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Extremal Equilibria of Oligopolistic Supergames*

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General propositions established in Abreu (Ph.D. thesis, Princeton University, October 1983) are applied to the analysis of *optimal* punishments and constrained Pareto optimal paths of symmetric oligopolistic supergames. A remarkably simple 2-dimensional *stick-and-carrot* characterization of optimal symmetric punishments is obtained. An analogous result holds for the general case of asymmetric punishments, motivating the study of asymmetric Pareto optimal paths. The latter turn out to have a highly non-stationary dynamic structure which sometimes entails intertemporal reversals of relative payoffs between firms. *Journal of Economic Literature* Classification Numbers: 022, 611.

1. INTRODUCTION

The classic model of oligopoly theory is the one-shot game with quantity setting firms proposed by Cournot. The non-cooperative equilibrium of this game provides the traditional description of how firms behave in the absence of binding contracts and in the context of regulations which prohibit explicit collusion. In recent times this model has been criticized for being too *static*, and thereby yielding predictions which are misleadingly competitive. An important feature of reality is that oligopolistic firms interact repeatedly. This provides a setting in which tacit collusion may emerge, propped up by *credible* threats of retaliation for defections from the implicitly collusive arrangements in question. These possibilities have led economists to the study of *oligopolistic* supergames; they provide a formal structure in which collusive behavior may be supported by *non-cooperative* equilibria. While it is well known (see, for example, Luce and Raiffa [6] and Friedman [4]) that more profitable (i.e., collusive) paths

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than the single period Cournot–Nash equilibrium (CNE) may be supportable by the *threat* of reverting to single period CNE behavior if a deviation from the collusive path occurs, there has been no systematic attempt to study the *maximal* degree of collusion sustainable by credible threats for arbitrary values of the discount factor.[1] In view of the motivation for moving from static to repeated models, this has some claims to being the essential question at issue.

The "credibility" criterion adopted is *subgame perfection* (Selten [8, 9]); the reader may verify that this is equivalent to sequential equilibrium (Kreps and Wilson [5]) in the informational setting of the repeated game presented below. Attention is restricted to pure strategies. This qualification should be understood in all that follows and applies to the various optimality propositions stated.

Abreu [1] introduces a general *optimizing* approach to this class of models, in particular emphasizing the central role of *optimal punishments*. As argued there, the fundamental determinant of the limits of collusion is the severity of punishments with which potential deviants from cooperative behavior can credibly be threatened. Accordingly, this paper concentrates principally on characterizing strategy profiles which yield optimal (in the sense of most severe) punishments. The approach taken also proves fruitful in studying the polar opposite case of (constrained) Pareto optimal behavior. The model studied is a symmetric Cournot oligopoly.

I now summarize a set of results which greatly simplify the present investigation. These are derived from more general propositions established in [1]. In the symmetric context of the present paper they imply that in looking for worst credible equilibria, attention may be restricted to strategy profiles defined by a single outcome path. (An outcome path is an infinite stream of actions, one for each player, i.e., it is an infinite stream of action *profiles.*) For instance, among the subgame perfect equilibrium strategy profiles which give firm 1 the lowest possible payoff, there exists one which has the following elementary structure: it specifies that firms play according to some outcome path $\underline{Q}$ unless some deviation occurs. If firm 1 has deviated, it specifies that $\underline{Q}$ simply be restarted. If firm i has deviated, the strategy profile now requires that $\underline{Q}(i \mid 1)$ be played, where $\underline{Q}(i \mid 1)$ is identical to $\underline{Q}$ except that the roles of firm 1 and firm i are interchanged. (Simultaneous deviations by two or more firms are ignored.) At any point in the game, if player j deviated from the currently specified path $\underline{Q}(i \mid 1)$, $\underline{Q}(j \mid 1)$ is the new path specified. Denote the strategy profile just described by $\sigma(\underline{Q})$. For arbitrary Q, $\sigma(Q)$ may be defined analogously to $\sigma(\underline{Q})$. Then

[1] In a pioneering paper Friedman [4] elegantly argues that CNE reversion will support Pareto efficient payoffs when the discount factor is "large." Clearly in many applications it is unreasonable to suppose that players are sufficiently patient, relative to the myopic incentives to cheat, for more severe threats to be redundant.

$\sigma(\underline{Q}(i \mid 1))$ is a subgame perfect equilibrium (P.E.) which gives firm i its lowest possible P.E. payoff. Since the single outcome path $\underline{Q}$ and its permutations define the worst P.E. for firm i, for each i, we refer to $\underline{Q}$ as an optimal punishment and shift attention from strategy profiles to outcome paths. Our problem then reduces to finding Q such that the discounted sum of payoffs it yields firm 1 is lowest among all Q such that $\sigma(Q)$ is a P.E.

In Section 4 I analyze optimal *symmetric* punishments, i.e., punishments which are optimal among the class of symmetric punishments. (An outcome path or punishment is symmetric if it requires all firms to produce the same output streams.) Although in general imposing symmetry is restrictive, it leads to strong characterization theorems which provide a number of important insights. A particular class of paths called two-phase punishments play a central role. A two-phase punishment is symmetric; in addition it is stationary after the first period, i.e., in the second phase. I show that the optimal two-phase punishment is:

(1) *Globally* optimal for a certain range of parameter values.

(2) An optimal symmetric punishment. The class of symmetric punishments is of course much larger than, and contains, the class of two-phase punishments.

(3) More severe than Cournot–Nash reversion.

(4) Easily calculated. It is completely characterized by a pair of simultaneous equations. In addition,

(5) The second phase of the optimal two-phase punishment is the most collusive symmetric output level which can be sustained by the optimal two-phase punishment itself.

(3) of course implies that optimal two-phase punishments support more collusive output levels than Cournot–Nash reversion; by (4), use of the latter cannot be justified on pragmatic grounds. (5) says that the optimal two-phase punishment consists of a stick and a carrot[2]; furthermore, the carrot phase is the most attractive collusive regime which can credibly be offered when optimal two-phase punishments are used to deter defections. Thus when we solve for optimal two-phase punishments, we simultaneously determine the maximal degree of collusion sustainable by optimal symmetric punishments. It is worth remarking that the stick-and-carrot property is not a curiosum, but arises naturally from the structure of the problem.

Section 5 treats asymmetric paths. I show that optimal punishments *must*

[2] I started investigating stick-and-carrot punishments at David Pearce's instigation. He suggested that they might fare better than Cournot–Nash punishments; as it turned out, his intuition was richly confirmed.

be asymmetric whenever symmetric punishments yield players strictly positive discounted payoffs. (Note that firms can guarantee themselves a zero payoff by producing nothing forever.) Optimal asymmetric punishments have a rather complicated structure and thus far elude description as complete as that provided for optimal symmetric punishments in the earlier section.

A smoothing result reduces the dimensions of asymmetry by allowing attention to be restricted to paths along which all firms other than the deviant produce the same output in any given period. For the propositions considered this means that there is no loss of generality involved in analyzing the two-firm case only. Results analogous to the Pareto optimality of the second phase of the optimal symmetric punishment hold in the asymmetric case. In particular, the total discounted payoffs provided in the second phase of an optimal asymmetric punishment cannot be interior to the set of P.E. payoffs. Indeed they will in many circumstances be "locally Pareto optimal" (as defined in Sect. 5).

A number of results for asymmetric Pareto optimal paths are obtained. Primary among these is that Pareto optimal paths in general have a non-trivial dynamic structure. Indeed, for a range of cases the non-stationarity is extreme: the better-off firm today must be worse off tomorrow both in terms of one-period payoffs and the present discounted value of its entire stream of payoffs.

An exploration of the bounds of credible collusion revolves around the study of optimal punishments. The latter lead directly (for certain parameter values) to a consideration of asymmetric punishments and asymmetric constrained Pareto optima. The richness of the incentive structures encountered along these asymmetric paths and the novelty of some of the arguments which suggest themselves provide a measure of consolation for the slower progress made once symmetry is abandoned.

2. The Model

N identical firms play an infinitely repeated oligopolistic game with discounting. In each period they *simultaneously* choose *quantities*. When making its quantity decision in period t, each firm knows and therefore can condition upon what every other firm has produced in all previous periods. Detailed assumptions are specified below. They have been chosen to allow as simple an expression of the main ideas as possible.

The One-Shot Game

(A1). Firms are identical, quantity setting, and produce a homogeneous product at constant marginal cost $c > 0$.

q_i denotes the output produced by firm i and $p(\cdot)$ the industry inverse demand function. The payoff functions of firms are defined in the usual way:

$$\Pi_i(q_1,..., q_N) = \left(p\left(\sum_{j=1}^{N} q_j \right) - c \right) q_i.$$

(A2) $p: R_+ \to R_+$ is strictly monotonic and continuous.

(A3) $p(0) > c$ and $\lim_{z\to\infty} p(z) = 0$. S_i is the ith firm's strategy set.

(A4) $S_i = [0, \bar{M}(\delta)]$, where $\delta \in (0, 1)$ is the discount factor and $\bar{M}(\delta) \in R_+$ satisfies

$$-\Pi_1(\bar{M}(\delta), 0,..., 0) > \frac{\delta}{1-\delta} \sup_z \Pi_1(z, 0,..., 0).$$

(A1), ($c > 0$), and (A3) imply that $\bar{M}(\delta)$ is well defined and strictly positive for all $\delta \in (0, 1)$.

Even in the context of repeated play, it can never be in a firm's interest to produce any output greater than $\bar{M}(\delta)$ in any period. If it does so, the losses it incurs are so large (even if all other firms produce nothing) that they can never be recouped even if the firm receives monopoly profits forever after. There is no loss of generality in restricting attention to the bounded strategy sets S_i.

Let $\pi(x) = \Pi_1(x,..., x) = (p(Nx) - c)\, x$. $\pi(x)$ is per firm profits when all firms produce x units of output each.

(A5) $\operatorname{argmax}\{\pi(x) \mid x \in S_1\}$ is a singleton. x^m denotes its unique element. $\pi(x)$ declines strictly monotonically as output increases beyond x^m or falls below x^m.

Let $G = (S_i, \Pi_i; i = 1,..., N)$ denote the one-shot oligopolistic game.

(A6) G has a symmetric pure strategy Nash equilibrium.

The Repeated Game $G^\infty(\delta)$

$G^\infty(\delta)$ denotes the supergame with discounting obtained by repeating G infinitely often, and evaluating payoffs in terms of the discount factor $\delta \in (0, 1)$. σ_i denotes a *pure* strategy for player i. It is a sequence of functions $\sigma_i(1)$, $\sigma_i(2)$,..., $\sigma_i(t)$,..., one for each period t. The function for period t determines player i's action at t as a function of the actions of all players in all previous periods. Formally, $\sigma_i(1) \in S_i$ and for $t = 2, 3,..., \sigma_i(t): S^{t-1} \to S_i$, where $S \equiv S_1 \times \cdots \times S_N$.

Let $q \equiv (q_1, q_2,..., q_N)$. Typically, when player subscripted symbols are used, the corresponding unsubscripted symbol refers to a Cartesian product or a vector, depending on context. A *stream* of action *profiles* $\{q(t)\}_{t=1}^{\infty}$ is referred to as an *outcome path* or *punishment* and is denoted by Q. $\Omega \equiv S^{\infty}$ is the set of outcome paths. Any strategy profile σ generates an outcome path $Q(\sigma) = \{q(\sigma)(t)\}_{t=1}^{\infty}$ defined inductively as

$$q(\sigma)(1) = \sigma(1)$$

$$q(\sigma)(t) = \sigma(t)(q(\sigma)(1),..., q(\sigma)(t-1)).$$

$v_i: \Omega \to R$ defines the ith player's payoff from an outcome path $Q = \{q(t)\}_{t=1}^{\infty} \in \Omega$:

$$v_i(Q) = \sum_{t=1}^{\infty} \delta^t \Pi_i(q(t)).$$

$$v(Q) = (v_1(Q),..., v_N(Q)).$$

Note that I am discounting to the beginning of period 1, and that period t payoffs are received at the end of period t.

3. Summary of Some Relevant Results

In the Introduction I reviewed results obtained in Abreu [1] which underlie the approach adopted here. These are presented below, keeping in mind the prior discussion and the availability of a detailed treatment in [1]. The notation defined below is used frequently:

D1. $\Omega^0 = \{Q(\sigma) \mid \sigma \text{ is a P.E.}\}$. Ω^0 is the set of P.E. outcome paths.

D2. $V = \{v(Q) \mid Q \in \Omega^0\}$. V is the set of P.E. payoff vectors. (A6) guarantees that V is nonempty; the strategy profile which specifies CNE outputs in all contingencies is a P.E.

D3. $\mathbf{v} = \min\{v_1(Q) \mid Q \in \Omega^0\}$. $\mathbf{v}$ is the lowest possible payoff for firm 1 among all its P.E. payoffs. Since the game is symmetric, nothing depends upon the choice of firm 1 as the firm whose payoff is being minimized.

D4. $\Pi_i^*(q_1,..., q_N) = \max\{\Pi_i(x, q_{-i}) \mid x \in S_i\}$, where $q_{-i} = (q_1,..., q_{i-1}, q_{i+1},..., q_N)$. By (A1)–(A4), $\Pi_i^*(q)$ exists for all $q \in S$.

D5. $v_i(Q; t+1) = \sum_{s=1}^{\infty} \delta^s \Pi_i(q(t+s))$. $v_i(Q; t+1)$ is the present discounted value (pdv) of firm i's payoffs, in periods $(t+1)$ to ∞, along the path Q.

Recall from the Introduction the definition of a strategy profile $\sigma(Q)$ for any $Q \in \Omega$. $\sigma(Q)$ specifies that Q be played initially and that any deviation by a single firm from a previously specified path be responded to by playing Q or a permutation[3] of Q, depending on which firm deviated. Strategy profiles of the form $\sigma(Q)$ have a particularly simple structure and play a central role in what follows.

Let $q^{cn} \in S$ denote a symmetric CNE and Q^{cn} the corresponding CNE path, i.e., $Q^{cn} = \{q'(t)\}_{t=1}^{\infty}$, where $q'(t) = q^{cn}$, $t = 1, 2, \ldots$. The results established in [1], applied to the present symmetric context, imply the following:

PROPOSITION 1. *$\mathbf{v} = \min\{v_1(Q) \mid Q \in \Omega^0\}$ exists.*

Let $\underline{Q} \in \Omega^0$ satisfy $v_1(\underline{Q}) = \mathbf{v}$.

PROPOSITION 2. *$\sigma(\underline{Q})$ is a* P.E.

PROPOSITION 3. *Suppose $\Psi \subseteq \Omega$ is compact and $Q^{cn} \in \Psi$. Then $\mathbf{v}(\Psi) = \min\{v_1(Q) \mid Q \in \Psi$ and $\sigma(Q)$ is a* P.E.$\}$ *exists.*

Propositions 2 and 3 imply $\mathbf{v}(\Omega) = \mathbf{v}$.

D6. $Q = \{q(t)\}_{t=1}^{\infty} \in \Omega$ is *supportable by* $w_1 \in R$ if and only if

$$\Pi_i^*(q(t)) - \Pi_i(q(t)) \leqslant v_i(Q; t+1) - w_1$$

for all $i = 1, \ldots, N$ and $t = 1, 2, \ldots$.

PROPOSITION 4. *$\sigma(Q)$ is a* P.E. *if and only if Q is supportable by $v_1(Q)$.*

Optimal punishments are important because they provide a simple characterization of the set of P.E. outcome paths:

PROPOSITION 5. *$Q \in \Omega^0$ if and only if Q is supportable by $\mathbf{v}(\Omega)$.*

4. SYMMETRIC PATHS

In this section I analyze the problem $\min\{v_1(Q) \mid Q \in \Psi$ and $\sigma(Q)$ is a P.E.$\}$ for the case $\Psi = \Gamma$, where Γ is the set of symmetric paths.

D7. $\Gamma = \{Q = \{q(t)\}_{t=1}^{\infty} \mid q_i(t) = q_1(t) \in S_1$ for all $i = 2, \ldots, N$ and $t = 1, 2, \ldots\}$.

[3] Let $q(t) \equiv (q_1(t), \ldots, q_N(t))$. The permutation $q(i \mid 1)(t) \equiv (q_1(i \mid 1)(t), \ldots, q_N(i \mid 1)(t))$ is defined by $q_i(i \mid 1)(t) = q_1(t)$, $q_1(i \mid 1)(t) = q_i(t)$, $q_j(i \mid 1)(t) = q_j(t)\, j \neq 1, i$. Then $Q(i \mid 1) = \{q(i \mid 1)(t)\}_{t=1}^{\infty}$. Thus the permutation $Q(i \mid 1)$ is identical to Q except that the roles of firm 1 and firm i have been interchanged.

Any solution to the problem specified above will be referred to as an *optimal symmetric punishment* and the minimized value denoted $\mathbf{v}(\Gamma)$.

THEOREM 1. *An optimal symmetric punishment exists.*

Proof. By (A6) a symmetric CNE path Q^{cn} exists. Since $Q^{cn} \in \Gamma$, Theorem 1 follows directly from Proposition 3. Q.E.D.

Even in the context of a symmetric game the restriction to symmetric paths is neither natural nor in principle innocuous. Global optimality might well require that a deviant play differently from a set of punishers. However, the symmetric class turns out to have a surprising amount of structure; it yields powerful results and important insights. This section may be viewed as a first step away from the conventional case of Cournot–Nash reversion upon which the literature so far appears to have relied. As the reader will, I hope, be convinced, this first step carries us quite far.

The detailed discussion of results in the Introduction should be used as a guide through the formal presentation below.

Lemma 2 provides the elementary structural property (apart from symmetry) upon which the results of this section depend. It simply says that the larger the aggregate output of other firms, the smaller are a firm's one period *best response* profits. In other words, the further to the left the derived demand curve facing the firm is, the lower are its maximal profits.

D8. For $z \in R_+$, $\Pi^*(z) = \max_{x \in S_1} (p(x+z) - c)\, x$.

LEMMA 2. *Let* $z_2 > z_1 \geqslant 0$. *Then*

$$\Pi^*(z_1) > \Pi^*(z_2) \qquad \textit{or} \qquad \Pi^*(z_1) = \Pi^*(z_2) = 0.$$

Proof. See Appendix.

D9. $\pi(x) = \Pi_1(x, \ldots, x)$ and $\pi^*(x) = \Pi^*((N-1)\, x)$. $\pi(x)$ and $\pi^*(x)$ are profits and best-reponse profits respectively when all other firms produce x units *each.*

COROLLARY 3. *Let* $x_2 > x_1 \geqslant 0$. *Then*

$$\pi^*(x_1) > \pi^*(x_2) \qquad \textit{or} \qquad \pi^*(x_1) = \pi^*(x_2) = 0.$$

The central result of this section is that there exists a stick-and-carrot punishment which is an optimal symmetric punishment. An intuitive account of why the result works is attempted below. By Theorem 1 an optimal symmetric punishment Y exists. Let Y be defined by the (per-firm) output stream $\{y_t\}_{t=1}^{\infty}$. We may represent Y in a diagrammatic way, as in Fig. 1. We wish to replace Y by the stick-and-carrot path defined by $\bar{x}$ and $\tilde{x}$, where $\tilde{x}$ is the most collusive output level supportable by $v_1(Y)$ and $\bar{x}$ is yet to be defined. The path $\tilde{x}$ forever yields at least as high a payoff as

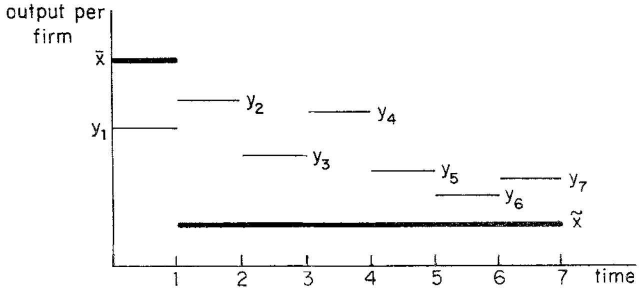

FIGURE 1

$\{y_2, y_3, \ldots\}$. Hence if the path defined by $(\bar{x}, \tilde{x})$ is to yield the same payoff as Y, profits at $\bar{x}$ cannot be larger than those at y_1. It is always possible (A3) to find an $\bar{x}$ at least as large as y_1 such that the two paths have the same payoff. Now we need only argue that if the punishment Y defines a P.E. then so does the stick-and-carrot punishment $(\bar{x}, \tilde{x})$. Recall from Proposition 4 the conditions under which Y defines a P.E. and note that by construction the stick-and-carrot punishment $(\bar{x}, \tilde{x})$ yields the same payoff as Y. Since $\tilde{x}$ is by definition supportable by $v_1(Y)$, to complete the proof we need only check that $\bar{x}$ satisfies the conditions of D6. Now compare cheating against $\bar{x}$ to cheating against y_1. The former is clearly less favorable. On the other hand, conformity with Y yields exactly the same payoff as conformity with the stick-and-carrot path. Consequently, if deviating from y_1 is not lucrative (and it is not, since Y defines a P.E.), then neither is deviating from $\bar{x}$, and we are essentially done.

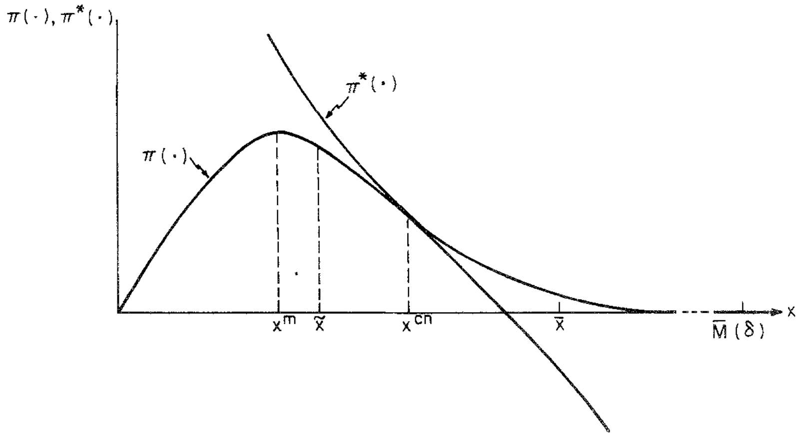

FIG. 2. $x^m \equiv$ monopoly output; $x^{cn} \equiv$ CNE output; $\bar{x}$ may for different specifications appear anywhere to the right of x^{cn}.

This discussion and Fig. 2 are perhaps most useful when used in combination with the formal arguments which follow.

A number of preliminary lemmas and definitions lead to Theorem 9, a result on symmetric Pareto optimal paths.

D10. $r \equiv (1-\delta)/\delta$. r is the "rate of interest."

Per firm payoffs from a stationary path along which firms produce x units each are $\pi(x)/r$. Recall that by (A6), a CNE exists.

D11. $x^{cn} \in S_1$ is per firm output in a symmetric CNE. $v^{cn} \equiv \pi(x^{cn})/r$.

LEMMA 4. (i) $\pi(x^m) \geqslant \pi(x^{cn}) > 0$,

(ii) $x^{cn} \geqslant x^m > 0$,

(iii) $v^{cn} \geqslant \mathbf{v}(\Gamma) \geqslant \mathbf{v}(\Omega) \geqslant 0$.

Proof. See Appendix.

LEMMA 5. *Let* $B^* = \{x \in S_1 \mid \pi^*(x) - \pi(x) \leqslant \pi(x)/r - \mathbf{v}(\Gamma)\}$. B^* *is nonempty and compact.*

Proof. Since $\mathbf{v}(\Gamma) \leqslant \pi(x^{cn})/r$, $x^{cn} \in B$. By (A2) and (A4), $\pi(\cdot)$ and $\pi^*(\cdot)$ are continuous. Hence B^* is closed, and by (A4), bounded. Q.E.D.

D12. $\tilde{v} = \max\{(\pi(x)/r) \mid x \in B^*\}$.

LEMMA 6. *Let* $B = \{x \in S_1 \mid \pi^*(x) - \pi(x) \leqslant \tilde{v} - \mathbf{v}(\Gamma)\}$. *$B$ is nonempty and compact.*

Proof. Since $B^* \subseteq B$, B is nonempty. Compactness is established as in the earlier proof. Q.E.D.

D13. $\bar{x} = \max B$, $\underline{x} = \min B$,

LEMMA 7. $\max\{x^m, \underline{x}\} = \operatorname{argmax}\{\pi(x) \mid x \in B\}$.

Proof. Suppose $\underline{x} > x^m$. Then by (A5), $\pi(x) < \pi(\underline{x})$, all $x > \underline{x}$. Now suppose $\underline{x} \leqslant x^m$. By Lemma 2, $\pi^*(x^m) \leqslant \pi^*(\underline{x})$ and by (A5), $\pi(x^m) > \pi(x)$, all $x \neq x^m$. Hence $\underline{x} \in B$ implies $x^m \in B$ and the proof is complete. Q.E.D.

D14. $\tilde{x} = \operatorname{argmax}\{\pi(x) \mid x \in B\}$.

LEMMA 8.

$$\pi^*(\bar{x}) - \pi(\bar{x}) = \tilde{v} - \mathbf{v}(\Gamma),$$

$$\pi^*(\tilde{x}) - \pi(\tilde{x}) = \tilde{v} - \mathbf{v}(\Gamma) \qquad \text{if} \quad \tilde{x} \neq x^m,$$

$$\pi^*(x^m) - \pi(x^m) \leqslant \tilde{v} - \mathbf{v}(\Gamma) \qquad \text{if} \quad \tilde{x} = x^m.$$

Proof. Using (A1)–(A4), since $\tilde{x} \in B$, $-\pi(\tilde{x}) \leqslant \tilde{v} - \mathbf{v}(\Gamma) - \pi^*(\tilde{x}) \leqslant \tilde{v} < N\pi(x^m)/r < -\Pi_1(\bar{M}(\delta), 0,..., 0) < -\pi(\bar{M}(\delta))$. Hence $\tilde{x} < \bar{M}(\delta)$. By the previous lemma, $\tilde{x} = x^m$ or $\tilde{x} = \underline{x} > x^m > 0$. The result now follows from D13 and the continuity of $\pi(\cdot)$, $\pi^*(\cdot)$. Q.E.D.

Constrained Pareto optimal paths play a central role in the description of optimal punishments and are of independent interest. Theorem 9 establishes that a unique symmetric Pareto optimal (PO) path (relative to $\mathbf{v}(\Gamma)$) exists; furthermore, such a path must be stationary. That stationarity might be an issue is attested to by the nonstationarity results obtained in Section 5 for *asymmetric* PO paths.

D15. $Q \in \Omega$ is Pareto optimal relative to $w_1 \in R$ if and only if

(i) Q is supportable by w_1,

(ii) $\nexists Q' \in \Omega$ supportable by w_1 such that $v_i(Q') > v_i(Q)$, $i = 1,..., N$.

In the remainder of this section $w_1 = \mathbf{v}(\Gamma)$. The qualifiers "by $\mathbf{v}(\Gamma)$" and "relative to $\mathbf{v}(\Gamma)$" will frequently be taken to be understood. For $x \in S_1$, let $x \cdot e$ denote the N-vector $(x, x,..., x) \in S_1^N$.

D16. $\tilde{X} = \{x_t \cdot e\}_{t=1}^{\infty}$, $x_t = \tilde{x} \in S_1$, $t = 1, 2,....$

It may be easily checked that $\max\{\pi(x)/r \mid x \in B\} = \max\{\pi(x)/r \mid x \in B^*\}$. Hence $\tilde{v} = v_1(\tilde{X}) = \pi(\tilde{x})/r$.

THEOREM 9. *The* stationary *path $\tilde{X}$ is the unique symmetric Pareto optimal path relative to* $\mathbf{v}(\Gamma)$.

Proof. Since $\tilde{x} \in B^*$, $\tilde{X}$ is supportable (by $\mathbf{v}(\Gamma)$). Let $Y = \{y_t \cdot e\}_{t=1}^{\infty}$ be any symmetric supportable path. I first argue that $\tilde{v} \geqslant v_1(Y)$, which establishes that $\tilde{X}$ is PO.

Since S_1 is compact and $\pi(\cdot)$ continuous ((A2)), there exists $\bar{y} \in$ closure $\{y_t \mid t = 1, 2,...\}$ such that $\pi(\bar{y}) \geqslant \pi(y_t)$, $t = 1, 2,....$ Clearly $v_1(Y; t+1) \leqslant \pi(\bar{y})/r$, $t = 0, 1, 2,....$ By D6, $\pi^*(y_t) - \pi(y_t) \leqslant \pi(\bar{y})/r - \mathbf{v}(\Gamma)$, $t = 1, 2,...,$ where I have replaced $v_1(Y; t+1)$ by $\pi(\bar{y})/r$. Continuity of $\pi(\cdot)$, $\pi^*(\cdot)$ ((A2, (A4)) therefore imply $\pi^*(\bar{y}) - \pi(\bar{y}) \leqslant (\pi(\bar{y})/r) - \mathbf{v}(\Gamma)$. Thus $\bar{y} \in B^* \subseteq B$ and $\pi(\tilde{x}) > \pi(\bar{y})$ if $\bar{y} \neq \tilde{x}$. Hence $\tilde{v} \geqslant \pi(\bar{y})/r \geqslant v_1(Y)$, and $\tilde{X}$ is PO.

Suppose finally that $v_1(Y) = \tilde{v}$, so that Y is PO also. Then $\pi(y_t) = \pi(\bar{y}) = \pi(\tilde{x})$, $t = 1, 2,....$ By D6, $y_t \in B$ and by Lemma 7, $y_t = \tilde{x}$, $t = 1, 2,....$ Q.E.D.

The reader is reminded that I use the terms path and punishment interchangeably. The rest of this section is devoted to *two-phase* punishments. These are symmetric punishments which are also stationary after the first period.

D17. A *two-phase punishment* $X(x_1, x_2)$ is the path $\{x'_t \cdot e\}_{t=1}^{\infty}$, $x'_1 = x_1$, $x'_t = x_2$, $t = 2, 3, \ldots$.

Abusing notation I define

D18. $v_1(x_1, x_2) \equiv \delta(\pi(x_1) + (\delta/(1-\delta))\, \pi(x_2)) \equiv v_1(X(x_1, x_2))$,

$$\sigma(x_1, x_2) \equiv \sigma(X(x_1, x_2)).$$

Theorem 13 is the main result of this section. It asserts that there exists a two-phase punishment with a stick-and-carrot structure (i.e., the second phase is PO relative to $\mathbf{v}(\Gamma)$) which is an optimal symmetric punishment, and thereby reduces an infinite dimensional optimizing problem to a 2-dimensional one. This is a very strong result in view of the fact that symmetric punishments may, in general, have a very complicated intertemporal structure. In addition, the stick-and-carrot characterization completely trivializes the 2-dimensional problem we are left with. The latter essentially reduces to solving a pair of simultaneous equations. On this, see Theorem 15.

I proceed to Theorem 13 via three lemmas. Note that $v_1(x, \tilde{x}) = \delta(\pi(x) + \tilde{v})$.

LEMMA 10. *For all $x \in B$, $X(x, \tilde{x})$ is supportable by $\mathbf{v}(\Gamma)$.*

Proof. Since $\tilde{x} \in B$ and $\tilde{v} = \pi(\tilde{x})/r$, the result follows directly from D6 and the definition of B. Q.E.D.

LEMMA 11. *For all $x \in B$, $v_1(\bar{x}, \tilde{x}) \leqslant v_1(x, \tilde{x})$.*

Proof. Suppose not, and $\pi(\bar{x}) > \pi(x)$, $\bar{x} > x$. By Lemma 2, $\pi^*(x) \geqslant \pi^*(\bar{x})$. Hence $\pi^*(\bar{x}) - \pi(\bar{x}) < \pi^*(x) - \pi(x) \leqslant \tilde{v} - \mathbf{v}(\Gamma)$. This contradicts Lemma 8. Q.E.D.

LEMMA 12. *Suppose $Y = \{y_t \cdot e\}_{t=1}^{\infty}$ is supportable by $\mathbf{v}(\Gamma)$. Then there exists $x \in B$ such that $v_1(x, \tilde{x}) = v_1(Y)$.*

Proof. Let $Y(2) = \{y_t \cdot e\}_{t=2}^{\infty}$. Since Y is supportable, so is $Y(2)$ and by Theorem 9, $v_1(Y(2)) \leqslant \tilde{v}$. By D6, $\pi^*(y_1) - \pi(y_1) \leqslant v_1(Y(2)) - \mathbf{v}(\Gamma)$. By (A2)–(A4), there exists $x \in S_1$ such that $x \geqslant y_1$ and $v_1(x, \tilde{x}) = v_1(Y)$. By Lemma 2, $\pi^*(x) \leqslant \pi^*(y_1)$. Hence $\pi^*(x) + \mathbf{v}(\Gamma) \leqslant \pi(y_1) + v_1(Y(2)) = \pi(x) + \tilde{v}$. Thus $x \in B$ and the proof is complete. Q.E.D.

THEOREM 13. *$X(\bar{x}, \tilde{x})$ is an optimal symmetric punishment.*

Proof. By Theorem 1, an optimal symmetric punishment exists. Denote it Y. By Proposition 4, Y is supportable by $v_1(Y) = \mathbf{v}(\Gamma)$. Hence by Lemmas 11 and 12, $v_1(\bar{x}, \tilde{x}) \leqslant \mathbf{v}(\Gamma)$. By Lemma 10, $X(\bar{x}, \tilde{x})$ is supportable by $\mathbf{v}(\Gamma)$ and therefore by $v_1(\bar{x}, \tilde{x})$. Thus by Proposition 4, $\sigma(\bar{x}, \tilde{x})$ is a P.E., and the proof is complete. Q.E.D.

D19. $(\bar{x}, \tilde{x})$ denotes the *stick-and-carrot punishment*.

Theorem 14 provides conditions under which an optimal symmetric punishment is *unique*. It implies that if $\mathbf{v}(\Gamma) > 0$, i.e., if incentive constraints "bite," an optimal symmetric punishment must be a two-phase punishment with a stick-and-carrot structure.

THEOREM 14. *If* $\mathbf{v}(\Gamma) > 0$ *the optimal symmetric punishment is unique.*

Proof. Let Y be an optimal punishment and $Y(2) = \{y_t \cdot e\}_{t=2}^{\infty}$. By arguments provided in Lemma 12, $y_1 \in B$. Hence $\bar{x} \geqslant y_1$. If $\bar{x} = y_1$, then $v_1(Y(2)) = \tilde{v}$ and by Theorem 9, $Y(2) = \tilde{X}$. Now suppose $\bar{x} > y_1$. By Lemma 2, $\pi^*(\bar{x}) < \pi^*(y_1)$ or $\pi^*(\bar{x}) = \pi^*(y_1) = 0$. In the former case, $\pi^*(\bar{x}) + \mathbf{v}(\Gamma) < \pi^*(y_1) + \mathbf{v}(\Gamma) \leqslant \pi(y_1) + v_1(Y(2)) = \pi(\bar{x}) + \tilde{v}$. In the latter, $\mathbf{v}(\Gamma) > 0$ and $\delta \in (0, 1)$ imply $\delta(\pi^*(\bar{x}) + \mathbf{v}(\Gamma)) = \delta\mathbf{v}(\Gamma) < \mathbf{v}(\Gamma) = \delta(\pi(\bar{x}) + \tilde{v})$. Thus, in either case Lemma 8 is contradicted. Q.E.D.

An immediate implication of Theorem 14 is that so long as a symmetric CNE punishment supports some (more) collusive path, it cannot be an optimal symmetric punishment. If it were, then $\mathbf{v}(\Gamma) = v^{cn} > 0$, which implies that a stick-and-carrot structure is necessary for optimality. Theorem 17 is another result in this vein.

As indicated earlier, stick-and-carrot punishments may be easily computed:

THEOREM 15. *Let* $(\bar{x}, \tilde{x})$ *be the optimal stick-and-carrot punishment. Then*

$$\pi^*(\bar{x}) - \pi(\bar{x}) = \delta(\pi(\tilde{x}) - \pi(\bar{x})), \tag{1}$$

$$\pi^*(\tilde{x}) - \pi(\tilde{x}) = \delta(\pi(\tilde{x}) - \pi(\bar{x})) \qquad \text{if} \quad \tilde{x} \neq x^m, \tag{2}$$

$$\pi^*(\tilde{x}) - \pi(\tilde{x}) \leqslant \delta(\pi(\tilde{x}) - \pi(\bar{x})) \qquad \text{if} \quad \tilde{x} = x^m. \tag{3}$$

Proof. Observe that

$$\tilde{v} - \mathbf{v}(\Gamma) = (\delta/(1-\delta))\, \pi(\tilde{x}) - \delta(\pi(\bar{x}) + (\delta/(1-\delta))\, \pi(\tilde{x})) = \delta(\pi(\tilde{x}) - \pi(\bar{x})).$$

Now use Lemma 8. Q.E.D.

To determine the optimal stick-and-carrot punishment we should first check if $\tilde{x} = x^m$ by setting $\tilde{x} = x^m$ and seeing if the largest $\bar{x}$ which satisfies

Eq. (1) also satisfies Eq. (3). If not, among all pairs $(\bar{x}, \tilde{x})$ which satisfy Eqs. (1) and (2), pick the pair for which $\bar{x}$ is largest and $\tilde{x}$ smallest.

Do the above equations have non-trivial solutions? That is, are $\bar{x}$, $\tilde{x}$ distinct from x^{cn} and can we prove that Cournot–Nash reversion is not optimal? It turns out that under fairly weak conditions we can. See also the remark which follows Theorem 14.

THEOREM 16. *Let x^{cn} be per firm output in a* CNE. *If $\pi(\cdot)$, $\pi^*(\cdot)$ are continuously differentiable at x^{cn}, then*

$$\bar{x} > x^{cn},$$

$$\tilde{x} < x^{cn},$$

$$\mathbf{v}(\Gamma) < v^{cn}.$$

Proof. By Lemmas 4 through 8 (which are also used below), $\bar{M}(\delta) > \bar{x} \geqslant x^{cn} > 0$, and $\pi(x^{cn}) = \pi^*(x^{cn}) > 0$. Let $\rho(x) = (\pi^*(x) - \pi(x))$. Since by definition $\rho(x) \geqslant 0$ all $x \in S_1$ and $\rho(x^{cn}) = 0$, $(d\rho(x)/dx)\,|_{x=x^{cn}} = 0$. Hence at $x = x^{cn}$, $d\pi(x)/dx = d\pi^*(x)/dx < 0$, where the last inequality follows from Lemma 2 and $\pi^*(x^{cn}) > 0$[4]. Hence $x^{cn} \neq x^m$ and thus by Lemma 4, $x^{cn} > x^m$. Also there exists $\theta > 0$ such that $\pi(x^{cn} - \theta) > \pi(x^{cn})$, and $\rho(x^{cn} - \theta) \leqslant (\pi(x^{cn} - \theta)/r) - (\pi(x^{cn})/r) \leqslant (\pi(x^{cn} - \theta)/r) - \mathbf{v}(\Gamma)$. Hence $(x^{cn} - \theta) \in B^* \subseteq B$ and by Lemma 7, $\tilde{x} < x^{cn}$. Thus $\tilde{v} > v^{cn} \geqslant \mathbf{v}(\Gamma)$ and therefore by Lemma 8, $\bar{x} \neq x^{cn}$. Hence $\bar{x} > x^{cn}$. Finally, by Theorem 14, if $\mathbf{v}(\Gamma) = v^{cn} > 0$ (this implies that the CNE punishment is an optimal punishment), the optimal symmetric punishment is *unique*, and must be the stick-and-carrot punishment $(\bar{x}, \tilde{x}) \neq (x^{cn}, x^{cn})$, a contradiction. Hence $\mathbf{v}(\Gamma) < v^{cn}$. Q.E.D.

Proposition 4 of Section 3 says that $\sigma(Q)$ is a P.E. if no *one-shot* deviations from Q yield a higher payoff. The power of this result is that we may ignore complex, possibly infinite patterns of deviations, and the arbitrary histories which the perfection criterion in general requires us to consider. Applied to two-phase punishments it yields

LEMMA 17. *$\sigma(x_1, x_2)$ is a* P.E. *if and only if*

$$\pi^*(x_1) - \pi(x_1) \leqslant \delta(\pi(x_2) - \pi(x_1)), \tag{4}$$

$$\pi^*(x_2) - \pi(x_2) \leqslant \delta(\pi(x_2) - \pi(x_1)). \tag{5}$$

Proof. Let X_2 denote the symmetric stationary path defined by x_2. As in the proof of Theorem 15, $v_1(X_2) - v_1(x_1, x_2) = \delta(\pi(x_2) - \pi(x_1))$. Now apply Proposition 4. Q.E.D.

[4] Strictly speaking, Lemma 21 is also needed.

To see this more directly, note that if a firm deviates once from a two-phase punishment and conforms thereafter, all firms produce x_1 rather than x_2 in the period after the deviation, and return to the predeviation path thereafter. Thus the future loss consequent on a one-shot deviation is $\delta(\pi(x_2)-\pi(x_1))$, and, given Proposition 4, Eqs. (4) and (5) follow immediately.

The next two theorems are concerned with the question of when optimal symmetric punishments are *globally* optimal in the sense that $\mathbf{v}(\Gamma)=\mathbf{v}(\Omega)$. I show that there exists a lower bound $\underline{\delta}<1$ on the discount factor such that for $\delta \geqslant \underline{\delta}$, $\mathbf{v}(\Gamma)=0$. Since a firm's minmax payoff in the component game is zero, global optimality is clearly implied. As the next paragraph clarifies, an interesting feature of the optimal two-phase punishment when $\mathbf{v}(\Gamma)=0$ is that *all* firms *simultaneously* minmax one another in the first phase. It should be emphasized that Theorems 18 and 19 are not "asymptotic" results. This point should be clear from the method of proof and is underlined by the numerical calculations reported at the end of Theorem 19.

The way the proof works is as follows: Consider two-phase punishments. The only way $\mathbf{v}(\Gamma)$ can equal zero and $\sigma(x_1, x_2)$ be a P.E. is if $\pi^*(x_1)=0$, since a firm can deviate in the first period and keep doing so every time the punishment is reimposed. Hence the total output produced by $(N-1)$ firms must be large enough that $p((N-1)\,x_1)\leqslant c$, which sets a lower bound on x_1 independent of δ. Hence profits in the first period must be *negative*, but not so large that they cannot be recouped by a collusive output level supportable by a zero punishment in the future. Therefore the future must be sufficiently important, and a lower bound on δ is clearly called for.

THEOREM 18. *There exists a lower bound $\underline{\delta}<1$ such that for $\delta \geqslant \underline{\delta}$ the optimal symmetric punishment is globally optimal.*

Proof. By Lemma 4, $\mathbf{v}(\Omega)\geqslant 0$. Now see Theorem 19. Q.E.D.

THEOREM 19. *There exists a lower bound $\underline{\delta}<1$ such that an optimal symmetric punishment yields a zero payoff if and only if $\delta \geqslant \underline{\delta}$.*

Proof. For any $(x_1, x_2)\in S_1^2$, $v_1(x_1, x_2)=0$ if and only if

$$-\pi(x_1)=\delta(\pi(x_2)-\pi(x_1)). \tag{6}$$

By Lemma 17, if $\sigma(x_1, x_2)$ satisfies Eq. (6), $\sigma(x_1, x_2)$ is a P.E. if and only if

$$\pi^*(x_1)=0, \tag{7}$$

$$\pi^*(x_2)\leqslant \pi(x_2)-\pi(x_1). \tag{8}$$

Let $D = \{\delta \in [0, 1] \mid (\delta, x_1, x_2)$ satisfies Eqs. (6)–(8) for some $(x_1, x_2) \in S_1^2\}$. Continuity of $\pi(\cdot)$, $\pi^*(\cdot)$ implies that D is closed. By (A2)–(A4), there exists $x^0 \in S_1$ such that $p((N-1)\,x^0) = c$. By (A2), $\pi(x^0) < 0$ and $\pi^*(x^0) = 0$. Observe that by Lemma 4 $\pi(x^{cn}) > 0$, and let $\delta' = -\pi(x^0)/(\pi(x^{cn}) - \pi(x^0)) < 1$. Then (δ', x^0, x^{cn}) satisfies Eqs. (6)–(8), and D is nonempty. Let $\underline{\delta} = \min D$. Then for $\delta < \underline{\delta}$ there exists no (δ, x_1, x_2) such that $\sigma(x_1, x_2)$ is a P.E. and $v_1(x_1, x_2) = 0$. Together with Theorem 13 this implies that the "only if" part of the proof is complete.

Let (x_1^*, x_2^*) satisfy Eqs. (6)–(8) for $\delta = \underline{\delta}$. Now consider $\delta'' \geqslant \underline{\delta}$ and let $x_1'' \in S_1$ satisfy Eq. (6) for $x_2 = x_2^*$ and $\delta = \delta''$. By (A2)–(A4), x_1'' exists, $-\pi(x_1'') \geqslant -\pi(x_1^*)$, and $x_1'' \geqslant x_1^* \geqslant x^0$, so that (δ'', x_1'', x_2^*) satisfies Eqs. (6)–(8). Hence $\sigma(x_1'', x_2^*)$ is a P.E. and yields a zero payoff. Q.E.D.

Note that $\underline{\delta}$ need not be "large." An example illustrates this point. If

$$p(z) = \begin{cases} \alpha - z & z \leqslant \alpha, \\ 0 & z > \alpha, \end{cases}$$

and $\alpha - c = 1$,

$$\underline{\delta} = \frac{4N}{(N+1)^2} \qquad N \leqslant 5,$$

$$\underline{\delta} = \left(\frac{(N+1)}{2(N-1)}\right)^2 \qquad N \geqslant 6.$$

For a wide range of economically relevant cases the global optimizing problem is therefore completely solved.

This concludes my analysis of symmetric punishments. The results obtained are surprising in their completeness and simplicity and embody an important insight: the most efficient way to provide low payoffs, in terms of incentives to cheat, is to combine a grim present with a credibly rosy future. In so doing, one simultaneously worsens the environment in which firms contemplate cheating today while providing them with an attractive non-deviation future. This very basic feature of the oligopoly problem has its sharpest expression in the symmetric context of the present section.

5. Asymmetric Paths

I now analyze the *unrestricted* problem $\min\{v_1(Q) \mid Q \in \Omega$ and $\sigma(Q)$ is a P.E.$\}$. A solution to this problem is called an *optimal punishment* and the minimized value denoted $\mathbf{v}(\Omega)$.

THEOREM 20. *An optimal punishment exists.*

Proof. See proof of Theorem 1 and replace Γ by Ω. Q.E.D.

By Proposition 2, $\mathbf{v}(\Omega) = \mathbf{v} = \min\{v_1 \mid (v_1,..., v_n) \in V\}$.

Subsection 5.1 provides a smoothing result which permits me to restrict attention to the two firm case. Subsection 5.2 motivates this section by establishing that optimal symmetric punishments are globally optimal only if $\mathbf{v}(\Gamma) = 0$. Thus the lower bound on δ of Theorem 18 is both *necessary* and sufficient for the global optimality of $(\bar{x}, \tilde{x})$. A generalization of the stick-and-carrot characterization is then obtained for the asymmetric case. This leads to an investigation of the structure of asymmetric Pareto optimal paths in Subsection 5.3. The flip–flop pattern such paths exhibit is perhaps the most interesting result of this section. The reader is referred to the end of the introduction for more in the nature of a preview.

5.1. *A Smoothing Result*

I show here that if an outcome path Q is supportable (by w_1), then any outcome path obtained by "averaging" or "smoothing" Q *across players* is also supportable; any uniform (over time) reduction in the inequality of the allocation of total market output among firms, at each point along the path Q, yields a path which is supportable. Thus, in looking for optimal punishments we may restrict attention to paths which specify identical behavior for all players other than the player being punished. There is only one dimension of asymmetry we need deal with—that between the deviant and the *class* of punishers. A second important implication of the theorem is that if a path $Q = \{q(t)\}$ is supportable, then the associated symmetric path $X = \{x_t \cdot e\}$, $x_t = (1/N) \sum_{i=1}^{N} q_i(t)$, which specifies the same level of total industry output as Q in each period, is also supportable. *Asymmetric paths are "less incentive compatible" than symmetric one.* These results are not surprising but turn out to be extremely helpful; asymmetric paths are very elusive and tools of any general applicability somewhat scarce.

Recall from D7 that $\Pi^*(z)$ denotes a firm's one-period best response profits when all other firms together produce z units.

LEMMA 21. *$\Pi^*(\cdot)$ is convex.*

Proof. See Appendix.

THEOREM 22. *Suppose $Q = \{q(t)\}_{t=1}^{\infty}$ is supportable by $w_1 \in R_+$. Then $\hat{Q} = \{\hat{q}(t)\}_{t=1}^{\infty}$ is supportable by w_1, where $\hat{q}(t) = q(t)\,A$ and $A = ((a_{ij}))$ is any N-dimensional bistochastic matrix.*[5]

[5] A is bistochastic if $A \geqslant 0$ and for all $i, j = 1,..., N$, $\sum_{j=1}^{N} a_{ij} = \sum_{i=1}^{N} a_{ij} = 1$.

Proof. First note that $\hat{q}_j(t)=\sum_i a_{ij} q_i(t)$, $\sum_j \hat{q}_j(t)=\sum_i q_i(t)$, and $p(\sum_j \hat{q}_j(t))=p(\sum_i q_i(t))$. Since marginal costs are constant and equal, $\Pi_j(\hat{q}(t))=\sum_i a_{ij}\Pi_i(q(t))$ and $v_j(\hat{Q}; t+1)=\sum_i a_{ij} v_i(Q; t+1)$. Since Q is supportable by w_1, $\Pi^*(\sum_k q_k(t)-q_i(t))-\Pi_i(q(t))\leqslant v_i(Q; t+1)-w_1$, and to complete the proof we need only show that $\sum_i a_{ij}\Pi^*(\sum_k q_k(t)-q_i(t))\geqslant \Pi^*(\sum_k \hat{q}_k(t)-\hat{q}_j(t))$. Since $\Pi^*(\cdot)$ is convex (Lemma 21) and $\sum_i a_{ij}(\sum_k q_k(t)-q_i(t))=\sum_k \hat{q}_k(t)-\hat{q}_j(t)$, this is indeed the case. Q.E.D.

THEOREM 23. *If $v\in V$, then $vA\in V$, where A is any N-dimensional bistochastic matrix.*

Proof. Consider $Q\in\Omega^0$ such that $v(Q)=v$. By Proposition 5, Q is supportable by $\mathbf{v}(\Omega)$. Let $\hat{Q}$ be defined as in the earlier theorem. By Theorem 22 and Proposition 5, $\hat{Q}\in\Omega^0$. Finally, note that $v(\hat{Q})=vA$.
Q.E.D.

THEOREM 24. *There exists a globally optimal punishment $\underline{Q}=\{\underline{q}(t)\}_{t=1}^{\infty}$ such that $\underline{q}_i(t)=\underline{q}_2(t)$, $i=2, 3,..., N$* and $t=1, 2,...$.

Proof. By Theorem 20, an optimal punishment exists. Denote it $Q=\{q(t)\}_{t=1}^{\infty}$. By Proposition 4, Q is supportable by $v_1(Q)$ and by definition $v_1(Q)=\mathbf{v}(\Omega)$. Let $\underline{q}(t)=q(t)A$, where $A=((a_{ij}))$ and $a_{11}=1$, $a_{i1}=a_{1j}=0$, all $i, j\neq 1$, $a_{ij}=1/(N-1)$ otherwise. By Theorem 22, $\underline{Q}=\{\underline{q}(t)\}_{t=1}^{\infty}$ is supportable by $v_1(Q)$, and by Proposition 4, is an optimal punishment if $v_1(\underline{Q})=v_1(Q)=\mathbf{v}(\Omega)$. Since by construction $v_1(\underline{Q})=v_1(Q)$ and $\underline{q}_i(t)=\underline{q}_2(t)$, $i=2, 3,..., N$, the proof is complete. Q.E.D.

THEOREM 25. *Suppose that $Q=\{q(t)\}_{t=1}^{\infty}$ is supportable by $w_1\in R_+$. Then the* symmetric *path $X=\{x_t\cdot e\}_{t=1}^{\infty}$ is supportable by w_1, where $x_t=(1/N)\sum_i q_i(t)$.*

Proof. Consider $A=((a_{ij}))$, $a_{ij}=1/N$, all i, j, and apply Theorem 22.
Q.E.D.

For the purposes of this section, the above results imply that there is no loss of generality in considering only the two firm case. *Henceforth I assume $N=2$.*

5.2. *Asymmetric Optimal Punishments*

Theorem 35 establishes that if an optimal symmetric punishment yields firms positive payoffs (i.e., more than their minmax payoffs), then it is not *globally* optimal. The rest of this subsection is devoted to extending the stick-and-carrot characterization obtained in Section 4, i.e., the second phase of an optimal symmetric punishment must be Pareto optimal relative to itself. I show that the second phase of an asymmetric optimal punishment must yield a payoff which is a *boundary point* of the set of P.E. payoffs V. For a range of cases this result can be sharpened considerably. The second phase payoff (which in general will be asymmetric) must be *locally Pareto optimal*; that is, in the second phase it is not possible to make both firms strictly better off locally.

The results of Section 4 depended critically on Lemma 2. Lemmas 26 to 30 provide structural properties of the one-shot game which play a similar role here. By symmetry, results stated for firm 1 extend in an obvious way to firm 2.

D20.
$$\alpha_1(q_1, q_2) = \Pi^*(q_2) - \Pi_1(q_1, q_2),$$
$$\alpha_2(q_1, q_2) = \Pi^*(q_1) - \Pi_2(q_1, q_2).$$

$\alpha_i(q_1, q_2)$ is the difference between the ith firm's one-period best response payoff and its non-deviation payoff. I will sometimes refer to this difference as the ith firm's one-period *incentive to cheat.*

Lemmas 26 to 28 essentially say that if a firm wants to increase its output, then its one-period incentive to cheat is increased if its output is reduced and the other firm's output increased by a corresponding amount. Alternatively, if it wants to decrease its output, its incentives to cheat increase if its output is increased and the other firm's output reduced by an equal amount. This implies that if the higher output firm wants to increase output, then so does the lower output firm, and the incentive to cheat of the lower output firm is larger than that of the higher output firm. On the other hand, if the lower output firm wants to reduce its output, then so does the higher output firm, and its incentives to cheat are greater.

D21. $b: R_+ \to S_1$ satisfies $b(z) \in \operatorname{argmax}\{\Pi_1(x, z) \mid x \in S_1\}$, all $z \in R_+$. $b(z)$ denotes a firm's one-period best response output when the total output of all other firms is z.

LEMMA 26. *For all* $(q_1, q_2) \in R^2_+$, *if* $q_1 < b(q_2)$, *then* $\alpha_1(q_1, q_2) < \alpha_1(q_1 - \theta, q_2 + \theta)$ *for all* $0 < \theta \leqslant q_1$; *if* $q_1 > b(q_2)$, *then* $\alpha_1(q_1, q_2) < \alpha_1(q_1 + \theta, q_2 - \theta)$ *for all* $0 < \theta \leqslant q_2$.

Proof. See Appendix.

LEMMA 27. *Suppose* $p(\cdot)$ *is differentiable. Then for all* $q_2 \in R_+$, *if* $b(q_2) > 0$,

$$\alpha_1(b(q_2) - \theta, q_2 + \theta) > 0, \qquad \textit{all} \quad \theta \neq 0,\ -q_2 \leqslant \theta < b(q_2).$$

Proof. See Appendix.

Remark. Lemma 27 implies that in a symmetric game (assuming differentiability), all CNE must be symmetric.

(A7). $p(\cdot)$ and $\Pi^*(\cdot)$ are continuously differentiable.

As in Section 4, assumptions once presented do not appear explicitly in the statement of results.

LEMMA 28. *For all* $(q_1, q_2) \in R^2_+$, *if* $q_1 \leqslant b(q_2)$ *and* $q_1 > q_2$, *then* $\alpha_1(q) < \alpha_2(q)$ *and* $q_2 < b(q_1)$; *if* $q_1 \geqslant b(q_2)$ *and* $q_1 < q_2$, *then* $\alpha_1(q) < \alpha_2(q)$ *and* $q_2 > b(q_1)$.

Proof. See Appendix.

(A8) requires that if a firm wants to increase output then its incentive to cheat falls if the other firm's output increases. The opposite is true if it wants to decrease output. The content of the assumption is that incentives to cheat change monotonically in the appropriate direction. Linear demand provides an example for which (A8) is satisfied. While it is a very natural regularity condition, it is certainly quite strong and far more demanding than anything assumed earlier. For instance, (A8) implies that reaction functions are weakly downward sloping. It has been chosen to highlight the essential trade-offs which extremal asymmetric paths exhibit, and cuts down the profusion of possible cases which would otherwise have to be confronted in every proof.

(A8). For all $(q_1, q_2) \in R^2_+$,

$$\text{if} \quad q_1 < b(q_2) \qquad \text{then} \quad \frac{\partial \alpha_1(q_1, q_2)}{\partial q_2} < 0;$$

$$\text{if} \quad q_1 > b(q_2) \qquad \text{then} \quad \frac{\partial \alpha_1(q_1, q_2)}{\partial q_2} > 0.$$

(A8) and the envelope theorem imply that argmax $\{\Pi_1(q_1, q_2) \mid q_1 \in S_1\}$ is a singleton.

Lemma 29 establishes monotonicity properties of $\alpha_1(\cdot, q_2)$, $\Pi_1(\cdot, q_2)$ with respect to own output, in the spirit of (A8).

LEMMA 29. *For all* $(q_1, q_2) \in S_1^2$,

$$\textit{if} \quad q_1 < b(q_2) \qquad \textit{then} \quad \frac{\partial \alpha_1(q)}{\partial q_1} < 0 \qquad \textit{and} \qquad \frac{\partial \Pi_1(q)}{\partial q_1} > 0;$$

$$\textit{if} \quad q_1 > b(q_2) \qquad \textit{then} \quad \frac{\partial \alpha_1(q)}{\partial q_1} > 0 \qquad \textit{and} \qquad \frac{\partial \Pi_1(q)}{\partial q_1} < 0.$$

Proof. See Appendix

LEMMA 30. *For all $q_2 \in S_1$ and $\theta > 0$, $b(q_2) \geqslant b(q_2 + \theta)$.*

Proof. See Appendix.

A final tool is provided by Theorem 31, which is akin to Proposition 2 (Factorization) of [2], and yields a characterization of V in terms of itself and a one-period incentive compatibility problem.

D22. A pair (q, u) is supportable if and only if $(q, u) \in S_1^2 \times V$, and

$$\alpha_1(q) \leqslant u_1 - \mathbf{v}(\Omega),$$
$$\alpha_2(q) \leqslant u_2 - \mathbf{v}(\Omega).$$

THEOREM 31. *$v \in V$ if and only if there exists a supportable pair (q, u) such that $v_i = \delta(\Pi_i(q) + u_i)$, $i = 1, 2$.*

Proof. Suppose $v \in V$ and consider $Q = \{q(t)\}_{t=1}^{\infty} \in \Omega^0$ such that $v(Q) = v$. Let $q = q(1)$, $Q(2) = \{q(t)\}_{t=2}^{\infty}$, and $u = v(Q(2)) = v(Q; 1)$. Clearly $v_i = \delta(\Pi_i(q) + u_i)$, $i = 1, 2$. By Proposition 5, $Q(2) \in \Omega^0$ so that $u \in V$. Also by Proposition 5, $\alpha_i(q) \leqslant u_i - \mathbf{v}(\Omega)$, $i = 1, 2$. Hence (q, u) is supportable, and the proof of the "only if" part is complete.

To establish the converse, consider $Q(2) = \{\hat{q}(t)\}_{t=2}^{\infty} \in \Omega^0$ such that $v(Q(2)) = u$. By Proposition 5, $Q(2)$ is supportable by $\mathbf{v}(\Omega)$. Let $Q = \{q(t)\}_{t=1}^{\infty}$, where $q(1) = q$ and $q(t) = \hat{q}(t)$, $t = 2, 3, \ldots$. Clearly $v(Q) = v$. Since the pair (q, u) and the path $Q(2)$ are supportable, so also is Q. Hence $Q \in \Omega^0$ and $v \in V$. Q.E.D.

COROLLARY 32. *There exists no supportable pair (q, u) such that $v_1 = \delta(\Pi_1(q) + u_1) < \mathbf{v}(\Omega)$.*

D23. $V^* = \{v \in V \mid \nexists v' \in V \text{ such that } v' >> v\}$. V^* is the set of Pareto optimal payoff vectors.

D24. $\bar{v}_1 = \max\{v_1 \mid (v_1, v_1) \in V\}$. $\bar{v}_1$ differs from $\tilde{v}$ of Section 4 in that it is Pareto optimal relative to $\mathbf{v}(\Omega)$, and not $\mathbf{v}(\Gamma)$.

The next assumption permits a simple proof of Theorem 35 *but may be dispensed with* at the cost of a clumsy argument which the author will reluctantly provide on request. It does play a *minor* role in Theorem 48 and this will be remarked on later.

(A9). There exist $\varepsilon > 0$ and a differentiable function $F\colon (\bar{v}_1 - \varepsilon, \bar{v}_1 + \varepsilon) \to R_+$ such that $(v_1, F(v_1)) \in V^*$, all $v_1 \in (\bar{v}_1 - \varepsilon, \bar{v}_1 + \varepsilon)$.

Note that $F(\bar{v}_1) = \bar{v}_1$ and by symmetry $F'(\bar{v}_1) = -1$. Recall from Section 4 that $(\bar{x}, \tilde{x})$ denotes an optimal stick-and-carrot punishment.

LEMMA 33. *If* $\mathbf{v}(\Gamma)>0$ *then* $\pi^*(\bar{x})>0$.

Proof. See the proof of Theorem 14. It is argued there that the two conditions $\pi^*(\bar{x})=0$ and $\mathbf{v}(\Gamma)>0$ together contradict Lemma 8. Q.E.D.

LEMMA 34. *For all* $z\in R_+$, *if* $\Pi^*(z)>0$, *then* $(d\Pi^*(z)/dz)<0$.

Proof. Follows directly from Lemma 2 and Lemma 21. Q.E.D.

THEOREM 35. *If* $\mathbf{v}(\Gamma)>0$, *the optimal symmetric punishment is not* globally *optimal, i.e.,* $\mathbf{v}(\Gamma)>\mathbf{v}(\Omega)$.

Proof. Consider $(\bar{x}, \bar{x})$. By Theorem 16, $\bar{x}>x^{cn}$, and hence by Lemma 30, $b(\bar{x})<\bar{x}$. Suppose $\mathbf{v}(\Gamma)=\mathbf{v}(\Omega)=\mathbf{v}$. Then $\tilde{v}=\bar{v}_1$, and by Theorem 31, $q_1=q_2=\bar{x}$, $u_1=\bar{v}_1$ is a strictly interior solution to the problem

$$\min_{q_1,q_2,u_1} \Pi_1(q_1, q_2)+u_1,$$

subject to

$$\alpha_1(q)\leqslant u_1-\mathbf{v},$$

$$\alpha_2(q)\leqslant F(u_1)-\mathbf{v}.$$

The relevant Lagrangean is

$$\mathscr{L}=\Pi_1(q_1, q_2)+u_1+\lambda_1(\alpha_1(q)-u_1+\mathbf{v})+\lambda_2(\alpha_2(q)-F(u_1)+\mathbf{v})$$

and at $q_1=q_2=\bar{x}$, $u_1=\bar{v}_1$ if constraint qualification holds, we necessarily have

$$(1-\lambda_1)\frac{\partial \Pi_1}{\partial q_1}+\lambda_2\frac{\partial \alpha_2}{\partial q_1}=0, \tag{9}$$

$$\frac{\partial \Pi_1}{\partial q_2}+\lambda_1\frac{\partial \alpha_1}{\partial q_2}+\lambda_2\frac{\partial \alpha_2}{\partial q_2}=0, \tag{10}$$

$$1-\lambda_1+\lambda_2=0, \tag{11}$$

where I have used $\partial\alpha_1/\partial q_1=-\partial\Pi_1/\partial q_1$ and $F'(\bar{v}_1)=-1$.

Since $b(\bar{x})<\bar{x}$, by (A8) and Lemma 29, $\partial\alpha_i/\partial q_j>0$, i, $j=1, 2$. Also $F'(\bar{v}_1)=-1$. Hence the gradients of the constraints are linearly independent and constraint qualification holds. Now suppose $\lambda_2=0$. Then $\lambda_1=1$, and noting that $\partial\alpha_1/\partial q_2 = d\Pi^*(q_2)/dq_2 - \partial\Pi_1/\partial q_2$, Eq. (10) reduces to $d\Pi^*/dq_2 = 0$. Since $\Pi^*(q_2) = \pi^*(\bar{x})$, this contradicts Lemmas 33 and 34. Hence $\lambda_2 > 0$. Equations (9) and (11) now imply $\partial\Pi_1/\partial q_1 = \partial\alpha_2/\partial q_1$. By Lemma 29, since $b(\bar{x}) < \bar{x}$, $\partial\Pi_1/\partial q_1 < 0$. As noted previously, $\partial\alpha_2/\partial q_1 > 0$, a contradiction. Q.E.D.

D25. (q, u) is an *optimal punishment pair* (OPP) if (q, u) is supportable, and $\delta(\Pi_1(q) + u_1) = \mathbf{v}(\Omega)$.

Theorem 36 establishes that the u of D25 cannot be an interior point of V. When q_1 and $\Pi_1(q)$ are both strictly positive, Theorem 41 provides a tighter result: u must be locally Pareto optimal. In between is some needed machinery, and at least one proposition (Theorem 39) of independent interest.

THEOREM 36. *Suppose* $\mathbf{v}(\Omega) > 0$, *and let* (q, u) *be an* OPP. *Then* u *is a boundary point of* V.

Proof. Suppose u is an interior point of V.

Case 1. $\Pi^*(q_2) > 0$. Define $q_1(\varepsilon) = q_1$, $q_2(\varepsilon) = q_2 + \varepsilon$, $u_2(\varepsilon) = u_2 + (\Pi_2(q) - \Pi_2(q(\varepsilon))$. Let $u_1(\varepsilon)$ solve

$$(\Pi_1(q) + u_1) - (\Pi_1(q(\varepsilon)) + u_1(\varepsilon)) = \Pi^*(q_2) - \Pi^*(q_2(\varepsilon)).$$

By assumption there exists $\varepsilon > 0$ such that $u(\varepsilon) \in V$. By construction, $(q(\varepsilon), u(\varepsilon))$ is supportable. By Lemma 2, $\Pi^*(q_2) > \Pi^*(q_2(\varepsilon))$. Hence, $\mathbf{v}(\Omega) = \delta(\Pi_1(q) + u_1) > \delta(\Pi_1(q(\varepsilon)) + u_1(\varepsilon))$, which contradicts Corollary 32.

Case 2. $\Pi^*(q_2) = 0$. Since $\mathbf{v}(\Omega) > 0$, $\delta(\Pi_1(q) + u_1) = \mathbf{v}(\Omega) > \delta(\Pi^*(q_2) + \mathbf{v}(\Omega))$. Hence there exists $\varepsilon > 0$ such that $(q, (u_1 - \varepsilon, u_2))$ is supportable, a contradiction. Q.E.D.

I now show that the punisher must produce a strictly higher output than the firm being punished, in the first period of an optimal punishment.

LEMMA 37. Suppose $Q = \{q(t)\}_{t=1}^{\infty}$ *is an optimal punishment and* $\mathbf{v}(\Gamma) > 0$. *Then* $q_2(1) > q_1(1)$.

Proof. Suppose not, and let $x_t = \frac{1}{2}(q_1(t) + q_2(t))$. By assumption, $x_1 \geqslant q_2(1)$, and hence by Lemma 2, $\pi^*(x_1) \leqslant \Pi^*(q_2(1))$. By Proposition 5, Q is supportable by $\mathbf{v}(\Omega)$. Hence by Theorem 25, $X = \{x_t \cdot e\}_{t=1}^{\infty}$ is supportable (by $\mathbf{v}(\Omega)$). Also, $v_1(X) = \frac{1}{2}(v_1(Q) + v_2(Q)) \geqslant \mathbf{v}(\Omega) = v_1(Q)$. By (A2)–(A4) there exists $\theta_1 \geqslant 0$ such that $\delta\pi(x_1 + \theta_1) + \sum_{t=2}^{\infty} \delta^t \pi(x_t) = \mathbf{v}(\Omega) = \delta(\Pi_1(q(1)) + v_1(Q; 2))$. Let $y_1 = x_1 + \theta_1$ and $y_t = x_t$, $t = 2, 3,\ldots$ and consider the symmetric path $Y = \{y_t \cdot e\}_{t=1}^{\infty}$. Since X is supportable, Y is supportable if $\pi^*(y_1) + \mathbf{v}(\Omega) \leqslant \pi(y_1) + v_1(Y; 2) = \Pi_1(q(1)) + v_1(Q; 2)$. But this follows from $\pi^*(y_1) \leqslant \pi^*(x_1) \leqslant \Pi^*(q_2(1))$, and the fact that Q is supportable. Since $v_1(Y) = \mathbf{v}(\Omega)$, Proposition 4 implies that Y is an optimal punishment. Since Y is also symmetric, we have contradicted Theorem 35. Q.E.D.

LEMMA 38. *Suppose* $\mathbf{v}(\Gamma) > 0$. *Then for any* $v \in V$, $v_1 + v_2 > 2\mathbf{v}(\Omega)$.

Proof. Suppose not, and consider $Q = \{q(t)\}_{t=1}^{\infty} \in \Omega^0$ such that $v = v(Q)$. Let $x_t = \frac{1}{2}(q_1(t) + q_2(t))$ and $X = \{x_t \cdot e\}_{t=1}^{\infty}$. By Theorem 25, X is supportable by $\mathbf{v}(\Omega)$. Since $v_1(X) \leqslant \mathbf{v}(\Omega)$, Proposition 4 implies that X is an optimal punishment. But this contradicts Lemma 37. Q.E.D.

(A10). $\mathbf{v}(\Gamma) > 0$.

As noted earlier, assumptions such as $\mathbf{v}(\Omega) > 0$ or $\mathbf{v}(\Gamma) > 0$ are made to ensure that the relevant incentive constraints are binding and thereby permit tighter characterizations than would otherwise be possible.

D26. (q, u) is a *regular optimal punishment pair* (ROPP) if (q, u) is an optimal punishment pair and $q_1 > 0$.

Optimality might require that $q_1 = 0$ and an ROPP might not exist. It seems clear that this is a very special case and is ignored in the interests of brevity.

The next result is accorded the status of a theorem because it demonstrates an important general principle: optimality might require that a player "cooperate" (in a one-period sense) in his own punishment. Indeed here cooperation is *necessary*. When it is recognized that the relevant notion of cooperation is an intertemporal one, this point is no longer counterintuitive, but appears to have been overlooked. As remarked earlier, the discounting literature has concentrated on Cournot–Nash reversion; in the more sophisticated no-discounting literature on the other hand, the future looms so large that the possibility of cooperative deviants is a redundant nicety.

THEOREM 39. *Let* (q, u) *be an ROPP. Then* $q_1 \neq b(q_2)$ *and* $q_2 > b(q_1)$.

Proof. See Appendix.

LEMMA 40. *Let* (q, u) *be an ROPP. If* $\Pi_1(q) > 0$, *then* $q_1 < b(q_2)$.

Proof. See Appendix.

D27. $u \in V$ is *locally Pareto optimal* if there do not exist differentiable functions h_i: $[0, 1] \to R_+$, $i = 1, 2$ such that $h(0) = u$, $h(\theta) \in V$, all $\theta \in [0, 1]$, and $dh_1/d\theta$, $dh_2/d\theta > 0$ at $\theta = 0$.

Obviously a Pareto optimal payoff vector is also locally Pareto optimal. If V is convex the converse is also true.

Whether or not an OPP yields firm 1 positive profits in the first period depends on the parameters of the one-shot game (the demand function, c, N), and the size of the discount factor δ. Theorem 41 assumes $\Pi_1(q) > 0$. This case is not unusual and must occur, for example, when δ is "low." On the other hand $\Pi_1(q) \leqslant 0$ is not an exceptional possibility and will arise when $\delta \geqslant \underset{\sim}{\delta}$, where $\underset{\sim}{\delta}$ is the lower bound of Theorem 19. It is natural in this section, however, to focus on the positive profit case since for "high" δ, Theorems 18 and 19 assure us that symmetry is unrestrictive.

THEOREM 41. *If* (q, u) *is an* ROPP *and* $\Pi_1(q) > 0$, *then* u *is locally Pareto optimal.*

Proof. Define $q_1(\theta) = q_1 - \theta$ and $q_2(\theta) = q_2 + k_1\theta$. Since $q_1 < b(q_2)$ and $q_2 > b(q_1)$ (Theorem 39 and Lemma 40), by (A8) and Lemma 29 there exists $k_1 > 0$ such that $d\alpha_2(q(\theta))/d\theta < 0$ and $d\Pi_1(q(\theta))/d\theta < 0$ at $\theta = 0$. Since $q_1 \neq b(q_2)$, $\Pi_1^*(q) > \Pi_1(q) > 0$, and by Lemma 34, $d\Pi_1^*/d\theta < 0$. Suppose u is not locally Pareto optimal, and let h_1, h_2 be as in D27. By the above discussion there exists $k_2 > 0$ such that $0 > d\Pi_1/d\theta + k_2(dh_1/d\theta) > d\Pi_1^*/d\theta$. Noting that $\partial h_i(k_2\theta)/\partial\theta = k_2(dh_i/d\theta)$ and using the inequalities derived earlier it follows that there exists $\theta_1 > 0$ such that

$$\mathrm{v}(\Omega) = \Pi_1(q) + u_1 > \Pi_1(q(\theta_1)) + h_1(k_2\theta_1) > \Pi_1^*(q(\theta_1)) + \mathrm{v}(\Omega),$$

$$\alpha_2(q(\theta_1)) < h_2(k_2\theta_1) - \mathrm{v}(\Omega)$$

Thus $(q(\theta_1), h(k_2\theta_1))$ is a supportable pair, and Corollary 32 is contradicted. Q.E.D.

5.3. *Asymmetric Pareto Optimal Paths*

A wide class of (locally) Pareto optimal paths exhibit a dramatic form of nonstationarity. The better-off player today must be worse off tomorrow. This description is unambiguous: the better-off player in terms of current payoffs is also better-off in terms of the discounted sum of current and future payoffs. The theorem requires that the better-off firm today wants to increase its output, i.e., if $q_2 > q_1$, $q_2 < b(q_1)$. This condition is natural since Pareto optimal paths may be presumed to be "collusive." The essence of the argument is as follows: let (q, u) be a supportable pair and assume that firm 2 is better off. By Lemma 28, if it wants to increase its output, $\alpha_1(q) > \alpha_2(q)$, i.e., the one-period incentive to cheat of the lower output firm is larger. Therefore if $u_2 > u_1$, $\alpha_2(q) < u_2 - \mathbf{v}(\Omega)$; the better-off firm is strictly incentive compatible. Now consider the following perturbation: Increase firm 2's output by θ and reduce firm 1's by θ. Firm 2's current profits rise and firm 1's fall by a corresponding amount. At the same time decrease u_2 and increase u_1 by equal amounts so that for each firm the combined changes in current profits and future payoffs cancel one another. Denote the resultant pair $(q(\theta), u(\theta))$. For small θ, firm 2 remains strictly incentive compatible and by Lemma 2, firm 1 becomes strictly incentive compatible. Also $u_2(\theta) > u_1(\theta)$, and therefore by the smoothing results $u(\theta) \in V$. Now both firms can be made strictly better off.

While this investigation is most directly motivated by Theorem 41, the principal result is of independent interest. It provides an important insight about the structure of incentives along extremal equilibrium paths which does not emerge in repeated games without discounting.[6] In the latter class Pareto optimal paths exhibit no essential nonstationarity.

D28. (q, u) is a *regular locally Pareto optimal pair* (RLPP) if (q, u) is supportable, $q_i > 0$, $i = 1, 2$, and $v = \delta(\Pi(q) + u)$ is locally Pareto optimal.

LEMMA 42. *Let (q, u) be an RLPP. Then $\Pi_i(q) > 0$, $i = 1, 2$.*

Proof. See Appendix.

In Section 4, I argued (Lemma 5–Theorem 9) that there was a unique symmetric path which was Pareto optimal relative to $\mathbf{v}(\Gamma)$. This path was stationary and was denoted $\tilde{X}$. It may easily be checked that the proofs provided there would be unaltered if $\mathbf{v}(\Gamma)$ were replaced by $\mathbf{v}(\Omega)$. Accordingly, let $\tilde{x}(\Omega)$ denote the most collusive symmetric output level supportable by $\mathbf{v}(\Omega)$, and $\tilde{X}(\Omega)$ the associated stationary path.

(A11). $\tilde{x}(\Omega) > x^m$.

(A11) ensures that incentive constraints "bite."

LEMMA 43. *If (q, u) is supportable, $q_1 + q_2 > 2x^m$.*

Proof. By the "smoothing" results, the symmetric pair (x, v), where $x_1 = x_2 = \frac{1}{2}(q_1 + q_2)$, $v_1 = v_2 = \frac{1}{2}(u_1 + u_2)$, is supportable. Hence $v_1 \leqslant v_1(\tilde{X}(\Omega))$. Suppose $x_1 \leqslant x^m$. Then by Lemma 2 and the fact that (x, v) is supportable,

$$\pi^*(x^m) + \mathbf{v}(\Omega) < \pi^*(x_1) + \mathbf{v}(\Omega) \leqslant \pi(x_1) + v_1 < \pi(x^m) + (\pi(x^m))/r.$$

[6] See, for instance, Aumann and Shapley [3] and Rubinstein [7].

Hence $X^m = \{x_t \cdot e\}_{t=1}^{\infty}$, $x_t = x^m$, $t = 1, 2,...$ is supportable, which contradicts (A11). Q.E.D.

Replace (A5) by

(A12). $d\pi(x)/dx > 0$, all $0 \leqslant x < x^m$, and

$d\pi(x)/dx < 0$, all $x^m < x \leqslant \bar{M}(\delta)$.

I use $\mathbf{v}$ and $\mathbf{v}(\Omega)$ interchangeably in the rest of this section.

LEMMA 44. *Suppose (q, u) is supportable and $\alpha_i(q) < u_i - \mathbf{v}$, $i = 1, 2$. Then (q, u) is not an RLPP.*

Proof. By the definition of an RLPP, only the case $q_i > 0$, $i = 1, 2$ need be considered. Define $q_1(\theta) = q_1 - k\theta$, $q_2(\theta) = q_2 - (1-k)\theta$. By Lemma 43 and (A12),

$$\frac{d(\Pi_1(q(\theta)) + \Pi_2(q(\theta)))}{d\theta} > 0.$$

Let k be defined by

$$\frac{d\Pi_1(q(\theta))}{d\theta} = \frac{d\Pi_2(q(\theta))}{d\theta} \quad \text{at} \quad \theta = 0.$$

Then there exists $\theta_1 > 0$ such that $(q(\theta), u)$ is a supportable pair and $d\Pi_i(q(\theta))/d\theta > 0$ all $0 \leqslant \theta \leqslant \theta_1$ and $i = 1, 2$. Q.E.D.

The next lemma is a key result. It says that the better-off firm tomorrow must be just incentive-compatible today, i.e., its incentive constraint must be binding.

LEMMA 45. *Let (q, u) be an RLPP. If $\alpha_i(q) < u_i + \mathbf{v}$, then $u_{-i} \geqslant u_i$.*

Proof. Suppose not, and without loss of generality let $i = 1$. Let $q_1(\theta) = q_1 + \theta$, $q_2(\theta) = q_2 - \theta$, $u_1(\theta) = u_1 - (\Pi_1(q(\theta)) - \Pi_1(q))$, $u_2(\theta) = u_1 + u_2 - u_1(\theta)$. Consider $0 < \theta < q_2$. By Lemma 42, $\Pi_1(q(\theta)) > \Pi_1(q)$ and $\Pi_2^*(q) > 0$. Hence by Lemma 2, $\alpha_2(q(\theta)) < u_2(\theta) + \mathbf{v}$. By Theorem 23, $u(\theta) \in V$, all θ such that $u_1(\theta) \geqslant u_2(\theta)$. Thus if the lemma is false there exists $0 < \theta_1 < q_2$ such that $(q(\theta_1), u(\theta_1))$ is a supportable pair and $\alpha_i(q(\theta_1)) < u_i(\theta_1) + \mathbf{v}$, $i = 1, 2$. Since $\delta(\Pi(q(\theta_1)) + u(\theta_1)) = \delta(\Pi(q) + u)$, Lemma 44 implies that $(q(\theta), u(\theta))$, and therefore (q, u) is not an RLPP. Q.E.D.

I now establish that the better-off firm in terms of current payoffs must also be better off overall, i.e., in terms of the sum of current and future payoffs, and vice-versa. The term "better-off firm today" may therefore be used without further qualification. It is interesting that this very natural result is proved using the non-trivial Lemma 45.

LEMMA 46. *Let (q, u) be an RLPP. Then $\Pi_2(q) > \Pi_1(q)$ if and only if $\delta(\Pi_2(q) + u_2) > \delta(\Pi_1(q) + u_1)$.*

Proof. Suppose $\Pi_2(q) > \Pi_1(q)$ and $\delta(\Pi_2(q) + u_2) \leqslant \delta(\Pi_1(q) + u_1)$. Then $u_1 > u_2$, and by Lemma 42, $q_2 > q_1$. By Lemma 2, since $\Pi_2(q) > 0$, $\Pi_1^*(q) < \Pi_2^*(q)$, and $\Pi_1^*(q) + \mathbf{v} < \Pi_2^*(q) + \mathbf{v} \leqslant \Pi_2(q) + u_2 \leqslant \Pi_1(q) + u_1$. Hence $\alpha_1(q) < u_1 - \mathbf{v}$ and $u_1 > u_2$, which contradicts Lemma 45. The proof of the converse is analogous. Q.E.D.

Theorem 47 shows that the set of symmetric payoffs is closed and *convex*, and is used in the proof of Case 2 of Theorem 48.

D29. $W = \{w \mid (w, w) \in V\}$.

THEOREM 47. *W is a closed interval.*

Proof. The proof uses arguments developed in Section 4, in particular Lemmas 5, 10, 12. As the reader may easily check, these results continue to hold if $\mathbf{v}(\Gamma)$ is replaced by $\mathbf{v}(\Omega)$.

Let $B(\Omega) = \{x \in S_1 \mid \pi^*(x) - \pi(x) \leqslant \tilde{v}(\Omega) - \mathbf{v}(\Omega)\}$. For convenience, the argument Ω is dropped below. For the usual reasons, B is compact and $\underline{x} = \min B \leqslant x^{cn} \leqslant \max B = \bar{x}$. By Lemma 30, for all $x < x^{cn}$, $x < b(x)$, and for all $x > x^{cn}$, $x > b(x)$. Let $\rho(x) \equiv \pi^*(x) - \pi(x)$. By (A8) and Lemma 29, $\rho(x) \leqslant \rho(\underline{x})$, all $x \in [\underline{x}, x^{cn}]$, and $\rho(x) \leqslant \rho(\bar{x})$, all $x \in [x^{cn}, \bar{x}]$. Hence $B = [\underline{x}, \bar{x}]$. By Lemma 10, $X(x, \tilde{x})$ is supportable by $\mathbf{v}(\Omega)$ for all $x \in B$. Hence by Lemma 12, $W = \{\delta(\pi(x) + \tilde{v}) \mid x \in [\underline{x}, \bar{x}]\}$. Since $\pi(\cdot)$ is continuous, the proof is complete. Q.E.D.

Remark. I am grateful to a referee for pointing out that Theorems 23 and 47 imply that V is path-connected.

(A9) was used for convenience in the proof of Theorem 35. It does play a small role in the next proof, however; it is used to rule out the case $u_1 = u_2$.

Theorem 48 should be read: "Let (q, u) be an RLPP in which firm 2 is strictly better off (than firm 1) in period 1. Then if firm 2 wants to increase output in period 1, it must be strictly worse off in period 2."

THEOREM 48. *Let (q, u) be an RLPP and $q_2 > q_1$. If $q_2 < b(q_1)$, then $u_1 > u_2$.*

Proof. By Lemma 28, $q_2 > q_1$ and $q_2 < b(q_1)$ imply $\alpha_1(q) > \alpha_2(q)$ and $q_1 < b(q_2)$. Suppose $u_2 \geqslant u_1$.

Case 1. $u_2 > u_1$. $\alpha_1(q) \leqslant u_1 - \mathbf{v}$, $\alpha_1(q) > \alpha_2(q)$, and $u_2 > u_1$ together imply $\alpha_2(q) < u_2 - \mathbf{v}$. Lemma 45 is therefore contradicted.

Case 2. $u_1 = u_2$. By Theorem 47, I need only consider $u_1 = u_2 = \bar{v}_1$. Let $\varepsilon = \frac{1}{2}(u_2 - \mathbf{v} - \alpha_2(q))$. As before, $\alpha_2(q) < u_2 - \mathbf{v}$. Therefore by Lemma 44 (A9), $x_1 = q_1$, $x_2 = q_2$, and $v_1 = u_1 = \bar{v}_1$ are an interior solution to

$$\operatorname*{Min}_{x_1, x_2, v_1} \Pi^*(x_2)$$

subject to

$$\Pi_1(x) + v_1 = k_1 = \Pi_1(q) + u_1,$$
$$\Pi_2(x) + F(v_1) = k_2 = \Pi_2(q) + u_2,$$
$$\alpha_2(x) \leqslant F(v_1) - \mathbf{v} - \varepsilon,$$

since $\Pi^*(x_2) + \mathbf{v} < \Pi^*(q_2) + \mathbf{v} \leqslant \Pi_1(q) + u_1 = \Pi_1(x) + v_1$ imply $\alpha_1(x) < v_1 - \mathbf{v}$.

At the proposed solution, the third constraint is slack. Dropping it I obtain the Lagrangean

$$\mathscr{L} = \Pi^*(x_2) + \lambda_1(\Pi_1(x) + v_1 - k_1) + \lambda_2(\Pi_2(x) + F(v_1) - k_2).$$

At $x = q$, by Lemma 42, $p(q_1 + q_2) - c > 0$. Also by Lemma 43 and (A12), $\partial(\Pi_1 + \Pi_2)/\partial q_1 = p'(q_1 + q_2) + (p - c) < 0$. The reader may check that these inequalities together imply that constraint qualification holds. Using $F'(\bar{v}_1) = -1$, the following necessary conditions obtain

$$\lambda_1 \frac{\partial \Pi_1}{\partial q_1} + \lambda_2 \frac{\partial \Pi_2}{\partial q_1} = 0,$$
$$\frac{d\Pi^*}{dq_2} + \lambda_1 \frac{\partial \Pi_1}{\partial q_2} + \lambda_2 \frac{\partial \Pi_2}{\partial q_2} = 0,$$
$$\lambda_1 = \lambda_2.$$

If $\lambda_1 = \lambda_2 = 0$, then $d\Pi^*(q_2)/dq_2 = 0$, which contradicts Lemma 34. Hence $\partial(\Pi_1 + \Pi_2)/\partial q_1 = 0$. But this contradicts Lemma 43 and (A12). Q.E.D.

Note that if (q, u) is an RLPP then u must be locally Pareto optimal, and hence the flip–flopping between firm 1 and firm 2 could go on indefinitely. Indeed I conjecture that if $Q = \{q(t)\}_{t=1}^{\infty}$ yields a locally Pareto optimal payoff and (i) $q_2(1) > q_1(1) > 0$, (ii) $q_2(1) < b(q_1(1))$, then

$$
\begin{aligned}
&q_2(t) > q_1(t) && t = 1, 3, 5,\ldots \\
&q_1(t) > q_2(t) && t = 2, 4, 6,\ldots \\
&\left.\begin{aligned} q_1(t+2) > q_1(t) \\ q_2(t+2) < q_2(t) \end{aligned}\right\} && t = 1, 3, 5,\ldots \\
&\left.\begin{aligned} q_1(t+2) < q_1(t) \\ q_2(t+2) > q_2(t) \end{aligned}\right\} && t = 2, 4, 6,\ldots \\
&\lim_{t \to \infty} q_1(t) = \lim_{t \to \infty} q_2(t) = \tilde{x}(\Omega)
\end{aligned}
$$

at least for sufficiently regular problems. Of course a conclusive answer must await further investigation.

6. Conclusion

This paper applies an approach to repeated games with discounting introduced in Abreu [1] and summarized (for symmetric games) by Propositions 1 through 5 of Section 3. These results emphasize the central role of optimal punishments and provide tools for their computation. They motivate the question I address here: what is the structure of optimal punishments in symmetric oligopolistic supergames? The answer to this question leads, as it turns out, to another: what do constrained (in particular, asymmetric) Pareto optimal paths look like?

Section 4 analyzes symmetric paths and provides a *stick-and-carrot* characterization for optimal symmetric punishments. The second phase of an optimal stick-and-carrot punishment is the most collusive symmetric output level which the optimal punishment can itself support. The pair $(\bar{x}, \tilde{x}) \in S_1^2$ which defines this punishment is essentially determined as the solution to a pair of simultaneous equations. This characterization is remarkably simple and embodies a significant insight about the efficient pattern of incentives along punishment paths. A uniqueness result emphasizes that the stick-and-carrot structure is essential. For a range of discount factors *global* optimality is also proved. Outside this range, symmetry is indeed restrictive, and this leads to the investigation of asymmetric paths in Section 5.

I show that under certain conditions the second phase of an optimal (asymmetric) punishment must be (locally) Pareto optimal. This is, of course, a natural generalization of the stick-and-carrot result of the earlier section. I then prove that a class of Pareto optimal paths exhibit a rather drastic kind of nonstationarity involving a complete reversal of roles between firms: the better-off firm today must be worse off tomorrow. This is in my opinion the most interesting and unexpected result of Section 5. The analysis of asymmetric paths is far more involved than that of Section 4 and the results obtained not as sharp as those established earlier. However a number of general principles emerge which suggest that a more extended investigation would be fruitful. I hope to return to these questions in the future.

This paper demonstrates that the *optimizing* perspective and the general framework developed in Abreu [1] are operational and useful. Viewed as an exercise in repeated games *with discounting*, it emphasizes how different the latter are from their much analyzed "no discounting" counterparts. In particular, confining attention to stationary paths might be very restrictive; extremal equilibria will typically have a non-trivial dynamic structure. Furthermore, in the context of deterring deviations, optimality might require that a player cooperate (in a one-period sense) in his own punishment. In relation to the discounting literature, Section 4 suggests that the somewhat unimaginative reliance on Cournot–Nash reversion (in contexts in which the severity of the available punishment is a binding constraint) cannot be defended on pragmatic grounds; it seems plausible that most problems will have enough shape to permit at least quasi-optimal punishments to be derived which are both easily computed and more severe than Cournot–Nash threats. Despite the fact that the results obtained here are special to oligopolistic supergames, they depend on simple structural properties, analogs of which presumably obtain in a wide class of game situations. Moreover they involve a mode of analysis and develop a point of view which may be profitably adapted to the particular features of other economic examples.

APPENDIX

Proofs omitted from the text are provided below.

LEMMA 2. *Let* $z_2 > z_1 \geqslant 0$. *Then* $\Pi^*(z_1) > \Pi^*(z_2)$ *or* $\Pi^*(z_1) = \Pi^*(z_2) = 0$.

Proof. Let $x_2^* \in \operatorname{argmax}_{x \in S_1} (p(x+z_2)-c)\,x$ and $m(z_2) = (p(x_2^*+z_2)-c)$. By definition, $\Pi^*(z_2) = m(z_2)\,x_2^*$. If $m(z_2)=0$, the lemma is proved. Suppose $m(z_2)>0$. Then $(x_2^*+z_2) < \bar{M}(\delta)$. Let $\hat{x}_1 = x_2^* + z_2 - z_1$. Since $\hat{x}_1 \in S_1$ and $\hat{x}_1 > x_2^*$, $\Pi^*(z_1) \geqslant (p(\hat{x}_1+z_1)-c)\,\hat{x}_1 = m(z_2)\,\hat{x}_1 > m(z_2)\,x_2^* = \Pi^*(z_2)$. Q.E.D.

LEMMA 4. (i) $\pi(x^m) \geqslant \pi(x^{cn}) > 0$

(ii) $x^{cn} \geqslant x^m > 0$

(iii) $v^{cn} \geqslant \mathbf{v}(\Gamma) \geqslant \mathbf{v}(\Omega) \geqslant 0.$

Proof. (i) follows directly from (A3) and implies $x^m > 0$. If $x^m > x^{cn}$, $\Pi_1((Nx^m - (N-1)\,x^{cn}), x^{cn},..., x^{cn}) = ((Nx^m - (N-1)\,x^{cn})/x^m)\,\pi(x^m) > \pi(x^m) \geqslant \pi(x^{cn}) = \pi^*(x^{cn})$, a contradiction. Since $Q^{cn} \in \Gamma$ and $\sigma(Q^{cn})$ is a P.E., $v^{cn} \geqslant \mathbf{v}(\Gamma)$. Clearly $\mathbf{v}(\Gamma) \geqslant \mathbf{v}(\Omega)$. $\mathbf{v}(\Omega) \geqslant 0$ because a firm can guarantee itself a zero payoff by producing nothing forever. Q.E.D.

DEFINITION. For all $x \in R_+$, $m(x) = (p(x) - c)$. $m(x)$ is the mark-up or profit per unit when total market output is x.

LEMMA 21. *$\Pi^*(\cdot)$ is convex.*

Proof. For $z_1, z_2 \in R_+$, and $\lambda \in [0, 1]$, let $z = \lambda z_1 + (1-\lambda)\,z_2$. Suppose $m(b(z)+z) \leqslant 0$. Then $\lambda \Pi^*(z_1) + (1-\lambda)\,\Pi^*(z_2) \geqslant 0 = \Pi^*(z)$. Now suppose $m(b(z)+z) > 0$. Since $(b(z)+z-z_i)$ is a feasible response to z_i or is negative,

$$\Pi^*(z_i) \geqslant m(b(z)+z-z_i+z_i)\cdot(b(z)+z-z_i), \qquad i = 1, 2.$$

Hence in this case also,

$$\lambda \Pi^*(z_1) + (1-\lambda)\,\Pi^*(z_2) \geqslant m(b(z)+z)\cdot b(z) = \Pi^*(z). \qquad \text{Q.E.D.}$$

LEMMA 26. *For all $(q_1, q_2) \in R^2_+$, if $q_1 < b(q_2)$, then $\alpha_1(q_1, q_2) < \alpha_1(q_1 - \theta, q_2 + \theta)$ for all $0 < \theta \leqslant q_1$; if $q_1 > b(q_2)$, then $\alpha_1(q_1, q_2) < \alpha_1(q_1 + \theta, q_2 - \theta)$ for all $0 < \theta \leqslant q_2$.*

Proof. Suppose $q_1 < b(q_2)$. Let $\theta^* = b(q_2) - q_1 > 0$, $m^* = m(b(q_2) + q_2)$, $m^0 = m(q_1 + q_2)$. By (A2), $m^0 > m^*$. Since $(q_1 - \theta + \theta^*)$ is a feasible response to $(q_2 + \theta)$, and $m(q_1 - \theta + \theta^* + q_2 + \theta) = m^*$,

$$\begin{aligned}\alpha_1(q_1 - \theta, q_2 + \theta) &\geqslant m^* \times (q_1 - \theta + \theta^*) - m^0 \times (q_1 - \theta)\\ &= m^*\theta^* - (m^0 - m^*)(q_1 - \theta)\\ &> m^*\theta^* - (m^0 - m^*)\,q_1 = \alpha_1(q_1, q_2).\end{aligned}$$

Now suppose $q_1 > b(q_2)$. Let $\theta^* = q_1 - b(q_2) > 0$, and m^*, m^0 be defined as above. By (A2), $m^* > m^0$. Proceeding as before,

$$\begin{aligned}\alpha_1(q_1 + \theta, q_2 - \theta) &\geqslant -m^*\theta^* + (m^* - m^0)(q_1 + \theta)\\ &> -m^*\theta^* + (m^* - m^0)\,q_1 = \alpha_1(q_1, q_2). \qquad \text{Q.E.D.}\end{aligned}$$

LEMMA 27. *Suppose $p(\cdot)$ is differentiable. Then for all $q_2 \in R_+$, if $b(q_2) > 0$,*

$$\alpha_1(b(q_2) - \theta, q_2 + \theta) > 0, \qquad \textit{all} \quad \theta \neq 0, \quad -q_2 \leqslant \theta < b(q_2).$$

Proof. We need only show that $\alpha_1(b(q_2) - \theta,\ q_2 + \theta) \neq 0$. Since $b(q_2) > 0$,

$$q_1(\partial p(q_1 + q_2)/\partial q_1) + p(q_1 + q_2) - c = 0, \qquad \text{at} \quad q_1 = b(q_2).$$

By (A2), (A3) and the fact that $b(q_2)$ is a best-response, $(p - c) > 0$. Hence $\partial p/\partial q_1 < 0$ and it is clear that the equation above cannot also be satisfied at $(b(q_2) - \theta,\ q_2 + \theta)$. Q.E.D.

LEMMA 28. For all $(q_1, q_2) \in S_1^2$, *if $q_1 \leqslant b(q_2)$ and $q_1 > q_2$, then $\alpha_1(q) < \alpha_2(q)$ and $q_2 < b(q_1)$; if $q_1 \geqslant b(q_2)$ and $q_1 < q_2$, then $\alpha_1(q) < \alpha_2(q)$ and $q_2 > b(q_1)$.*

Proof. Observe that $\alpha_2(q_1, q_2) = \alpha_1(q_1 - \theta,\ q_2 + \theta)$, where $\theta = q_1 - q_2$. By Lemmas 26 and 27, $0 < q_1 \leqslant b(q_2)$ and $q_1 > q_2$ imply $0 \leqslant \alpha_1(q_1, q_2) < \alpha_2(q_1, q_2)$. Hence $q_2 \neq b(q_1)$. Suppose finally that $q_2 > b(q_1)$. Then by Lemma 26, $\alpha_2(q_1, q_2) = \alpha_1(q_2, q_1) < \alpha_1(q_2 + (q_1 - q_2),\ \ q_1 - (q_1 - q_2)) = \alpha_1(q_1, q_2)$, a contradiction. This proves the first part of the proposition. Now suppose $q_2 > q_1 \geqslant b(q_2) \geqslant 0$. If $0 < q_2 \leqslant b(q_1)$, then applying the earlier argument symmetrically, $q_1 < b(q_2)$, a contradiction. Hence $q_2 > b(q_1)$. Now $q_1 \geqslant b(q_2)$ and Lemma 26 yield $\alpha_1(q_1, q_2) < \alpha_2(q_1, q_2)$. Q.E.D.

LEMMA 29. *For all $(q_1, q_2) \in S_1^2$,*

$$\textit{if} \quad q_1 < b(q_2) \qquad \textit{then} \quad \frac{\partial \alpha_1(q)}{\partial q_1} < 0 \qquad \textit{and} \qquad \frac{\partial \Pi_1(q)}{\partial q_1} > 0,$$

$$\textit{if} \quad q_1 > b(q_2) \qquad \textit{then} \quad \frac{\partial \alpha_1(q)}{\partial q_1} > 0 \qquad \textit{and} \qquad \frac{\partial \Pi_1(q)}{\partial q_1} < 0.$$

Proof. Suppose $q_1 < b(q_2)$. Let $\theta^* = b(q_2) - q_1$. By Lemma 26, $\alpha_1(q_1 + \theta, q_2) < \alpha_1(q_1, q_2 + \theta)$, for all $0 < \theta \leqslant \theta^*$. Hence $\partial \alpha_1(q)/\partial q_1 \leqslant \partial \alpha_1(q)/\partial q_2 < 0$, where the last inequality follows from (A8). Since $\partial \Pi^*(q_2)/\partial q_1 = 0$, $\partial \Pi_1(q)/\partial q_1 > 0$ follows directly. Now suppose $q_1 > b(q_2)$, and let $\theta^* = q_1 - b(q_2)$. By a similar argument, $\alpha_1(q_1 - \theta, q_2) < \alpha_2(q_1, q_2 + \theta)$, all $0 < \theta \leqslant \theta^*$, $\partial \alpha_1/\partial q_1 \geqslant \partial \alpha_1/\partial q_2 > 0$, and $\partial \Pi_1/\partial q_1 < 0$. Q.E.D.

LEMMA 30. *For all $q_2 \in S_1$ and $\theta > 0$, $b(q_2) \geqslant b(q_2 + \theta)$.*

Proof. Consider $\theta > 0$ and suppose $q_1 = b(q_2 + \theta) > b(q_2)$. By (A8), $\partial \alpha_1(q_1, q_2)/\partial q_2 > 0$. Hence $\alpha_1(b(q_2 + \theta),\ q_2 + \theta) > 0$, a contradiction. Q.E.D.

THEOREM 39. *Let (q, u) be an ROPP. Then $q_1 \neq b(q_2)$ and $q_2 > b(q_1)$.*

Proof. The proof proceeds by considering a series of separate cases, each of which contradicts Corollary 32. The first two cases use max $\{u_1, u_2\} > \mathbf{v}(\Omega)$, which follows from Lemma 38.

Case 1. $q_1 = b(q_2)$ and $u_1 = \mathbf{v}(\Omega)$. By Lemma 37 and Lemma 28, $q_2 > b(q_1)$. Let $q(\theta) = (q_1 - \theta, q_2 + k_1\theta)$. By (A8) there exists $k_1 > 0$ such that $d\alpha_2(q(\theta))/d\theta = -\partial\alpha_2/\partial q_1 + k_1(\partial\alpha_2/\partial q_2) < 0$. Clearly $d\alpha_1/d\theta = 0$ and $d\Pi_1/d\theta < 0$. Define $u_1(\theta) = u_1 + k_2(\alpha_2(q) - \alpha_2(q(\theta)))$, $u_2(\theta) = u_1 + u_2 - u_1(\theta)$, where $k_2 \in (0, 1)$ is chosen to satisfy $d(\Pi_1 + u_1)/d\theta < 0$. Since $k_2 \in (0, 1)$, we also have $d(u_1 - \alpha_1)/d\theta = -k_2(d\alpha_2/d\theta) > 0$ and $d(u_2 - \alpha_2)/d\theta = -(1 - k_2)(d\alpha_2/d\theta) > 0$. By Theorem 23, since $u_2 > u_1 = \mathbf{v}(\Omega)$, $u(\theta) \in V$, all θ such that $u_2(\theta) \geqslant u_1(\theta)$. All this implies that there exists $\theta_1 > 0$ such that $(q(\theta_1), u(\theta_1))$ is supportable and $\Pi_1(q(\theta_1)) + u_1(\theta_1) < \Pi_1(q) + u_1$, a contradiction.

Case 2. $q_2 = b(q_1)$ and $u_2 = \mathbf{v}(\Omega)$. Let $q(\theta) = (q_1, q_2 + \theta)$. By Lemma 37 and Lemma 28, $q_1 < b(q_2)$, and by (A8), $d\alpha_1/d\theta < 0$. Clearly $d\alpha_2/d\theta = 0$. Define $u_2(\theta) = u_2 + \frac{1}{2}(\alpha_1(q) - \alpha_1(q(\theta)))$, $u_1(\theta) = u_1 + u_2 - u_2(\theta)$. Then $d(u_1 - \alpha_1)/d\theta$, $d(u_2 - \alpha_2)/d\theta > 0$, and $d(\Pi_1 + u_1)/d\theta < 0$. By Theorem 23, $u(\theta) \in V$, all θ such that $u_1(\theta) \geqslant u_2(\theta)$. Now proceed as in Case 1.

Case 3. $q_1 = b(q_2)$ and $u_1 > \mathbf{v}(\Omega)$. Let $q(\theta) = (q_1 - \theta, q_2)$. For by now familiar reasons $d\alpha_2/d\theta < 0$. Also, $\alpha_1(q) = 0 < u_1 - \mathbf{v}(\Omega)$. Hence there exists $\theta_1 > 0$ such that $(q(\theta_1), u)$ is supportable. Since $q_1 - \theta_1 \neq b(q_2)$, $\Pi_1(q(\theta_1)) < \Pi_1(q)$ and we have a contradiction as before.

Case 4. $q_2 < b(q_1)$. By Lemma 37 and Lemma 28, $q_1 < b(q_2)$. Let $q(\theta) = (q_1, q_2 + \theta)$. By (A8) and Lemma 29 there exists $\theta_1 > 0$ such that $(q(\theta), u)$ is supportable and $\Pi_1(q(\theta_1)) < \Pi_1(q)$.

Case 5. $q_2 = b(q_1)$ and $u_2 > \mathbf{v}(\Omega)$. As in Case 4, $q_1 < b(q_2)$. Furthermore, $\alpha_2(q) = 0 < u_2 - \mathbf{v}(\Omega)$. Now proceed as in Case 4. Q.E.D.

LEMMA 40. *Let (q, u) be an ROPP. If $\Pi_1(q) > 0$ then $q_1 < b(q_2)$.*

Proof. By Theorem 39, $q_1 \neq b(q_2)$. Suppose $q_1 > b(q_2)$. By Lemma 37 and Lemma 28, $q_2 > q_1$ and $q_2 > b(q_1)$. Define $q_1(\theta) = q_1 - \theta$, $q_2(\theta) = \max\{b(q_1(\theta)), q_2\}$. By Lemma 28, for all $0 \leqslant \theta \leqslant q_1 - b(q_2)$, $q_2 > b(q_1(\theta))$. Hence $\Pi_1(q(q_1 - b(q_2))) = \Pi_1(b(q_2), q_2) > \Pi_1(q)$. Since $\Pi_1(q) > 0$, by (A2)–(A4), there exists $\theta_1 > q_1 - b(q_2)$ such that $\Pi_1(q(\theta_1)) = \Pi_1(q)$.

Case 1. $q_2(\theta_1) = q_2 > b(q_1(\theta_1))$. Clearly $q_1(\theta_1) < b(q_2(\theta_1)) = b(q_2)$ and $\alpha_1(q(\theta_1)) = \alpha_1(q)$. By Lemma 30, $q_2 > b(q_1(\theta))$, all $0 \leqslant \theta \leqslant \theta_1$. Hence by (A8), $\alpha_2(q(\theta_1)) < \alpha_2(q)$. Define $q(\theta, \varepsilon) = (q_1(\theta), q_2(\theta) + \varepsilon)$. By (A8) again there exists $\varepsilon > 0$ such that $(q(\theta_1, \varepsilon), u)$ is a supportable pair. Since $\Pi_1(q(\theta_1, \varepsilon)) < \Pi_1(q(\theta_1)) = \Pi_1(q)$, we have contradicted Corollary 32.

Case 2. $q_2(\theta_1) = b(q_1(\theta_1)) \geqslant q_2$. Clearly $\alpha_1(q(\theta_1)) \leqslant \alpha_1(q)$. Since $q_2(\theta_1) > q_1(\theta_1)$, Lemma 28 implies $q_1(\theta_1) < b(q_2(\theta_1))$. Also $\alpha_2(q(\theta_1)) = 0 < \alpha_2(q)$. Now proceed as in Case 1. Q.E.D.

LEMMA 42. *Let (q, u) be an RLPP. Then $\Pi_i(q) > 0$, $i = 1, 2$.*

Proof. Suppose not. Then $q_1 > b(q_2)$ and $q_2 > b(q_1)$. Let $q_1(\theta) = q_1 - \theta$ and $q_2(\theta) = q_2$. By (A8) and Lemma 29, there exists $\theta_1 > 0$ such that $(q(\theta), u)$ is a supportable pair all $0 \leqslant \theta \leqslant \theta_1$. Since $d\Pi_i(q(\theta))/d\theta > 0$ (Lemma 29), we are done. Q.E.D.

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