ANNUAL REVIEWS

Annual Review of Statistics and Its Application

Innovation Diffusion Processes: Concepts, Models, and Predictions

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Annu. Rev. Stat. Appl. 2023. 10:451–73

First published as a Review in Advance on November 18, 2022

The *Annual Review of Statistics and Its Application* is online at statistics.annualreviews.org

https://doi.org/10.1146/annurev-statistics-040220-091526

Keywords

competition, critical diffusion, innovation diffusion model, nonlinear regression, structured intervention, word-of-mouth

Abstract

Innovation diffusion processes have attracted considerable research attention for their interdisciplinary character, which combines theories and concepts from disciplines such as mathematics, physics, statistics, social sciences, marketing, economics, and technological forecasting. The formal representation of innovation diffusion processes historically used epidemic models borrowed from biology, departing from the logistic equation, under the hypothesis that an innovation spreads in a social system through communication between people like an epidemic through contagion. This review integrates basic innovation diffusion models built upon the Bass model, primarily from the marketing literature, with a number of ideas from the epidemiological literature in order to offer a different perspective on innovation diffusion by focusing on critical diffusions, which are key for the progress of human communities. The article analyzes three key issues: barriers to diffusion, centrality of word-of-mouth, and the management of policy interventions to assist beneficial diffusions and to prevent harmful ones. We focus on deterministic innovation diffusion models described by ordinary differential equations.

1. INTRODUCTION

Modeling the diffusion of innovations uses mathematical models to describe and predict the temporal growth of an innovation entering a social system. The term innovation covers a broad spectrum of phenomena: An innovation may be a new product, technology, service, idea, or behavior. This theme has attracted many scholars to combine theories, concepts, and models from both the natural and the social sciences. The famous treatise on innovation diffusion by Rogers (2003), first published in 1962, proposed a rich conceptual framework for diffusion processes and their determinants, such as characteristics of innovation, type of adopters, drivers, and obstacles, with a wide variety of examples. Rogers (2003) stressed that this research grew from a series of independent studies combining apparently separate sciences whose common factor was that an innovation spreads into society through communication and imitation, just like a new virus circulates among people through contagion. The mathematical laws governing biological processes, such as the spread of a virus or the growth of an organism, are similar to those acting in a social system when an innovation, social norm, or behavior starts to spread and is adopted by individuals. This has been confirmed by scholars of modern socio-technical systems, who emphasized the need to develop appropriate theories and methods for understanding and controlling contagion dynamics (Vespignani 2012). Marchetti (1980) theorized that society is a learning system and proposed the logistic equation—which Verhulst (1838) introduced and McKendrick & Pai (1912) reformulated as an infection transmission model—as a mathematical principle able to capture this learning behavior.

The logistic equation fueled a rich variety of contributions in diverse fields, from mathematical epidemiology (Keeling & Rohani 2011) to sociology (Tuma & Hannan 1984) and economics (Mansfield 1961). In particular, the marketing and management literatures on innovation diffusion have dramatically benefited from the publication of the Bass model (BM) (Bass 1969), one of the most successful modifications of the logistic model. The BM is a simple aggregate dynamic model that explains the temporal diffusion of an innovation, from its onset up to the final saturation of its market potential, by the action of the available sources of social communication within the relevant population (or social system). These sources are identified in external communication, provided by the media or institutions, and in private communication flowing through the spontaneous interactions between individuals over their social networks, labeled word-of-mouth or imitation. Imitation is a broad concept referring to the many situations where the transmission of information or behavior between people is based not only on explicit interpersonal communication but also on social signals that individuals can receive and use to imitate others' behavior. This is typical of social media dynamics where the mere observation of posted contents can trigger imitation. Although several growth models have been proposed for describing the typical s-shaped cumulative profiles of innovation diffusions, such as generalized logistic, Gompertz-like, and Richards-like models (Seber & Wild 1989), the BM has gradually emerged as the standard model of innovation diffusion in these disciplines owing to its appealing mix of simplicity and parsimony, the fact that it is mechanistic with simple causal factors, and its excellent fit to many diffusion datasets. The central role played by the BM has been discussed in reviews by Mahajan et al. (1990), Parker (1994), Mahajan et al. (1995), Meade & Islam (1995), Meade & Islam (1998), Meade & Islam (2006), Hauser et al. (2006), Chandrasekaran & Tellis (2007) and Peres et al. (2010). Over time, the focus of these reviews shifted from the need to develop a new generation of innovation models by relaxing the BM assumptions (Mahajan et al. 1990) to the modeling needs of the typical management and marketing viewpoints (Hauser et al. 2006) to the statistical issues (inference and forecasting) related to the empirical use of diffusion models (Meade & Islam 2006) to emerging issues such as the power of communication, consumer heterogeneity, extensions to nondurable goods, and competition between products and brands (Peres et al. 2010).

Owing to the aforementioned characteristics, the BM has often been perceived as a general tool for any kind of life cycle. However, expert commentators (see Mahajan et al. 1990) have often remarked that diffusion models are especially useful when the diffusion is completed, to quantitatively identify its dominant drivers. This may indicate that the BM tells only the part of the story after the take-off, when the diffusion is ongoing because barriers to adoption (costs, social norms, logistical issues) have been removed. This suggests the need to reconsider the BM's hypotheses and behavioral determinants. This revision is necessary if one aims to respond to further questions, such as whether a diffusion will take off, stagnate at low levels for a long time, or be vital—that is, whether it is able to travel on its own legs or needs external support. These questions become relevant when the focus is shifted from commercial goods, whose diffusion might be supported by advertising, to diffusions that are critically important from a societal perspective, i.e., critical diffusions such as the success of an energy transition program, the diffusion of best environmental practices, or compliance with large-scale public health programs.

This article reviews a class of aggregate models for innovation diffusion and their applications by exploiting the knowledge of other disciplines—first of all, mathematical epidemiology. To do so, we start from the classic BM and its main developments in the innovation diffusion literature, such as the role of marketing mix and external shocks and the possibility of a dynamic market potential. Next, we discuss the role of word-of-mouth diffusions by exploiting the richness of the epidemiological literature. Here, we rely on the empirical fact that, in most innovation diffusions, word-of-mouth—the internal or epidemiological component—exceeds the external component, which is often negligible or absent (Bass 1969, Guidolin & Mortarino 2010, Rao & Kishore 2010, Bunea et al. 2020). This implies that the forces driving many real-world diffusions are essentially epidemiological. With this in mind, we discuss innovation processes wherein the basic logistic equation [i.e., the susceptible-infective (SI) model] is amended to account for the possibility that active spreading of information by adopters ends at some time [the susceptible-infective-removed (SIR) model]; active spreading needs regular boosting [the susceptible-infective-susceptible (SIS) model]; individuals are not homogeneous, but superspreaders coexist with rather inactive individuals (the SI and SIR models with heterogeneities); and the diffusion is affected by behavioral drivers. This yields a richer catalogue of possible dynamics whose knowledge may be important for innovation diffusion scholars in marketing, statistics, and social sciences, but also for policy-makers. For completeness, we then review some selected materials on multivariate diffusion models accounting for the presence of competition. We end with a discussion of applications.

The remainder of the article contains the following: motivating examples (Section 2); a review of the main mathematical features of innovation diffusion models, for both univariate and multivariate processes (Sections 3–5); basic statistical techniques involved in model estimation and their applications (Section 6); and future perspectives and concluding remarks (Section 7).

2. INNOVATION DIFFUSION: MOTIVATING EXAMPLES

Diffusion processes may range from the spread of new epidemics (starting from the still ongoing COVID-19 pandemic) to the adoption of new products, technologies and life-styles. In this section, four examples are presented to illustrate common traits and specificities of applications related to different contexts. **Figure 1*a*** displays global quarterly sales of the Apple iPhone from quarter 3 of 2007 to quarter 4 of 2018 (all generations of iPhone are included). Apart from evident seasonal peaks, the series exhibits a clear nonlinear trend, typical of products with a finite life cycle and characterized by phases of launch, growth, maturity, and decline. The iPhone series appears to have just reached the maturity phase. **Figure 1*b*** shows the yearly consumption of solar energy in Japan from 2001 to 2020. In this case, the technology appears to be in the growth phase of its

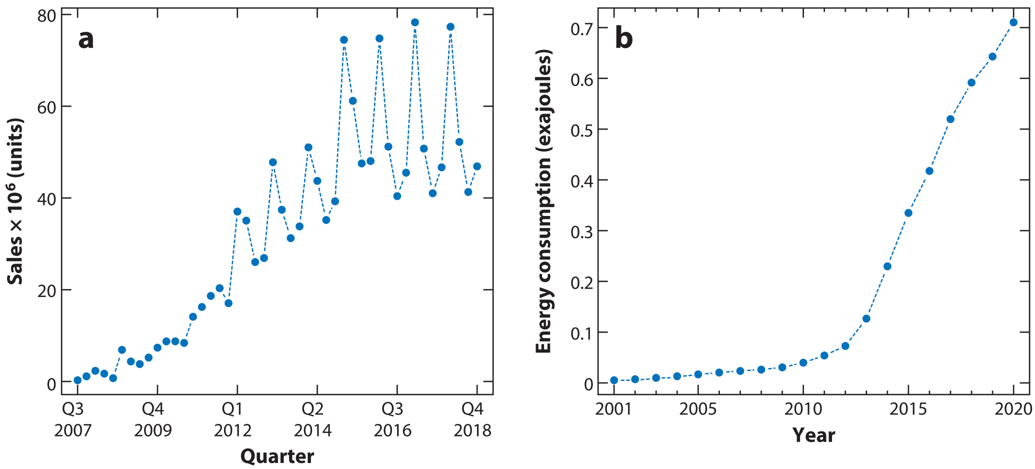

Figure 1

(*a*) Quarterly global sales of Apple iPhone (in millions of units). (*b*) Annual solar energy consumption in Japan (exajoules). Data in panel *a* are from Apple, Inc. (**https://www.apple.com**), and data in panel *b* are from British Petroleum (2021) (**https://www.bp.com**).

life cycle, again with a marked nonlinear trend. Here we follow the hypothesis made by Marchetti (1977, 1980) and then developed in the energy research literature (see Bessi et al. 2021): that energy sources are comparable to commercial products with finite life cycles that initially must be accepted by a market niche and then experience phases of growth, maturity (with a peak), and decline. This has been historically observed because newer energy technologies have gradually supplanted existing ones. **Figure 2*a*** displays daily new cases of COVID-19 infections from February 19 to June 1, 2020, in Italy. The reported series represents the first wave of the COVID-19 pandemic, with its initial growth phase followed by the recession phase forced by control measures. Similar patterns were observed in most countries. This first wave suggests a finite cycle and shows an asymmetric shape reflecting an initial phase of free epidemic growth, followed by a slow-down, peak,

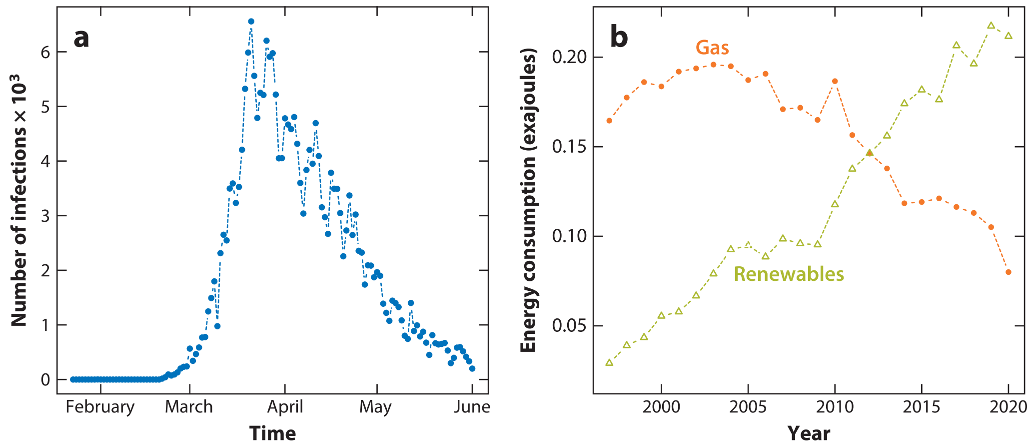

Figure 2

(*a*) Daily infections from COVID-19 in Italy (first wave). (*b*) Annual gas and renewable energy consumption in Denmark (exajoules). Data in panel *a* are from **https://www.worldometers.info/coronavirus/country/italy/** and data in panel *b* are from British Petroleum (2021) (**https://www.bp.com**).

and decline due to the measures and their inertia. **Figure 2*b*** shows the growing trend of renewable energy consumption and the parallel decrease of gas consumption in Denmark from 1997 to 2020 (British Petroleum 2021). Taking the same perspective adopted for Japan in considering energy technologies as products with a finite life cycle, this example is useful to show the possible competition and substitution dynamics between energy technologies, where the newer technology is able to subtract market share from the older one. The diffusion of renewables as a key component of the energy transition process (Geels et al. 2017) represents a major example of a critical diffusion.

Though they represent markedly different situations, these examples share the common trait of an evident nonlinear growth, typical of diffusion processes. The underlying hypothesis is that these dynamics are fueled by individuals' behavior, by means of communication (word-of-mouth) and contagion. However, these different cases suggest the need for external management aimed at controlling the speed and scale of diffusions. This is evident in the case of renewable energy in Japan and Denmark and COVID-19 in Italy, where public policy actions were implemented to control the process, although to achieve opposite results. In the case of renewables, incentive measures such as feed-in tariffs have been implemented to stimulate the growth of this technology, whose widespread adoption may be hindered by important barriers. On the contrary, a strong policy measure (a two-month lockdown) was put in place in Italy to contain the diffusion of COVID-19. In this case, reaching the peak (in the first wave), after which the process started to decline, was clearly a desirable outcome.

The modeling approach proposed in this article, based on aggregate diffusion models, assumes that individuals' behavior is the driver of any diffusion process, but management through suitable policies appears to be inescapable, both to stimulate beneficial diffusions and to control harmful ones.

3. UNIVARIATE DIFFUSION MODELS à la BASS

3.1. Innovation and Imitation: The Bass Model

The BM (Bass 1969) describes the life cycle of an innovation, capturing its typical phases of launch, growth, maturity and decline. Originally conceived in the marketing domain to model the growth over time of a new durable good as a result of purchases of potential adopters, its applications have expanded well beyond commercial durables, becoming a paradigm for representing life cycles of many other types of innovations. The adoption decisions involved are assumed to be influenced by two sources of information: an external one, such as mass media and advertising, and an internal one, by social interactions, termed imitation or word-of-mouth. If we consider a deterministic function $z(t)$ representing the cumulative number of adopters (and of adoptions, in view of the first-purchase nature of the model) at time t, the BM is described by the ordinary differential equation

$$z'(t) = \frac{dz(t)}{dt} = \left\{p + q\frac{z(t)}{m}\right\}\{m - z(t)\}, \quad p, q > 0, \quad t > 0; \qquad 1.$$

here and below, we let $g'(t)$ denote the derivative of a function $g(t)$.

Equation 1 relates the variation over time of adopters, $z'(t)$ (instantaneous adoptions), to the residual market, $m - z(t)$, where m is the market potential, and to the overall hazard rate of adoption $p + qz(t)/m$, where the parameter p, termed the innovation rate, is a positive constant tuning the intensity of the external source of information, while q, termed the imitation rate, is a positive constant tuning the intensity of the internal source, whose influence is modulated by the ratio $z(t)/m$, giving rise to the word-of-mouth effect. In particular, the market potential represents the maximum number of adoptions within the life cycle, i.e., the market carrying capacity, whose value is assumed to be constant along the whole life cycle. Equation 1 is typically coupled with the initial condition $z(0) = 0$, corresponding to the situation where no adopters exist at launch time.

The BM can also be represented as a hazard rate model because the growth of adopters over time occurs by way of irreversible transition events, i.e., adoptions, which are equivalently described as the elimination over time of a cohort of susceptible individuals, the potential adopters, having initial size m. Letting $x(t)$ denote the size of the population of potential adopters at time t, the identity $x(t) + z(t) = m$ yields the equation $x'(t) = -\{p + qz(t)/m\}x(t)$, with initial condition $x(0) = m$. By the normalization $x(t)/m = \mathcal{P}$, the model takes the standard hazard rate form,

$$\mathcal{P}'(t) = -\left\{p + qF(t)\right\}\mathcal{P}(t), \qquad 2.$$

where $F(t) = z(t)/m$ is the cumulative adoption probability at time t, $\mathcal{P}(t) = 1 - F(t)$ is the corresponding survival function, and

$$\lambda(t) = \frac{F'(t)}{1 - F(t)} = p + qF(t) \qquad 3.$$

is the hazard rate of adoption at time t. This hazard is the sum of the competing risks p and $q(z/m)$ of two (almost) incompatible events, namely adoption through the external or internal sources, respectively.

The closed-form solution of the BM under the initial condition $z(0) = 0$ is

$$z(t) = m\,\frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}, \quad t > 0. \qquad 4.$$

The cumulative adoptions $z(t)$ follow a monotonically increasing and saturating function of time whose shape depends on the relative effects of p and q, while m acts as a scale parameter.

The resulting solution for instantaneous adoptions is

$$z'(t) = m\frac{(p+q)^2}{p}\,\frac{1 - e^{-(p+q)t}}{\{1 + \frac{q}{p}e^{-(p+q)t}\}^2}, \quad t > 0. \qquad 5.$$

The first derivative $z'(t)$ of cumulative adoptions describes the (absolute) number of adoptions over a very narrow time interval. Normalizing by the market potential yields the relative probability density function of adoption, $f(t) = z'(t)/m$. Adoptions over the interval $(t, t+h)$ can be computed as the differences $z(t+h) - z(t)$.

3.1.1. Media and word-of-mouth: external and internal models. Appreciation of previous results and their meaning requires a deeper analysis of the external and the internal submodels of the BM. These are hazard models describing the elimination over time of a cohort of susceptible individuals of initial size m who are homogeneously exposed to only one of the two communication sources in the BM, either the external source or word-of-mouth.

The external model is represented by the constant hazard rate equation $x'(t) = -px(t)$, $x(0) = m$, $p > 0$, yielding the adopters' equation

$$z'(t) = p\{m - z(t)\}, \quad z(0) = 0, \quad t > 0. \qquad 6.$$

Potential adopters decay exponentially over time as $x(t) = me^{-pt}$, so the solution for $z(t)$ is $z(t) = m - x(t) = m(1 - e^{-pt})$. This corresponds to the growth of adopters until saturation of the market potential with a concave shape (see the related curve in **Figure 3*a***). In particular, the innovation rate p represents the model's hazard rate. The ensuing curve of instantaneous adoptions is $z'(t) = mpe^{-pt}$, corresponding to the exponential adoption density $f(t) = pe^{-pt}$. This suggests that diffusions sustained by constant media or institutional effort are decreasing rather than bell shaped.

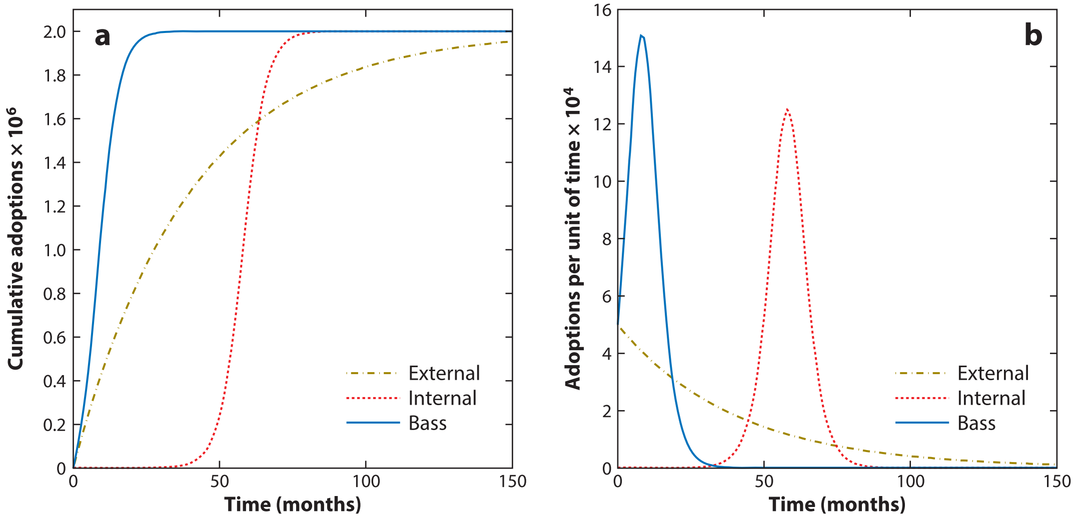

Figure 3

Bass-type diffusion compared with that of its external and internal components. (*a*) Cumulative adoptions $z(t)$. (*b*) Instantaneous adoptions $z'(t)$. Parameter values: innovation parameter, $p = 0.025$; imitation parameter, $q = 0.25$; and market potential, $m = 2$ million. Bass and external models initialized from $z(0) = 0$, internal model from $z(0) = 1$.

The internal model is represented by the hazard rate equation $x'(t) = -q\{z(t)/m\}x(t)$, $x(0) = m - 1$, $q > 0$, yielding the adopters' equation

$$z'(t) = q\frac{z(t)}{m}\{m - z(t)\}, \quad z(0) = 1, \quad t > 0. \qquad 7.$$

This is an epidemiological equation that McKendrick & Pai (1912) introduced, representing a pure transmission process whereby adopters at time t, $z(t)$, are infectious individuals actively transmitting an infection (or a behavior, etc.) to the uninfected, also termed susceptible, individuals. This is termed an SI model, and the imitation rate q is termed a transmission rate. This is a composite parameter that depends on two main processes, namely, the social interaction process among individuals over the relevant social network and the contagion process occurring when an infective and a susceptible individual meet. Therefore, $q = \beta = \alpha C$, where C represents the average number of social contacts per unit of time for a generic member of the population, and α is the transmission probability per single encounter between a susceptible and an infective individual. In particular, $z'(t)$ represents the incidence of new infections and is obtained by noting that, at time t, each infective individual will have C social contacts, of which a proportion $x(t)/m$ are with susceptible people if social contacts are homogeneous (i.e., random), thus infecting $\alpha Cx(t)/m$ individuals. Summing over all infected individuals leads to the second term of Equation 7.

Unlike the external model, the initial condition in Equation 7 must be nonzero because no transmission chain can be started in the absence of infective individuals. Indeed, if $z(0) = 0$, Equation 7 gives $z' = 0$ ($z = 0$ is an equilibrium of the model, the infection-free equilibrium). For any positive initial number of infective individuals, the solution to Equation 7 is represented by the logistic curve (**Figure 3*a***), with an s-shaped saturation of the market potential m, which represents a globally stable equilibrium of the model. In particular, the initial phase of growth is exponential with instantaneous rate q, as is easily seen by taking $z(0)$ as small compared with m, which yields $z'(t) \approx qz(t)$. The resulting instantaneous adoption curve is bell shaped (**Figure 3*b***).

Once the behavior of the external and internal components is known, that of the BM straightforwardly follows. This is clear from **Figure 3**, drawn for the most frequent case where communication is word-of-mouth dominated ($q \gg p$), yielding a shape of the BM cumulative solution very similar to its underlying logistic component (**Figure 3*a***). Even if $z(0) = 0$, so that the word-of-mouth mechanism is initially inactive, external communication rapidly creates an initial cohort of early adopters who initialize word-of-mouth diffusion. To sum up, the external model starts fast but progresses slowly, the internal model starts slowly and proceeds virally only after the market has grown sufficiently, and the BM enjoys the initial push of the external component, which immediately sets into motion the viral behavior due to word-of-mouth. Word-of-mouth transmission emerges as the big gift of the community to the market. The corresponding patterns of the instantaneous curves are displayed in **Figure 3*b***.

3.1.2. The innovators and imitators debate.

Compared with the logistic diffusion due to internal information only, the inclusion of external information allows a Bass-type diffusion to take off even for $z(0) = 0$, i.e., in the absence of initial spreaders, also called innovators. There exists a wide body of literature on the role of innovators, identified as early adopters by Rogers (2003). Bass (1969) called innovators those who adopt the innovation through external influence (the p component), i.e., independently of peer pressure. Later research clarified that the nature of the BM as a hazard model with two competing risks implies that, at any time t, potential adopters are exposed to the overall risk $\lambda(t) = p + qF(t)$. Therefore, adoptions always occur due to the joint action of both information sources, and, unless further information is available, one can only make a posteriori assessments on which of the two sources was more likely to prevail based on actual p and q values. Nonetheless, the BM shows that Rogers's and Bass's definitions of innovators agree during the very early phase of the life cycle. Indeed, for $t \approx 0$, the (instantaneous) number of new adoptions is about $z'(0) = \{(p + qz(t)/m)(m - z(t))\}_{t=0} = pm$, i.e., all adoptions actually occurred due to external influence because word-of-mouth is not active yet (Mahajan et al. 1990). The BM originated a still-active debate with several developments summarized in the above-cited reviews. The next subsections review two main areas of development, but there are many more, including models for successive generations of the same product (Norton & Bass 1987, Jiang & Jain 2012).

3.2. Decision Variables and Structured Interventions: The Generalized Bass Model

Reviews on diffusion models (Mahajan et al. 1990, Peres et al. 2010) pointed out that a drawback of the BM was the absence of marketing mix variables, such as price strategies and advertising. Additionally, the shortening of life cycles due to successive generations increased the need for models incorporating control variables. Bass et al. (2000) reviewed attempts to introduce control variables into diffusion models and summarized some desirable properties the resulting models should have: they should be empirically supported, be managerially useful, allow a direct interpretation of parameters and comparisons with other situations, have a closed-form solution, and be easy to implement. A model fulfilling these properties is the generalized BM (GBM) (Bass et al. 1994). Conceived to take into account both price and advertising strategies, the GBM extends the BM by multiplying its basic structure by a general intervention function $\chi(t)$:

$$z'(t) = \left\{p + q\frac{z(t)}{m}\right\}\{m - z(t)\}\chi(t), \quad t > 0, \quad \chi(t) > 0. \qquad 8.$$

The specification of $\chi(t)$ originally proposed by Bass et al. (1994) included the rate of change of prices, $P(t)$, and advertising efforts, $A(t)$, according to $\chi(t) = 1 + P'(t)/P(t) + A'(t)/A(t)$. The GBM reduces to the BM when $\chi(t) = 1$, i.e., when the rates of change of prices and advertising are constant. Equation 8 has the following closed-form solution:

$$z(t) = m\,\frac{1 - e^{-(p+q)\int_0^t \chi(\tau)d\tau}}{1 + \frac{q}{p}e^{-(p+q)\int_0^t \chi(\tau)d\tau}}, \quad t > 0. \qquad 9.$$

As evidenced in Equation 9, the model's internal parameters m, p, and q are not modified by these external actions—i.e., $\chi(t)$ acts on the shape of diffusion, modifying its temporal structure by accelerating or delaying adoptions, but does not affect its final size. Hence, $\chi(t)$ allows the GBM to represent all those strategies controlling the timing of a diffusion process but not its size, allowing testing of the effect of marketing mix strategies and making scenario simulations based on the modulation of function $\chi(t)$.

Guseo et al. (2007) proposed a wider perspective of the GBM in order to use $\chi(t)$ for detecting patterned, or structured, shocks in an adoption process (Guseo et al. 2007, Guidolin & Mortarino 2010). A structured shock may take an exponential form, $\chi(t) = 1 + c_1 e^{b_1(t-a_1)} I_{t\geq a_1}$, $b_1 < 0$, representing a shock starting at time a_1, with intensity c_1 (regardless of the sign), and reabsorbed into the overall trajectory with time persistency (or memory) tuned by $1/b_1$. The use of exponential shocks, as well as other structured shocks, is suitable for identifying the positive effect of marketing strategies or incentive measures to stimulate the diffusion process. The possibility of defining flexible functions $\chi(t)$ has opened a larger perspective on the usability of the GBM, making it an efficient tool for detecting external actions that affect a diffusion process in a structured form. This approach has proven relevant in cases in which critical diffusions are significantly influenced by institutional aspects, policies, and cultural and economic factors (Guseo et al. 2007, Guidolin & Mortarino 2010, Bunea et al. 2020).

3.3. Dynamic Market Potential

A typical assumption of the BM relates to the market potential m, which is assumed to be determined at the time of introduction of the new product and remains constant during diffusion. Mahajan et al. (1990) observed that, in principle, there is no rationale for this simplifying assumption, and on the contrary, a dynamic market seems reasonable owing to a number of factors, the first of which is the decline of prices in competitive markets as diffusion proceeds. The literature has addressed the possibility of dynamic potential since the 1970s. Some works introduced a variable structure by modifying the residual market $m(t) - z(t)$ (Mahajan & Peterson 1978, Kamakura & Balasubramanian 1988, Horsky 1990, Mesak & Darrat 2002). Others considered a modification of the word-of-mouth component (Sharif & Ramanathan 1981, Jain & Rao 1990, Goldenberg et al. 2010) or assumed exogenous demographic dynamics (Centrone et al. 2007).

Following Guseo & Guidolin (2009), a benchmark generalization of the BM, considering a dynamic market potential, $m(t)$, can be obtained by postulating a separate dynamics of $m(t)$, with respect to that tuning the diffusion, allowing cumulative adoptions $F(t) = z(t)/m(t)$ to maintain the standard Bass form. A time differentiation of $F(t)$ implies that

$$z'(t) = \left\{p + q\frac{z(t)}{m}\right\}\{m - z(t)\} + z(t)\frac{m'(t)}{m(t)}, \quad t > 0, \qquad 10.$$

which yields

$$F'(t) = p + F(t)y(t)\{1 - F(t)\}, \quad t > 0, \qquad 11.$$

confirming that relative adoptions obey the standard BM. This generalization of the BM with variable $m(t)$ has closed-form solution

$$z(t) = m(t)\,\frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}, \quad t > 0, \qquad 12.$$

where $m(t)$ is a free function that may depend on modeling choices. Guseo & Guidolin (2009) made $m(t)$ dependent on a supercommunication process about the new product, also obeying a Bass-type diffusion. The resulting Guseo–Guidolin model (GGM) is

$$z(t) = K\sqrt{\frac{1-e^{-(p_c+q_c)t}}{1+\frac{q_c}{p_c}e^{-(p_c+q_c)t}}}\,\frac{1-e^{-(p_s+q_s)t}}{1+\frac{q_s}{p_s}e^{-(p_s+q_s)t}}, \quad t>0. \qquad 13.$$

In Equation 13, the cumulative adoptions $z(t)$ depend on two sets of parameters, namely K, p_c, and q_c, which describe the dynamics of communication aimed to create the market potential, and p_s and q_s, which refer to the adoption phase of the product. The BM emerges as a special case where the spread of market-building information is fast enough that all potential adopters are ready to purchase as soon as the product enters the market, i.e., $m(t) = K$.

4. WORD-OF-MOUTH DIFFUSION: THE LESSONS OF EPIDEMIOLOGY

In this section, we exploit the concepts of mathematical epidemiology to highlight the complexities of critical diffusion phenomena, starting from the hypotheses of the BM. This postulates an ideal situation where an external source of communication is activated at some time and remains active, and adopters are permanent spreaders of information. However, what happens if these hypotheses are removed? Two ideas are useful. The first is that, in most observed diffusions, including critical ones, the external component is estimated to be negligible or absent, and actual diffusions are purely word-of-mouth processes. The second is that the external component, though conceptually appealing, only initiates the process and could be muted after the initial phase without substantial effects on the diffusion. The next subsections discuss diffusion processes lacking external support, i.e., purely epidemiological processes.

4.1. Removal of Active Spreaders and Take-Off Thresholds: The Susceptible-Infective-Removed Model

The fact that, in the BM, the market take-off is granted by hypothesis may have led the diffusion literature to overlook this aspect. One detailed study of the subject by Golder & Tellis (1997) focused on the quantitative characterization of observed take-offs. Here we highlight that in pure word-of-mouth diffusions, the occurrence of the take-off ceases to be necessary if we assume that, at some stage, adopters cease to be active spreaders of information. This removal of active spreaders is a well-known epidemiological phenomenon resulting from infection recovery and subsequent immunity. Such effects are expected in many socio-dynamic processes due to psychological mechanisms (e.g., loss of enthusiasm, perhaps enhanced by lack of satisfaction with the choice, or a switch toward other priorities). However, though documented for critical diffusion processes, such as solar energy adoption (Graziano & Gillingham 2015), these effects do not seem to have received attention from the marketing and management literature. In the presence of removal, the SI pure transmission model evolves into an SIR epidemic model, whose simplest form is (Anderson & May 1992, Keeling & Rohani 2011) as follows:

$$\begin{aligned} S'(t) &= -\frac{\beta}{m}SI, \\ I'(t) &= \frac{\beta}{m}SI - \gamma I, \quad t>0, \\ R'(t) &= \gamma I. \end{aligned} \qquad 14.$$

In Equation 14, the variables S, I, and R denote the numbers of individuals who are susceptible, infective, and removed, respectively, $(S + I + R = m)$, $\beta > 0$ is the transmission rate, and $\gamma > 0$ is the removal rate. In this model, the identity between individuals and adoptions, characteristic of the BM, is lost. Cumulative adoptions can be counted by introducing a counting auxiliary variable $z(t)$, such that $z'(t) = (\beta/m)SI$.

The removal of active spreaders has notable effects on the properties of the BM: First, diffusion take-off is no longer necessary, but it is subject to a precise threshold condition; second, even if take-off occurs, the diffusion never saturates its market potential. The first result follows by noting that, even under the most favorable conditions for diffusion, when the entire population is initially susceptible ($S_0 = m$), the initial dynamics are approximately governed by the equation $I'(t) = (\beta - \gamma)I$, whose solution $I(t) = I_0 e^{(\beta - \gamma)t}$ will increase if and only if $\beta - \gamma > 0$, i.e., if the intensity of transmission, β, exceeds that of removal, γ. This can be expressed as $\beta/\gamma > 1$, which has a noteworthy substantive interpretation. Indeed, recalling that $1/\gamma$ represents the average duration of the infective phase, the ratio β/γ represents the (expected) number of secondary infections caused by an infective individual during his infective phase in a wholly susceptible population. Therefore, β/γ represents a natural threshold quantity termed the basic reproduction number of the infection, denoted by $\mathcal{R}_0$ (Anderson & May 1992, Keeling & Rohani 2011).

Based on the previous threshold principle, if $\mathcal{R}_0 < 1$, diffusion will never occur, and if $\mathcal{R}_0 > 1$, an epidemic will start, initially growing at rate $r_0 = \gamma(\mathcal{R}_0 - 1)$. This growth will, however, be limited by removal. As time passes, the number of susceptible individuals will deplete and the true dynamics will be governed by $I'(t) = \gamma\{\mathcal{R}_0 S(t) - 1\}I$, where the product $\mathcal{R}_E(t) = \mathcal{R}_0 S(t)$ defines the effective reproduction number. This suggests that diffusion will occur at the gradually reducing rate $r(t) = \gamma\{\mathcal{R}_E(t) - 1\}$, so growth will continue up to when $\mathcal{R}_E(t) = 1$ (i.e., $S = 1/\mathcal{R}_0$), which represents the peak, after which the epidemic will start to decline until complete extinction of the infective population. The overall fraction of the population that becomes infected, $u_\infty = R_\infty/m$, obeys the equation $1 - u_\infty = e^{-\mathcal{R}_0 u_\infty}$ (Hethcote 1989). For $\mathcal{R}_0 > 1$, this admits a unique solution, which strictly increases in $\mathcal{R}_0$ but is always lower than m, meaning that no epidemic, however intense, is able to infect the entire at-risk population. This is the so-called incomplete contagion principle. This can be understood by noting that when the epidemic goes to extinction, the infection hazard $\lambda(t) = (\beta/m)I$ also goes to zero, so individuals who are still susceptible no longer face becoming infected. **Figure 4*a*** illustrates the temporal trends of the state variables S, I, and R and of cumulative infections (i.e., adoptions) $z(t)$.

4.1.1. External communication in the susceptible-infective-removed model: the Bass–susceptible-infective-removed model. The negative effects of removal for adoption dynamics can be partly reduced by introducing external communication that will allow the market to always grow and saturate, yielding a Bass–SIR model. In this case, $S'(t) < -(p + (\beta/m)I)S < -pS$, suggesting that susceptible individuals will be depleted, as in the external model. However, if the word-of-mouth component is not vital (i.e., for $\mathcal{R}_0 < 1$), this will be of little use because the diffusion will occur at the declining pace ensured by the presence of external communication only. Fibich (2016) provided an extension of these ideas in a network framework via the Bass–SIR model, whose behavior is illustrated in **Figure 4*b*,*c***.

4.2. Loss of Motivation: The Susceptible-Infective-Susceptible Model

In some cases, infective individuals return to the susceptible state where they can again be infected. This situation is not rare in socioeconomic dynamics: For example, a socially relevant behavior, such as a best environmental practice, may be initially adopted but subsequently weakened or forgotten (at some rate γ) and therefore needs boosting to be relearned. This is described by the

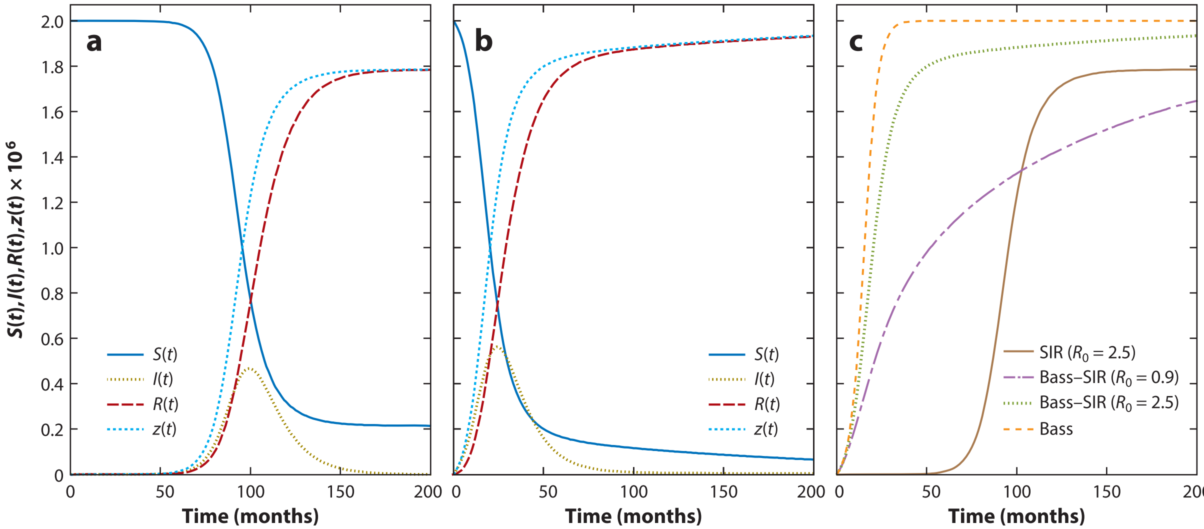

Figure 4

Temporal dynamics of SIR (susceptible-infective-removed) and Bass–SIR models. (*a*) Trends of state variables $S(t)$, $I(t)$, and $R(t)$ and cumulative adoptions $z(t)$ in an SIR epidemic with $q = \beta = 0.25$, $m = 2$ million, and a duration of infective phase $1/\gamma = 10$ months, yielding $\mathcal{R}_0 = 2.5$. Failure to saturate the market potential is apparent. (*b*) Trends of $S(t)$, $I(t)$, $R(t)$, and $z(t)$ in a Bass–SIR model adding external information ($p = 0.005$) to the SIR model of panel *a*, showing epidemic acceleration and saturation. (*c*) Comparison of cumulative adoptions $z(t)$ between the SIR model of panel *a* ($\mathcal{R}_0 = 2.5$); the Bass–SIR model of panel *b*; the Bass–SIR model with $\mathcal{R}_0 = 0.9$, behaving as an external model; and the underlying Bass model with $p = 0.005$, $q = 0.25$.

SIS model in which $S'(t) = -\beta SI/m + \gamma I$, and $I'(t) = \beta SI/m - \gamma I$. The identity $S = m - I$ yields

$$I'(t) = \left\{\frac{\beta}{m}(m - I) - \gamma\right\} I = \left\{(\beta - \gamma) - \frac{\beta}{m} I\right\} I = \gamma \left\{(\mathcal{R}_0 - 1) - \frac{\mathcal{R}_0}{m} I\right\} I. \qquad 15.$$

Compared with the SI model, in addition to the infection-free equilibrium $I = 0$, there is the positive (but not saturating) endemic equilibrium $I = m(1 - 1/\mathcal{R}_0)$. Concerning the SIR model, the presence of removal yields a threshold $\mathcal{R}_0 = \beta/\gamma > 1$ for diffusion. Below this threshold, no diffusion will occur and any initial infective cohort will disappear, converging to $I = 0$ (which is globally asymptotically stable), while above the threshold, the diffusion will occur and its temporal shape will be a logistically increasing curve (note the formal similarity with Equation 7), eventually converging to the endemic equilibrium. Comparing the main outcomes of the SIR model (a fast epidemic conferring permanent immunity always dies out) with those of the SIS model, we learn that return to susceptibility is a basic factor of endemic persistence of an infection. The inclusion of an external component of diffusion into the SIS model (i.e., a Bass–SIS model) removes the threshold effect but cannot ensure saturation.

Concerning critical diffusions, the main lesson from the SIS model is that, other things being equal, policy interventions should be able to extend the duration of the infective phase in order to increase $\mathcal{R}$. As a result, the intensity of diffusion will eventually lead to a higher endemic fraction of people adhering to a socially relevant behavior.

4.3. Relaxing Homogeneity: Highly Active and Mildly Active Agents

There are several studies on heterogeneities in Bass-type models. Bemmaor (1992) highlighted a bridge between Bass-type and Gompertz-type growth, showing that Bass-type trends can arise as the outcome of the shifted-Gompertz growth in a heterogeneous population with a negative exponentially distributed shifting parameter. Bemmaor & Lee (2002) considered the bias induced by heterogeneities in the estimates of Bass parameters, while Van den Bulte & Joshi (2007) considered a model with heterogeneous segments called the influential-imitators model. Here, we discuss the issue, starting from the observation that individuals neither behave nor interact homogeneously. Sticking to deterministic diffusion models, one convenient approach is that of genetic (or unobserved) heterogeneity, by which one gives up on explaining the determinants of heterogeneity and simply postulates it by subdividing the population into different groups.

We start by briefly illustrating a basic external model with genetic heterogeneity in the innovation rate p, leading to the main principle, the variance effect. Then we analyze the effects of heterogeneity in word-of-mouth by considering an SI model with heterogeneous contact patterns.

4.3.1. Genetic heterogeneity in external communication. Manfredi et al. (1998) considered an item of news spreading only through the media into a population composed of n separate groups, whose sizes $m_1, \ldots, m_n$ sum to m and that have different innovation rates, p_i. This requires a system of n replicates of the basic external model, with independent equations $z_i'(t) = p_i\{m_i - z_i(t)\}, z_i(0) = 0$, where $p_1 < \cdots < p_n$ are the group-specific innovation rates. The aggregate behavior of this system is not trivial, as can be appreciated by summing previous equations to obtain the dynamics of aggregate adopters $z(t) = \sum_i z_i(t)$,

$$z'(t) = \overline{p}(t)\{m - z(t)\}, \quad z(0) = 0, \quad t > 0, \tag{16.}$$

where $\overline{p}(t)$ is an aggregate quantity representing the mean innovation rate, defined as the weighted mean of group-specific innovation rates weighted with the corresponding residual market weights,

$$\overline{p}(t) = \sum_{i=1}^{n} p_i \frac{m_i - z_i(t)}{m - z(t)}. \tag{17.}$$

From the identity $x_i(t) = m_i - z_i(t)$, the residual market weights can be defined as $\omega_i(t) = x_i(t)/x(t)$. A time differentiation of $p(t)$ leads to the main result:

$$\overline{p}'(t) = -\sum_{i=1}^{n} \{p_i - \overline{p}(t)\}^2 \omega_i = -V_p(t) < 0, \tag{18.}$$

say, where $V_p(t)$ is the descriptive variance of the p_is, measuring the variability of group-specific innovation rates. This implies that the aggregate innovation rate will decline monotonically over time from its initial value,

$$p(0) = \sum_i p_i \frac{x_i(0)}{x(0)} = \sum_i p_i \frac{m_i}{m}, \tag{19.}$$

down to its long-term minimum of p_1, the innovation rate of the slowest group, for t tending to ∞. That is, the mean innovation rate of the market will continuously decline over time until the $n - 1$ fastest groups have adopted and only the slowest group remains, so that the timescale of the market saturation process will coincide with that of the slowest group. This has important implications: When the population is heterogeneous, any estimate of the innovation rate based on a homogeneous model and aggregate data only will always be biased because the size of faster groups is monotonically decreasing over time. This will always lead to overestimation of the aggregate innovation rate and a consequent underestimation of saturation times.

4.3.2. Heterogeneity in contact patterns. A result analogous to that in Section 4.3.1 also holds for genetic heterogeneities in the imitation rate, when n groups with rates q_i mix homogeneously. However, this is trivial because q (i.e., β) depends on both social contacts and contagion (see Section 3.1.1). Heterogeneity might appear in both processes. Here, we consider the most important case where social contact patterns are (genetically) heterogeneous, i.e., we subdivide the population into n groups having size m_i, each characterized by a specific daily contact rate $C_1, \ldots, C_n$ (Anderson & May 1992). This implies that group i contributes $C_i m_i$ to the total social activity of the community $\sum_i C_i m_i$. To complete the description of social interaction, it is then necessary to specify the $n \times n$ matrix of who contacts whom, which specifies the proportions of contacts of individuals in group i, which are undertaken with individuals in group j. Such proportions are termed the mixing proportions, often denoted as $\rho_{i,j}$, and the products $C_{i,j} = C_i \rho_{i,j}$ define the contact matrix, whose elements represent the number of contacts that an average individual in group i undertakes with individuals in group j.

Although we cannot enter into the details of the analysis and estimation of contact matrices, we describe a simple model that provides some insight. This model arises when we extend to a heterogeneous context the hypothesis that individuals of the various groups mix at random, as was the case for all previous models. In this case, the mixing and contact matrices simplify to the proportionate mixing hypothesis,

$$\rho_{i,j} = \frac{C_j m_j}{\sum_h C_h m_h}, \quad C_{i,j} = C_i \rho_{i,j}, \quad i, j = 1, \ldots, n, \tag{20.}$$

where the group sizes are biased by the group-specific number of contacts. When the previous hypothesis is embedded in the simplest word-of-mouth diffusion model, the SI model (Section 3.1.1), by assuming a constant infectivity α per contact, the within- and between-groups diffusion will be driven by an imitation matrix having elements $q_{i,j} = \beta C_{i,j}$. Compared with the model in Section 4.3.1, this model not only exhibits the long-term variance effect by which the slowest group will drive saturation but additionally shows an initial variance effect. Indeed, after an initial transient, a phase of early exponential growth will emerge with growth rate

$$q_{\text{initial}} = \alpha \overline{C} \left\{ 1 + \frac{V_c}{\overline{C}^2} \right\}, \tag{21.}$$

where $\overline{C} = \sum_{j=1}^{n} C_j m_j / m$ and $V_c = \sum_{j=1}^{n} (C_j - \overline{C})^2 m_j / m$, where V_c represents the descriptive variance of the social activity of the community. Compared with the basic homogeneous SI model for word-of-mouth diffusion in Section 3.1.1, where the growth rate of the initial exponential phase was simply the imitation rate $q = \alpha C$, the proportionate mixing hypothesis yields an initially high-speed phase whose rate is not simply the average imitation rate, $\alpha\overline{C}$, but is augmented by the square of the coefficient of variation of the contact distribution.

4.4. Including Behavioral Determinants in Word-of-Mouth Diffusion

Previous models only incorporate the kinetic factors of the diffusion—namely, the social contacts between infective and susceptible individuals that are transformed into new infections by the transmission/imitation rate q. For costly decisions or investments, such as solar photovoltaic panels or an electric car, individuals or families will adopt expensive innovations only if some further conditions are met, including a positive cost-benefit evaluation. The missing factor is the net perceived payoff of the investment, which should be added to the kinetic factors. Letting $\Delta P(t)$ denote the net perceived payoff at time t and getting back to the simplest SI case, the rate of change of the diffusion will include $\Delta P(t)$ as a further factor, yielding

$$z'(t) = q\frac{z(t)}{m}\{m - z(t)\}\Delta P(t), \quad z(0) = 1, \quad t > 0. \tag{22}$$

Appropriately closing the model requires specifying $\Delta P(t)$ as a function of the unique state variable, $z(t)$. Since a net payoff is the balance of the underlying benefit and cost, we can write $\Delta P(t) = \Delta P\{z(t)\} = b(z(t)) - c(z(t))$. Even in a simple case, where both benefits and costs are constant, there are differences with the standard SI model: Diffusion will occur, and logistic saturation will appear only if the benefit of adoption is larger than its cost. More reasonably, standard economic considerations suggest that costs should be lower when the diffusion is larger, $c'(z) < 0$, due to competitive pressure on markets, while benefits should be increasing, $b'(z) > 0$, due to social acceptance or legitimation (Hannan & Freeman 1984, Rogers 2003, Tuma & Hannan 1984). These assumptions imply that, even with a negative net payoff at the beginning of the diffusion, $b(0) - c(0) < 0$, there will be a level z^* at which benefits exceed costs. However, the break-even point z^* is actually a further equilibrium point of the model, $z = z^*$, break-even equilibrium. In particular, if the break-even equilibrium is lower than the saturation level, $z^* < m$, the diffusion will be successful, and will continue until saturation of m, only if the initial number of adopters exceeds the break-even equilibrium, $z(0) > z^*$, while diffusion will fail when $z(0) < z^*$.

Even if this model may appear simplistic, it provides a rationale for interventions supporting costly diffusions in their initial phase and also an explanation for failures. A simple example deals with incentives that have been persistently used for supporting solar photovoltaic markets. The model outcome shows that an incentive should not purely lower the cost for some people but should shock the market, creating an initial cohort of adopters in excess of the break-even equilibrium. The model in Equation 22, though purely internal, shares a formal similarity with the GBM [we might call it a generalized internal model (GIM)], suggesting that the inclusion of behavioral determinants could be a promising upgrade of the GBM for predictive and policy-making purposes.

5. MULTIVARIATE DIFFUSION MODELS

A further point to consider is the possible presence of competitors. Accounting for this may be very important, since the actions of competitors may be an important barrier to diffusion: new products, technologies, and ideas entering a new market or social system soon gain competitors. To date, competition modeling in diffusion has essentially been limited to duopolistic situations, where no more than two diffusion processes are simultaneously modeled. This is partly due to the complexity of systems of differential equations that account for the possible interactions among market players. As the number of equations grows, so does the number of parameters involved, making the use of these systems practically infeasible. A traditional approach to diffusion modeling under competition relied on Lotka–Volterra models, which are based on the independent contributions of Lotka (1920) and Volterra (1926). These equations, originally employed in the natural sciences for describing interactions between species and especially the so-called predator-prey relationship, have also been widely applied in technology. From the first contributions by Abramson & Zanette (1998), Baláž & Williams (2012), and Morris & Pratt (2003), the literature on Lotka–Volterra models has expanded to analyze competitive dynamics in innovative markets (Chakrabarti 2016, Guidolin & Guseo 2015, Guidolin et al. 2019, Kreng & Wang 2009, Tseng et al. 2014).

By generalizing the basic structure of the BM to the bivariate case, diffusion models under duopolistic competition were developed by, for instance, Guseo & Mortarino (2012), Guseo & Mortarino (2014), Guseo & Mortarino (2015), Krishnan et al. (2000), Laciana et al. (2014), and Savin & Terwiesch (2005). A common feature of these models is accounting for the interplay between products by splitting the imitation effect into the within-product imitation, due to a product's specific sales, and the cross-product imitation, due to sales of the competitor. In addition, competitors may enter the market at the same time so that their life cycles are essentially simultaneous, or more usually, a product starts as a monopolist and gains competitors along the way. Sequential market entry, also called diachronic competition, is common in reality but less addressed in the literature. A diffusion in competition may be described according to the model of Guseo & Mortarino (2014), with a system of differential equations where $z_1'(t)$ and $z_2'(t)$ indicate instantaneous adoptions of the first and second market players:

$$z_1'(t) = \left\{p_1 + (q_1 + \delta)\frac{z_1(t)}{m} + q_1\frac{z_2(t)}{m}\right\}\{m - z(t)\},$$
$$z_2'(t) = \left\{p_2 + (q_2 - \gamma)\frac{z_1(t)}{m} + q_2\frac{z_2(t)}{m}\right\}\{m - z(t)\}, \quad t > 0. \qquad 23.$$

In System of Equation 23, the residual market $m - z(t)$ is assumed to be common, where $z(t) = z_1(t) + z_2(t)$ denotes the total cumulative adoptions. The growth of the first product, $z_1'(t)$, is governed by the innovation coefficient p_1 and the imitation coefficient, which is divided into two parts: the within-imitation coefficient $(q_1 + \delta)$, measuring internal growth through the ratio z_1/m, and the cross-imitation coefficient q_1, which is powered by z_2/m and measures the effect of the diffusion of the second product on the first. The same is true for the growth of the second product, $z_2'(t)$.

Competition models have been usefully employed to study the competition between energy sources in energy transition processes; readers are directed to, for instance, Guidolin & Guseo (2016), Guidolin & Alpcan (2019), Furlan & Mortarino (2018), and Bessi et al. (2021). Generalizations of the model by Guseo & Mortarino (2014) have been proposed by Guidolin & Guseo (2015, 2020) and Guseo & Mortarino (2015).

6. STATISTICAL INFERENCE FOR DIFFUSION MODELS AND APPLICATIONS TO DATA

The statistical estimation of diffusion models à la Bass is quite sensitive to the amount of data available, and reliable estimates are obtained if noncumulative data include the peak (Srinivasan & Mason 1986), which clearly reduces model usefulness for predictive purposes. Mahajan et al. (1990, p. 9) stated that "parameter estimation for diffusion models is primarily of historical interest: by the time sufficient observations are available for reliable estimation, it is too late to use the estimates for forecasting." Van den Bulte & Lilien (1997) considered some issues in parameter estimation, including the tendency to underestimate the market potential, whose value is generally estimated to be close to the latest observed data. Estimation aspects were also discussed by Venkatesan & Kumar (2002), Venkatesan et al. (2004), and Jiang et al. (2006). Empirical experience demonstrated that ordinary least squares applied to discrete versions of the model (as in Bass 1969) has a number of shortcomings. Since the work of Srinivasan & Mason (1986), the nonlinear least squares (NLS) method has been accepted and used as more reliable. Following Seber & Wild (1989), the structure of a nonlinear regression model may be considered to be

$$w(t) = \eta(\vartheta, t) + \varepsilon(t), \qquad 24.$$

where $w(t)$ is the observed response; $\eta(\vartheta, t)$ is the deterministic component describing instantaneous or cumulative processes, depending on parameter vector ϑ and time t; and $\varepsilon(t)$ is the error term, typically assumed to follow standard hypotheses (Seber & Wild 1989). Traditionally, simple univariate models like the BM and the GBM are estimated on cumulative data by using the model closed-form solutions. More complicated models for which no closed-form solution is available, such as competition models, are estimated by numerical fitting of the underlying ordinary differential equations. Global goodness of fit may be evaluated through the R^2 (Bass 1969, Bass et al. 1994). For more details on nonlinear regression, including inferential aspects, we refer readers to Seber & Wild (1989).

Notwithstanding the central role of NLS estimation in fitting the BM and extensions, the standard approach based on additive normal homoscedastic errors has a number of shortcomings. This suggests that the empirical modeling of diffusions should start from the nature of these phenomena as counting processes. Maximum likelihood approaches based, for example, on Poisson or other counting populations, possibly combined with MCMC procedures (Robert & Casella 2004), would represent a desirable improvement for this class of models.

6.1. Applications to Data

We now apply some of the models discussed in previous sections to the series described as motivating examples in order to show their ease of use and interpretation. The common hypothesis made for using the BM and extensions is the presence of a finite life cycle with phases of introduction, growth, maturity, and decline. For each application we report the NLS estimated parameters. In the case of the BM, GGM, and GBM, given that a closed-form solution is available, the model is estimated on cumulative data, following common practice in the literature (Mahajan et al. 1990). For the competition model, for which there is no closed-form solution, the model is estimated directly by applying System of Equation 23 to instantaneous data. The performance of each model in graphical terms is displayed through the corresponding instantaneous fit.

6.1.1. Bass model and Guseo–Guidolin model. We first fit a standard BM (Equation 4) to the Apple iPhone sales data. The results are displayed in **Table 1** and **Figure 5*a***, which illustrates instantaneous predictions realized with the BM. The mean trajectory is well captured. The seasonal component is obviously not described through this model and may be treated in different ways according to modeling needs, either by prior smoothing or by describing it at a second stage. Parameter estimates in **Table 1** describe a typical case for a standard BM, with the imitation coefficient q being much larger than innovation coefficient p. This reflects most products' typical behavior, where the imitative component dominates the diffusion process. According to its structure, the BM predicts a maximum, after which the series starts to decline. The last data points are not well-fitted with the BM, which tends to underestimate them. A dynamic market

Table 1 Fits of BM and GGM for global sales of iPhone

BM		GGM	
Parameter	**Estimate**	**Parameter**	**Estimate**
m	1,823	K	2,116
$p \times 10^3$	1.41	$p_c \times 10^3$	5.92
$q \times 10$	1.26	$q_c \times 10$	2.05
		$p_s \times 10^3$	2.12
		$q_s \times 10$	1.00

Parameters listed in the table are K, the asymptotic market potential; m, the market potential; p, the innovation parameter; p_c, the innovation parameter in communication; p_s, the innovation parameter in adoption; q, the imitation parameter; q_c, the imitation parameter in communication; and q_s, the imitation parameter in adoption. Abbreviations: BM, Bass model; GGM, Guseo–Guidolin model.

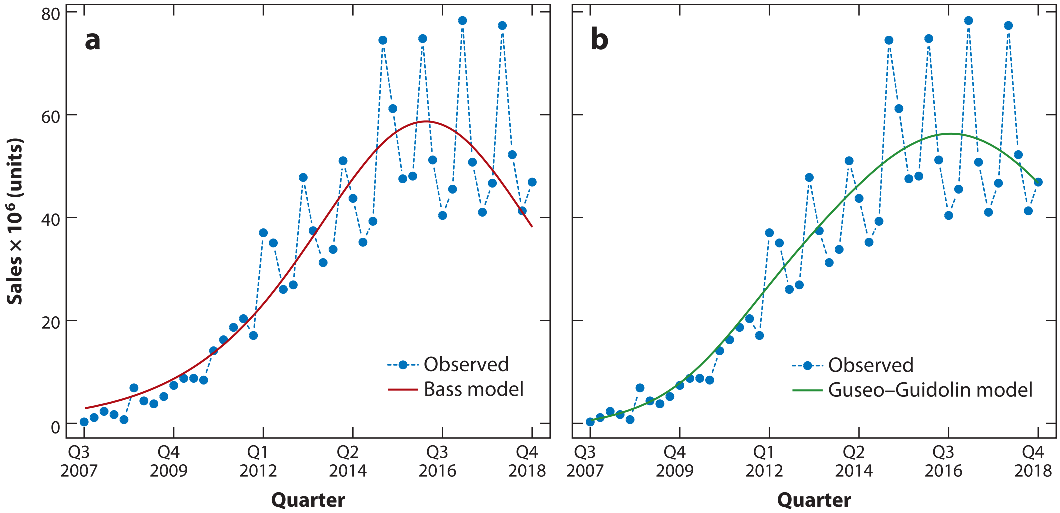

Figure 5

(*a*) Global iPhone sales observed data and Bass model (BM) instantaneous fitted values. (*b*) Global iPhone sales observed data and Guseo–Guidolin model (GGM) instantaneous fitted values.

potential model, the GGM (Equation 10), applied to the series of iPhone sales data, gives the results presented in **Table 1** and in **Figure 5*b*** (instantaneous fit). According to the model, the diffusion process is characterized by two phases: one for the spread of information regarding the product, creating the market potential, and one for the adoption process. Both are governed by imitative behavior, i.e., the parameters q_c and q_s. The model captures the data better than the simple BM, especially at the beginning and the end.

6.1.2. Generalized Bass model and competition modeling. To provide a description of solar energy consumption in Japan, a BM and a GBM with one exponential shock were fitted (Equation 9). The results are presented in **Table 2** and **Figure 6*a***. The coefficients concerning the exponential shock in the GBM, a_1, b_1, and c_1, highlight the positive effect of an external perturbation on consumption dynamics. Parameter $a_1 = 13.08$ suggests that the positive shock to consumption ($c_1 = 1.06$), connected with policy support for solar energy, started in 2013 and rapidly led to an acceleration of solar energy diffusion in Japan. However, parameter $b_1 = -0.243$

Table 2 Fit of BM and GBM for solar energy consumption in Japan

BM		GBM	
Parameter	**Estimate**	**Parameter**	**Estimate**
m	6.99	m	10.37
$p \times 10^4$	1.72	$p \times 10^4$	4.53
$q \times 10$	3.99	$q \times 10$	2.45
		a_1	13.08
		$b_1 \times 10$	−2.43
		c_1	1.06

Parameters listed in the table are a_1, the starting time of shock; b_1, the memory of shock; c_1, the intensity of shock; m, the market potential; p, the innovation parameter; and q, the imitation parameter. Abbreviations: BM, Bass model; GBM, generalized Bass model.

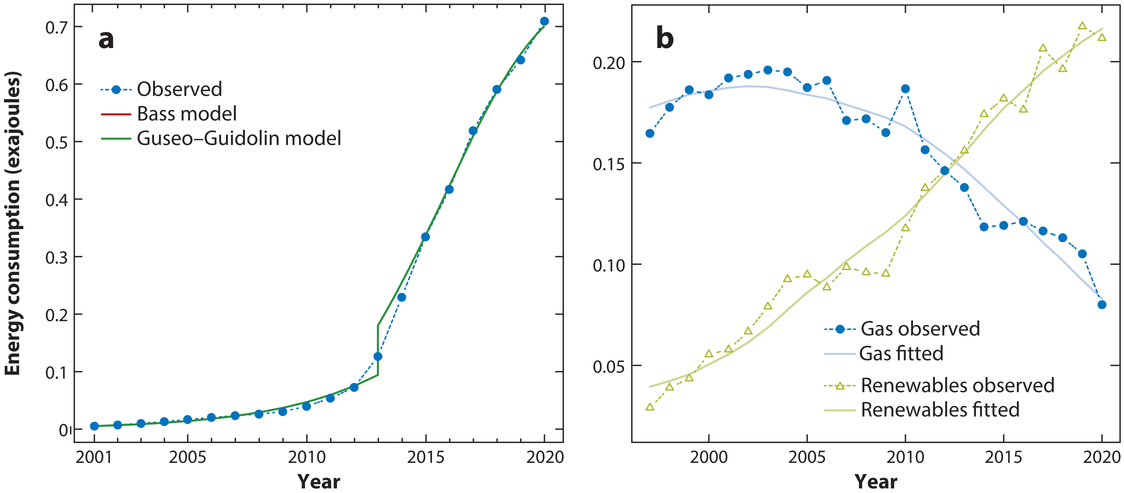

Figure 6

(*a*) Solar energy consumption in Japan: observed data and instantaneous fitted values with a Bass model (BM) and generalized Bass model (GBM). (*b*) Gas and renewables in Denmark: observed data and instantaneous fitted values with competition model.

suggests that the effect of this structured shock has been absorbed and the diffusion process has returned to its normal behavior. The fitted GBM outperforms the BM, with a reduction in residual sum of squares of 83% (**Figure 6*a***).

Finally, the results of modeling competition, according to System of Equation 23, between gas and renewables in Denmark are reported in **Table 3** and **Figure 6*b***. The within-imitation coefficients $q_1 + \delta = 0.053$ and $q_2 = 0.171$ show positive dynamics of growth, whereas the cross-imitation ones, $q_1 = -0.068$ and $q_2 - \gamma = -0.017$, are both negative, indicating a significant competitive relationship between gas and renewables, with a higher effect of renewables on gas.

7. FUTURE PERSPECTIVES AND CONCLUDING REMARKS

This review aims to provide a broad perspective on innovation diffusion processes starting from the standard model of marketing and management sciences, the BM, along with its extensions, empirical use, and criticism. The BM represents the foundation of a wide variety of

Table 3 Fit of competition model for gas and renewable energy in Denmark

Parameter	Estimate
$p_1 \times 10^2$	1.39
$p_2 \times 10^3$	3.05
$q_1 \times 10^2$	−6.84
$q_2 \times 10$	1.71
$\delta \times 10$	1.21
$\gamma \times 10$	1.89
m	12.44

Parameters listed in the table are $q_2 - \gamma$, the second product cross-imitation parameter; $q_1 + \delta$, the first product within-imitation parameter; m, the market potential; p_1, the first product innovation parameter; p_2, the second product innovation parameter; q_1, the first product cross-imitation parameter; and q_2, the second product within-imitation parameter.

developments, applications, and reviews. However, the diffusion of innovation goes beyond applications to commercial goods and durables: The diffusion of new ideas, social norms, and practices has led throughout history to the reshaping of human communities. In this sense, the BM depicts a part of the story because, due to its hypotheses, it can only deal with diffusions whose take-off has occurred and growth is ongoing because barriers to adoption have been removed.

We have aimed to offer a somewhat different point of view from other reviews of the BM, with a focus on three key issues: barriers to diffusion, centrality of word-of-mouth compared with external communication, and the role of policy interventions to promote critical diffusions. Consequently, we followed two main directions. The first, based on the BM literature, highlighted the centrality of the GBM as a general tool for monitoring diffusion processes subjected to structured shocks; the concept of market potential as a dynamic, rather than static, construct; and competition between innovations/technologies as crucial elements in critical diffusions. The second direction relied on ideas of mathematical epidemiology to highlight the variety and the policy implications of pure word-of-mouth processes. This allowed us to recall some principles that policy-makers should consider when facing critical diffusions: the threshold principle, whereby diffusions where the effects of word-of-mouth are moderated by removal of active spreaders might not take off; the incomplete contagion principle, whereby even if the diffusion takes off, the market will never saturate; the effects of behavioral heterogeneities, whereby diffusions might be severely slowed down in populations with large proportions of slow adopters; and the central role of behavioral variables in determining whether a diffusion is possible and how to address incentives.

As a limitation of this work, we mostly focused on simple deterministic diffusion models, represented through ordinary differential equations. Such models are appropriate to describe the mean behavior of the true underlying stochastic processes in large populations (indeed, most applications of innovation diffusion models consider large communities, typically nationwide) but could be poor in very early phases of diffusion, where randomness is crucial.

In the past two decades, the theory of social networks has been used in essentially every field of socio-dynamics (Pastor-Satorras et al. 2015), including diffusion processes. Complex social network theory allows detailed representations of systems of relationships arising in the social space (Newman 2018, Pastor-Satorras et al. 2015). In this perspective, the increasing availability of data at the individual level owing, for example, to the enormous amount of information collected from social networks (big data) may support finer modeling of adoption behavior, allowing a better understanding of diffusion dynamics. A possible achievement in this sense could be the use of the aggregate diffusion models proposed in this review on the wide range of adoption curves available from social media (stratified by type of platform, space, age, etc.), to identify common patterns and differences between individuals. Taking advantage of new data availability, we believe that the aggregate models proposed in this review, which have proven so useful despite their often simplified structure, will still represent a necessary reference point in the modeling of diffusion processes in time.

DISCLOSURE STATEMENT

The authors are not aware of any affiliations, memberships, funding, or financial holdings that might be perceived as affecting the objectivity of this review.

ACKNOWLEDGMENTS

We warmly thank two anonymous reviewers for constructive comments and criticisms that greatly improved the article, and for having pointed out to us a number of relevant articles.

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