CHAPTER 4

Stochastic processes and linear dynamic system models

4.1 INTRODUCTION

This chapter adds dynamics to the system model developed in Chapter 3, thereby allowing consideration of a much larger class of problems of interest. First, Sections 4.2 and 4.3 characterize stochastic processes in general. Section 4.4 presents the motivation and conceptual framework for developing stochastic linear dynamic system models. State equations in the form of linear stochastic differential or difference equations are developed in Sections 4.5–4.9, and 4.10 adds the description of measured outputs to complete the overall system model. Finally, Sections 4.11–4.13 develop practical system models designed to duplicate (to the extent possible) the characteristics of processes observed empirically.

4.2 STOCHASTIC PROCESSES

Let Ω be a fundamental sample space and T be a subset of the real line denoting a time set of interest. Then a *stochastic process* [1–14] can be defined as a real-valued function $\mathbf{x}(\cdot,\cdot)$ defined on the product space $T \times \Omega$ (i.e., a function of two arguments, the first of which is an element of T and the second an element of Ω), such that for any fixed $t \in T$, $\mathbf{x}(t,\cdot)$ is a random variable. A scalar random process assumes values $\mathrm{x}(t,\omega) \in R^1$, whereas a vector random process assumes values $\mathbf{x}(t,\omega) \in R^n$. In other words, $\mathbf{x}(\cdot,\cdot)$ is a stochastic process if all sets of the form

$$A = \{\omega \in \Omega : \mathbf{x}(t,\omega) \le \xi\} \tag{4-1}$$

for any $t \in T$ and $\xi \in R^n$ (R^1 for a scalar random process) are in the underlying σ-algebra $\mathscr{F}$. If we fix the second argument instead of the first, we can say that to each point $\omega_i \in \Omega$ there can be associated a time function $\mathbf{x}(\cdot, \omega_i) = \mathbf{x}(\cdot)$, each of which is a *sample* from the stochastic process.

Although the definition of a stochastic process can be generalized to T being a subset of R^n (as for a process as a function of spatial coordinates), we will be interested in $T \subset R^1$ with elements of T being time instants. Two particular forms of T will be important. If T is a sequence $\{t_1, t_2, t_3, \ldots\}$, not necessarily equally spaced, then $\mathbf{x}(t_1, \cdot)$, $\mathbf{x}(t_2, \cdot)$, $\mathbf{x}(t_3, \cdot), \ldots$ becomes a sequence of random variables. This $\mathbf{x}(\cdot, \cdot)$ is then called a discrete-parameter stochastic process, or a *discrete-time stochastic process.* A sample from such a process is depicted in Fig. 4.1; different ω values then generate different samples from the process. If T is instead an interval of R^1, then $\mathbf{x}(\cdot, \cdot)$ becomes a continuous-parameter family of random variables, or a *continuous-time stochastic process.* For each ω, the sample is a function defined on the interval T, as portrayed in Fig. 4.2.

If T is of the discrete form of a finite sequence of N points along the real line, the set of random variables $\mathbf{x}(t_1, \cdot)$, $\mathbf{x}(t_2, \cdot), \ldots, \mathbf{x}(t_N, \cdot)$ can be characterized by

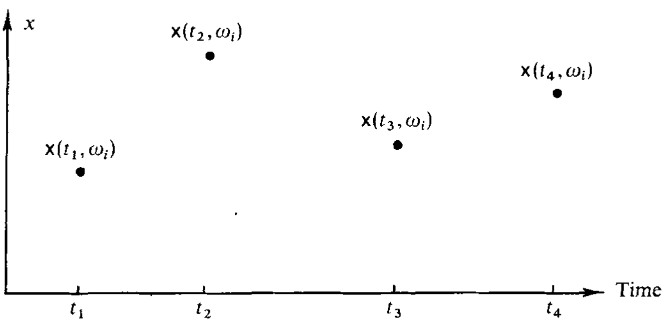

FIG. 4.1 Sample from a discrete-time stochastic process.

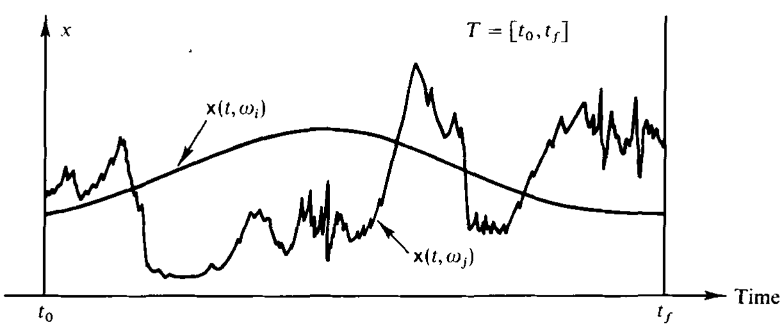

FIG. 4.2 Samples from a continuous-time stochastic process.

the joint probability distribution function

$$F_{\mathbf{x}(t_1),\ldots,\mathbf{x}(t_N)}(\xi_1,\ldots,\xi_N) \triangleq P(\{\omega:\mathbf{x}(t_1,\omega)\le\xi_1,\ldots,\mathbf{x}(t_N,\omega)\le\xi_N\}) \quad (4\text{-}2)$$

or the joint density function (if it exists):

$$f_{\mathbf{x}(t_1),\ldots,\mathbf{x}(t_N)}(\xi_1,\ldots,\xi_N) = \frac{\partial^{Nn}F_{\mathbf{x}(t_1),\ldots,\mathbf{x}(t_N)}(\xi_1,\ldots,\xi_N)}{\partial\xi_{11}\cdots\partial\xi_{1n}\cdots\partial\xi_{N1}\cdots\partial\xi_{Nn}} \quad (4\text{-}3)$$

That is to say, knowledge of such a joint distribution function or the associated joint density function completely describes the set of random variables.

If we want to characterize a continuous-time stochastic process completely, we would require knowledge of the joint probability distribution function $F_{\mathbf{x}(t_1),\ldots,\mathbf{x}(t_N)}(\xi_1,\ldots,\xi_N)$ for *all* possible sequences $\{t_1,t_2,\ldots\}$. Again, if it exists, the associated set of joint density functions for all possible time sequences would provide the same complete description. Consider Fig. 4.3. The distribution function $F_{\mathbf{x}(t_1)}(\xi_1)$ establishes the probability of the set of $\omega\in\Omega$ that gives rise to random process samples that assume values less than or equal to ξ_1 at time t_1. Similarly, $f_{\mathbf{x}(t_1)}(\xi_1)$ reveals the probability of the set of samples, out of the entire ensemble of process samples, that assume values between ξ_1 and $\xi_1 + d\xi_1$ at time t_1. The joint distribution $F_{\mathbf{x}(t_1),\mathbf{x}(t_2)}(\xi_1,\xi_2)$ indicates the probability of the set of samples that not only take on values less than or equal to

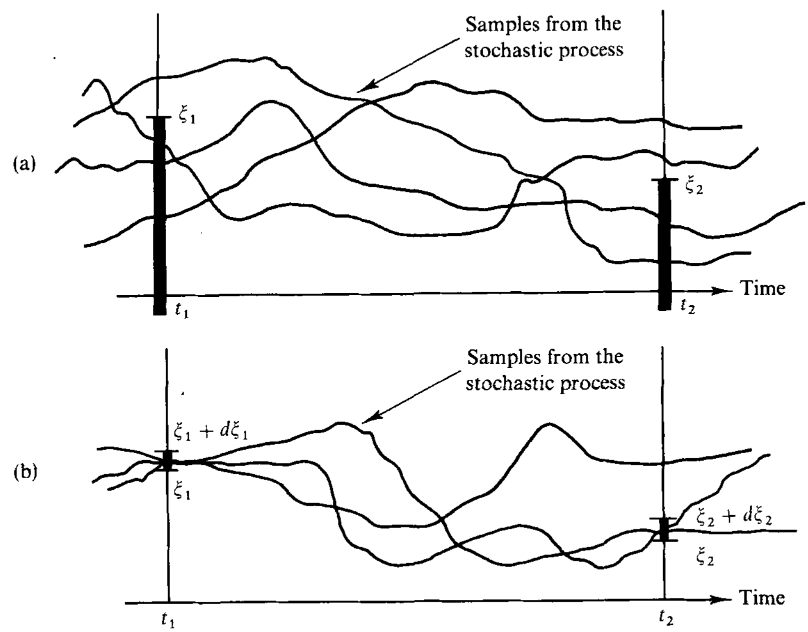

FIG. 4.3 Characterization of stochastic processes. (a) Joint distribution functions. (b) Joint density functions.

ξ_1 at t_1, but also take on values less than or equal to ξ_2 at t_2, and similarly for $f_{\mathbf{x}(t_1),\mathbf{x}(t_2)}(\xi_1, \xi_2)$. These functions more fully characterize the manner in which process samples can change values over time, but even a specification of such functions for all possible t_1 and t_2 would not provide a complete description. That would require an evaluation of all first and second order functions as described, plus all higher order functions for all choices of t_1, $t_2, \ldots$, etc.: conceptually satisfying but infeasible practically.

As can be seen from the definition of a stochastic process and the preceding discussion, the concepts and tools of probability theory from the previous chapter will readily apply to an investigation of stochastic processes. This is particularly true with respect to gaining at least a partial description of a stochastic process through a finite number of moments. Rather than trying to generate explicit relations for joint distributions (or densities if they exist), it is often convenient, especially computationally, to describe these functions to some degree by the associated first two moments. In the case of Gaussian processes, such information will completely characterize the joint distribution or density functions, and thus completely characterize the process itself. Note that in the following descriptions the ω argument will be deleted as in the previous chapter, but the boldface sans serif typeface will be maintained to demark stochastic processes: $\mathbf{x}(\cdot,\cdot)$ will be written as $\mathbf{x}(\cdot)$, $\mathbf{x}(t,\cdot)$ becomes the random variable $\mathbf{x}(t)$, and $\mathbf{x}(t,\omega_i) = \mathbf{x}(t)$ will be written as $\mathbf{x}(t)$.

The *mean value function* or mean $\mathbf{m}_x(\cdot)$ of the process $\mathbf{x}(\cdot)$ is defined for all $t \in T$ by

$$\mathbf{m}_x(t) \triangleq E\{\mathbf{x}(t)\} \tag{4-4}$$

i.e., the average value $\mathbf{x}(\cdot)$ assumes at time t, where the average is taken over the entire ensemble of samples from the process. An indication of the spread of values about the mean at time t, $\mathbf{m}_x(t)$, is provided by the second central moment, or *covariance matrix*, $\mathbf{P}_{xx}(\cdot)$, defined by

$$\mathbf{P}_{xx}(t) \triangleq E\{[\mathbf{x}(t) - \mathbf{m}_x(t)][\mathbf{x}(t) - \mathbf{m}_x(t)]^{\mathrm{T}}\} \tag{4-5}$$

A useful generalization of this, containing additional information about how fast $\mathbf{x}(t)$ sample values can change in time, is the *covariance kernel* $\mathbf{P}_{xx}(\cdot,\cdot)$, defined for all $t_1, t_2 \in T$ as

$$\mathbf{P}_{xx}(t_1, t_2) \triangleq E\{[\mathbf{x}(t_1) - \mathbf{m}_x(t_1)][\mathbf{x}(t_2) - \mathbf{m}_x(t_2)]^{\mathrm{T}}\} \tag{4-6}$$

The nature of the information embodied in (4-6) that is not available in (4-5) will be made more explicit in the example to follow. From (4-5) and (4-6), the covariance matrix can be defined as

$$\mathbf{P}_{xx}(t) = \mathbf{P}_{xx}(t, t) \tag{4-7}$$

i.e., $\mathbf{P}_{xx}(\cdot,\cdot)$ with both arguments the same time. The second noncentral moment concept generalizes to the *correlation kernel* $\boldsymbol{\Psi}_{xx}(\cdot,\cdot)$ defined for all $t_1, t_2 \in T$ as

$$\boldsymbol{\Psi}_{xx}(t_1,t_2) \triangleq E\{\mathbf{x}(t_1)\mathbf{x}(t_2)^{\mathrm{T}}\} \tag{4-8}$$

The *correlation matrix* would then be $\boldsymbol{\Psi}_{xx}(t,t)$, composed of individual correlations of components of $\mathbf{x}(t)$: its i–j component would be $E\{\mathrm{x}_i(t)\mathrm{x}_j(t)\}$, and the diagonal is made up of mean squared values of individual random variables $\mathrm{x}_i(t)$. From (4-6) and (4-8), it can be seen that

$$\boldsymbol{\Psi}_{xx}(t_1,t_2) = \mathbf{P}_{xx}(t_1,t_2) + \mathbf{m}_x(t_1)\mathbf{m}_x(t_2)^{\mathrm{T}} \tag{4-9}$$

and thus if $\mathbf{x}(\cdot)$ is a zero-mean process, $\boldsymbol{\Psi}_{xx}(t_1,t_2) = \mathbf{P}_{xx}(t_1,t_2)$.

EXAMPLE 4.1 Consider two scalar zero-mean processes $\mathrm{x}(\cdot)$ and $\mathrm{y}(\cdot)$ with

$$\Psi_{xx}(t_1,t_2) = P_{xx}(t_1,t_2) = \sigma^2 e^{-|t_1-t_2|/T}, \qquad \Psi_{yy}(t_1,t_2) = P_{yy}(t_1,t_2) = \sigma^2 e^{-|t_1-t_2|/10T}$$

where these two correlations are plotted as a function of the time difference $(t_1 - t_2)$ in Fig. 4.4. For a given value of $(t_1 - t_2) \neq 0$, there is a higher correlation between the values of $\mathrm{y}(t_1)$ and $\mathrm{y}(t_2)$ than between $\mathrm{x}(t_1)$ and $\mathrm{x}(t_2)$. Physically one would then expect a typical sample $\mathrm{x}(\cdot,\omega_i)$ to exhibit more rapid variations in magnitude than $\mathrm{y}(\cdot,\omega_i)$, as also depicted in Fig. 4.4. Note that such information is not contained in $P_{xx}(t)$ and $P_{yy}(t)$, or $\Psi_{xx}(t)$ and $\Psi_{yy}(t)$, all of which are the same value for this example, σ^2, as seen by evaluating the preceding expressions for $t_1 = t_2$. ■

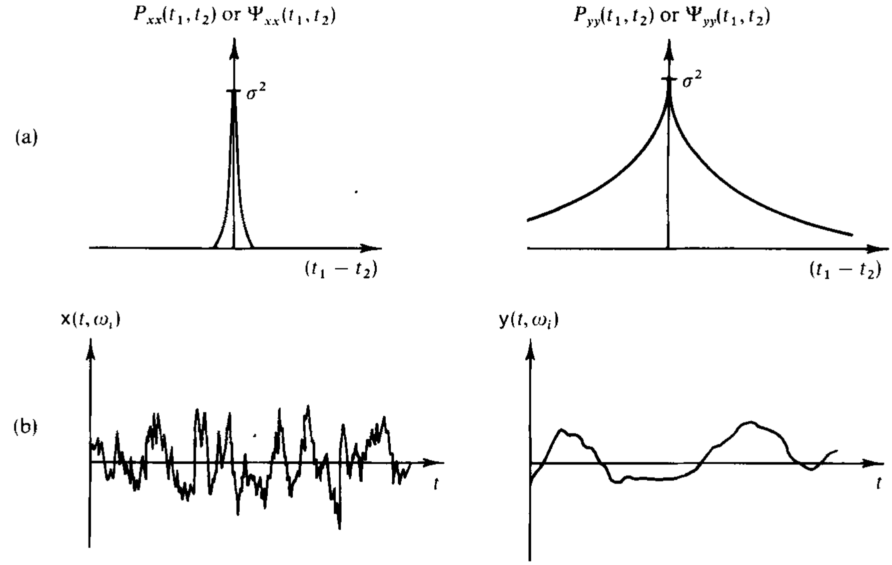

FIG. 4.4 Second moment information about stochastic processes. (a) Correlation or variance kernels. (b) Typical samples from the stochastic processes.

For characterizing the interrelationship between two stochastic processes $\mathbf{x}(\cdot)$ and $\mathbf{y}(\cdot)$, the preceding second moment concepts generalize to the *cross-covariance kernel* of $\mathbf{x}(\cdot)$ and $\mathbf{y}(\cdot)$, $\mathbf{P}_{xy}(\cdot,\cdot)$ defined for all $t_1, t_2 \in T$ as

$$\mathbf{P}_{xy}(t_1,t_2) \triangleq E\{[\mathbf{x}(t_1) - \mathbf{m}_x(t_1)][\mathbf{y}(t_2) - \mathbf{m}_y(t_2)]^{\mathrm{T}}\} \tag{4-10}$$

the *cross-covariance matrix*:

$$\mathbf{P}_{xy}(t) = \mathbf{P}_{xy}(t,t) \tag{4-11}$$

and the *cross-correlation kernel* and associated matrix:

$$\mathbf{\Psi}_{xy}(t_1,t_2) \triangleq E\{\mathbf{x}(t_1)\mathbf{y}(t_2)^{\mathrm{T}}\} \tag{4-12}$$

$$= \mathbf{P}_{xy}(t_1,t_2) + \mathbf{m}_x(t_1)\mathbf{m}_y{}^{\mathrm{T}}(t_2) \tag{4-13}$$

Other concepts also readily translate from probability theory, but care must be taken to avoid such ambiguities as the meaning of "independent processes" and "uncorrelated processes." A process $\mathbf{x}(\cdot,\cdot)$ is *independent* (*in time*) or *white* if, for any choice of $t_1,\ldots,t_N \in T$, $\mathbf{x}(t_1),\ldots,\mathbf{x}(t_N)$ are a set of independent random vectors; i.e.,

$$P(\{\omega:\mathbf{x}(t_1,\omega) \le \boldsymbol{\xi}_1,\ldots,\mathbf{x}(t_N,\omega) \le \boldsymbol{\xi}_N\}) = \prod_{i=1}^{N} P(\{\omega:\mathbf{x}(t_i,\omega) \le \boldsymbol{\xi}_i\}) \tag{4-14}$$

or equivalently,

$$F_{\mathbf{x}(t_1),\ldots,\mathbf{x}(t_N)}(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_N) = \prod_{i=1}^{N} F_{\mathbf{x}(t_i)}(\boldsymbol{\xi}_i) \tag{4-15}$$

or, if the densities exist,

$$f_{\mathbf{x}(t_1),\ldots,\mathbf{x}(t_N)}(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_N) = \prod_{i=1}^{N} f_{\mathbf{x}(t_i)}(\boldsymbol{\xi}_i) \tag{4-16}$$

On the other hand, two processes $\mathbf{x}(\cdot,\cdot)$ and $\mathbf{y}(\cdot,\cdot)$ are said to be *independent* (*of each other*) if, for any $t_1,\ldots,t_N \in T$,

$$\begin{aligned} P(\{\omega:\mathbf{x}(t_1,\omega) &\le \boldsymbol{\xi}_1,\ldots,\mathbf{x}(t_N,\omega) \le \boldsymbol{\xi}_N, \mathbf{y}(t_1,\omega) \le \boldsymbol{\rho}_1,\ldots,\mathbf{y}(t_N,\omega) \le \boldsymbol{\rho}_N\}) \\ &= P(\{\omega:\mathbf{x}(t_1,\omega) \le \boldsymbol{\xi}_1,\ldots,\mathbf{x}(t_N,\omega) \le \boldsymbol{\xi}_N\}) \\ &\quad \cdot P(\{\omega:\mathbf{y}(t_1,\omega) \le \boldsymbol{\rho}_1,\ldots,\mathbf{y}(t_N,\omega) \le \boldsymbol{\rho}_N\}) \end{aligned} \tag{4-17}$$

Thus, "two independent processes" could mean two processes, each of which were independent in time, or two processes independent of each other, or some combination of these. We will use the term "white" to clarify this issue.

In a similar manner, a process $\mathbf{x}(\cdot,\cdot)$ is *uncorrelated* (*in time*) if, for all $t_1, t_2 \in T$ except for $t_1 = t_2$,

$$\mathbf{\Psi}_{xx}(t_1,t_2) \triangleq E[\mathbf{x}(t_1)\mathbf{x}^{\mathrm{T}}(t_2)] = E[\mathbf{x}(t_1)]E[\mathbf{x}^{\mathrm{T}}(t_2)] \tag{4-18a}$$

or

$$\mathbf{P}_{xx}(t_1,t_2) = \mathbf{0} \tag{4-18b}$$

By comparison, two processes $\mathbf{x}(\cdot,\cdot)$ and $\mathbf{y}(\cdot,\cdot)$ are *uncorrelated* with each other if, for all $t_1, t_2 \in T$ (including $t_1 = t_2$),

$$\mathbf{\Psi}_{xy}(t_1,t_2) \triangleq E[\mathbf{x}(t_1)\mathbf{y}^{\mathrm{T}}(t_2)] = E[\mathbf{x}(t_1)]E[\mathbf{y}^{\mathrm{T}}(t_2)] \tag{4-19a}$$

or

$$\mathbf{P}_{xy}(t_1, t_2) = \mathbf{0} \tag{4-19b}$$

As shown previously, independence implies uncorrelatedness (which restricts attention to only the second moments), but the opposite implication is not true, except in such special cases as Gaussian processes, to be discussed. Note that "white" is often accepted to mean uncorrelated in time rather than independent in time; the distinction between these definitions disappears for the important case of white Gaussian processes.

In the previous chapter, much attention was devoted to Gaussian random variables, motivated by both practical justification (central limit theorem) and mathematical considerations (such as the first two moments completely describing the distribution and Gaussianness being preserved through linear operations). Similarly, Gaussian processes will be of primary interest here. A process $\mathbf{x}(\cdot,\cdot)$ is a *Gaussian process* if all finite joint distribution functions for $\mathbf{x}(t_1,\cdot)$, $\mathbf{x}(t_2,\cdot), \ldots, \mathbf{x}(t_N,\cdot)$ are Gaussian for any choice of $t_1, t_2, \ldots, t_N$. For instance, if $\mathbf{x}(\cdot,\cdot)$ is Gaussian and the appropriate densities exist, then for any choice of $t_1, t_2 \in T$,

$$F_{\mathbf{x}(t_1),\mathbf{x}(t_2)}(\boldsymbol{\xi}) = [(2\pi)^n|\mathbf{P}|^{1/2}]^{-1} \exp\{-\tfrac{1}{2}(\boldsymbol{\xi} - \mathbf{m})^{\mathrm{T}}\mathbf{P}^{-1}(\boldsymbol{\xi} - \mathbf{m})\} \tag{4-20}$$

where

$$\mathbf{m} = \left[\begin{array}{c} \mathbf{m}_x(t_1) \\ \hline \mathbf{m}_x(t_2) \end{array}\right] = \left[\begin{array}{c} E[\mathbf{x}(t_1)] \\ E[\mathbf{x}(t_2)] \end{array}\right] \tag{4-21a}$$

$$\mathbf{P} = \left[\begin{array}{c|c} E[\mathbf{x}(t_1)\mathbf{x}^{\mathrm{T}}(t_1)] - \mathbf{m}_x(t_1)\mathbf{m}_x^{\mathrm{T}}(t_1) & E[\mathbf{x}(t_1)\mathbf{x}^{\mathrm{T}}(t_2) - \mathbf{m}_x(t_1)\mathbf{m}_x^{\mathrm{T}}(t_2) \\ \hline E[\mathbf{x}(t_2)\mathbf{x}^{\mathrm{T}}(t_1)] - \mathbf{m}_x(t_2)\mathbf{m}_x^{\mathrm{T}}(t_1) & E[\mathbf{x}(t_2)\mathbf{x}^{\mathrm{T}}(t_2) - \mathbf{m}_x(t_2)\mathbf{m}_x^{\mathrm{T}}(t_2) \end{array}\right]$$

$$= \left[\begin{array}{c|c} E[\mathbf{x}(t_1)\mathbf{x}^{\mathrm{T}}(t_1)] & E[\mathbf{x}(t_1)\mathbf{x}^{\mathrm{T}}(t_2)] \\ \hline E[\mathbf{x}(t_2)\mathbf{x}^{\mathrm{T}}(t_1)] & E[\mathbf{x}(t_2)\mathbf{x}^{\mathrm{T}}(t_2)] \end{array}\right] - \mathbf{m}\mathbf{m}^{\mathrm{T}} \tag{4-21b}$$

Analogous statements could then be made about density functions of any order, corresponding to any choice of N time points instead of just two.

4.3 STATIONARY STOCHASTIC PROCESSES AND POWER SPECTRAL DENSITY

One particularly pertinent characterization of a stochastic process is whether or not it is stationary. In this regard, there is a strict sense of stationarity, concerned with all moments, and a wide sense of stationarity, concerned only with the first two moments. A process $\mathbf{x}(\cdot,\cdot)$ is *strictly stationary* if, for all sets $t_1, \ldots, t_N \in T$ and any $\tau \in T$ [supposing that $(t_i + \tau) \in T$ also], the joint distribution of $\mathbf{x}(t_1 + \tau), \ldots, \mathbf{x}(t_N + \tau)$ does not depend on the time shift τ: i.e.,

$$\begin{aligned} &P(\{\omega : \mathbf{x}(t_1 + \tau, \omega) \le \xi_1, \ldots, \mathbf{x}(t_N + \tau, \omega) \le \xi_N\}) \\ &\quad = P(\{\omega : \mathbf{x}(t_1, \omega) \le \xi_1, \ldots, \mathbf{x}(t_N, \omega) \le \xi_N\}) \end{aligned} \tag{4-22}$$

A process $\mathbf{x}(\cdot,\cdot)$ is *wide-sense stationary* if, for all $t, \tau \in T$, the following three criteria are met:

(i) $E\{\mathbf{x}(t)\mathbf{x}(t)^{\mathrm{T}}\}$ is finite.
(ii) $E\{\mathbf{x}(t)\}$ is a constant.
(iii) $E\{[\mathbf{x}(t) - \mathbf{m}_x][\mathbf{x}(t+\tau) - \mathbf{m}_x]^{\mathrm{T}}\}$ depends only on the time difference τ [and thus $\mathbf{\Psi}_{xx}(t) = \mathbf{\Psi}_{xx}(t,t)$ and $\mathbf{P}_{xx}(t) = \mathbf{P}_{xx}(t,t)$ are constant].

Thus, in general, a strictly stationary process is wide-sense stationary if and only if it has finite second moments, and wide-sense stationarity does not imply strict-sense stationarity. However, in the special case of Gaussian processes, a wide-sense stationary process is strict-sense stationary as well.

By definition, the correlation and covariance kernels $\mathbf{\Psi}_{xx}(t,t+\tau)$ and $\mathbf{P}_{xx}(t,t+\tau)$ for wide-sense stationary $\mathbf{x}(\cdot,\cdot)$ are functions only of the time difference τ; this is often made explicit notationally by writing these as functions of a single argument, the time difference τ:

$$\mathbf{\Psi}_{xx}(t,t+\tau) \to \mathbf{\Psi}_{xx}(\tau) \tag{4-23a}$$

$$\mathbf{P}_{xx}(t,t+\tau) \to \mathbf{P}_{xx}(\tau) \tag{4-23b}$$

To avoid confusion between these relations and $\mathbf{\Psi}_{xx}(t)$ and $\mathbf{P}_{xx}(t)$ as defined in (4-7), a single argument τ will be reserved to denote a time *difference* when discussing functions $\mathbf{\Psi}_{xx}(\cdot)$ or $\mathbf{P}_{xx}(\cdot)$. A further characterization of these functions can be made as well: not only are the diagonal terms of $\mathbf{\Psi}_{xx}$ functions only of τ, but they are *even* functions of τ that assume their maximum value at $\tau = 0$.

In the case of wide-sense stationary processes, Fourier transform theory can be exploited to generate a frequency-domain characterization of processes in the form of power spectral densities. If a scalar time function $y(\cdot)$ is Fourier transformable, the relation between it and its Fourier transform $\bar{y}(\cdot)$, as a function of frequency ω, is given by

$$\bar{y}(\omega) = a\int_{-\infty}^{\infty} y(t)e^{-j\omega t}\,dt \tag{4-24a}$$

$$y(t) = b\int_{-\infty}^{\infty} \bar{y}(\omega)e^{j\omega t}\,d\omega \tag{4-24b}$$

where a and b are scalars such that their product is $1/(2\pi)$:

$$ab = 1/(2\pi) \tag{4-24c}$$

Power spectral density of a scalar wide-sense stationary process $\mathrm{x}(\cdot,\cdot)$ is defined as the Fourier transform of the correlation function $\Psi_{xx}(\tau)$. Since samples from a wide-sense stationary process must be visualized as existing for all negative and positive time if τ is to be allowed to assume any value in R^1, the two-sided Fourier transform, i.e., integrating from $t = -\infty$ to $t = \infty$, does make sense conceptually in the definition.

Some useful properties of Fourier transforms of real-valued functions include the following:

(1) $\bar{y}(\omega)$ is complex in general, with $\bar{y}(-\omega) = \bar{y}^*(\omega)$, where * denotes complex conjugate.

(2) Thus, $\bar{y}(\omega)\bar{y}(-\omega) = \bar{y}(\omega)\bar{y}^*(\omega) = |\bar{y}(\omega)|^2$.

(3) The real part of $\bar{y}(\cdot)$ is an even function of ω, and the imaginary part is odd in ω.

(4) Although $\delta(t)$ is not Fourier transformable, its transform can be defined formally through (4-24a) as

$$\bar{\delta}(\omega) = a \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = a \tag{4-25a}$$

so that (4-24b) yields, formally,

$$\delta(t) = b \int_{-\infty}^{\infty} a e^{j\omega t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t}\, d\omega \tag{4-25b}$$

Unfortunately, there are a number of conventions on the choice of a and b in (4-24) for defining power spectral density. The most common convention is

$$\bar{\Psi}_{xx}(\omega) = \int_{-\infty}^{\infty} \Psi_{xx}(\tau) e^{-j\omega\tau}\, d\tau \tag{4-26a}$$

$$\Psi_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\Psi}_{xx}(\omega) e^{j\omega\tau}\, d\omega \tag{4-26b}$$

Note that if frequency is expressed in hertz rather than rad/sec, using $f = \omega/(2\pi)$, then there is a unity coefficient for both defining equations:

$$\bar{\Psi}_{xx}(f) = \int_{-\infty}^{\infty} \Psi_{xx}(\tau) e^{-j2\pi f\tau}\, d\tau \tag{4-27a}$$

$$\Psi_{xx}(\tau) = \int_{-\infty}^{\infty} \bar{\Psi}_{xx}(f) e^{j2\pi f\tau}\, df \tag{4-27b}$$

Using this convention, power spectral density is typically specified in units of (quantity)^2/hertz. Since $\Psi_{xx}(0)$ is just the mean squared value of $\mathrm{x}(t)$, it can be obtained by integrating the power spectral density function:

$$E\{\mathrm{x}(t)^2\} = \Psi_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\Psi}_{xx}(\omega)\, d\omega = \int_{-\infty}^{\infty} \bar{\Psi}_{xx}(f)\, df \tag{4-28}$$

Furthermore, we can use Euler's identity to write (4-26a) as

$$\bar{\Psi}_{xx}(\omega) = \int_{-\infty}^{\infty} \Psi_{xx}(\tau)[\cos \omega\tau - j \sin \omega\tau]\, d\tau$$

and, since $\Psi_{xx}(\tau)$ and $\cos\omega\tau$ are even functions of τ and $\sin\omega\tau$ is odd in τ, this becomes

$$\bar{\Psi}_{xx}(\omega) = \int_{-\infty}^{\infty} \Psi_{xx}(\tau)\cos\omega\tau\, d\tau \tag{4-29a}$$

$$= 2\int_{0}^{\infty} \Psi_{xx}(\tau)\cos\omega\tau\, d\tau \qquad [\text{if } \Psi_{xx}(0) \text{ is finite}] \tag{4-29b}$$

Thus, the power spectral density function is a *real*, *even* function of ω. It can be shown to be a pointwise *positive* function of ω as well. Analogously to (4-29), the correlation function becomes

$$\Psi_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\Psi}_{xx}(\omega)\cos\omega\tau\, d\omega \tag{4-30a}$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \bar{\Psi}_{xx}(\omega)\cos\omega\tau\, d\omega \qquad [\text{if } \bar{\Psi}_{xx}(0) \text{ is finite}] \tag{4-30b}$$

Another common convention for power spectral density is

$$\bar{\Psi}'_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_{xx}(\tau)e^{-j\omega\tau}\, d\tau \tag{4-31a}$$

$$\Psi_{xx}(\tau) = \int_{-\infty}^{\infty} \bar{\Psi}'_{xx}(\omega)e^{j\omega\tau}\, d\omega \tag{4-31b}$$

with units as $(\text{quantity})^2/(\text{rad/sec})$, motivated by the property that the mean squared value becomes

$$E\{\mathrm{x}(t)^2\} = \int_{-\infty}^{\infty} \bar{\Psi}'_{xx}(\omega)\, d\omega \tag{4-32}$$

i.e., without the $1/(2\pi)$ factor. A third convention encountered commonly is

$$\bar{\Psi}''_{xx}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \Psi_{xx}(\tau)e^{-j\omega\tau}\, d\tau \tag{4-33a}$$

$$\Psi_{xx}(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} \bar{\Psi}''_{xx}(\omega)e^{j\omega\tau}\, d\omega \tag{4-33b}$$

also with units as $(\text{quantity})^2/(\text{rad/sec})$, such that

$$E\{\mathrm{x}(t)^2\} = \int_{0}^{\infty} \bar{\Psi}''_{xx}(\omega)\, d\omega \tag{4-34}$$

using only *positive* values of ω to correspond to a physical interpretation of frequencies being nonnegative.

The name power spectral density can be motivated by interpreting "power" in the generalized sense of expected squared values of the members of an ensemble. $\bar{\Psi}_{xx}(\omega)$ is a spectral density for the power in a process $\mathrm{x}(\cdot)$ in that integration of $\bar{\Psi}_{xx}$ over the frequency band from ω_1 to ω_2 yields the mean squared value of the process which consists only of those harmonic com-

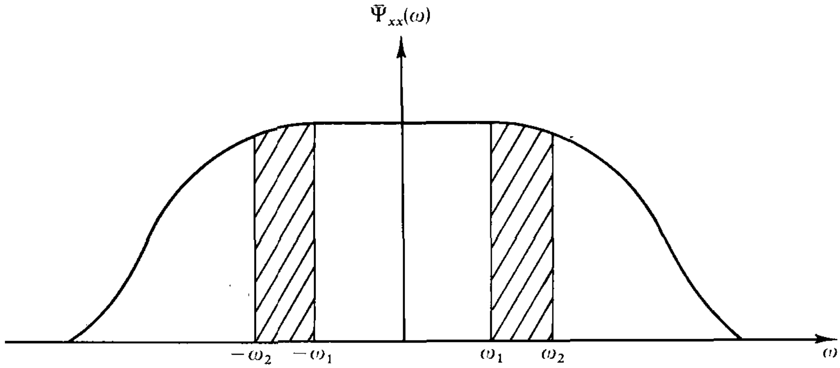

FIG. 4.5 Power spectral density.

ponents of $x(t)$ that lie between ω_1 and ω_2, as shown by the shaded region in Fig. 4.5. (This is in fact another means of defining power spectral density.) In particular, the mean squared value of $x(t)$ itself is given by an integration of $\Psi_{xx}(\omega)$ over the full range of possible frequencies ω.

EXAMPLE 4.2 Figure 4.6 depicts the autocorrelation functions and power spectral density functions (using the most common convention of definition) of a white process, an exponentially time-correlated process, and a random bias. Note that a white noise is uncorrelated in time, yielding an impulse at $\tau = 0$ in Fig. 4.6a; the corresponding power spectral density is flat over all ω—equal power content over all frequencies. Figure 4.6b corresponds to an exponentially time-correlated process with correlation time T, as discussed in Example 4.1. Heuristically, these

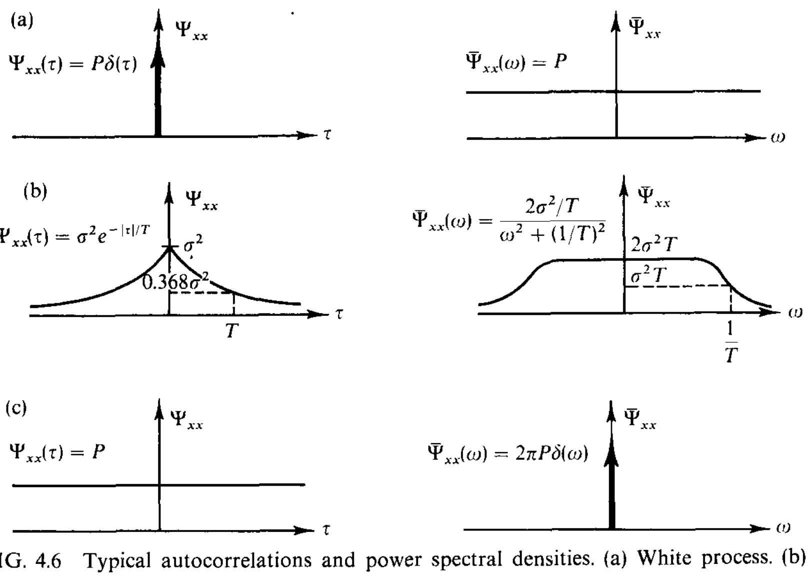

FIG. 4.6 Typical autocorrelations and power spectral densities. (a) White process. (b) Exponentially time-correlated process. (c) Random bias.

plots converge to those of (a) as $T \to 0$, and to those of (c) as $T \to \infty$. Figure 4.6c corresponds to a random bias process, the samples of which are constant in time—thus, there is constant correlation over all time differences τ, and all of the process power is concentrated at the zero frequency. Another process with nonzero power at a *discrete* frequency would be the process composed of sinusoids at a known frequency ω_0 and of uniformly distributed phase, with power spectral density composed of two impulses, at ω_0 and $-\omega_0$, and a cosinusoidal autocorrelation. ■

The *cross-power spectral density* of two wide-sense stationary scalar processes $\mathsf{x}(\cdot,\cdot)$ and $\mathsf{y}(\cdot,\cdot)$ is the Fourier transform of the associated cross-correlation function:

$$\bar{\Psi}_{xy}(\omega) = \int_{-\infty}^{\infty} \Psi_{xy}(\tau) e^{-j\omega\tau}\, d\tau \tag{4-35}$$

This is, in general, a complex function of ω, and, since $\Psi_{xy}(\tau) = \Psi_{yx}(-\tau)$, the following relations are valid.

$$\bar{\Psi}_{xy}(\omega) = \bar{\Psi}_{yx}^{*}(\omega) \tag{4-36}$$

$$\tfrac{1}{2}[\bar{\Psi}_{xy}(\omega) + \bar{\Psi}_{yx}(\omega)] = \mathrm{Real}\{\bar{\Psi}_{xy}(\omega)\} = \mathrm{Real}\{\bar{\Psi}_{yx}(\omega)\} \tag{4-37}$$

One subclass of real strictly stationary processes of particular interest is the set of ergodic processes. A process is *ergodic* if any statistic calculated by averaging over all members of the ensemble of samples at a fixed time can be calculated equivalently by time-averaging over any single representive member of the ensemble, except possibly a single member out of a set of probability zero. Not all stationary processes are ergodic: the ensemble of constant functions is an obvious counterexample, in that a time average of each sample will yield that particular constant value rather than the mean value for the entire ensemble.

There is no readily applied condition to ensure ergodicity in general. For a scalar stationary Gaussian process $\mathsf{x}(\cdot,\cdot)$ defined on $T = (-\infty, \infty)$, a sufficient condition does exist: $\mathsf{x}(\cdot,\cdot)$ is ergodic if $\int_{-\infty}^{\infty} |P_{xx}(\tau)|\, d\tau$ is finite [however, $P_{xx}(\tau)$ itself requires an ensemble average]. In practice, empirical results for stationary processes are often obtained by time-averaging of a single process sample, under the *assumption* of ergodicity, such as

$$m_x = E[\mathsf{x}(t,\cdot)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathsf{x}(t,\omega_i)\, dt \tag{4-38a}$$

$$\begin{aligned} \Psi_{xx}(\tau) &\triangleq E[\mathsf{x}(t,\cdot)\mathsf{x}(t+\tau,\cdot)] \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathsf{x}(t,\omega_i)\mathsf{x}(t+\tau,\omega_i)\, dt \end{aligned} \tag{4-38b}$$

$$\begin{aligned} \Psi_{xy}(\tau) &= E[\mathsf{x}(t,\cdot)\mathsf{y}(t+\tau,\cdot)] \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathsf{x}(t,\omega_i)\mathsf{y}(t+\tau,\omega_i)\, dt \end{aligned} \tag{4-38c}$$

Moreover, these are only approximately evaluated due to the use of finite length, rather than infinite, samples.

4.4 SYSTEM MODELING: OBJECTIVES AND DIRECTIONS

Suppose we are given a physical system that can be subjected to known controls and to inputs beyond our own direct control, typically wideband noises (noises with instantaneous power over a wide range of frequencies), although narrow band noises and other forms are also possible. Further assume that we want to characterize certain outputs of the system, for instance, by depicting their mean and covariance kernel for all time values. Such a characterization would be necessary to initiate designs of estimators or controllers for the system, and a prerequisite to a means of analyzing the performance capabilities of such devices as well.

The objective of a mathematical model would be to generate an adequate, tractable representation of the behavior of all outputs of interest from the real physical system. Here "adequate" is subjective and is a function of the intended use of this representation. For example, if a gyro were being tested in a laboratory, one would like to develop a mathematical model that would generate outputs whose characteristics were identical to those actually observed empirically. Since no model is perfect, one really attempts to generate models that closely approximate the behavior of observed quantities.

From deterministic modeling, we gain the insight that a potentially useful model form would be a linear state equation and sampled data output relation *formally* written as

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{n}_1(t) \tag{4-39a}$$

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{n}_2(t_i) \tag{4-39b}$$

These are direct extensions of Eqs. (2-35) and (2-60) obtained by adding a noise process $\mathbf{n}_1(\cdot,\cdot)$ to the dynamics equation and $\mathbf{n}_2(\cdot,\cdot)$ to the output equation, with $\mathbf{n}_1(t,\omega_k) \in R^s$ and $\mathbf{n}_2(t_i,\omega_j) \in R^m$. [$\mathbf{G}(t)$ is n-by-s to be compatible with the state dimension n, and $\mathbf{n}_2(t_i,\omega_j)$ is of dimension m, the number of measurements available.] Note that (4-39b) could be extended to allow direct feedthrough of $\mathbf{u}$ and $\mathbf{n}_1$, but this will not be pursued here. Again it is emphasized that (4-39) is just a formal extension; for instance, how would $\dot{\mathbf{x}}(t)$ be interpreted fundamentally?

Unfortunately, a model of this generality is not directly exploitable. We would like to evaluate the joint probability distribution or density function for $\mathbf{x}(t_1,\cdot),\ldots,\mathbf{x}(t_N,\cdot)$, and, through this evaluation, to obtain the corresponding joint functions for $\mathbf{z}(t_1,\cdot),\ldots,\mathbf{z}(t_N,\cdot)$. This is generally infeasible. For example, if $\mathbf{n}_1(t)$ were uniformly distributed for all $t \in T$, one can say very little about these joint probability functions. However, if $\mathbf{n}_1(\cdot,\cdot)$ and $\mathbf{n}_2(\cdot,\cdot)$ were assumed *Gaussian*, then all distribution or density functions of interest might be shown to be Gaussian as well, completely characterized by the corresponding first two moments. If observed quantities are not Gaussian, one can still seek to construct a model that provides a Gaussian output whose first two moments duplicate the first two moments of the empirically observed data.

Complete depiction of a joint distribution or density is still generally intractable however. In order to achieve this objective, we will restrict attention to system inputs describable as *Markov processes*. Let $\mathbf{x}(\cdot,\cdot)$ be a random process and consider

$$F_{\mathbf{x}(t_i)|\mathbf{x}(t_{i-1}),\mathbf{x}(t_{i-2}),\ldots,\mathbf{x}(t_j)}(\boldsymbol{\xi}_i|\mathbf{x}_{i-1},\mathbf{x}_{i-2},\ldots,\mathbf{x}_j)$$

i.e., the probability distribution function of $\mathbf{x}(t_i)$ as a function of the n-vector $\boldsymbol{\xi}_i$, given that $\mathbf{x}(t_{i-1},\omega_k)=\mathbf{x}_{i-1}$, $\mathbf{x}(t_{i-2},\omega_k)=\mathbf{x}_{i-2},\ldots,\mathbf{x}(t_j,\omega_k)=\mathbf{x}_j$. If

$$\begin{aligned}F_{\mathbf{x}(t_i)|\mathbf{x}(t_{i-1}),\mathbf{x}(t_{i-2}),\ldots,\mathbf{x}(t_j)}&(\boldsymbol{\xi}_i|\mathbf{x}_{i-1},\mathbf{x}_{i-2},\ldots,\mathbf{x}_j)\\ &=F_{\mathbf{x}(t_i)|\mathbf{x}(t_{i-1})}(\boldsymbol{\xi}_i|\mathbf{x}_{i-1})\end{aligned}\tag{4-40}$$

for any countable choice of values i and j and for all values of $\mathbf{x}_{i-1},\ldots,\mathbf{x}_j$, then $\mathbf{x}(\cdot,\cdot)$ is a Markov process. Thus, the Markov property for stochastic processes is conceptually analogous to the ability to define a system *state* for deterministic processes. The value that the process $\mathbf{x}(\cdot,\cdot)$ assumes at time t_{i-1} provides as much information about $\mathbf{x}(t_i,\cdot)$ as do the values of $\mathbf{x}(\cdot,\cdot)$ at time t_{i-1} and all previous time instants: the value assumed by $\mathbf{x}(t_{i-1},\cdot)$ embodies all information needed for propagation to time t_i, and the past history leading to $\mathbf{x}_{i-1}$ is of no consequence. In the context of linear system models, the Markov assumption will be shown equivalent to the fact that the continuous-time process $\mathbf{n}_1(\cdot,\cdot)$ and the discrete-time process $\mathbf{n}_2(\cdot,\cdot)$ in (4-39) are expressible as the outputs of linear state-described models, called "shaping filters," driven only by deterministic inputs and *white noises*. A *Gauss–Markov process* is then a process which is both Gaussian and Markov.

Thus, the form of the system model depicted in Fig. 4.7 is motivated. A linear model of the physical system is driven by deterministic inputs, white Gaussian noises, and Gauss–Markov processes. As discussed in Section 1.4, the white noises are chosen as adequate representations of wideband noises with essentially constant power density over the system bandpass. The other Markov processes are time-correlated processes for which a white model would be inadequate. However, these can be generated by passing white noise through linear shaping filters. Consequently, one can consider the original system model and the shaping filters as a single "augmented" linear system, driven *only* by deterministic inputs and white Gaussian noises. This can be described through a restricted form of (4-39):

$$\dot{\mathbf{x}}(t)=\mathbf{F}(t)\mathbf{x}(t)+\mathbf{B}(t)\mathbf{u}(t)+\mathbf{G}(t)\mathbf{w}(t)\tag{4-41a}$$

$$\mathbf{z}(t_i)=\mathbf{H}(t_i)\mathbf{x}(t_i)+\mathbf{v}(t_i)\tag{4-41b}$$

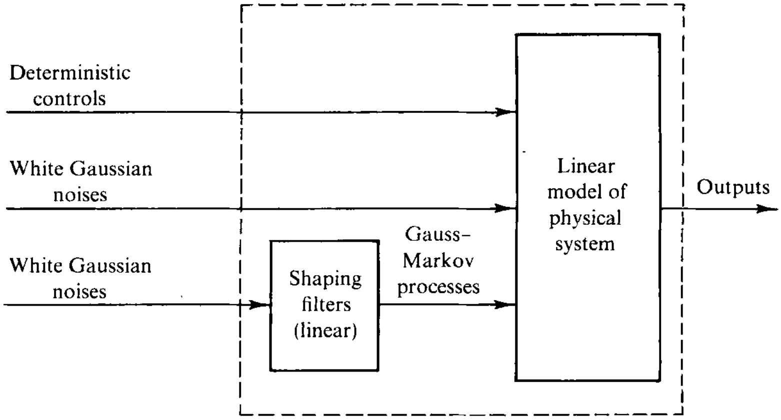

FIG. 4.7 Linear system model.

where $\mathbf{x}(t)$ is now the augmented system state, and $\mathbf{w}(t)$ and $\mathbf{v}(t_i)$ are white Gaussian noises, assumed independent of each other and of the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{x}_0$ is a Gaussian random variable. These noises model not only the disturbances and noise corruption that affect the system, but also the uncertainty inherent in the mathematical models themselves.

Using the insights from deterministic system theory, one seeks a solution to (4-41a): formally, one could write

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{G}(\tau)\mathbf{w}(\tau)\,d\tau \quad (4\text{-}42)$$

If this were a valid result, then it would be possible to describe the first two moments of $\mathbf{x}(\cdot, \cdot)$ and thus totally characterize this Gaussian stochastic process. However, the last term in (4-42) cannot be evaluated properly, and thus (4-42) has no real meaning at all. The remainder of this chapter is devoted to (1) a proper development of the mathematics as motivated by this section and (2) the practical application of the results to useful model formulations.

4.5 FOUNDATIONS: WHITE GAUSSIAN NOISE AND BROWNIAN MOTION

The previous section motivated the use of white Gaussian noise models as the only stochastic inputs to a linear system model, and this warrants further attention. First, a process $\mathbf{x}(\cdot, \cdot)$ is a *white Gaussian process* if, for any choice of $t_1, \ldots, t_N \in T$, the N random vectors $\mathbf{x}(t_1, \cdot), \ldots, \mathbf{x}(t_N, \cdot)$ are independent Gaussian random vectors. If the time set of interest, T, is a set of discrete time points, this is conceptually straightforward and implies that

$$\mathbf{P}_{xx}(t_i, t_j) = \mathbf{0} \qquad \text{if} \quad i \neq j \quad (4\text{-}43)$$

Such a discrete-time process can in fact be used to drive a difference equation model of a system with no theoretical difficulties.

However, if T is a time interval, then the definition of a white Gaussian process implies that there is no correlation between $\mathbf{x}(t_i, \cdot)$ and $\mathbf{x}(t_j, \cdot)$, even for t_i and t_j separated by only an infinitesimal amount:

$$E\{\mathbf{x}(t_i)\mathbf{x}^{\mathrm{T}}(t_j)\} = \mathbf{\Psi}_{xx}(t_i)\,\delta(t_i - t_j) \tag{4-44}$$

This is contrary to the behavior exhibited by any processes observed empirically. If we consider stationary white Gaussian noise [not wide sense to be precise, since $E\{\mathbf{x}(t_i)\mathbf{x}^{\mathrm{T}}(t_i)\}$ is not finite], then the power spectral density of such a process would be constant over *all* frequencies, and thus it would be an infinite power process: thus it cannot exist. Moreover, if we were to construct a continuous-time system model in the form of a linear differential equation driven by such a process, then a solution to the differential equation could *not* be obtained rigorously, as pointed out in the last section.

Brownian motion (or the "Wiener process") [8, 13] will serve as a basic process for continuous-time modeling. Through it, system models can be *properly* developed in the form of stochastic differential equations whose solutions *can* be obtained. Scalar constant-diffusion Brownian motion will be discussed first, and then extensions made to the general case of a vector time-varying-diffusion Brownian motion process.

To discuss Brownian motion, we first need a definition of a *process with independent increments*. Let $t_0 < t_1 < \cdots < t_N$ be a partition of the time interval T. If the "increments" of the process $\mathbf{x}(\cdot,\cdot)$, i.e., the set of N random variables

$$\begin{aligned}
\boldsymbol{\delta}_1(\cdot) &= [\mathbf{x}(t_1,\cdot) - \mathbf{x}(t_0,\cdot)]\\
\boldsymbol{\delta}_2(\cdot) &= [\mathbf{x}(t_2,\cdot) - \mathbf{x}(t_1,\cdot)]\\
&\ \ \vdots\\
\boldsymbol{\delta}_N(\cdot) &= [\mathbf{x}(t_N,\cdot) - \mathbf{x}(t_{N-1},\cdot)]
\end{aligned} \tag{4-45}$$

are mutually independent for *any* such partition of T, then $\mathbf{x}(\cdot,\cdot)$ is said to be a process with independent increments.

The process $\beta(\cdot,\cdot)$ is defined to be a *scalar constant-diffusion Brownian motion process* if

(i) it is a process with independent increments,

(ii) the increments are Gaussian random variables such that, for t_1 and t_2 any time instants in T,

$$E\{[\beta(t_2) - \beta(t_1)]\} = 0 \tag{4-46a}$$

$$E\{[\beta(t_2) - \beta(t_1)]^2\} = q|t_2 - t_1| \tag{4-46b}$$

(iii) $\beta(t_0, \omega_i) = 0$ for all $\omega_i \in \Omega$, except possibly a set of ω_i of probability zero (this is by *convention*).

Such a definition provides a mathematical abstraction of empirically observed Brownian motion processes, such as the motion of gas molecules. Figure 4.8 depicts some samples from a Brownian motion process $\beta(\cdot,\cdot)$. Note that for all samples shown, $\beta(t_0,\omega_i) = 0$. Specific realizations of increments are also shown, for instance,

$$\begin{aligned} \Delta_1 &= \delta_1(\omega_2) = [\beta(t_2,\omega_2) - \beta(t_1,\omega_2)] \\ \Delta_2 &= \delta_2(\omega_2) = [\beta(t_3,\omega_2) = \beta(t_2,\omega_2)] \end{aligned} \tag{4-47}$$

The random variables $\delta_1(\cdot)$ and $\delta_2(\cdot)$ are independent.

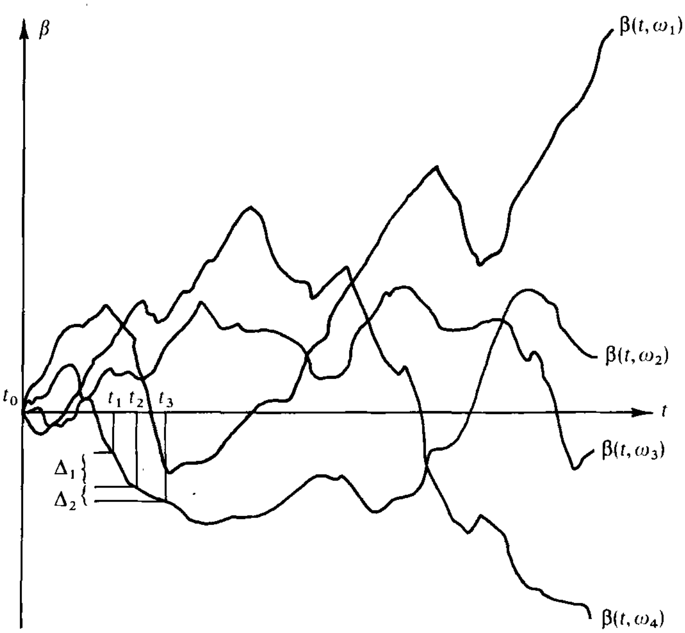

FIG. 4.8 Samples from a Brownian motion process.

The parameter q in (4-46b) is called the *diffusion* of the process. Note that the variance of the change in value of $\beta(\cdot,\cdot)$ between any time t_1 and any later time t_2 is a *linear* function of the *time difference* $(t_2 - t_1)$. For that reason, constant-diffusion Brownian motion is sometimes termed (misleadingly) "stationary" Brownian motion, but such terminology will be avoided here.

Since $\beta(t_1,\cdot)$ for given $t_1 \in T$ is a random variable composed of a sum of independent Gaussian increments, it is also Gaussian, with statistics

$$m_\beta(t_i) = E[\beta(t_i)] = 0 \tag{4-48a}$$

$$P_{\beta\beta}(t_i) = E[\beta(t_i)^2] = q[t_i - t_0] \tag{4-48b}$$

Thus q indicates how fast the mean square value of $\beta(\cdot,\cdot)$ diverges from its initial value of zero at time t_0.

To characterize the scalar constant-diffusion Brownian motion completely, we can explicitly generate the joint density functions for any finite set of random variables $\beta(t_0,\cdot),\ldots,\beta(t_N,\cdot)$ as a Gaussian density function, with zero mean and covariance composed of diagonal terms as $P_{\beta\beta}(t_i,t_i) = P_{\beta\beta}(t_i)$ as in (4-48b). The off-diagonal terms can be specified by considering $t_j > t_i$ and writing

$$\beta(t_j) = \beta(t_i) + [\beta(t_j) - \beta(t_i)]$$

so that the off-diagonal elements become

$$\begin{aligned} E\{\beta(t_i)\beta(t_j)\} &= E\{\beta(t_i)^2\} + E\{\beta(t_i)[\beta(t_j) - \beta(t_i)]\} \\ &= E\{\beta(t_i)^2\} + E\{\beta(t_i)\}E\{[\beta(t_j) - \beta(t_i)]\} \\ &= E\{\beta(t_i)^2\} = q(t_i - t_0) \end{aligned} \tag{4-49}$$

where the second equality follows from independence of $\beta(t_i)$ and $[\beta(t_j) - \beta(t_i)]$ for $t_j > t_i$, and then both separate expectations are zero.

Although scalar constant-diffusion Brownian motion was just described completely in a probabilistic sense, a further characterization in terms of such concepts as continuity and differentiability is desirable, since these will directly influence the development and meaning of stochastic differential equations. For a deterministic function f, such concepts are straightforwardly approached by asking if the number $f(t_2)$ converges to the number $f(t_1)$ in the limit as t_2 approaches t_1, or similarly if an appropriate difference quotient converges to some limit. For stochastic processes, one needs to conceive of what "convergence" itself means. There are three *concepts of convergence* [4, 9, 13] of use to us: (1) mean square convergence, (2) convergence in probability, and (3) convergence almost surely (with probability one).

A sequence of random variables, $\mathrm{x}_1, \mathrm{x}_2, \ldots$, is said to *converge in mean square* (or sometimes, to converge "in the mean") to the random variable x if $E[\mathrm{x}_k{}^2]$ is finite for all k, and $E[\mathrm{x}^2]$ is finite, and

$$\lim_{k\to\infty} E[(\mathrm{x}_k - \mathrm{x})^2] = 0 \tag{4-50}$$

Thus, we are again concerned with the convergence of a sequence of *real numbers* $E[(\mathrm{x}_k - \mathrm{x})^2]$ in order to establish convergence in mean square. If (4-50) holds, then one often writes

$$\underset{k\to\infty}{\text{l.i.m.}}\ \mathrm{x}_k = \mathrm{x} \tag{4-51}$$

where l.i.m. denotes limit in the mean. This conception of convergence will provide the basis for defining stochastic integrals subsequently.

A sequence of random variables $\mathrm{x}_1, \mathrm{x}_2, \ldots$ is said to *converge in probability* to x if, for all $\varepsilon > 0$,

$$\lim_{k\to\infty} P(\{\omega : |\mathrm{x}_k(\omega) - \mathrm{x}(\omega)| \geq \varepsilon\}) = 0 \tag{4-52}$$

Here the sequence of real numbers is a sequence of individual probabilities, which is to converge to zero no matter how small ε might be chosen.

A sequence of random variables $\mathrm{x}_1, \mathrm{x}_2, \ldots$ is said to *converge almost surely* (a.s.) or to *converge with probability one* (w.p.1) to x if

$$\lim_{k\to\infty} |\mathrm{x}_k(\omega) - \mathrm{x}(\omega)| = 0 \tag{4-53}$$

for "almost all" realizations: if (4-53) holds for all ω except possibly a set A of ω whose probability is zero, $P(A) = 0$. Unlike the two previous concepts, this idea of convergence directly considers the convergence of every sequence of *realizations* of the random variables involved, rather than ensemble averages or probabilities.

Convergence in probability is the weakest of the three concepts, and it can be shown to be implied by the others:

$$[\text{Convergence in mean square}] \to [\text{Convergence in probability}] \tag{4-54a}$$

$$[\text{Convergence almost surely}] \to [\text{Convergence in probability}] \tag{4-54b}$$

Relation (4-54a) follows directly from the *Chebychev inequality*:

$$P(\{\omega : |\mathrm{x}_k(\omega) - \mathrm{x}(\omega)| \geq \varepsilon\}) \leq E\{[\mathrm{x}_k(\cdot) - \mathrm{x}(\cdot)]^2\}/\varepsilon^2 \qquad (\text{all } \varepsilon > 0) \tag{4-55}$$

since, if the mean square limit exists, then $\lim_{k\to\infty} E\{[\mathrm{x}_k - \mathrm{x}]^2\} = 0$, and so $\lim_{k\to\infty} P(\{\omega : |\mathrm{x}_k(\omega) - \mathrm{x}(\omega)| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$. Convergence in mean square does not imply, and is not implied by, convergence almost surely.

The *continuity of Brownian motion* can now be described. (For proof, see [4].) Let $\beta(\cdot,\cdot)$ be a Brownian motion process defined on $T \times \Omega$ with $T = [0, \infty)$. Then to each point $t \in T$ there corresponds two random variables $\beta^-(t,\cdot)$ and $\beta^+(t,\cdot)$ such that

$$\operatorname*{l.i.m.}_{t' \uparrow t} \beta(t',\cdot) = \beta^-(t,\cdot) \tag{4-56a}$$

$$\operatorname*{l.i.m.}_{t' \downarrow t} \beta(t',\cdot) = \beta^+(t,\cdot) \tag{4-56b}$$

where $t' \uparrow t$ means in the limit as t' approaches t from below and $t' \downarrow t$ means as t' approaches t from above. Furthermore, for each $t \in T$,

$$\beta^-(t,\cdot) = \beta(t,\cdot) = \beta^+(t,\cdot) \tag{4-56c}$$

almost surely. Equation (4-56b) states that as we let time t' approach time t from above, the value of $E\{[\beta(t',\cdot) - \beta^+(t,\cdot)]^2\}$ converges to zero: the variance describing the spread of values of realizations of $\beta(t',\cdot)$ from realizations of $\beta^+(t,\cdot)$ goes to zero in the limit. Moreover, (4-56c) dictates that all realizations $\beta^+(t,\omega_i)$ equal $\beta(t,\omega_i)$, except possibly for a set of ω_i's whose total probability is zero. Similar results are obtained by letting t' approach t from below as well.

This result implies that $\beta(\cdot,\cdot)$ is also continuous in probability through the Chebychev inequality:

$$P(\{\omega:|\beta(t',\cdot) - \beta(t,\cdot)| \geq \varepsilon\}) \leq E\{[\beta(t',\cdot) - \beta(t,\cdot)]^2\}/\varepsilon^2 = q|t' - t|/\varepsilon^2 \tag{4-57}$$

for all $\varepsilon > 0$. Consequently, the limit of the probability in (4-57) is zero as t' approaches t from above or below.

Moreover, Brownian motion can be shown to be continuous almost surely. In other words, almost all samples from the process (except possibly a set of samples of probability zero) are themselves continuous.

Brownian motion is nondifferentiable in the mean square and almost sure senses. A process $x(\cdot,\cdot)$ is mean square differentiable if the limit

$$\underset{\Delta t \to 0}{\text{l.i.m.}}\ \frac{x(t + \Delta t,\cdot) - x(t,\cdot)}{\Delta t}$$

exists, and then this limit defines the mean square derivative, $\dot{x}(t,\cdot)$, at $t \in T$. However, for Brownian motion $\beta(\cdot,\cdot)$,

$$E\left\{\left[\frac{\beta(t + \Delta t,\cdot) - \beta(t,\cdot)}{\Delta t}\right]\right\} = 0 \tag{4-58a}$$

$$E\left\{\left[\frac{\beta(t + \Delta t,\cdot) - \beta(t,\cdot)}{\Delta t}\right]^2\right\} = \frac{q\,\Delta t}{\Delta t^2} = \frac{q}{\Delta t} \tag{4-58b}$$

Thus, as $\Delta t \to 0$, the variance of the difference quotient used to define the mean square derivative becomes infinite. This can be used to show that

$$\lim_{\Delta t \to 0} P\left(\left\{\omega: \left|\frac{\beta(t + \Delta t,\omega) - \beta(t,\omega)}{\Delta t}\right| \leq B\right\}\right) = 0 \tag{4-59}$$

for any finite choice of bound B. Thus, the difference quotient for defining the derivative of each sample function has no finite limit for any $t \in T$ and almost all $\omega \in \Omega$ (except possibly for a set of probability zero): Brownian motion is nondifferentiable almost surely.

Thus, almost all sample functions from a Brownian motion process are continuous but nondifferentiable. Heuristically they are continuous, but have "corners" everywhere. Moreover, it can be shown that these sample functions are also of unbounded variation with probability one. It is this property especially that precludes a fruitful development of stochastic integrals in an almost sure sense. Instead, we will pursue a mean square approach to stochastic integral and differential equations.

Having described scalar constant-diffusion Brownian motion, it is now possible to investigate continuous-time *scalar stationary white Gaussian noise.* Let us assume (incorrectly) that the Brownian motion process $\beta(\cdot,\cdot)$ is differentiable, and that there is an integrable process $\mathrm{w}(\cdot,\cdot)$ such that, for $t, \tau \in T$,

$$\beta(t,\cdot) = \int_{t_0}^{t} \mathrm{w}(\tau,\cdot)\,d\tau \tag{4-60}$$

where the integral is to be understood in some as yet unspecified sense. In other words, we assume that $\mathrm{w}(\cdot,\cdot)$ is the derivative of Brownian motion,

$$\mathrm{w}(t,\cdot) = d\beta(t,\cdot)/dt \tag{4-61}$$

a derivative that does not really exist. Formal procedures based on this incorrect assumption will reveal that $\mathrm{w}(\cdot,\cdot)$ is, in fact, white Gaussian noise.

Let us calculate the mean and variance kernel for this fictitious process. First consider two disjoint time intervals, $(t_1, t_2]$ and $(t_3, t_4]$, so that, formally,

$$\beta(t_2) - \beta(t_1) = \int_{t_1}^{t_2} \mathrm{w}(t)\,dt \tag{4-62a}$$

$$\beta(t_4) - \beta(t_3) = \int_{t_3}^{t_4} \mathrm{w}(t)\,dt \tag{4-62b}$$

Use the properties of Brownian motion and (4-62a) to write

$$E\left\{\int_{t_1}^{t_2} \mathrm{w}(t)\,dt\right\} = E\{[\beta(t_2) - \beta(t_1)]\} = 0$$

If the preceding formal integral has the properties of regular integrals, then the expectation operation can be brought inside the time integrals, to yield

$$\int_{t_1}^{t_2} E\{\mathrm{w}(t)\}\,dt = 0$$

Since t_1 and t_2 can be arbitrary, this could only be true if

$$E\{\mathrm{w}(t)\} = 0 \qquad \text{for all} \quad t \in T \tag{4-63}$$

To establish the variance kernel, consider the two disjoint intervals, and use the property of independent increments of Brownian motion to write:

$$E\{[\beta(t_4) - \beta(t_3)][\beta(t_2) - \beta(t_1)]\} = 0$$

Using (4-62), this yields, formally

$$\begin{aligned} 0 &= E\left\{\int_{t_3}^{t_4} \mathrm{w}(t)\,dt \int_{t_1}^{t_2} \mathrm{w}(t')\,dt'\right\} \\ &= E\left\{\int_{t_3}^{t_4} \int_{t_1}^{t_2} \mathrm{w}(t)\mathrm{w}(t')\,dt'\,dt\right\} \\ &= \int_{t_3}^{t_4} \int_{t_1}^{t_2} E\{\mathrm{w}(t)\mathrm{w}(t')\}\,dt'\,dt \end{aligned}$$

Since $(t_1, t_2]$ and $(t_3, t_4]$ are arbitrary disjoint intervals, this implies

$$E\{\mathrm{w}(t)\mathrm{w}(t')\} = 0 \qquad \text{for} \quad t \neq t' \tag{4-64}$$

To establish $E\{\mathrm{w}(t)^2\}$, perform the same steps, but using a single interval. By the fact that $\beta(t)$ is Brownian motion,

$$E\{[\beta(t_2) - \beta(t_1)]^2\} = q[t_2 - t_1] = \int_{t_1}^{t_2} q\,dt$$

Combining this and (4-62a) yields

$$\begin{aligned}\int_{t_1}^{t_2} q\,dt &= E\left\{\int_{t_1}^{t_2} \mathrm{w}(t)\,dt \int_{t_1}^{t_2} \mathrm{w}(t')\,dt'\right\}\\ &= \int_{t_1}^{t_2}\int_{t_1}^{t_2} E\{\mathrm{w}(t)\mathrm{w}(t')\}\,dt'\,dt\end{aligned}$$

or, rewriting,

$$\int_{t_1}^{t_2}\left[\int_{t_1}^{t_2} E\{\mathrm{w}(t)\mathrm{w}(t')\}\,dt' - q\right]dt = 0$$

Since this is true for an arbitrary interval $(t_1, t_2]$, this implies

$$\int_{t_1}^{t_2} E\{\mathrm{w}(t)\mathrm{w}(t')\}\,dt' = q \tag{4-65}$$

for $t \in (t_1, t_2]$.

Now (4-64) and (4-65) together yield the definition of a delta function, so that we can write (4-63)–(4-65) as

$$E\{\mathrm{w}(t)\} = 0 \tag{4-66a}$$

$$E\{\mathrm{w}(t)\mathrm{w}(t')\} = q\,\delta(t - t') \tag{4-66b}$$

Furthermore, $\mathrm{w}(\cdot,\cdot)$ can be shown to be Gaussian, and thus, is a zero-mean, white Gaussian noise process of *strength* q. Heuristically, one can generate Brownian motion of diffusion q by passing white Gaussian noise of strength q through an integrator, as depicted in Fig. 4.9.

The preceding discussion can be generalized to the case of *scalar time-varying-diffusion Brownian motion* by redefining

$$E\{[\beta(t_2) - \beta(t_1)]^2\} = \int_{t_1}^{t_2} q(t)\,dt \tag{4-67}$$

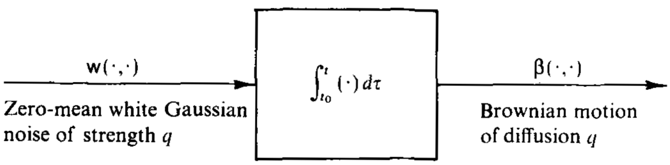

FIG. 4.9 White Gaussian noise and Brownian motion.

for $t_2 \geq t_1$ and $q(t) \geq 0$ for all $t \in T$, instead of $q[t_2 - t_1]$ as in (4-46b). If we assume $q(\cdot)$ to be at least piecewise continuous, which is very nonrestrictive, then no problems are encountered in making the extension. The corresponding *scalar nonstationary white Gaussian noise* $\mathrm{w}(\cdot,\cdot)$ would be described by the statistics

$$E\{\mathrm{w}(t)\} = 0 \tag{4-68a}$$

$$E\{\mathrm{w}(t)\mathrm{w}(t')\} = q(t)\,\delta(t - t') \tag{4-68b}$$

for all $t, t' \in T$. This extension will be essential to an adequate description of a "wideband" noise process whose strength can vary with time, typical of many problems of interest. [Note that frequency domain concepts such as a frequency bandwidth cannot be handled rigorously for nonstationary processes, but in many applications $q(\cdot)$ varies slowly with time, and quasi-static methods can be applied.]

EXAMPLE 4.3 The accuracy of position data available to an aircraft from radio navigation aids such as radar, VOR/DME, or TACAN, varies with the range from the aircraft to the navigation aid station. This range data is corrupted by wideband noise, and a reasonable model for indicated range $r_{\text{indicated}}$ is a stochastic process model defined for $t \in T$ and $\omega \in \Omega$ through

$$\mathrm{r}_{\text{indicated}}(t,\omega) = r_{\text{true}}(t) + \mathrm{w}(t,\omega)$$

where $\mathrm{w}(\cdot,\cdot)$ is zero-mean white Gaussian noise, with $q(\cdot)$ reaching a minimum when the aircraft is at minimum distance from the station. The $q(\cdot)$ function of time can be established by knowing the nominal flight path as a function of time. ■

Vector Brownian motion is a further extension, defined as an n-vector stochastic process, $\boldsymbol{\beta}(\cdot,\cdot)$, that has independent Gaussian increments with:

$$E\{\boldsymbol{\beta}(t)\} = \mathbf{0} \tag{4-69a}$$

$$E\{[\boldsymbol{\beta}(t_2) - \boldsymbol{\beta}(t_1)][\boldsymbol{\beta}(t_2) - \boldsymbol{\beta}(t_1)]^{\mathrm{T}}\} = \int_{t_1}^{t_2} \mathbf{Q}(t)\,dt \tag{4-69b}$$

for $t_2 \geq t_1$, and $\mathbf{Q}(t)$ is symmetric and positive semidefinite for all $t \in T$ and $\mathbf{Q}(\cdot)$ is at least piecewise continuous. The corresponding *vector white Gaussian noise* would be the hypothetical time derivative of this vector Brownian motion: a Gaussian process $\mathbf{w}(\cdot,\cdot)$ with

$$E\{\mathbf{w}(t)\} = \mathbf{0} \tag{4-70a}$$

$$E\{\mathbf{w}(t)\mathbf{w}^{\mathrm{T}}(t')\} = \mathbf{Q}(t)\delta(t - t') \tag{4-70b}$$

for all $t, t' \in T$, with the same description of $\mathbf{Q}(\cdot)$. Note that (4-70b) indicates that $\mathbf{w}(\cdot,\cdot)$ is uncorrelated in time, which implies that $\mathbf{w}(\cdot,\cdot)$ is white (independent in time) because it is Gaussian. However, this does *not* mean to say that the components of $\mathbf{w}(\cdot,\cdot)$ are uncorrelated with each other at the same time instant: $\mathbf{Q}(t)$ *can* have nonzero off-diagonal terms.

EXAMPLE 4.4 The radio navigation aids described in Example 4.3 provide bearing information as well as range. If b denotes bearing, the data available to the aircraft at time t can be modeled as

$$\mathrm{r}_{\text{indicated}}(t,\omega) = r_{\text{true}}(t) + \mathrm{w}_1(t,\omega)$$
$$\mathrm{b}_{\text{indicated}}(t,\omega) = b_{\text{true}}(t) + \mathrm{w}_2(t,\omega)$$

with $\mathbf{w}(\cdot,\cdot)$ a zero-mean white Gaussian noise to model the actual wideband noise corruption. The 2-by-2 matrix $\mathbf{Q}(t)$ is composed of variances $\sigma_{w_1}^2(t)$ and $\sigma_{w_2}^2(t)$ along the diagonal, with an off-diagonal term of $E\{\mathrm{w}_1(t)\mathrm{w}_2(t)\}$, generally nonzero. ■

4.6 STOCHASTIC INTEGRALS

In Section 4.4, a formal approach to stochastic differential equations led to a solution form (4-42) involving an integral $\int_{t_0}^{t} \mathbf{\Phi}(t,\tau)\mathbf{G}(\tau)\mathbf{w}(\tau)\,d\tau$ with $\mathbf{w}(\cdot,\cdot)$ white Gaussian noise, to which no meaning could be attributed rigorously. From Section 4.5, especially (4-61), one perceives that it may however be possible to give meaning to $\int_{t_0}^{t} \mathbf{\Phi}(t,\tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau)$ in some manner, thereby generating proper solutions to stochastic differential equations. Consequently, we have to consider the basics of defining integrals as the limit of sums, being careful to establish the conditions under which such a limit in fact exists. To do this properly will require certain concepts from functional analysis, which will be introduced heuristically rather than rigorously. First the simple scalar case is developed in detail, then the general case can be understood as an extension of the same basic concepts.

If $a(\cdot)$ is a known, piecewise continuous scalar function of time and $\beta(\cdot,\cdot)$ is a scalar Brownian motion of diffusion $q(t)$ for all $t \in T = [0,\infty)$, then we want to give meaning to

$$\mathrm{I}(t,\cdot) \triangleq \int_{t_0}^{t} a(\tau)\,\mathrm{d}\beta(\tau,\cdot) \tag{4-71}$$

called a *scalar stochastic integral* [1, 3, 5, 13, 14]. The notation provides the insight that for a particular time t, $\mathrm{I}(t,\cdot)$ will be a random variable, so that considered as a function of both t and ω, $\mathrm{I}(\cdot,\cdot)$ will be a stochastic process. In order to give meaning to this expression, we will require that the Riemann integral $\int_{t_0}^{t} a(\tau)^2 q(\tau)\,d\tau$ be finite; the need for this assumption will be explained subsequently. Note that we could extend this to the case of stochastic, rather than deterministic, $\mathrm{a}(\cdot,\cdot)$ if we desired to develop stochastic integrals appropriate for solutions to *nonlinear* stochastic differential equations; this will be postponed until Chapter 11 (Volume 2).

First partition the time interval $[t_0, t]$ into N steps, not necessarily of equal length, with $t_0 < t_1 < t_2 < \cdots < t_N = t$, and let the maximum time increment be denoted as Δt_N:

$$\Delta t_N = \max_{i=1,\ldots,N} \{(t_i - t_{i-1})\} \tag{4-72}$$

Now define a special function $a_N(\cdot)$, called a "simple function," through the relation

$$a_N(t) = \begin{cases} a(t_0) & t \in [t_0, t_1) \\ a(t_1) & t \in [t_1, t_2) \\ \vdots & \vdots \\ a(t_N) & t \in [t_{N-1}, t_N) \end{cases} \tag{4-73}$$

This is a piecewise constant approximation to the known function $a(\cdot)$, as depicted in Fig. 4.10. For this simple function, the stochastic integral can be defined constructively as the sum of N increment random variables:

$$\mathsf{I}_N(t,\cdot) \triangleq \sum_{i=0}^{N-1} a_N(t_i)[\beta(t_{i+1},\cdot) - \beta(t_i,\cdot)] \triangleq \int_{t_0}^{t} a_N(\tau)\mathrm{d}\beta(\tau,\cdot) \tag{4-74}$$

Let us characterize the random variable $\mathsf{I}_N(t,\cdot)$ probabilistically. Since $\mathsf{I}_N(t,\cdot)$ is composed of the sum of independent Gaussian increments, $\mathsf{I}_N(t,\cdot)$ itself is *Gaussian*. Its mean is

$$\begin{aligned} E\{\mathsf{I}_N(t)\} &= E\left\{\sum_{i=0}^{N-1} a_N(t_i)[\beta(t_{i+1}) - \beta(t_i)]\right\} \\ &= \sum_{i=0}^{N-1} a_N(t_i)E\{[\beta(t_{i+1}) - \beta(t_i)]\} = 0 \end{aligned} \tag{4-75}$$

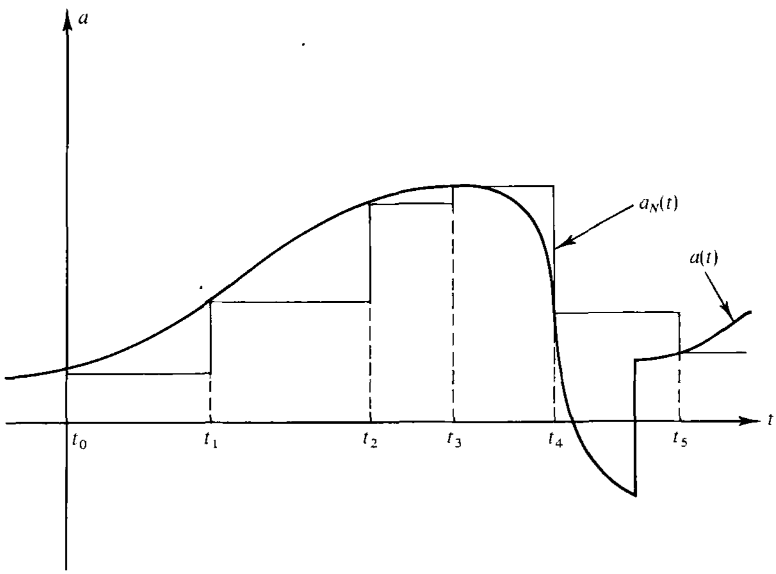

FIG. 4.10 Simple function $a_N(\cdot)$.

which results from $E\{\cdot\}$ being linear and $a_N(\cdot)$ being deterministic (so it can be brought out of the expectations), and then the N separate expectations are zero by the properties of Brownian motion. Its variance can be generated as

$$E\{\mathrm{I}_N(t)^2\} = E\left\{\left[\sum_{i=0}^{N-1} a_N(t_i)[\beta(t_{i+1}) - \beta(t_i)]\right]^2\right\}$$
$$= \sum_{i=0}^{N-1} a_N(t_i)^2 E\{[\beta(t_{i+1}) - \beta(t_i)]^2\}$$

where the reduction from N^2 to N separate expectations is due to the independence of Brownian motion increments. Thus,

$$E\{I_N(t)^2\} = \sum_{i=0}^{N-1} a_N(t_i)^2 \int_{t_i}^{t_{i+1}} q(\tau)\,d\tau = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} a_N(t_i)^2 q(\tau)\,d\tau$$
$$= \int_{t_0}^{t} a_N(\tau)^2 q(\tau)\,d\tau \tag{4-76}$$

Now it is desired to extend the definition of a stochastic integral in (4-74), valid only for piecewise constant $a_N(\cdot)$, to the case of piecewise continuous $a(\cdot)$. We will consider partitioning the time interval into finer and finer steps, and see if the sequence of random variables $\mathrm{I}_N(t,\cdot)$ so formed will converge to some limit as $N \to \infty$.

To motivate this development, consider the set of deterministic functions of time $a(\cdot)$ defined on $[t_0, t]$ such that $\int_{t_0}^{t} a^2(\tau)q(\tau)\,d\tau$ is finite for piecewise continuous $q(\cdot)$, identical to the assumption made at the beginning of this section. On this set of functions, called a *Hilbert space* (or a "complete inner product space"), the "distance" between two functions $a_N(\cdot)$ and $a_P(\cdot)$ can be defined properly by the scalar quantity $\|a_N - a_P\|$ where

$$\|a_N - a_P\|^2 = \int_{t_0}^{t} [a_N(\tau) - a_P(\tau)]^2 q(\tau)\,d\tau \tag{4-77}$$

It can be shown that if $a(\cdot)$ is an element of this set, i.e., if $\int_{t_0}^{t} a^2(\tau)q(\tau)\,d\tau$ is finite, then there exists a sequence of simple functions $a_k(\cdot)$ in this set that converges to $a(\cdot)$ as $k \to \infty$ [i.e., as the number of partitions of the time interval goes to ∞ and Δt_N, the maximum time increment, defined in (4-72), goes to zero], where convergence is in the sense that the "distance" between a_k and a converges to zero:

$$\lim_{k\to\infty} \|a - a_k\|^2 = 0 \tag{4-78}$$

Moreover, if a sequence of elements from this "Hilbert space" is a "Cauchy" sequence (for an arbitrarily small $\varepsilon > 0$, there exists an integer K such that for all $i > K$ and $j > K$, $\|a_i - a_j\| < \varepsilon$, which heuristically means that the members of the sequence become progressively closer together), then the sequence does in fact converge to a limit $a(\cdot)$. The limit is itself a member of that Hilbert space, which is assured by the "completeness" of the space.

To consider the convergence of a sequence of stochastic integrals of the form (4-74), define another stochastic integral of this form, but based upon P time partitions, with $P > N$:

$$\mathsf{I}_P(t,\cdot) \triangleq \sum_{j=0}^{P-1} a_P(t_j)[\beta(t_{j+1},\cdot) - \beta(t_j,\cdot)] \tag{4-79}$$

The difference between $\mathsf{I}_N(t,\cdot)$ and $\mathsf{I}_P(t,\cdot)$ is then

$$[\mathsf{I}_N(t) - \mathsf{I}_P(t)] = \sum_{i=0}^{N-1} a_N(t_i)[\beta(t_{i+1}) - \beta(t_i)] - \sum_{j=0}^{P-1} a_P(t_j)[\beta(t_{j+1}) - \beta(t_j)]$$

Since $a_N(\cdot)$ and $a_P(\cdot)$ are piecewise constant, their difference must be piecewise constant, with at most $N + P$ points of discontinuity. Thus, for some $K \le (N + P)$

$$\begin{aligned}[\mathsf{I}_N(t) - \mathsf{I}_P(t)] &= \sum_{k=0}^{K-1} [a_N(t_k) - a_P(t_k)][\beta(t_{k+1}) - \beta(t_k)] \\ &\triangleq \int_{t_0}^{t} [a_N(\tau) - a_P(\tau)]\, d\beta(\tau)\end{aligned} \tag{4-80}$$

serves to define the integral $\int_{t_0}^{t} [a_N(\tau) - a_P(\tau)]\, d\beta(\tau)$. The mean square value of this difference is

$$E\{[\mathsf{I}_N(t) - \mathsf{I}_P(t)]^2\} = E\left\{\left[\int_{t_0}^{t} [a_N(\tau) - a_P(\tau)]\, d\beta(\tau)\right]^2\right\}$$

and, since $[a_N(\cdot) - a_P(\cdot)]$ is piecewise constant, this can be shown equal to the *ordinary Riemann integral*:

$$E\{[\mathsf{I}_N(t) - \mathsf{I}_P(t)]^2\} = \int_{t_0}^{t} [a_N(\tau) - a_P(\tau)]^2 q(\tau)\, d\tau \tag{4-81}$$

Under the assumptions made previously, essentially that the random variables under consideration are zero mean and of finite second moments, we can now speak of the *Hilbert space of random variables* $\mathsf{I}(t,\cdot)$, with "distance" between random variables $\mathsf{I}_N(t,\cdot)$ and $\mathsf{I}_P(t,\cdot)$ defined as $\|\mathsf{I}_N(t) - \mathsf{I}_P(t)\|$, where

$$\|\mathsf{I}_N(t) - \mathsf{I}_P(t)\|^2 = E\{[\mathsf{I}_N(t) - \mathsf{I}_P(t)]^2\} \tag{4-82}$$

Combined with (4-81), this yields a distance measure identical to (4-77). A sequence of random variables $\mathsf{I}_1(t,\cdot)$, $\mathsf{I}_2(t,\cdot)$, ..., generated by taking finer partitions of $[t_0, t]$ such that Δt_k converges to zero, will be a Cauchy sequence, and thus will converge to a limit in that space, denoted as $\mathsf{I}(t,\cdot)$. By Eqs. (4-78) and (4-82), this assured convergence is in the *mean square sense*:

$$\lim_{k\to\infty} \|\mathsf{I}(t) - \mathsf{I}_k(t)\|^2 = \lim_{k\to\infty} E\{[I(t) - I_k(t)]^2\} = 0 \tag{4-83}$$

Thus, we can define the *scalar stochastic integral* $\mathsf{I}(\cdot,\cdot)$ properly through

$$\begin{aligned}\mathsf{I}(t,\cdot) &= \int_{t_0}^{t} a(\tau)\,\mathrm{d}\beta(\tau,\cdot)\\ &\doteq \underset{N\to\infty}{\text{l.i.m.}}\ \mathsf{I}_N(t,\cdot) \triangleq \underset{N\to\infty}{\text{l.i.m.}} \int_{t_0}^{t} a_N(\tau)\,\mathrm{d}\beta(\tau)\end{aligned} \tag{4-84}$$

Since Brownian motion is Gaussian and only linear operations on $\beta(\cdot,\cdot)$ were used in this development, $\mathsf{I}(t,\cdot)$ can be shown to be *Gaussian* with mean and variance

$$E\{\mathsf{I}(t)\} = \lim_{N\to\infty} E\{\mathsf{I}_N(t)\} = 0 \tag{4-85a}$$

$$E\{\mathsf{I}(t)^2\} = \lim_{N\to\infty} E\{\mathsf{I}_N(t)^2\} = \int_{t_0}^{t} a(\tau)^2 q(\tau)\,d\tau \tag{4-85b}$$

Stochastic integrals exhibit the usual *linear properties* of ordinary integrals:

$$\int_{t_0}^{t_2} a(\tau)\,\mathrm{d}\beta(\tau) = \int_{t_0}^{t_1} a(\tau)\,\mathrm{d}\beta(\tau) + \int_{t_1}^{t_2} a(\tau)\,\mathrm{d}\beta(\tau) \tag{4-86a}$$

$$\int_{t_0}^{t} [a(\tau) + a'(\tau)]\,\mathrm{d}\beta(\tau) = \int_{t_0}^{t} a(\tau)\,\mathrm{d}\beta(\tau) + \int_{t_0}^{t} a'(\tau)\,\mathrm{d}\beta(\tau) \tag{4-86b}$$

$$\int_{t_0}^{t} a(\tau)\,\mathrm{d}[\beta(\tau) + \beta'(\tau)] = \int_{t_0}^{t} a(\tau)\,\mathrm{d}\beta(\tau) + \int_{t_0}^{t} a(\tau)\,\mathrm{d}\beta'(\tau) \tag{4-86c}$$

$$\int_{t_0}^{t} c a(\tau)\,\mathrm{d}\beta(\tau) = c\int_{t_0}^{t} a(\tau)\,\mathrm{d}\beta(\tau) = \int_{t_0}^{t} a(\tau)\,\mathrm{d}[c\beta(\tau)] \tag{4-86d}$$

Integration by parts is also valid:

$$\int_{t_0}^{t} a(\tau)\,\mathrm{d}\beta(\tau) = a(\tau)\beta(\tau)\Big|_{t_0}^{t} - \int_{t_0}^{t} \beta(\tau)\,da(\tau) \tag{4-87}$$

where the last integral term is not a stochastic integral, but an *ordinary* Stieltjes integral definable for each sample of $\beta(\cdot,\cdot)$ if $a(\cdot)$ is of bounded variation.

Viewed as a stochastic process, the stochastic integral can be shown to be *mean square continuous*: $[\mathsf{I}(t_2) - \mathsf{I}(t_1)]$ is zero-mean and

$$\begin{aligned}E\{[\mathsf{I}(t_2) - \mathsf{I}(t_1)]^2\} &= E\left\{\left[\int_{t_0}^{t_2} a(\tau)\,\mathrm{d}\beta(\tau) - \int_{t_0}^{t_1} a(\tau)\,\mathrm{d}\beta(\tau)\right]^2\right\}\\ &= E\left\{\left[\int_{t_1}^{t_2} a(\tau)\,\mathrm{d}\beta(\tau)\right]^2\right\} = \int_{t_1}^{t_2} a(\tau)^2 q(\tau)\,d\tau\end{aligned} \tag{4-88}$$

The limit of this ordinary integral as $t_1 \to t_2$ is zero, thereby demonstrating mean square continuity.

Now consider two disjoint intervals $(t_1, t_2]$ and $(t_3, t_4]$, and form

$$[\mathrm{I}(t_2) - \mathrm{I}(t_1)] = \int_{t_1}^{t_2} a(\tau)\, d\beta(\tau), \qquad [\mathrm{I}(t_4) - \mathrm{I}(t_3)] = \int_{t_3}^{t_4} a(\tau)\, d\beta(\tau)$$

Since the intervals are disjoint, the independent increments of $\beta(\cdot,\cdot)$ in $(t_1, t_2]$ are independent of the increments in $(t_3, t_4]$. Thus, $[\mathrm{I}(t_2) - \mathrm{I}(t_1)]$ and $[\mathrm{I}(t_4) - \mathrm{I}(t_3)]$ are themselves independent, zero-mean, Gaussian increments of the $\mathrm{I}(\cdot,\cdot)$ process: *the* $\mathrm{I}(\cdot,\cdot)$ *process is itself a Brownian motion process with rescaled diffusion*, as seen by comparing (4-88) with (4-67).

Extension to the vector case is straightforward. Recall that an s-dimensional vector Brownian motion $\boldsymbol{\beta}(\cdot,\cdot)$ is a Gaussian process composed of independent increments, with statistics

$$E\{\boldsymbol{\beta}(t)\} = \mathbf{0} \tag{4-69a}$$

$$E\{[\boldsymbol{\beta}(t_2) - \boldsymbol{\beta}(t_1)][\boldsymbol{\beta}(t_2) - \boldsymbol{\beta}(t_1)]^{\mathrm{T}}\} = \int_{t_1}^{t_2} \mathbf{Q}(\tau)\, d\tau \tag{4-69b}$$

with the s-by-s diffusion matrix $\mathbf{Q}(t)$ symmetric and positive semidefinite and $\mathbf{Q}(\cdot)$ a matrix of piecewise continuous functions. If $\mathbf{A}(\cdot)$ is an n-by-s matrix of piecewise continuous time functions, then a development analogous to the scalar case yields a definition of an n-dimensional *vector-valued* stochastic integral

$$\mathbf{I}(t,\cdot) = \int_{t_0}^{t} \mathbf{A}(\tau)\, \mathbf{d}\boldsymbol{\beta}(\tau) \tag{4-89}$$

by means of a mean square limit:

$$\mathbf{I}(t,\cdot) \triangleq \underset{N\to\infty}{\mathrm{l.i.m.}}\ \mathbf{I}_N(t,\cdot) \triangleq \underset{N\to\infty}{\mathrm{l.i.m.}} \int_{t_0}^{t} \mathbf{A}_N(\tau)\, \mathbf{d}\boldsymbol{\beta}(\tau) \tag{4-90}$$

The random vector $\mathbf{I}(t,\cdot)$ is *Gaussian*, with statistics

$$E\{\mathbf{I}(t)\} = \mathbf{0} \tag{4-91a}$$

$$E\{\mathbf{I}(t)\mathbf{I}^{\mathrm{T}}(t)\} = \int_{t_0}^{t} \mathbf{A}(\tau)\mathbf{Q}(\tau)\mathbf{A}^{\mathrm{T}}(\tau)\, d\tau \tag{4-91b}$$

In such a development, the appropriate "distance" measure $\|\mathbf{I}_N(t) - \mathbf{I}_P(t)\|$ to replace that defined in (4-82) would be

$$\|\mathbf{I}_N(t) - \mathbf{I}_P(t)\|^2 = \operatorname{tr} E\{[\mathbf{I}_N(t) - \mathbf{I}_P(t)][\mathbf{I}_N(t) - \mathbf{I}_P(t)]^{\mathrm{T}}\} \tag{4-92}$$

where tr denotes trace.

Viewed as a function of both $t \in T$ and $\omega \in \Omega$, *the stochastic process* $\mathbf{I}(\cdot,\cdot)$ *is itself a Brownian motion process with rescaled diffusion*:

$$E\{[\mathbf{I}(t_2) - \mathbf{I}(t_1)][\mathbf{I}(t_2) - \mathbf{I}(t_1)]^{\mathrm{T}}\} = \int_{t_1}^{t_2} \mathbf{A}(\tau)\mathbf{Q}(\tau)\mathbf{A}^{\mathrm{T}}(\tau)\, d\tau \tag{4-93}$$

4.7 STOCHASTIC DIFFERENTIALS

Given a stochastic integral of the form

$$\mathbf{I}(t) = \mathbf{I}(t_0) + \int_{t_0}^{t} \mathbf{A}(\tau)\,\mathbf{d\beta}(\tau) \tag{4-94}$$

the *stochastic differential* of $\mathbf{I}(t)$ can be defined as

$$\mathbf{dI}(t) = \mathbf{A}(\tau)\,\mathbf{d\beta}(\tau) \tag{4-95}$$

Notice that the differential is defined in terms of the stochastic integral form, and not through an alternate definition in terms of a derivative, since Brownian motion is nondifferentiable. The $\mathbf{dI}(t)$ in (4-95) is thus a differential in the sense that if it is integrated over the entire interval from t_0 to a fixed time t, it yields the random variable $[\mathbf{I}(t) - \mathbf{I}(t_0)]$:

$$\int_{t_0}^{t} \mathbf{dI}(t) = \mathbf{I}(t) - \mathbf{I}(t_0) \tag{4-96}$$

Viewed as a function of t, this yields the stochastic process $[\mathbf{I}(\cdot) - \mathbf{I}(t_0)]$. *Heuristically*, it can be interpreted as an infinitesimal difference

$$\mathbf{dI}(t) = \mathbf{I}(t + dt) - \mathbf{I}(t) \tag{4-97}$$

One particular form required in the next section is the differential of the product of a time function and a stochastic integral. Suppose $\mathbf{s}(\cdot,\cdot)$ is a stochastic integral (which can also be regarded as a Brownian motion) defined through

$$\mathbf{s}(t) = \mathbf{s}(t_0) + \int_{t_0}^{t} \mathbf{A}(\tau)\,\mathbf{d\beta}(\tau) \tag{4-98}$$

Further suppose that $\mathbf{D}(\cdot)$ is a known matrix of differentiable functions, and a random process $\mathbf{y}(\cdot,\cdot)$ were defined by

$$\mathbf{y}(t) = \mathbf{D}(t)\mathbf{s}(t) \tag{4-99}$$

If the time interval $[t_0, t]$ were partitioned into N steps, one could write, assuming $t_{i+1} > t_i$,

$$\mathbf{s}(t) = \mathbf{s}(t_0) + \sum_{i=0}^{N-1} [\mathbf{s}(t_{i+1}) - \mathbf{s}(t_i)], \qquad \mathbf{D}(t) = \mathbf{D}(t_0) + \sum_{i=0}^{N-1} [\mathbf{D}(t_{i+1}) - \mathbf{D}(t_i)]$$

Substituting these back into (4-99) and rearranging yields

$$\mathbf{y}(t) = \sum_{i=0}^{N-1} [\mathbf{D}(t_{i+1}) - \mathbf{D}(t_i)]\mathbf{s}(t_{i+1}) + \sum_{i=0}^{N-1} \mathbf{D}(t_i)[\mathbf{s}(t_{i+1}) - \mathbf{s}(t_i)] + \mathbf{D}(t_0)\mathbf{s}(t_0)$$

Since $\mathbf{D}(\cdot)$ is assumed differentiable, the mean value theorem can be used to write $[\mathbf{D}(t_{i+1}) - \mathbf{D}(t_i)]$ as $\dot{\mathbf{D}}(\tau_i)[t_{i+1} - t_i]$ for some $\tau_i \in (t_i, t_{i+1})$. Putting this into the preceding expression, and taking the mean square limit as $N \to \infty$ yields

$$\mathbf{y}(t) = \int_{t_0}^{t} \dot{\mathbf{D}}(\tau)\mathbf{s}(\tau)\,d\tau + \int_{t_0}^{t} \mathbf{D}(\tau)\,\mathbf{ds}(\tau) + \mathbf{D}(t_0)\mathbf{s}(t_0) \tag{4-100}$$

Note that the first term could be interpreted as an ordinary Riemann integral for each sample function $\mathbf{s}(\cdot,\omega_i)$ of $\mathbf{s}(\cdot,\cdot)$ (i.e., it can be defined in the almost sure sense as well as the mean square sense), but the second term is a stochastic integral which is defined properly only in the mean square sense. From (4-100) and the definition of a stochastic differential, it can be seen that for $\mathbf{y}(t,\cdot)$ given by (4-99),

$$\mathbf{dy}(t) = \dot{\mathbf{D}}(t)\mathbf{s}(t)\,dt + \mathbf{D}(t)\,\mathbf{ds}(t) \tag{4-101}$$

Equation (4-101) reveals that the stochastic differential of the linear form in (4-99) obeys the same formal rules as the deterministic total differential of a corresponding $\mathbf{y}(t) = \mathbf{D}(t)\mathbf{s}(t)$. This will *not* be the case for nonlinear forms defined in terms of Itô stochastic integrals, as will be seen in Chapter 11 (Volume 2).

4.8 LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

Section 2.3 developed the solution to linear deterministic state differential equations of the form:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{4-102}$$

with $\mathbf{u}(\cdot)$ a known function of time. Now we would *like* to generate a system model of the form

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{w}(t) \tag{4-103}$$

where $\mathbf{w}(\cdot,\cdot)$ is a white Gaussian noise process of mean zero and strength $\mathbf{Q}(t)$ for all $t \in T$, and analogously develop its solution. (Deterministic driving terms will be admitted subsequently.) However, (4-103) cannot be used rigorously, since its solution cannot be generated.

It *is* possible to write the *linear stochastic differential equation*

$$\mathbf{dx}(t) = \mathbf{F}(t)\mathbf{x}(t)\,dt + \mathbf{G}(t)\,\mathbf{d\beta}(t) \tag{4-104}$$

where $\mathbf{G}(\cdot)$ is a known n-by-s matrix of piecewise continuous functions, and $\boldsymbol{\beta}(\cdot,\cdot)$ is an s-vector-valued Brownian motion process of diffusion $\mathbf{Q}(t)$ for all $t \in T$ [1, 5, 13, 14]. For engineering applications, Eq. (4-103) will often be used to describe a system model, but it is to be interpreted in a rigorous sense as a representation of the more proper relation, (4-104).

Now we seek the solution to (4-104). Recalling the interpretation of stochastic differentials from the last section, we equivalently want to find the $\mathbf{x}(\cdot,\cdot)$ process that satisfies the integral equation

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{F}(\tau)\mathbf{x}(\tau)\,d\tau + \int_{t_0}^{t} \mathbf{G}(\tau)\,\mathbf{d\beta}(\tau) \tag{4-105}$$

The last term is a stochastic integral to be understood in the mean square sense; the second term can also be interpreted as a mean square Riemann integral, or

$\int_{t_0}^{t} \mathbf{F}(\tau)\mathbf{x}(\tau, \omega_i)\,d\tau$ can be considered an *ordinary* Riemann integral for a particular sample from the $\mathbf{x}(\cdot,\cdot)$ process.

Let us propose as a solution to (4-104) the process $\mathbf{x}(\cdot,\cdot)$ defined by

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{y}(t) \tag{4-106a}$$

where $\mathbf{\Phi}(t, t_0)$ is the state transition matrix that satisfies $\dot{\mathbf{\Phi}}(t, t_0) = \mathbf{F}(t)\mathbf{\Phi}(t, t_0)$ and $\mathbf{\Phi}(t_0, t_0) = \mathbf{I}$, and $\mathbf{y}(t)$ is defined by

$$\mathbf{y}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t_0, \tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau) \tag{4-106b}$$

where the order of the time indices in evaluating the preceding $\mathbf{\Phi}(\cdot,\cdot)$ are to be noted (indicative of backward transitions). It must now be demonstrated that this proposed solution satisfies both the initial condition and differential equation (or integral equation). First, it satisfies the initial condition

$$\mathbf{x}(t_0) = \mathbf{\Phi}(t_0, t_0)\mathbf{y}(t_0) = \mathbf{I}\mathbf{x}(t_0) = \mathbf{x}(t_0) \tag{4-107}$$

The assumed solution (4-106a) is of the same form as (4-99), so the corresponding $\mathbf{dx}(t)$ can be written from (4-101) as

$$\mathbf{dx}(t) = \frac{d\mathbf{\Phi}(t, t_0)}{dt}\mathbf{y}(t)\,dt + \mathbf{\Phi}(t, t_0)\,\mathbf{dy}(t) \tag{4-108}$$

But, from (4-106b) and the definition of a stochastic differential, $\mathbf{dy}(t)$ is just $\mathbf{\Phi}(t_0, t)\mathbf{G}(t)\,\mathbf{d\beta}(t)$, so that (4-108) can be used to write

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{dx}(\tau) \\
&= \mathbf{x}(t_0) + \int_{t_0}^{t} [\mathbf{F}(\tau)\mathbf{\Phi}(\tau, t_0)]\mathbf{y}(\tau)\,d\tau + \int_{t_0}^{t} \mathbf{\Phi}(\tau, t_0)\mathbf{\Phi}(t_0, \tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau) \\
&= \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{F}(\tau)\mathbf{x}(\tau)\,d\tau + \int_{t_0}^{t} \mathbf{G}(\tau)\,\mathbf{d\beta}(\tau)
\end{aligned} \tag{4-109}$$

Thus, the proposed solution form does satisfy the given differential equation and initial condition.

From (4-106), the *solution of the linear stochastic differential equation* (4-104) is given by the stochastic process $\mathbf{x}(\cdot,\cdot)$ defined by

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau) \tag{4-110}$$

The extensive development in Sections 4.4–4.7 was required in order to give meaning to this solution form. Note that the Gaussian property of Brownian motion $\mathbf{\beta}(\cdot,\cdot)$ was never needed in the development: (4-110) is a valid solution form for any input process having independent increments.

Having obtained the solution as a stochastic process, it is desirable to characterize the statistical properties of that process. First, the *mean* $\mathbf{m}_x(\cdot)$

is described for all $t \in T$ as

$$\mathbf{m}_x(t) = E\{\mathbf{x}(t)\} = \mathbf{\Phi}(t, t_0)E\{\mathbf{x}(t_0)\} + E\left\{\int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau)\right\}$$

$$\mathbf{m}_x(t) = \mathbf{\Phi}(t, t_0)\mathbf{m}_x(t_0) \tag{4-111}$$

since the stochastic integral is of mean zero. To obtain the mean squared value of $\mathbf{x}(t)$, Eq. (4-110) can be used to write $E\{\mathbf{x}(t)\mathbf{x}^{\mathrm{T}}(t)\}$ as the sum of four separate expectations. However, Brownian motion is implicitly independent of $\mathbf{x}(t_0)$ by its definition, so the two cross terms are $\mathbf{0}$, such as:

$$E\left\{\left[\int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau)\right][\mathbf{x}^{\mathrm{T}}(t_0)]\right\} = E\left\{\int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau)\right\}E\{\mathbf{x}^{\mathrm{T}}(t_0)\} = \mathbf{0}$$

Thus, the *mean squared value* or *correlation matrix* of $\mathbf{x}(t)$ is

$$\begin{aligned} E\{\mathbf{x}(t)\mathbf{x}^{\mathrm{T}}(t)\} &= \mathbf{\Phi}(t, t_0)E\{\mathbf{x}(t_0)\mathbf{x}^{\mathrm{T}}(t_0)\}\mathbf{\Phi}^{\mathrm{T}}(t, t_0) \\ &\quad + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\mathbf{\Phi}^{\mathrm{T}}(t, \tau)\,d\tau \end{aligned} \tag{4-112}$$

where $\mathbf{Q}(t)$ is the diffusion of the Brownian motion $\boldsymbol{\beta}(\cdot, \cdot)$ at time t. The form of the *ordinary* Riemann integral in this expression is derived directly from Eq. (4-93).

The *covariance* can be derived directly from the mean square value by substituting

$$E\{\mathbf{x}(t)\mathbf{x}^{\mathrm{T}}(t)\} = \mathbf{P}_{xx}(t) + \mathbf{m}_x(t)\mathbf{m}_x^{\mathrm{T}}(t) \tag{4-113a}$$

$$E\{\mathbf{x}(t_0)\mathbf{x}^{\mathrm{T}}(t_0)] = \mathbf{P}_{xx}(t_0) + \mathbf{m}_x(t_0)\mathbf{m}_x^{\mathrm{T}}(t_0) \tag{4-113b}$$

into (4-112) and incorporating (4-111) to yield

$$\mathbf{P}_{xx}(t) = \mathbf{\Phi}(t, t_0)\mathbf{P}_{xx}(t_0)\mathbf{\Phi}^{\mathrm{T}}(t, t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\mathbf{\Phi}^{\mathrm{T}}(t, \tau)\,d\tau \tag{4-114}$$

From (4-110) it can be seen that, if $\mathbf{x}(t_0)$ is a Gaussian random variable or if it is nonrandom (known exactly), then $\mathbf{x}(t)$ for any fixed t is a *Gaussian* random variable. Thus, the first order density $f_{\mathbf{x}(t)}(\boldsymbol{\xi})$ is completely determined by the mean and covariance in (4-111) and (4-114) as

$$f_{\mathbf{x}(t)}(\boldsymbol{\xi}) = [(2\pi)^{n/2}|\mathbf{P}_{xx}(t)|^{1/2}]^{-1}\exp\{-\tfrac{1}{2}[\boldsymbol{\xi} - \mathbf{m}_x(t)]^{\mathrm{T}}\mathbf{P}_{xx}^{-1}(t)[\boldsymbol{\xi} - \mathbf{m}_x(t)]\} \tag{4-115}$$

Moreover, because the stochastic integral in (4-110) is composed of independent Gaussian increments, $\mathbf{x}(\cdot, \cdot)$ is a *Gaussian process*. Thus, the joint density $f_{\mathbf{x}(t_1),\mathbf{x}(t_2),\ldots,\mathbf{x}(t_N)}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_N)$ is a Gaussian density for any choice of $t_1, t_2, \ldots, t_N$. Its mean components and covariance block-diagonal terms are depicted by (4-111) and (4-114) for $t = t_1, t_2, \ldots, t_N$. To completely specify this density requires an expression for the covariance kernel $\mathbf{P}_{xx}(t_i, t_j)$, to be derived next.

For $t_2 \ge t_1 \ge t_0$, $\mathbf{x}(t_2)$ can be written as

$$\begin{aligned}\mathbf{x}(t_2) &= \mathbf{\Phi}(t_2, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t_2} \mathbf{\Phi}(t_2, \tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau) \\ &= \mathbf{\Phi}(t_2, t_1)\mathbf{\Phi}(t_1, t_0)\mathbf{x}(t_0) + \mathbf{\Phi}(t_2, t_1)\int_{t_0}^{t_1} \mathbf{\Phi}(t_1, \tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau) \\ &\quad + \int_{t_1}^{t_2} \mathbf{\Phi}(t_2, \tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau) \\ &= \mathbf{\Phi}(t_2, t_1)\mathbf{x}(t_1) + \int_{t_1}^{t_2} \mathbf{\Phi}(t_2, \tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau) \end{aligned} \tag{4-116}$$

Since the increments of $\boldsymbol{\beta}(\cdot,\cdot)$ over $[t_1, t_2)$ are independent of both the increments over $[t_0, t_1)$ and $\mathbf{x}(t_0)$, the *autocorrelation* $E\{\mathbf{x}(t_2)\mathbf{x}^{\mathrm{T}}(t_1)\}$ can be written as

$$\begin{aligned} E\{\mathbf{x}(t_2)\mathbf{x}^{\mathrm{T}}(t_1)\} &= \mathbf{\Phi}(t_2, t_1)E\{\mathbf{x}(t_1)\mathbf{x}^{\mathrm{T}}(t_1)\} + E\left\{\int_{t_1}^{t_2} \mathbf{\Phi}(t_2, \tau)\mathbf{G}(\tau)\,\mathbf{d\beta}(\tau)\mathbf{x}^{\mathrm{T}}(t_1)\right\} \\ &= \mathbf{\Phi}(t_2, t_1)E\{\mathbf{x}(t_1)\mathbf{x}^{\mathrm{T}}(t_1)\} \end{aligned} \tag{4-117}$$

Then (4-113) can be used to show that the desired *covariance kernel* for $t_2 \ge t_1$ is

$$\mathbf{P}_{xx}(t_2, t_1) = \mathbf{\Phi}(t_2, t_1)\mathbf{P}_{xx}(t_1, t_1) = \mathbf{\Phi}(t_2, t_1)\mathbf{P}_{xx}(t_1) \tag{4-118}$$

Close inspection of Eq. (4-116) reveals the fact that $\mathbf{x}(\cdot,\cdot)$ is not only a Gaussian process, but a *Gauss–Markov process* as described in Section 4.4. The probability law that describes the process evolution in the future depends only on the present process description (at time t_1 for instance) and not upon the history of the process evolution (before time t_1).

Equations (4-111), (4-114), and (4-118) are the fundamental characterization of the Gauss–Markov process solution (4-110) to the linear stochastic differential equation (4-104). However, it is often convenient to utilize the equivalent set of *differential equations for* $\mathbf{m}_x(t)$ *and* $\mathbf{P}_{xx}(t)$ to describe their evolution in time. Differentiating (4-111) yields the *mean time propagation* as

$$\begin{aligned} \dot{\mathbf{m}}_x(t) &= \dot{\mathbf{\Phi}}(t, t_0)\mathbf{m}_x(t_0) = \mathbf{F}(t)\mathbf{\Phi}(t, t_0)\mathbf{m}_x(t_0) \\ \dot{\mathbf{m}}_x(t) &= \mathbf{F}(t)\mathbf{m}_x(t) \end{aligned} \tag{4-119}$$

Since the stochastic driving term in (4-104) has zero-mean, the mean of $\mathbf{x}(t)$ satisfies the homogeneous form of the state equation. Differentiating (4-114) yields, using Leibnitz' rule,

$$\begin{aligned} \dot{\mathbf{P}}_{xx}(t) &= \mathbf{F}(t)\mathbf{\Phi}(t, t_0)\mathbf{P}_{xx}(t_0)\mathbf{\Phi}^{\mathrm{T}}(t, t_0) + \mathbf{\Phi}(t, t_0)\mathbf{P}_{xx}(t_0)\mathbf{\Phi}^{\mathrm{T}}(t, t_0)\mathbf{F}^{\mathrm{T}}(t) \\ &\quad + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t) + \int_{t_0}^{t} \mathbf{F}(t)\mathbf{\Phi}(t, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\mathbf{\Phi}^{\mathrm{T}}(t, \tau)\,d\tau \\ &\quad + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\mathbf{\Phi}^{\mathrm{T}}(t, \tau)\mathbf{F}^{\mathrm{T}}(t)\,d\tau \end{aligned}$$

Taking $\mathbf{F}(t)$ and $\mathbf{F}^{\mathrm{T}}(t)$ out of the integrals since they are not functions of τ, and rearranging, yields

$$\dot{\mathbf{P}}_{xx}(t) = \mathbf{F}(t)\mathbf{P}_{xx}(t) + \mathbf{P}_{xx}(t)\mathbf{F}^{\mathrm{T}}(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t) \tag{4-120}$$

These are general relationships in that they allow time-varying system models and Brownian motion diffusion as well as time-invariant parameters.

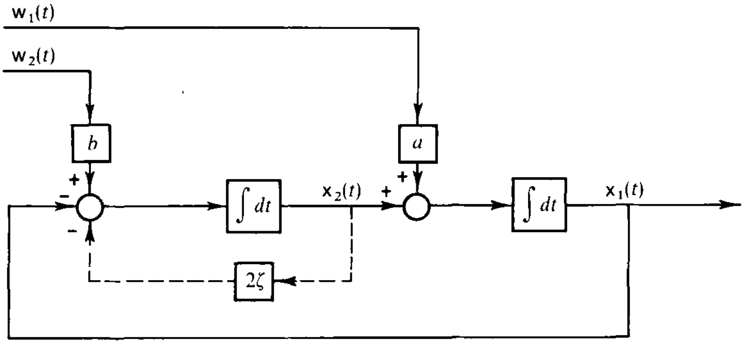

FIG. 4.11 Second order system model.

EXAMPLE 4.5 Consider a 1 rad/sec oscillator driven by white Gaussian noises, as depicted in Fig. 4.11 for $\zeta = 0$. The state equations can be written in the nonrigorous white noise notation as

$$\begin{bmatrix} \dot{\mathrm{x}}_1(t) \\ \dot{\mathrm{x}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{x}_1(t) \\ \mathrm{x}_2(t) \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \mathrm{w}_1(t) \\ \mathrm{w}_2(t) \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) \quad = \quad \mathbf{F} \quad \mathbf{x}(t) \quad + \quad \mathbf{G} \quad \mathbf{w}(t)$$

Suppose the initial conditions at time $t = 0$ are that the oscillator is known to start precisely at $x_1(0) = 1$, $x_2(0) = 3$. Let $\mathrm{w}_1(\cdot)$ and $\mathrm{w}_2(\cdot)$ be independent, zero-mean, and of strength one and two, respectively:

$$E\{\mathrm{w}_1(t)\mathrm{w}_1(t+\tau)\} = 1\delta(\tau), \qquad E\{\mathrm{w}_2(t)\mathrm{w}_2(t+\tau)\} = 2\delta(\tau), \qquad E\{\mathrm{w}_1(t)\mathrm{w}_2(t+\tau)\} = 0$$

Now we want to derive expressions for $\mathbf{m}_x(t)$ and $\mathbf{P}_{xx}(t)$ for all $t \geq 0$.

Since the initial conditions are known without uncertainty, $\mathbf{x}(t_0)$ can be modeled as a Gaussian random variable with zero covariance:

$$\mathbf{m}_x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad \mathbf{P}_{xx}(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Furthermore, from the given information, $\mathbf{Q}$ can be identified as

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

To use (4-111) requires knowledge of the state transition matrix, the solution to $\dot{\mathbf{\Phi}}(t, t_0) = \mathbf{F}(t)\mathbf{\Phi}(t, t_0)$, $\mathbf{\Phi}(t_0, t_0) = \mathbf{I}$. Since the system is time invariant, Laplace transform techniques could be used also. The result is

$$\mathbf{\Phi}(t, t_0) = \mathbf{\Phi}(t - t_0) = \begin{bmatrix} \cos(t - t_0) & \sin(t - t_0) \\ -\sin(t - t_0) & \cos(t - t_0) \end{bmatrix}$$

Thus, (4-111) yields $\mathbf{m}_x(t)$ as

$$\mathbf{m}_x(t) = \mathbf{\Phi}(t,0)\mathbf{m}_x(0) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \cos t + 3\sin t \\ -\sin t + 3\cos t \end{bmatrix}$$

$\mathbf{P}_{xx}(t)$ is generated from (4-114) as

$$\begin{aligned}\mathbf{P}_{xx}(t) &= \int_0^t \mathbf{\Phi}(t,\tau)\mathbf{GQG}^{\mathrm{T}}\mathbf{\Phi}^{\mathrm{T}}(t,\tau)\,d\tau \\ &= \int_0^t \begin{bmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{bmatrix}\begin{bmatrix} a^2 & 0 \\ 0 & 2b^2 \end{bmatrix}\begin{bmatrix} \cos(t-\tau) & -\sin(t-\tau) \\ \sin(t-\tau) & \cos(t-\tau) \end{bmatrix} d\tau \\ &= \begin{bmatrix} \left(\dfrac{a^2+2b^2}{2}\right)t + \left(\dfrac{a^2-2b^2}{4}\right)\sin 2t & \left(\dfrac{2b^2-a^2}{2}\right)\sin^2 t \\ \left(\dfrac{2b^2-a^2}{2}\right)\sin^2 t & \left(\dfrac{a^2+2b^2}{2}\right)t + \left(\dfrac{2b^2-a^2}{4}\right)\sin 2t \end{bmatrix}\end{aligned}$$

Note that the covariance is diverging: the diagonal terms grow linearly with time with a sinusoid superimposed. Thus a single sample from the $\mathbf{x}(\cdot,\cdot)$ process would be expected to be divergent as well as oscillatory. ■

EXAMPLE 4.6 Consider the same second order system as in Fig. 4.11, but with damping added by letting the damping ratio ζ be nonzero. $\mathbf{F}$ then becomes

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix}$$

and the same calculations can be performed, or (4-119) and (4-120) used to write

$$\dot{\mathbf{m}}_x(t) = \begin{bmatrix} \dot{m}_{x_1}(t) \\ \dot{m}_{x_2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix}\begin{bmatrix} m_{x_1}(t) \\ m_{x_2}(t) \end{bmatrix} = \begin{bmatrix} m_{x_2}(t) \\ -m_{x_1}(t) - 2\zeta m_{x_2}(t) \end{bmatrix}$$

$$\begin{aligned}\dot{\mathbf{P}}_{xx}(t) &= \begin{bmatrix} \dot{P}_{11}(t) & \dot{P}_{12}(t) \\ \dot{P}_{12}(t) & \dot{P}_{22}(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix}\begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} + \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -2\zeta \end{bmatrix} + \begin{bmatrix} a^2 & 0 \\ 0 & 2b^2 \end{bmatrix} \\ &= \begin{bmatrix} 2P_{12}(t) + a^2 & -P_{11}(t) - 2\zeta P_{12}(t) + P_{22}(t) \\ -P_{11}(t) - 2\zeta P_{12}(t) + P_{22}(t) & -2P_{12}(t) - 4\zeta P_{22}(t) + 2b^2 \end{bmatrix}\end{aligned}$$

This covariance does *not* grow without bound, and a steady state value can be found by evaluating $\dot{\mathbf{P}}_{xx}(t) = \mathbf{0}$, to yield

$$\mathbf{P}_{xx}(t \to \infty) = \begin{bmatrix} \dfrac{a^2(1-4\zeta^2)+2b^2}{4\zeta} & -\dfrac{a^2}{2} \\ -\dfrac{a^2}{2} & \dfrac{a^2+2b^2}{4\zeta} \end{bmatrix} \qquad ■$$

Deterministic control inputs can be added to the system model (4-103) or (4-104) without contributing any substantial complexity to the previous development. Let the linear stochastic differential equation be written as

$$\mathbf{dx}(t) = \mathbf{F}(t)\mathbf{x}(t)\,dt + \mathbf{B}(t)\mathbf{u}(t)\,dt + \mathbf{G}(t)\,\mathbf{d}\boldsymbol{\beta}(t) \tag{4-121a}$$

or, in the less rigorous white noise notation,

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t) \tag{4-121b}$$

where $\mathbf{u}(t)$ is an r-dimensional vector of deterministic control inputs applied at time t, and $\mathbf{B}(t)$ is an n-by-r control input matrix. The solution to (4-121) is the process $\mathbf{x}(\cdot,\cdot)$ defined by

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau + \int_{t_0}^{t} \boldsymbol{\Phi}(t,\tau)\mathbf{G}(\tau)\,\mathbf{d}\boldsymbol{\beta}(\tau) \tag{4-122}$$

The only difference between this and (4-110) is the addition of the ordinary Riemann integral $\int_{t_0}^{t} \boldsymbol{\Phi}(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau$, a *known n-vector* for fixed time t. Since this contributes no additional uncertainty, *only the mean of* $\mathbf{x}(t)$ *is affected* by this addition: $\mathbf{P}_{xx}(t)$ is still propagated by (4-114) or (4-120), while $\mathbf{m}_x(t)$ is propagated by

$$\mathbf{m}_x(t) = \boldsymbol{\Phi}(t, t_0)\mathbf{m}_x(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau \tag{4-123a}$$

or

$$\dot{\mathbf{m}}_x(t) = \mathbf{F}(t)\mathbf{m}_x(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{4-123b}$$

EXAMPLE 4.7 Consider Example 4.6 but with $\mathrm{w}_1(t)$ changed to $[u(t) + \mathrm{w}_1(t)]$, where $u(t) = u$ = constant for all $t \geq 0$. Then the state equation can be written as

$$\begin{bmatrix} \dot{\mathrm{x}}_1(t) \\ \dot{\mathrm{x}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix}\begin{bmatrix} \mathrm{x}_1(t) \\ \mathrm{x}_2(t) \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix} u + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} \mathrm{w}_1(t) \\ \mathrm{w}_2(t) \end{bmatrix}$$

so that $\dot{\mathbf{m}}_x(t)$ is given by

$$\begin{bmatrix} \dot{m}_{x_1}(t) \\ \dot{m}_{x_2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix}\begin{bmatrix} m_{x_1}(t) \\ m_{x_2}(t) \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix} u = \begin{bmatrix} m_{x_2}(t) + au \\ -m_{x_1}(t) - 2\zeta m_{x_2}(t) \end{bmatrix}$$

There is a steady state value of $\mathbf{m}_x(t)$, and it can be found by setting $\dot{\mathbf{m}}_x(t) = \mathbf{0}$. Unlike Example 4.6, for which $\mathbf{m}_x(t \to \infty) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, this yields

$$\mathbf{m}_x(t \to \infty) = \begin{bmatrix} 2\zeta a u \\ -au \end{bmatrix}$$

The covariance relations remain unchanged from Example 4.6. ■

The results of this section can be obtained using *formal procedures* on linear differential equations driven by white Gaussian noise. If $\boldsymbol{\beta}(\cdot,\cdot)$ is Brownian motion of diffusion $\mathbf{Q}(t)$ for all time $t \in T$, then a stochastic integral $\mathbf{I}(t,\cdot) = \int_{t_0}^{t} \mathbf{A}(\tau)\mathbf{d}\boldsymbol{\beta}(\tau,\cdot)$ can be defined properly in the mean square sense, with mean zero and

$$E\{\mathbf{I}(t)\mathbf{I}^{\mathrm{T}}(t)\} = \int_{t_0}^{t} \mathbf{A}(\tau)\mathbf{Q}(\tau)\mathbf{A}^{\mathrm{T}}(\tau)\,d\tau$$

as found previously. If the *formal* (nonexistent) derivative of the Brownian motion is taken, white Gaussian noise $\mathbf{w}(\cdot,\cdot)$ results, with mean zero and strength $\mathbf{Q}(t)$: $E\{\mathbf{w}(t)\mathbf{w}^{\mathrm{T}}(t')\} = \mathbf{Q}(t)\,\delta(t-t')$. The results gained by assuming the existence of this formal derivative will be *consistent* with the results that were derived *properly*. Based on this formal definition, we could write

$$\mathbf{I}(t) \stackrel{\mathrm{f}}{=} \int_{t_0}^{t} \mathbf{A}(\tau)\mathbf{w}(\tau)\,d\tau$$

where $\stackrel{\mathrm{f}}{=}$ denotes "formally." Then the mean and covariance would be

$$E\{\mathbf{I}(t)\} \stackrel{\mathrm{f}}{=} \int_{t_0}^{t} \mathbf{A}(\tau)E\{\mathbf{w}(\tau)\}\,d\tau = \mathbf{0}$$

$$\begin{aligned}
E\{\mathbf{I}(t)\mathbf{I}^{\mathrm{T}}(t)\} &\stackrel{\mathrm{f}}{=} E\left\{\left[\int_{t_0}^{t} \mathbf{A}(\tau_1)\mathbf{w}(\tau_1)\,d\tau_1\right]\left[\int_{t_0}^{t} \mathbf{A}(\tau_2)\mathbf{w}(\tau_2)\,d\tau_2\right]^{\mathrm{T}}\right\} \\
&\stackrel{\mathrm{f}}{=} \int_{t_0}^{t}\int_{t_0}^{t} \mathbf{A}(\tau_1)E\{\mathbf{w}(\tau_1)\mathbf{w}^{\mathrm{T}}(\tau_2)\}\mathbf{A}^{\mathrm{T}}(\tau_2)\,d\tau_2\,d\tau_1 \\
&\stackrel{\mathrm{f}}{=} \int_{t_0}^{t}\left[\int_{t_0}^{t} \mathbf{A}(\tau_1)\mathbf{Q}(\tau_1)\,\delta(\tau_1-\tau_2)\mathbf{A}^{\mathrm{T}}(\tau_2)\,d\tau_2\right]d\tau_1 \\
&\stackrel{\mathrm{f}}{=} \int_{t_0}^{t} \mathbf{A}(\tau_1)\mathbf{Q}(\tau_1)\mathbf{A}^{\mathrm{T}}(\tau_1)\,d\tau_1
\end{aligned}$$

so, at least formally, this is consistent. In terms of this white noise notation, the state equation can be written as in (4-121b) and the solution written formally as in (4-122), with identical statistical characteristics.

What is gained by avoiding the simplistic approach? First, such an approach does not force one to ask himself some fundamental questions, the answers to which provide significant insights into the nature of stochastic processes themselves. Second, basing an entire development of estimators and controllers upon an improperly defined model will make the validity of everything that follows subject to doubt. Finally, such an approach is totally misleading, in that when nonlinear stochastic differential equations are considered, formal procedures *will not* provide results consistent with those obtained properly through the Itô stochastic integral.

4.9 LINEAR STOCHASTIC DIFFERENCE EQUATIONS

Consider the concept of an *equivalent discrete-time system model* motivated by eventual digital computer implementations of algorithms, as introduced in Section 2.4. Suppose we obtain discrete-time measurements from a continuous-time system described by Eq. (4-121), with $\mathbf{u}(t)$ held constant over each sample period from sample time t_i to t_{i+1}. At the discrete time t_{i+1}, the solution can be written as

$$\mathbf{x}(t_{i+1}) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{x}(t_i) + \left[\int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{B}(\tau)\, d\tau\right]\mathbf{u}(t_i)$$
$$+ \left[\int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{G}(\tau)\, \mathbf{d\beta}(\tau)\right] \tag{4-124}$$

This can be written as an equivalent stochastic difference equation, i.e., an equivalent discrete-time model, as:

$$\mathbf{x}(t_{i+1}) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{x}(t_i) + \mathbf{B}_{\mathrm{d}}(t_i)\mathbf{u}(t_i) + \mathbf{w}_{\mathrm{d}}(t_i) \tag{4-125}$$

where $\mathbf{B}_{\mathrm{d}}(t_i)$ is the discrete-time input matrix defined by

$$\mathbf{B}_{\mathrm{d}}(t_i) = \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{B}(\tau)\, d\tau \tag{4-126}$$

and $\mathbf{w}_{\mathrm{d}}(\cdot,\cdot)$ is an n-vector-valued white Gaussian discrete-time stochastic process with statistics duplicating those of $\int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{G}(\tau)\mathbf{d\beta}(\tau)$ for all $t_i \in T$,

$$E\{\mathbf{w}_{\mathrm{d}}(t_i)\} = \mathbf{0} \tag{4-127a}$$

$$E\{\mathbf{w}_{\mathrm{d}}(t_i)\mathbf{w}_{\mathrm{d}}^{\mathrm{T}}(t_i)\} = \mathbf{Q}_{\mathrm{d}}(t_i) = \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\mathbf{\Phi}^{\mathrm{T}}(t_{i+1}, \tau)\, d\tau \tag{4-127b}$$

$$E\{\mathbf{w}_{\mathrm{d}}(t_i)\mathbf{w}_{\mathrm{d}}^{\mathrm{T}}(t_j)\} = \mathbf{0}, \qquad t_i \neq t_j \tag{4-127c}$$

Thus, (4-125) defines a discrete-time stochastic process which has identical characteristics to the result of sampling the solution process to (4-121) at the discrete times $t_0, t_1, t_2, \ldots$. The subscript d denotes discrete-time, to avoid confusion between $\mathbf{B}(\cdot)$ and $\mathbf{B}_{\mathrm{d}}(\cdot)$, etc.

Computationally, it is often more convenient to specify differential equations to solve over an interval rather than an integral relation as in (4-126) or (4-127b). To accomplish this, first define for any $t \in [t_i, t_{i+1})$

$$\overline{\mathbf{B}}(t, t_i) = \int_{t_i}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\, d\tau \tag{4-128a}$$

$$\overline{\mathbf{Q}}(t, t_i) = \int_{t_i}^{t} \mathbf{\Phi}(t, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^{\mathrm{T}}(\tau)\mathbf{\Phi}^{\mathrm{T}}(t, \tau)\, d\tau \tag{4-128b}$$

Taking the time derivative of these relations yields the desired result: the differential equations to be solved over each interval $[t_i, t_{i+1}) \in T$ to generate $\mathbf{\Phi}(t_{i+1}, t_i)$, $\mathbf{B}_{\mathrm{d}}(t_i)$, and $\mathbf{Q}_{\mathrm{d}}(t_i)$, which completely describe the equivalent discrete-time model (4-125) corresponding to (4-121) [12]:

$$\dot{\mathbf{\Phi}}(t, t_i) = \mathbf{F}(t)\mathbf{\Phi}(t, t_i) \tag{4-129a}$$

$$\dot{\overline{\mathbf{B}}}(t, t_i) = \mathbf{F}(t)\overline{\mathbf{B}}(t, t_i) + \mathbf{B}(t) \tag{4-129b}$$

$$\dot{\overline{\mathbf{Q}}}(t, t_i) = \mathbf{F}(t)\overline{\mathbf{Q}}(t, t_i) + \overline{\mathbf{Q}}(t, t_i)\mathbf{F}^{\mathrm{T}}(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^{\mathrm{T}}(t) \tag{4-129c}$$

These are integrated forward from the initial conditions

$$\mathbf{\Phi}(t_i, t_i) = \mathbf{I}; \qquad \bar{\mathbf{B}}(t_i, t_i) = \mathbf{0}; \qquad \bar{\mathbf{Q}}(t_i, t_i) = \mathbf{0} \tag{4-130}$$

to the time t_{i+1}, to yield the desired $\mathbf{\Phi}(t_{i+1}, t_i)$ and

$$\mathbf{B}_{\mathrm{d}}(t_i) \triangleq \bar{\mathbf{B}}(t_{i+1}, t_i) \tag{4-131a}$$

$$\mathbf{Q}_{\mathrm{d}}(t_i) \triangleq \bar{\mathbf{Q}}(t_{i+1}, t_i) \tag{4-131b}$$

In the general case, these integrations must be carried out separately for each sample period. However, many practical cases involve time-invariant system models and stationary noise inputs, for which a single set of integrations suffices for all sample periods. Moreover, for time-invariant or slowly varying $\mathbf{F}(\cdot)$, $\mathbf{B}(\cdot)$, and $[\mathbf{G}(\cdot)\mathbf{Q}(\cdot)\mathbf{G}^{\mathrm{T}}(\cdot)]$, if the sample period is short compared to the system's natural transients, a first order approximation to the solution of (4-129)–(4-131) can often be used [12]; namely,

$$\mathbf{\Phi}(t_{i+1}, t_i) \cong \mathbf{I} + \mathbf{F}(t_i)[t_{i+1} - t_i] \tag{4-132a}$$

$$\mathbf{B}_{\mathrm{d}}(t_i) \cong \mathbf{B}(t_i)[t_{i+1} - t_i] \tag{4-132b}$$

$$\mathbf{Q}_{\mathrm{d}}(t_i) \cong \mathbf{G}(t_i)\mathbf{Q}(t_i)\mathbf{G}^{\mathrm{T}}(t_i)[t_{i+1} - t_i] \tag{4-132c}$$

Equation (4-125) is a particular case of a *linear stochastic difference equation* of the general form

$$\mathbf{x}(t_{i+1}) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{x}(t_i) + \mathbf{B}_{\mathrm{d}}(t_i)\mathbf{u}(t_i) + \mathbf{G}_{\mathrm{d}}(t_i)\mathbf{w}_{\mathrm{d}}(t_i) \tag{4-133}$$

where $\mathbf{w}_{\mathrm{d}}(\cdot,\cdot)$ is an s-vector-valued discrete-time white Gaussian noise process, with mean zero and covariance kernel

$$E\{\mathbf{w}_{\mathrm{d}}(t_i)\mathbf{w}_{\mathrm{d}}^{\mathrm{T}}(t_j)\} = \begin{cases} \mathbf{Q}_{\mathrm{d}}(t_i) & t_i = t_j \\ \mathbf{0} & t_i \neq t_j \end{cases} \tag{4-134}$$

and $\mathbf{G}_{\mathrm{d}}(t_i)$ is an n-by-s noise input matrix for all $t_i \in T$. In (4-134), $\mathbf{Q}_{\mathrm{d}}(t_i)$ is a real, symmetric, positive semidefinite s-by-s matrix for all $t_i \in T$. Sometimes difference equations are written in terms of the argument i, for instant, rather than time t_i. Recall from Section 2.4 that if (4-133) did not arise from discretizing a a continuous-time model, there is no longer any assurance that $\mathbf{\Phi}(t_{i+1}, t_i)$ is always nonsingular.

The mean and covariance of the $\mathbf{x}(\cdot,\cdot)$ process defined by (4-133) propagate as

$$\mathbf{m}_x(t_{i+1}) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{m}_x(t_i) + \mathbf{B}_{\mathrm{d}}(t_i)\mathbf{u}(t_i) \tag{4-135a}$$

$$\mathbf{P}_{xx}(t_{i+1}) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{P}_{xx}(t_i)\mathbf{\Phi}^{\mathrm{T}}(t_{i+1}, t_i) + \mathbf{G}_{\mathrm{d}}(t_i)\mathbf{Q}_{\mathrm{d}}(t_i)\mathbf{G}_{\mathrm{d}}^{\mathrm{T}}(t_i) \tag{4-135b}$$

EXAMPLE 4.8 Consider a digital simulation of a first order lag as depicted in Fig. 4.12. Let $\mathrm{w}(\cdot,\cdot)$ be a white Gaussian noise with zero mean and

$$E[\mathrm{w}(t)\mathrm{w}(t+\tau)] = Q\,\delta(\tau)$$

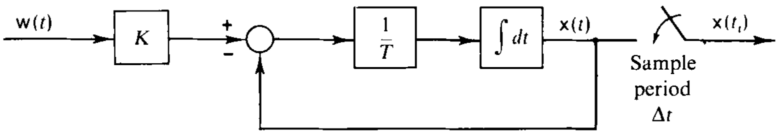

FIG. 4.12 First order lag system model.

and let the interval between sample times be a constant, $(t_{i+1} - t_i) = \Delta t$. From the figure, the state equation can be written as

$$\dot{\mathrm{x}}(t) = -(1/T)\mathrm{x}(t) + (K/T)\mathrm{w}(t)$$

Now we desire an equivalent discrete-time model. The state transition matrix is

$$\Phi(t, \tau) = \Phi(t - \tau) = e^{-(t-\tau)/T}$$

Therefore, the desired model is defined by

$$\mathrm{x}(t_{i+1}) = e^{-\Delta t/T}\mathrm{x}(t_i) + \mathrm{w}_{\mathrm{d}}(t_i)$$

where $\mathrm{w}_{\mathrm{d}}(\cdot,\cdot)$ is a white Gaussian discrete-time process with mean zero and

$$E\{\mathrm{w}_{\mathrm{d}}(t_i)^2\} = \int_{t_i}^{t_{i+1}} \Phi^2(t_{i+1}, \tau)\left[\frac{K}{T}\right]^2 Q\, d\tau = \frac{QK^2}{2T}[1 - e^{-2\Delta t/T}] = Q_{\mathrm{d}}$$

Steady state performance is reached, and can be found from either the continuous-time or discrete-time model. For the continuous-time model, in general, set

$$\dot{\mathbf{P}}(t) = \mathbf{F}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^{\mathrm{T}} + \mathbf{G}\mathbf{Q}\mathbf{G}^{\mathrm{T}} = \mathbf{0}$$

which here becomes

$$\dot{P}(t) = -(2/T)P(t) + (K^2Q/T^2) = 0$$

so that

$$P(t \to \infty) = QK^2/(2T)$$

For the discrete-time model, the same result can be obtained by setting

$$\mathbf{P}(t_{i+1}) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{P}(t_i)\mathbf{\Phi}^{\mathrm{T}}(t_{i+1}, t_i) + \mathbf{G}_{\mathrm{d}}\mathbf{Q}_{\mathrm{d}}\mathbf{G}_{\mathrm{d}}^{\mathrm{T}} = \mathbf{P}(t_i)$$

or

$$P(t_{i+1}) = e^{-2\Delta t/T}P(t_i) + (QK^2/(2T))[1 - e^{-2\Delta t/T}] = P(t_i) = P$$

so that

$$P[1 - e^{-2\Delta t/T}] = (QK^2/(2T))[1 - e^{-2\Delta t/T}]$$

or

$$P = QK^2/(2T)$$

Assume that $m_x(t_0) = 0$, so that

$$E\{\mathrm{x}(t_{i+1})\mathrm{x}(t_i)\} = \Phi(t_{i+1}, t_i)P(t_i) = e^{-\Delta t/T}P(t_i)$$

which converges to $e^{-\Delta t/T}P$ in steady state. Thus, if the sample period Δt is long compared to the first order lag time constant T, then there is very little correlation between $\mathrm{x}(t_{i+1})$ and $\mathrm{x}(t_i)$. ■

Monte Carlo simulations of systems will be discussed in detail in Section 6.8. Moreover, Problem 7.14 will describe means of generating samples of $\mathbf{w}_d(\cdot,\cdot)$, required for simulations of (4-133) for the general case in which $\mathbf{Q}_d(t_i)$ in (4-134) is nondiagonal.

4.10 THE OVERALL SYSTEM MODEL

The previous two sections developed continuous-time and discrete-time state propagation models. To develop an overall system model, the measurements available from a system must be described. We will be interested mostly in data samples rather than continuously available outputs. At time t_i, the measurements can be described through the m-dimensional random vector $\mathbf{z}(t_i,\cdot)$:

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \tag{4-136}$$

where the number of measurements m is typically smaller than the state dimension n, $\mathbf{H}(t_i)$ is an m-by-n measurement matrix, and $\mathbf{v}(t_i)$ is an m-dimensional vector of additive noise. Thus, each of the m measurements available at time t_i is assumed to be expressible as a linear combination of state variables, corrupted by noise. The physical data (i.e., numbers) from measuring devices are then *realizations* of (4-136):

$$\mathrm{z}(t_i) = \mathbf{z}(t_i,\omega_k) = \mathbf{H}(t_i)\mathbf{x}(t_i,\omega_k) + \mathbf{v}(t_i,\omega_k) \tag{4-137}$$

The noise $\mathbf{v}(\cdot,\cdot)$ will be modeled as a white Gaussian discrete-time stochastic process, with

$$E\{\mathbf{v}(t_i)\} = \mathbf{0} \tag{4-138a}$$

$$E\{\mathbf{v}(t_i)\mathbf{v}^{\mathrm{T}}(t_j)\} = \begin{cases}\mathbf{R}(t_i) & t_i = t_j\\ \mathbf{0} & t_i \neq t_j\end{cases} \tag{4-138b}$$

It will also be assumed that $\mathbf{v}(t_i,\cdot)$ is independent of both the initial condition $\mathbf{x}(t_0)$ and the dynamic driving noise $\boldsymbol{\beta}(t_j,\cdot)$ or $\mathbf{w}_d(t_j,\cdot)$ for all $t_i, t_j \in T$. A generalization allowing correlation between these various random variables is possible, but this will be pursued later.

Thus, there will be two system models of fundamental interest to us. First there is the continuous-time system dynamics model

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t) \tag{4-139a}$$

or, more properly,

$$\mathbf{dx}(t) = \mathbf{F}(t)\mathbf{x}(t)\,dt + \mathbf{B}(t)\mathbf{u}(t)\,dt + \mathbf{G}(t)\,\mathbf{d}\boldsymbol{\beta}(t) \tag{4-139b}$$

from which sampled-data measurements are available at times $t_1, t_2, \ldots$ as

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \tag{4-140}$$

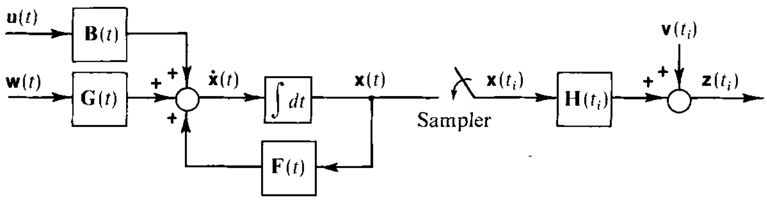

FIG. 4.13 Continuous-time dynamics/discrete-time measurement model.

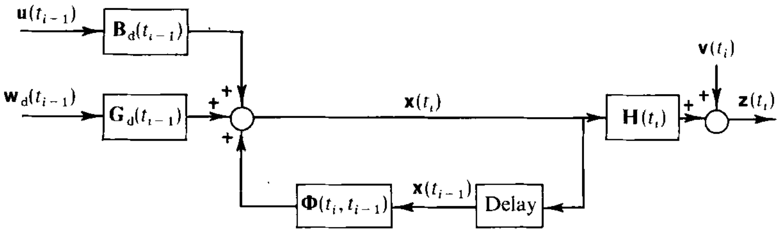

FIG. 4.14 Discrete-time dynamics/discrete-time measurement model.

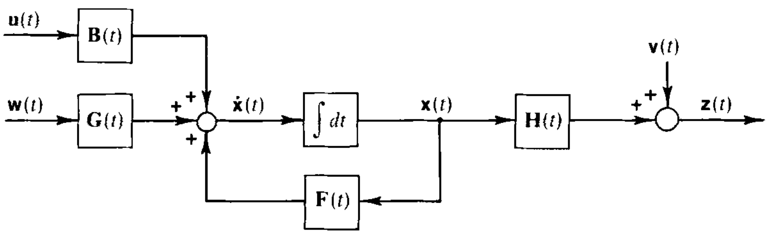

FIG. 4.15 Continuous-time dynamics/continuous-time measurement model.

This model is portrayed schematically in Fig. 4.13. The second model of interest is depicted in Fig. 4.14: a discrete-time dynamics model

$$\mathbf{x}(t_{i+1}) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{x}(t_i) + \mathbf{B}_d(t_i)\mathbf{u}(t_i) + \mathbf{G}_d(t_i)\mathbf{w}_d(t_i) \tag{4-141}$$

with measurements available at discrete times $t_1, t_2, \ldots,$ of the same form as in (4-140). Note that if a model is derived originally in the first form, then an "equivalent discrete-time model" can be generated as in Fig. 4.14, but with $\mathbf{G}_d(t_i)$ equal to an n-by-n identity matrix for all times $t_i \in T$.

A third possible model formulation is depicted in Fig. 4.15, consisting of a continuous-time dynamics model (4-139), with measurements continuously available as

$$\mathbf{z}(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}(t) \tag{4-142}$$

with $\mathbf{H}(\cdot)$ an m-by-n matrix of piecewise continuous functions, and $\mathbf{v}(\cdot,\cdot)$ an m-vector-valued continuous-time noise process. This additive noise would be modeled as a white Gaussian noise with zero mean and covariance kernel

$$E\{\mathbf{v}(t)\mathbf{v}^{\mathrm{T}}(t')\} = \mathbf{R}_c(t)\,\delta(t - t') \tag{4-143}$$

It would further be assumed that $\mathbf{v}(t)$ is independent of $\mathbf{x}(t_0)$ and $\mathbf{w}(t')$ or $\boldsymbol{\beta}(t')$ for all $t, t' \in T$ (such an assumption could be relaxed, as mentioned before). Although such a model is of theoretical interest, the sampled-data measurement models are more significant practically, since virtually all estimators and stochastic controllers are implemented on digital computers.

In fact, our attention will be focused upon the continuous-time dynamics/discrete-time measurement model, since this will be the most natural description of the majority of problems of interest. For such a model, the statistics of the system outputs can be calculated explicitly in terms of corresponding state characteristics, which have been described previously. The *mean* of the measurement process at time t_i is

$$\begin{aligned}
\mathbf{m}_z(t_i) &= E\{\mathbf{z}(t_i)\} = E\{\mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i)\} \\
&= \mathbf{H}(t_i)E\{\mathbf{x}(t_i)\} + E\{\mathbf{v}(t_i)\} \\
\mathbf{m}_z(t_i) &= \mathbf{H}(t_i)\mathbf{m}_x(t_i)
\end{aligned} \tag{4-144}$$

The output autocorrelation is generated as

$$\begin{aligned}
E\{\mathbf{z}(t_j)\mathbf{z}^{\mathrm{T}}(t_i)\} = {} & E\{\mathbf{H}(t_j)\mathbf{x}(t_j)\mathbf{x}^{\mathrm{T}}(t_i)\mathbf{H}^{\mathrm{T}}(t_i)\} + E\{\mathbf{v}(t_j)\mathbf{v}^{\mathrm{T}}(t_i)\} \\
& + E\{\mathbf{H}(t_j)\mathbf{x}(t_j)\mathbf{v}^{\mathrm{T}}(t_i)\} + E\{\mathbf{v}(t_j)\mathbf{x}^{\mathrm{T}}(t_i)\mathbf{H}^{\mathrm{T}}(t_i)\}
\end{aligned}$$

But, the third and fourth terms are zero, since we can write

$$\mathbf{x}(t_j) = \mathbf{\Phi}(t_j, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t_j} \mathbf{\Phi}(t_j, \tau)\mathbf{G}(\tau)\,\mathbf{d}\boldsymbol{\beta}(\tau)$$

or, for the equivalent discrete-time system,

$$\mathbf{x}(t_j) = \mathbf{\Phi}(t_j, t_0)\mathbf{x}(t_0) + \sum_{k=1}^{j} \mathbf{\Phi}(t_j, t_k)\mathbf{G}_d(t_{k-1})\mathbf{w}_d(t_{k-1})$$

i.e., in either case as the sum of terms, all of which are independent of $\mathbf{v}(t_i)$. Thus, $\mathbf{x}(t_j)$ and $\mathbf{v}(t_i)$ are independent, and so uncorrelated, so that the third term in the previous expression becomes

$$\mathbf{H}(t_j)E\{\mathbf{x}(t_j)\mathbf{v}^{\mathrm{T}}(t_i)\} = \mathbf{H}(t_j)E\{\mathbf{x}(t_j)\}E\{\mathbf{v}^{\mathrm{T}}(t_i)\} = \mathbf{0}$$

since $\mathbf{v}(t_i)$ is assumed zero-mean and similarly for the fourth term. Therefore, the output *autocorrelation* is

$$\mathbf{\Psi}_{zz}(t_j, t_i) = E\{\mathbf{z}(t_j)\mathbf{z}^{\mathrm{T}}(t_i)\} = \begin{cases} \mathbf{H}(t_j)E\{\mathbf{x}(t_j)\mathbf{x}^{\mathrm{T}}(t_i)\}\mathbf{H}^{\mathrm{T}}(t_i) & t_j \neq t_i \\ \mathbf{H}(t_i)E\{\mathbf{x}(t_i)\mathbf{x}^{\mathrm{T}}(t_i)\}\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i) & t_j = t_i \end{cases} \tag{4-145}$$

Similarly, the *covariance kernel* is

$$\mathbf{P}_{zz}(t_j, t_i) = \begin{cases} \mathbf{H}(t_j)\mathbf{P}_{xx}(t_j, t_i)\mathbf{H}^{\mathrm{T}}(t_i) & t_j \neq t_i \\ \mathbf{H}(t_i)\mathbf{P}_{xx}(t_i, t_i)\mathbf{H}^{\mathrm{T}}(t_i) + \mathbf{R}(t_i) & t_j = t_i \end{cases} \tag{4-146}$$

Note that for (4-145) and (4-146), expressions for $E\{\mathbf{x}(t_j)\mathbf{x}^{\mathrm{T}}(t_i)\}$ and $\mathbf{P}_{xx}(t_j, t_i)$ were derived previously for $t_j \geq t_i$ as

$$E\{\mathbf{x}(t_j)\mathbf{x}^{\mathrm{T}}(t_i)\} = \mathbf{\Phi}(t_j, t_i)E\{\mathbf{x}(t_i)\mathbf{x}^{\mathrm{T}}(t_i)\} \qquad \mathbf{P}_{xx}(t_j, t_i) = \mathbf{\Phi}(t_j, t_i)\mathbf{P}_{xx}(t_i)$$

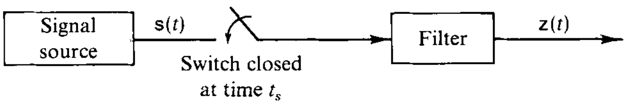

FIG. 4.16 Schematic for Example 4.9.

EXAMPLE 4.9 The following problem, first suggested by Deyst [3], incorporates many of the concepts that have been discussed. Consider Fig. 4.16. A signal source generates a scalar output which can be described as a stationary zero-mean process $\mathsf{s}(\cdot,\cdot)$, which is exponentially time-correlated with correlation time T:

$$E\{\mathsf{s}(t)\} = 0$$
$$E\{\mathsf{s}(t)\mathsf{s}(t+\tau)\} = \sigma^2 e^{-|\tau|/T}$$

That is to say, the signal source is started at some time t_0, and the transients are allowed to decay so as to achieve a steady state output process. Then, at time t_s, the switch is closed. It is assumed that at the time just before switch closure, the filter output is zero: $z(t_s^-) = 0$. The filter is a lead-lag type, with a transfer function

$$\frac{z(s)}{s(s)} = \frac{s+a}{s+b}$$

Its amplitude ratio (Bode) plot as a function of frequency is depicted in Fig. 4.17 for the case of $a < b$ ($b < a$ is also possible—this yields a low-pass lead-lag). The objective is to derive an expression for the autocorrelation function of the filter output $\mathsf{z}(\cdot,\cdot)$, valid for both time arguments assuming values greater than or equal to t_s.

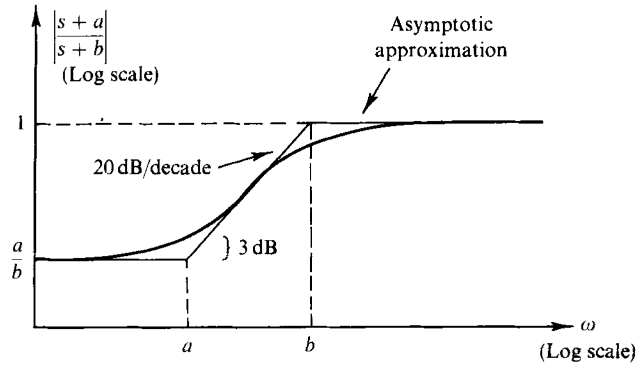

FIG. 4.17 Lead-lag filter amplitude ratio plot.

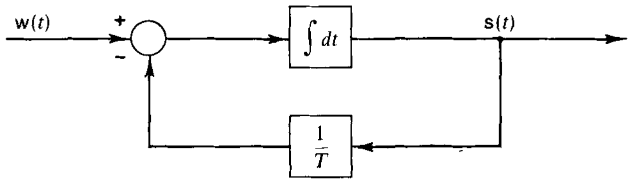

FIG. 4.18 Model of signal generator.

First, we must generate a model of the signal source. A system that generates a process duplicating the characteristics of $s(\cdot,\cdot)$ is depicted in Fig. 4.18: a first order lag driven by zero-mean white Gaussian noise. The output s satisfies the differential equation

$$\dot{s}(t) = -(1/T)s(t) + w(t)$$

where the strength of $w(\cdot,\cdot)$ is as yet unknown: it must be determined so as to yield the appropriate steady state variance of $s(\cdot,\cdot)$. The state transition matrix for this system model is

$$\Phi(t, t') = e^{-(t-t')/T}$$

Therefore, the statistics of $s(\cdot,\cdot)$, assuming $E\{s(t_0)\} = 0$, are given by (4-111) and (4-114) as

$$m_s(t) = E\{s(t)\} = 0$$

$$P_{ss}(t) = E\{s^2(t)\} = e^{-2(t-t_0)/T}E\{s^2(t_0)\} + \int_{t_0}^{t} e^{-2(t-\tau)/T}\, Q\, d\tau$$
$$= e^{-2(t-t_0)/T}E\{s^2(t_0)\} + \tfrac{1}{2}QT[1 - e^{-2(t-t_0)/T}]$$

where we are seeking the appropriate value of Q: $E\{w(t)w(t+\tau)\} = Q\,\delta(\tau)$. To obtain stationary characteristics of s, Q must be constant and we must let the transient die out by letting $t_0 \to -\infty$, yielding

$$E\{s^2(t)\} \to \tfrac{1}{2}QT$$

But, from the given autocorrelation function $\sigma^2 e^{-|\tau|/T}$, this is supposed to equal σ^2, so the desired value of Q is

$$Q = (2/T)\sigma^2$$

Note that the model output obeys (4-118):

$$E\{s(t+\tau)s(t)\} = \Phi(t+\tau, t)E\{s^2(t)\} \to e^{-\tau/T}\sigma^2 \qquad \tau > 0$$

which is the desired form of autocorrelation. In fact, identifying the constant part of the given $\sigma^2 e^{-|\tau|/T}$ as the steady state mean squared value, and the function of τ as the state transition matrix, gives the initial insight that a first order lag is the appropriate system model to propose.

Another procedure for obtaining the appropriate Q would be to seek the steady state solution to (4-120), which for this case becomes the scalar equation

$$\dot{P}_{ss}(t) = -(1/T)P_{ss}(t) - (1/T)P_{ss}(t) + Q = 0$$

Since we desire $P_{ss}(t \to \infty) = \sigma^2$, this again yields $Q = (2/T)\sigma^2$.

For the filter, a state model can be generated as in Example 2.6. Thus, an overall model of the situation is depicted in Fig. 4.19. Note the choice of integrator outputs as state variables x_1 and x_2. Once the switch is closed, the state equations become

$$\dot{x}_1(t) = -(1/T)x_1(t) + w(t), \qquad \dot{x}_2(t) = -bx_2(t) + x_1(t)$$

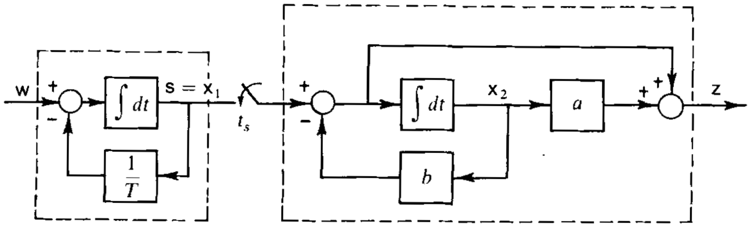

FIG. 4.19 Overall state model for Example 4.9.

and the output equation becomes

$$z(t_i) = a x_2(t_i) + \dot{x}_2(t_i) = a x_2(t_i) - b x_2(t_i) + x_1(t_i) = [a - b] x_2(t_i) + x_1(t_i)$$

In vector notation, this can be written as $\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}w(t)$, $z(t_i) = \mathbf{H}\mathbf{x}(t_i)$:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1/T & 0 \\ 1 & -b \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t)$$

$$z(t_i) = \begin{bmatrix} 1 & (a-b) \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix}$$

Furthermore, the uncertainties can be described through

$$\mathbf{G}Q\mathbf{G}^{\mathrm{T}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} Q \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, \qquad R(t_i) = 0$$

Now (4-145) can be used to write the desired autocorrelation for $t_j \geq t_i$ as

$$\begin{aligned} E\{z(t_j)z(t_i)\} &= \mathbf{H}E\{\mathbf{x}(t_j)\mathbf{x}^{\mathrm{T}}(t_i)\}\mathbf{H}^{\mathrm{T}} \\ &= \mathbf{H}\mathbf{\Phi}(t_j, t_i)[\mathbf{P}_{xx}(t_i) + \mathbf{m}_x(t_i)\mathbf{m}_x^{\mathrm{T}}(t_i)]\mathbf{H}^{\mathrm{T}} \end{aligned}$$

But, $\mathbf{m}_x(t)$ is zero for all time $t \geq t_s$: by time t_s, all transients in $x_1(\cdot)$ have died out, and it is driven by zero-mean noise; $x_2(\cdot)$ starts at $x_2(t_s) = 0$ since $z(t_s^-) = 0$, and it is driven by the zero-mean process $x_1(\cdot)$. Thus, the desired result is

$$E\{z(t_j)z(t_i)\} = \mathbf{H}\mathbf{\Phi}(t_j, t_i)\mathbf{P}_{xx}(t_i)\mathbf{H}^{\mathrm{T}}$$

and so $\mathbf{\Phi}(t_j, t_i)$ and $\mathbf{P}_{xx}(t_i)$ must be generated explicitly.

The state transition matrix can be derived using Laplace transforms because of the time-invariant nature of the system:

$$\mathbf{\Phi}(s) = [s\mathbf{I} - \mathbf{F}]^{-1} = \begin{bmatrix} s + (1/T) & 0 \\ -1 & s + b \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{s + (1/T)} & 0 \\ \dfrac{1}{(s + (1/T))(s + b)} & \dfrac{1}{s + b} \end{bmatrix}$$

so that

$$\mathbf{\Phi}(t, t') = \begin{bmatrix} e^{-(t-t')/T} & 0 \\ (b - (1/T))^{-1}[e^{-(t-t')/T} - e^{-b(t-t')}] & e^{-b(t-t')} \end{bmatrix}$$

The covariance matrix $\mathbf{P}_{xx}(t_i)$ can be written for any $t_i \geq t_s$ through (4-114) as

$$\mathbf{P}_{xx}(t_i) = \mathbf{\Phi}(t_i, t_s)\mathbf{P}_{xx}(t_s)\mathbf{\Phi}^{\mathrm{T}}(t_i, t_s) + \int_{t_s}^{t_i} \mathbf{\Phi}(t_i, \tau)\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}\mathbf{\Phi}^{\mathrm{T}}(t_i, \tau)\,d\tau$$

The initial condition is specified by noting $\mathsf{x}_1(t_s) = \mathsf{s}(t_s)$, and $\mathsf{x}_2(t_s)$ is known exactly:

$$\mathbf{P}_{xx}(t_s) = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$$

Substituting these into the expression for $E\{\mathsf{z}(t_j)\mathsf{z}(t_i)\}$ yields the final result as

$$\begin{aligned} E\{\mathsf{z}(t_j)\mathsf{z}(t_i)\} &= \sigma^2[c_1{}^2 \exp[-(t_i + t_j - 2t_s)/T] + c_2{}^2 \exp[-b(t_i + t_j - 2t_s)] \\ &\quad - c_1 c_2(\exp[-b(t_j - t_s) - (t_i - t_s)/T] + \exp[-(t_j - t_s)/T - b(t_i - t_s)])] \\ &\quad + (2\sigma^2/T)\int_{t_s}^{t_i} [c_1{}^2 \exp[-(t_i + t_j - 2\tau)/T] + c_2{}^2 \exp[-b(t_i + t_j - 2\tau)] \\ &\quad - c_1 c_2(\exp[-b(t_j - \tau) - (t_i - \tau)/T] + \exp[-(t_j - \tau)/T - b(t_i - \tau)])]\,d\tau \end{aligned}$$

where

$$c_1 = \frac{a + b - (2/T)}{b - (1/T)}, \qquad c_2 = \frac{a - (1/T)}{b - (1/T)}$$

Note that the result is a *nonstationary* autocorrelation. To obtain a stationary output, we require not only a time-invariant system driven only by stationary inputs, but also must consider the output only after sufficient time for transients to die out. This stationary result as $t_s \to -\infty$ can also be obtained by convolutions, but the initial transient after time t_s *cannot* be generated through such an approach. ■

4.11 SHAPING FILTERS AND STATE AUGMENTATION

In many instances, the use of white Gaussian noise models to describe all noises in a real system may not be adequate. It would be desirable to be able to generate empirical autocorrelation or power spectral density data, and then to develop a mathematical model that would produce an output with duplicate characteristics. If observed data were in fact samples from a Brownian motion or stationary Gaussian process with a known rational power spectral density (or corresponding known autocorrelation or covariance kernel), then a linear time-invariant system, or *shaping filter*, driven by stationary white Gaussian noise, provides such a model. If the power spectral density is not rational, it can be approximated as closely as desired by a rational model, and the same procedure followed. Furthermore, if all one knows are the first and second order statistics of a wide-sense stationary process (which is often the case), then a *Gaussian* process with the same first and second order statistics can always be generated via a shaping filter. Time-varying shaping filters are also possible, but we will focus mostly upon models for stationary processes.

The previous example in fact demonstrated the use of such a shaping filter, a first order lag, to duplicate the observed process $\mathsf{s}(\cdot,\cdot)$. This section will formalize and extend that development.

Suppose that a system of interest is described by

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{n}(t) \tag{4-147a}$$

$$\mathbf{z}(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}(t) \tag{4-147b}$$

where $\mathbf{n}(\cdot,\cdot)$ is a nonwhite, i.e., time-correlated, Gaussian noise. Also, suppose that the noise $\mathbf{n}(\cdot,\cdot)$ can be generated by a linear shaping filter:

$$\dot{\mathbf{x}}_{\mathrm{f}}(t) = \mathbf{F}_{\mathrm{f}}(t)\mathbf{x}_{\mathrm{f}}(t) + \mathbf{G}_{\mathrm{f}}(t)\mathbf{w}(t) \tag{4-148a}$$

$$\mathbf{n}(t) = \mathbf{H}_{\mathrm{f}}(t)\mathbf{x}_{\mathrm{f}}(t) \tag{4-148b}$$

where the subscript f denotes filter, and $\mathbf{w}(\cdot,\cdot)$ is a *white* Gaussian noise process. Then the filter output in (4-148b) can be used to drive the system, as shown in Fig. 4.20. Now define the *augmented state vector* process $\mathbf{x}_{\mathrm{a}}(\cdot,\cdot)$ through

$$\mathbf{x}_{\mathrm{a}}(\cdot,\cdot) = \begin{bmatrix} \mathbf{x}(\cdot,\cdot) \\ \mathbf{x}_{\mathrm{f}}(\cdot,\cdot) \end{bmatrix} \tag{4-149}$$

to write (4-147) and (4-148) as an augmented state equation

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \hline \dot{\mathbf{x}}_{\mathrm{f}}(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{F}(t) & \mathbf{G}(t)\mathbf{H}_{\mathrm{f}}(t) \\ \hline \mathbf{0} & \mathbf{F}_{\mathrm{f}}(t) \end{array}\right] \begin{bmatrix} \mathbf{x}(t) \\ \hline \mathbf{x}_{\mathrm{f}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \hline \mathbf{G}_{\mathrm{f}}(t) \end{bmatrix} \mathbf{w}(t) \tag{4-150a}$$

$$\dot{\mathbf{x}}_{\mathrm{a}}(t) = \mathbf{F}_{\mathrm{a}}(t) \quad \mathbf{x}_{\mathrm{a}}(t) + \mathbf{G}_{\mathrm{a}}(t) \quad \mathbf{w}(t) \tag{4-150b}$$

and associated output equation

$$\mathbf{z}(t) = \left[\begin{array}{c|c} \mathbf{H}(t) & \mathbf{0} \end{array}\right] \begin{bmatrix} \mathbf{x}(t) \\ \hline \mathbf{x}_{\mathrm{f}}(t) \end{bmatrix} + \mathbf{v}(t) \tag{4-150c}$$

$$\mathbf{z}(t) = \mathbf{H}_{\mathrm{a}}(t) \quad \mathbf{x}_{\mathrm{a}}(t) + \mathbf{v}(t) \tag{4-150d}$$

This is again in the form of an overall (augmented) linear system model driven only by *white* Gaussian noise.

An analogous development is possible for the case of time-correlated measurement corruption noise. Let a system of interest be described by

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{w}(t) \tag{4-151a}$$

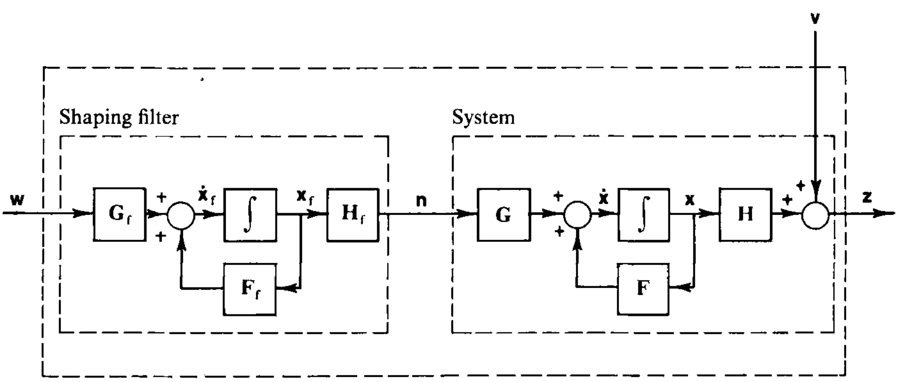

FIG. 4.20 Shaping filter generating dynamic driving noise.

$$\mathbf{z}(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{n}(t) + \mathbf{v}(t) \tag{4-151b}$$

where $\mathbf{w}(\cdot,\cdot)$ and $\mathbf{v}(\cdot,\cdot)$ are white noises and $\mathbf{n}(\cdot,\cdot)$ is nonwhite. Generate $\mathbf{n}(\cdot,\cdot)$ as the output of a shaping filter driven by white Gaussian $\mathbf{w}_f(\cdot,\cdot)$, as depicted in Fig. 4.21:

$$\dot{\mathbf{x}}_f(t) = \mathbf{F}_f(t)\mathbf{x}_f(t) + \mathbf{G}_f(t)\mathbf{w}_f(t) \tag{4-152a}$$

$$\mathbf{n}(t) = \mathbf{H}_f(t)\mathbf{x}_f(t) \tag{4-152b}$$

The augmented state can be defined as in (4-149) to yield an augmented system description as

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \hline \dot{\mathbf{x}}_f(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{F}(t) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{F}_f(t) \end{array}\right] \begin{bmatrix} \mathbf{x}(t) \\ \hline \mathbf{x}_f(t) \end{bmatrix} + \left[\begin{array}{c|c} \mathbf{G}(t) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{G}_f(t) \end{array}\right] \begin{bmatrix} \mathbf{w}(t) \\ \hline \mathbf{w}_f(t) \end{bmatrix} \tag{4-153a}$$

$$\mathbf{z}(t) = \left[\begin{array}{c|c} \mathbf{H}(t) & \mathbf{H}_f(t) \end{array}\right] \begin{bmatrix} \mathbf{x}(t) \\ \hline \mathbf{x}_f(t) \end{bmatrix} + \mathbf{v}(t) \tag{4-153b}$$

which is again in the form of a linear system model driven only by white Gaussian noises. Obvious extensions can allow time-correlated components of both the dynamic driving noise and the measurement corruption noise for a given system.

Certain shaping filter configurations are recurrent and useful enough for process modeling to be discussed individually. These are depicted in Fig. 4.22.

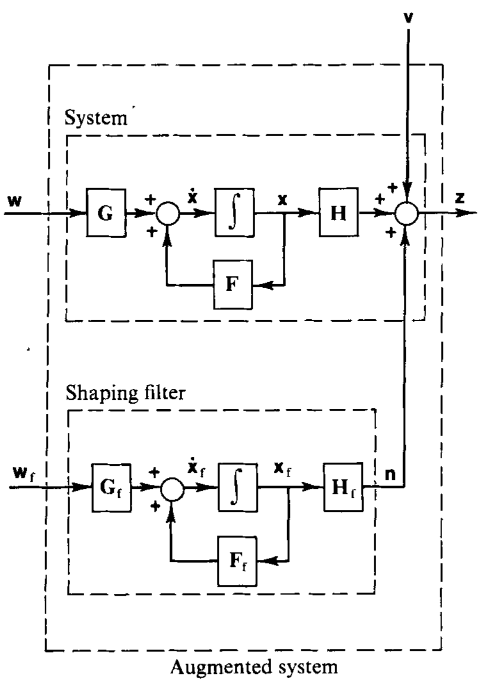

FIG. 4.21 Shaping filter generating measurement corruption noise.

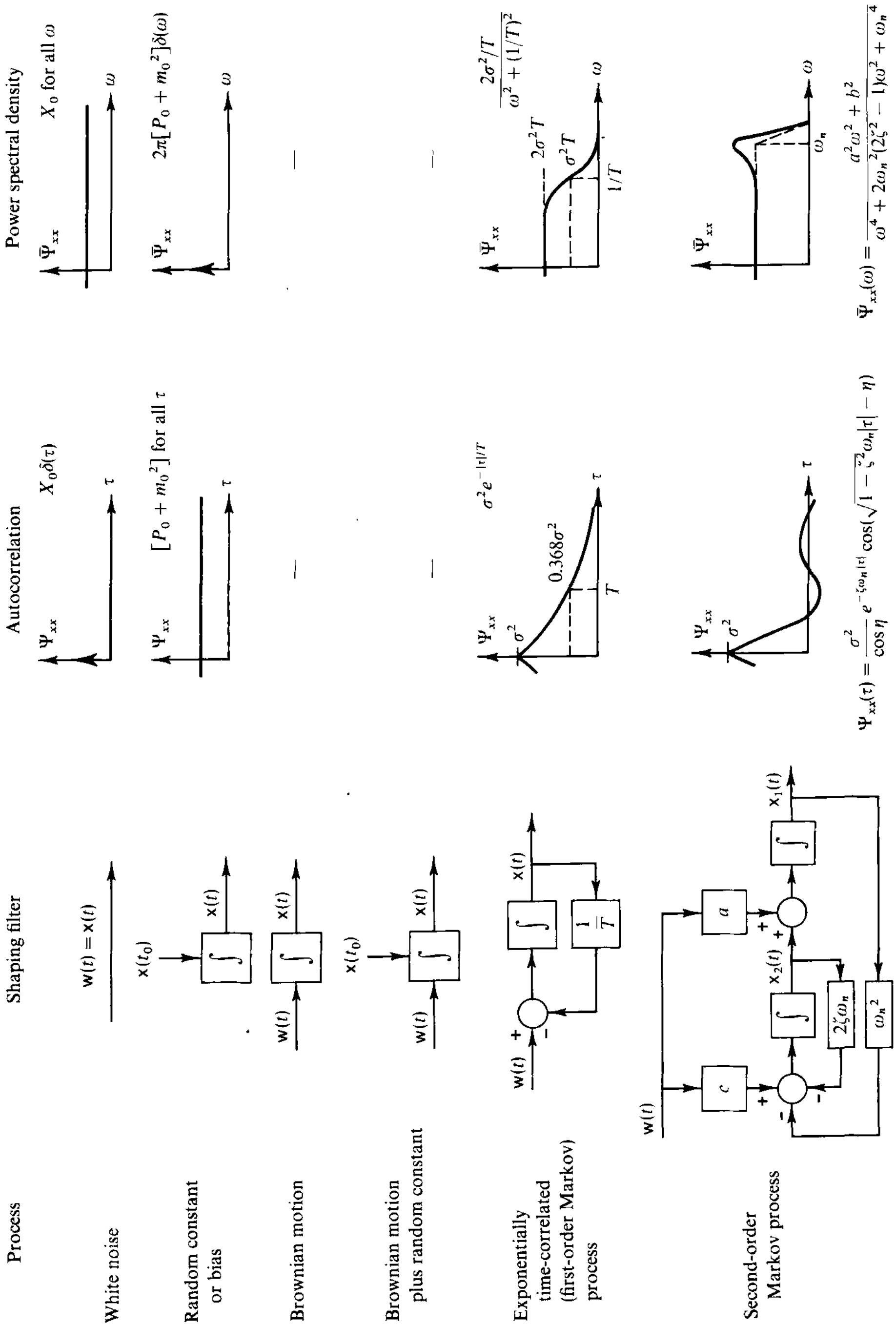

FIG. 4.22 Useful shaping filters.

The trivial case is stationary *white Gaussian noise* itself, of mean m_0 and autocorrelation

$$E\{\mathrm{w}(t)\mathrm{w}(t+\tau)\} = [P_0 + m_0{}^2]\,\delta(\tau) = X_0\,\delta(\tau) \tag{4-154}$$

Second, there is the *random constant* or *bias* model, generated as the output of an integrator with no input, but with an initial condition modeled as a Gaussian random variable $\mathrm{x}(t_0)$ with specified mean m_0 and variance P_0. Thus, the defining relationship is

$$\dot{\mathrm{x}}(t) = 0 \tag{4-155a}$$

starting from the given initial condition $\mathrm{x}(t_0)$. Note that this is a degenerate form of shaping filter, in that no noise drives the filter state equation. Since the samples are constant in time, the autocorrelation is constant over all τ, resulting in an impulse power spectral density (all power concentrated at $\omega = 0$):

$$\Psi_{xx}(\tau) = E\{\mathrm{x}(t)\mathrm{x}(t+\tau)\} = [P_0 + m_0{}^2] \tag{4-155b}$$

$$\bar{\Psi}_{xx}(\omega) = 2\pi[P_0 + m_0{}^2]\,\delta(\omega) \tag{4-155c}$$

This is a good model for such phenomena as turnon-to-turnon nonrepeatability biases of rate gyros and other sensors: from one period of operation to another, the bias level can change, but it remains constant while the instrument is turned on. Care must be exercised in using such a model in developing an optimal estimator. This model indicates that although you do not know the bias magnitude a priori, you *know* that it does not change value in time. As a result, an optimal filter will estimate its magnitude using initial data, but will essentially disregard all measurements that come later. If it is desired to maintain a viable estimate of a bias that *may* vary slowly (or unexpectedly, as due to instrument failure or degradation), the following shaping filter is a more appropriate bias model.

Brownian motion (random walk) is the output of an integrator driven by white Gaussian noise (heuristically, in view of the development of the previous sections):

$$\dot{\mathrm{x}}(t) = \mathrm{w}(t); \qquad \mathrm{x}(t_0) \triangleq 0 \tag{4-156}$$

where $\mathrm{w}(\cdot,\cdot)$ has mean zero and $E\{\mathrm{w}(t)\mathrm{w}(t+\tau)\} = Q\,\delta(\tau)$. The mean equation would be the same as for the random constant, $\dot{m}_x(t) = 0$ or $m_x(t) = m_x(t_0)$, but the second order statistics are different: $\dot{P}_{xx}(t) = Q$ instead of $\dot{P}_{xx}(t) = 0$, so that the mean squared value grows linearly with time, $E\{\mathrm{x}^2(t)\} = Q[t - t_0]$. The random walk and random constant can both be represented by the use of only one state variable, essentially just adding the capability of generalizing a random walk to the case of nonzero mean or nonzero initial variance.

Exponentially time-correlated (first order Markov) process models are first order lags driven by zero-mean white Gaussian noise of strength Q. As shown in Example 4.9, to produce an output with autocorrelation

$$\Psi_{xx}(\tau) = E\{\mathrm{x}(t)\mathrm{x}(t+\tau)\} = \sigma^2 e^{-|\tau|/T} \tag{4-157a}$$

i.e., of correlation time T and mean squared value σ^2 (and mean zero), the model is described by

$$\dot{\mathrm{x}}(t) = -(1/T)\mathrm{x}(t) + \mathrm{w}(t) \tag{4-157b}$$

where $Q = 2\sigma^2/T$, or, in other words, $E\{\mathrm{x}^2(t)\} = QT/2$. The associated power spectral density is

$$\bar{\Psi}_{xx}(\omega) = \frac{2\sigma^2/T}{\omega^2 + (1/T)^2} \tag{4-157c}$$

This is an especially useful shaping filter, providing an adequate approximation to a wide variety of empirically observed band-limited (wide or narrow band) noises.

A *second order Markov process* provides a good model of oscillatory random phenomena, such as vibration, bending, and fuel slosh in aerospace vehicles. The general form of autocorrelation is

$$\Psi_{xx}(\tau) = E\{\mathrm{x}(t)\mathrm{x}(t+\tau)\} = \frac{\sigma^2}{\cos\eta} e^{-\zeta\omega_n|\tau|}\cos(\sqrt{1-\zeta^2}\,\omega_n|\tau| - \eta) \tag{4-158a}$$

as depicted in Fig. 4.22 [note that the autocorrelation periodically is negative, i.e., if you knew the value of $x(t)$, then you expect $x(t+\tau)$ to be of opposite sign for those values of time difference τ]. This can be generated by passing a stationary white Gaussian noise $\mathrm{w}(\cdot,\cdot)$ of strength $Q = 1$ through a second order system, having a transfer function most generally expressed as

$$G(s) = \frac{as+b}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4-158b}$$

or a state description as depicted in Fig. 4.22:

$$\begin{bmatrix}\dot{\mathrm{x}}_1(t)\\ \dot{\mathrm{x}}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\omega_n^2 & -2\zeta\omega_n\end{bmatrix}\begin{bmatrix}\mathrm{x}_1(t)\\ \mathrm{x}_2(t)\end{bmatrix} + \begin{bmatrix}a\\ c\end{bmatrix}\mathrm{w}(t) \tag{4-158c}$$

where $\mathrm{x}_1(t)$ is the system output and

$$a = [(2\sigma^2/\cos\eta)\omega_n \sin(\alpha - \eta)]^{1/2}, \qquad b = [(2\sigma^2/\cos\eta)\omega_n^3 \sin(\alpha + \eta)]^{1/2},$$
$$c = b - 2a\zeta\omega_n, \qquad \alpha = \tan^{-1}[\zeta/\sqrt{1-\zeta^2}].$$

In practice, σ^2, η, ζ, and ω_n are chosen to fit empirical data. (See Problem 4.21.) For instance, ζ is chosen to fit the observed resonant peak in power spectral density, from the condition of no peak ($\zeta \in (0.707,1]$) to extreme resonance ($\zeta \ll 1$). The extreme case of $\zeta = 1$ and $a = 0$ is not periodic at all, but provides a steeper rolloff of power spectral density than a first order Markov model.

4.12 POWER SPECTRUM CONCEPTS AND SHAPING FILTERS

It is useful to be able to express the power spectral density of a wide-sense stationary output of a system directly in terms of the power spectral density of the input and the description of the system itself. Inherent in such a concept are the facts that the input is stationary, the system is time invariant, and we are interested only in a steady state description of the output (i.e., $t_0 \to -\infty$). Not only would such a relationship indicate the effect of a given system on the statistics of a signal, it also will yield an expedient means of synthesizing a shaping filter to duplicate a desired power spectral density from a white noise input. This section develops the desired relationship and then directly applies it to shaping filter design.

The input–output relationship of a single input–single output linear system can be described through a time domain analog of a transfer function, called an impulse response function $G_t(\cdot,\cdot)$, where $G_t(t,t')$ is the system output response at time t due to a unit impulse input applied at time t'. In terms of this function, the output $z(t)$ can be expressed in terms of the input $n(t')$ for $t' \in (-\infty, t]$ as

$$z(t) = \int_{-\infty}^{t} G_t(t,t')n(t')\,dt' \tag{4-159}$$

For any physically realizable system, $G_t(t,t') \equiv 0$ for $t < t'$: the system does not respond to an input before it arrives (the system is "nonanticipative"). If the system is time invariant, then the impulse response function is a function only of the time difference $(t - t')$, and not of t and t' separately, denoted by $G_t(t,t') \triangleq G_t(t - t')$, so that (4-159) becomes

$$z(t) = \int_{-\infty}^{t} G_t(t - t')n(t')\,dt' \tag{4-160a}$$

By defining a change of variables, $(t - t') = \tau$ (so that $dt' = -d\tau$), this becomes

$$z(t) = \int_{0}^{\infty} G_t(\tau)n(t - \tau)\,d\tau \tag{4-160b}$$

This is an ordinary convolution integral relation, the Laplace transform of which yields the multiplicative transfer function relation of Eq. (2-2). Note that physical realizability requires $G_t(\tau) = 0$ for $\tau < 0$ in this time-invariant case.

Now (4-160b) can be used to write the stationary statistics of the output process $\mathsf{z}(\cdot,\cdot)$ in terms of those of the input process $\mathsf{n}(\cdot,\cdot)$:

$$E\{\mathsf{z}(t)\} = E\{\mathsf{n}(t)\}\int_{0}^{\infty} G_t(\tau)\,d\tau = \text{const} \tag{4-161a}$$

$$\Psi_{zz}(\tau) = \int_{0}^{\infty}\int_{0}^{\infty} G_t(\tau_1)G_t(\tau_2)\Psi_{nn}(\tau + \tau_1 - \tau_2)\,d\tau_1\,d\tau_2 \tag{4-161b}$$

$$E\{\mathsf{z}^2(t)\} = \Psi_{zz}(0) = \int_{0}^{\infty}\int_{0}^{\infty} G_t(\tau_1)G_t(\tau_2)\Psi_{nn}(\tau_1 - \tau_2)\,d\tau_1\,d\tau_2 \tag{4-161c}$$

$$\Psi_{nz}(\tau) = \int_{0}^{\infty} G_t(\tau_1)\Psi_{nn}(\tau - \tau_1)\,d\tau_1 \tag{4-161d}$$

Define $G(\cdot)$ to be the Fourier transform of $G_t(\cdot)$:

$$G(\omega) = \int_{-\infty}^{\infty} G_t(\tau)e^{-j\omega\tau}\,d\tau$$

By considering the Fourier transform of (4-161), the convolutions become product relations.

First, due to physical realizability, $\int_0^\infty G_t(\tau)\,d\tau = \int_{-\infty}^{\infty} G_t(\tau)\,d\tau$ in (4-161a). This is then recognized as the Fourier transform evaluated at $\omega = 0$:

$$E\{\mathrm{z}(t)\} = E\{\mathrm{n}(t)\}G(0) = \text{const} \tag{4-162a}$$

The Fourier transform of (4-161b) is

$$\begin{aligned}\bar{\Psi}_{zz}(\omega) &= \int_{-\infty}^{\infty} \Psi_{zz}(\tau)e^{-j\omega\tau}\,d\tau\\ &= \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} G_t(\tau_1)G_t(\tau_2)\Psi_{nn}(\tau + \tau_1 - \tau_2)\,d\tau_1\,d\tau_2\,e^{-j\omega\tau}\,d\tau\end{aligned}$$

But the orders of integration can be changed since $\Psi_{nn}(\cdot)$ is assumed Fourier transformable, so

$$\begin{aligned}\bar{\Psi}_{zz}(\omega) &= \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} G_t(\tau_1)G_t(\tau_2)\Psi_{nn}(\tau + \tau_1 - \tau_2)e^{-j\omega\tau}\,d\tau\,d\tau_2\,d\tau_1\\ &= \int_0^{\infty} G_t(\tau_1)e^{j\omega\tau_1}\,d\tau_1 \int_0^{\infty} G_t(\tau_2)e^{-j\omega\tau_2}\,d\tau_2\\ &\quad\times \int_{-\infty}^{\infty} \Psi_{nn}(\tau + \tau_1 - \tau_2)e^{-j\omega(\tau+\tau_1-\tau_2)}\,d\tau\\ &= G(-\omega)G(\omega)\bar{\Psi}_{nn}(\omega)\end{aligned}$$

where use has been made of the fact that $G_t(\tau) = 0$ for $\tau < 0$. Remembering that $G(-\omega) = G^*(\omega)$, the final result is

$$\bar{\Psi}_{zz}(\omega) = G(\omega)G(-\omega)\bar{\Psi}_{nn}(\omega) = |G(\omega)|^2\bar{\Psi}_{nn}(\omega) \tag{4-162b}$$

Note that this depends only on the magnitude, and not the phase, of $G(\omega)$.

Similarly, (4-161c) and (4-161d) become

$$\begin{aligned}E\{\ {}^2(t)\} = \text{const} = \Psi_{zz}(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\Psi}_{zz}(\omega)\,d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)G(-\omega)\bar{\Psi}_{nn}(\omega)\,d\omega\end{aligned} \tag{4-162c}$$

$$\bar{\Psi}_{nz}(\omega) = G(\omega)\bar{\Psi}_{nn}(\omega) \tag{4-162d}$$

In much analytical work with linear systems, control engineers use the Laplace transform instead of the Fourier transform. Rather than considering two-sided Laplace transforms, we will treat these as Fourier transforms with a change of variable, replacing ω by s/j, i.e., by letting $s = j\omega$. Since power spectral densities are even functions of ω, there are no odd powers in ω, so $\bar{\Psi}_{zz}(s)$ is always a *real* function of s: a rational $\bar{\Psi}_{zz}(s)$ can be obtained from a rational $\bar{\Psi}_{zz}(\omega)$ by replacing all powers of ω^2 by $(s/j)^2 = -s^2$. The results of (4-162) can then be written directly in terms of functions of s instead of ω, with the only structural change being

$$E\{z^2(t)\} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} G(s)G(-s)\bar{\Psi}_{nn}(s)\,ds \tag{4-162c$'$}$$

Now we can consider shaping filter design: suppose we have a stationary stochastic process $n(\cdot,\cdot)$ with known power spectral density $\bar{\Psi}_{nn}(\cdot)$ (without loss of generality, a white process), and we want to generate from it a process $z(\cdot,\cdot)$ with a desired $\bar{\Psi}_{zz}(\cdot)$. If both spectra are rational (the ratio of polynomials in ω), then the design of the linear time-invariant shaping filter is straightforward, once spectral factorization is understood.

If $\bar{\Psi}_{zz}(\cdot)$ is rational, then it can be written, since it is also even, as

$$\bar{\Psi}_{zz}(\omega) = \frac{a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \cdots}{b_0 + b_1\omega^2 + b_2\omega^4 + b_3\omega^6 + \cdots} \tag{4-163a}$$

or

$$\bar{\Psi}_{zz}(s) = \frac{a_0 - a_1 s^2 + a_2 s^4 - a_3 s^6 + \cdots}{b_0 - b_1 s^2 + b_2 s^4 - b_3 s^6 + \cdots} \tag{4-163b}$$

$\bar{\Psi}_{zz}(s)$ can then always be factored into the form

$$\bar{\Psi}_{zz}(s) = K\,\frac{(c_1 - s^2)(c_2 - s^2)\cdots}{(d_1 - s^2)(d_2 - s^2)\cdots} \tag{4-163c}$$

Since the coefficients a_i and b_j are all real numbers, the c_i's and d_j's must either be real or occur in complex conjugate pairs. For real positive d_i, the poles are at $\pm\sqrt{d_i}$, as shown in Fig. 4.23a. For complex d_i, the poles are at $\pm\sqrt{d_i} = \pm(e + jf)$, as in Fig. 4.23b. In such a case, there is another d_j that is the complex conjugate of d_i, $d_j = d_i{}^*$, with roots at $\pm\sqrt{d_j} = \pm\sqrt{d_i{}^*} = \pm(e - jf)$, as in Fig. 4.23c. Thus, complex roots occur in quadruplets, symmetric about both the real and imaginary axes of the s plane, as in Fig. 4-23d. Similarly, any pure imaginary poles, the case for real negative d_i, appear as doubles; this can be viewed as the case above in the limit of zero separation, and is depicted in Fig. 4.23e. Zeros corresponding to the c_i in (4-163c) are treated analogously.

Now collect all factors of $\bar{\Psi}_{zz}(s)$ which define poles or zeros in the left half s plane, and denote the product of all of these factors and $\sqrt{K}$ [recall (4-163c)] as $\bar{\Psi}_{zz}(s)_{\mathrm{L}}$. Correspondingly generate $\bar{\Psi}_{zz}(s)_{\mathrm{R}}$ of right half plane factors and $\sqrt{K}$. Since pure imaginary roots always appear as doubles, one would be associated with $\bar{\Psi}_{zz}(s)_{\mathrm{L}}$ and the other with $\bar{\Psi}_{zz}(s)_{\mathrm{R}}$. This yields the *spectral factorization*:

$$\bar{\Psi}_{zz}(s) = \bar{\Psi}_{zz}(s)_{\mathrm{L}}\bar{\Psi}_{zz}(s)_{\mathrm{R}} \tag{4-164}$$

where all poles and zeros of $\bar{\Psi}_{zz}(s)_{\mathrm{L}}$ are in the left half plane, and all of $\bar{\Psi}_{zz}(s)_{\mathrm{R}}$ are in the right half plane. Note that, because of the symmetry about both

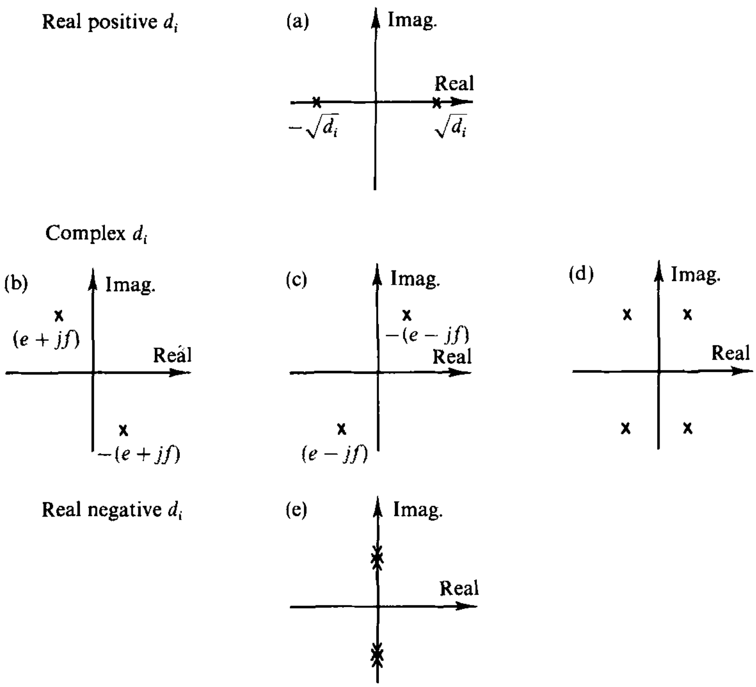

FIG. 4.23 s-Plane plots of roots of rational $\Psi_{zz}(s)$.

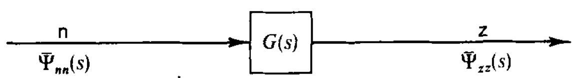

FIG. 4.24 Shaping filter design.

coordinate axes,

$$\Psi_{zz}(s)_R = \Psi_{zz}(-s)_L \tag{4-165}$$

Now we have the situation depicted in Fig. 4.24: we know $\Psi_{nn}(s)$ and $\Psi_{zz}(s)$ for all s, and wish to determine the appropriate $G(s)$ to describe the shaping filter. From (4-162b),

$$\Psi_{zz}(s) = G(s)G(-s)\Psi_{nn}(s)$$

which can be written in factored form as

$$\Psi_{zz}(s)_L\Psi_{zz}(s)_R = G(s)G(-s)\Psi_{nn}(s)_L\Psi_{nn}(s)_R$$

or, equivalently in view of (4-165),

$$\Psi_{zz}(s)_L\Psi_{zz}(-s)_L = G(s)G(-s)\Psi_{nn}(s)_L\Psi_{nn}(-s)_L$$

Thus, by letting the shaping filter be described by

$$G(s) = \Psi_{zz}(s)_L/\Psi_{nn}(s)_L \tag{4-166}$$

it will have all of its poles in the left half plane (and thus be *stable*) and all of its zeros also in the left half plane (and thus be *minimum phase*: for a given amplitude-versus-frequency Bode plot, this form has the least phase lag).

EXAMPLE 4.10 Use a white (Gaussian) noise with $\Psi_{nn}(\omega) = Q$ for all ω to generate an exponentially time-correlated (Gaussian) noise with

$$\Psi_{zz}(\omega) = \frac{2\sigma^2/T}{\omega^2 + (1/T)^2}$$

First replace ω^2 by $(-s^2)$ to generate $\bar{\Psi}_{zz}(s)$:

$$\bar{\Psi}_{zz}(s) = \frac{2\sigma^2/T}{(1/T)^2 - s^2} = \frac{2\sigma^2/T}{[(1/T) - s][(1/T) + s]}$$

Perform spectral factorization to obtain

$$\bar{\Psi}_{zz}(s) = \frac{\sqrt{2/T}\sigma}{(1/T) + s}\,\frac{\sqrt{2/T}\sigma}{(1/T) - s} = [\bar{\Psi}_{zz}(s)_{\rm L}][\bar{\Psi}_{zz}(s)_{\rm R}]$$

and similarly $\bar{\Psi}_{nn}(s)_{\rm L} = \sqrt{Q}$. The desired shaping filter can then be expressed as

$$G(s) = \frac{\bar{\Psi}_{zz}(s)_{\rm L}}{\bar{\Psi}_{nn}(s)_{\rm L}} = \frac{\sqrt{2/TQ}\sigma}{s + (1/T)}$$

Note that if we let $\mathrm{n}(\cdot,\cdot)$ be a white noise with $Q = 2\sigma^2/T$, then passing it through a first order lag, $G(s) = 1/[s + (1/T)]$, yields the desired result. This agrees with the results found previously by time-domain shaping filter design techniques. ■

EXAMPLE 4.11 To illustrate the case of poles or zeros on the imaginary axis, consider generating a signal $\mathrm{z}(\cdot,\cdot)$ with

$$\Psi_{zz}(\omega) = \frac{a\omega^2}{b^2 + \omega^2}$$

from white noise $\mathrm{n}(\cdot,\cdot)$ with $\Psi_{nn}(\omega) = Q$. Replace ω^2 by $(-s^2)$ and perform spectral factorization:

$$\begin{aligned}\bar{\Psi}_{zz}(s) &= \frac{-as^2}{b^2 - s^2} = \lim_{\varepsilon\to 0}\frac{\varepsilon^2 - as^2}{b^2 - s^2} = \lim_{\varepsilon\to 0}\frac{\varepsilon + \sqrt{a}s}{b + s}\,\frac{\varepsilon - \sqrt{a}s}{b - s}\\ &= \left[\frac{\sqrt{a}s}{b + s}\right]\left[\frac{-\sqrt{a}s}{b - s}\right] = [\bar{\Psi}_{zz}(s)_{\rm L}][\bar{\Psi}_{zz}(s)_{\rm R}]\end{aligned}$$

where the use of ε allows proper assignment of signs. The shaping filter can be expressed as

$$G(s) = \frac{\bar{\Psi}_{zz}(s)_{\rm L}}{\bar{\Psi}_{nn}(s)_{\rm L}} = \frac{(\sqrt{a}s)/(b + s)}{\sqrt{Q}} = \sqrt{\frac{a}{Q}}\,\frac{s}{s + b} \qquad ■$$

4.13 GENERATING PRACTICAL SYSTEM MODELS

In order to generate a model of the errors and uncertainty in a physical device (sensor, actuator, etc.) or any other type of system, empirical test data is collected. For example, a gyro output signal might be recorded over a period of hours or days when subjected to known (as especially zero) inputs. Or, a number of gyros might be tested in this manner to obtain "population statistics" on all gyros of that particular type.

Conceptually, sample autocorrelations can be generated by taking one set of data and generating all products of the form $x(t_i)x(t_j)$ for all discrete times t_i and t_j of interest. This would correspond to $\mathrm{x}(t_i, \omega_k) \cdot \mathrm{x}(t_j, \omega_k)$ for a single $\omega_k \in \Omega$. Then other sets of data would provide similar product values for different values of k. Averaging over N sets of data would yield the approximate *sample autocorrelation* $\tilde{\Psi}_{xx}(t_i, t_j; N)$, where

$$\tilde{\Psi}_{xx}(t_i, t_j; N) \triangleq \frac{1}{N} \sum_{k=1}^{N} \mathrm{x}(t_i, \omega_k)\mathrm{x}(t_j, \omega_k) \tag{4-167}$$

If the process $\mathrm{x}(\cdot,\cdot)$ is stationary, then the sample autocorrelation is a function of only the time difference $(t_i - t_j)$, and so fewer products are required to define the function totally: $\tilde{\Psi}_{xx}(t_i - t_j; N)$ could be evaluated by using (4-167) for any single value of t_i. Often the ergodic assumption is made in this case, yielding another *sample autocorrelation* $\Psi'_{xx}(m\,\Delta t; N)$ for $m = 0, \pm 1, \ldots, \pm(N-1)$ as the *time* average

$$\Psi'_{xx}(m\,\Delta t; N) \triangleq \frac{1}{N} \sum_{i=0}^{N-|m|-1} \mathrm{x}(t_i, \omega_k)\mathrm{x}(t_i + |m|\,\Delta t, \omega_k) \tag{4-168}$$

This uses N samples from a *single* realization of $\mathrm{x}(\cdot,\cdot)$, equally spaced at a sample period of Δt, to evaluate an approximate autocorrelation as an average of all possible products of samples separated by $m\,\Delta t$ seconds [there are few such products for m almost as large as N, and $\Psi'_{xx}(m\,\Delta t; N)$ is assumed to be zero for $m \geq N$)]. The corresponding *estimate of the power spectral density function* can be defined as the discrete Fourier transform of $\Psi'_{xx}(\cdot\,; N)$:

$$\bar{\Psi}'_{xx}(\omega; N) = \sum_{m=-\infty}^{\infty} \Psi'_{xx}(m\,\Delta t; N)e^{-jm\omega\Delta t} \tag{4-169}$$

Prior to the advent of fast Fourier transform (FFT) algorithms, (4-168) and (4-169) were used directly for generating approximate autocorrelations and power spectral densities from sampled data signals of the form $x(t_0), x(t_1), \ldots, x(t_{N-1})$. Now a somewhat different sequence of computations is performed to generate the desired sample functions [7, 10]. First, the fast Fourier transform of the *data* is generated for $i = 0, 1, 2, \ldots, N-1$ as

$$\begin{aligned}\bar{x}(i\omega_s; N) &= \sum_{k=0}^{N-1} x(t_0 + k\,\Delta t)e^{-jk\Delta t\, i\omega_s} \\ &= \sum_{k=0}^{N-1} x(t_0 + k\,\Delta t)(e^{-j2\pi/N})^{ki}\end{aligned} \tag{4-170}$$

where the frequency spacing ω_s between the discrete Fourier transform values is chosen to be $\omega_s = 2\pi/(N\,\Delta t)$ to provide adequate frequency spacing according to the "sampling theorem." Second, a power spectral density estimate is computed from the result of (4-170) and

$$\bar{\Psi}'_{xx}(i\omega_s; N) = \frac{1}{N}|\bar{x}(i\omega_s; N)|^2 \tag{4-171}$$

Computing this for $i = 0, 1, \ldots, N - 1$ requires $2N$ multiplications, since real and imaginary parts are squared separately. The autocorrelation $\Psi'_{xx}(m\,\Delta t; N)$ for $m = 0, 1, \ldots, N - 1$ is computed as the inverse fast Fourier transform of (4-171). To smooth the result, decreasing the distortion due to only a finite set of data being used, this $\Psi'_{xx}(m\,\Delta t; N)$ is multiplied by a "window function," i.e., each of its N values is multiplied by a predetermined coefficient, to yield a better (in terms of bias and variance properties) estimate of the autocorrelation, denoted as $\Psi''_{xx}(m\,\Delta t; N)$ for $m = 0, 1, \ldots, N - 1$. Finally, a better estimate of power spectral density, $\bar{\Psi}''_{xx}(i\omega_s; N)$, $i = 0, 1, \ldots, N - 1$, is computed as the fast Fourier transform of $\Psi''_{xx}(\cdot\,; N)$. Sometimes additional processing, as averaging of data subsets and "prewhitening," are employed to produce even better estimates of autocorrelation and/or power spectral density. Although this procedure involves a substantial number of computations, it is significantly faster than the straightforward use of (4-168) and (4-169), due to the capabilities of FFT algorithms.

Once such empirical data is generated, shaping filters can be produced by curve-fitting these sample functions. Any degree of complexity of the filters can be provided, depending on the complexity of the curves used to fit the data. This is illustrated by the following two examples.

EXAMPLE 4.12 Suppose laboratory tests of a gyro yield empirical drift rate autocorrelation and power spectral density data as portrayed in Figs. 4.25a and b. A reasonable fit to the autocorrelation data would be a curve of the form

$$\Psi_{xx}(\tau) = \sigma^2 e^{-|\tau|/T} + B$$

which describes a combination of a random bias and an exponentially time-correlated component. The values of the parameters σ, T, and B can be determined so as to provide the best fit of the assumed model to the data.

Looking at the power spectral density data indicates that an additional wideband component (modeled as white) should also be included [the corresponding narrow pulse of $\Psi_{xx}(\tau)$ at $\tau = 0$ is hard to discern from data]. A reasonable curve-fit here is

$$\bar{\Psi}_{xx}(\omega) = \frac{2\sigma^2/T}{\omega^2 + (1/T)^2} + Q$$

(Note that the bias is difficult to discern accurately from this data.)

This yields a gyro drift model as depicted in Fig. 25c, with defining statistics

$$E\{\mathrm{x}_1{}^2(t_0)\} = \sigma^2, \qquad E\{\mathrm{w}_1(t)\mathrm{w}_1(t+\tau)\} = [2\sigma^2/T]\delta(\tau)$$
$$E\{\mathrm{x}_2{}^2(t_0)\} = B, \qquad E\{\mathrm{w}_2(t)\mathrm{w}_2(t+\tau)\} = Q\delta(\tau)$$

Naturally, more complicated fitted curves would yield more complex drift models. ■

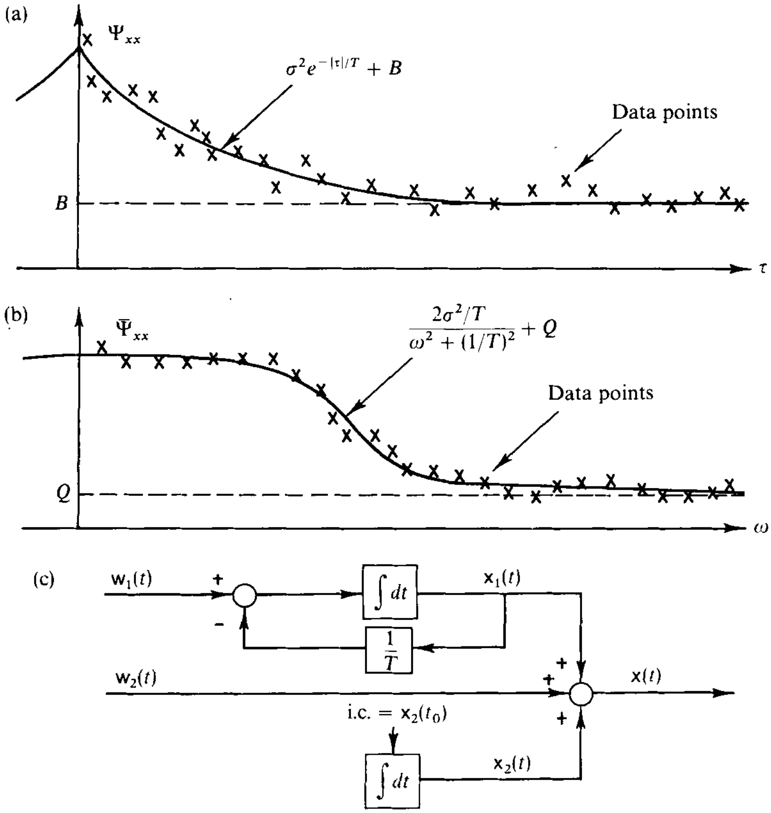

FIG. 4.25 Gyro drift model. (a) Curve-fitting empirical autocorrelation data. (b) Curve-fitting empirical power spectral density data. (c) Shaping filter.

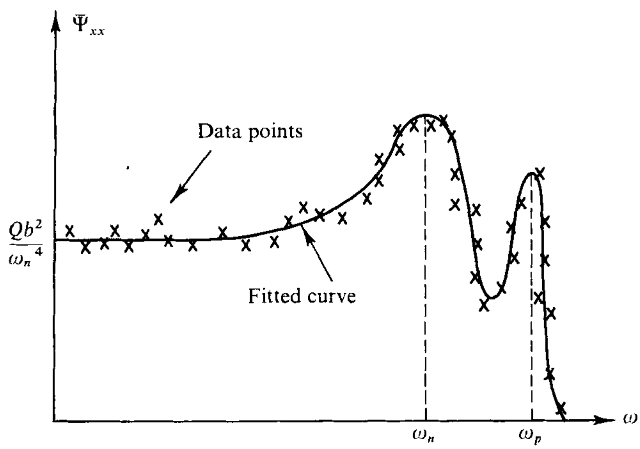

FIG. 4.26 Power spectral density function for Example 4.13.

EXAMPLE 4.13 Suppose vibration testing of a structure produced an acceleration power spectral density at a certain location as in Fig. 4.26. Except for the peaking at $\omega = \omega_p$, the data are well fit by

$$\Psi_{xx}(\omega) = \frac{Qb^2}{\omega^4 + 2\omega_n^2(2\zeta^2 - 1)\omega^2 + \omega_n^4}$$

which is generated by passing white noise of strength Q through a second order shaping filter described through

$$G_1(s) = \frac{b}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The peak at ω_p can be generated by cascading a "notch filter" of the form

$$G_2(s) = \frac{s^2 + 2\zeta_1\omega_p s + \omega_p^2}{s^2 + 2\zeta_2\omega_p s + \omega_p^2}$$

with the shaping filter described by $G_1(s)$. The size of the resonant peak at ω_p is then adjusted by controlling the magnitude of ζ_1 and ζ_2. Unlike most applications of notch filters, we want to accentuate the signal content in the "notch" region rather than attenuate it, so we require $\zeta_2 < \zeta_1$. ■

4.14 SUMMARY

Stochastic processes were defined and then characterized through an infinite array of joint distribution functions. A practical, though generally only partial, characterization was then developed in terms of the first two moments: the mean value function and the correlation or covariance kernel. This statistical knowledge was shown to be complete for Gaussian processes, and very readily generated for Gauss–Markov processes. For wide-sense stationary processes, another useful characterization was developed in the form of power spectral density.

A basic system model structure in the form of linear state dynamics driven only by known inputs and white Gaussian noise, with a linear measurement corrupted by additive white Gaussian noise, was motivated and shown to be widely applicable. With such a structure in mind, linear stochastic differential equations (4-121) and their solutions (4-122) were developed properly through stochastic integrals and Brownian motion. The Gauss–Markov state stochastic process could then be characterized by its mean (4-123), covariance (4-114) and (4-120), and covariance kernel (4-118).

With measurements typically available on a sampled-data basis as in (4-136) an overall system model was developed. An equivalent discrete-time system model (4-125) was also developed to describe such an output process from a physical system. Moreover, the output process characteristics were defined in terms of the state process description obtained previously: (4-144)–(4-146) portray the mean, correlation, and covariance kernel for this output process.

Finally, the concept of a shaping filter applied the proposed general model structure to the problem of synthesizing a mathematical model to duplicate the characteristics of empirically observed processes. Both time-domain and power spectral density techniques were exploited in this synthesis procedure. The generation of empirical autocorrelation and power spectrum data was discussed, and curve-fitting these data then allowed complete definition of the appropriate shaping filter.

At this point, we have adequate linear stochastic models for both static and dynamic systems. These will be exploited extensively in estimation and control and will also provide insights into nonlinear models and associated estimation and control algorithms.

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PROBLEMS

4.1 Consider the variable $\mathrm{x}(t)$ defined by

$$\mathrm{x}(t) = \sum_{i=0}^{N} \Delta\mathrm{x}(t_i)$$

where the variables $\Delta\mathrm{x}(t_i)$ are independent and Gaussian scalars with statistics

$$E\{\Delta\mathrm{x}(t_i)\} = 0, \qquad E\{\Delta\mathrm{x}^2(t_i)\} = q[t_{i+1} - t_i]$$

Find the characteristic function, $\phi_{\mathrm{x}}(\mu, t)$, of $\mathrm{x}(t)$ and use it to determine the first two moments of $\mathrm{x}(t)$.

4.2 Verify the integral below where $\beta(t)$ is Brownian motion:

$$\int_{t_0}^{t} m(\tau)\, d\beta(\tau) = m(t)\beta(t) - m(t_0)\beta(t_0) - \int_{t_0}^{t} \beta(\tau)\, dm(\tau)$$

4.3 Prove that Eq. (4-100) is valid by using fundamental definitions for a finite partitioning of the time axis, and then taking the limit as the time cuts become infinitesimally fine.

4.4 In deriving Eq. (4-145), the following relation was used:

$$\mathbf{x}(t_j) = \mathbf{\Phi}(t_j, t_0)\mathbf{x}(t_0) + \sum_{k=1}^{j} \mathbf{\Phi}(t_j, t_k)\mathbf{G}_d(t_{k-1})\mathbf{w}_d(t_{k-1})$$

Show that this is the solution to the stochastic difference equation (4-133) with $\mathbf{u}(t_i) \equiv \mathbf{0}$ for all time.

4.5 Let $\beta(\cdot,\cdot)$ be a scalar Brownian motion process with statistics

$$E\{\beta(t)\} = 0, \qquad E\{\beta(t)^2\} = t$$

The process $x(\cdot,\cdot)$ satisfies the stochastic differential equation

$$dx(t) = \beta(t)\cos t\, dt + \sin t\, d\beta(t)$$

Determine the variance of $x(t)$.

Explain how you would determine the variance if the equation were, instead,

$$dx(t) = x(t)\cos t\, dt + \sin t\, d\beta(t)$$

4.6 Let $\mathbf{x}(\cdot,\cdot)$ be a discrete-time process satisfying

$$\mathbf{x}(t_i) = \mathbf{\Phi}(t_i, t_{i-1})\mathbf{x}(t_{i-1}) + \mathbf{G}_d(t_{i-1})\mathbf{w}_d(t_{i-1})$$

where $\mathbf{w}_d(\cdot,\cdot)$ is a white Gaussian process with mean $\bar{\mathbf{w}}_d(t_i)$ for all t_i, and covariance kernel $\mathbf{Q}_d(t_i)\delta_{ij}$. Show that $\mathbf{x}(\cdot,\cdot)$ can also be generated by

$$\mathbf{x}(t_i) = \mathbf{\Phi}(t_i, t_{i-1})\mathbf{x}(t_{i-1}) + \mathbf{G}_d(t_{i-1})\mathbf{u}(t_{i-1}) + \mathbf{G}_d(t_{i-1})\mathbf{w}_d'(t_{i-1})$$

where the deterministic input $\mathbf{u}(t_i) \triangleq \bar{\mathbf{w}}_d(t_i)$ for all t_i and $\mathbf{w}_d'(\cdot,\cdot)$ is a zero-mean white Gaussian noise with covariance kernel $\mathbf{Q}_d(t_i)\delta_{ij}$.

4.7 An engineer had the task of simplifying a system model for eventual filter implementation. He was given a basic model of a noise input $n(\cdot,\cdot)$ into a physical system as in Fig. 4.P1a, where $w_1(\cdot,\cdot)$ and $w_2(\cdot,\cdot)$ are independent white Gaussian noises, each zero-mean and of unit strength (variance kernel of one times the delta function). He has proposed that the model depicted in Fig. 4.P1b is equivalent to the original, and it requires one less noise source in the overall system model. Do you agree that this is equivalent? Explain.

4.8 Suppose you are investigating the system modeled as in Fig. 4.P2. The noises n_1, n_2, n_3, and n_4 are zero-mean white Gaussian noises independent of x_1 and x_2 histories with variances

$$E\{n_i(t)n_i(t+\tau)\} = N_i\,\delta(\tau)$$

for $i = 1, 2, 3, 4$ and N_1, N_2, N_3, and N_4 specified values,

$$E\{n_2(t)n_3(t+\tau)\} = K_{23}\,\delta(\tau)$$

and other cross-correlations zero.

The two sampling devices are corrupted by zero-mean white Gaussian noise sequences d_1 and d_2 with

$$E\{d_1(t_i)\,d_1(t_j)\} = D_1\,\delta_{ij}, \qquad E\{d_2(t_i)\,d_2(t_j)\} = D_2\,\delta_{ij}, \qquad E\{d_1(t_i)\,d_2(t_j)\} = D_3\,\delta_{ij}$$

and the d_i's are independent of the x_i's and n_i's.

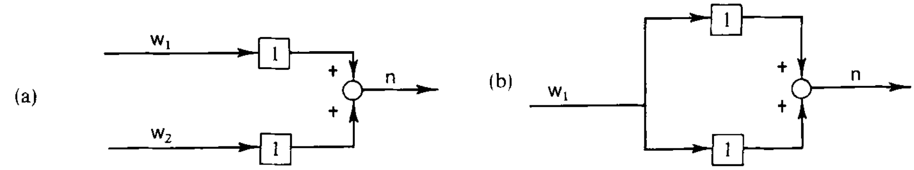

FIG. 4.P1

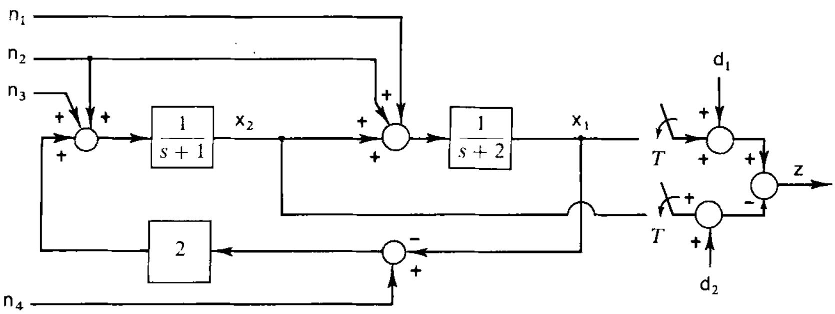

FIG. 4.P2

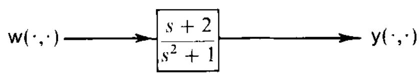

FIG. 4.P3

Develop the linear state variable stochastic differential equations to describe the system, using x_1 and x_2 as states.

Obtain the differential equations for the elements of the covariance matrix to describe the evolution of the second moment of $\mathbf{x} = [x_1 x_2]^T$. Assuming that the solution to the differential equation is available, can you generate an expression for the time-varying autocorrelation function of the output, $E\{z(t_i)z(t_j)\}$ in terms of the elements of this covariance matrix solution?

Can you generate an equivalent system description with fewer noise inputs than the six originally defined? *Specifically* describe such a system model and the statistics of the noises used to replace the original six.

4.9 The transfer function model for a system is given as in Fig. 4.P3, where $w(\cdot,\cdot)$ is a white Gaussian noise with

$$E\{w(t)\} = 0, \qquad E\{w(t)w(t+\tau)\} = \delta(\tau)$$

If the system starts at rest at $t = 0$, determine the variances of $y(t)$ and $\dot{y}(t)$ for $t \geq 0$. (Modeling suggestion: Standard controllable form is convenient.)

4.10 A random process with power spectral density

$$\Psi_{xx}(\omega) = A/(a^2 + \omega^2)$$

drives a first order lag, described by

$$T(s) = 1/(1 + \tau s)$$

What is the steady state mean squared value of $y(t)$, the output of the lag? Note that an expression of the form

$$E\{y^2\} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Psi_{yy}(s)\,ds$$

can be evaluated by the Cauchy residue theorem as

$$E\{y^2\} = \sum \{\text{residues of } \Psi_{yy}(s) \text{ in left half plane}\}$$

The result of such calculations is often tabulated in control texts and handbooks.

4.11 A lead-lag network is driven by a scalar white Gaussian noise process $w(\cdot,\cdot)$. The network transfer function is

$$x(s)/w(s) = (s+a)/(s+b)$$

Statistics of the input $w(\cdot,\cdot)$ are

$$E\{w(t)\} = 0, \qquad E\{w(t)w(t+\tau)\} = \delta(\tau)$$

All initial conditions are zero at $t = 0$. Find the nonstationary autocorrelation function for the output $x(\cdot,\cdot)$, $E\{x(t_2)x(t_1)\}$ for all t_1 and t_2, $0 \le t_1 \le t_2$. Note that

$$\int_{t_1}^{t_2} A(\tau)\,\delta(\tau - t_1)\,d\tau = \tfrac{1}{2}A(t_1)$$

for $A(\cdot)$ a general time function, and the t_1 in the argument of the delta function equal to the lower limit of integration.

4.12 Given the stochastic *vector* differential equation

$$\mathbf{dx}(t) = \mathbf{x}(t)\,dt + \mathbf{d\beta}(t)$$

with the initial conditions

$$E\{\mathbf{x}(t_0)\} = \mathbf{0}, \qquad E\{\mathbf{x}(t_0)\mathbf{x}^{\mathrm{T}}(t_0)\} = \mathbf{P}_0$$

and where $\boldsymbol{\beta}(\cdot,\cdot)$ is a vector Brownian motion with

$$E\{\boldsymbol{\beta}(t)\} = \mathbf{0}, \qquad E\{[\boldsymbol{\beta}(t) - \boldsymbol{\beta}(t')][\boldsymbol{\beta}(t) - \boldsymbol{\beta}(t')]^{\mathrm{T}}\} = \mathbf{Q}|t - t'|$$

Determine $E\{\mathbf{x}(t_2)\mathbf{x}^{\mathrm{T}}(t_1)\}$ for $t_0 < t_1 < t_2$.

4.13 Consider the discrete-time process $x(\cdot,\cdot)$ defined by

$$x(i+1) = [(i+1)/(i+2)]x(i)$$

with $x(0)$ a Gaussian random variable with mean $\hat{x}_0$ and variance P_0. Determine the mean and mean square functions, and variance and correlation kernels for the process. Repeat this for

$$x(i+1) = [(i+1)/(i+2)]x(i) + w_d(i)$$

where $w_d(i)$ is zero-mean white Gaussian noise of strength Q_d for all i.

4.14 A stationary process $x(\cdot,\cdot)$ with zero mean and autocorrelation $e^{-\alpha|\tau|}$ is applied at $t = 0$ to a linear system with impulse response $h(t) = e^{-\beta t}U(t)$, where $U(t)$ is the unit step. Find the autocorrelation $P_{yy}(t_1, t_2)$ of the resulting output $y(\cdot,\cdot)$.

Also generate the mean squared value of $y(t)$ for all t. Use both spectral analysis and state space analysis (generating $x(\cdot,\cdot)$ as the output of a shaping filter) to solve this problem.

4.15 Now suppose that the process $x(\cdot,\cdot)$ of the previous problem is applied instead to an integrator starting at rest at $t = 0$. Show that the variance of the output of the integrator is

$$P_{yy}(t) = (2/\alpha^2)(\alpha t - 1 + e^{-\alpha t})$$

and thus never converges to a stationary process as $t \to \infty$.

4.16 Einstein in 1905 gave a solution to the Brownian motion problem. He assumed the visible particles were large compared to the mean free path of the molecules of the fluid, so that the equations of motion of a visible particle would be well approximated by

$$\dot{x}(t) = v(t), \qquad m\dot{v}(t) = -cv(t) + f(t)$$

where $x(t)$ = position of particle, $v(t)$ = velocity of particle, m = mass of particle, c = Stokes's viscous force coefficient (constant), and $f(t)$ = random force due to collision with molecules. The mean time between collisions is very short and $f(t)$ is well approximated by Gaussian white noise, with

$$E\{f(t)\} = 0, \qquad E\{f(t)f(t+\tau)\} = q\,\delta(\tau), \qquad q = \text{const}$$

Assuming that

$$E\{x(0)\} = E\{v(0)\} = E\{x(0)^2\} = E\{v(0)^2\} = E\{x(0)v(0)\} = 0$$

determine expressions for

$$E\{v(t)^2\}, \quad E\{v(t)x(t)\}, \quad \text{and} \quad E\{x(t)^2\}$$

4.17 Recall Problem 2.7 concerning a single-axis stable platform system. It is desired to determine the response of this system to a random gyro drift driving function. Gyro drift is usually modeled as having a random component plus components that depend both linearly and quadratically on the acceleration of the instrument. In this problem we model only the random component. The random drift rate can be modeled in different ways, but for this problem, to simplify the analysis, let the gyro drift rate be an unbiased Gaussian white noise with

$$E\{\omega_{\text{drift}}(t)\} = 0, \qquad E\{\omega_{\text{drift}}(t)\omega_{\text{drift}}(t+\tau)\} = N\,\delta(\tau)$$

Assume that the interfering torque can be modeled as a white Gaussian noise with

$$E\{M_{\text{intf}}(t)\} = 0, \qquad E\{M_{\text{intf}}(t)M_{\text{intf}}(t+\tau)\} = M\,\delta(\tau)$$

For this problem let $F_c(p) = 1$.

Determine the expression for $E\{\omega_{\text{im}}^2(t)\}$ as a function of time, assuming M and N to be constant.

4.18 Consider a pendulum of length l with a bob of mass m as shown in Fig. 4.P4. Horizontal winds perturb the pendulum from its equilibrium position, and the perturbing force is proportional to the *relative* wind velocity on the bob, with proportionality constant a:

$$(\text{perturbing force}) = a \cdot (\text{relative wind})$$

The wind velocity is a white noise process with statistics

$$E\{w(t)\} = 0, \qquad E\{w(t)w(t+\tau)\} = b\,\delta(\tau)$$

Develop the linearized state variable stochastic differential equations for the system: for sufficiently small angles θ, write $m\ddot{x}$ = sum of forces, or equivalently, $ml^2\ddot{\theta}$ = sum of torques.

Obtain the differential equations for the elements of the appropriate covariance matrix. Determine an expression for the variance of the bob displacement, $x(t)$, when the system has reached stationary operation.

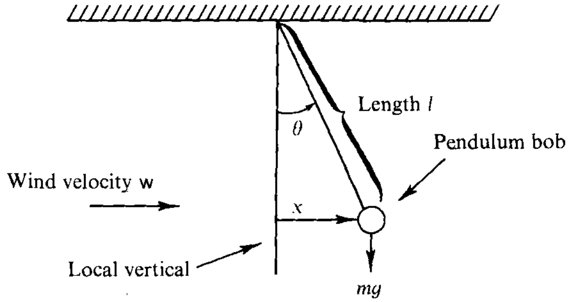

FIG. 4.P4

4.19 Given the *scalar* stochastic differential equation

$$\mathrm{d}\mathrm{x}(t) = [\mathrm{x}(t)/t]\,dt + \mathrm{d}\beta(t), \qquad t > 0$$

where

$$E\{\mathrm{x}(t_0)\} = 0, \qquad E\{\mathrm{x}(t_0)^2\} = a, \qquad t_0 > 0$$

and where $\beta(\cdot,\cdot)$ is Brownian motion with

$$E\{\beta(t)\} = 0, \qquad E\{[\beta(t) - \beta(t')]^2\} = q|t - t'|$$

Determine the variance of $\mathrm{x}(t)$ for $0 < t_0 < t$.

If $\mathrm{y}(t)$ is defined for all t as

$$\mathrm{y}(t) = \sqrt{t}\,\mathrm{x}(t)$$

what is $E\{\mathrm{y}(t_2)\mathrm{y}(t_1)\}$ for $0 < t_0 \le t_1 \le t_2$?

4.20 Consider the scalar process $\mathrm{x}(\cdot,\cdot)$ defined on $[t_0, t_f]$ by

$$\mathrm{d}\mathrm{x}(t) = -(1/T)\mathrm{x}(t)\,dt + \mathrm{d}\beta(t)$$

where $\beta(\cdot,\cdot)$ is Brownian motion of constant diffusion parameter Q, and $\mathrm{x}(t_0)$ is a Gaussian random variable independent of $\beta(\cdot,\cdot)$, with mean $\hat{x}_0$ and variance P_0. What must be true of $\hat{x}_0$, P_0, and Q for the $\mathrm{x}(\cdot,\cdot)$ process to be wide-sense stationary over the interval $[t_0, t_f]$? Strict-sense stationary?

4.21 (a) Without assuming a priori that $Q = 1$, derive the appropriate Q value for the strength of the white Gaussian noise to drive a second order shaping filter, (4-158b) or (4-158c), so as to generate a second order Markov process with autocorrelation as in (4-158a) with $\eta = 0$. Generate expressions for Q in terms of parameters σ^2, ζ, and ω_n, and show that reduction yields $Q = 1$. Obtain the result in both the time and frequency domains. Note that $\eta = 0$ iff $b^2 - a^2\omega_n{}^2 =$ (0 or $N\pi$, $N =$ integer), which specifies the location of the zero of $G(s)$ in (4-158b).

(b) Show that η shifts the zeros of $\Psi_{xx}(\tau)$ depicted in the bottom plot of Fig. 4.22: that the first zero occurs at $\tau = (0.5\pi + \eta)/\omega_d$ and successive zeros are spaced a distance of $\Delta\tau = \pi/\omega_d$ apart, where $\omega_d = \sqrt{1 - \zeta^2}\,\omega_n$. Show that the slope $d\Psi_{xx}(\tau)/d\tau$ at $\tau = 0^+$ is given by $\sigma^2(\omega_d \tan\eta - \zeta\omega_n)$.

Thus, given an empirical autocorrelation function, the parameters required in (4-158b,c) can be established as follows. First $\sigma^2 = \Psi_{xx}(0)$. Then ω_d and η are set by the first two zeros of $\Psi_{xx}(\tau)$. The slope at $\tau = 0^+$ is used to obtain $[\zeta\omega_n] = \omega_d \tan\eta - [d\Psi_{xx}(0^+)/d\tau]/\sigma^2$. Then ζ is found by solving $\zeta/\sqrt{1 - \zeta^2} = [\zeta\omega_n]/\omega_n$. Finally, α, a, b, and c are computed as given after Eq. (4-158c).

4.22 Design the shaping filter to generate a random process having the power spectral density

$$\Psi_{xx}(\omega) = a(\omega^2 + b^2)/(\omega^4 + 4c^4)$$

from a white noise with autocorrelation $E\{w(t)w(t+\tau)\} = Q\,\delta(\tau)$. Using the state model of the filter, calculate the mean squared value of $x(t)$ for $t \in [0, \infty)$, assuming that the filter starts from rest at $t = 0$.

4.23 The shaping filter depicted in Fig. 4.P5 is meant to generate a stationary output process $x_1(\cdot,\cdot)$ in steady state operation. Show that, if $w(\cdot,\cdot)$ is zero-mean white Gaussian noise of strength Q, then in steady state,

$$E\{x_1^2(t)\} \to Q/[4\zeta\omega_n^3], \qquad E\{x_2^2(t)\} \to Q/[4\zeta\omega_n]$$

If this is an adequate system model, is $E\{\dot{x}_2^2(t)\}$ finite or not?

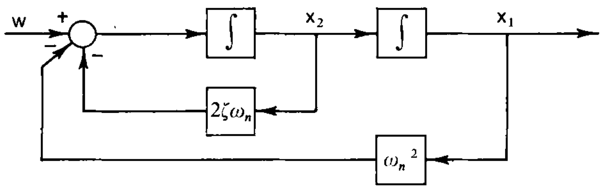

FIG. 4.P5

4.24 Design the shaping filter to generate a signal with an autocorrelation function of

$$E\{x(t)x(t+\tau)\} = K[(1/a)e^{-a|\tau|} - (1/b)e^{-b|\tau|}]$$

from an input of white noise with power spectral density value of Ψ_0.

4.25 An autocorrelation function curve-fitted to certain empirical data is of the form

$$E\{x(t)x(t+\tau)\} = \sigma^2[c_1 + c_2 e^{-|\tau|/T} + c_3 \cos \omega\tau]$$

where the positive constants c_1, c_2, c_3, σ^2, T, and ω are obtained through the curve-fitting process. Generate the state model for a shaping filter that would produce such an output process.

4.26 Two proposed models for a shaping filter to generate a process $x(\cdot,\cdot)$ with a specified autocorrelation function are as depicted in Fig. 4.P6, where $w_1(\cdot,\cdot)$ and $w_2(\cdot,\cdot)$ are white Gaussian noises of zero mean and variance kernels

$$E\{w_1(t)w_1(t+\tau)\} = q_1\,\delta(\tau), \qquad E\{w_2(t)w_2(t+\tau)\} = q_2\,\delta(\tau)$$

If these are to provide identical x process statistics in steady state operation, what is the relationship required between q_1 and q_2? If $x(\cdot,\cdot)$ is to have zero mean and mean squared value K in steady state, what values of q_1 and q_2 are required?

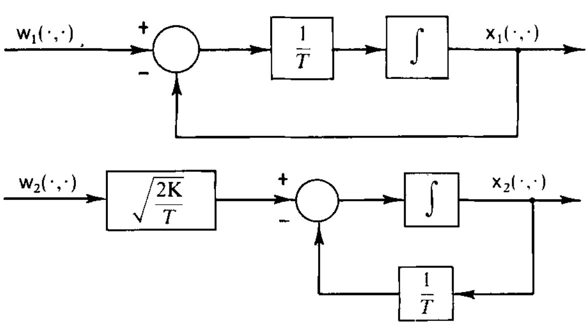

FIG. 4.P6

In terms of either q_1 or q_2, what is the autocorrelation function for the process $\mathrm{x}(\cdot,\cdot)$ in steady state?

4.27 It can be shown that a zero-mean Gaussian process with power spectral density of $\{Q_0 + 2b\sigma^2/(\omega^2 + b^2)\}$ can be generated by summing the output of a first order lag $1/(s+b)$ driven by zero-mean white Gaussian noise of strength $(2b\sigma^2)$ with a second, independent, zero-mean white Gaussian noise of strength Q_0. Can the process also be generated as the output of a lead-lag $(s+a)/(s+b)$ driven by a single zero-mean white Gaussian noise? If so, what are the strength of the noise and the value of the parameter a?

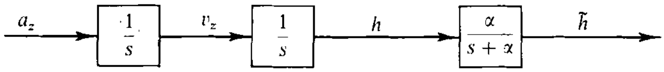

FIG. 4.P7

4.28 Let a linear model of vertical motion of an aircraft and barometric altimeter output be as depicted in Fig. 4.P7. Vertical acceleration is integrated twice to obtain altitude, and then the altitude indicated by the barometric altimeter, $\tilde{h}$, is the output of a first order lag (because of the inherent lag in the device). Derive the state equations appropriate to this system. Now let the vertical acceleration be modeled as a wideband (approximated as white) Gaussian noise of strength q_w, with an additional time-correlated component whose autocorrelation function can be approximated as

$$E\{\mathrm{x}(t)\mathrm{x}(t+\tau)\} = p\exp(-\zeta\omega_n|\tau|)\cos[\omega_n(1-\zeta^2)^{1/2}|\tau|]$$

Let the altimeter output be corrupted by

(1) a bias modeled as a process with constant samples, whose values can be described at any time t through a Gaussian random variable with mean zero and variance b^2,

(2) a wideband (approximated as white) Gaussian noise of strength R_w,

(3) an additional low frequency noise component, modeled as exponentially time-correlated noise whose mean square value is S and whose correlation time is T.

Assume all noises have zero mean.

Derive the state equations and output equation appropriate for the augmented system description.

4.29 A first order linear system is driven by white Gaussian noise. The statistics of the output $\mathrm{x}(\cdot,\cdot)$ are, for $t \geq 0.2$ sec,

$$E\{\mathrm{x}(t)\} = 0, \qquad E\{\mathrm{x}^2(t)\} = 1 + e^{\cos t}(5t - 1)$$

and the statistics of the input $\mathrm{w}(\cdot,\cdot)$ are

$$E\{\mathrm{w}(t)\} = 0, \qquad E\{\mathrm{w}(t)\mathrm{w}(t+\tau)\} = [5e^{\cos t} + \sin t]\,\delta(\tau)$$

Find the differential equation describing the system.