} 4 - 3\cos\omega t & (\sin\omega t)/\omega & 0 & 2(1-\cos\omega t)/\omega \\ 3\omega\sin\omega t & \cos\omega t & 0 & 2\sin\omega t \\ 6(-\omega t + \sin\omega t) & -2(1-\cos\omega t)/\omega & 1 & (-3\omega t + 4\sin\omega t)/\omega \\ 6\omega(-1+\cos\omega t) & -2\sin\omega t & 0 & -3+4\cos\omega t \end{bmatrix}$$

2.14 Given that **F** is a 2×2 constant matrix and given that

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t)$$

Suppose that if

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \text{then} \quad \mathbf{x}(t) = \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$$

and if

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{then} \quad \mathbf{x}(t) = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$$

Determine the transition matrix for the system and the matrix **F**.

Problem 2.14 is from *Finite Dimensional Linear Systems* by R. Brockett. © 1970. Used with permission of John Wiley & Sons, Inc.

2.15 (a) Show that, for all t_0, t_1, and t,

$$\boldsymbol{\Phi}(t,t_0) = \boldsymbol{\Phi}(t,t_1)\boldsymbol{\Phi}(t_1,t_0)$$

by showing that both quantities satisfy the same linear differential equation and "initial condition" at time t_1. Thus, the solution to $\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t)$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ (i.e., $\boldsymbol{\Phi}(t,t_0)\mathbf{x}_0$) at any time t_2 can be obtained by forming $\mathbf{x}(t_1) = \boldsymbol{\Phi}(t_1,t_0)\mathbf{x}_0$ and using it to generate $\mathbf{x}(t_2) = \boldsymbol{\Phi}(t_2,t_1)\mathbf{x}(t_1)$.

(b) Since it can be shown that $\boldsymbol{\Phi}(t,t_0)$ is nonsingular, show that the above "semigroup property" implies that

$$\boldsymbol{\Phi}^{-1}(t,t_0) = \boldsymbol{\Phi}(t_0,t)$$

2.16 Let $\boldsymbol{\Phi}(t,t_0)$ be the state transition matrix associated with $\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$. Consider a change of variables,

$$\mathbf{x}^*(t) = \mathbf{T}(t)\mathbf{x}(t)$$

where $\mathbf{T}(\cdot)$ is differentiable and $\mathbf{T}^{-1}(t)$ exists for all time t. Show that the state transition matrix associated with the transformed state variables, $\boldsymbol{\Phi}^*(t,t_0)$, is the solution to

$$\dot{\boldsymbol{\Phi}}^*(t,t_0) = [\mathbf{T}(t)\mathbf{F}(t)\mathbf{T}^{-1}(t) + \dot{\mathbf{T}}(t)\mathbf{T}^{-1}(t)]\boldsymbol{\Phi}^*(t,t_0)$$
$$\boldsymbol{\Phi}^*(t_0,t_0) = \mathbf{I}$$

and that

$$\boldsymbol{\Phi}(t,t_0) = \mathbf{T}^{-1}(t)\boldsymbol{\Phi}^*(t,t_0)\mathbf{T}(t_0)$$

Note that if **T** is constant, this yields a similarity transformation useful for evaluation of the state transition matrix.

2.17 Let $\mathbf{F}$ be constant. Then the evaluation of $\boldsymbol{\Phi}(t, t_0) = \boldsymbol{\Phi}(t - t_0)$ can be obtained by
(a) approximation through truncation of series definition of matrix exponential, $e^{\mathbf{F}(t-t_0)}$:

$$e^{\mathbf{F}(t-t_0)} = \mathbf{I} + \mathbf{F}(t - t_0) + \frac{1}{2!}\mathbf{F}^2(t - t_0)^2 + \cdots$$

(b) Laplace methods of solving $\dot{\boldsymbol{\Phi}}(t - t_0) = \mathbf{F}\boldsymbol{\Phi}(t - t_0)$, $\boldsymbol{\Phi}(0) = \mathbf{I}$:

$$\boldsymbol{\Phi}(t - t_0) = \mathscr{L}^{-1}\{[s\mathbf{I} - \mathbf{F}]^{-1}\}|_{(t-t_0)}$$

where $\mathscr{L}^{-1}\{\cdot\}|_{(t-t_0)}$ denotes inverse Laplace transform evaluated with time argument equal to $(t - t_0)$.
(c) Cayley–Hamilton theorem (for $\mathbf{F}$ with nonrepeated eigenvalues)

$$\boldsymbol{\Phi}(t - t_0) = \alpha_0\mathbf{I} + \alpha_1\mathbf{F} + \alpha_2\mathbf{F}^2 + \cdots + \alpha_{n-1}\mathbf{F}^{n-1}$$

To solve for the n functions of $(t - t_0)$, $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, the n eigenvalues of $\mathbf{F}$ are determined as $\lambda_1, \ldots, \lambda_n$. Then

$$e^{\lambda(t-t_0)} = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_{n-1}\lambda^{n-1}$$

must be satisfied by each of the eigenvalues, yielding n equations for the n unknown α_i's.
(d) Sylvester expansion theorem (for $\mathbf{F}$ with nonrepeated eigenvalues)

$$\boldsymbol{\Phi}(t - t_0) = \mathbf{F}_1 e^{\lambda_1(t-t_0)} + \mathbf{F}_2 e^{\lambda_2(t-t_0)} + \cdots + \mathbf{F}_n e^{\lambda_n(t-t_0)}$$

where λ_i is the ith eigenvalue of $\mathbf{F}$ and $\mathbf{F}_i$ is given as the following product of $(n - 1)$ factors:

$$\mathbf{F}_i = \left[\frac{\mathbf{F} - \lambda_1\mathbf{I}}{\lambda_i - \lambda_1}\right]\cdots\left[\frac{\mathbf{F} - \lambda_{i-1}\mathbf{I}}{\lambda_i - \lambda_{i-1}}\right]\left[\frac{\mathbf{F} - \lambda_{i+1}\mathbf{I}}{\lambda_i - \lambda_{i+1}}\right]\cdots\left[\frac{\mathbf{F} - \lambda_n\mathbf{I}}{\lambda_i - \lambda_n}\right]$$

Use the four methods to evaluate $\boldsymbol{\Phi}(t - t_0)$ if $\mathbf{F}$ is given by

$$\mathbf{F} = \begin{bmatrix} 0 & 6 \\ -1 & -5 \end{bmatrix}$$

Let $(t - t_0) = 0.1$ sec; if the series method of part (a) is truncated at the first order term, what is the greatest percentage error (i.e., [(calculated value − true)/true value] · 100%) committed in approximating the four elements of $\boldsymbol{\Phi}(0.1)$? What if it were truncated at the second order term?

2.18 Given a homogeneous linear differential equation $\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t)$, the associated "adjoint" differential equation is the differential equation for the n-vector $\mathbf{p}(t)$ such that the inner product of $\mathbf{p}(t)$ with $\mathbf{x}(t)$ is constant for all time:

$$\mathbf{x}(t)^{\mathrm{T}}\mathbf{p}(t) = \text{const}$$

(a) Take the derivative of this expression to show that the adjoint equation associated with $\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t)$ is

$$\dot{\mathbf{p}}(t) = -\mathbf{F}^{\mathrm{T}}(t)\mathbf{p}(t)$$

(b) If $\boldsymbol{\Phi}_x(t, t_0)$ is the state transition matrix associated with $\mathbf{F}(t)$ and $\boldsymbol{\Phi}_p(t, t_0)$ is the state transition matrix associated with $[-\mathbf{F}^{\mathrm{T}}(t)]$, then show that

$$\boldsymbol{\Phi}_p(t, t_0) = \boldsymbol{\Phi}_x^{\mathrm{T}}(t_0, t) = [\boldsymbol{\Phi}_x^{\mathrm{T}}(t, t_0)]^{-1}$$

To do this, show that $[\boldsymbol{\Phi}_p^{\mathrm{T}}(t, t_0)\boldsymbol{\Phi}_x(t, t_0)]$ and $\mathbf{I}$ satisfy the same differential equation and initial condition.

(c) Show that, as a function of its second argument, $\mathbf{\Phi}_x(t, \tau)$ must satisfy

$$\partial[\mathbf{\Phi}_x(t, \tau)]/\partial\tau = -\mathbf{\Phi}_x(t, \tau)\mathbf{F}(\tau)$$

or, in other words,

$$\partial[\mathbf{\Phi}_x{}^{\mathrm{T}}(t, \tau)]/\partial\tau = [-\mathbf{F}(\tau)^{\mathrm{T}}]\mathbf{\Phi}_x{}^{\mathrm{T}}(t, \tau)$$

2.19 The Euler equations for the angular velocities of a rigid body are

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + u_1 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_1\omega_3 + u_2 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + u_3 \end{aligned}$$

Here ω is the angular velocity in a body-fixed coordinate system coinciding with the principal axes, $\mathbf{u}$ is the applied torque, and I_1, I_2, and I_3 are the principal moments of inertia. If $I_1 = I_2$, we call the body symmetrical. In this case, linearize these equations about the solution $\mathbf{u} = \mathbf{0}$,

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} \cos \omega_0 (I_2 - I_3)/I_1 \\ \sin \omega_0 (I_2 - I_3)/I_1 \\ \omega_0 \end{bmatrix}$$

2.20 Consider a system with input $r(t)$ and output $c(t)$ modeled by the nonlinear differential equation

$$\ddot{c}(t) + c^3(t)\dot{c}^2(t) + \sin[c(t)] - t^2c(t) = r(t)$$

Determine the linear perturbation equations describing the system's behavior near nominal trajectories of

(1) $c(t) = r(t) = 0$,

(2) $c(t) = t$, $r(t) = \sin t$.

Describe how you would obtain the equivalent discrete-time model for the perturbation equations in the two preceding cases, to describe the perturbed output Δc at discrete time points:

$$\Delta c(kT), \qquad k = 0, 1, 2, \ldots, \qquad \text{for given } T$$

2.21 Consider the system configuration of Fig. 2.P5. The sampler has a period of T seconds and generates the sequence $\{e_1(0), e_1(T), e_1(2T), \ldots\}$. The algorithm implemented in the digital computer is a first-order difference approximation to differentiation:

$$e_2(t_i) = [e_1(t_i) - e_1(t_{i-1})]/T$$

Finally, the zero-order hold (ZOH) generates a piecewise constant output by holding the value of its input over the ensuing sample period:

$$u(t) = e_2(t_i) \qquad \text{for all} \quad t \in [t_i, t_{i+1})$$

(a) Generate the equivalent discrete-time model for the "plant."

(b) Develop the discrete-time state equation and output relation model of the entire system configuration, and thus describe $c(t_i) = c(iT)$ for $i = 0, 1, 2, \ldots$.

(c) Determine the eigenvalues of the overall system state transition matrix; if the magnitude of any of these is greater than one, the system is unstable.

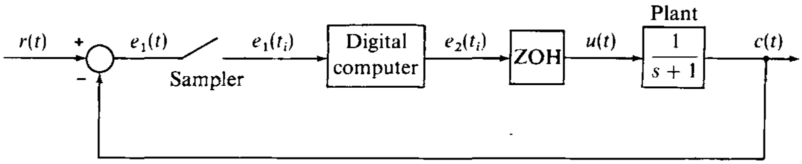

FIG. 2.P5 Block diagram for Problem 2.21.

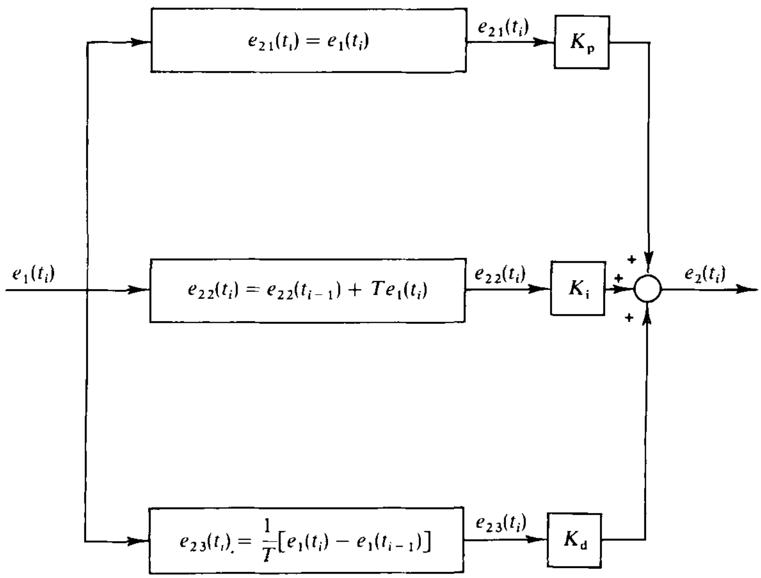

FIG. 2.P6 Digital PID controller.

2.22 A PID (proportional plus integral plus derivative) controller is a device which accepts a signal $e_1(t)$ as input and generates an output as

$$e_2(t) = K_p e_1(t) + K_i \int_{t_0}^{t} e_1(\tau)\,d\tau + K_d \dot{e}_1(t)$$

where the coefficients K_p, K_i, and K_d are adjusted to obtain desirable behavior from the closed loop system generated by using a PID controller for feedback.

A simple digital PID controller for use with an iteration period of T seconds is shown in Fig. 2.P6. Show that the difference equation approximation to the integration operation is the result of Euler integration, and that this channel can be described by the discrete-time state and output model:

$$x_i(t_{i+1}) = x_i(t_i) + e_1(t_i), \qquad e_{22}(t_i) = Tx_i(t_i) + Te_1(t_i)$$

Similarly show that the "derivative" first order difference approximation can be represented as

$$x_d(t_{i+1}) = e_1(t_i), \qquad e_{23}(t_i) = -[x_d(t_i)/T] + e_1(t_i)/T$$

Generate the state vector difference equation and output relation model for this digital PID controller. In general, the relationships among scalar difference equations, discrete-time state equations and output relations, and Z-transform transfer functions (not discussed in this book) are analogous to the relationships among scalar differential equations, differential state models, and Laplace transform transfer functions (where applicable), and methods analogous to those of this chapter yield means of generating one form from the other.

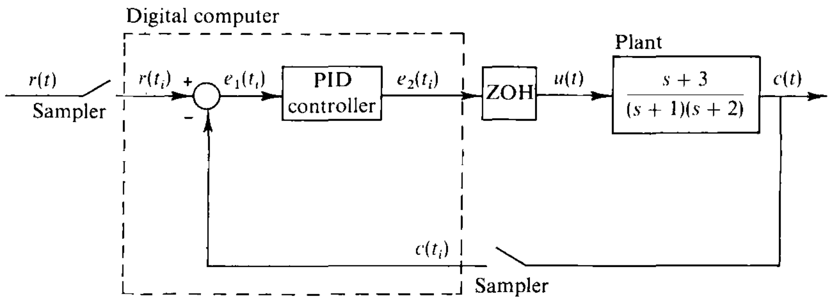

FIG. 2.P7 Control system configuration for Problem 2.23.

2.23 Consider the application of the PID controller of the previous problem applied to the system configuration depicted in Fig. 2.P7. Note that the samplers are synchronized and have sample period T. The zero-order hold (ZOH) generates

$$u(t) = e_2(t_i) \qquad \text{for all} \quad t \in [t_i, t_{i+1})$$

(a) Develop the equivalent discrete-time model for the "plant."

(b) Develop the discrete-time state model of the entire control system configuration, and thereby means of generating

$$c(iT) \qquad \text{for} \quad i = 0, 1, 2, \ldots$$

(c) To evaluate adequacy of controller performance, one desires knowledge of $c(t)$ for all time between sample times as well. Generate the relations necessary to evaluate $c(t)$ for all $t \in [iT, (i+1)T]$.